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Théorie de Hodge mixte et variétés des représentations des groupes fondamentaux des variétés algébriques complexes

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Résumé

La théorie de Hodge mixte de Deligne fournit des structures supplémentaires sur les groupes de cohomologie des variétés algébriques complexes. Depuis, des structures de Hodge mixtes ont été construites sur les groupes d'homotopie rationnels de telles variétés par Morgan et Hain.
Dans cette lignée, nous construisons des structures de Hodge mixtes sur des invariants associés aux représentations linéaires des groupes fondamentaux des variétés algébriques complexes lisses. Le point de départ est la théorie de Goldman et Millson qui relie la théorie des déformations de telles représentations à la théorie des déformations via les algèbres de Lie différentielles graduées. Ceci a été relu par P. Eyssidieux et C. Simpson dans le cas des variétés kählériennes compactes. Dans le cas non compact, et pour des représentations d'image finie, Kapovich et Millson ont construit seulement des graduations non canoniques.
Pour construire des structures de Hodge mixtes dans le cas non compact et l'unifier avec le cas compact traité par Eyssidieux-Simpson, nous ré-écrivons la théorie de Goldman-Millson classique en utilisant des idées plus modernes de la théorie des déformations dérivée et une construction d'algèbres L_∞ due à Fiorenza et Manetti. Notre structure de Hodge mixte provient alors directement du H^0 d'un complexe de Hodge mixte explicite, de façon similaire à la méthode de Hain pour le groupe fondamental, et dont la fonctorialité apparaît clairement.

Mots-clés Géométrie algébrique complexe, Théorie de Hodge, Groupes fondamentaux, Variétés des représentations, Théorie des déformations formelles, Algèbres L_∞.

Abstract

The mixed Hodge theory of Deligne provides additional structures on the cohomology groups of complex algebraic varieties. Since then, mixed Hodge structures have been constructed on the rational homotopy groups of such varieties by Morgan and Hain.
In this vein, we construct mixed Hodge structures on invariants associated to linear representations of fundamental groups of smooth complex algebraic varieties. The starting point is the theory of Goldman and Millson that relates the deformation theory of such representations to the deformation theory via differential graded Lie algebras. This was reviewed by P. Eyssidieux and C. Simpson in the case of compact Kähler manifolds. In the non-compact case, and for representations with finite image, Kapovich and Millson constructed only non-canonical gradings.
In order to construct mixed Hodge structures in the non-compact case and unify it with the compact case treated by Eyssidieux-Simpson, we re-write the classical Goldman-Millson theory using more modern ideas from derived deformation theory and a construction of L_∞ algebras due to Fiorenza and Manetti. Our mixed Hodge structure comes then directly from the H^0 of an explicit mixed Hodge complex, in a similar way as the method of Hain for the fundamental group, and whose functoriality appears clearly.

Keywords Complex algebraic geometry, Hodge theory, Fundamental groups, Representation varieties, Formal deformation theory, L_∞ algebras.

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Introduction (version française)

A Contexte

A.1 Théorie de Hodge

La théorie de Hodge a pour but l'étude des structures additionnelles présentes sur les invariants topologiques usuels des variétés kählériennes compactes et des variétés algébriques complexes reflétant leur nature analytique ou algébrique complexe. Le premier exemple d'interaction entre la topologie algébrique et la structure complexe de telles variétés a été étudié par Hodge.

Théorème (Hodge)**.** *Si X est une variété kählérienne compacte, sa cohomologie à coefficients complexes se décompose en chaque degré n en une somme directe*

$$H^n(X,\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}$$

où la conjugaison complexe échange $H^{p,q}(X)$ et $H^{q,p}(X)$.

Avec ce théorème on obtient facilement des restrictions sur le topologie possible d'une variété kählérienne compacte. Par exemple, la décomposition de Hodge en degré 1 implique que la dimension de $H^1(X,\mathbb{C})$ est toujours paire. Puisque ce dernier est aussi égal à

$$H^1(X,\mathbb{C}) = \mathrm{Hom}(\pi_1(X),\mathbb{C})$$

on en déduit que $\mathbb{Z}$ ne peut jamais être le groupe fondamental $\pi_1(X)$ d'une telle variété X. La question de décrire tous les groupes qui peuvent apparaître comme groupes fondamentaux de variétés kählériennes compactes (resp. de variétés complexes projectives lisses, resp. de variétés algébriques complexes lisses quasi-projectives) est connue sous le nom de *problème de Serre*. Il est toujours largement ouvert, bien que de nombreux types de restrictions soient connus. La plus simple est que ces groupes doivent être finiment présentables. Mais pour les variétés non compactes, au delà de la présentabilité finie, aucune de ces restrictions n'est aussi facile à décrire que pour les variétés compactes.

Cette structure sur la cohomologie a été abstraite en la notion de *structure de Hodge*. Depuis, les structures de Hodge ont été reconnues comme un outil utile pour étudier la topologie des variétés complexes et leurs modules. Nous nous référons aux livres de Voisin [Voi02] et Peters-Steenbrink [PS08] pour la théorie de Hodge, et à [ABC$^+$96] pour le problème de Serre.

A.2 Théorie de Hodge mixte

La notion de structure de Hodge pure a ensuite été généralisée par P. Deligne [Del71a], [Del71b], [Del74]. Il montre que la cohomologie d'une variété non compacte se comporte comme une extension itérée de groupes de cohomologie de variétés compactes, provenant d'une compactification par un diviseur à croisements normaux, et définit ceci comme étant une structure de Hodge mixte.

Définition (Def. 2.15)**.** Une *structure de Hodge mixte* sur un espace vectoriel de dimension finie K sur un sous-corps $\mathbf{k} \subset \mathbb{R}$ est la donnée d'une filtration croissante W de K appelée la *filtration par le poids* et d'une filtration décroissante F de $K_{\mathbb{C}}$ appelée la *filtration de Hodge* telles que pour tout $k \in \mathbb{Z}$ la partie graduée de poids k

$$\mathrm{Gr}_k^W(K) := W_k K / W_{k-1} K$$

avec la filtration induite F sur $\mathbb{C}$ forme une structure de Hodge pure de poids k, i.e. se décompose en une somme directe de termes $K^{p,q}$ pour $p + q = k$ avec la conjugaison complexe qui échange $K^{p,q}$ et $K^{q,p}$ et avec F qui filtre par rapport à p.

Depuis, des structures de Hodge mixtes ont été construites sur beaucoup d'autres invariants topologiques des variétés algébriques et ont été utilisées pour restreindre leur topologie. Toutes ces méthodes utilisent la théorie de l'homotopie rationnelle pour extraire des informations sur le type d'homotopie d'une variété à partir de son algèbre de formes différentielles. Voir le livre de Griffiths-Morgan [GM81] pour ces questions. Pour travailler sur le corps des rationnels on doit construire une algèbre différentielle graduée commutative définie sur $\mathbb{Q}$ qui calcule l'algèbre de cohomologie à coefficients rationnels d'une variété donnée. Cela n'est pas facile car sur le complexe de cochaînes singulières usuel le cup-produit n'est pas commutatif.

Pour des raisons de théorie de Hodge, il est montré par Deligne, Griffiths, Morgan et Sullivan [DGMS75] que le type d'homotopie réelle d'une variété kählérienne compacte est assez simple : le groupe fondamental réel est déterminé par l'algèbre de cohomologie.

En combinant ces idées d'homotopie rationnelle et de théorie de Hodge mixte, J. Morgan [Mor78] a construit des structures de Hodge mixtes sur les groupes d'homotopie d'une variété algébrique lisse complexe. Ceci peut être utilisé pour exhiber le premier exemple d'un groupe de présentation finie qui n'est pas le groupe fondamental d'une variété algébrique, un groupe avec deux générateurs x, y et une relation donnée par un commutateur itéré

$$[x, [x, [\cdots, [x, y] \cdots]]]$$

de longueur au moins cinq. Pour un tel commutateur de longueur au moins trois, ceci n'est pas le groupe fondamental d'une variété kählérienne compacte par [DGMS75]. Cependant la construction de Morgan n'est pas complètement fonctorielle et a été suivie d'une correction [Mor86].

Une autre approche a été donnée par R. Hain [Hai87] utilisant la construction d'un certain complexe de cochaînes dont la cohomologie donne directement et fonctoriellement les groupes d'homotopie rationnels. Pour construire des structures de Hodge mixtes sur sa cohomologie, il suffit de construire sur ce complexe une structure de *diagramme de Hodge mixte* (voir la section 2.1.2 et la Definition 2.27) et de vérifier les axiomes correspondant.

Ces méthodes ont aussi été revues par V. Navarro Aznar [Nav87]. Il construit des diagrammes de Hodge mixtes *fonctoriels* qui calculent la cohomologie d'une variété algébrique complexe. Ceci doit être associé à des méthodes d'algèbre homotopique pour comprendre la catégorie homotopique des diagrammes de Hodge mixtes et améliorer la théorie de Deligne, Morgan et Hain. Voir par exemple le travail de J. Cirici [Cir15] et ses collaborateurs.

Dans cette lignée, dans cette thèse nous étudions la théorie de Hodge mixte des représentations linéaires des groupes fondamentaux des variétés kählériennes compactes et des variétés algébriques.

A.3 La théorie de Goldman et Millson

Fixons une variété complexe X qui est soit kählérienne compacte, soit algébrique lisse. Dans les deux cas son groupe fondamental est finiment présentable. Fixons un point base $x \in X$. Soit G un groupe algébrique linéaire sur un sous-corps $\mathbf{k} \subset \mathbb{R}$ d'algèbre de Lie $\mathfrak{g}$.

Définition (Sect. 1.1.2 et Thm. 1.26)**.** L'ensemble des représentations de $\pi_1(X,x)$ dans $G(\mathbf{k})$, i.e. de morphismes de groupes

$$\rho : \pi_1(X,x) \longrightarrow G(\mathbf{k}),$$

a la structure d'ensemble des points sur $\mathbf{k}$ d'un schéma affine qu'on note $\mathrm{Hom}(\pi_1(X,x),G)$. On l'appelle la *variété des représentations* de $\pi_1(X,x)$ dans G. On note $\widehat{\mathcal{O}}_\rho$ le complété de l'anneau local de la variété des représentations en un point ρ.

L'anneau local $\widehat{\mathcal{O}}_\rho$ sera l'un de nos principaux objets d'étude. C'est en effet l'espace de module formel pour les déformations infinitésimales de ρ et son type de singularité décrit les possibilités de déformer des représentations et d'étendre des déformations données. Son étude a été faite par Goldman et Millson [GM88] quand X est une variété kählérienne compacte.

Théorème (Goldman-Millson)**.** *Si X est une variété kählérienne compacte et $\rho : \pi_1(X,x) \to G(\mathbb{R})$ a son image contenue dans un sous-groupe compact alors $\widehat{\mathcal{O}}_\rho$ a une présentation quadratique.*

Ceci peut être vu comme une version de [DGMS75] pour les représentations du groupe fondamental. Le résultat exprime que la variété des représentations a des singularités assez simples en ρ et donc que le groupe fondamental ne peut pas être trop compliqué.

Décrivons brièvement les méthodes de preuve. À ρ est associé, via la monodromie de sa représentation adjointe, un système local d'algèbres de Lie de fibre $\mathfrak{g}$. L'algèbre des formes différentielles réelles à valeurs dans ce système local est une algèbre de Lie différentielle graduée (Definition 1.13) qu'on note L. La construction principale relie le foncteur des déformations associé à ρ au foncteur des déformations associé à L.

Définition (Def. 1.17)**.** Le *foncteur de déformation* d'une algèbre de Lie DG L sur un corps $\mathbf{k}$ de caractéristique zéro est le foncteur Def_L donnée pour une algèbre locale artinienne $(A, \mathfrak{m}_A)$ sur $\mathbf{k}$ par le quotient de l'ensemble des *éléments de Maurer-Cartan* de $L \otimes \mathfrak{m}_A$

$$\mathrm{MC}(L \otimes \mathfrak{m}_A) := \left\{ x \in L^1 \otimes \mathfrak{m}_A \;\middle|\; d(x) + \frac{1}{2}[x, x] = 0 \right\}$$

par l'action de $L^0 \otimes \mathfrak{m}_A$ par *transformations de jauge* (Definition 1.16).

Le principe fondamental de la théorie des déformations énonce que *tout problème de déformation est contrôlé par une algèbre de Lie DG* (i.e. le foncteur de déformation associé est isomorphe à un foncteur Def_L comme ci-dessus) et que *deux algèbres de Lie DG quasi-isomorphes ont des foncteurs de déformation isomorphes.* Ceci a été inspiré par une lettre de Deligne [Del86] aux auteurs.

Dans la situation de Goldman-Millson, par la théorie de Hodge, L est quasi-isomorphe à une algèbre de Lie DG avec différentielle nulle — cette propriété, appelée *formalité*, est la même qu'utilisée dans [DGMS75] — pour laquelle le foncteur de déformation prend une forme simplifiée et permet de donner une description simple de $\widehat{\mathcal{O}}_\rho$.

Par formalité encore, en utilisant la théorie de Hodge non abélienne, le théorème de Goldman-Millson a été étendu par C. Simpson [Sim92, § 2] au cas des représentations complexes semi-simples.

Ensuite, la description de $\widehat{\mathcal{O}}_\rho$ pour des cas de représentations de groupes fondamentaux de variétés algébriques non compactes a été effectuée par Kapovich et Millson [KM98].

Théorème (Kapovich-Millson)**.** *Si X est une variété algébrique lisse et $\rho : \pi_1(X, x) \to G(\mathbb{R})$ est d'image finie, alors $\widehat{\mathcal{O}}_\rho$ a une présentation homogène à poids avec des générateurs de poids $1, 2$ et des relations de poids $2, 3, 4$.*

Ceci est obtenu en étudiant la théorie de Hodge mixte de L : sa cohomologie a une structure de Hodge mixte avec seuls poids possibles $1, 2$ sur H^1 et $2, 3, 4$ sur H^2. Au lieu d'utiliser la formalité, ils utilisent le travail de Morgan et remplacent L à quasi-isomorphisme près par une algèbre de Lie DG M portant une structure de Hodge mixte et étudient les conséquence de la graduation par le poids sur le foncteur de déformation de M.

Dans leur article, ils obtiennent de nouveaux exemples de groupes de présentation finie qui ne sont pas des groupes fondamentaux de variétés algébriques complexes lisses, par une étude complète de leur combinatoire et de leurs représentations finies.

Le travail de Goldman-Millson a été revu par P. Eyssidieux et C. Simpson [ES11] et interprété en termes de théorie de Hodge mixte. Supposons que ρ soit la monodromie d'une variation de structure de Hodge (Definition 3.1 et Definition 3.8). A l'intérieur de $\mathrm{Hom}(\pi_1(X, x), G)$ se trouve l'orbite Ω_ρ de ρ sous l'action de G par conjugaison, qui est un sous-schéma réduit. Son germe formel en ρ est un schéma formel $\widehat{\Omega}_\rho$ défini par un idéal $\mathfrak{j} \subset \widehat{\mathcal{O}}_\rho$.

Théorème (Eyssidieux-Simpson)**.** *Si X est une variété kählérienne compacte et ρ est la monodromie d'une variation de structure de Hodge complexe polarisée sur X, alors $\widehat{\mathcal{O}}_\rho$ a une structure de Hodge mixte complexe fonctorielle. La filtration par le poids est indexée en degré négatifs et est donnée par les puissances de l'idéal $\mathfrak{j}$. La partie graduée de poids zéro est $\widehat{\mathcal{O}}_\rho/\mathfrak{j}$, l'anneau local formel de $\widehat{\Omega}_\rho$.*

Dans leur article, ceci est utilisé pour construire des variations de structure de Hodge mixte intéressantes sur X. C'est aussi un outil important dans la preuve de la conjecture de Shafarevich [EKPR12] sur les revêtements universels des variétés projectives lisses dont le groupe fondamental admet une représentation linéaire fidèle. De telles structures de Hodge mixtes associées à une représentation du groupe fondamental de X ont aussi été construites par Hain dans [Hai98].

A.4 Théorie des déformations dérivée

L'utilisation des algèbres de Lie DG en théorie des déformations, et la philosophie selon laquelle *en caractéristique zéro, tout problème de déformation est contrôlé par une algèbre de Lie DG*, s'est beaucoup développée depuis le travail original de Goldman-Millson et la lettre de Deligne. D'une part cela a été appliqué à beaucoup d'autres situations, voir par exemple les notes de Kontsevich [Kon94]. D'autre part, plusieurs personnes ont construit une équivalence entre la catégorie des algèbres de Lie DG à quasi-isomorphismes près et une catégorie à définir des problèmes de déformation. Il faut citer les articles originaux de Manetti [Man02], Hinich [Hin01], le travail de Pridham [Pri10] qui unifie les deux, le tout culminant dans sa forme la plus générale dans la théorie de Lurie [Lur11]. Ainsi cette philosophie est maintenant appelée le théorème de Pridham-Lurie. On se réfère aussi aux notes de B. Toën [Toë17].

Il est donc très intéressant de relire la théorie originale de Goldman et Millson à l'aide de ces outils nouveaux et puissants. Le fait particulièrement important pour nous est que le foncteur de déformation d'une algèbre de Lie DG L est représenté (sous l'hypothèse que $H^n(L) = 0$ pour $n \leq 0$) par une algèbre locale complète qui est obtenue comme le H^0 d'une certain complexe très explicite $\mathscr{C}(L)$, fonctoriel, invariant par quasi-isomorphismes. On l'appelle aussi la *construction bar* sur L.

Cette construction n'est pas possible à comprendre sans en appeler à la théorie des déformations dérivée. Cela nécessite au moins d'étendre les foncteurs de déformation depuis les algèbres artiniennes vers certaines catégories d'anneaux artiniens DG, à valeurs non plus dans les ensembles mais dans les groupoïdes ou dans les ensembles simpliciaux. Ces idées apparaissent pour la première fois dans une lettre de Drinfeld [Dri88]. Ces principes n'étaient pas connus à l'époque du premier travail de Goldman et Millson. Encore une fois, par formalité, tout ceci se simplifie beaucoup quand on travaille dans le cas compact mais n'est pas du tout trivial dans le cas général.

De plus, d'autres outils ont été développés pour comprendre la catégorie homotopique des algèbres de Lie DG (c'est à dire la catégorie des algèbres de Lie DG à quasi-isomorphismes près) : les algèbres L_∞. Le bon cadre est la théorie des opérades pour laquelle on se réfère au livre de Loday-Valette [LV12].

Brièvement, les algèbres L_∞ sont des versions affaiblies des algèbres de Lie DG équipées d'opérations supérieures et dans lesquelles l'identité de Jacobi a lieu seulement *à homotopie près* donnée par les opérations supérieures, satisfaisant elles-mêmes des lois de cohérence supérieures. Les algèbres de Lie DG sont exactement les algèbres L_∞ avec les opérations supérieures nulles. Puisque les algèbres L_∞ ont naturellement un foncteur de déformation associé qui étend celui des algèbres de Lie DG et qui est invariant par quasi-isomorphismes, et puisqu'un quasi-isomorphisme entre algèbres L_∞ admet automatiquement un quasi-isomorphisme inverse, ceci forme une catégorie très pratique pour étudier la théorie des déformations.

Bien que l'on sache par la théorie très abstraite que tout problème de déformation est contrôlé par une algèbre de Lie DG, ce qui est compte est de trouver la bonne avec des bonnes propriétés qui permettent de mieux comprendre le problème de déformation donné. Puisque les algèbres L_∞ sont des objets plus fins, il se peut qu'un problème de déformation soit contrôlé plus naturellement par une algèbre L_∞ que par une algèbre de Lie DG. Beaucoup d'exemples de problèmes de déformation contrôlés par des algèbres L_∞ ont été étudiés par M. Manetti et ses collaborateurs et on renvoie aux notes [Man04].

B Résultats

B.1 Travail préliminaire

Notre premier travail préliminaire, qui est maintenant un article publié [Lef17], est reproduit dans l'appendice A.

Ceci concerne des cas particuliers du théorème de Kapovich-Millson où on trouve plus de restrictions sur les poids possibles sur $\widehat{\mathcal{O}}_\rho$ de telle façon qu'il se comporte comme dans le cas compact, obtenu en analysant la preuve et d'où ces restrictions proviennent.

Théorème A (Thm. A.2)**.** *Soit X une variété complexe lisse quasi-projective et soit $\rho : \pi_1(X, x) \to G(\mathbb{R})$ une représentation d'image finie. Supposons que le revêtement fini $Y \to X$ associé au sous-groupe* $\mathrm{Ker}(\rho) \subset \pi_1(X, x)$ *ait une compactification lisse $\overline{Y}$ avec premier nombre de Betti $b_1(\overline{Y}) = 0$. Alors $\widehat{\mathcal{O}}_\rho$ a une présentation quadratique.*

La motivation principale vient du cas des complémentaires d'arrangements projectifs d'hyperplans pour la représentation triviale, pour lequel cette notion est connue sous le nom de 1-formalité, voir [DPS09], [PS09]. Ceci devrait aussi être relié à l'énoncé que *la pureté impliqué la formalité*, voir par exemple le travail [Dup15] dont nous n'étions pas informés.

Dans ce même article on donne aussi des exemples où l'hypothèse du théorème s'applique pour *toutes* les représentations finies. On en trouve parmi les familles de variétés abéliennes, en lien avec les divers résultats de rigidité, et parmi les espaces hermitiens localement symétriques, en lien avec la propriété (T) de Kazhdan.

B.2 Résultat principal

Notre but principal est d'étendre le résultat d'Eyssidieux-Simpson au cas non compact, en construisant une structure de Hodge mixte fonctorielle sur $\widehat{\mathcal{O}}_\rho$ et en retrouvant le résultat de Kapovich-Millson.

Ceci amène plusieurs difficultés importantes. D'un côté le travail d'Eyssidieux-Simpson utilise fortement la géométrie kählérienne, les laplaciens pour les formes différentielles, la formalité, et est impossible à adapter directement au cas non compact. De l'autre côté le travail de Kapovich-Millson repose sur la théorie de Morgan qui utilise des modèles minimaux pour les diagrammes de Hodge mixtes et, même si on pourrait améliorer la graduation sur $\widehat{\mathcal{O}}_\rho$ en une structure de Hodge mixte, celle-ci ne serait pas du tout fonctorielle et n'aurait pas le bon comportement.

On doit donc trouver une preuve complètement différente. Au passage, on souhaite que la structure de Hodge mixte sur $\widehat{\mathcal{O}}_\rho$ provienne dans les deux cas directement et fonctoriellement d'un diagramme de Hodge mixte approprié dont la construction dépende de la situation géométrique.

Donnons d'abord le résultat. Fixons un corps $\mathbf{k}$ qui est soit un sous-corps de $\mathbb{R}$ soit le corps $\mathbb{C}$, nous permettant de parler en même temps des structures de Hodge mixtes usuelles et des structures de Hodge mixtes complexes. On rappelle que G est un groupe algébrique linéaire sur $\mathbf{k}$.

Théorème B (Théorème principal)**.** *Soit X une variété kählérienne compacte et soit $\rho : \pi_1(X,x) \to G(\mathbf{k})$ la monodromie d'une variation de structure de Hodge polarisée définie sur $\mathbf{k}$ sur X. Alors il y a une structure de Hodge mixte fonctorielle définie sur $\mathbf{k}$ sur $\widehat{\mathcal{O}}_\rho$ avec poids négatifs et dont la partie graduée de poids zéro est l'anneau local formel de l'orbite de ρ.*

Si X est une variété complexe lisse quasi-projective et $\rho : \pi_1(X,x) \to G(\mathbf{k})$ est une représentation d'image finie, alors $\widehat{\mathcal{O}}_\rho$ a une structure de Hodge mixte fonctorielle à poids négatifs. Les poids induits sur l'espace cotangent sont $-1, -2$.

Pour le moment, il n'est pas prouvé que notre structure de Hodge mixte sur $\widehat{\mathcal{O}}_\rho$ dans le cas compact est la même que celle construite par Eyssidieux-Simpson, bien qu'il y ait de fortes indications pour : elle se comporte de la même façon, a la même description sur l'espace cotangent, et la même partie graduée de poids zéro. Dans le cas non compact on ne retrouve pas complètement le résultat de Kapovich-Millson : la structure de Hodge mixte sur l'espace cotangent peut être scindée sur $\mathbb{C}$ et des éléments de base peuvent être relevés en des générateurs homogènes à poids de $\widehat{\mathcal{O}}_\rho$, retrouvant ainsi les générateurs de poids $1, 2$, mais jusque là nous ne retrouvons pas les relations homogènes à poids avec nos méthodes. On aimerait les retrouver d'une façon canonique à partir d'un idéal de relations portant une structure de Hodge mixte avec les poids $2, 3, 4$ et nous sommes dans l'incapacité d'obtenir ceci.

B.3 Plan de preuve

Indiquons maintenant les théorèmes principaux nécessaires à la preuve. Dans la théorie de Goldman-Millson l'algèbre de Lie DG contrôlante L est équipée d'une augmentation $\varepsilon_x : L \to \mathfrak{g}$ qui est simplement l'évaluation des formes de degré zéro au point base x. Dans le travail d'Eyssidieux-Simpson est introduit un *foncteur de déformation augmenté* (Definition 1.21), qui est une petite variation du foncteur de déformation Def_L, et il est expliqué qu'il s'agit du bon objet à considérer pour contrôler la théorie de déformations de ρ et pour comprendre son orbite.

Dans le cas compact, L est une algèbre de Lie DG de formes différentielles à coefficients dans une variation de structure de Hodge, donc par le travail de Zucker [Zuc79] a une structure de diagramme de Hodge mixte d'algèbres de Lie. Dans la situation de Kapovich-Millson, L est obtenue à partir de l'algèbre des formes différentielles sur le revêtement fini correspondant à $\mathrm{Ker}(\rho)$, sur lequel la représentation tirée en arrière est triviale. Donc, en suivant leurs idées et en les ré-écrivant avec les constructions de diagrammes de Hodge mixtes fonctoriels de Navarro Aznar, on trouve encore un diagramme de Hodge mixte d'algèbres de Lie qui est quasi-isomorphe à L. Résumons les structures qu'on obtient par les situations géométriques.

Théorème C (Chapitre 3)**.** *Dans toutes les situations du théorème ci-dessus, la théorie des déformations de ρ est contrôlée par un* diagramme de Hodge mixte augmenté d'algèbres de Lie *(Definition 2.33 et Definition 2.37) qui est fonctoriel à quasi-isomorphisme près.*

Puis on souhaite appliquer la construction bar appropriée provenant de la théorie des déformations dérivée à un tel diagramme L et montrer que ceci définit un diagramme de Hodge mixte. Ainsi on obtiendra une structure de Hodge mixte fonctorielle sur un objet qui représente le foncteur de déformation de L, qui par le lemme de Yoneda est canoniquement isomorphe à $\widehat{\mathcal{O}}_\rho$.

Cependant, malgré nos efforts, nous avons été dans l'incapacité d'accéder à $\widehat{\mathcal{O}}_\rho$ en utilisant cette stratégie naïve puisqu'on doit réellement travailler avec L et son augmentation ε_x. Les possibilités suivantes sont naturelles :

1. On peut travailler avec L tout entier puis plus tard essayer d'extraire l'information provenant de l'augmentation. Cependant L n'a pas $H^0(L) = 0$ ce qui signifie que son foncteur de déformation n'est pas représentable. De plus, la construction bar ne se comporte pas bien si $H^0(L) \neq 0$ (les éléments de degrés 0 de L produisent des éléments de degré négatif dans $\mathscr{C}(L)$).
2. On peut travailler avec $L' := \mathrm{Ker}(\varepsilon_x) \subset L$. C'est bien une algèbre de Lie DG avec $H^0(L') = 0$. Mais ce n'est plus un diagramme de Hodge mixte, puisque les axiomes de diagrammes de Hodge mixtes sont très forts et on ne peut pas a priori y considérer des noyaux.
3. À la place, on peut travailler avec le cône de ε_x. C'est l'opération naturelle dans les diagrammes de Hodge mixtes qui remplace le noyau et il a bien $H^0 = 0$, étant en fait quasi-isomorphe à L'. Cependant ce n'est plus une algèbre de Lie DG.

La solution à ce problème, nous l'avons trouvée en travaillant avec des algèbres L_∞. Le cône d'un morphisme entre algèbres de Lie DG a été étudié par Fiorenza-Manetti [FM07]. Ils montrent que le cône a une structure naturelle d'algèbre L_∞ et décrivent le foncteur de déformation associé. Nous faisons la remarque très simple mais fondamentale (pour nous) que lorsqu'on l'applique à l'augmentation ε le foncteur de déformation associé est le même que le foncteur de déformation augmenté introduit indépendamment par Eyssidieux-Simpson.

Lemme D (Observation fondamentale, Lem. 1.55, combiner avec Thm. 1.54)**.** *Le foncteur de déformation augmenté d'une algèbre de Lie DG augmentée $\varepsilon : L \to \mathfrak{g}$ introduit par Goldman-Millson et par Eyssidieux-Simpson est le foncteur de déformation associé à une structure d'algèbre L_∞ sur le cône de ε étudiée par Fiorenza-Manetti.*

Ceci nous permet de mener à bien notre stratégie de preuve. Les opérations supérieures d'algèbre L_∞ sur le cône ont des formules algébriques très explicites et c'est un calcul direct de vérifier leur compatibilité avec la structure de diagramme de Hodge mixte. On appelle diagramme de Hodge mixte d'algèbres L_∞ l'objet qui en résulte.

Théorème E (Sect. 2.2.1 et Thm. 2.38)**.** *Soit $\varepsilon : L \to \mathfrak{g}$ un diagramme de Hodge mixte augmenté d'algèbres de Lie provenant du Théorème C. Les opérations d'algèbre L_∞ de Fiorenza-Manetti sur le cône C de ε donnent à C la structure de* diagramme de Hodge mixte d'algèbres L_∞ *(Definition 2.34).*

Puis on applique la construction bar (foncteur $\mathscr{C}$) au sens des algèbres L_∞. Ceci donne naturellement une coalgèbre DG de laquelle on extrait (via son H^0 puis en dualisant) une algèbre locale complète qui, par les principes de théorie des déformations dérivée, représente le foncteur de déformation de l'algèbre L_∞. Donc on doit montrer que $\mathscr{C}$ peut être appliqué au diagramme de Hodge mixte C et qu'on obtient un diagramme de Hodge mixte de coalgèbres. Ceci est très proche de la *construction bar sur des diagrammes de Hodge mixtes* (d'algèbres commutatives, ou de modules sur elles, avec beaucoup de variations) de Hain, utilisée dans [Hai87] et dans de nombreux autres articles.

Théorème F (Sect. 2.2.2 et Thm. 2.44)**.** *Le foncteur $\mathscr{C}$ peut être appliqué au diagramme de Hodge mixte d'algèbres L_∞ C du Théorème E et donne un diagramme de Hodge mixte de coalgèbres.*

De ceci, on extrait directement une structure de Hodge mixte sur son H^0 qui est invariante par quasi-isomorphismes. Ceci donne la structure de Hodge mixte fonctorielle sur $\widehat{\mathcal{O}}_\rho$ dans toutes les situations du Théorème B.

B.4 Perspectives

Notre stratégie de preuve a été développée avec l'objectif constant de séparer les constructions géométriques de diagrammes de Hodge mixtes et la machinerie algébrique qui nous donne la structure de Hodge mixte. Nous pensons fortement qu'elle nous aidera à construire une structure de Hodge mixte sur $\widehat{\mathcal{O}}_\rho$ dans des cas non compacts plus généraux : quand ρ est la monodromie d'une variation de structure de Hodge mixte, et même quand X est singulière. Combiné avec une étude plus approfondie de théorie des groupes, ceci pourrait aboutir à des nouvelles restrictions sur le problème de Serre.

C Organisation du travail

C.1 Plan

Ce travail est organisé comme suit. On le divise en trois chapitres, chacun formant une partie de la preuve de notre Théorème B. Puisque le travail préliminaire est indépendant et est la reproduction d'un article publié, on le place en appendice A. Aussi, on reporte toutes les constructions géométriques au chapitre 3, développant notre machinerie d'abord au travers des chapitres 1 et 2.

Donc, le chapitre 1 est centré sur la théorie de Goldman-Millson et la théorie des déformations mais sans théorie de Hodge. Dans sa première partie on revoit tout le matériel nécessaire pour comprendre proprement la théorie de Goldman-Millson, du point de vue classique. La section 1.1.1 est purement algébrique et catégorique. On y introduit soigneusement le foncteur de déformation augmenté d'une algèbre de Lie DG. Dans la section 1.1.2 on introduit la variété des représentations et son étude est reliée à des constructions géométriques. Dans la deuxième partie, on expose la théorie des algèbres L_∞ et les théorèmes de théorie des déformations dont on a besoin. Puisque ceci est technique et puisque nous avons besoin des détails, on dédie toute la section 1.2.1 à l'introduction des algèbres L_∞. Puis dans la section 1.2.2 on décrit la structure d'algèbre L_∞ de Fiorenza-Manetti. Notre unique contribution est la très simple observation du Lemme D. Enfin dans la section 1.2.3 on étudie plus en détail le foncteur de déformation d'une algèbre L_∞ et on extrait de la littérature le théorème de pro-représentabilité qu'on veut appliquer.

Puis le chapitre 2 est dédié à la théorie de Hodge. Dans sa première partie on y expose soigneusement toutes les définitions dont on a besoin : les structures de Hodge mixtes dans la section 2.1.1, les diagrammes de Hodge mixtes dans la section 2.1.2. À chaque étape on montre comment ces définitions se comportent si on ne travaille pas seulement avec des espaces vectoriels mais avec d'autres types d'algèbres. La deuxième partie contient le cœur de ce travail. Dans la section 2.2.1 on donne la définition de diagramme de Hodge mixte d'algèbres L_∞ et on démontre le Théorème E. Dans la section 2.2.2 on démontre le Théorème F et on étudie cette construction.

Enfin dans le chapitre 3 on étudie beaucoup de situations géométriques différentes dans lesquelles on construit des diagrammes de Hodge mixtes augmentés d'algèbres de Lie contrôlants, démontrant le Théorème C. On sépare le cas compact et non compact. Dans le cas compact, pour une représentation à valeurs dans une variation de structure de Hodge réelle, une telle construction est immédiate. Elle sert donc de modèle de preuve et d'application de notre méthode et est écrite en détail dans la section 3.1.1. Il est ensuite facile d'adapter la méthode à plusieurs variations de ce cas, ce que nous faisons dans la section 3.1.2. De même dans le cas non compact, on prend pour modèle détaillé les représentations réelles, dans la section 3.2.1, et on y ajoute plus tard la structure rationnelle, section 3.2.2. Pour traiter du cas non compact on expose brièvement la construction de Navarro Aznar et on l'adapte à nos besoins.

C.2 Conventions et notations

Les variétés que nous étudions sont soit kählériennes compactes (ceci inclut la classe des variétés algébriques complexes lisses projectives) soit algébriques complexes lisses quasi-projectives. Elles sont bien entendu connexes. Ceci est le cadre naturel dans lequel la théorie de Hodge (resp. de Hodge mixte) s'applique le plus directement et où on peut directement citer la littérature existante.

On note toujours $\mathbf{k}$ un corps de caractéristique zéro. Dans les situations géométriques c'est la plupart du temps $\mathbb{Q}$, $\mathbb{R}$ ou $\mathbb{C}$.

On utilise partout l'abréviation *DG* pour *différentiel gradué*. Nos espaces vectoriels DG sur $\mathbf{k}$ sont aussi connus sous le nom de *complexes de cochaînes*. On note toujours la graduation par un indice en exposant (i.e. $V = \bigoplus V^n$) et la différentielle d est de degré $+1$ (i.e. $d : V^n \to V^{n+1}$).

On utilise le symbole $\simeq$ pour les isomorphismes et $\approx$ pour les équivalences faibles (quasi-isomorphismes entre espaces vectoriels DG, équivalences de catégories ou de foncteurs vers les groupoïdes). Les catégories sont notées avec des lettres grasses, par exemple : $\mathbf{Set}$, $\mathbf{Alg_k}$, $\mathbf{DG{-}Vect_k}$.

Introduction (English version)

A Context

A.1 Hodge theory

Hodge theory is concerned with the study of the additional structures on the usual topological invariants of compact Kähler manifolds and complex algebraic varieties reflecting their complex analytic and algebraic nature. The first example of interaction between the algebraic topology and the complex structure of such manifolds was studied by Hodge.

Theorem (Hodge)**.** *If X is a compact Kähler manifold, then its cohomology with complex coefficients decomposes in each degree n as a direct sum*

$$H^n(X,\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}$$

with the complex conjugation exchanging $H^{p,q}(X)$ and $H^{q,p}(X)$.

From this theorem one obtains easily restrictions on the possible topology of a compact Kähler manifold. For example, the Hodge decomposition in degree 1 implies that the dimension of $H^1(X,\mathbb{C})$ is always even. Since this also equals

$$H^1(X,\mathbb{C}) = \mathrm{Hom}(\pi_1(X),\mathbb{C})$$

then $\mathbb{Z}$ can never be the fundamental group $\pi_1(X)$ of such a manifold X. The question of describing all groups that can appear as fundamental groups of compact Kähler manifolds (resp. smooth complex projective varieties, resp. smooth complex quasi-projective algebraic varieties) is known as *Serre's problem*. It is still widely open, though many different kinds of restrictions are known. The most simple one is that these groups must be finitely presentable. But for non-compact varieties, besides finite presentability none of these restrictions is as easy to describe as the above one for compact varieties.

This structure on the cohomology has been abstracted into the notion of *Hodge structure*. Then, Hodge structures have proven to be a useful tool for studying the topology of complex varieties and their moduli. We refer to the books by Voisin [Voi02] and Peters-Steenbrink [PS08] for Hodge theory, and to [ABC$^+$96] for Serre's problem.

A.2 Mixed Hodge theory

The notion of pure Hodge structure was generalized by P. Deligne [Del71a], [Del71b], [Del74]. He shows that the cohomology groups of a non-compact algebraic variety behave like an iterated extension of cohomology groups of compact varieties, coming from of a compactification by a divisor with normal crossings, and defines this to be a mixed Hodge structure.

Definition (Def. 2.15)**.** A *mixed Hodge structure* on a finite-dimensional vector space K over a subfield $\mathbf{k} \subset \mathbb{R}$ is the data of an increasing filtration W of K called the *weight filtration* and a decreasing filtration F of $K_{\mathbb{C}}$ called the *Hodge filtration* such that for each $k \in \mathbb{Z}$ the graded part of weight k

$$\mathrm{Gr}_k^W(K) := W_k K / W_{k-1} K$$

with the induced filtration F over $\mathbb{C}$ forms a pure Hodge structure of weight k, i.e. decomposes as a direct sum of terms $K^{p,q}$ for $p+q=k$ with the complex conjugation exchanging $K^{p,q}$ and $K^{q,p}$ and with F filtering with respect to p.

Then, mixed Hodge structures have been constructed on many other topological invariants of algebraic varieties and used to restrict their topology. All these methods use rational homotopy theory to extract information about the homotopy type of a variety from its algebra of differential forms. See the book by Griffiths-Morgan [GM81] for these questions. To work over the rational numbers one needs to construct a commutative differential graded algebra defined over $\mathbb{Q}$ whose cohomology computes the rational cohomology algebra of a given variety. This is not easy since on the usual singular cochain complex the cup-product is not commutative.

For Hodge-theoretical reasons, the real homotopy type of compact Kähler manifolds is shown to be quite simple by Deligne, Griffiths, Morgan and Sullivan [DGMS75]: the real fundamental group is determined by the cohomology algebra.

Combining these ideas of rational homotopy and mixed Hodge theory, J. Morgan [Mor78] constructed mixed Hodge structures on the homotopy groups of a smooth complex algebraic variety. It can be used to exhibit the first example of a finitely presented group which is not the fundamental group of an algebraic variety, a group with two generators x, y and one relation given by an iterated commutator

$$[x, [x, [\cdots, [x, y] \cdots]]]$$

of length at least five. For such a commutator of length at least three, this is not the fundamental group of a compact Kähler manifold by [DGMS75]. The construction of Morgan, however, lacks some functoriality and is followed by a correction [Mor86].

Another approach was given by R. Hain [Hai87] using the construction of a certain cochain complex whose cohomology gives directly and functorially the rational homotopy groups. To construct mixed Hodge structures on its cohomology, it is enough to give this complex the structure of a *mixed Hodge diagram* (see section 2.1.2 and Definition 2.27) and check the corresponding axioms.

These methods were also reviewed by V. Navarro Aznar [Nav87]. He constructs *functorial* mixed Hodge diagrams computing the cohomology of a complex algebraic variety. This has to be associated with some methods of homotopical algebra to understand the homotopy category of mixed Hodge diagrams and improve the theory of Deligne, Morgan and Hain. See for instance the work of J. Cirici [Cir15] and her collaborators.

In this vein, in this thesis we study the mixed Hodge theory of linear representations of fundamental groups of compact Kähler manifolds and algebraic varieties.

A.3 The theory of Goldman and Millson

Let us fix a complex manifold X which is either compact Kähler, either algebraic smooth. In both cases its fundamental group is finitely presentable. Fix a base point $x \in X$. Let G be a linear algebraic group over a subfield $\mathbf{k} \subset \mathbb{R}$ with Lie algebra $\mathfrak{g}$.

Definition (Sect. 1.1.2 and Thm. 1.26)**.** The set of representations of $\pi_1(X,x)$ into $G(\mathbf{k})$, i.e. of group morphisms

$$\rho : \pi_1(X,x) \longrightarrow G(\mathbf{k}),$$

has the structure of points over $\mathbf{k}$ of an affine scheme that we denote by $\mathrm{Hom}(\pi_1(X,x),G)$. It is called the *representation variety* of $\pi_1(X,x)$ into G. We denote by $\widehat{\mathcal{O}}_\rho$ the complete local ring of the representation variety at a point ρ.

The local ring $\widehat{\mathcal{O}}_\rho$ will be one of the main objects of study. Namely it is the formal moduli space for infinitesimal deformations of ρ and its singularity type tells about the possibility of deforming representations and extending given deformations. His study was done by Goldman and Millson [GM88] when X is a compact Kähler manifold.

Theorem (Goldman-Millson)**.** *If X is a compact Kähler manifold and $\rho : \pi_1(X,x) \to G(\mathbb{R})$ has image contained in a compact subgroup, then $\widehat{\mathcal{O}}_\rho$ has a quadratic presentation.*

This can be seen as a version of [DGMS75] for representations of the fundamental group. It states that the representation variety has quite simple singularities at ρ and thus that the fundamental group cannot be too complicated.

Let us describe briefly the methods of proof. To ρ is associated, via the monodromy of its adjoint representation, a local system of Lie algebras with fiber $\mathfrak{g}$. The algebra of real differential forms with values in this local system is then a differential graded Lie algebra (Definition 1.13), that we denote by L. The main construction relates the deformation functor associated with ρ to the deformation functor associated with L.

Definition (Def. 1.17)**.** The *deformation functor* of a DG Lie algebra L over a field $\mathbf{k}$ of characteristic zero is the functor Def_L given on a local Artin algebra $(A,\mathfrak{m}_A)$ over $\mathbf{k}$ as the quotient of the set of *Maurer-Cartan elements* of $L \otimes \mathfrak{m}_A$

$$\mathrm{MC}(L \otimes \mathfrak{m}_A) := \left\{ x \in L^1 \otimes \mathfrak{m}_A \;\middle|\; d(x) + \frac{1}{2}[x,x] = 0 \right\}$$

by the action of $L^0 \otimes \mathfrak{m}_A$ by *gauge transformations* (Definition 1.16).

The fundamental principle of deformation theory states that *every deformation problem is controlled by a DG Lie algebra* (i.e. the associated deformation functor is isomorphic to some Def_L as above) and that *two quasi-isomorphic DG Lie algebras have isomorphic deformation functors*. This was inspired by a letter of Deligne [Del86] to the authors.

So in the situation of Goldman-Millson, by using Hodge theory, L is quasi-isomorphic to a DG Lie algebra with zero differential — this property, called *formality*, is the same as in [DGMS75] — for which the deformation functor takes a simplified form and allows us to give a simple description of $\widehat{\mathcal{O}}_\rho$.

By formality again, using non-abelian Hodge theory, the theorem of Goldman-Millson was extended by C. Simpson [Sim92, § 2] to the case of complex semi-simple representations.

Later, the description of $\widehat{\mathcal{O}}_\rho$ for some cases of representations of fundamental groups of non-compact algebraic varieties was done by Kapovich and Millson [KM98].

Theorem (Kapovich-Millson)**.** *If X is a smooth algebraic variety and $\rho : \pi_1(X, x) \to G(\mathbb{R})$ has finite image, then $\widehat{\mathcal{O}}_\rho$ has a weighted-homogeneous presentation, with generators of weight $1, 2$ and relations of weight $2, 3, 4$.*

This is obtained by studying the mixed Hodge theory of L: its cohomology has a mixed Hodge structure with only possible weights $1, 2$ on H^1 and $2, 3, 4$ on H^2. Instead of using formality, they use the work of Morgan and replace L up to quasi-isomorphism by a DG Lie algebra M having a mixed Hodge structure, and study the consequence of the grading by weight on the deformation functor of M.

In their article, they obtain new examples of finitely presented groups that are not fundamental groups of smooth complex algebraic varieties, by a whole study of their combinatorics and their finite representations.

The work of Goldman-Millson was reviewed by P. Eyssidieux and C. Simpson [ES11] and interpreted in terms of mixed Hodge theory. Assume that ρ is the mononodromy of a variation of Hodge structure (Definition 3.1 and Definition 3.8). Inside $\mathrm{Hom}(\pi_1(X, x), G)$ lies the orbit Ω_ρ of ρ under the action of G by conjugation, which is a reduced subscheme. Its formal germ at ρ is a formal scheme $\widehat{\Omega}_\rho$ defined by an ideal $\mathfrak{j} \subset \widehat{\mathcal{O}}_\rho$.

Theorem (Eyssidieux-Simpson)**.** *If X is a compact Kähler manifold and ρ is the monodromy of a complex polarized variation of Hodge structure over X, then $\widehat{\mathcal{O}}_\rho$ has a functorial complex mixed Hodge structure. The weight filtration is indexed in non-positive degree and is given by the powers of the ideal $\mathfrak{j}$. The weight zero graded piece is $\widehat{\mathcal{O}}_\rho/\mathfrak{j}$, the formal local ring of $\widehat{\Omega}_\rho$..*

In their article, this is used to construct interesting variations of mixed Hodge structures on X. This is an important tool in the proof of the Shafarevich conjecture [EKPR12] on the universal cover of smooth projective varieties whose fundamental group admits a faithful linear representation. Such mixed Hodge structures associated to a representation of the fundamental group of X where also constructed by Hain in [Hai98].

A.4 Derived deformation theory

The use of DG Lie algebras in deformation theory, and the philosophy according to which *in characteristic zero, every deformation problem is controlled by a DG Lie algebra*, has received many developments since the original work of Goldman-Millson and the letter of Deligne. On the one hand it has been applied to many other situations, see for instance the lecture notes of Kontsevich [Kon94]. On the other hand several people have constructed an equivalence between the category of DG Lie algebras up to quasi-isomorphisms and some to-be-defined category of deformation problems. We have to cite the original articles by Manetti [Man02], Hinich [Hin01], the work of Pridham [Pri10] that unifies both, all this culminating in the most general form in the theory of Lurie [Lur11]. Therefore this philosophy is now called the Pridham-Lurie theorem. We refer also to the survey of B. Toën [Toë17].

It is then very interesting to review the original theory of Goldman and Millson with these new and powerful tools. Of particular importance for us is the fact that the deformation functor of a DG Lie algebra L is represented (under the assumption $H^n(L) = 0$ for $n \leq 0$) by a complete local algebra which is obtained as the H^0 of a very explicit complex $\mathscr{C}(L)$, functorial, invariant under quasi-isomorphism. It is also called the *bar construction* on L.

This construction is not possible to understand without appealing to derived deformation theory. This requires at least extending deformation functors from Artin algebras to some category of DG Artin algebras, taking values not any more into sets but into groupoids or simplicial sets. These ideas appear first in a letter of Drinfeld [Dri88]. These principles were not known at the time of the first work of Goldman and Millson. Again by the formality property all this simplifies much when working in the compact case, but it is not trivial at all in the general case.

Furthermore, other tools have been developed to understand the homotopy category of DG Lie algebras (that is, the category of DG Lie algebras up to quasi-isomorphisms): L_∞ algebras. The right framework is operad theory, for which we refer to the book by Loday-Valette [LV12].

Briefly, L_∞ algebras are weakened versions of DG Lie algebras equipped with higher operations in which the Jacobi identity only holds *up to homotopy* given by the higher operations, satisfying themselves higher coherence laws. DG Lie algebras are exactly the L_∞ algebras with zero higher operations. Since L_∞ algebras naturally have an associated deformation functor that extends the one of DG Lie algebras and that is invariant by quasi-isomorphisms, and since a quasi-isomorphism between L_∞ algebras automatically admits an inverse quasi-isomorphism, this forms a convenient category for studying deformation theory.

Though it is known by very abstract theory that every deformation problem is controlled by a DG Lie algebra, what matters is to find the right one with good properties that allows us to understand better the given deformation problem. Since L_∞ algebras are finer objects, it can happen that a deformation problem is controlled more naturally by a L_∞ algebra than by a DG Lie algebra. Many examples of deformation problems controlled by L_∞ algebras have been studied by M. Manetti and his collaborators and we refer to the lectures notes [Man04].

B Results

B.1 Preliminary work

Our first preliminary work, which is now a published article [Lef17], is reproduced in the appendix A.

It concerns special cases of the theorem of Kapovich-Millson where we find more restrictions on the possible weights on $\widehat{\mathcal{O}}_\rho$ such that it behaves as in the compact case, obtained by analyzing the proof and where these restrictions come from.

Theorem A (Thm. A.2)**.** *Let X be a smooth complex quasi-projective algebraic variety and let $\rho : \pi_1(X, x) \to G(\mathbb{R})$ be a representation with finite image. Assume that the finite cover $Y \to X$ corresponding to the subgroup $\mathrm{Ker}(\rho) \subset \pi_1(X, x)$ has a smooth compactification $\overline{Y}$ with first Betti number $b_1(\overline{Y}) = 0$. Then $\widehat{\mathcal{O}}_\rho$ has a quadratic presentation.*

The main motivation comes from the case of complements of projective arrangements of hyperplanes at the trivial representation, for which this notion was known under the name of 1-formality, see [DPS09], [PS09]. This should also be related to the statement that *purity implies formality*, see for example the work [Dup15] that we were not aware of.

In this same article we also give examples where the hypothesis of the theorem is satisfied with respect to *every* finite representation. We find some among families of abelian varieties, related to the various rigidity results, and among hermitian locally symmetric spaces, related to Kazhdan's property (T).

B.2 Main result

Our main goal is to extend the result of Eyssidieux-Simpson to the non-compact case, constructing a functorial mixed Hodge structure on $\widehat{\mathcal{O}}_\rho$ and recovering the result of Kapovich-Millson.

This leads to several important difficulties. On one hand the work of Eyssidieux-Simpson makes strong use of Kähler geometry, laplacians for differential forms, formality, and is impossible to adapt directly in the non-compact case. On the other hand, the work of Kapovich-Millson relies on the theory of Morgan using minimal models for mixed Hodge diagrams and, though the grading on $\widehat{\mathcal{O}}_\rho$ could be improved to a mixed Hodge structure, this would not at all be functorial and well-behaved.

So one has to find a completely different proof. By the way, we want the mixed Hodge structure on $\widehat{\mathcal{O}}_\rho$ to come in both cases directly and functorially from an appropriate mixed Hodge diagram, whose construction depends on the geometric situation.

Let us state first the result. We fix a field $\mathbf{k}$ which is either a subfield of $\mathbb{R}$ or the field $\mathbb{C}$, allowing us to speak both of usual mixed Hodge structures and of complex mixed Hodge structures, and recall that G is a linear algebraic group over $\mathbf{k}$.

Theorem B (Main theorem)**.** *Let X be a compact Kähler manifold and let $\rho : \pi_1(X, x) \to G(\mathbf{k})$ be the monodromy of a polarized variation of Hodge structure defined over $\mathbf{k}$ on X. Then there is a functorial mixed Hodge structure over $\mathbf{k}$ on $\widehat{\mathcal{O}}_\rho$ with non-positive weights and whose weight zero graded piece is the formal local ring of the orbit of ρ.*

If X is a smooth complex quasi-projective algebraic variety and $\rho : \pi_1(X, x) \to G(\mathbf{k})$ is a representation with finite image, then $\widehat{\mathcal{O}}_\rho$ has a functorial mixed Hodge structure with non-positive weights. The induced weights on the cotangent space are $-1, -2$.

At this time, it is not proved that our mixed Hodge structure on $\widehat{\mathcal{O}}_\rho$ in the compact case is exactly the same as the one constructed by Eyssidieux-Simpson, though there are strong indications for: it behaves the same way, has the same description on the cotangent space, and the same weight zero graded piece. In the non-compact case we do not recover completely the result of Kapovich-Millson: the mixed Hodge structure on the cotangent space can be splitted over $\mathbb{C}$ and basis elements can be lifted to weighted homogeneous generators of $\widehat{\mathcal{O}}_\rho$, thus recovering the generators of weight $1, 2$, but until now we do not recover the weighted-homogeneous relations with our methods. We would like to recover them in some canonical way from an ideal of relations carrying a mixed Hodge structure with weights $2, 3, 4$ and we are unable to get this.

B.3 Plan of proof

Now let us indicate the main theorems needed for the proof. In the theory of Goldman-Millson the controlling DG Lie algebra L is equipped with an augmentation $\varepsilon_x : L \to \mathfrak{g}$ which simply evaluates degree zero forms at the base point x. In the work of Eyssidieux-Simpson is introduced an *augmented deformation functor* (Definition 1.21), which is a slight variation of the usual deformation functor Def_L, and it is shown that this is the right object to consider in order to control the deformation theory of ρ and to understand its orbit.

In the compact case, L is a DG Lie algebra of differential forms with coefficients in a variation of Hodge structure, thus by the work of Zucker [Zuc79] it has the structure of a mixed Hodge diagram of Lie algebras. In the situation of Kapovich-Millson, L is obtained from the algebra of differential forms on the finite cover corresponding to $\mathrm{Ker}(\rho)$, on which the pulled-back representation is trivial. So, following their ideas and re-writing them with the functorial construction of mixed Hodge diagrams of Navarro Aznar, we find again a mixed Hodge diagram of Lie algebras that is quasi-isomorphic to L. We sum up the structures we get from the geometric situations:

Theorem C (Chapter 3)**.** *In all situations of the above theorem, the deformation theory of ρ is controlled by an* augmented mixed Hodge diagram of Lie algebras *(Definition 2.33 and Definition 2.37) that is functorial up to quasi-isomorphism.*

Then we want to apply the appropriate bar construction coming from derived deformation theory to such a diagram L and show that it defines a mixed Hodge diagram. This way we will get a functorial mixed Hodge structure on an object representing the deformation functor of L, which by the Yoneda lemma is canonically isomorphic to $\widehat{\mathcal{O}}_\rho$.

However, despite our efforts, we were not able to access to $\widehat{\mathcal{O}}_\rho$ using this naive strategy since one really needs to work with L together with its augmentation ε_x. The following possibilities are the natural ones:

1. One can work with the full L and then later try to extract the information coming from the augmentation. However, L does not have $H^0(L) = 0$ which means that its deformation functor is not representable. Also, the bar construction doesn't behave so well if $H^0(L) \neq 0$ (elements of degree 0 in L produce elements of negative degree in $\mathscr{C}(L)$).
2. One can work with $L' := \mathrm{Ker}(\varepsilon_x) \subset L$. This is a DG Lie algebra which has $H^0(L') = 0$. But it is not anymore a mixed Hodge diagram, since the axioms of mixed Hodge diagrams are very strong and one cannot a priori take kernels.

3. Instead, one can work with the mapping cone of ε_x. This is the natural operation in mixed Hodge diagrams that replaces the kernel and it has $H^0 = 0$, in fact it is quasi-isomorphic to L'. However, it is not anymore a DG Lie algebra.

The solution to this problem we find by working with L_∞ algebras. The mapping cone of a morphism between DG Lie algebras was studied by Fiorenza-Manetti [FM07]. They show that it carries a natural L_∞ algebra structure and describe the associated deformation functor. We make the very simple but fundamental claim (for us) that when applied to the augmentation ε_x the associated deformation functor is the same as the augmented deformation functor introduced independently by Eyssidieux-Simpson.

Lemma D (Fundamental observation, Lem. 1.55, combine with Thm. 1.54)**.** *The augmented deformation functor of an augmented DG Lie algebra $\varepsilon : L \to \mathfrak{g}$ introduced by Goldman-Millson and Eyssidieux-Simpson is the deformation functor associated with the L_∞ algebra structure on the mapping cone of ε studied by Fiorenza-Manetti.*

This allows us to carry out our strategy of proof. The higher operations of L_∞ on the mapping cone have very explicit algebraic formulas and it is a direct calculation to check the compatibility with the structure of mixed Hodge diagram. The resulting object we call a *mixed Hodge diagram of L_∞ algebras.*

Theorem E (Sect. 2.2.1 and Thm. 2.38)**.** *Let $\varepsilon : L \to \mathfrak{g}$ be an augmented mixed Hodge diagram of Lie algebras coming from Theorem C. The operations of L_∞ algebra of Fiorenza-Manetti on the mapping cone C of ε give C the structure of mixed Hodge diagram of L_∞ algebras (Definition 2.34).*

Then we apply the bar construction (functor $\mathscr{C}$) in the sense of L_∞ algebras. This gives naturally a DG coalgebra from which we extract (via its H^0 and then dualize) a complete local algebra that, by the principles of derived deformation theory, represents the deformation functor of the L_∞ algebra. So we show that $\mathscr{C}$ can be applied to the mixed Hodge diagram C and that we get a mixed Hodge diagram of coalgebras. This is very close to Hain's *bar construction of mixed Hodge diagrams* (of commutative algebras, or modules over them, with many variations) used in [Hai87] and several other articles.

Theorem F (Sect. 2.2.2 and Thm. 2.44)**.** *The functor $\mathscr{C}$ can be applied to the mixed Hodge diagram of L_∞ algebras C of Theorem E and gives a mixed Hodge diagram of coalgebras.*

From this, one extracts directly a mixed Hodge structure on its H^0 that is invariant under quasi-isomorphisms. This gives the functorial mixed Hodge structure on $\widehat{\mathcal{O}}_\rho$ in all situations of Theorem B.

B.4 Perspectives

Our strategy of proof was developed with the constant objective of separating the geometrical construction of mixed Hodge diagrams and the algebraic machinery giving us mixed Hodge structures. We strongly believe it will help us to construct a mixed Hodge structure on $\widehat{\mathcal{O}}_\rho$ in more general non-compact cases: when ρ is the monodromy of a variation of mixed Hodge structure, and even when X is singular. Combined with a deeper study of group theory, this could lead to new restrictions on Serre's problem.

C Organization of the work

C.1 Plan

This work is organized as follows. We separate it into three chapters, each forming pieces of the proof our Theorem B. Since the preliminary work is independent and is the reproduction of a published article, we put in the appendix A. Also, we postpone all the geometric constructions to the chapter 3, developing our machinery before through chapters 1 and 2.

So chapter 1 is centered around Goldman-Millson theory and deformation theory but without Hodge theory. In its first part we review all the necessary material in order to understand properly the theory of Goldman-Millson, from a classical point of view. Section 1.1.1 is purely algebraic and categorical. There we introduce carefully the augmented deformation functor of a DG Lie algebra. In section 1.1.2 is introduced the representation variety and its study is related to geometric constructions. In the second part, we expose the theory of L_∞ algebras and the theorems of deformation theory we need. Since this is technical and since we need the details, we devote the whole section 1.2.1 to introducing L_∞ algebras. Then in section 1.2.2 we describe the L_∞ algebra structure of Fiorenza-Manetti. Our only contribution is the simple observation of Lemma D. Finally in section 1.2.3 we study more in detail the deformation functor of a L_∞ algebra and we extract from the literature the pro-representability theorem we want to apply.

Then chapter 2 is devoted to Hodge theory. In its first part we expose carefully all the definitions we need: the mixed Hodge structures in section 2.1.1, the mixed Hodge diagrams in section 2.1.2. At each step we show how these definitions behave if one works not only with vector spaces but with other kinds of algebras. The second part contains the heart of the present work. In section 2.2.1 we give the definition of mixed Hodge diagram of L_∞ algebras and we prove Theorem E. In section 2.2.2 we prove Theorem F and study this construction.

Finally in chapter 3 we study many different situations in which we construct controlling augmented mixed Hodge diagrams of Lie algebras, proving Theorem C. We separate the compact and the non-compact case. In the compact case, for representations with values in a real variation of Hodge structure, such a construction is straightforward. So it serves as a model of proof and of application of our method, and is written down in detail in section 3.1.1. It is then easy to adapt the method to several variations of this case, which we do in section 3.1.2. Similarly in the non-compact case, we take as detailed model the real representations, in section 3.2.1, and later we add the rational structure, section 3.2.2. To treat the non-compact case we expose briefly the construction of Navarro Aznar and we adapt it to our needs.

C.2 Conventions and notations

The manifolds we study are either compact Kähler manifolds (this includes the class of smooth complex projective algebraic varieties) or smooth complex quasi-projective algebraic varieties. They are of course connected. This is the natural setting in which Hodge (resp. mixed Hodge) theory applies in the most direct way and where we can directly cite the existing literature.

We always denote by $\mathbf{k}$ a field of characteristic zero. In geometric situations this is mostly $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.

We use everywhere the abbreviation *DG* for *differential graded*. Our DG vector spaces over $\mathbf{k}$ are also known under the name of *cochain complexes*. We always denote the grading by an upper index (i.e. $V = \bigoplus V^n$) and the differential d is of degree $+1$ (i.e. $d : V^n \to V^{n+1}$).

We use the symbol $\simeq$ for isomorphisms and $\approx$ for weak equivalences (quasi-isomorphisms between DG vector spaces, equivalences of categories or functors to groupoids). Categories are denoted with bold letters, for example: $\mathbf{Set}$, $\mathbf{Alg_k}$, $\mathbf{DG{-}Vect_k}$.

Chapter 1

Deformation theory

The first chapter is dedicated to the study of deformation theory. We first expose all the ingredients we need, then we show how to re-write and improve the classical theory of Deligne-Goldman-Millson using tools from derived deformation theory: L_∞ algebras. These tools will be combined with mixed Hodge theory in the next chapter.

1.1 Classical deformation theory

This section is an expository section of the theory of Goldman and Millson. It contains no new result. To understand this theory it is necessary to review first groupoids, Artin algebras and DG Lie algebras.

1.1.1 Deligne-Goldman-Millson classical setting

The classical deformation theory is formulated in terms of functors from local Artin algebras to groupoids or sets.

Groupoids

Definition 1.1. A *groupoid* is a small category $\mathcal{G}$ all of whose morphisms are invertible. We denote by $\mathcal{G}_0$ the set of objects and by $\mathcal{G}_1$ the set of morphisms. We denote by $\pi_0(\mathcal{G})$ the set of *isomorphism classes* of objects of $\mathcal{G}$ and, for an object x, by $\pi_1(\mathcal{G}, x)$ for the group of automorphisms of x.

Example 1.2. If X is a topological space, there is a groupoid $\pi_{\leq 1}(X)$ (the *fundamental groupoid*) with set of objects X and where the morphisms from x to y are given by homotopy classes of paths

$$\gamma : [0,1] \longrightarrow X$$

such that

$$\gamma(0) = x, \quad \gamma(1) = y.$$

In this case

$$\pi_0(\pi_{\leq 1}(X)) = \pi_0(X)$$

is the set of path-components of X and

$$\pi_1(\pi_{\leq 1}(X), x) = \pi_1(X, x)$$

is the fundamental group of X based at x. Whence our notations.

Groupoids form a 2-category **Gpd** (see [Mac88, § XII.3]) and π_0 is a functor

$$\pi_0 : \mathbf{Gpd} \longrightarrow \mathbf{Set} \tag{1.1}$$

to the category of sets. Let us give one useful construction of groupoids.

Definition 1.3. If a group G acts on the left on a set X, one gets a groupoid $[X/G]$ called the *action groupoid* with set of objects

$$[X/G]_0 := X \tag{1.2}$$

and morphisms

$$\operatorname*{Hom}_{[X/G]}(x, y) := \{g \in G \mid g.x = y\} \tag{1.3}$$

(in this case $[X/G]_1 = X \times G$). Then $\pi_0([X/G])$ is the quotient set X/G and $\pi_1(X, x)$ is the stabilizer of x in G.

Definition 1.4 ([GM88, § 3.7])**.** If a group G acts on a set X, so that there is an action groupoid $[X/G]$, and G acts on another set Y, we define a new groupoid

$$[X/G] \bowtie Y := [X \times Y/G]. \tag{1.4}$$

Lemma 1.5 ([GM88, 3.8])**.** *Let G be a group that acts on a set X, G' a group that acts on a set X', and let*

$$\varphi : [X/G] \longrightarrow [X'/G']$$

be a morphism of action groupoids (i.e. induced by a morphism of groups $\psi : G \to G'$ and a ψ-equivariant map $X \to X'$). Let Y be another set on which G' acts. Then G also acts on Y via ψ and there is an induced morphism of groupoids

$$\varphi \bowtie Y : [X/G] \bowtie_\psi Y \longrightarrow [X'/G'] \bowtie Y$$

(where the subscript ψ indicates that G acts on Y via ψ) which is an equivalence if φ is.

Local Artin algebras

From now on we work over a fixed field $\mathbf{k}$ of characteristic zero. Our algebras are always assumed to be associative, unital and commutative.

Definition 1.6. A *local Artin algebra* over $\mathbf{k}$ is an algebra A over $\mathbf{k}$, local with maximal ideal $\mathfrak{m}_A$, with residue field $A/\mathfrak{m}_A = \mathbf{k}$, and finite-dimensional over $\mathbf{k}$. These form a category $\mathbf{Art_k}$, where the morphisms are morphisms of algebras over $\mathbf{k}$ that are required to preserve the maximal ideals.

Remark 1.7. The finite-dimensionality is a consequence of the artinian condition on A stating that every descending sequence of ideals stabilizes, see for example [AM69, § 8] for more details. It then implies that $\mathfrak{m}_A$ is a nilpotent ideal, so A is local complete.

Example 1.8. The most basic example is

$$A := \frac{\mathbf{k}[\varepsilon]}{(\varepsilon^n)} \tag{1.5}$$

with maximal ideal $\mathfrak{m}_A = (\varepsilon)$ and $(\mathfrak{m}_A)^n = 0$.

Definition 1.9. A *pro-Artin algebra* over $\mathbf{k}$ is a complete local algebra R over $\mathbf{k}$ such that all quotients $R/(\mathfrak{m}_R)^n$ are local Artin algebras (so R is a projective limit of local Artin algebras). These form a category $\mathbf{ProArt_k}$ where morphisms have to preserve the maximal ideal.

Definition 1.10. The *cotangent space* to a pro-Artin algebra R is the vector space

$$\mathfrak{m}_R/(\mathfrak{m}_R)^2 \tag{1.6}$$

over $\mathbf{k}$.

Example 1.11. The most basic examples are the formal power series algebras

$$\mathbf{k}[[X_1, \ldots, X_r]]. \tag{1.7}$$

For $r = 1$ the quotients by the powers of the maximal ideal are precisely the local Artin algebras of Example 1.8. The cotangent space is the vector space of dimension r spanned by $X_1, \ldots, X_r$.

The category $\mathbf{Art_k}$ is a full subcategory of $\mathbf{ProArt_k}$, which is itself a full subcategory of the category of complete local algebras with residue field $\mathbf{k}$. Namely in all these three categories, morphisms are morphisms of algebras that furthermore preserve the maximal ideal, thus commute with the augmentation to $\mathbf{k}$ obtained by quotienting by the maximal ideal. The difference between complete local algebras and those which are pro-Artin (both are always assumed to have residue field $\mathbf{k}$) is simply the finite-dimensionality of the cotangent space, as expressed by the following lemma.

Lemma 1.12. *For a complete local algebra R over $\mathbf{k}$, the following conditions are equivalent:*

(1) R is pro-Artin.

(2) R is Noetherian.

(3) $R/(\mathfrak{m}_R)^n$ is finite-dimensional for all $n \geq 1$.

(4) $\mathfrak{m}_R/(\mathfrak{m}_R)^2$ is finite-dimensional.

Proof. Condition (1) is equivalent to (3) almost by definition, and (3) implies (4). Consider the associated graded ring

$$G(R) := \bigoplus_{n\geq 0} \frac{(\mathfrak{m}_R)^n}{(\mathfrak{m}_R)^{n+1}}. \tag{1.8}$$

Then $G(R)$ is naturally an algebra over $R/(\mathfrak{m}_R) = \mathbf{k}$ generated by its component $(\mathfrak{m}_R)/(\mathfrak{m}_R)^2$. If R is assumed to be Noetherian, then $G(R)$ is also Noetherian ([AM69, 10.22]) which implies that $G(R)$ is finitely generated as algebra over $\mathbf{k}$ ([AM69, 10.7]). So all terms are finite-dimensional, and using the exact sequence

$$0 \longrightarrow \frac{(\mathfrak{m}_R)^n}{(\mathfrak{m}_R)^{n+1}} \longrightarrow \frac{R}{(\mathfrak{m}_R)^{n+1}} \longrightarrow \frac{R}{(\mathfrak{m}_R)^n} \longrightarrow 0$$

ends the proof that the condition (2) also implies (3) and (4). Conversely, if $\mathfrak{m}_R/(\mathfrak{m}_R)^2$ is finite-dimensional then $G(R)$ is finitely generated as algebra over $\mathbf{k}$. This implies that $G(R)$ is Noetherian and then that R is ([AM69, 10.25]). Finally if we assume (4) then $R/(\mathfrak{m}_R)^n$ is again Noetherian, and of dimension zero, so it is Artinian ([AM69, 8.5]) and this proves that condition (4) implies all the others. Alternatively, the Cohen structure theorem states that if R is a complete local algebra over $\mathbf{k}$ which is Noetherian then R is a quotient of some formal power series algebra $\mathbf{k}[[X_1, \dots, X_r]]$, from which we see that R is pro-Artin with our definition. □

DG Lie algebras

We start using DG vector spaces and related kinds of DG algebras. A *DG vector space* over $\mathbf{k}$ is a vector space V over $\mathbf{k}$ with a direct sum decomposition

$$V = \bigoplus_{n\in\mathbb{Z}} V^n$$

and a differential

$$d_V^n : V^n \longrightarrow V^{n+1}$$

satisfying $d_V^{n+1} \circ d_V^n = 0$. The *tensor product* of two DG vector spaces V, W is the vector space $V \otimes W$ with the grading

$$(V \otimes W)^n := \bigoplus_{i+j=n} V^i \otimes W^j \tag{1.9}$$

and the differential

$$d^n_{V\otimes W} := \sum_{i+j=n} d_V^i \otimes \mathrm{id}_W + (-1)^i \,\mathrm{id}_V \otimes d_W^j. \tag{1.10}$$

The *degree* of a homogeneous element x is denoted by $|x|$. There is a canonical isomorphism

$$\begin{array}{rcl} \sigma_{V,W} : V \otimes W & \xrightarrow{\simeq} & W \otimes V \\ x \otimes y & \longmapsto & (-1)^{|x|\cdot|y|}\, y \otimes x. \end{array} \tag{1.11}$$

By forgetting the differential, we get the category of *graded vector spaces* $\mathbf{G{-}Vect_k}$. The above isomorphism (1.11) plays a crucial role when dealing with graded vector spaces.

Definition 1.13. A *DG Lie algebra* over $\mathbf{k}$ is the data of a DG vector space L together with a morphism called *Lie bracket*

$$[-,-] : L \otimes L \longrightarrow L, \tag{1.12}$$

given by a collection of morphisms

$$[-,-] : L^i \otimes L^j \longrightarrow L^{i+j}, \tag{1.13}$$

satisfying the *anti-symmetry*

$$[x,y]+(-1)^{|x|\cdot|y|}[y,x]=0 \tag{1.14}$$

and the *Jacobi identity*

$$(-1)^{|x|\cdot|z|}[x,[y,z]]+(-1)^{|x|\cdot|y|}[y,[z,x]]+(-1)^{|y|\cdot|z|}[z,[x,y]]=0. \tag{1.15}$$

The fact that the bracket is a morphism of DG vector spaces automatically implies that d is a *derivation*, meaning that

$$d([x,y])=[d(x),y]+(-1)^{|x|}[x,d(y)]. \tag{1.16}$$

This last one is also called *Leibniz' rule.* A *morphism* between DG Lie algebras

$$f:(L,[-,-]_L)\longrightarrow(M,[-,-]_M)$$

is given by a morphism $f:L\to M$ of DG vector spaces such that

$$f([x,y]_L)=[f(x),f(y)]_M. \tag{1.17}$$

DG Lie algebras over $\mathbf{k}$ form a category $\mathbf{DG{-}Lie_k}$.

The category of Lie algebras over $\mathbf{k}$ in the usual sense, $\mathbf{Lie_k}$, is a full subcategory of $\mathbf{DG{-}Lie_k}$ corresponding to DG Lie algebras concentrated in degree zero.

Definition 1.14. If L is a DG Lie algebra, the set of *Maurer-Cartan elements* is

$$\mathrm{MC}(L):=\left\{x\in L^1 \;\middle|\; d(x)+\frac{1}{2}[x,x]=0\right\}. \tag{1.18}$$

From it we construct the deformation functor associated to a DG Lie algebra L over $\mathbf{k}$. Let A be a local Artin algebra. Then $L\otimes\mathfrak{m}_A$ is a *nilpotent* DG Lie algebra with the bracket

$$[u\otimes a,v\otimes b]:=[u,v]\otimes ab,\quad u,v\in L,\ a,b\in\mathfrak{m}_A \tag{1.19}$$

and the differential

$$d(u\otimes a):=d(u)\otimes a. \tag{1.20}$$

Namely the nilpotency means that there is an order r such that all iterated brackets

$$[x_1,[x_2,[\dots,[x_{r-1},x_r]\dots]]]$$

of elements $x_1,\dots,x_r\in L\otimes\mathfrak{m}_A$ vanish; since $x_i\in L\otimes\mathfrak{m}_A$ then the above iterated bracket is an element in $L\otimes(\mathfrak{m}_A)^r$ and vanishes for r such that $(\mathfrak{m}_A)^r=0$.

Definition 1.15. On $L^0\otimes\mathfrak{m}_A$, which is a nilpotent Lie algebra, it is possible to define a group structure denoted by $*$ using the Baker-Campbell-Hausdorff formula (see [Man04, § V.2] for all the details) and we denote this group by

$$\exp(L^0\otimes\mathfrak{m}_A) \tag{1.21}$$

whose elements are simply denoted by e^α, $\alpha\in L^0\otimes\mathfrak{m}_A$.

The Baker-Campbell-Hausdorff formula starts with

$$e^{\alpha} * e^{\beta} := e^{\alpha+\beta+\frac{1}{2}[\alpha,\beta]+\cdots}$$

so that if $[\alpha,\beta]=0$ then $e^{\alpha} * e^{\beta} = e^{\beta} * e^{\alpha}$, and $e^{-\alpha} = (e^{\alpha})^{-1}$. Denote by $\mathrm{ad}(\alpha)$ the usual endomorphism of L defined by

$$\mathrm{ad}(\alpha)(x) := [\alpha, x]. \tag{1.22}$$

Definition 1.16. The *gauge action* of $\exp(L^0 \otimes \mathfrak{m}_A)$ on $\mathrm{MC}(L \otimes \mathfrak{m}_A)$ is the group action given by the formula

$$e^{\alpha}.x := x + \frac{e^{\mathrm{ad}(\alpha)} - \mathrm{id}}{\mathrm{ad}(\alpha)}\left([\alpha, x] - d(\alpha)\right) = x + \sum_{i=0}^{+\infty} \frac{\mathrm{ad}(\alpha)^i}{(i+1)!}\left([\alpha, x] - d(\alpha)\right) \tag{1.23}$$

where the right-hand side is obtained by a power series expansion and the sum is finite because $\mathrm{ad}(\alpha)$ is nilpotent.

The infinitesimal generator for this action is

$$\alpha.x := [\alpha, x] - d(\alpha).$$

This is a formal version of the infinitesimal action of the gauge group on the space of flat connections on a principal bundle, namely the formula (1.23) can also be written as

$$e^{\alpha}.x := e^{\alpha} x e^{-\alpha} - d(e^{\alpha}) e^{-\alpha}$$

for the appropriate way of expanding into power series. We see that for this construction it is really important for $\mathbf{k}$ to have characteristic zero.

Definition 1.17. Let L be a DG Lie algebra and let A be a local Artin algebra over $\mathbf{k}$. Define the *Deligne-Goldman-Millson groupoid* $\mathrm{Def}(L, A)$ to be the action groupoid (Definition 1.3)

$$\mathrm{Def}(L, A) := \left[\frac{\mathrm{MC}(L \otimes \mathfrak{m}_A)}{\exp(L^0 \otimes \mathfrak{m}_A)}\right] \tag{1.24}$$

and the associated *deformation functor* (in the sense of [Man99, § 2])

$$\begin{array}{rcl} \mathrm{Def}_L : \mathbf{Art}_{\mathbf{k}} & \longrightarrow & \mathbf{Set} \\ A & \longmapsto & \pi_0(\mathrm{Def}(L, A)) = \frac{\mathrm{MC}(L \otimes \mathfrak{m}_A)}{\exp(L^0 \otimes \mathfrak{m}_A)}. \end{array} \tag{1.25}$$

This is our first example of a deformation functor. We will not need to give a precise definition of a deformation functor but at the very least a classical deformation functor should be a functor from $\mathbf{Art}_{\mathbf{k}}$ to $\mathbf{Set}$, with axioms expressing some compatibility with the fiber product of local Artin algebras and the surjective morphisms. This can be extended in several ways, to DG Artin algebras on one side, to groupoids or simplicial sets on the other side.

Definition 1.18. If a functor $F : \mathbf{Art}_{\mathbf{k}} \to \mathbf{Set}$ is isomorphic to Def_L for some DG Lie algebra L, we say that L *controls* the deformation problem F.

The fundamental theorem in the classical deformation theory is:

Theorem 1.19 ([GM88, 2.4]). *Let $\varphi : L \xrightarrow{\approx} M$ be a quasi-isomorphism between DG Lie algebras over* $\mathbf{k}$ *(i.e. φ induces an isomorphism on cohomology). Then there is an induced equivalence of groupoids*

$$\mathrm{Def}(L, A) \xrightarrow{\approx} \mathrm{Def}(M, A), \quad A \in \mathbf{Art_k}$$

inducing an isomorphism of deformation functors

$$\mathrm{Def}_L \xrightarrow{\simeq} \mathrm{Def}_M\,.$$

Remark 1.20. The above theorem is stated with a priori non-bounded-below DG Lie algebras. However in our concrete situations these are non-negatively graded. It is then enough for φ to be a 1-quasi-isomorphism (i.e. φ induces an isomorphism on H^0, H^1, and is injective on H^2). This shows that the deformation theory of L is essentially controlled by $H^1(L)$ and $H^2(L)$. However all our arguments will apply without discussing restrictions on the grading of L.

To deal properly will the Goldman-Millson theory, Eyssidieux and Simpson introduce the following construction.

Definition 1.21 ([ES11, § 2.1.1]). Given a DG Lie algebra L, an augmentation

$$\varepsilon : L \longrightarrow \mathfrak{g} \tag{1.26}$$

to a Lie algebra $\mathfrak{g}$ and a local Artin algebra A, define the *augmented Deligne-Goldman-Millson groupoid* $\mathrm{Def}(L, \varepsilon, A)$ with set of objects

$$\mathrm{Def}(L, \varepsilon, A)_0 := \left\{ (x, e^\alpha) \in (L^1 \otimes \mathfrak{m}_A) \times \exp(\mathfrak{g} \otimes \mathfrak{m}_A) \;\middle|\; d(x) + \frac{1}{2}[x, x] = 0 \right\} \tag{1.27}$$

and morphisms given by

$$\mathrm{Hom}((x, e^\alpha), (y, e^\beta)) := \left\{ \lambda \in L^0 \otimes \mathfrak{m}_A \;\middle|\; e^\lambda.x = y,\ e^\beta = e^\alpha * e^{-\varepsilon(\lambda)} \right\}. \tag{1.28}$$

We denote by $\mathrm{Def}_{L,\varepsilon}$ the associated deformation functor

$$\begin{array}{rcl} \mathrm{Def}_{L,\varepsilon} : \mathbf{Art_k} & \longrightarrow & \mathbf{Set} \\ A & \longmapsto & \pi_0(\mathrm{Def}(L, \varepsilon, A)). \end{array} \tag{1.29}$$

Remark 1.22. Using the operation $\bowtie$ (Definition 1.4) one sees that

$$\mathrm{Def}(L, \varepsilon, A) = \mathrm{Def}(L, A) \bowtie \exp(\mathfrak{g} \otimes \mathfrak{m}_A)$$

where the group $\exp(L^0 \otimes \mathfrak{m}_A)$ acts on $\exp(\mathfrak{g} \otimes \mathfrak{m}_A)$ via

$$e^\lambda.e^\alpha := e^\alpha * e^{-\varepsilon(\lambda)}, \quad e^\lambda \in \exp(L^0 \otimes \mathfrak{m}_A),\ e^\alpha \in \exp(\mathfrak{g} \otimes \mathfrak{m}_A).$$

Since in the group $\exp(\mathfrak{g} \otimes \mathfrak{m}_A)$ the element $e^{-\varepsilon(\lambda)}$ is the inverse of $e^{\varepsilon(\lambda)}$, this is well-defined as a left action. This is not exactly the same action as in [ES11], but we make this choice so as to be compatible with the construction developed in section 1.2.2.

The fundamental theorem has a small variation for the augmented deformation functor.

Lemma 1.23. *Let $\varphi : L \xrightarrow{\approx} L'$ be a quasi-isomorphism of augmented DG Lie algebras commuting with the augmentations as in the diagram*

$$\begin{array}{ccc} L & \xrightarrow[\varphi]{\approx} & L' \\ & {\scriptstyle\varepsilon}\searrow \quad \swarrow{\scriptstyle\varepsilon'} & \\ & \mathfrak{g}. & \end{array}$$

Then φ induces and equivalence of groupoids

$$\mathrm{Def}(L, \varepsilon, A) \xrightarrow{\approx} \mathrm{Def}(L', \varepsilon', A), \quad A \in \mathbf{Art_k}.$$

Proof. Writing it with the operation $\bowtie$ as in Remark 1.22, this is an immediate combination of the fundamental Theorem 1.19 with the Lemma 1.5. □

Pro-representability

Definition 1.24. A functor $F : \mathbf{Art_k} \to \mathbf{Set}$ is said to be *pro-representable* if there is a pro-Artin algebra R such that F is isomorphic to

$$A \in \mathbf{Art_k} \longmapsto \operatorname*{Hom}_{\mathbf{ProArt}}(R, A). \tag{1.30}$$

We also say that R *controls* the deformation problem F (indeed such a pro-representable functor is always a deformation functor).

Theorem 1.25 (Pro-Yoneda Lemma, see [Sch68, § 2], [GM88, 3.1])**.** *If $F : \mathbf{Art_k} \to \mathbf{Set}$ is pro-represented by R, then R is unique up to a unique isomorphism. More precisely, a morphism of functors on $\mathbf{Art_k}$*

$$\varphi : \operatorname*{Hom}_{\mathbf{ProArt}}(R, -) \longrightarrow F$$

is determined uniquely by the family of objects

$$\varphi\Big(R \twoheadrightarrow R/(\mathfrak{m}_R)^n\Big) \in F\Big(R/(\mathfrak{m}_R)^n\Big).$$

It is an isomorphism if and only if there exists a compatible *family*

$$\Big(\xi_n \in F(R/(\mathfrak{m}_R)^n)\Big)_{n \geq 0},$$

where compatible means that

$$F\Big(R/(\mathfrak{m}_R)^p \twoheadrightarrow R/(\mathfrak{m}_R)^n\Big)(\xi_p) = \xi_n, \quad p \geq n$$

such that any other compatible family (x_n) is obtained via a unique map

$$f : (\xi_n) \longrightarrow (x_n)$$

as $(x_n) = F(f)(\xi_n)$.

In others words, an algebra that pro-represents a functor is unique up to a unique isomorphism and one can describe explicitly how, exactly as in the classical Yoneda lemma, the difference being that A lives in a smaller category than R. But since R is a projective limit of objects of the category where A lives, the proof is essentially the same as for the classical Yoneda lemma combined with an induction over the tower of $R/(\mathfrak{m}_R)^n$.

1.1.2 Goldman-Millson theory of deformations of representations of the fundamental group

Let G be a linear algebraic group over $\mathbf{k} = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. We think of G as a representable functor

$$G : \mathbf{Alg_k} \longrightarrow \mathbf{Grp} \tag{1.31}$$

from the category of algebras over $\mathbf{k}$ to the category of groups. We are interested in studying a group Γ by looking at its representations into $G(\mathbf{k})$, and actually into all $G(A)$ for varying algebras A over $\mathbf{k}$.

Variety of representations

Theorem 1.26 ([LM85])**.** *For any finitely generated group* Γ *the functor*

$$\begin{array}{rcl} \mathrm{Hom}(\Gamma, G) : \mathbf{Alg_k} & \longrightarrow & \mathbf{Set} \\ A & \longmapsto & \mathrm{Hom}_{\mathbf{Grp}}(\Gamma, G(A)) \end{array} \tag{1.32}$$

is represented by an affine scheme of finite type over $\mathbf{k}$*. We denote it again by* $\mathrm{Hom}(\Gamma, G)$ *(we think of it as a scheme structure on* $\mathrm{Hom}(\Gamma, G(\mathbf{k}))$*). It is called the* representation variety *of* Γ *into* G*.*

Example 1.27. If Γ is a free group on r generators, then by its universal property

$$\mathrm{Hom}(\Gamma, G(A)) \simeq G(A)^r$$

(a map from Γ to $G(A)$ is given by the image of each generator, without relations) so as schemes

$$\mathrm{Hom}(\Gamma, G) \simeq G^r.$$

More generally, the choice of a presentation of Γ on r generators embeds naturally $\mathrm{Hom}(\Gamma, G)$ as a closed subscheme of G^r, the relations in Γ being translated into algebraic equations in G^r.

Any representation $\rho : \Gamma \to G(\mathbf{k})$ can thus be seen as a point of $\mathrm{Hom}(\Gamma, G)$ over $\mathbf{k}$. We denote by $\widehat{\mathcal{O}}_\rho$ the complete local ring of $\mathrm{Hom}(\Gamma, G)$ at ρ. It is a pro-Artin algebra whose corresponding pro-representable functor on local Artin algebras over $\mathbf{k}$

$$\begin{array}{rcl} \mathrm{Hom}(\widehat{\mathcal{O}}_\rho, -) : \mathbf{Art_k} & \longrightarrow & \mathbf{Set} \\ A & \longmapsto & \mathrm{Hom}_{\mathbf{ProArt}}(\widehat{\mathcal{O}}_\rho, A) \end{array} \tag{1.33}$$

is canonically isomorphic to the functor of *formal deformations of* ρ

$$\mathrm{Def}_0(\rho, -) : A \longmapsto \left\{ \tilde{\rho} : \Gamma \to G(A) \;\middle|\; \tilde{\rho} = \rho \bmod \mathfrak{m}_A \right\}. \tag{1.34}$$

By the pro-Yoneda Lemma 1.25, this is equivalent to saying that there is a *universal* (or *tautological*) representation

$$\rho^u : \Gamma \longrightarrow G(\widehat{\mathcal{O}}_\rho), \tag{1.35}$$

equivalently a compatible family of representations

$$\rho^u_n : \Gamma \longrightarrow G(\widehat{\mathcal{O}}_\rho / \mathfrak{m}^n), \tag{1.36}$$

such that any deformation $\tilde{\rho}$ of ρ over $A \in \mathbf{Art_k}$ is obtained from ρ^u by a unique map $f : \widehat{\mathcal{O}}_\rho \to A$ as

$$\tilde{\rho} : \Gamma \xrightarrow{\rho^u} G(\widehat{\mathcal{O}}_\rho) \xrightarrow{f} G(A).$$

Define G^0 to be the functor

$$\begin{array}{rcl} G^0 : \mathbf{Art_k} & \longrightarrow & \mathbf{Grp} \\ A & \longrightarrow & \left\{ g \in G(A) \;\middle|\; g = 1_{G(\mathbf{k})} \bmod \mathfrak{m}_A \right\} \end{array} \tag{1.37}$$

then $G^0(A)$ acts on $\mathrm{Def}_0(\rho, A)$ by conjugation because if $\tilde{\rho} = \rho$ and $g = 1$ modulo $\mathfrak{m}_A$ then

$$g \cdot \tilde{\rho} \cdot g^{-1} = \rho \bmod \mathfrak{m}_A.$$

This action is functorial in A. If $\mathfrak{g}$ is the Lie algebra of G then $G^0(A)$ has $\mathfrak{g} \otimes \mathfrak{m}_A$ as Lie algebra and actually one can construct it as

$$G^0(A) = \exp(\mathfrak{g} \otimes \mathfrak{m}_A). \tag{1.38}$$

Definition 1.28. Let A be a local Artin algebra. Define the groupoid of *deformations of* ρ over A to be

$$\mathrm{Def}(\rho, A) := \left[\frac{\{\tilde{\rho} : \Gamma \to G(A) \mid \tilde{\rho} = \rho \bmod \mathfrak{m}_A\}}{G^0(A)} \right] \tag{1.39}$$

with associated deformation functor

$$\begin{array}{rcl} \mathrm{Def}_\rho : \mathbf{Art_k} & \longrightarrow & \mathbf{Set} \\ A & \longmapsto & \pi_0(\mathrm{Def}(\rho, A)) = \frac{\mathrm{Def}_0(\rho, A)}{G^0(A)}. \end{array} \tag{1.40}$$

Goldman-Millson construction

Let now X be a manifold whose fundamental group is finitely presentable. This hypothesis encompasses both the compact Kähler manifolds and the smooth algebraic varieties over $\mathbb{C}$. Let x be a base point of X. We are interested in the deformation theory of representations of $\pi_1(X, x)$. We assume that we work with a fix field either $\mathbb{R}$ or $\mathbb{C}$ and we write G for $G(\mathbf{k})$ which is either a real or complex Lie group. Let

$$\rho : \pi_1(X, x) \to G(\mathbf{k}) \tag{1.41}$$

be a representation. Let $\mathfrak{g}$ be the Lie algebra of G. Let $P \to X$ be the flat principal G-bundle given by the holonomy of ρ. Recall that concretely P can be constructed as

$$P := \widetilde{X} \times_{\pi_1(X,x)} G \tag{1.42}$$

where $\widetilde{X}$ is the universal cover of X, seen as a principal $\pi_1(X, x)$-bundle, and $\pi_1(X, x)$ acts on $\widetilde{X}$ on the right and on G by left multiplication via ρ. Let

$$\mathrm{Ad}(P) := P \times_G \mathfrak{g} = \widetilde{X} \times_{\pi_1(X,x)} \mathfrak{g} \tag{1.43}$$

be the adjoint bundle (G acts on $\mathfrak{g}$ via Ad, and so $\pi_1(X, x)$ acts via $\mathrm{Ad} \circ \rho$). Let

$$L := \mathscr{E}^\bullet(X, \mathrm{Ad}(P)) \tag{1.44}$$

be the DG Lie algebra of $\mathcal{C}^\infty$ differential forms with values in the local system of Lie algebras $\mathrm{Ad}(P)$. Locally, an element of L is given by a sum $\sum \alpha_i \otimes u_i$ where α_i is a differential form and u_i is an element of $\mathfrak{g}$. The differential is given by

$$d(\alpha \otimes u) := d(\alpha) \otimes u \tag{1.45}$$

(this is because P is a *flat* bundle) and the Lie bracket is given by

$$[\alpha \otimes u, \beta \otimes v] := (\alpha \wedge \beta) \otimes [u, v]. \tag{1.46}$$

As we constructed it, there is a canonical identification of fibers $\mathrm{Ad}(P)_x \simeq \mathfrak{g}$.

One part of the main theorem of Goldman and Millson can be stated as follows.

Theorem 1.29 ([GM88])**.** *There is a canonical equivalence of functors from local Artin algebras to groupoids*

$$\mathrm{Def}(L, -) \xrightarrow{\approx} \mathrm{Def}(\rho, -) \tag{1.47}$$

given by holonomy.

Sketch of proof. Roughly, a representation ρ of $\pi_1(X, x)$ into G modulo conjugation is interpreted as a flat principal G-bundle P modulo gauge transformations via the usual holonomy construction. Recall that connections on P form an affine space over $\mathscr{E}^1(X, \mathrm{Ad}(P))$, and that a flat connection on P defines a differential on $L = \mathscr{E}^\bullet(X, \mathrm{Ad}(P))$ giving it the structure of a DG Lie algebra. Then flat connections correspond to solutions of the Maurer-Cartan equation in L and L^0 is the Lie algebra of the gauge group; the exponential action of L^0 is the infinitesimal action of the gauge group. In their article, Goldman and Millson interpret deformations of ρ into $G(A)$ (which is again a Lie group over $\mathbf{k}$) as flat A-linear connections on the principal $G(A)$-bundle $P \times_{G(\mathbf{k})} G(A)$ which in some sense restrict to the given flat connection on P, and again the conjugation of deformed representations corresponds to the gauge equivalence of deformed connections. This gives the equivalence (1.47). Of course it induces an isomorphism between the deformation functors Def_L and Def_ρ, which are the sets of isomorphism classes of these groupoids. □

However, we are really interested in the functor $\mathrm{Def}_0(\rho, -)$ (1.34) (which is the set of objects of the groupoid $\mathrm{Def}(\rho, -)$) on the right side because it is isomorphic to the pro-representable functor associated with $\widehat{\mathcal{O}}_\rho$ (1.33) so it allows us to access, via the pro-Yoneda Lemma 1.25, to $\widehat{\mathcal{O}}_\rho$ itself. This works by replacing the deformation functor of L by its augmented version, which is some sort of gauge fixing procedure.

For this, let

$$\varepsilon_x : L = \mathscr{E}^\bullet(X, \mathrm{Ad}(P)) \longrightarrow \mathrm{Ad}(P)_x \simeq \mathfrak{g} \tag{1.48}$$

be the augmentation of L given by evaluating 0-forms at x and higher-degree forms to zero.

Theorem 1.30 ([GM88])**.** *The equivalence* (1.47) *induces an isomorphism of functors from local Artin algebras to sets*

$$\mathrm{Def}_{L,\varepsilon_x} \xrightarrow{\simeq} \mathrm{Def}_0(\rho, -). \tag{1.49}$$

Proof. Given $A \in \mathbf{Art_k}$, start from the equivalence

$$\operatorname{Def}(L, A) \xrightarrow{\approx} \operatorname{Def}(\rho, A) \tag{1.50}$$

where $\operatorname{Def}(L, A)$ is an action groupoid for $\exp(L^0 \otimes \mathfrak{m}_A)$, $\operatorname{Def}(\rho, A)$ is an action groupoid for $G^0(A) = \exp(\mathfrak{g} \otimes \mathfrak{m}_A)$ (see (1.38)), and the morphism

$$\exp(L^0 \otimes \mathfrak{m}_A) \longrightarrow \exp(\mathfrak{g} \otimes \mathfrak{m}_A)$$

is induced by ε_x. Then $\exp(\mathfrak{g} \otimes \mathfrak{m}_A)$ acts on itself on the left via

$$e^\alpha . e^\beta := e^\beta * e^{-\alpha}$$

(in the exponential group, $e^{-\alpha}$ is the inverse for e^α) such that the pull-back of this action via ε_x to an action of $\exp(L^0 \otimes \mathfrak{m}_A)$ on $\exp(\mathfrak{g} \otimes \mathfrak{m}_A)$ is precisely the action appearing in Remark 1.22. One then applies the operation $\bowtie G^0(A)$ (Definition 1.4) to both sides of the equivalence (1.50) to get by Lemma 1.5 an equivalence

$$\operatorname{Def}(L, \varepsilon_x, A) \xrightarrow{\approx} \operatorname{Def}(\rho, A) \bowtie G^0(A). \tag{1.51}$$

Now check carefully that there is an equivalence of groupoids

$$\operatorname{Def}_0(\rho, A) \xrightarrow{\approx} \operatorname{Def}(\rho, A) \bowtie G^0(A) \tag{1.52}$$

where the left-hand side is a set considered as a discrete groupoid (i.e. with only identity morphisms), sending $\tilde{\rho}$ to $(\tilde{\rho}, 1_{G^0(A)})$. Then apply π_0 to both sides of 1.51 gives the desired isomorphism. □

Remark 1.31. The whole construction we described is compatible with the change of coefficients from $\mathbb{R}$ to $\mathbb{C}$. If G is defined over $\mathbb{R}$ and

$$\rho : \pi_1(X, x) \longrightarrow G(\mathbb{R})$$

is a real representation, giving us a real DG Lie algebra L, then one can also consider ρ as a representation into $G(\mathbb{C})$ which has as Lie algebra $\mathfrak{g} \otimes \mathbb{C}$. It gives us a complex principal bundle $P_\mathbb{C}$ and a complex DG Lie algebra $L_\mathbb{C}$ which is simply $L \otimes \mathbb{C}$. On one side the associated deformation functors are defined on local Artin algebras over $\mathbb{R}$ and allow us to access to $\widehat{\mathcal{O}}_\rho$, and on the other side there are deformation functors on local Artin algebras over $\mathbb{C}$ that allow us to access to $\widehat{\mathcal{O}}_\rho \otimes \mathbb{C}$.

Now we see how to use this theorem to get information and structure on $\widehat{\mathcal{O}}_\rho$. It amounts to a careful study of L and its augmentation, or by Lemma 1.23 to a DG Lie algebra M which is quasi-isomorphic to L and thus controls the same deformation problem, and to a study of how the deformation functor of M is pro-represented. Furthermore, one can try to do Hodge theory by comparing controlling algebras over $\mathbb{R}$ and over $\mathbb{C}$, leading to the Hodge theory of $\widehat{\mathcal{O}}_\rho$.

1.2 Derived deformation theory

We start introducing much more powerful tools for working with deformation theory. We saw in the preceding section that we have to work with a deformation functor associated to an augmented DG Lie algebra

$$\varepsilon : L \longrightarrow \mathfrak{g}.$$

We are going to show that the functor $\mathrm{Def}(L, \varepsilon, -)$ of Definition 1.21 is in fact the deformation functor associated with an L_∞ algebra, following the article of Fiorenza-Manetti [FM07].

This L_∞ algebra is an algebraic structure on the mapping cone of ε. By the way, L_∞ algebras naturally have a deformation functor which extends the deformation functor for DG Lie algebras and this has been intensively studied in the literature since the first examples of deformation problems controlled by DG Lie algebras appeared. From this literature, we extract a functorial construction of an algebra that pro-represents the augmented deformation functor and is invariant under quasi-isomorphisms. This is a very strong result that improves much the classical theory and is an important step to re-write it.

The main result of this section is the very simple Lemma 1.55 that claims that the augmented deformation functor defined by Eyssidieux-Simpson was also worked out independently by Manetti and put in his very general framework of L_∞ algebras and extended deformation functors. This simple and crucial observation is the main motivation for re-writing the classical theory with L_∞ algebras. Also this L_∞ algebra structure is very explicit and thus will have good compatibility with Hodge theory.

In this whole section $\mathbf{k}$ is a fix field of characteristic zero and we work with various kind of algebras and coalgebras over $\mathbf{k}$.

1.2.1 L_∞ algebras

First we need to recall more about graded vector spaces and algebras. L_∞ algebras are better described in terms of codifferentials on conilpotent graded coalgebras. For these two parts, we refer to the lecture notes of Manetti [Man04, § VIII–IX] and the book of Loday-Valette [LV12] that are very well written and contain all the technical details.

Coalgebras

Definition 1.32. A *DG coalgebra* over $\mathbf{k}$ is the data of a DG vector space X together with a morphism called *comultiplication*

$$\Delta : X \longrightarrow X \otimes X. \tag{1.53}$$

The fact that Δ is a morphism of DG vector spaces, together with the construction of the differential on the tensor product (1.10), implies that d is a *coderivation* meaning that

$$\Delta \circ d = (d \otimes \mathrm{id} + \mathrm{id} \otimes d) \circ \Delta. \tag{1.54}$$

Since d is of degree 1, if one writes $\Delta(x) = \sum u_i \otimes v_i$ for $x \in X$ this last equation has to be written

$$\Delta(d(x)) = \sum \Big(d(u_i) \otimes v_i + (-1)^{|u_i|}\, u_i \otimes d(v_i) \Big). \tag{1.55}$$

The coalgebra is said to be *coassociative* if the bracket satisfies

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta : X \longrightarrow X^{\otimes 3}. \tag{1.56}$$

A *counit* is a morphism

$$\varepsilon : X \longrightarrow \mathbf{k} \tag{1.57}$$

such that

$$(\mathrm{id} \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} : X \longrightarrow X, \tag{1.58}$$

in particular it is identified with a morphism of DG coalgebras for the trivial comultiplication on $\mathbf{k}$ given by

$$\Delta(x) = x \otimes 1 = 1 \otimes x \tag{1.59}$$

and ε commutes with d. The coalgebra X is said to be *counital* if it is equipped with a counit. It is *cocommutative* if the bracket satisfies

$$\sigma_{X,X} \circ \Delta = \Delta \tag{1.60}$$

where σ is the interchange map (1.11). A *morphism* of DG coalgebras

$$f : (X, \Delta_X) \longrightarrow (Y, \Delta_Y)$$

is given by a morphism $f : X \to Y$ of DG vector spaces such that

$$(f \otimes f) \circ \Delta_X = \Delta_Y \circ f. \tag{1.61}$$

If X, Y are equipped with counits then we also require f to commute with the counits. Our coalgebras are always assumed to be coassociative and cocommutative (but *not* counital) and these form a category $\mathbf{DG{-}CoAlg_k}$. Forgetting the differential, one gets the full subcategory of *graded coalgebras* denoted by $\mathbf{G{-}CoAlg_k}$ and forgetting the grading one gets the full subcategory of coalgebras $\mathbf{CoAlg_k}$.

For the moment we work only with graded coalgebras and we will put a differential later on them.

Definition 1.33. A graded coalgebra X has a canonical filtration over $\mathbb{N}$ by sub-graded coalgebras given by

$$X_n := \mathrm{Ker}\left(\Delta^n : X \to X^{\otimes(n+1)}\right) \tag{1.62}$$

where Δ^n is the iterated comultiplication defined inductively by

$$\Delta^n := \left(\mathrm{id}^{\otimes i} \otimes \Delta \otimes \mathrm{id}^{\otimes(n-i-1)}\right) \circ \Delta^{n-1} : X \longrightarrow X^{\otimes(n+1)} \tag{1.63}$$

independently of i by coassociativity (see [Man04, VIII.10]). The graded coalgebra X is said to be *conilpotent* (terminology of [LV12, 1.2.4]) if its canonical filtration is exhaustive, i.e.

$$X = \bigcup_{n \geq 0} X_n. \tag{1.64}$$

This condition is also called *connected* in [Qui69, B.3] and *locally nilpotent* in [Man04, VIII.13] (where *nilpotent* means that the canonical filtration is finite). Up to adding a counit, this is the same condition as the *unital* coalgebras in [Hin01, 2.1.1].

Remark 1.34. We see that this definition doesn't make sense if X has a counit: by the axiom (1.58) if $\Delta(x) = 0$ then

$$(\varepsilon \otimes \mathrm{id})(\Delta(x)) = x = 0.$$

Our notions for coalgebras are much easier stated when we work without counits. Anyway one can always add a counit as does Hinich [Hin01, § 2.1], see also [LV12, § 1.2.1]. Let us describe this procedure: if X is a graded coalgebra with a counit ε, a *unit* for X is an element $u \in X$ of degree zero such that $\Delta(u) = u \otimes u$ and $\varepsilon(u) = 1$. Equivalently, it is a morphism of graded coalgebras $\mathbf{k} \to X$ for the trivial coalgebra structure on $\mathbf{k}$ (1.59). Then there is a canonical decomposition

$$X = \mathbf{k}.u \oplus \overline{X}, \quad \overline{X} := \mathrm{Ker}(\varepsilon)$$

where $\overline{X}$ is the *reduced coalgebra* equipped with the *reduced coproduct*

$$\overline{\Delta} := \Delta - \mathrm{id} \otimes u - u \otimes \mathrm{id}.$$

Then $\overline{X}$ has no unit nor counit. And by the same formulas one can add u to $\overline{X}$ to recover X. As is expressed in the next lemma, this is dual to working with augmented algebras or only with their maximal ideal: to an algebra $\mathfrak{m}$ without unit, one can consider $A := \mathbf{k} \oplus \mathfrak{m}$ as an algebra with unit $1 \in \mathbf{k}$ (which determines in an obvious way how to extend the multiplication to A) and augmentation the projection to $\mathbf{k}$. Then $\mathfrak{m}$ is the corresponding maximal ideal of A. Since we are more used to work with algebras rather than coalgebras, we choose to work with coalgebras without units but add them to their dual algebras.

Lemma 1.35. *The linear dual*

$$X^* := \underset{\mathbf{Vect}}{\mathrm{Hom}}(X, \mathbf{k}) \tag{1.65}$$

of a conilpotent coalgebra X is the maximal ideal of a complete local algebra.

Proof. First X^* has a multiplication. Namely the comultiplication

$$\Delta : X \longrightarrow X \otimes X$$

dualizes to a linear map

$$\Delta^* : (X \otimes X)^* \longrightarrow X^*$$

that one can pre-compose by the canonical map

$$\begin{array}{ccc} X^* \otimes X^* & \longrightarrow & (X \otimes X)^* \\ \varphi \otimes \psi & \longmapsto & \big((x, y) \mapsto \varphi(x)\psi(y)\big) \end{array} \tag{1.66}$$

to get a multiplication μ in $\mathfrak{m} := X^*$. See [Swe69, § 1.1–1.3]. The axioms of coassociativity and cocommutativity for (X, Δ) tell us respectively that $(\mathfrak{m}, \mu)$ is associative and commutative. Consider $A := \mathbf{k} \oplus \mathfrak{m}$, with the multiplication induced by μ such that $1 \in \mathbf{k}$ is a unit for A. Then $\mathfrak{m}$ is a maximal ideal of A. The dual of the conilpotency condition tells us precisely that A is complete with respect to $\mathfrak{m}$. And then A is necessarily local. □

But for the moment our interest in conilpotent coalgebras lies also in the fact that there is an explicit description of the cofree objects. For this we need to describe the symmetric algebra in the differential graded setting.

Definition 1.36 ([Man04, § VIII.1])**.** Let $\mathfrak{S}(r)$ be the symmetric group on r elements. Given a DG vector space V, homogeneous elements $v_1, \dots, v_r$, and a permutation $\tau \in \mathfrak{S}(r)$, the *Koszul sign*

$$\varepsilon(\tau; v_1, \dots, v_r) \tag{1.67}$$

is defined to be the sign such that the canonical isomorphism

$$V_1 \otimes \cdots \otimes V_r \xrightarrow{\simeq} V_{\tau(1)} \otimes \cdots \otimes V_{\tau(r)}$$

(where $V_1, \dots, V_r$ are r copies of V), given by using the isomorphisms (1.11) in any order, is given by

$$v_1 \otimes \cdots \otimes v_r \longmapsto \varepsilon(\tau; v_1, \dots, v_r)\, v_{\tau(1)} \otimes \cdots \otimes v_{\tau(r)}. \tag{1.68}$$

Thus by definition the sign of the transposition of v_1 and v_2 is

$$\varepsilon(\tau; v_1, v_2) = (-1)^{|v_1| \cdot |v_2|}.$$

We also denote by $\varepsilon(\tau)$ the *signature* of τ. By definition if τ is the transposition of v_1 and v_2 then $\varepsilon(\tau) = -1$.

Definition 1.37 ([Man04, VIII.2])**.** Inside $\mathfrak{S}(r)$, the set of *unshuffles* of type (p, q) (for $p + q = r$), denoted by $\mathfrak{S}(p, q)$, is the set of permutations whose restrictions to

$$\{1, \dots, p\}, \quad \{p+1, \dots, p+q\}$$

are increasing.

Definition 1.38. Let V be a DG vector space. Denote by $V^{\otimes r}$ its r-th tensor power

$$V^{\otimes r} := V \otimes \cdots \otimes V \quad (r \geq 0). \tag{1.69}$$

For $v \in V$ the element $v \otimes \cdots \otimes v \in V^{\otimes r}$ is denoted by $v^{\otimes r}$. By definition the r-th *symmetric power* of V is the quotient of $V^{\otimes r}$ by the ideal generated by elements

$$v_1 \otimes \cdots \otimes v_n - \varepsilon(\tau; v_1, \dots, v_n)\, v_{\tau(1)} \otimes \cdots \otimes v_{\tau(r)}, \quad \tau \in \mathfrak{S}(r). \tag{1.70}$$

We denote it by $V^{\odot r}$. The *symmetric algebra* on V is

$$\mathrm{Sym}(V) := \bigoplus_{r \geq 0} V^{\odot r} \tag{1.71}$$

and the *reduced symmetric algebra* is

$$\mathrm{Sym}^+(V) := \bigoplus_{r \geq 1} V^{\odot r} \tag{1.72}$$

Similarly, the r-th *exterior power* of V is the quotient of $V^{\otimes r}$ by the ideal generated by elements

$$v_1 \otimes \cdots \otimes v_n - \varepsilon(\tau)\, \varepsilon(\tau; v_1, \dots, v_n)\, v_{\tau(1)} \otimes \cdots \otimes v_{\tau(r)}, \quad \tau \in \mathfrak{S}(r). \tag{1.73}$$

We denote it by $V^{\wedge r}$, and the *exterior algebra* is

$$\Lambda(V) := \bigoplus_{r\geq 0} V^{\wedge r}. \tag{1.74}$$

There is also a reduced exterior algebra $\Lambda^+(V)$.

For the moment we only need this for graded vector spaces, forgetting the differential.

Definition 1.39. Let V be a graded vector space over $\mathbf{k}$. The *cofree conilpotent graded coalgebra* on V is the graded vector space $\mathrm{Sym}^+(V)$ turned into a graded coalgebra with the comultiplication (see [LV12, 1.2.9] and [Man04, VIII.24])

$$\Delta(v_1 \odot \cdots \odot v_r) := \\ \sum_{p=0}^{r} \sum_{\tau\in\mathfrak{S}(p,r-p)} \varepsilon(\tau; v_1, \ldots, v_r) \left(v_{\tau(1)} \odot \cdots \odot v_{\tau(p)}\right) \otimes \left(v_{\tau(p+1)} \odot \cdots \odot v_{\tau(r)}\right). \tag{1.75}$$

This defines a functor

$$\mathscr{F}^c_{\textbf{G-CoAlg}} : \mathbf{G{-}Vect_k} \longrightarrow \mathbf{G{-}CoAlg_k} \tag{1.76}$$

having the adjunction property

$$\operatorname*{Hom}_{\mathbf{G-CoAlg}} \left(X, \mathscr{F}^c_{\textbf{G-CoAlg}}(V)\right) = \operatorname*{Hom}_{\mathbf{G-Vect}}(X, V) \tag{1.77}$$

if X is conilpotent. This last one is the definition of being *cofree*, which is being a *right* adjoint to the forgetful functor from graded coalgebras to DG vector spaces, whereas a *free object* corresponds to a *left* adjoint.

The canonical filtration is simply given by

$$\mathscr{F}^c_{\textbf{G-CoAlg}}(V)_n := \bigoplus_{r=1}^{n} V^{\odot r} \tag{1.78}$$

which is clearly respected by the comultiplication.

Remark 1.40. Here it is really important that the object we construct is cofree in the category of *conilpotent* DG coalgebras. The general cofree coalgebra exists but is much harder to describe, see [Swe69, 6.4.1].

Lemma 1.41 ([LV12, 1.2.2])**.** *A coderivation Q on the cofree conilpotent graded coalgebra on V is uniquely determined by its composition with the projection* $\mathrm{Sym}^+(V) \to V$*, that is, by a sequence of linear maps of degree* 1

$$q_r : V^{\odot r} \longrightarrow V \quad (r \geq 1). \tag{1.79}$$

One recovers Q by the formula (see [Man04, VIII.34])

$$Q(v_1 \odot \cdots \odot v_r) = \\ \sum_{p=1}^{r} \sum_{\tau\in\mathfrak{S}(p,r-p)} \varepsilon(\tau, v_1, \ldots, v_r)\, q_p(v_{\tau(1)} \odot \cdots \odot v_{\tau(p)}) \odot v_{\tau(p+1)} \odot \cdots \odot v_{\tau(r)}. \tag{1.80}$$

Remark that if V had a differential d, the coderivation Q induced by d only is the same as the differential of $\mathrm{Sym}^+(V)$ induced by the symmetric product of DG vector spaces.

L_∞ algebras

We are going to apply the preceding formulas, combined with shifts of DG vector spaces.

Definition 1.42. Let V be a DG vector space over $\mathbf{k}$. Define the *r-shift* of V to be the DG vector space $V[r]$ with components

$$V[r]^n := V^{n+r} \tag{1.81}$$

and differential

$$d_{V[r]} := (-1)^r \, d_V. \tag{1.82}$$

For $r = 1$ this is called the *suspension* of V and for $r = -1$ the *desuspension*.

Summing up everything, the definition of L_∞ algebra can be given in a very short way.

Definition 1.43. Let L be a graded vector space over $\mathbf{k}$. A L_∞ *algebra structure* on L is the data of a codifferential Q on the cofree conilpotent graded coalgebra $\mathscr{F}^c_{\mathbf{G\text{-}CoAlg}}(L[1])$.

Let us be more explicit. By Lemma 1.41, the codifferential Q is defined uniquely by a family of linear maps of degree 1

$$q_r : (L[1])^{\odot r} \longrightarrow L[1], \quad r \geq 1 \tag{1.83}$$

with all axioms encoded in the condition $Q \circ Q = 0$. If $v \in L$ is of degree n we denote by $v[1]$ the same element viewed of degree $n-1$ in $L[1]$ (since $L[1]^{n-1} = L^n$). If $\mathbf{k}[1]$ is the DG vector space with $\mathbf{k}$ in degree -1, with its canonical element $1[1]$, there is a canonical isomorphism

$$\begin{array}{rcl} L \otimes \mathbf{k}[1] & \xrightarrow{\sim} & L[1] \\ v \otimes (1[1]) & \longmapsto & (-1)^{|v|} \, v[1] \end{array} \tag{1.84}$$

inducing canonical isomorphisms (see [Man04, § IX.1])

$$(L \otimes \mathbf{k}[1])^{\otimes r} \simeq (L[1])^{\otimes r} \xrightarrow{\simeq} L^{\otimes r}[r] \tag{1.85}$$

given by

$$v_1[1] \otimes \cdots \otimes v_r[1] \longmapsto (-1)^{\sum_{j=1}^r (r-j)\cdot|v_j|} \, (v_1 \otimes \cdots \otimes v_r)[r] \tag{1.86}$$

and in particular isomorphisms for maps

$$[1] : \mathrm{Hom}^{2-r}(L^{\otimes r}, L) \xrightarrow{\simeq} \mathrm{Hom}^1((L[1])^{\otimes r}, L[1]) \tag{1.87}$$

given by

$$f[1](v_1[1] \otimes \cdots \otimes v_r[1]) = (-1)^{r + \sum_{j=1}^r (r-j)\cdot|v_j|} \, f(v_1 \otimes \cdots \otimes v_r)[1]. \tag{1.88}$$

This last isomorphism exchanges symmetric and exterior products, inducing

$$[1] : \mathrm{Hom}^{2-r}(L^{\wedge r}, L) \xrightarrow{\simeq} \mathrm{Hom}^1((L[1])^{\odot r}, L[1]). \tag{1.89}$$

Thus, the operations q_r can be seen (via desuspension) as anti-symmetric linear maps of degree $2-r$

$$\ell_r : L^{\wedge r} \longrightarrow L, \quad r \geq 1 \tag{1.90}$$

but in this forms the axioms of L_∞ algebras are not so easy to write down and involve some combinatorics (see[Man99, § IX.2]). In degree 1, the relation between q_1 and ℓ_1 is simply

$$q_1(v[1]) := -\ell_1(v)[1] \tag{1.91}$$

and the axiom $Q \circ Q = 0$ translates into

$$\ell_1 \circ \ell_1 = 0 \tag{1.92}$$

so that (L, ℓ_1) can be considered as a DG vector space; we also denote ℓ_1 by d. In degree 2, the relation between q_2 and ℓ_2 is

$$q_2(u[1] \odot v[1]) := (-1)^{|u|}\, \ell_2(u \wedge v)[1] \tag{1.93}$$

and the condition $Q \circ Q = 0$ becomes

$$\ell_1(\ell_2(u \wedge v)) - \ell_2(\ell_1(u) \wedge v) - (-1)^{|u|}\, \ell_2(u \wedge \ell_1(v)) = 0 \tag{1.94}$$

which means that ℓ_1 is a derivation for ℓ_2. In degree 3 one gets

$$\begin{aligned} (-1)^{|u|\cdot|w|}\, \ell_2(\ell_2(u \wedge v) \wedge w) + (-1)^{|v|\cdot|w|}\, \ell_2(\ell_2(w \wedge u) \wedge v) + (-1)^{|u|\cdot|v|}\, \ell_2(\ell_2(v \wedge w) \wedge u) \\ = (-1)^{|u|\cdot|w|+1} \Big(\ell_1 \circ \ell_3(u \wedge v \wedge w) + \ell_3(\ell_1(u) \wedge v \wedge w) \\ + (-1)^{|u|}\, \ell_3(u \wedge \ell_1(v) \wedge w) + (-1)^{|u|+|v|}\, \ell_3(u \wedge v \wedge \ell_1(w)) \Big) \end{aligned} \tag{1.95}$$

and this is the *Jacobi identity up to homotopy.* If $\ell_3 = 0$ then ℓ_2 is a Lie bracket and we denote it also by $[-,-]$.

From these conditions, one sees that any DG Lie algebra can be seen as a L_∞ algebra with ℓ_1 being the differential and ℓ_2 the Lie bracket, and with $\ell_r = 0$ for $r \geq 3$. Conversely, L_∞ algebras with $\ell_r = 0$ for $r \geq 3$ are exactly DG Lie algebras. However L_∞ algebras contain the same information as DG Lie algebras at the level of cohomology.

Proposition 1.44. *The cohomology of a L_∞ algebra L is a graded Lie algebra.*

Proof. On $H(L)$, $d = \ell_1$ is zero and in (1.95) the defect of ℓ_2 to satisfy the Jacobi identity involves coboundaries and becomes zero in cohomology. Then one forgets the higher operations and $(H(L), [-,-] = \ell_2)$ is a graded Lie algebra. □

Now let us introduce the morphisms.

Definition 1.45. If L, M are L_∞ algebras, a (strong) *morphism* from L to M is a morphism of graded vector spaces $L \to M$ commuting with all the operations ℓ_r $(r \geq 1)$. It is called a *quasi-isomorphism* if it induces a quasi-isomorphism

$$(L, \ell_1) \xrightarrow{\approx} (M, \ell_1). \tag{1.96}$$

We denote by $\mathbf{L}_{\infty,\mathbf{k}}$ the category of L_∞ algebras over $\mathbf{k}$ with these morphisms.

Definition 1.46. If a graded vector space L is equipped with a L_∞ algebra structure (given as either the maps ℓ_r or q_r) we denote by $\mathscr{C}(L)$ the graded coalgebra $\mathscr{F}^c_{\mathbf{G\text{-}CoAlg}}(L[1])$ equipped with the codifferential Q. The assignment $L \mapsto \mathscr{C}(L)$ is a functor

$$\mathscr{C} : \mathbf{L}_{\infty,\mathbf{k}} \longrightarrow \mathbf{DG\text{-}CoAlg}_{\mathbf{k}} \tag{1.97}$$

that is also called the *bar construction.* The canonical filtration of the conilpotent coalgebra $\mathscr{C}(L)$ is given here by

$$\mathscr{C}_s(L) := \bigoplus_{r=1}^{s} (L[1])^{\odot r} \tag{1.98}$$

and is a filtration by sub-DG coalgebras called the *bar filtration.*

Remark 1.47. If L is a DG Lie algebra this functor $\mathscr{C}$ is the functor considered already by Quillen [Qui69, § B], see also [Hin01, § 2.2].

The fact that the bar filtration is a filtration by sub-DG coalgebras appears clearly from the formulas for the coproduct on a cofree coalgebra (1.75) and for the extension of a codifferential to the whole cofree coalgebra (1.80).

Remark 1.48. The notion of morphism we introduced is called *strong morphism.* It is quite clear that such a morphism f induces a morphism of DG coalgebras

$$\mathscr{C}(f) : \mathscr{C}(L) \longrightarrow \mathscr{C}(M)$$

and it is the strongest notion of morphism one can imagine. A *weak morphism* or L_∞-*morphism* (Definition 1.66) from L to M is simply a morphism of DG coalgebras from $\mathscr{C}(L)$ to $\mathscr{C}(M)$. We will see in the next section that a quasi-isomorphism of L_∞ algebras induces via $\mathscr{C}$ a quasi-isomorphism of DG coalgebras, so that this notion of quasi-isomorphism is also the strongest one.

The following theorem is one of the main theorems of deformation theory with L_∞ algebras (compare to the fundamental Theorem 1.19). We will develop it much in section 1.2.3.

Theorem 1.49 (See sect. 1.2.3)**.** *To any L_∞ algebra L over $\mathbf{k}$ is associated a deformation functor*

$$\mathrm{Def}_L : \mathbf{Art}_{\mathbf{k}} \longrightarrow \mathbf{Set} \tag{1.99}$$

which restricts for DG Lie algebras to the usual Deligne-Goldman-Millson deformation functor. A quasi-isomorphism of L_∞ algebras induces an isomorphism between their associated deformation functors.

Let us just say for the moment that this deformation functor is a quotient of the Maurer-Cartan functor, where the Maurer-Cartan equation for $x \in L^1 \otimes \mathfrak{m}_A$ ($A \in \mathbf{Art}_{\mathbf{k}}$) is

$$0 = \sum_{r=1}^{\infty} \frac{\ell_r(x^{\wedge r})}{r!}.$$

This is well-defined since $\ell_r(x^{\wedge r}) = 0$ for r such that $(\mathfrak{m}_A)^r = 0$. One sees that if L is a DG Lie algebra then

$$\ell_1(x) = d(x), \quad \ell_2(x \wedge x) = [x, x], \quad \ell_{r\geq 3} = 0$$

so that one recovers the usual Maurer-Cartan equation.

1.2.2 L_∞ algebra structure on the mapping cone

Now we use the preceding theory. We will see that L_∞ algebras arise naturally when dealing with augmented DG Lie algebras via their mapping cone. Then we show how one can re-write the theory of Goldman and Millson with L_∞ algebras.

Naive bracket on the mapping cone

Definition 1.50. Let $f : V \to W$ be a morphism between DG vector spaces. One constructs the *mapping cone* of f which is the DG vector space $\mathrm{Cone}(f)$ with

$$\mathrm{Cone}(f)^n := V^{n+1} \oplus W^n \tag{1.100}$$

and differential

$$d^n_{\mathrm{Cone}(f)}(x,y) := \Big(- d_V^{n+1}(x),\ d_W^n(y) - f(x)\Big). \tag{1.101}$$

It is sometimes more natural to work with the *desuspended mapping cone* $\mathrm{Cone}(f)[-1]$, which has

$$\mathrm{Cone}(f)[-1]^n = V^n \oplus W^{n-1} \tag{1.102}$$

and differential

$$d^n_{\mathrm{Cone}(f)[-1]}(x,y) = \Big(d_V^n(x),\ f(x) - d_W^{n-1}(y)\Big). \tag{1.103}$$

Lemma 1.51. *If $f : V \to W$ is a morphism of DG vector spaces, there is a short exact sequence*

$$0 \longrightarrow W \longrightarrow \mathrm{Cone}(f) \longrightarrow V[1] \longrightarrow 0 \tag{1.104}$$

inducing a long exact sequence

$$\cdots \longrightarrow H^n(V) \longrightarrow H^n(W) \longrightarrow H^n(\mathrm{Cone}(f)) \longrightarrow H^{n+1}(V) \longrightarrow H^{n+1}(W) \longrightarrow \cdots \tag{1.105}$$

and one can replace $H^n(\mathrm{Cone}(f))$ by $H^{n+1}(\mathrm{Cone}(f)[-1])$.

Consider a morphism $\varepsilon : L \to M$ between DG Lie algebras. Let C be the desuspended mapping cone of ε, $\mathrm{Cone}(\varepsilon)[-1]$. Later on we will consider the case where M is concentrated in degree zero, and we will write $M = \mathfrak{g}[0]$ where $\mathfrak{g}$ is a Lie algebra.

Definition 1.52. On C, define the *naive bracket* to be the bilinear map

$$\Big[(x,u),(y,v)\Big] := \left([x,y],\ \frac{1}{2}\,[u,\varepsilon(y)] + \frac{(-1)^{|x|}}{2}\,[\varepsilon(x),v] \right) \tag{1.106}$$

for $x, y \in L$, $u, v \in M$.

Lemma 1.53. *The naive bracket* (1.106) *is anti-symmetric and satisfies Leibniz' rule with respect to the differential of the desuspended mapping cone.*

Proof. For the anti-symmetry write

$$\Big[(y,v),(x,u)\Big] = \left(-(-1)^{|y|\cdot|x|}\,[x,y],\ -\frac{(-1)^{|v|\cdot|x|}}{2}\,[\varepsilon(x),v] - \frac{(-1)^{|y|+|y|\cdot|u|}}{2}\,[u,\varepsilon(y)] \right)$$

and equal it with

$$-(-1)^{|x|\cdot|y|}\Big[(x,u),(y,v)\Big]$$

using $|u| = |x| - 1$ and $|v| = |y| - 1$. To check Leibniz' rule, compute

$$\begin{aligned} d\Big[(x,u),(y,v)\Big] = \Bigg(&[d(x),y] + (-1)^{|x|}[x,d(y)],\ [\varepsilon(x),\varepsilon(y)] \\ &-\frac{1}{2}[d(u),\varepsilon(y)] - \frac{(-1)^{|u|}}{2}[u,d\varepsilon(y)] - \frac{(-1)^{|x|}}{2}[d\varepsilon(x),v] - \frac{1}{2}[\varepsilon(x),d(v)]\Bigg) \end{aligned}$$

and verify that it is the same thing as

$$\begin{aligned} &\Big[d(x,u),(y,v)\Big] + (-1)^{|x|}\Big[(x,u),d(y,v)\Big] \\ =& \Big[(d(x),\ \varepsilon(x) - d(u)),\ (y,v)\Big] + (-1)^{|x|}\Big[(x,u),\ (d(y),\ \varepsilon(y) - d(v))\Big] \\ =& \left([d(x),y],\ \frac{1}{2}[\varepsilon(x) - d(u),\ \varepsilon(y)] + \frac{(-1)^{|d(x)|}}{2}[\varepsilon d(x),v]\right) \\ &+ (-1)^{|x|}\left([x,d(y)],\ \frac{1}{2}[u,\varepsilon d(y)] + \frac{(-1)^{|x|}}{2}[\varepsilon(x),\ \varepsilon(y) - d(v)]\right) \end{aligned}$$

using $\varepsilon d = d\varepsilon$ and $|d(x)| = |x| + 1$. □

However if one tries to check the Jacobi identity by computing the double bracket

$$\begin{aligned} &\Big[(x,u),\Big[(y,v),(z,w)\Big]\Big] = \\ &\left([x,[y,z]],\ \frac{1}{2}[u,[\varepsilon(y),\varepsilon(z)]] + \frac{(-1)^{|x|}}{4}[\varepsilon(x),[v,\varepsilon(z)]] + \frac{(-1)^{|x|+|y|}}{4}[\varepsilon(x),[\varepsilon(y),w]]\right) \end{aligned}$$

one sees that it is clearly satisfied in the L part but a priori not in M and thus the naive bracket may not be a Lie bracket. It will satisfy only the Jacobi identity *up to homotopy* and this is the notion of L_∞ algebra structure on C.

The construction of Fiorenza-Manetti

Now the main motivation for this section is the following theorem and construction.

Theorem 1.54 ([FM07]). *If $\varepsilon : L \to M$ is a morphism between DG Lie algebras, then the desuspended mapping cone C of ε has a canonical L_∞ algebra structure such that ℓ_1 is the usual differential of C and ℓ_2 is the naive bracket* (1.106). *The associated deformation functor on $\mathbf{Art_k}$ is isomorphic to the functor of isomorphism classes of the groupoid with set of objects (for $A \in \mathbf{Art_k}$)*

$$\left\{(x,e^\alpha) \in (L^1 \otimes \mathfrak{m}_A) \times \exp(M^0 \otimes \mathfrak{m}_A)\ \middle|\ d(x) + \frac{1}{2}[x,x] = 0,\ e^\alpha * \varepsilon(x) = 0\right\} \quad (1.107)$$

and morphisms given by

$$\begin{aligned} &\operatorname{Hom}((x,e^\alpha),(y,e^\beta)) := \\ &\quad\left\{(\lambda,\mu) \in (L^0 \otimes \mathfrak{m}_A) \times (M^{-1} \otimes \mathfrak{m}_A)\ \middle|\ e^\lambda.x = y,\ e^\beta = e^{d(\mu)} * e^\alpha * e^{-\varepsilon(\lambda)}\right\}. \quad (1.108) \end{aligned}$$

This L_∞ algebra structure is functorial from the category of morphisms between DG Lie algebras to the category of L_∞ algebras with their strong morphisms.

As promised, the following very simple remark is one of the main motivations for this thesis.

Lemma 1.55. *Applied to an augmentation $\varepsilon : L \to \mathfrak{g}$ to a Lie algebra $\mathfrak{g}$, the above deformation functor associated with the L_∞ algebra structure on the desuspended mapping cone of ε coincides with the augmented deformation functor $\mathrm{Def}_{L,\varepsilon}$ of Definition 1.21.*

Proof. Apply the theorem with $M = \mathfrak{g}[0]$. Then for degree reasons, for $x \in L^1 \otimes \mathfrak{m}_A$ we have $\varepsilon(x) = 0$ and also $\mu = 0$ so $e^{d(\mu)}$ is the identity of $\exp(L^0 \otimes \mathfrak{m}_A)$. Then we see immediately that the set of objects (1.27) and (1.107) are the same, and the set of morphisms (1.28) and (1.108) are also the same. □

Let us give one simple application of working with the L_∞ algebra structure on the mapping cone by re-writing our Lemma 1.23.

Lemma 1.56. *Let $\varphi : L \xrightarrow{\approx} L'$ be a quasi-isomorphism of augmented DG Lie algebras commuting with the augmentations as in the diagram*

$$\begin{array}{ccc} L & \xrightarrow[\varphi]{\approx} & L' \\ & {\scriptstyle\varepsilon}\searrow \quad \swarrow{\scriptstyle\varepsilon'} & \\ & \mathfrak{g}. & \end{array}$$

Then φ induces a quasi-isomorphism of the mapping cones C, C' and an isomorphism of deformation functors

$$\mathrm{Def}_{L,\varepsilon} \xrightarrow{\simeq} \mathrm{Def}_{L,\varepsilon'}\,.$$

Proof. Write the morphism induced by φ between the long exact sequences for the mapping cones:

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & H^n(L) & \longrightarrow & H^n(\mathfrak{g}[0]) & \longrightarrow & H^{n+1}(C) & \longrightarrow & H^{n+1}(L) & \longrightarrow & H^{n+1}(\mathfrak{g}[0]) & \longrightarrow \cdots \\ & \varphi\downarrow\simeq & & \mathrm{id}\downarrow\simeq & & \downarrow & & \varphi\downarrow\simeq & & \mathrm{id}\downarrow\simeq & \\ \cdots \longrightarrow & H^n(L') & \longrightarrow & H^n(\mathfrak{g}[0]) & \longrightarrow & H^{n+1}(C) & \longrightarrow & H^{n+1}(L') & \longrightarrow & H^{n+1}(\mathfrak{g}[0]) & \longrightarrow \cdots \end{array}$$

By the five lemma, φ induces a quasi-isomorphism from C to C', and by functoriality it is a morphism of L_∞ algebras. Then quasi-isomorphic L_∞ algebras have isomorphic deformation functors (Theorem 1.49). □

Now we want to describe explicitly the higher operations on the mapping cone. Let us take back our morphism $\varepsilon : L \to M$ of DG Lie algebras, C the desuspended mapping cone and $C[1]$ the (usual) mapping cone.

Theorem 1.57 ([FM07, 5.5])**.** *The higher brackets of the L_∞ algebra C, given as the components q_r of a codifferential Q on $\mathscr{C}(C)$, are given as follows (we write $x, y, \dots$ for elements of L and $u, v, \dots$ for elements of M). First for the operations q_1 and q_2:*

$$\begin{aligned} q_1(x,u) &= (-d(x),\ d(u) - \varepsilon(x)) \\ q_2(x \odot y) &= (-1)^{|x|}\,[x,y] \\ q_2(u \otimes y) &= \frac{(-1)^{|u|+1}}{2}\,[u,\varepsilon(y)] \\ q_2(u \odot v) &= 0. \end{aligned}$$

From here we see that q_1, or its associated operation ℓ_1, corresponds to the differential of the mapping cone. Via the rule of signs (1.93) *q_2 and its associated operation ℓ_2 correspond to the naive bracket.*

For the higher brackets, if $r + k \geq 3$ and $k \neq 1$

$$q_{r+k}(u_1 \odot \cdots \odot u_r \otimes x_1 \odot \cdots \odot x_k) = 0$$

and for $r \geq 2$

$$q_{r+1}(u_1 \odot \cdots \odot u_r \otimes x) = \\ -(-1)^{\sum_{i=1}^{r} |u_i|} \frac{B_r}{r!} \sum_{\tau \in \mathfrak{S}(r)} \varepsilon(\tau, u_1, \ldots, u_r) \, [u_{\tau(1)}, [u_{\tau(2)}, \ldots, [u_{\tau(r)}, \varepsilon(x)] \ldots]] \tag{1.109}$$

where the B_n are the Bernouilli numbers *([FM07, 5.4]).*

Remark 1.58. In the case $M = \mathfrak{g}[0]$ is concentrated in degree 0 these relations simplify but there still remains a priori non-zero higher operations in each order. First

$$q_1(x, u) = (-d(x), -\varepsilon(x)).$$

Then $q_2(u \otimes y)$ is zero if y is not of degree 0. Finally the last relation for q_{r+1} where all terms u_i have degree 0 is zero if x is not of degree 0 and else

$$q_{r+1}(u_1 \odot \cdots \odot u_r \otimes x) = -\frac{B_r}{r!} \sum_{\tau \in \mathfrak{S}(r)} \varepsilon(\tau) \, [u_{\tau(1)}, [u_{\tau(2)}, \ldots, [u_{\tau(r)}, \varepsilon(x)] \ldots]].$$

1.2.3 Deformation functor of a L_∞ algebra

This whole section is devoted to explaining the main Theorem 1.49. The deformation functor of L_∞ or DG Lie algebras has been studied a lot in the literature. Furthermore we want the pro-representability of this functor. Under the assumption that a L_∞ algebra L over $\mathbf{k}$ has $H^n(L) = 0$ for $n \leq 0$ we will get that the deformation functor

$$\mathrm{Def}_L : \mathbf{Art_k} \longrightarrow \mathbf{Set}$$

is isomorphic to

$$A \longmapsto \operatorname*{Hom}_{\mathbf{CoAlg}} \Big((\mathfrak{m}_A)^*, H^0(\mathscr{C}(L)) \Big)$$

and under some finite-dimensionality hypothesis the dual of $H^0(\mathscr{C}(L))$ is the maximal ideal of a pro-Artin algebra, to which we add a unit to form the algebra $\mathbf{k} \oplus H^0(\mathscr{C}(L))^*$, and one can dualize to get

$$A \longmapsto \operatorname*{Hom}_{\mathbf{ProArt}} \Big(\mathbf{k} \oplus H^0(\mathscr{C}(L))^*, A \Big).$$

Recalling the pro-Yoneda Lemma 1.25, this says that there is a functorial pro-representing algebra for Def_L.

Construction of the deformation functor

If L is a L_∞ algebra and A is a local Artin algebra with maximal ideal $\mathfrak{m}_A$, one can extend the operations ℓ_r to $L \otimes \mathfrak{m}_A$ simply by the rule

$$\ell_r((x_1 \otimes a_1) \wedge \cdots \wedge (x_r \otimes a_r)) := \ell_r(x_1 \wedge \cdots \wedge x_r) \otimes (a_1 \cdots a_r) \tag{1.110}$$

for $x_1, \ldots, x_r \in L$ and $a_1, \ldots, a_r \in \mathfrak{m}_A$. If $x \in L^1 \otimes \mathfrak{m}_A$ then

$$x^{\wedge r} = x \wedge \cdots \wedge x \in L^r \otimes (\mathfrak{m}_A)^r$$

(the left exponent r is a grading, the right one is a power of the ideal) so that

$$\ell_r(x^{\wedge r}) \in L^2 \otimes (\mathfrak{m}_A)^r$$

and for r such that $(\mathfrak{m}_A)^r = 0$ then this is zero.

Definition 1.59. Let L be a L_∞ algebra over $\mathbf{k}$. Its associated *Maurer-Cartan functor* is the functor

$$\mathrm{MC}_L : \mathbf{Art_k} \longrightarrow \mathbf{Set} \tag{1.111}$$

defined by

$$\mathrm{MC}_L(A) := \left\{ x \in L^1 \otimes \mathfrak{m}_A \;\middle|\; \sum_{r=1}^{\infty} \frac{\ell_r(x^{\wedge r})}{r!} = 0 \right\}. \tag{1.112}$$

This sum is in fact finite since $\ell_r(x^{\wedge r}) = 0$ if $(\mathfrak{m}_A)^r = 0$.

Lemma 1.60 ([LV12, 13.2.9])**.** *The linear dual X^* of a conilpotent coalgebra X whose canonical filtration is finite and by finite-dimensional sub-coalgebras is the maximal ideal of a local Artin algebra. Conversely if A is a local Artin algebra then the dual $(\mathfrak{m}_A)^*$ is such a conilpotent coalgebra.*

Proof. Recall Lemma 1.35. Denote by $\mathfrak{m}$ the dual X^* of a conilpotent coalgebra X, then $A := \mathbf{k} \oplus \mathfrak{m}$ is a complete local algebra with maximal ideal $\mathfrak{m}$. Under the assumption that the canonical filtration of X is finite, then $\mathfrak{m}$ is in fact nilpotent, so A is a local Artin algebra.

Conversely, if A is a local Artin algebra with maximal ideal $\mathfrak{m}_A$ then $(\mathfrak{m}_A)^*$ is a coalgebra: the multiplication in A, that we restrict to $\mathfrak{m}_A$,

$$\mathfrak{m}_A \otimes \mathfrak{m}_A \longrightarrow \mathfrak{m}_A$$

dualizes to a linear map

$$(\mathfrak{m}_A)^* \longrightarrow (\mathfrak{m}_A \otimes \mathfrak{m}_A)^*$$

that one can compose by the inverse of the canonical map (1.66)

$$(\mathfrak{m}_A)^* \otimes (\mathfrak{m}_A)^* \longrightarrow (\mathfrak{m}_A \otimes \mathfrak{m}_A)^*$$

(which is invertible only if $\mathfrak{m}_A$ is finite-dimensional) to get a comultiplication in $(\mathfrak{m}_A)^*$. This comultiplication is coassociative and cocommutative if the multiplication of A satisfies the corresponding dual axioms. And since $\mathfrak{m}_A$ is nilpotent, the canonical filtration of X is finite. □

The following simple proposition is one important step for considering extended deformation functors, from categories of DG Artin algebras to sets or simplicial sets. It gives a more powerful point of view on the Maurer-Cartan equation.

Proposition 1.61. *If L is a L_∞ algebra and A is a local Artin algebra then*

$$\mathrm{MC}_L(A) = \operatorname*{Hom}_{\mathbf{DG-CoAlg}}\Big((\mathfrak{m}_A)^*, \mathscr{C}(L)\Big) \tag{1.113}$$

(by the previous lemma $(\mathfrak{m}_A)^*$ is a conilpotent coalgebra).

Proof. By the cofree property (adjunction (1.77)), a morphism of graded coalgebras

$$\varphi : (\mathfrak{m}_A)^* \longrightarrow \mathscr{C}(L)$$

is determined uniquely by a morphism of graded vector spaces

$$\psi : (\mathfrak{m}_A)^* \longrightarrow L[1]$$

that one can see as an element $x \in L^1 \otimes \mathfrak{m}_A$. The Maurer-Cartan equation is then the condition for φ to respect the codifferential (see [Man04, § IX.4]) written in terms of x. □

Roughly, the deformation functor of a L_∞ algebra is the quotient of the Maurer-Cartan functor by the homotopy equivalences. However, there is a technical subtle point in the definition of homotopy equivalence for these Maurer-Cartan elements related to the fact that the Maurer-Cartan equation is defined only for elements $x \in L^1 \otimes \mathfrak{m}_A$, since they have $\ell_r(x^{\wedge r}) = 0$ for $r \gg 0$ and the Maurer-Cartan equation is a finite sum. So we give first the definition of homotopy equivalence in the case of DG Lie algebras.

Definition 1.62 (See [BG76, § 2])**.** Denote by $\Omega_{\mathbf{k}}(\Delta)$ the simplicial DG algebra of polynomial differential forms on the standard simplex over $\mathbf{k}$. It has components given explicitly by

$$\Omega_{\mathbf{k}}(\Delta^n) := \frac{\mathbf{k}[t_0, \dots, t_n, dt_0, \dots, dt_n]}{(1 - \sum t_i, \sum dt_i)} \tag{1.114}$$

where t_i has degree 0, dt_i has degree 1 and $d(t_i) = dt_i$. The 1-dimensional component of $\Omega_{\mathbf{k}}(\Delta)$ is more simply written as $\mathbf{k}[t, dt]$, and the tensor product with a DG algebra A is written $A[t, dt]$. The two face maps $\mathbf{k}[t, dt] \to \mathbf{k}$ are simply the maps evaluating t to 0 (resp. 1) and dt to 0.

Definition 1.63. Let L be a DG Lie algebra over $\mathbf{k}$ and let A be a local Artin algebra. A *homotopy* between two elements x, y of $\mathrm{MC}(L \otimes \mathfrak{m}_A)$ is an element

$$H \in \mathrm{MC}\Big(L \otimes \mathfrak{m}_A \otimes \mathbf{k}[t, dt]\Big) \tag{1.115}$$

such that $H_{|t=0} = x$ and $H_{|t=1} = y$.

This corresponds exactly to the notion of homotopy (or, path) in the simplicial set

$$\mathrm{MC}\Big(L \otimes \mathfrak{m}_A \otimes \Omega_{\mathbf{k}}(\Delta)\Big)$$

which has for simplices of dimension n

$$\mathrm{MC}\left(L \otimes \mathfrak{m}_A \otimes \Omega_{\mathbf{k}}(\Delta^n)\right)$$

between the two 0-simplices x and y. See the references [Hov99] and [GJ99] for simplicial sets and their use in homotopy theory. The important fact is that this recovers the notion that we wrote first with gauge transformations:

Proposition 1.64 ([Man99, 5.5])**.** *Let L be a DG Lie algebra and let A be a local Artin algebra. Then two elements of $\mathrm{MC}(L \otimes \mathfrak{m}_A)$ are homotopic (as above) if and only if they are gauge equivalent (i.e. equivalent under the gauge action of $\exp(L^0 \otimes \mathfrak{m}_A)$).*

The formula (1.113) shows that one can extend the functor MC_L for L_∞ algebras to the category of DG algebras A augmented over $\mathbf{k}$ (such an algebra splits canonically as $A = \mathbf{k} \oplus \mathfrak{m}_A$) that are globally finite-dimensional over $\mathbf{k}$ and such that the augmentation ideal $\mathfrak{m}_A$ is nilpotent; because then $(\mathfrak{m}_A)^*$ is a conilpotent DG coalgebra. This is one possible definition of the category of *DG Artin algebras.* If A is a local Artin algebra, then $A[t, dt]$ is an increasing union of sub-DG algebras $A[t, dt]_\varepsilon$ over $\varepsilon > 0$ (see [Man04, § IX.5]) that are DG Artin algebras: for $\varepsilon > 0$ and $n > 0$ denote by $A^{\lceil n\varepsilon \rceil}$ the sub-DG algebra of A generated by products of at least $\lceil n\varepsilon \rceil$ elements of $\mathfrak{m}_A$ and then

$$A[t, dt]_\varepsilon := A \oplus \bigoplus_{n>0} \left(A^{\lceil n\varepsilon \rceil} t^n \oplus A^{\lceil n\varepsilon \rceil} t^{n-1} dt\right) \subset A[t, dt] \tag{1.116}$$

and it appears clearly that $A[t, dt]$ is the union of all $A[t, dt]_\varepsilon$ over $\varepsilon > 0$.

Definition 1.65. Let L be a L_∞ algebra over $\mathbf{k}$. Let A be a local Artin algebra. Two elements x, y of $\mathrm{MC}_L(A)$ are said to be *homotopic* if there exists $\varepsilon > 0$ and an element

$$H \in \mathrm{MC}_L\left(A[t, dt]_\varepsilon\right) \tag{1.117}$$

such that $H_{|t=0} = x$ and $H_{|t=1} = y$. The *deformation functor* Def_L of L is the quotient of MC_L by the equivalence relation generated by homotopies.

The Proposition 1.64 proves that this extends the deformation functor for DG Lie algebras.

Invariance under quasi-isomorphisms

The fact that a quasi-isomorphism of L_∞ algebras induces an isomorphism of their deformation functor is explained in [Man04, IX.22]. We will review this and the notion of L_∞-morphism between L_∞ algebras. A good references for this is the chapters 10 and 11 of [LV12]: everything is well-written in the general theory of operads.

Definition 1.66. Let L, M be L_∞ algebras. A L_∞*-morphism* from L to M is the data of a morphism of conilpotent DG coalgebras

$$f : \mathscr{C}(L) \longrightarrow \mathscr{C}(M). \tag{1.118}$$

Such a morphism automatically preserves the canonical filtrations and by the cofree property (adjunction (1.77)) it can be seen as a collection of linear maps of degree 0

$$f_r[1] : (L[1])^{\odot r} \longrightarrow M[1], \quad r \geq 1 \tag{1.119}$$

or equivalently by desuspension as linear maps of degree $r-1$

$$f_r : L^{\wedge r} \longrightarrow M \tag{1.120}$$

satisfying some relations with the operations ℓ_r that are not so easy to write in this form (see [LV12, 10.2.7]). We write such a morphism

$$f : L \rightsquigarrow M. \tag{1.121}$$

The map f_1, which is a morphism of DG vector spaces (i.e. of degree 0 and commutes with $d = \ell_1$), is called the *linear part* of f; equivalently it is $f_1[1]$ which commutes with q_1.

Remark 1.67. A strong morphism from L to M induces a morphism of DG coalgebras from $\mathscr{C}(L)$ to $\mathscr{C}(M)$ so that strong morphisms correspond to L_∞-morphisms with $f_r = 0$ for $r > 1$ (see [Man04, IX.4]), also called *linear morphisms*.

Definition 1.68. A L_∞-morphism $f : L \rightsquigarrow M$ between L_∞ algebras is called a *L_∞-quasi-isomorphism* if its linear part f_1 is a quasi-isomorphism

$$f_1 : (L, \ell_1) \xrightarrow{\approx} (M, \ell_1), \tag{1.122}$$

equivalently

$$f_1[1] : (L[1], q_1) \xrightarrow{\approx} (M[1], q_1) \tag{1.123}$$

is a quasi-isomorphism.

From this description one sees that a L_∞-quasi-isomorphism which is induced by a strong morphism is the same thing as a quasi-isomorphism as in Definition 1.45. Then one of the main theorems, for which we reproduce a complete proof, is:

Theorem 1.69 ([Man04, IX.9])**.** *A L_∞-quasi-isomorphism $f : L \rightsquigarrow M$ (a fortiori, a strong quasi-isomorphism) between L_∞ algebras induces a quasi-isomorphism of DG coalgebras*

$$\mathscr{C}(f) : \mathscr{C}(L) \xrightarrow{\approx} \mathscr{C}(M). \tag{1.124}$$

The essential tool of the proof, to which we will come back several times, is the computation of the spectral sequence for the bar filtration.

Theorem 1.70 (Convergence theorem for spectral sequences [LV12, 1.5.1], [God58, § I.4.3])**.** *Let V be a DG vector space with a decreasing filtration $F^\bullet$ (by sub-DG vector spaces). Assume that the filtration F is bounded above (i.e. for each n there exists N such that $F^pV^n = 0$ for $p \geq N$) and exhaustive (i.e. $V^n = \bigcup_p F^pV^n$). Then for fixed q the term $E_0^{p,q}$ is zero for $p \gg 0$ and for fixed (p,q) the differential $d_r^{p,q}$ is zero for $r \gg 0$. There is a canonical isomorphism*

$$E_\infty^{p,q} \xrightarrow{\simeq} \frac{F^pH^{p+q}(V)}{F^{p+1}H^{p+q}(V)}. \tag{1.125}$$

where $E_\infty^{p,q}$ is the inductive limit of the terms $E_r^{p,q}$.

Concretely, for fixed (p, q), the sequence $E_r^{p,q}$ may not stabilize but for r big enough it is obtained only by taking quotients. There are canonical maps

$$E_r^{p,q} \longrightarrow E_s^{p,q}, \quad s \geq r \gg 0$$

whose limit defines $E_\infty^{p,q}$.

This applies in particular for $\mathscr{C}(L)$ with the bar filtration of Definition 1.46 turned into a decreasing filtration: it is bounded above since the bar filtration is bounded below, and exhaustive since $\mathscr{C}(L)$ is the union of the $\mathscr{C}_s(L)$ over $s \geq 1$.

Proof of Theorem 1.69. Of course, the statement for L_∞-quasi-isomorphisms implies the one for strong quasi-isomorphisms. So assume that f has components f_r and that f_1 is a quasi-isomorphism. Let $\mathscr{C}_s$ be the bar filtration By definition the graded pieces are given by

$$\operatorname{Gr}_s^{\mathscr{C}} \mathscr{C}(L) = \frac{\mathscr{C}_s(L)}{\mathscr{C}_{s-1}(L)} = (L[1])^{\odot s}, \quad s \geq 1$$

with differential induced by q_1 only. So the induced map between the graded pieces is

$$\operatorname{Gr}_s^{\mathscr{C}}(f) = (f_1[1])^{\odot s} : (L[1], q_1)^{\odot s} \longrightarrow (M[1], q_1)^{\odot s}. \tag{1.126}$$

By hypothesis

$$f_1[1] : (L[1], q_1) \xrightarrow{\simeq} (M[1], q_1)$$

is a quasi-isomorphism, and by the Künneth formula

$$H((L[1], q_1)^{\odot s}) \simeq (H(L[1], q_1))^{\odot s}$$

for any $s \geq 1$. This follows from the usual Künneth formula for the tensor product and the fact that the symmetric product is a quotient of the tensor product by a finite group. So combining these shows that the map $\operatorname{Gr}_s^{\mathscr{C}}(f)$ of (1.126) is a quasi-isomorphism.

Now compute the spectral sequence. By definition (since $\mathscr{C}$ is increasing we work with $-s$), for all integer q,

$${}_{\mathscr{C}}E_0^{-s,q}(\mathscr{C}(L)) = \operatorname{Gr}_s^{\mathscr{C}} \mathscr{C}(L)^q$$

and

$${}_{\mathscr{C}}E_1^{-s,q}(\mathscr{C}(L)) = H^{-s+q}(\operatorname{Gr}_s^{\mathscr{C}} \mathscr{C}(L)).$$

So by the previous analysis there is an isomorphism

$${}_{\mathscr{C}}E_1^{-s,q}(\mathscr{C}(L)) \xrightarrow{\simeq} {}_{\mathscr{C}}E_1^{-s,q}(\mathscr{C}(M)).$$

Now we use the Theorem 1.70: this induces an isomorphism at the inductive limit

$${}_{\mathscr{C}}E_\infty^{-s,q}(\mathscr{C}(L)) \xrightarrow{\simeq} {}_{\mathscr{C}}E_\infty^{-s,q}(\mathscr{C}(M))$$

which gives

$$\operatorname{Gr}_s^{\mathscr{C}} H^{-s+q}(\mathscr{C}(L)) \xrightarrow{\simeq} \operatorname{Gr}_s^{\mathscr{C}} H^{-s+q}(\mathscr{C}(M)), \quad s \geq 1$$

from which we deduce, step by step by induction on s (since the bar filtration is bounded below), an isomorphism

$$H(\mathscr{C}(L)) \xrightarrow{\simeq} H(\mathscr{C}(M)).$$

See also [God58, I.4.3.1] for this argument. □

This has the following important consequence, which is the derived version of the classical Theorem 1.19. See the cited lectures notes and the original article [Man02] (Corollary 3.3 combined with section 5) for full proofs.

Theorem 1.71 ([Man04, IX.22])**.** *A L_∞-quasi-isomorphism $f : L \rightsquigarrow M$ between L_∞ algebras induces an isomorphism*

$$\mathrm{Def}_L \xrightarrow{\simeq} \mathrm{Def}_M$$

between their associated deformation functors.

Before coming to the pro-representability, let us add another point of view on L_∞ algebras. This is called the *rectification*, and is well-known from the general theory of DG operads.

Theorem 1.72 ([LV12, 11.4.6])**.** *Any L_∞ algebra is L_∞-quasi-isomorphic to a DG Lie algebra.*

Remark 1.73. Here we don't need to precise the direction or zig-zag of morphisms because L_∞-quasi-isomorphisms always admit an inverse ([LV12, 10.4.4]) and being L_∞-quasi-isomorphic is an equivalence relation. So at this point we see that working with L_∞ algebras up to L_∞-quasi-isomorphisms is much more convenient and flexible than working with DG Lie algebras up to quasi-isomorphisms. However there is an equivalence of homotopy categories between these ([LV12, 11.4.8]) expressing the fact that these two approaches contain the same information from the point of view of deformation theory.

Remark 1.74. This result combined to the main Theorem 1.71 implies that there is another description of the deformation functor of a L_∞ algebra L. Namely take any DG Lie algebra M which is L_∞-quasi-isomorphic to L and define the deformation functor of L to be Def_M (in the classical sense); up to isomorphism this does not depend on M. Actually, the article of Pridham [Pri10] shows that all these approaches to the deformation functor of a DG Lie or L_∞ algebra are equivalent (see in particular Remark 4.28 therein).

Pro-representability

The pro-representability theorem we use is the one proved by Hinich. This appears also earlier in the lectures notes of Kontsevich [Kon94].

Theorem 1.75 ([Hin01, § 9.3])**.** *Let L be a DG Lie algebra over $\mathbf{k}$ with $H^n(L) = 0$ for all $n \leq 0$. Then there is a canonical isomorphism*

$$\mathrm{Def}_L(A) = \operatorname*{Hom}_{\mathbf{CoAlg}}\Big((\mathfrak{m}_A)^*, H^0(\mathscr{C}(L))\Big), \quad A \in \mathbf{Art}_{\mathbf{k}}. \tag{1.127}$$

Remark 1.76. Actually, Hinich works with unital coalgebras. So we extract (1.127) simply by replacing his A by $\mathfrak{m}_A$. Conversely, from this formula, one can always add a unit and a counit to $H^0(\mathscr{C}(L))$ as in Remark 1.34. Then write this equality using the Hom set of unital coalgebras (i.e. requiring morphisms to preserve the units and counits). However this is easier to do after passing to the dual algebra.

Now we deduce the dualization. This step is, to our knowledge, never treated exactly as we need in the literature. It will follow from two lemmas.

Lemma 1.77. *The dual X^* of a conilpotent coalgebra X whose canonical filtration is by finite-dimensional sub-coalgebras is the maximal ideal of a pro-Artin algebra. Conversely if R is a pro-Artin algebra then the dual $(\mathfrak{m}_R)^*$ is a conilpotent coalgebra whose canonical filtration is by finite-dimensional sub-coalgebras. In particular if X is such a coalgebra and A is a local Artin algebra then*

$$\operatorname*{Hom}_{\mathbf{CoAlg}}((\mathfrak{m}_A)^*, X) = \operatorname*{Hom}_{\mathbf{ProArt}}(\mathbf{k} \oplus X^*, A). \tag{1.128}$$

Proof. This is just the extension of Lemma 1.60 to infinite filtrations, combined with the Lemma 1.12. Namely if X is a conilpotent coalgebra, then X^* is the maximal ideal $\mathfrak{m}_R$ of a complete local algebra $R = \mathbf{k} \oplus \mathfrak{m}_R$ and the canonical filtration of X is dual to the powers of $\mathfrak{m}_R$. When this filtration is by finite-dimensional sub-coalgebras, then R is pro-Artin. Conversely if R is a pro-Artin algebra then $R/(\mathfrak{m}_R)^n$ is a local Artin algebra for each n, so $(\mathfrak{m}_R/(\mathfrak{m}_R)^n)^*$ is a conilpotent coalgebra which is finite-dimensional, and R^* is the increasing union of these.

One deduce the equality (1.128) simply by linear algebra. Since $\mathfrak{m}_A$ is finite-dimensional, a morphism of coalgebras

$$f : (\mathfrak{m}_A)^* \longrightarrow X$$

factorizes through some finite-dimensional sub-coalgebra of X, that is, through some step X_n of the canonical filtration. Then by linear algebra for finite-dimensional vector spaces, f dualizes perfectly to a linear map

$$f^* : X_n^* \longrightarrow \mathfrak{m}_A$$

which is a morphism of algebras (without units). Then simply add the units: $\mathbf{k} \oplus X^*$ has unit $1 \in \mathbf{k}$ and f can also be seen as a morphism of algebras with units and preserving the maximal ideals (that is, a morphism of local algebras, a fortiori here a morphism of local Artin algebras)

$$f^* : \mathbf{k} \oplus X_n^* \longrightarrow A.$$

By definition this constructs a morphism

$$f^* : \mathbf{k} \oplus X^* \longrightarrow A$$

in the category of pro-Artin algebras. Going to the reverse direction is also easy: simply note that a morphism of pro-Artin algebra

$$g : R \longrightarrow A$$

factorizes though some quotient $R/(\mathfrak{m}_R)^n$ since the maximal ideal of A is nilpotent, and induces a morphism of algebras without units

$$g : \mathfrak{m}_R/(\mathfrak{m}_R)^* \longrightarrow \mathfrak{m}_A.$$

Then dualize as above, these algebras being finite-dimensional. □

Remark 1.78. We warn the reader of the following possible confusion. It is known that *any* coalgebra X is the colimit of its finite-dimensional sub-coalgebras (this is sometimes called the fundamental theorem of coalgebras, see [Swe69, § 2.2]) so that the dual algebra X^* is always a projective limit of finite-dimensional algebras. However in our Definition 1.9 a pro-Artin algebra is a projective limit of local Artin algebras in a very particular way and that is the difference with the treatment of this duality in the literature.

Lemma 1.79. *Let L be a L_∞ algebra with $H^n(L) = 0$ for $n \leq 0$ and $H^1(L)$ finite-dimensional. Then the filtration of $H^0(\mathscr{C}(L))$ induced by the bar filtration, which is the canonical filtration turning $H^0(\mathscr{C}(L))$ into a conilpotent coalgebra, is by finite-dimensional sub-coalgebras.*

Proof. First $\mathscr{C}(L)$ is a conilpotent coalgebra and the bar filtration is its canonical filtration: the comultiplication of $H^0(\mathscr{C}(L))$ is induced by the one of $\mathscr{C}(L)$, from which we see clearly that it is conilpotent and its canonical filtration is induced by the bar filtration.

So, as in the proof of Theorem 1.69, we compute the spectral sequence for the bar filtration for L. It starts with

$$\mathrm{Gr}^{\mathscr{C}}_s \mathscr{C}(L) = (L[1])^{\odot s}, \quad s \geq 1$$

with the differential induced by $\ell_1[1] = q_1$. By the hypothesis that $H^n(L) = 0$ for $n \leq 0$, with $H^n(L[1]) = H^{n+1}(L)$, the Künneth formula simply states here that

$$H^0(\mathrm{Gr}^{\mathscr{C}}_s \mathscr{C}(L)) = \bigoplus_{t_1+\cdots+t_s=0} \Big(H^{t_1}(L[1]) \odot \cdots \odot H^{t_s}(L[1])\Big) = (H^0(L[1]))^{\odot s} = (H^1(L))^{\odot s}$$

which is finite-dimensional if $H^1(L)$ is. This is also ${}_{\mathscr{C}}E_1^{-s,s}(\mathscr{C}(L))$. So we deduce that each terms ${}_{\mathscr{C}}E_r^{-s,s}(\mathscr{C}(L))$ as well as ${}_{\mathscr{C}}E_\infty^{-s,s}(\mathscr{C}(L))$ (defined as in the convergence Theorem 1.70) are finite-dimensional, being sub-quotients of ${}_{\mathscr{C}}E_1^{-s,s}(\mathscr{C}(L))$. And so is $\mathrm{Gr}^{\mathscr{C}}_s H^0(\mathscr{C}(L))$.

Since the bar filtration is bounded below this in turns implies (via an induction on s) that the induced filtration given by

$$H^0(\mathscr{C}(L))_s := \mathrm{Im}\Big(H^0(\mathscr{C}_s(L)) \to H^0(\mathscr{C}(L))\Big) \tag{1.129}$$

is by finite-dimensional sub-coalgebras. □

So we deduce:

Theorem 1.80 (Pro-representability)**.** *Let L be a L_∞ algebra with $H^n(L) = 0$ for $n \leq 0$ and $H^1(L)$ finite-dimensional. Then $\mathbf{k} \oplus H^0(\mathscr{C}(L))^*$ is a pro-Artin algebra that pro-represents Def_L. A quasi-isomorphism between such L_∞ algebras*

$$f : L \xrightarrow{\approx} M$$

induces an isomorphism between their deformation functors and their pro-representing objects making commutative the following diagram of isomorphism of functors from $\mathbf{Art_k}$ to **Set**

$$\begin{array}{ccc} \mathrm{Def}_L & = & \mathrm{Hom}\Big(\mathbf{k} \oplus H^0(\mathscr{C}(L))^*, -\Big) \\ f \downarrow \simeq & & \simeq \downarrow f \\ \mathrm{Def}_M & = & \mathrm{Hom}\Big(\mathbf{k} \oplus H^0(\mathscr{C}(M))^*, -\Big). \end{array} \tag{1.130}$$

Proof. Combine the preceding theorems. The L_∞-quasi-isomorphisms preserve our conditions on cohomology. So one can assume that L is a DG Lie algebra (as in Remark 1.74) and apply the representability theorem of Hinich 1.75. Then combine the two preceding lemmas to get that $\mathbf{k} \oplus H^0(\mathscr{C}(L))^*$ is a pro-Artin algebra and that one can dualize to get

$$\mathrm{Def}_L(A) = \operatorname*{Hom}_{\mathbf{ProArt}}\Big(\mathbf{k} \oplus H^0(\mathscr{C}(L))^*, A\Big), \quad A \in \mathbf{Art_k}. \tag{1.131}$$

Both sides of this equation are functorial in L and invariant under quasi-isomorphisms. □

Remark 1.81. This theorem, though expressed as a theorem in classical deformation theory, was clearly unknown at the time of the first article of Goldman and Millson [GM88]. It doesn't seem possible to understand it without appealing to L_∞ algebras and extending deformation functors to some categories of DG Artin algebras, which gives the natural interpretation of the Maurer-Cartan elements as in Proposition 1.61. In [GM90], they proved that there is a complete local algebra that pro-represents the functor Def_L and that is invariant under quasi-isomorphisms, but the construction is not functorial. It was also known by them, in [KM98, § 14], that this algebra can be constructed as a quotient of the formal power series on $H^1(L)$, which is the tangent space to the deformation functor Def_L ([Man99, § 3.c] for the classical point of view). But in the form we give, the pro-representability theorem is a great improvement of the theory.

Examples

Let us give simple examples of computation with the functor $\mathscr{C}$.

Example 1.82. Assume that $L = \mathfrak{g}[0]$ is a Lie algebra, concentrated in degree zero. Then $L[1]$ is concentrated in degree -1 and

$$\mathrm{Sym}^+(L[1]) = \bigoplus_{r \geq 1} (\Lambda^r \mathfrak{g})[r].$$

So $\mathscr{C}(L)$ is the usual Chevalley-Eilenberg complex computing the Lie algebra homology of $\mathfrak{g}$ (it is in negative degrees with a differential of degree $+1$, so it really computes homology) and its dual computes the Lie algebra cohomology.

Example 1.83. Assume that $L = V[-1]$ is a vector space concentrated in degree 1, with zero bracket and zero differential. Then $L[1] = V[0]$ and

$$\mathrm{Sym}^+(L[1]) = \mathrm{Sym}^+(V)[0]$$

with zero differential. Its dual, to which we add $\mathbf{k}$, is the algebra of formal power series on V. This is coherent with the fact that the deformation functor is simply

$$\mathrm{Def}_L(A) = V \otimes \mathfrak{m}_A$$

which is smooth.

Example 1.84. Assume that L has $d = 0$ and $L^n = 0$ for $n \leq 0$. In this case the deformation functor is simply given by the Maurer-Cartan equation $[x, x] = 0$ for $x \in L^1$. Then $L[1]$ is concentrated in non-negative degree and the part of degree 0 of $\mathscr{C}(L)$ is

$\mathrm{Sym}^+(L^1)$. The differential is given by the Lie bracket. So we see that $H^0(\mathscr{C}(L))$ is dual (up to adding $\mathbf{k}$) to the algebra of power series at 0 of the quadratic cone defined by the Maurer-Cartan equation.

In particular this applies if L is formal, that is, quasi-isomorphic to a DG Lie algebra with $d = 0$, and with $H^0(L) = 0$: up to quasi-isomorphism one can assume that L is of the above form.

Chapter 2

Hodge theory

In this chapter we combine Hodge theory with the algebraic constructions we described in the preceding chapter. The geometric situation of Goldman and Millson will not only give us an augmented DG Lie algebra, to which we associated a L_∞ algebra in section 1.2.2, but actually a more complicated object having both the structures of mixed Hodge complex and of augmented DG Lie algebra. Since we report all the geometric constructions to the chapter 3 we deal here only with the algebraic part of these.

So we have to study first mixed Hodge structures and mixed Hodge complexes. These last ones are more subtle to work with but are for us the essential tools for constructing mixed Hodge structures. So we need to introduce them and fix our point of view in section 2.1.

Then section 2.2 contains the heart of the work. There we define and study the structure of *augmented mixed Hodge diagram of Lie algebras*. The main results are Theorem 2.38, stating that from this one obtains a *mixed Hodge diagram of L_∞ algebras* via the construction of Fiorenza-Manetti, and the Theorem 2.44 saying that one can apply the functor $\mathscr{C}$ (the bar construction) on these and get again a mixed Hodge diagram, this time of conilpotent DG coalgebras. So its H^0 carries a mixed Hodge structure and we will combine this with the pro-representability Theorem 1.80.

2.1 Mixed Hodge structures and mixed Hodge complexes

We adopt all the notations and classical constructions from the original articles of Deligne [Del71b] and [Del74] concerning filtrations and Hodge theory. We also refer to the book of Peters-Steenbrink [PS08]. This first sections contains nothing new except that we fix some point of view on mixed Hodge complexes.

So let us describe some conventions and notations. All our filtrations are indexed by $\mathbb{Z}$. The use of upper indices $F^\bullet$ will denote a decreasing filtration with its graded parts $\mathrm{Gr}_F^\bullet$ and lower indices $W_\bullet$ correspond to an increasing filtration with graded parts $\mathrm{Gr}_\bullet^W$. One can always turn an increasing filtration $W_\bullet$ into a decreasing one by letting

$$W^k := W_{-k} \tag{2.1}$$

and thus we will mainly state results involving only one filtration for an increasing filtration W. This comes from the fact that in defining mixed Hodge complexes, one complex carries the weight filtration W and one other carries the two filtrations W, F, thus results holding for one filtration appear first for W. We say that an element x of a filtered vector space $(K, W_\bullet)$ is of *weight* k if k is the smallest of the integers i such that $x \in W_i$.

We assume that all our filtrations of vector spaces are *finite* ($W_k K = 0$ for $k \ll 0$ and $W_k K = K$ for $k \gg 0$) and filtrations of complexes are *biregular* (on each component K^n, the induced filtration is finite) unless we explicitly state that we work with inductive or projective limits of such objects.

2.1.1 Filtrations and mixed Hodge structures

In this section, if $\mathbf{k}$ is a field, $\mathbf{L}/\mathbf{k}$ a field extension and $K_\mathbf{k}$ a vector space over $\mathbf{k}$, we will always denote by

$$K_\mathbf{L} := K_\mathbf{k} \otimes_\mathbf{k} \mathbf{L} \tag{2.2}$$

the extension of scalars of $K_\mathbf{k}$ from $\mathbf{k}$ to $\mathbf{L}$.

In section 1.1 of [Del71b] it is recalled how to construct naturally filtrations on sub-objects, quotients, direct sums, tensor products etc. We review briefly what we need.

Filtered and bifiltered DG vector spaces

Definition 2.1. A *filtered DG vector space* over $\mathbf{k}$ is a DG vector space K over $\mathbf{k}$ with an increasing filtration $W_\bullet$ by sub-DG vector spaces. This implies that

$$W_k K = \bigoplus_n W_k K^n \tag{2.3}$$

and that

$$d(W_k K^n) \subset W_k K^{n+1}. \tag{2.4}$$

A *bifiltered DG vector space* is a DG vector space K with an increasing filtration $W_\bullet$ and a decreasing filtration $F^\bullet$, such that K is filtered for W and for F. Writing $W_k F^p := W_k \cap F^p$ this implies that

$$W_k F^p K = \bigoplus_n W_k F^p K^n. \tag{2.5}$$

Morphisms are required to preserve the filtrations. We denote by $\mathbf{Fil{-}DG{-}Vect_k}$ the category of filtered DG vector spaces and by $\mathbf{Fil^2{-}DG{-}Vect_k}$ the category of bifiltered DG vector spaces over $\mathbf{k}$.

Definition 2.2. Let $(K, W_\bullet)$ and $(L, W_\bullet)$ be filtered DG vector spaces. The following linear algebraic constructions have canonical filtrations:

1. The direct sum $K \oplus L$ with

$$W_k(K \oplus L) := (W_k K) \oplus (W_k L). \tag{2.6}$$

2. The tensor product $K \otimes L$ (defined in (1.9)) with

$$W_k(K \otimes L) := \bigoplus_{i+j=k} (W_i K) \otimes (W_j L). \tag{2.7}$$

We call it the *multiplicative extension* of the filtrations on K and on L.

3. By the same multiplicative extension rule, the symmetric and exterior powers also have natural filtrations. Using the direct sum, one gets filtrations on the whole tensor, symmetric and exterior algebras.
4. The linear dual K^* which has

$$(K^*)^n := \mathrm{Hom}(K^{-n}, \mathbf{k}) \tag{2.8}$$

is filtered with

$$W_k(K^*) := (W_{-k}K)^* \tag{2.9}$$

(so this is more naturally a decreasing filtration if W is increasing).
5. The cohomology $H(K)$ gets an induced filtration

$$W_k H(K) = \mathrm{Im}\Big(H(W_k K) \to H(K)\Big). \tag{2.10}$$

6. Finally the field $\mathbf{k}$ has a trivial filtration with

$$W_{-1}(\mathbf{k}) = 0, \quad W_0(\mathbf{k}) = \mathbf{k}. \tag{2.11}$$

Of course, all these constructions carry over directly to bifiltered DG vector spaces.

The last operation we will be interested in is the shift. But here there are two ways of shifting filtrations and when working with bifiltered DG vector spaces we will use both.

Definition 2.3. Given a filtered DG vector space $(K, W_\bullet)$ and an integer r, the r-shift (Definition 1.42) $K[r]$ has an induced filtration given by

$$W_k(K[r]^n) = W_k(K^{n+r}) \subset K^{n+r} = K[r]^n. \tag{2.12}$$

Similarly, if $(K, W_\bullet, F^\bullet)$ is bifiltered then $K[r]$ has induced filtrations W, F.

Definition 2.4. Given a filtered DG vector space $(K, W_\bullet)$ and an integer r, the r-shift of W is the filtration $W[r]$ defined by

$$W[r]_k K := W_{k-r} K. \tag{2.13}$$

The need for these two shifts is expressed by the following result.

Lemma 2.5. *If $(K, W_\bullet)$ is a filtered DG vector space then*

$$\mathrm{Gr}_k^{W[r]}(K[r]) = (\mathrm{Gr}_{k-r}^W K)[r]. \tag{2.14}$$

Remark 2.6. Because of this, there are two natural choices of induced filtration on $K[r]$. One is called W and the other one is $W[r]$. If $(K, W_\bullet, F^\bullet)$ is bifiltered, then there are two choices we will use for inducing two filtrations on $K[r]$, denoted by $(K[r], W_\bullet, F^\bullet)$ and by $(K[r], W[r]_\bullet, F^\bullet)$. So we warn the reader that in a bifiltered DG vector space the two filtrations do not play the same role and the algebraic constructions for bifiltered DG vector spaces are *not all* obtained by applying two times the corresponding algebraic constructions for filtered DG vector spaces, as we will see right now.

Definition 2.7. Let $f : (K, W_\bullet) \to (L, W_\bullet)$ be a morphism between filtered DG vector spaces. Its mapping cone $\operatorname{Cone}(f)$ (Definition 1.50) is filtered with

$$W_k \operatorname{Cone}(f)^n := W_{k-1}K^{n+1} \oplus W_k L^n. \tag{2.15}$$

The desuspended mapping cone $\operatorname{Cone}(f)[-1]$ is filtered with

$$W_k \operatorname{Cone}(f)[-1]^n := W_k K^n \oplus W_{k+1} L^{n-1} \tag{2.16}$$

which is precisely $W[-1]$ of the filtration W on the mapping cone. If $(K, W_\bullet, F^\bullet)$ and $(L, W_\bullet, F^\bullet)$ are bifiltered and f respects also the filtration F, then the filtration F on the mapping cone is given by

$$F^p \operatorname{Cone}(f)^n := F^p K^{n+1} \oplus F^p L^n \tag{2.17}$$

and on the desuspended mapping cone it is simply

$$F^p \operatorname{Cone}(f)[-1]^n := F^p K^n \oplus F^p L^{n-1}. \tag{2.18}$$

Also, for the needs of mixed Hodge theory, Deligne introduces the décalage filtration.

Definition 2.8 ([Del71b, 1.3.3])**.** Let $(K, W_\bullet)$ be a filtered DG vector space. The *décalage* filtration of K is the filtration $(\operatorname{Dec} W)_\bullet$ defined by

$$(\operatorname{Dec} W)_k K^n := \left\{ x \in W_{k-n}K^n \;\middle|\; dx \in W_{k-n-1}K^{n+1} \right\}. \tag{2.19}$$

Its main properties are that for the induced filtration on cohomology

$$(\operatorname{Dec} W)_k H^n(K) = W_{k-n} H^n(K) = W[n]_k H^n(K) \tag{2.20}$$

and if K is concentrated in degree zero then $\operatorname{Dec} W = W$. For the induced spectral sequence there is an isomorphism

$${}_{\operatorname{Dec} W}E_r^{p,q} \xrightarrow{\simeq} {}_W E_{r+1}^{2p+q,-p}, \quad r \geq 1. \tag{2.21}$$

If a morphism of DG vector spaces $f : K \to L$ is a morphism of filtered DG vector spaces for some given filtrations on K and L we will briefly say that f is *compatible* with the filtrations.

Definition 2.9. We introduce the following categories of filtered algebras:

1. If A is a DG algebra with a filtration W as DG vector space, we say that it is a *filtered DG algebra* if its multiplication

$$\mu : A \otimes A \longrightarrow A$$

is compatible with the filtrations $W \otimes W$ induced on $A \otimes A$ and W on A. This reads concretely as

$$\mu(W_k A^n, W_\ell A^m) \subset W_{k+\ell} A^{n+m}. \tag{2.22}$$

If A has a unit one also requires the unit

$$1_A : \mathbf{k} \longrightarrow A$$

to be compatible with the trivial filtration on $\mathbf{k}$ (2.11).

2. Similarly if L is a DG Lie algebra with a filtration W as DG vector space, we say it is a *filtered DG Lie algebra* if its bracket

$$[-,-] : L \otimes L \longrightarrow L$$

is compatible with the induced filtrations. This reads as

$$[W_k L^n, W_\ell L^m] \subset W_{k+\ell} L^{n+m}. \tag{2.23}$$

3. If X is a coalgebra with a filtration W as DG vector space, we say it is a *filtered DG coalgebra* if its comultiplication

$$\Delta : X \longrightarrow X \otimes X$$

is compatible with the induced filtrations.

4. Finally if L is a L_∞ algebra with a filtration W as DG vector space, we say it is a *filtered L_∞ algebra* if all the operations in r variables of degree $2-r$

$$\ell_r : L^{\wedge r} \longrightarrow L, \quad r \geq 1$$

are compatible with the induced filtrations $W^{\wedge r}$ on $L^{\wedge r}$ (by the multiplicative rule as for the tensor product) and W on L (and *not* $W[2-r]$, see the remark below).

Remark 2.10. Let L be a L_∞ algebra with a filtration W as DG vector space. Then $d = \ell_1$ can be seen as a morphism of graded vector spaces

$$d \in \operatorname*{Hom}_{\mathbf{G-Vect}}(L, L[1]).$$

However d is *not* a morphism of filtered graded vector spaces if we put the filtration $W[1]$ on $L[1]$, namely this would mean

$$d(W_k L^n) \subset W_{k-1} L^{n+1}$$

but W is increasing so this has no reason to hold. That is why in the above definition of filtered L_∞ algebra, when ℓ_r is of degree $2-r$, we work with the filtration W and not $W[2-r]$ on the right-hand side.

Filtered quasi-isomorphisms

For a morphism between filtered DG vector spaces the condition of being a filtered quasi-isomorphism is slightly stronger that the condition of being a quasi-isomorphism compatible with the filtrations. In the former case one requires the morphism to identify all the graded pieces via a quasi-isomorphism. This notion behaves much better when working with mixed Hodge theory and spectral sequences.

Definition 2.11. A morphism of filtered DG vector spaces $f : (K, W_\bullet) \to (L, W_\bullet)$ is called a *filtered quasi-isomorphism* if for each k the induced morphism

$$\mathrm{Gr}_k^W(f) : \mathrm{Gr}_k^W(K) \longrightarrow \mathrm{Gr}_k^W(L) \tag{2.24}$$

is a quasi-isomorphism. If $f : (K, W_\bullet, F^\bullet) \to (L, W_\bullet, F^\bullet)$ is a morphism of bifiltered DG vector spaces, it is called a *bifiltered quasi-isomorphism* if for all k and p the induced morphism

$$\mathrm{Gr}_k^W \mathrm{Gr}_F^p(f) : \mathrm{Gr}_k^W \mathrm{Gr}_F^p(K) \longrightarrow \mathrm{Gr}_k^W \mathrm{Gr}_F^p(L) \tag{2.25}$$

is a quasi-isomorphism.

Remark 2.12. By the Zassenhauss lemma ([Del71b, 1.2.1]) the two graded pieces $\mathrm{Gr}_k^W \mathrm{Gr}_F^p$ and $\mathrm{Gr}_F^p \mathrm{Gr}_W^k$ are canonically isomorphic and one can invert them in the equation (2.25).

Lemma 2.13. *A filtered quasi-isomorphism as well as a bifiltered quasi-isomorphism is in particular a quasi-isomorphism.*

Proof. Do it first for the filtered case. Such a morphism $f : K \to L$ induces by definition an isomorphism at the first page of the spectral sequence for W:

$$ {}_W E_1(f) : {}_W E_1(K) \xrightarrow{\simeq} {}_W E_1(L). $$

Since the filtrations are biregular, these spectral sequences converge, in the sense that for fixed (k, n) the terms $E_r^{k,n}$ stabilize with r. So f induces an isomorphism

$$ {}_W E_\infty(f) : {}_W E_\infty(K) \xrightarrow{\simeq} {}_W E_\infty(L) $$

and in each degree n, f induces an isomorphism between the finitely many graded pieces

$$ \mathrm{Gr}_k^W(H^n(K)) \xrightarrow{\sim} \mathrm{Gr}_k^W(H^n(L)). $$

Then an induction on k concludes that $H^n(f)$ is an isomorphism: when W is a one-step filtration (such that there is only one non-zero graded piece) this is trivial and when it is two-step this is the three lemma.

The filtered case implies immediately the bifiltered case by treating one filtration at a time. □

Mixed Hodge structures

Definition 2.14. A (pure) *Hodge structure* of weight k over the field $\mathbf{k} \subset \mathbb{R}$ is the data of a finite-dimensional vector space $K_\mathbf{k}$ over $\mathbf{k}$ and a decreasing filtration $F^\bullet$ of $K_\mathbb{C}$ called the *Hodge filtration* such that F and its conjugate filtration $\overline{F}$ (defined with respect to the real structure coming from $K_\mathbf{k}$) are *k-opposed*, which means that if we define

$$ K^{p,q} := F^p K_\mathbb{C} \cap \overline{F}^q K_\mathbb{C} \tag{2.26} $$

then

$$ \overline{K^{p,q}} = K^{q,p} \tag{2.27} $$

and

$$ F^p K_\mathbb{C} = \bigoplus_{r \geq p} K^{r,q} \quad (K^{p,q} = 0 \text{ if } p + q \neq r). \tag{2.28} $$

This implies then that $\overline{F}^q K_\mathbb{C} = \bigoplus_{s \geq q} K^{p,s}$.

Definition 2.15. A *mixed Hodge structure* over $\mathbf{k}$ is the data of a finite-dimensional vector space $K_\mathbf{k}$ over $\mathbf{k}$ with an increasing filtration $W_\bullet$ called the *weight filtration* and a decreasing filtration $F^\bullet$ on $K_\mathbb{C}$ called the *Hodge filtration* such that for each k the graded piece

$$ \left(\mathrm{Gr}_k^W(K_\mathbf{k}),\ \mathrm{Gr}_k^W(K_\mathbf{k}) \otimes \mathbb{C} = \mathrm{Gr}_k^{W \otimes \mathbb{C}}(K_\mathbb{C}),\ F^\bullet \right) \tag{2.29} $$

(with the induced filtration F on $\mathrm{Gr}_k^W(K_\mathbb{C})$) forms a pure Hodge structure of weight k.

We will soon relax notations and simply write (K, W, F) or $(K_{\mathbf{k}}, K_{\mathbb{C}}, W, F)$ for such a mixed Hodge structure. The category of mixed Hodge structures is an abelian category with a tensor product. The morphisms are simply required to be induced by a morphism over $\mathbf{k}$ and to be compatible with both filtrations W, F.

Proposition 2.16. *If K, L are mixed Hodge structures, then all the algebraic operations described in Definition 2.2 applied to K, L seen as DG vector spaces concentrated in degree zero give again mixed Hodge structures.*

In particular, this gives automatically rise to the notion of algebra carrying a mixed Hodge structure. This is very similar to the filtered version of Definition 2.9 but concentrated in degree zero.

Definition 2.17. If A is an algebra (associative, commutative) with a mixed Hodge structure and such that the multiplication is a morphism of mixed Hodge structures, we say that A is an algebra with a mixed Hodge structure. If A has a unit 1 we also require the morphism $\mathbf{k} \to A$ to be a morphism of mixed Hodge structures, for the trivial mixed Hodge structure (of pure weight zero) on $K := \mathbf{k}$ with $K_{\mathbb{C}} = \mathbb{C}$ and $K^{p,q} = 0$ if $(p, q) \neq (0, 0)$.

Similarly if $\mathfrak{g}$ is a Lie algebra with a mixed Hodge structure such that the Lie bracket is a morphism of mixed Hodge structures, we say that $\mathfrak{g}$ is a Lie algebra with a mixed Hodge structure. And if X is a coalgebra (coassociative, cocommutative) with a mixed Hodge structure such that the comultiplication is a morphism of mixed Hodge structures, we say that X is a coalgebra with a mixed Hodge structure.

A mixed Hodge structure on K defines (in several ways) a bigrading of $K_{\mathbb{C}}$.

Definition 2.18 ([PS08, 3.4])**.** Let K be a mixed Hodge structure. Let $K^{p,q}$ be the (p, q)-component of $\mathrm{Gr}^W_{p+q}(K)$. Define the subspace of $K_{\mathbb{C}}$

$$I^{p,q} := F^p \cap W_{p+q} \cap \left(\overline{F}^q \cap W_{p+q} \cap \sum_{j \geq 2} \left(\overline{F}^{q-j+1} \cap W_{p+q-j} \right) \right). \tag{2.30}$$

Then this defines a bigrading

$$K_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q} \tag{2.31}$$

such that the canonical projection $K_{\mathbb{C}} \to \mathrm{Gr}^W_{p+q}(K_{\mathbb{C}})$ induces an isomorphism

$$I^{p,q} \simeq K^{p,q}. \tag{2.32}$$

We call it the *Deligne splitting*. It is functorial and compatible with duals and tensor products.

Finally we describe a notion that will be used only in the geometrical constructions.

Definition 2.19. If (K, F) is a Hodge structure of weight k, a *polarization* for K is a $\mathbf{k}$-bilinear form

$$Q : K \otimes K \longrightarrow \mathbf{k}$$

which is is symmetric if k is even, anti-symmetric if k is odd, and on $K_{\mathbb{C}}$ satisfies the two Riemann bilinear relations:

1. $Q(K^{p,q}, K^{r,s}) = 0$ if $(p, q) \neq (r, s)$,
2. $i^{p-q} Q(v, \bar{v}) > 0$ for $0 \neq v \in K^{p,q}$.

2.1.2 Mixed Hodge complexes

The notion of mixed Hodge complex is much more subtle that the one of mixed Hodge structure. From our point of view, this comes from the fact that the various pieces of a mixed Hodge complex live naturally in a filtered derived category. This last one is not easy to describe since the category of filtered DG vector spaces is not an abelian category.

So we prefer to fix the chain of filtered quasi-isomorphisms relating the various pieces. For this we adopt the point of view of Cirici in [Cir15] and [CG16].

In this section, contrary to the convention adopted in the previous section, if $\mathbf{k}$ is a field and $\mathbf{L}/\mathbf{k}$ is a field extension then $K_{\mathbf{k}}$ and $K_{\mathbf{L}}$ are different vector spaces, respectively over $\mathbf{k}$ and over $\mathbf{L}$. Then we will be interested in some comparison morphisms or quasi-isomorphisms between $K_{\mathbf{k}} \otimes_{\mathbf{k}} \mathbf{L}$ and $K_{\mathbf{L}}$. In the geometric constructions these part have very different origins.

Diagrams of filtered DG vector spaces

In a first step we work only with diagrams of filtered DG vector spaces. This is a more flexible category than the category of mixed Hodge complexes, for which one has to check additional non-trivial axioms.

So denote by **Cat** the category of all categories. We fix a finite category I of zig-zag type of length s, that is,

$$I = \Big\{0 \longrightarrow 1 \longleftarrow \cdots \longrightarrow s-1 \longleftarrow s\Big\}. \tag{2.33}$$

We call it our *index category.* For a functor $\mathcal{C} : I \to \mathbf{Cat}$, we write $\mathcal{C}_i$ for the category $\mathcal{C}(i)$ and for a morphism $u : i \to j$ we write u_* for $\mathcal{C}(u)$ and we call it a *comparison functor*.

Definition 2.20 ([Cir15, 4.1])**.** Given a functor $\mathcal{C} : I \to \mathbf{Cat}$, the *category of diagrams* associated with $\mathcal{C}$ is the category denoted by $\Gamma\mathcal{C}$ with:

— Objects given by families

$$K = (K_i, \varphi_u)$$

indexed by $i \in I$ and $u : i \to j$ where K_i is an object of $\mathcal{C}_i$ and

$$\varphi_u : u_*(K_i) \longrightarrow K_j$$

is a morphism in $\mathcal{C}_j$ called a *comparison morphism.*

— Morphisms

$$f : K = (K_i, \varphi_u) \longrightarrow L = (L_i, \psi_u)$$

given by families of morphisms

$$f = (f_i : K_i \to L_i)$$

in $\mathcal{C}_i$ such that for all $u : i \to j$ in I the diagram

$$\begin{array}{ccc} u_*(K_i) & \xrightarrow{u_*(f_i)} & u_*(L_i) \\ {\scriptstyle\varphi_u}\downarrow & & \downarrow{\scriptstyle\psi_u} \\ K_j & \xrightarrow[f_j]{} & L_j \end{array} \tag{2.34}$$

commutes in $\mathcal{C}_j$.

We see that this is the natural definition for a family of objects parametrized by I belonging to possibly different categories, with some given functors u_* to compare the different categories.

Now again, for doing Hodge theory, consider for $\mathbf{k}$ a subfield of $\mathbb{R}$.

Definition 2.21. Let the functor $\mathcal{V}_{\mathbf{k}} : I \to \mathbf{Cat}$ be defined as follows:

$$\begin{array}{ccc}
0 & \longmapsto & \mathbf{Fil{-}DG{-}Vect_k} \\
{\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v_*} \\
1 & \longmapsto & \mathbf{Fil{-}DG{-}Vect}_{\mathbb{C}} \\
\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\
\vdots & & \vdots \\
\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
s-1 & \longmapsto & \mathbf{Fil{-}DG{-}Vect}_{\mathbb{C}} \\
{\scriptstyle w}\uparrow & & \uparrow{\scriptstyle w_*} \\
s & \longmapsto & \mathbf{Fil^2{-}DG{-}Vect}_{\mathbb{C}}
\end{array} \tag{2.35}$$

where the comparison functor

$$v_*(K_{\mathbf{k}}, W) := (K_{\mathbf{k}}, W) \otimes_{\mathbf{k}} \mathbb{C} \tag{2.36}$$

is the extension of scalars from $\mathbf{k}$ to $\mathbb{C}$ and

$$w_*(K_{\mathbb{C}}, W_{\mathbb{C}}, F) := (K_{\mathbb{C}}, W_{\mathbb{C}}) \tag{2.37}$$

forgets the filtration F. The other comparison functors are identities. The category of objects $\Gamma\mathcal{V}_{\mathbf{k}}$ for this diagram is called the *category of diagrams of filtered DG vector spaces* over $\mathbf{k}$.

When writing a general object of $\Gamma\mathcal{V}_{\mathbf{k}}$ we will always write $(K_{\mathbf{k}}, W)$ for K_0 and $(K_{\mathbb{C}}, W, F)$ for K_s. Then K_i will denote any of the components of K, which always has the filtration W.

Definition 2.22. A *morphism* f between diagrams of filtered DG vector spaces $K = (K_i)$, $L = (L_i)$ is a morphism in the category $\Gamma\mathcal{V}_{\mathbf{k}}$ between K and L, thus is given by its components

$$f_i : K_i \longrightarrow L_i$$

commuting with the comparison morphisms. We say that f is a *quasi-isomorphism* if each component f_i is a filtered quasi-isomorphism and the last component $f_{\mathbb{C}}$ (between $K_{\mathbb{C}}$ and $L_{\mathbb{C}}$ which carry two filtrations) is a bifiltered quasi-isomorphism.

Linear algebraic constructions on filtered DG vector spaces that commute with the extension of scalars can be carried out directly to diagrams of filtered DG vector spaces.

Definition 2.23. If $K = (K_i)$, $L = (L_i)$ are diagrams of filtered DG vector spaces, then all linear algebraic constructions described in Definition 2.2 and Definition 2.3 can be done level-wise and give again diagrams of filtered DG vector spaces. In particular, the direct sum $K \oplus L$ has components

$$(K \oplus L)_i := K_i \oplus L_i, \tag{2.38}$$

the tensor product $K \otimes L$ has components

$$(K \otimes L)_i := K_i \otimes L_i, \tag{2.39}$$

the dual has components

$$(K^*)_i := (K_i)^*, \tag{2.40}$$

the cohomology $H(L)$ has components

$$(H(K))_i := H(K_i) \tag{2.41}$$

and the r-shift $K[r]$ (for $r \in \mathbb{Z}$) has components

$$(K[r])_i := K_i[r] \tag{2.42}$$

all with their naturally induced filtrations.

And there is another way of shiftings the complexes and filtrations (Remark 2.6).

Definition 2.24. If $K = (K_i)$ is a diagram of filtered DG vector spaces and $r \in \mathbb{Z}$, we denote by $(K[r], W[r], F)$ the diagram of filtered DG vector spaces with components

$$(K[r], W[r])_i := (K_i[r], W[r]) \tag{2.43}$$

and with

$$(K[r], W[r], F)_{\mathbb{C}} := (K_{\mathbb{C}}[r], W[r], F). \tag{2.44}$$

Because of these constructions, our favorite kind of algebras can also be defined internally to the category of diagrams of filtered DG vector spaces.

Definition 2.25. A *diagram of filtered DG algebras* is a diagram of filtered DG vector spaces where all the involved components have an algebra structure whose multiplication is compatible with the filtrations (as in Definition 2.9) and the comparison morphisms are morphisms of algebras. Similarly one defines *diagrams of filtered DG Lie algebras*, *diagrams of filtered L_∞ algebras* and *diagrams of filtered DG coalgebras.*

Finally the last construction we will use is the mapping cone.

Definition 2.26 (See [PS08, 3.22])**.** Let $f = (f_i : K_i \to L_i)$ be a morphism of diagrams of filtered DG vector spaces. The *mapping cone* of f is the diagram of filtered DG vector spaces $\mathrm{Cone}(f)$ given by the mapping cones of the f_i with their filtrations induced as in Definition 2.7.

Mixed Hodge complexes

In the point of view we adopted, a mixed Hodge complex is given by additional axioms on a diagram of filtered DG vector spaces. Then the main theorem of Deligne states that the cohomology of such a diagram, whose various pieces are identified via quasi-isomorphisms, carries a mixed Hodge structure. So for us this is just a list of axioms to be checked in order to get an induced mixed Hodge structure on cohomology.

We still work with diagrams over a fixed index category I (2.33) and with a field $\mathbf{k} \subset \mathbb{R}$.

Definition 2.27 (Compare [Del74, 8.1.5], [Cir15, 4.4], [Nav87, § 7.4])**.** A *mixed Hodge complex* (over the field $\mathbf{k}$, of shape given by the index category I) is a diagram of filtered DG vector spaces $K = (K_i, \varphi_u)$ over $\mathbf{k}$ satisfying the additional conditions:

1. All the DG vector spaces K_i are bounded below complexes (i.e. $K_i^n = 0$ for $n \ll 0$).
2. All comparison morphisms φ_u are filtered quasi-isomorphisms. This implies that the cohomologies $H^n(K_i)$ are identified and $H^n(K_{\mathbf{k}})$ gives a $\mathbf{k}$-structure (a fortiori, a real structure) on $H^n(K_{\mathbb{C}})$.
3. For all $n \in \mathbb{Z}$, $H^n(K)$ is finite-dimensional.
4. For all $k \in \mathbb{Z}$, the differential of $\mathrm{Gr}_k^W(K_{\mathbb{C}})$ is strictly compatible with the filtration F.
5. For all $n \in \mathbb{Z}$ and all $k \in \mathbb{Z}$, the filtration F induced on $H^n(\mathrm{Gr}_k^W(K_{\mathbb{C}}))$ and the form $H^n(\mathrm{Gr}_k^W(K_{\mathbf{k}}))$ over $\mathbf{k}$ are part of a pure Hodge structure of weight $k+n$ over $\mathbf{k}$ on $H^n(\mathrm{Gr}_k^W(K))$.

This forms a full subcategory of $\Gamma\mathcal{V}_{\mathbf{k}}$.

Example 2.28. If (K, W, F) is a mixed Hodge structure, one can see K as a mixed Hodge complex concentrated in degree 0. The comparison morphisms are simply the isomorphism $K_{\mathbb{C}} \simeq K_{\mathbf{k}} \otimes \mathbb{C}$. Conversely, if K is a mixed Hodge complex all of whose components are concentrated in degree 0 then all components K_i are finite-dimensional, the comparison morphisms φ_u identify them (compatibly with the filtrations) and K can be considered as a mixed Hodge structure.

The main theorem of Deligne is:

Theorem 2.29 (Deligne [Del74, 8.1.9])**.** *If K is a mixed Hodge complex over $\mathbf{k}$, then $H^n(K)$ endowed with the $\mathbf{k}$-structure coming from $H^n(K_{\mathbf{k}})$, the induced filtration F, and the shift $W[n]$ of the induced filtration W (alternatively, $H(K)$ with $\mathrm{Dec}\, W$ of Definition 2.8), is a mixed Hodge structure. The spectral sequence for F degenerates at E_1 and the spectral sequence for W degenerates at E_2.*

The category of mixed Hodge complexes is much more rigid than the one of mixed Hodge structures and there are only a limited number of algebraic operations that one can perform on them. But for us it will be enough to have the direct sum, the tensor product, and the mapping cone.

Proposition 2.30. *Let K, L be two mixed Hodge complexes. The direct sum $K \oplus L$ is again a mixed Hodge complex and on cohomology*

$$H^n(K \oplus L) \simeq H^n(K) \oplus H^n(L) \tag{2.45}$$

as mixed Hodge structures. Similarly ([Del74, 8.1.24], [PS08, 3.20]) the tensor product $K \otimes L$ *is a mixed Hodge complex and on cohomology, via the Künneth formula,*

$$H^n(K \otimes L) \simeq \bigoplus_{i+j=n} H^i(K) \otimes H^j(L) \tag{2.46}$$

as mixed Hodge structures (since K, L *are bounded below complexes there is no infinite sum involved here).*

Lemma 2.31. *If* (K, W, F) *is a mixed Hodge complex then* $(K[r], W[r], F)$ *is again a mixed Hodge complex.*

Proof. By Lemma 2.5, in each component i, $\mathrm{Gr}_k^{W[r]}(K_i[r]) = (\mathrm{Gr}_{k-r}^W K_i)[r]$ so that

$$H^n(\mathrm{Gr}_k^{W[r]}(K_i[r])) = H^{n+r}(\mathrm{Gr}_{k-r}^W(K_i)).$$

By the axiom 5 of mixed Hodge complexes for (K, W, F), gluing these terms for varying i gives a pure Hodge structure of weight $k + n$. And this checks this same axiom for $(K[r], W[r], F)$. The other axioms are then clearly satisfied. □

Proposition 2.32 ([PS08, 3.22])**.** *The mapping cone described in Definition 2.26 of a morphism* $f : K \to L$ *between mixed Hodge complexes is a mixed Hodge complex. For completeness, recall that it is given by the filtrations* W *on the component* i

$$W_k \operatorname{Cone}(f)_i^n := W_{k-1} K_i^{n+1} \oplus W_k L_i^n \tag{2.47}$$

and on the component $\mathbb{C}$ *carrying the filtration* F *it is given by*

$$F^p \operatorname{Cone}(f)_{\mathbb{C}}^n := F^p K_{\mathbb{C}}^{n+1} \oplus F^p L_{\mathbb{C}}^n. \tag{2.48}$$

For the desuspended mapping cone, which is $(\operatorname{Cone}(f)[-1], W[-1], F)$*, the structure of mixed Hodge complex is given by*

$$W_k(\operatorname{Cone}(f)[-1])_i^n := W_k K_i^n \oplus W_{k+1} L_i^{n-1} \tag{2.49}$$

and

$$F^p(\operatorname{Cone}(f)[-1])_{\mathbb{C}}^n := F^p K_{\mathbb{C}}^n \oplus F^p L_{\mathbb{C}}^{n-1}. \tag{2.50}$$

On cohomology the long exact sequence for the mapping cone

$$\cdots \longrightarrow H^n(K) \longrightarrow H^n(L) \longrightarrow H^n(\operatorname{Cone}(f)) \longrightarrow H^{n+1}(K) \longrightarrow H^{n+1}(L) \longrightarrow \cdots \tag{2.51}$$

(where one can replace $H^n(\operatorname{Cone}(f))$ *by* $H^{n+1}(\operatorname{Cone}(f)[-1])$*) becomes a long exact sequence of mixed Hodge structures.*

Remark that the suspension $(K[1], W[1], F)$ we described above ($r = 1$) is in particular the mapping cone of $K \to 0$.

Finally, our various kinds of DG algebras or coalgebras can also be defined internally to mixed Hodge complexes. In this situation, following the original terminology of Morgan [Mor78, 3.5], we call them *mixed Hodge diagrams*.

Definition 2.33. We define a *mixed Hodge diagrams of algebras* to be a diagram of filtered DG algebras which is also a mixed Hodge complex (for its underlying structure of diagram of filtered DG vector spaces). Similarly, one defines *mixed Hodge diagrams of Lie algebras*, *mixed Hodge diagrams of* L_∞ *algebras* and *mixed Hodge diagrams of coalgebras.*

These will be the main objects of study in the next section.

2.2 Mixed Hodge diagrams

Now we use all the definitions and constructions of the previous section together and relate them with the chapter 1. Our ultimate goal is to show, first that given a morphism of mixed Hodge diagrams of Lie algebras one gets a mixed Hodge diagram of L_∞ algebras L by the construction of Fiorenza-Manetti of section 1.2.2, then that the functor $\mathscr{C}$ applied to L gives a mixed Hodge diagram of coalgebras so as to define a mixed Hodge structure on $H^0(\mathscr{C}(L))$, with which as in section 1.2.3 we will get a mixed Hodge structure on a pro-Artin algebra pro-representing the deformation functor of L. We naturally separate these into two sections.

All our mixed Hodge complexes are indexed over a fixed category I (2.33) and defined over a field $\mathbf{k} \subset \mathbb{R}$.

2.2.1 Augmented mixed Hodge diagrams of Lie algebras and mixed Hodge diagrams of L_∞ algebras

First for completeness, let us write the full definition of mixed Hodge diagrams for Lie algebras and for L_∞ algebras. These are the objects we will work with.

Definition 2.34. A *diagram of filtered L_∞ algebras* is a diagram of filtered DG vector spaces $L = (L_i, \varphi_u)$ such that all the L_i have a L_∞ algebra structure which is compatible with the filtrations. For the operations in r variables of degree $2 - r$

$$\ell_r : L^{\wedge r} \longrightarrow L$$

in the component L_i one requires the compatibility with W

$$\ell_r \left((W_{k_1} L_i^{n_1}) \wedge \cdots \wedge (W_{k_r} L_i^{n_r}) \right) \subset W_{k_1+\cdots+k_r} L_i^{n_1+\cdots+n_r+2-r} \tag{2.52}$$

and in the component $L_\mathbb{C}$, carrying the filtration F, one requires

$$\ell_r \left((F^{p_1} L_\mathbb{C}^{n_1}) \wedge \cdots \wedge (F^{p_r} L_\mathbb{C}^{n_r}) \right) \subset F^{p_1+\cdots+p_r} L_\mathbb{C}^{n_1+\cdots+n_r+2-r}. \tag{2.53}$$

Furthermore the comparison morphisms φ_u are required to be strong morphisms of L_∞ algebras (Definition 1.45) respecting the filtrations. A *mixed Hodge diagram of L_∞ algebras* is a diagram of filtered L_∞ algebras that is also a mixed Hodge complex for the underlying structure of diagram of filtered DG vector spaces given by $d = \ell_1$. If we restrict to DG Lie algebras by assuming $\ell_r = 0$ for $r > 2$ in every component, then one gets *diagrams of filtered DG Lie algebras* and *mixed Hodge diagrams of Lie algebras.*

One of the results that we immediately get with this definition is:

Proposition 2.35. *If L is a mixed Hodge diagram of L_∞ algebras (a fortiori, of Lie algebras) then on the cohomology $H(L)$, which has the structure of a diagram of filtered graded Lie algebras (Proposition 1.44) and each term $H^n(L)$ has a mixed Hodge structure for the induced filtrations* $\operatorname{Dec} W, F$ *(Theorem 2.29), the induced Lie bracket is a morphism of mixed Hodge structures.*

Proof. By passing to cohomology we forget all the operations ℓ_r for $r \neq 2$ and ℓ_2 becomes a Lie bracket $[-,-]$. The statement that the Lie bracket respects F is clear because the condition

$$\ell_2\left((F^p L^n_{\mathbb{C}}) \wedge (F^q L_{\mathbb{C}})^m\right) \subset F^{p+q} L^{n+m}_{\mathbb{C}}$$

directly induces on cohomology

$$[F^p H^n(L_{\mathbb{C}}), F^q H^m(L_{\mathbb{C}})] \subset F^{p+q} H^{n+m}(L_{\mathbb{C}}).$$

For W, in any of the components L_i, one has to be more careful. Take cohomology classes

$$[u] \in (\operatorname{Dec} W)_k H^n(L_i), \quad [v] \in (\operatorname{Dec} W)_\ell H^m(L_i).$$

Then $[u]$ comes from an element $u \in W_{k-n} L^n_i$, and $[v]$ comes from $v \in W_{\ell-m} L^m_i$. So

$$\ell_2(u \wedge v) \in W_{(k-n)+(\ell-m)} L^{n+m}_i$$

and this corresponds to

$$[[u],[v]] \in (\operatorname{Dec} W)_{k+\ell} H^{n+m}(L_i).$$

This proves that the Lie bracket is a morphism of mixed Hodge structures. □

Recall also the notion of morphism. It is obviously obtained by our general definition of morphisms of diagrams of filtered DG vector spaces 2.22 and the definition of strong morphisms for L_∞ algebras 1.45.

Definition 2.36. A *morphism* f between mixed Hodge diagrams of L_∞ algebras L, M is given by a collection of morphisms

$$f_i : L_i \longrightarrow M_i \tag{2.54}$$

that are at the same time strong morphisms of L_∞ algebras and morphisms of diagrams of filtered DG vector spaces. It is said to be a *quasi-isomorphism* if each f_i is a filtered quasi-isomorphism and $f_{\mathbb{C}}$ is a bifiltered quasi-isomorphism.

The structure we will get from the geometric situation is:

Definition 2.37. An *augmented mixed Hodge diagram of Lie algebras* is the data of a mixed Hodge diagram of Lie algebras L and a Lie algebra $\mathfrak{g}$ carrying a mixed Hodge structure, seen as a mixed Hodge diagram of Lie algebras over I as in Example 2.28, together with a morphism

$$\varepsilon : L \longrightarrow \mathfrak{g} \tag{2.55}$$

of mixed Hodge diagrams of Lie algebras.

Now we can state and prove the main theorem of this section. It claims the compatibility between the formation of the mapping cone of mixed Hodge complexes and the L_∞ algebra structure of Fiorenza-Manetti.

Theorem 2.38. *Let $\varepsilon : L \to \mathfrak{g}$ be an augmented mixed Hodge diagram of Lie algebras. Assume that in each of the components L_i the filtration W has only non-negative weights (i.e. $W_k L_i = 0$ for $k < 0$) and that the mixed Hodge structure on $\mathfrak{g}$ is pure of weight zero. Then the desuspended mapping cone C of ε with its L_∞ algebra structure described in section 1.2.2 is a mixed Hodge diagram of L_∞ algebras.*

Proof. It is practical to consider $\mathfrak{g}$ as a mixed Hodge diagram of Lie algebras concentrated in degree 0. So we will write terms $\mathfrak{g}_i^n$ that are zero for $n \neq 0$. The structure of mixed Hodge complex on C is recalled in Proposition 2.32 (with $C = \mathrm{Cone}(\varepsilon)[-1]$), the axioms to be checked are in Definition 2.34 and the operations of L_∞ algebra of C are described in Theorem 1.57. It is clear that these operations commute with the change of coefficients from $\mathbf{k}$ to $\mathbb{C}$ so the difficult part is to check the compatibility with the filtrations.

Let Q be the codifferential on the cofree coalgebra on $C[1]$ which gives the structure of L_∞ algebra to C, with its components q_r ($r \geq 1$), and ℓ_r are the corresponding operations on C. Up to signs and shifts of gradings, the operations q_r and ℓ_r are given by the same algebraic formulas.

First check the compatibility for W in some component C_i.

For ℓ_1, which is the differential of the desuspended mapping cone: take $(x,u) \in W_k C_i^n$, so that $x \in W_k L_i^n$ and $u \in W_{k+1}\mathfrak{g}_i^{n-1}$. Then we know that

$$\ell_1(x,u) = (d(x),\ \varepsilon(x) - d(u))$$

(actually $d = 0$ on $\mathfrak{g}_i$). But $d(x) \in W_k L_i^{n+1}$, $\varepsilon(x) \in W_k \mathfrak{g}_i^n \subset W_{k+1}\mathfrak{g}_i^n$ and $d(u) \in W_{k+1}\mathfrak{g}_i^n$. So one sees that

$$\ell_1(x,u) \in W_k C_i^{n+1}.$$

For ℓ_2: take $(x,u) \in W_k C_i^n$, $(y,v) \in W_\ell C_i^m$, so that $x \in W_k L_i^n$, $u \in W_{k+1}\mathfrak{g}_i^{n-1}$, $y \in W_\ell L_i^m$, $v \in W_{\ell+1}\mathfrak{g}_i^{m-1}$. We want to show that

$$\ell_2((x,u) \wedge (y,v)) \in W_{k+\ell} C_i^{n+m}.$$

For the part $\ell_2(x \wedge y)$ this is given (up to sign) by $[x,y]$, and it is in $W_{k+\ell} L_i^{n+m}$. For $\ell_2(u \otimes y)$ this is given up to sign by $[u, \varepsilon(y)]$ which is in $W_{(k+1)+\ell}\mathfrak{g}_i^{(n-1)+m}$. This proves the compatibility for ℓ_2.

Now for the higher operations ℓ_r with $r \geq 3$ there is only one compatibility in the relation (1.109) to check, and up to sign this is just an iterated bracket. So take r elements

$$(x_j, u_j) \in W_{k_j} C_i^{n_j}, \quad j = 1, \ldots, r$$

so that $x_j \in W_{k_j} L^{n_j}$ and $u_j \in W_{k_j+1}\mathfrak{g}^{n_j-1}$. When computing $\ell_r((x_1,u_1) \wedge \cdots \wedge (x_r,u_r))$, the only nonzero part is when we multiply only one of the x_j with the others u_j; call it x_s. Since $\mathfrak{g}_i$ is concentrated in degree 0, this is zero if all u are not of degree 0 or if x_s is not of degree 0. So we can assume $n_s = 0$ and $n_j = 1$ for $j \neq s$. Then for the iterated bracket, and for a permutation $\{t_1, \ldots, t_{r-1}\}$ of $\{1, \ldots, \hat{s}, \ldots, r\}$,

$$\begin{aligned} [u_{t_1}, [u_{t_2}, \ldots, [u_{t_{r-1}}, \varepsilon(x_s)] \ldots]] &\in W_{(k_{t_1}+1)+\cdots+(k_{t_{r-1}}+1)+k_s}\mathfrak{g}_i^{(n_{t_1}-1)+\cdots+(n_{t_{r-1}}-1)+n_s} \\ &= W_{k_1+\cdots+k_r+(r-1)}\mathfrak{g}_i^{n_1+\cdots+n_r-r+1} = W_{k_1+\cdots+k_r+r-1}(\mathfrak{g}_i). \quad (2.56) \end{aligned}$$

One would like

$$\ell_r((x_1,u_1)\wedge\cdots\wedge(x_r,u_r))\in W_{k_1+\cdots+k_r}C_i^{n_1+\cdots+n_r+2-r}=W_{k_1+\cdots+k_r}C_i^1$$

so that the iterated bracket in (2.56) would land in $W_{k_1+\cdots+k_r+1}(\mathfrak{g}_i)$. But if we assume that $\mathfrak{g}_\mathfrak{i}$ has pure weight zero and since $r\geq 3$, the condition

$$W_{k_1+\cdots+k_r+r-1}(\mathfrak{g}_\mathfrak{i})\subset W_{k_1+\cdots+k_r+1}(\mathfrak{g}_\mathfrak{i})\subset\mathfrak{g}_\mathfrak{i}$$

is realized by an equality as soon as $k_1+\cdots+k_r+1\geq 0$. So, under the assumption that L_i has only non-negative weights, one can reduce the compatibility checking to $k_1,\ldots,k_r\geq 0$ and this equality is realized.

The condition to check for F on $C_\mathbb{C}$ is much easier because there is no shift in the filtration. One sees directly that $(x,u)\in F^pC_\mathbb{C}^n$ means $x\in F^pL_\mathbb{C}^n$, $u\in F^p\mathfrak{g}_\mathbb{C}^{n-1}$, so that $d(x)\in F^pL_\mathbb{C}^{n+1}$, $\varepsilon(x)\in F^p\mathfrak{g}_\mathbb{C}^n$ and $d(u)\in F^p\mathfrak{g}_\mathbb{C}^n$ so

$$\ell_1(x,u)\in F^pC_\mathbb{C}^{n+1}.$$

For ℓ_2 then $(x,u)\in F^pC_\mathbb{C}^n$, $(y,v)\in F^qC_\mathbb{C}^m$ means that $x\in F^pL_\mathbb{C}^n$, $u\in F^p\mathfrak{g}_\mathbb{C}^{n-1}$, $y\in F^qL_\mathbb{C}^m$, $v\in F^q\mathfrak{g}_\mathbb{C}^{m-1}$. So

$$\ell_2((x,u)\wedge(y,v))=\left([x,y],\ \frac{1}{2}\,[u,\varepsilon(y)]+\frac{(-1)^{|x|}}{2}\,[\varepsilon(x),v]\right)$$
$$\in F^{p+q}L_\mathbb{C}^{n+m}\oplus F^{p+q}\mathfrak{g}_\mathbb{C}^{n+m-1}=F^{p+q}C_\mathbb{C}^{n+m}.$$

Finally for the higher operations, take again $(x_j,u_j)\in F^{p_j}C_\mathbb{C}^{n_j}$ so that $x_j\in F^{p_j}L_\mathbb{C}^{n_j}$ and $u_j\in F^{p_j}\mathfrak{g}_\mathbb{C}^{n_j-1}$. Again, select one element x_s and consider a permutation $\{t_1,\ldots,t_{r-1}\}$ of $\{1,\ldots,\hat{s},\ldots,r\}$. Then

$$[u_{t_1},[u_{t_2},\ldots,[u_{t_{r-1}},\varepsilon(x_s)]\ldots]]\in F^{p_{t_1}+\cdots+p_{t_{r-1}}+p_s}\mathfrak{g}_\mathbb{C}^{(n_{t_1}-1)+\cdots+(n_{t_{r-1}}-1)+n_s}$$
$$=F^{p_1+\cdots+p_r}\mathfrak{g}_\mathbb{C}^{n_1+\cdots+n_r-r+1}. \quad (2.57)$$

This checks directly that

$$\ell_r((x_1,u_1)\wedge\cdots\wedge(x_r,u_r))\in F^{p_1+\cdots+p_r}C_\mathbb{C}^{n_1+\cdots+n_r+2-r}.$$

□

We see also that this construction is perfectly functorial. Furthermore we have the analogue of Lemma 1.56 for mixed Hodge diagrams:

Lemma 2.39 (Compare with Lem. 1.56). *Let $\varphi : L\xrightarrow{\approx}L'$ be a quasi-isomorphism of augmented mixed Hodge diagrams of Lie algebras over the same Lie algebra $\mathfrak{g}$ with a mixed Hodge structure, satisfying both the hypothesis of Theorem 2.38. Then the induced morphism $\psi : C\to C'$ between the desuspended mapping cones is a quasi-isomorphism of mixed Hodge diagrams of L_∞ algebras.*

Proof. By the functoriality of the L_∞ algebra structure on the mapping cone, it is already clear that ψ is a morphism of mixed Hodge diagrams of L_∞ algebras. We only have to show that it is a quasi-isomorphism. But in the component i, φ induces by definition a quasi-isomorphism

$$\mathrm{Gr}_k^W(\varphi_i) : \mathrm{Gr}_k^W(L_i) \xrightarrow{\approx} \mathrm{Gr}_k^W(L'_i)$$

augmented over $\mathfrak{g}$. So the same argument as in the cited lemma, using the induced morphism between the long exact sequences for the mapping cones of the augmentations, shows that there is a quasi-isomorphism

$$\mathrm{Gr}_k^W(\psi_i) : \mathrm{Gr}_k^W(C_i) \xrightarrow{\approx} \mathrm{Gr}_k^W(C'_i).$$

This proves that ψ_i is a filtered quasi-isomorphism. And similarly by applying $\mathrm{Gr}_k^W \mathrm{Gr}_F^p$ one proves that $\psi_{\mathbb{C}}$ is a bifiltered quasi-isomorphism. □

Remark 2.40. The proof of Theorem 2.38 shows actually the more general result that for *any* morphism $\varepsilon : L \to M$ between diagrams of filtered DG Lie algebras (a fortiori, between mixed Hodge diagrams of Lie algebras), without restrictive hypothesis on M, then on the mapping cone C the operations ℓ_1, ℓ_2 (that is, the differential and the naive bracket of Definition 1.52) respect the filtrations of C. The operations ℓ_r for $r \geq 3$ respect the filtration F of $C_{\mathbb{C}}$ and respect W up to some shift of $r-2$:

$$\begin{aligned}\ell_r(W_{k_1}C_i^{n_1} \wedge \cdots \wedge W_{k_r}C_i^{n_r}) &\subset W_{(k_1+1)+\cdots+(k_r+1)-1}M_i^{(n_1-1)+\cdots+(n_r-1)+1} \\ &= W_{k_1+\cdots+k_r+r-1}M_i^{n_1+\cdots+n_r-r+1} \subset W_{(k_1+\cdots+k_r)+(r-2)}C_i^{(n_1+\cdots+n_r)+(2-r)}. \quad (2.58)\end{aligned}$$

This should land in $W_{k_1+\cdots+k_r}C_i^{(n_1+\cdots+n_r)+(2-r)}$ if one wanted ℓ_r to respect perfectly the filtration W.

2.2.2 Bar construction on mixed Hodge diagrams

If L is a mixed Hodge diagram of L_∞ algebras, one can consider the construction $\mathscr{C}$ on L. It is simply defined as the construction $\mathscr{C}$ on each component of L, with the induced filtrations from linear algebraic operations. Since $\mathscr{C}$ preserves quasi-isomorphisms, $\mathscr{C}(L)$ will be a diagram of filtered DG coalgebras related by filtered quasi-isomorphisms. So when applying H^0 the various parts related by quasi-isomorphisms will be identified. Our goal is to construct a mixed Hodge structure on $H^0(\mathscr{C}(L))$. For this, we prove that $\mathscr{C}(L)$ is a mixed Hodge diagram of coalgebras. Our theory will be very close to what Hain does in [Hai87, § 3] for the bar construction on commutative DG algebras and we adopt his terminology of bar filtration. Let us be more precise.

Let L be a L_∞ algebra. Recall from Definition 1.46 that $\mathscr{C}(L)$, which is a conilpotent DG coalgebra (coassociative, cocommutative, but without counit nor unit) has an increasing filtration by sub-DG coalgebras given by

$$\mathscr{C}_s(L) := \bigoplus_{r=1}^{s}(L[1])^{\odot r}, \quad s \geq 1 \qquad (2.59)$$

called the *bar filtration* and that $\mathscr{C}(L)$ is the inductive limit (or, increasing union over s) of the terms $\mathscr{C}_s(L)$. Recall also (Theorem 1.69 and its proof) that $\mathscr{C}$ sends quasi-isomorphisms of L_∞ algebras to quasi-isomorphisms of DG coalgebras. Of course such a morphism is compatible with the bar filtration.

Since the beginning of section 2.1 we took the precaution of working only with biregular filtrations. A filtration of a L_∞ algebra L will induce a filtration on $\mathscr{C}(L)$, by the simple combination of the induced filtrations on symmetric powers and on direct sums, but this may not be biregular. However if we work with $\mathscr{C}_s(L)$, obtained from L by a finite number of symmetric powers, and if L is a bounded-below complex, then this will be biregular. Then we will consider $\mathscr{C}(L)$ as the inductive limit of the $\mathscr{C}_s(L)$ in the category of filtered DG coalgebras.

Definition 2.41. Let (L, W) be a filtered L_∞ algebra. The filtration induced by W on $\mathscr{C}(L)$, via $W[1]$ on $L[1]$ and then $(W[1])^{\odot r}$ on $(L[1])^{\odot r}$, is called the *bar-weight filtration*. We denote it by $\mathscr{C}W$.

The bar-weight filtration can be seen as a convolution of the weight filtration and the bar filtration (2.59). If (L, W, F) is bifiltered, F induces simply a filtration F on $L[1]$ and then $F^{\odot r}$ on $(L[1])^{\odot r}$, and we simply denote by F the induced filtration on $\mathscr{C}(L)$.

Proposition 2.42. *Let L be a diagram of filtered L_∞ algebras. Assume that in each component i, L_i is a bounded-below complex. Then for any $s \geq 1$, $\mathscr{C}_s(L)$ is a diagram of filtered DG coalgebras for the bar-weight filtration and $\mathscr{C}(L)$ is an inductive limit of diagrams of filtered DG coalgebras.*

Proof. Fix a component L_i of L. First work with W. As $\mathscr{C}W$ is induced by $W[1]$ on $L_i[1]$ and then by multiplicative extension, and seeing the algebraic formula for the coproduct of the cofree coalgebra (Definition 1.39), it is clear that $\mathscr{C}W$ is compatible with the graded coalgebra structure and gives $\mathscr{C}(L)$ the structure of an inductive limit of filtered graded coalgebra. Then we have to show that the codifferential Q of $\mathscr{C}_s(L_i)$ respects the filtration, and it is enough to check it for its components $q_r : (L[1])^{\odot r} \to L[1]$ $(r \geq 1)$ because of the explicit formula for recovering Q from its components in Lemma 1.41.

So take r elements

$$x_j[1] \in W[1]_{k_j} L_i[1]^{n_j}, \quad j = 1, \ldots, r$$

which means $x_j \in W_{k_j - 1} L_i^{n_j+1}$. Then

$$\begin{aligned} q_r(x_1[1] \odot \cdots \odot x_r[1]) = \pm \ell_r(x_1 \wedge \cdots \wedge x_r) &\in W_{(k_1-1)+\cdots+(k_r-1)} L_i^{(n_1+1)+\cdots+(n_r+1)+(2-r)} \\ &\subset W_{k_1+\cdots+k_r-1} L_i^{n_1+\cdots+n_r+2} = W[1]_{k_1+\cdots+k_r} L_i[1]^{n_1+\cdots+n_r+1}. \end{aligned} \tag{2.60}$$

Then check the compatibility for F on $L_\mathbb{C}$. This is easier, since it is induced directly by F (and not $F[1]$). Take r elements

$$x_j[1] \in F^{p_j} L_\mathbb{C}[1]^{n_j}, \quad j = 1, \ldots, r$$

which means $x_j \in F^{p_j} L_\mathbb{C}^{n_j+1}$ and then directly

$$\begin{aligned} q_r(x_1[1] \odot \cdots \odot x_r[1]) = \pm \ell_r(x_1 \wedge \cdots \wedge x_r) &\in F^{p_1+\cdots+p_r} L_\mathbb{C}^{(n_1+1)+\cdots+(n_r+1)+(2-r)} \\ &= F^{p_1+\cdots+p_r} L_\mathbb{C}^{n_1+\cdots+n_r+2} = F^{p_1+\cdots+p_r} L_\mathbb{C}[1]^{n_1+\cdots+n_r+1}. \end{aligned}$$

□

Remark 2.43. The above proposition is true more generally if L is a diagram of filtered L_∞ algebras whose higher operations shift the weight filtration as in Remark 2.40. In this case, for $r \geq 3$, the operation ℓ_r satisfies

$$\ell_r(W_{k_1}L_i^{n_1} \wedge \cdots \wedge W_{k_r}C_i^{n_r}) \subset W_{(k_1+\cdots+k_r)+(r-2)}L_i^{(n_1+\cdots+n_r)+(2-r)}$$

and in equation (2.60) of the preceding proof q_r satisfies

$$\begin{aligned} q_r(x_1[1]\odot\cdots\odot x_r[1]) &= \pm\ell_r(x_1\wedge\cdots\wedge x_r) \in W_{(k_1-1)+\cdots+(k_r-1)+(r-2)}L_i^{(n_1+1)+\cdots+(n_r+1)+(2-r)} \\ &= W_{k_1+\cdots+k_r+2}L_i^{n_1+\cdots+n_r+2} \subset W_{k_1+\cdots+k_r+1}L_i^{n_1+\cdots+n_r+2} = W[1]_{k_1+\cdots+k_r}L_i[1]^{n_1+\cdots+n_r+1}. \end{aligned} \tag{2.61}$$

We arrive finally at our main goal. For this we follow closely the method of Hain [Hai87, § 3] for commutative DG algebras, re-writing it for L_∞ algebras. See also [PS08, § 8.7].

Theorem 2.44. *Let L be a mixed Hodge diagram of L_∞ algebras. Assume that each component L_i is non-negatively graded ($L_i^n = 0$ for $n < 0$) and that $H^0(L) = 0$. Then $\mathscr{C}_s(L)$ is a mixed Hodge diagram of coalgebras for any $s \geq 1$ and $\mathscr{C}(L)$ is an inductive limit of mixed Hodge diagrams of coalgebras.*

Proof. The main statement to prove is that $\mathscr{C}_s(L)$ is a mixed Hodge complex and for this we need to check the axioms of Definition 2.27. We already know that it is a diagram of filtered DG coalgebras and then $\mathscr{C}(L)$ will obviously be an inductive limit in the category of mixed Hodge diagrams of coalgebras. So we fix $s \geq 1$ and we fix temporarily a component L_i.

Since we took the precaution of working with inductive limits, $\mathscr{C}_s(L_i)$ is a bounded-below complex because it is obtained by a finite number of symmetric powers and because the induced filtration W is biregular. This is the axiom 1.

The Lemma 1.79 and its proof almost check the axiom 3. Namely $H^n(L_i)$ is finite-dimensional for all n so we see from the cited proof that for the spectral sequence for $\mathscr{C}$

$$_{\mathscr{C}}E_1^{-s,n+s} = H^n((L_i[1])^{\odot s}) \Longrightarrow {}_{\mathscr{C}}E_\infty^{-s,n+s} = \mathrm{Gr}_s^{\mathscr{C}} H^n(\mathscr{C}(L_i)).$$

By the Künneth formula and since L_i is bounded below, $H^n((L_i[1])^{\odot s})$ is finite-dimensional. So at infinity the term $\mathrm{Gr}_s^{\mathscr{C}} H^n(\mathscr{C}(L_i))$ is also finite-dimensional and on $H^n(\mathscr{C}(L_i))$ the induced bar filtration is by finite-dimensional sub-vector spaces.

To check the other axioms we need to compute the first pages of the spectral sequence for the bar-weight filtration. Since W and $\mathscr{C}W$ are decreasing filtrations, we work with $-k$ instead of k. We denote by $\mathscr{C}_s(L_i)^m$ the component of total degree m in $\mathscr{C}_s(L_i)$. Then by definition of the spectral sequence

$$_{\mathscr{C}W}E_0^{-k,q}(\mathscr{C}_s(L_i)) = \mathrm{Gr}_k^{\mathscr{C}W} \mathscr{C}_s(L_i)^{-k+q}.$$

Since

$$\begin{aligned} \mathscr{C}W_k(L_i[1])^{\odot r} &= \bigoplus_{k_1+\cdots+k_s=k} W[1]_{k_1}L_i[1] \odot \cdots \odot W[1]_{k_s}L_i[1] \\ &= \bigoplus_{k_1+\cdots+k_r=k-r} W_{k_1}L_i[1] \odot \cdots \odot W_{k_s}L_i[1] \end{aligned}$$

it follows that

$$\mathrm{Gr}_k^{\mathscr{C}W}\,\mathscr{C}_s(L_i)^{-k+q} = \bigoplus_{r=1}^{s} \mathrm{Gr}_{k-r}^{W}((L_i[1])^{\odot r})^{-k+q} = \bigoplus_{r=1}^{s} \mathrm{Gr}_{k-r}^{W}(L_i^{\wedge r})^{-k+q+r}.$$

So we recognize

$$_{\mathscr{C}W}E_0^{-k,q}(\mathscr{C}_s(L_i)) = \bigoplus_{r=1}^{s} {}_W E_0^{-k+r,q}(L_i^{\wedge r}). \tag{2.62}$$

The differential d_0 on $_{\mathscr{C}W}E_0^{-k,\bullet}(\mathscr{C}_r(L_i))$ is induced by the codifferential $Q := \sum q_r$ of $\mathscr{C}(L_i)$. Crucial here is the equation (2.60) appearing in the preceding proof, which shows that q_r is zero on $\mathrm{Gr}_k^{\mathscr{C}W}\,\mathscr{C}(L_i)$ for all $r \geq 2$. Thus d_0 is only induced by q_1, which is up to sign $d[1]$, and it is the sum of the d_0's appearing on the right side of (2.62). But by Proposition 2.30 $L^{\wedge r}$ is a mixed Hodge complex and this right side is a direct sum of terms $_W E_0$ of mixed Hodge complexes.

This computation allows us to check the axioms 2 and 4 for mixed Hodge complexes. Let

$$\varphi_u : (L_i, W) \xrightarrow{\approx} (L_j, W)$$

be some comparison morphism between the two components L_i, L_j, which by hypothesis is a filtered quasi-isomorphism. By the Künneth formula (combined with the fact that we work with bounded-below complexes) φ_u induces a filtered quasi-isomorphism

$$((L_i)^{\wedge r}, W^{\wedge r}) \xrightarrow{\approx} ((L_j)^{\wedge r}, W^{\wedge r})$$

so it induces an isomorphism

$$_W E_0^{-k+r,q}(L_i^{\wedge r}) \xrightarrow{\simeq} {}_W E_0^{-k+r,q}(L_j^{\wedge r}).$$

So equation (2.62) tells us precisely that φ_u induces a filtered quasi-isomorphism

$$(\mathscr{C}_s(L_i), \mathscr{C}W) \xrightarrow{\approx} (\mathscr{C}_s(L_j), \mathscr{C}W)$$

and this is the axiom 2.

The axiom 4 has to be checked in the component over $\mathbb{C}$ carrying also the filtration F. By this axiom for $L_{\mathbb{C}}^{\wedge r}$ the differential of $_W E_0^{-k+r,\bullet}(L_{\mathbb{C}}^{\wedge r})$ is strictly compatible with the induced filtration F, so from equation (2.62) the differential of $_{\mathscr{C}W}E_0^{k,\bullet}(\mathscr{C}_r(L_{\mathbb{C}}))$ is the direct sum of these and is also strictly compatible with F.

To check the last axiom we compute the spectral sequence at the page E_1. By definition

$$_{\mathscr{C}W}E_1^{-k,q}(\mathscr{C}_s(L_i)) = H^{-k+q}(\mathrm{Gr}_k^{\mathscr{C}W}\,\mathscr{C}_s(L_i))$$

where the cohomology is computed with respect to d_0. Then using (2.62)

$$H^{-k+q}(\mathrm{Gr}_k^{\mathscr{C}W}\,\mathscr{C}_s(L_i)) = \bigoplus_{r=1}^{s} H^{-k+q}(\mathrm{Gr}_{k-r}^{W}(L_i^{\wedge r})^{\bullet+r}) = \bigoplus_{r=1}^{s} {}_W E_1^{-k+r,q}(L_i^{\wedge r}).$$

So, put together,

$$_{\mathscr{C}W}E_1^{-k,q}(\mathscr{C}_s(L_i)) = \bigoplus_{r=1}^{s} {}_W E_1^{-k+r,q}(L_i^{\wedge r}). \tag{2.63}$$

Since $L^{\wedge r}$ is a mixed Hodge complex, in equation (2.63) the terms on the right side

$$ {}_W E_1^{-k+r,q}(L_i^{\wedge r}) = H^{-k+r+q}(\mathrm{Gr}^W_{k-r} L_i^{\wedge r}) $$

glue to, when varying i, a pure Hodge structure of weight q. So does their direct sum and this proves that the terms

$$ {}_{\mathscr{C}W} E_1^{-k,q}(\mathscr{C}_s(L_i)) = H^{-k+q}(\mathrm{Gr}^{\mathscr{C}W}_k \mathscr{C}_s(L_i)) $$

define a pure Hodge structure of weight q. This checks the last axiom 5. □

We see that this construction is functorial. Furthermore we have the analogue of Theorem 1.69.

Lemma 2.45 (Compare with Thm. 1.69)**.** *Let $\varphi : L \xrightarrow{\approx} L'$ be a quasi-isomorphism of mixed Hodge diagrams of L_∞ algebras satisfying both the hypothesis of Theorem 2.44. Then the induced morphism $\mathscr{C}(\varphi)$ is a quasi-isomorphism of mixed Hodge diagrams of coalgebras.*

Proof. By the functoriality and explicit nature of the bar construction, it is already clear that $\mathscr{C}(\varphi)$ is a morphism of diagrams of filtered DG coalgebras, compatible with the bar filtration. Then, following the proof of the preceding theorem, we see that in the component i and in equation (2.63) the hypothesis tells us that φ_i induces an isomorphism on the right-hand side, so it induces an isomorphism on the left-hand side. Similarly for the bifiltered part we repeat the arguments replacing $L_{\mathbb{C}}$ by $\mathrm{Gr}^p_F(L_{\mathbb{C}})$. □

Corollary 2.46. *Under the hypothesis of Theorem 2.44, the sub-coalgebras of $H^0(\mathscr{C}(L))$ given by the canonical filtration carry a compatible mixed Hodge structure (Definition 2.17). The dual $H^0(\mathscr{C}(L))^*$, to which we add a unit with its trivial mixed Hodge structure to form a pro-Artin algebra R, is a projective limit of local Artin algebras $R/(\mathfrak{m}_R)^s$ carrying a mixed Hodge structure. A quasi-isomorphism $\varphi : L \xrightarrow{\approx} L'$ between such mixed Hodge diagrams of L_∞ algebras induces an isomorphism $R \xrightarrow{\simeq} R'$ between the corresponding algebras with their mixed Hodge structure.*

Proof. In Lemma 1.79 we proved that the canonical filtration of $H^0(\mathscr{C}(L))$ is induced by the bar filtration. So our main Theorem 2.44 proves that these sub-coalgebras, which are given by the image of $H^0(\mathscr{C}_s(L))$ in $H^0(\mathscr{C}(L))$ where $\mathscr{C}_s(L)$ is a mixed Hodge complex, have mixed Hodge structures. We see, from the various compatibilities with the filtrations, that this mixed Hodge structure is compatible with the coalgebra structure. In Lemma 1.77 we explained how the inductive limit of the $H^0(\mathscr{C}_s(L))$ is dual to the projective limit of the $H^0(\mathscr{C}_s(L))^*$, and the dual algebra of a coalgebra carries a natural mixed Hodge structure compatible with the multiplication. The base field $\mathbf{k}$ has a trivial mixed Hodge structure. So one can form the pro-Artin algebra

$$ R := \mathbf{k} \oplus H^0(\mathscr{C}(L))^* $$

and it is a projective limit of the $R/(\mathfrak{m}_R)^s$, dual to $\mathbf{k} \oplus H^0(\mathscr{C}_s(L))$, which are Artin algebras with a compatible mixed Hodge structure (alternatively, $\mathfrak{m}_R/(\mathfrak{m}_R)^s$ is dual to $H^0(\mathscr{C}_s(L))$).

This is clearly functorial in L and the statement for quasi-isomorphisms follows directly from Lemma 2.45. □

Chapter 3

Geometry

In the next sections we present several different geometric situations concerning a complex manifold X and a representation ρ of its fundamental group into a linear algebraic group G. In each of them we construct an appropriate augmented mixed Hodge diagram of Lie algebras that controls the deformation problem as in the classical Goldman-Millson theory. Then the machinery we developed in the two preceding chapters gives us directly and functorially a mixed Hodge structure on the complete local ring $\widehat{\mathcal{O}}_\rho$ of the representation variety $\mathrm{Hom}(\pi_1(X,x),G)$ at ρ. The section 3.1.1 has all details written and serves as a model of the construction. The core of the method is resumed in Theorem 3.3. The other sections present variations of this situation, in which we emphasize the construction of a mixed Hodge diagram.

We will always use the letter $\mathscr{E}$ to refer to $\mathcal{C}^\infty$ differential forms, Ω for holomorphic differential forms and $\mathscr{A}$ for analytic differential forms.

3.1 Compact case

The case where X is a compact Kähler manifold is much easier to deal with because the usual algebra of $\mathscr{C}^\infty$ differential forms on X already forms a mixed Hodge complex. The complex of differential forms with coefficients in a variation of Hodge structure is also known to form a mixed Hodge complex by the work of Deligne-Zucker. So one can directly study this case and apply for the first time our method. Recall that smooth complex projective algebraic varieties are examples of compact Kähler manifolds.

3.1.1 Representations with values in a real variation of Hodge structure

Let X be a compact Kähler manifold.

Main construction

Definition 3.1. A real *polarized variation of Hodge structure* of weight k on X is the data of a local system of finite-dimensional real vector spaces V on X with a decreasing filtration of the associated holomorphic vector bundle by holomorphic sub-vector bundles

$$\mathcal{F}^\bullet \subset V \otimes \mathcal{O}_X, \tag{3.1}$$

a flat bilinear map

$$Q : V \otimes V \longrightarrow \mathbb{R}, \tag{3.2}$$

and a flat connection

$$\nabla : V \otimes \mathcal{O}_X \longrightarrow V \otimes \Omega^1_X \tag{3.3}$$

such that at each point $x \in X$ the data

$$(V_x, \mathcal{F}^\bullet_x, Q_x) \tag{3.4}$$

forms a polarized Hodge structure of weight k (Definition 2.14 and Definition 2.19). Furthermore ∇ is required to satisfy Griffiths' transversality:

$$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega^1_X. \tag{3.5}$$

So let x be a base point of X. Let

$$\rho : \pi_1(X, x) \longrightarrow GL(V_x) \tag{3.6}$$

be a representation which is the monodromy of a real polarized variation of Hodge structure $(V, \mathcal{F}^\bullet, \nabla, Q)$ of weight k on X. The local system $\mathrm{Ad}(\rho)$ associated to ρ is now $\mathrm{End}(V)$, which by the usual linear algebraic constructions is a polarized variation of Hodge structure of weight 0. Explicitly

$$\mathcal{F}^p \,\mathrm{End}(V \otimes \mathcal{O}_X) = \Big\{ f : V \otimes \mathcal{O}_X \to V \otimes \mathcal{O}_X \ \Big|\ f(\mathcal{F}^\bullet) \subset \mathcal{F}^{\bullet + p} \Big\}. \tag{3.7}$$

One constructs a real mixed Hodge diagram as follows. Let

$$L_\mathbb{R} := \mathscr{E}^\bullet(X, \mathrm{End}(V)) \tag{3.8}$$

and

$$L_\mathbb{C} := \mathscr{E}^\bullet(X, \mathrm{End}(V \otimes \mathbb{C})). \tag{3.9}$$

One defines a filtration W which is the trivial one (everything has weight 0) and a filtration F on $L_\mathbb{C}$ by

$$F^p L^n_\mathbb{C} := \bigoplus_{r+s \geq p} \mathscr{E}^{r, n-r}(X, \mathcal{F}^s \,\mathrm{End}(V)). \tag{3.10}$$

It has a conjugate filtration

$$\overline{F}^q L^n_\mathbb{C} := \bigoplus_{r+s \geq q} \mathscr{E}^{n-r, r}(X, \overline{\mathcal{F}}^s \,\mathrm{End}(V)). \tag{3.11}$$

There is a DG Lie algebra structure as usual: locally, differential forms in L are sums of terms $\alpha \otimes u$ where α is a differential form on X and u is a section of $\mathrm{End}(V)$ and the Lie bracket is

$$[\alpha \otimes u, \beta \otimes v] := (\alpha \wedge \beta) \otimes [u, v] \tag{3.12}$$

where $\alpha \wedge \beta$ is the usual exterior product of differential forms and $[u, v]$ is the Lie bracket in $\mathrm{End}(V)$.

Then we define an augmentation ε_x from $(L_{\mathbb{R}}, L_{\mathbb{C}}, W, F)$ to the mixed Hodge diagram of Lie algebras formed by the Hodge structure on the Lie algebra $\mathfrak{g} := \mathrm{End}(V_x)$ with

$$\mathfrak{g}_{\mathbb{C}} = \mathrm{End}(V_x \otimes \mathbb{C}) = \mathrm{End}(V_x) \otimes \mathbb{C} \tag{3.13}$$

and the Hodge filtration is simply $F^\bullet = \mathcal{F}_x^\bullet$. It is given by

$$\begin{array}{lcl} \varepsilon_x : L & \longrightarrow & \mathfrak{g} \\ \omega \in L^0 & \longmapsto & \omega(x) \\ \omega \in L^{>0} & \longmapsto & 0. \end{array} \tag{3.14}$$

Theorem 3.2. *The data*

$$(L_{\mathbb{R}}, L_{\mathbb{C}}, W, F) \tag{3.15}$$

forms a real mixed Hodge diagram of Lie algebras. Together with $\varepsilon_x : L \to \mathfrak{g}$ this is an augmented mixed Hodge diagram of Lie algebras satisfying the hypothesis of Theorem 2.38.

More formally it is a diagram over the index category

$$I = \left\{ 0 \longrightarrow 1 \longleftarrow 2 \right\}$$

with

$$L_0 = (L_{\mathbb{R}}, W), \quad L_1 = (L_{\mathbb{C}}, W), \quad L_2 = (L_{\mathbb{C}}, W, F)$$

and the comparison morphisms are the obvious ones.

Proof. The fact that L is a real mixed Hodge complex is essentially the classical Hodge theory with values in a variation of Hodge structure, proved by Deligne and written by Zucker [Zuc79]. The condition that the differential be strictly compatible with F is known to be equivalent to the degeneration at E_1 of the F-spectral sequence and the last non-trivial axiom (axiom 5) requires $H^k(L)$ to carry a pure Hodge structure of weight k. Both of these statements are consequences of the Kähler identities for differential forms with twisted coefficients and this is explained in § 2 of loc. cit. We still have to check that the Lie bracket is compatible with the filtrations and this is clear by the formula (3.12) since both the wedge product of differential forms and the Lie bracket on $\mathrm{End}(V)$ are compatible with the filtration F.

By definition of the filtration $F^\bullet$ on $\mathfrak{g}_{\mathbb{C}}$, ε_x clearly preserves F. Then it is clear that ε_x is a morphism of mixed Hodge complexes and then that it preserves the Lie brackets. □

So we apply for the first time the method we developed.

Theorem 3.3. *If X is a compact Kähler manifold and ρ is the monodromy of a real polarized variation of Hodge structure on X, then there is a mixed Hodge structure on the complete local ring $\widehat{\mathcal{O}}_\rho$ of the representation variety* $\mathrm{Hom}(\pi_1(X, x), GL(V_x))$ *at ρ which is functorial in X, x, ρ.*

Proof. Over both fields $\mathbb{R}$ and $\mathbb{C}$, L and its augmentation control the deformation theory of ρ. This is the main theorem of Goldman-Millson (Theorem 1.30) and, combined with Remark 1.31 (change of base field), we say that the functor of deformations of ρ is controlled by the augmented mixed Hodge diagram of Lie algebras L. By Lemma 1.55, this deformation functor is associated with the L_∞ algebra structure on the desuspended mapping cone C of ε_x.

We need to check that $H^0(C) = 0$. By definition of the cone, a closed element of C^0 is given by a $\mathcal{C}^\infty$ section ω of $\mathrm{End}(V)$ such that $d(\omega) = 0$, so that ω is locally constant, and such that $\omega(x) = 0$. So $\omega = 0$ globally.

Then the deformation functor of ρ is pro-represented by a pro-Artin algebra that we denote by R as in Theorem 1.80. By the pro-Yoneda lemma (Theorem 1.25) this R is canonically isomorphic to $\widehat{\mathcal{O}}_\rho$. Again all this construction commutes with the change of base field so we work with C as mixed Hodge diagram. The augmented mixed Hodge diagram of Lie algebras we just have defined satisfies the hypothesis of our Theorem 2.38. So C is a mixed Hodge diagram of L_∞ algebras. Then we apply the main Theorem 2.44 (or its Corollary 2.46) to get a mixed Hodge structure on the pro-Artin algebra R, which as we said is canonically isomorphic to $\widehat{\mathcal{O}}_\rho$ (as pro-Artin algebra, so that each quotients by powers of the maximal ideals are isomorphic). And this induces the mixed Hodge structure on $\widehat{\mathcal{O}}_\rho$. □

By definition each quotient $\widehat{\mathcal{O}}_\rho/\mathfrak{m}^n$ carries a mixed Hodge structure (in the usual sense, finite-dimensional) and each map $\widehat{\mathcal{O}}_\rho/\mathfrak{m}^{n+1} \to \widehat{\mathcal{O}}_\rho/\mathfrak{m}^n$ is a morphism of mixed Hodge structures. The cotangent space $\mathfrak{m}/\mathfrak{m}^2$ also carries a mixed Hodge structure.

Description of this mixed Hodge structure

For the moment it is not proved that this mixed Hodge structure is the same as in [ES11] though there are strong indications for it. However describing the cotangent space (Definition 1.10) is easily tractable via the spectral sequence for the bar-weight filtration.

So let us describe the weight filtration. The first obvious thing is that it has non-positive weights because we work with coalgebras with non-negative weights and then we dualize (Definition 2.2.4).

The group $\pi_1(X, x)$ acts on $\mathrm{Hom}(\pi_1(X, x), GL(V_x))$ via ρ by conjugation. This action is algebraic. The orbit of ρ for this action has an induced reduced subscheme structure denoted by Ω_ρ. Its germ at ρ is a formal scheme $\widehat{\Omega}_\rho$ that is defined by an ideal $\mathfrak{j} \subset \widehat{\mathcal{O}}_\rho$. The quotient $\widehat{\mathcal{O}}_\rho/\mathfrak{j}$ is the algebra of formal functions of $\widehat{\Omega}_\rho$. In [ES11, § 2.2.2] it is explained that $\widehat{\mathcal{O}}_\rho/\mathfrak{j}$ is canonically isomorphic to the algebra of formal functions of the vector space $\mathfrak{g}/\varepsilon(H^0(L))$ at 0.

Proposition 3.4. *On the cotangent space to $\widehat{\mathcal{O}}_\rho$ the only weights of the induced mixed Hodge structure are $-1, 0$. The weight 0 part is $(\mathfrak{g}/\varepsilon(H^0(L)))^*$ and the weight -1 part is $(H^1(L))^*$. On $\widehat{\mathcal{O}}_\rho$ the weight 0 part is $\widehat{\mathcal{O}}_\rho/\mathfrak{j}$.*

Proof. We relax the notations, writing $L, \varepsilon, \ldots$ for any of the components of the mixed Hodge diagram. Let C be the desuspended mapping cone of ε. To access to the cotangent space to $\widehat{\mathcal{O}}_\rho$ we use the spectral sequence for the bar-weight filtration that we computed in the proof of Theorem 2.44 and in particular the equation (2.63). There, replacing L (which was an L_∞ algebra) by our C, we proved that

$$
{}_{\mathscr{C}W}E_1^{-k,q}(\mathscr{C}_s(C)) = \bigoplus_{r=1}^{s} H^{-k+r+q}(\mathrm{Gr}^W_{k-r}\, C^{\wedge r}). \tag{3.16}
$$

Letting $s = 1$ corresponds to computing the coalgebra which is dual to the cotangent space to $\widehat{\mathcal{O}}_\rho$. The bar-weight filtration is the natural weight filtration on $\mathscr{C}(C)$ and since $\widehat{\mathcal{O}}_\rho$ is obtained by an H^0, on which the construction Dec (Definition 2.8) doesn't shift the filtration, this is also (up to duality) the natural weight filtration of $\widehat{\mathcal{O}}_\rho$. Furthermore by the Theorem 2.29 of Deligne this spectral sequence degenerates at E_2. Recall that the weight filtration on C has $W_kC^n = W_kL^n$ for $n \neq 1$ and

$$
W_kC^1 = W_kL^1 \oplus W_{k+1}\mathfrak{g}.
$$

So $\mathrm{Gr}^W_0(C^n) = L^n$, $\mathrm{Gr}^W_{-1}(C^1) = \mathfrak{g}$ and all other terms $\mathrm{Gr}^W_k(C^n)$ vanish. The page E_1 of the spectral sequence looks like this

$$
\begin{array}{c|ccc}
q & \vdots & \vdots & \\
2 & H^2(L) & 0 & \\
1 & H^1(L) & 0 & \\
0 & H^0(L) & \mathfrak{g} & \\
\hline
 & -1 & 0 & -k
\end{array} \tag{3.17}
$$

with the differential d_1 going to the right and all outside terms, except on the column $-k = -1$, are zero. So d_1 has only one non-zero component which is exactly

$$
\varepsilon : H^0(L) \longrightarrow \mathfrak{g}.
$$

Thus on $H^0(\mathscr{C}_1(C))$ there are only two weight graded pieces, with

$$
\mathrm{Gr}^{\mathscr{C}W}_0\, H^0(\mathscr{C}_1(C)) = \mathfrak{g}/\varepsilon(H^0(L)), \quad \mathrm{Gr}^{\mathscr{C}W}_1\, H^0(\mathscr{C}_1(C)) = H^1(L).
$$

Going to the dual, these give the two weight graded pieces on the cotangent space to $\widehat{\mathcal{O}}_\rho$ as claimed.

Then we explain how $\widehat{\mathcal{O}}_\rho$ contains the ideal $\mathfrak{j}$ defining the orbit of ρ in weight zero. First observe that the inclusion $H^0(L) \subset L$ induces a morphism

$$
\mathrm{Cone}\Big(\varepsilon : H^0(L) \to \mathfrak{g}\Big)[-1] \longrightarrow \mathrm{Cone}\Big(\varepsilon : L \to \mathfrak{g}\Big)[-1] = C. \tag{3.18}
$$

Let D the desuspended mapping cone on the left-hand side with its L_∞ algebra structure. Then the map (3.18) induces an embedding

$$
\mathscr{C}(D) \longrightarrow \mathscr{C}(C)
$$

and gives by applying the pro-representability Theorem 1.80 a dual morphism

$$
\widehat{\mathcal{O}}_\rho \longrightarrow B := \mathbf{k} \oplus H^0(\mathscr{C}(D))^*. \tag{3.19}
$$

We are going to show that this defines the ideal $\mathfrak{j}$ by computing B. First D is the DG vector space with $D^0 = H^0(L)$, $D^1 = \mathfrak{g}$ and the only non-zero differential given by ε. This admits a morphism to the DG vector space E concentrated in degree 1

$$\begin{array}{llllll} \text{degree} & -1 & 0 & 1 & 2 & \\ D: \cdots \longrightarrow & 0 \longrightarrow & H^0(L) & \xrightarrow{\varepsilon} \mathfrak{g} & \longrightarrow 0 & \longrightarrow \cdots \\ & & \downarrow & \quad\downarrow & & \\ E: \cdots \longrightarrow & 0 \longrightarrow & 0 & \longrightarrow \mathfrak{g}/\varepsilon(H^0(L)) & \longrightarrow 0 & \longrightarrow \cdots \end{array} \tag{3.20}$$

which is a quasi-isomorphism. One can see it as a morphism of desuspended mapping cones induced by the morphisms from the central commutative square of (3.20). So

$$H^0(\mathscr{C}(D)) = H^0(\mathscr{C}(E)).$$

But the L_∞ algebra structure of Fiorenza-Manetti on

$$E = \mathfrak{g}/\varepsilon(H^0(L))[-1] = \operatorname{Cone}\Big(0 \to \mathfrak{g}/\varepsilon(H^0(L))\Big)[-1]$$

is trivial, with all operations ℓ_r for $r \geq 1$ (including the differential and the Lie bracket) being zero. Since this is concentrated in degree 1 (see also Example 1.83) it is straightforward to compute that

$$H^0(\mathscr{C}(E)) = \operatorname{Sym}^+(\mathfrak{g}/\varepsilon(H^0(L))) = H^0(\mathscr{C}(D))$$

and that

$$B = \mathbf{k} \oplus H^0(\mathscr{C}(D))^* = (\operatorname{Sym}(\mathfrak{g}/\varepsilon(H^0(L))))^*$$

which is the algebra of formal power series at 0 on the vector space $\mathfrak{g}/\varepsilon(H^0(L))$. The kernel of the map (3.19) is then the ideal $\mathfrak{j}$ and this map is a morphism of mixed Hodge structures, where B has pure weight 0. Using the strictness of the weight filtration for morphisms of mixed Hodge structures, this proves that B is also $\operatorname{Gr}_0^W(\widehat{\mathcal{O}}_\rho)$. □

3.1.2 Variations

There are many possible variations of the hypothesis of our main theorem, concerning the field of definition of the mixed Hodge structure and the group in which we look at representations. In each of theses cases the variations are in the definitions and the constructions of the mixed Hodge diagrams. The description of the mixed Hodge structure in Proposition 3.4 holds in a similar way. So we will write the proofs much more briefly.

The first possible variation is to work with a subgroup G of $GL(V_x)$. For example, one can take G to be the Zariski closure of the image of ρ in $GL(V_x)$.

Proposition 3.5. *Let G be a real linear algebraic group. Let $\rho : \pi_1(X, x) \to G(\mathbb{R})$ be a representation which is the monodromy of a real polarized variation of Hodge structure on X. Then there is a functorial mixed Hodge structure on the local ring $\widehat{\mathcal{O}}_\rho$ of the representation variety $\operatorname{Hom}(\pi_1(X, x), G)$ at ρ.*

Proof. Re-write the proof of Theorem 3.3 by replacing $GL(V_x)$ by G and $\mathrm{End}(V_x)$ by the Lie algebra $\mathfrak{g}$ of G. Then again one gets L that is an augmented mixed Hodge diagram of Lie algebras over $\mathfrak{g}$. □

The second possible variation is working with complex mixed Hodge structures. We define them only now since these are *not* "mixed Hodge structures over $\mathbb{C}$" in the naive sense and if we would have taken this into account the redaction of the whole chapter 2 would have been much heavier.

Definition 3.6 ([ES11, 1.1])**.** A *complex Hodge structure* of weight k is the data of a finite-dimensional vector space K over $\mathbb{C}$ with two filtrations $F, \overline{G}$ that are k-opposed.

A *polarization* of the complex mixed Hodge structure K is the data of a hermitian form Q on K such that for the direct sum decomposition

$$K = \bigoplus_{p+q=k} K^{p,q}, \quad K^{p,q} := F^p K \cap \overline{G}^q K \tag{3.21}$$

are satisfied the conditions:

1. $Q(K^{p,q}, K^{r,s}) = 0$ if $(p,q) \neq (r,s)$,
2. $(-1)^{p+k} Q$ is definite positive on $K^{p,q}$.

A *complex mixed Hodge structure* is the data of a finite-dimensional vector space K over $\mathbb{C}$ with two decreasing filtrations $F, \overline{G}$ and an increasing filtration W such that each graded part $\mathrm{Gr}_k^W(K)$ with its induced filtrations $F, \overline{G}$ is a complex Hodge structure of weight k.

For example if K is a mixed Hodge structure over $\mathbf{k}$ then $K \otimes \mathbb{C}$ is canonically a mixed Hodge structure over $\mathbb{C}$ with $\overline{G}$ being the conjugate filtration of F. It is polarized if K is.

Definition 3.7. A *complex mixed Hodge complex* is the data of a DG vector space K over $\mathbb{C}$ equipped with an increasing filtration W and two decreasing filtrations $F, \overline{G}$ such that:

1. K is a bounded-below complex.
2. For all $n \in \mathbb{Z}$, $H^n(K)$ is finite-dimensional.
3. For all $k \in \mathbb{Z}$, the differential of $\mathrm{Gr}_k^W(K)$ is strictly compatible with the two filtrations $F, \overline{G}$.
4. For all $n \in \mathbb{Z}$ and all $k \in \mathbb{Z}$, the filtrations $F, \overline{G}$ induced on $H^n(\mathrm{Gr}_k^W(K))$ define a complex Hodge structure of weight $k+n$.

Each term of the cohomology of such a complex carries a canonical complex mixed Hodge structure.

Definition 3.8 ([ES11, 1.8])**.** A *complex polarized variation of Hodge structure* of weight k on X is the data of a local system of finite-dimensional complex vector spaces V on X with a decreasing filtration of the associated holomorphic vector bundle by sub-holomorphic bundles

$$\mathcal{F}^\bullet \subset V \otimes \mathcal{O}_X \tag{3.22}$$

and a decreasing filtration of the associated anti-holomorphic vector bundle by anti-holomorphic sub-vector bundles

$$\overline{\mathcal{G}}^{\bullet} \subset V \otimes \overline{\mathcal{O}}_X, \tag{3.23}$$

with a flat connection

$$\nabla = \nabla^{1,0} + \nabla^{0,1} \tag{3.24}$$

such that

$$\nabla^{1,0} : V \otimes \mathcal{O}_X \longrightarrow V \otimes \Omega^1_X, \tag{3.25}$$

$$\nabla^{0,1} : V \otimes \overline{\mathcal{O}}_X \longrightarrow V \otimes \overline{\Omega}^1_X, \tag{3.26}$$

a flat hermitian form

$$Q : V \otimes V \longrightarrow \mathbb{C} \tag{3.27}$$

such at at each point $x \in X$ the data

$$(V_x, \mathcal{F}_x, \overline{\mathcal{G}}_x, Q_x) \tag{3.28}$$

forms a complex Hodge structure of weight k, and furthermore Griffiths' transversality

$$\nabla^{1,0}(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega^1_X, \tag{3.29}$$

$$\nabla^{0,1}(\overline{\mathcal{G}}^q) \subset \overline{\mathcal{G}}^{q-1} \otimes \overline{\Omega}^1_X, \tag{3.30}$$

is satisfied.

Finally, one can state the most general result:

Proposition 3.9. *Let G be a complex linear algebraic group. Let $\rho : \pi_1(X, x) \to G(\mathbb{C})$ be a representation which is the monodromy of a complex polarized variation of Hodge structure on X. Then there is a functorial complex mixed Hodge structure on the local ring $\widehat{\mathcal{O}}_\rho$ of the representation variety* $\mathrm{Hom}(\pi_1(X, x), G)$ *at ρ.*

Proof. Re-write everything with complex mixed Hodge structures and complex mixed Hodge complexes. □

The last possible addition to the main theorem is to show that the mixed Hodge structure on $\widehat{\mathcal{O}}_\rho$ is defined over a subfield $\mathbf{k} \subset \mathbb{R}$ if the variation of Hodge structure is.

Definition 3.10. The polarized variation of Hodge structure $(V, \mathcal{F}^\bullet, Q, \nabla)$ over X is said to be *defined over* $\mathbf{k}$ (where $\mathbf{k}$ is a subfield of $\mathbb{R}$) if V is a local system of finite-dimensional vector spaces over $\mathbf{k}$, Q is defined over $\mathbf{k}$, and at each point $x \in X$ the data $(V_x, \mathcal{F}^\bullet_x, Q_x)$ forms a polarized Hodge structure over $\mathbf{k}$.

Proposition 3.11 (See sect. 3.2.2)**.** *In Theorem 3.3, assume that the variation of Hodge structure is defined over* $\mathbf{k}$*. Then the mixed Hodge structure on $\widehat{\mathcal{O}}_\rho$ is defined over* $\mathbf{k}$*.*

This we will prove in the next section dealing with the non-compact case since it introduces the necessary DG algebra over $\mathbf{k}$ computing the cohomology of X that serves as the component over $\mathbf{k}$ of a mixed Hodge diagram.

3.2 Non-compact case

In the case where X is non-compact, the construction of an appropriate mixed Hodge diagram that computes the cohomology of X is already more difficult than in the compact case. Furthermore this construction depends on the choice of a compactification by a divisor with normal crossings and one has to be careful when studying the functoriality of this mixed Hodge diagram.

3.2.1 Real representations with finite image

Let X be a smooth quasi-projective algebraic variety over $\mathbb{C}$. Let x be a base point of X. Let G be a linear algebraic group over $\mathbb{R}$ with Lie algebra $\mathfrak{g}$. Let

$$\rho : \pi_1(X, x) \longrightarrow G(\mathbb{R}) \tag{3.31}$$

be a representation and assume that ρ has *finite* image. Under these hypothesis we want to construct an appropriate augmented mixed Hodge diagram of Lie algebras controlling the deformation theory of ρ. The construction of such an object is already done by Kapovich-Millson and entirely contained in [KM98, § 14–15], also reviewed as an essential tool in the sections A.2.2 and A.3.1 of the preliminary work reproduced in the appendix A. So we could invoke it and we would already get a mixed Hodge structure on $\widehat{\mathcal{O}}_\rho$ by the method of Theorem 3.3. However the functoriality of such a construction is not so clear: it uses the method of Morgan [Mor78, § 2–3] which involves many choices to construct a mixed Hodge diagram and then the quasi-isomorphisms to compare the resulting diagrams commute only *up to homotopy*. So we shall better re-write this using the construction of mixed Hodge diagrams of Navarro Aznar [Nav87] which is totally functorial on varieties with a given compactification.

Before entering the construction, let us explain the strategy and fix some notations. Let the group

$$\Phi := \frac{\pi_1(X, x)}{\mathrm{Ker}(\rho)} \simeq \rho(\pi_1(X, x)). \tag{3.32}$$

To $\mathrm{Ker}(\rho) \subset \pi_1(X, x)$ corresponds a finite étale Galois cover $Y \to X$ with automorphism group Φ that acts simply transitively on the fibers, and equipped with a fixed base point $y \in Y$ over x. It is known that Y is automatically a smooth quasi-projective algebraic variety. The flat principal bundle P induced by the holonomy of ρ is trivial when pulled-back to Y, as well as its adjoint bundle $\mathrm{Ad}(P)$. So the DG Lie algebra of Goldman-Millson is (over both $\mathbb{R}$ and $\mathbb{C}$)

$$L := \mathscr{E}^\bullet(X, \mathrm{Ad}(P)) = (\mathscr{E}^\bullet(Y, \pi^* \mathrm{Ad}(P)))^\Phi = (\mathscr{E}^\bullet(Y) \otimes \mathfrak{g})^\Phi \tag{3.33}$$

(where the exponent Φ denotes the invariants by the action of Φ). In order to construct a mixed Hodge diagram that is quasi-isomorphic to this we want to find a mixed Hodge diagram for Y equipped with an action of Φ, then tensor it with $\mathfrak{g}$, then take the invariants by Φ.

For the augmentation, since there is a canonical identification of fibers $\mathrm{Ad}(P)_x \simeq \mathfrak{g}$ one can define an augmentation

$$\varepsilon_x : \mathscr{E}^\bullet(X, \mathrm{Ad}(P)) \longrightarrow \mathfrak{g} \tag{3.34}$$

exactly as in the compact case (3.14) by evaluating degree zero forms at x and higher degree forms to zero. This augmentation can be lifted equivariantly to Y: ε_x comes from the augmentation

$$\eta_x : \mathscr{E}^\bullet(Y) \otimes \mathfrak{g} \longrightarrow \mathfrak{g} \tag{3.35}$$

defined by

$$\eta_x(\omega \otimes u) := \frac{1}{|\Phi|} \sum_{g \in \Phi} \varepsilon_y(g.(\omega \otimes u)) \tag{3.36}$$

where ε_y simply evaluates forms with values in $\mathfrak{g}$ at y. Observe the notations: ε_y depends on y but in the definition of η_x we sum over the whole (finite) fiber of π over x so η_x depends only on x. Then we see that η_x induces ε_x when restricted to the equivariant forms $(\mathscr{E}^\bullet(Y) \otimes \mathfrak{g})^\Phi$.

The construction of mixed Hodge diagrams of Navarro Aznar

Let Y be a smooth quasi-projective algebraic variety. We start by choosing a smooth compactification

$$j : Y \hookrightarrow \overline{Y} \tag{3.37}$$

such that the complement

$$D := \overline{Y} \setminus Y \tag{3.38}$$

is a divisor with simple normal crossings, all whose components are smooth. From this situation, Navarro Aznar constructs a functorial real mixed Hodge diagram that we will denote by $\mathrm{MHD}(\overline{Y}, D)_\mathbb{R}$ as follows.

The first step is to construct sheaves on $\overline{Y}$ whose cohomology will compute the cohomology of Y. These sheaves are called the *logarithmic Dolbeaut complexes* ([Nav87, § 8]) and are constructed from analytic differentials forms on Y, to which we refer by the letter $\mathscr{A}$.

The part over $\mathbb{R}$ that carries the filtration W is denoted by $\mathscr{A}^\bullet_{\overline{Y}}(\log D)_\mathbb{R}$. It is a sheaf of real filtered DG algebras. Locally near a point $y \in Y$, there are local coordinates $z_1, \ldots, z_N$ such that D has equation $z_1 \cdots z_r = 0$, meaning that there are r components of D crossing at y. Then $\mathscr{A}^\bullet_{\overline{Y}}(\log D)_\mathbb{R}$ is the sub-sheaf of $j_*\mathscr{A}^\bullet_{Y,\mathbb{R}}$ locally generated by the sections

$$\mathrm{Re}\left(\frac{dz_i}{z_i}\right), \quad \mathrm{Im}\left(\frac{dz_i}{z_i}\right), \quad \ln(|z_i|), \quad 1 \le i \le r \tag{3.39}$$

and

$$\mathrm{Re}(dz_i), \quad \mathrm{Im}(dz_i), \quad r+1 \le i \le N. \tag{3.40}$$

The filtration W is defined such that the sections in (3.39) have weight 1 and those in (3.40) have weight 0, extended multiplicatively.

The part over $\mathbb{C}$ is a sheaf of bidifferential bigraded algebras (i.e. admitting a bigrading and a differential that is a sum of two differentials of degrees respectively $(1,0)$ and $(0,1)$) denoted by $\mathscr{A}^{\bullet,\bullet}_{\overline{Y}}(\log D)_\mathbb{C}$ and carries two filtrations W, F. Locally, it is the sub-sheaf of $j_*\mathscr{A}^{\bullet,\bullet}_{Y,\mathbb{C}}$ generated by the sections

$$\frac{dz_i}{z_i}, \quad \frac{d\bar{z}_i}{\bar{z}_i}, \quad \ln(|z_i|), \quad 1 \le i \le r \tag{3.41}$$

and

$$dz_i, \quad d\bar{z}_i, \quad r+1 \le i \le N. \tag{3.42}$$

The filtration W is defined such that the sections in (3.41) have weight 1 and those in (3.42) have weight 0. The filtration F is defined as usual: sections in F^p have at least p terms dz. There is a canonical isomorphism

$$\mathscr{A}^\bullet_{\overline{Y}}(\log D)_{\mathbb{R}} \otimes \mathbb{C} \xrightarrow{\simeq} \mathscr{A}^\bullet_{\overline{Y}}(\log D)_{\mathbb{C}} \tag{3.43}$$

where in the right-hand side we consider the associated sheaf of (simple) DG algebras.

Now there are various filtered quasi-isomorphisms relating the Dolbeaut complex to other complexes that are well-known to compute the cohomology of Y by sheaf theory or Hodge theory. There is a bifiltered quasi-isomorphism

$$(\Omega^\bullet_{\overline{Y}}(\log D), W, F) \xrightarrow{\approx} (\mathscr{A}^\bullet_{\overline{Y}}(\log D)_{\mathbb{C}}, W, F). \tag{3.44}$$

that compares the Dolbeaut complex over $\mathbb{C}$ with the complex of holomorphic forms with logarithmic poles along D. This last one computes the cohomology of Y with complex coefficients via the inclusion

$$\Omega^\bullet_{\overline{Y}}(\log D) \hookrightarrow j_* \mathscr{E}^\bullet_{Y,\mathbb{C}} \tag{3.45}$$

which is a quasi-isomorphism (see for example [Voi02, § 8.2.3]), and the filtrations W, F induced on the cohomology are simply induced by the filtrations of the logarithmic complex. Similarly, there is a (longer !) fixed functorial chain of filtered quasi-isomorphisms showing that the real Dolbeaut complex computes the cohomology of Y with real coefficients and induces the filtration W over $\mathbb{R}$. Together with the canonical isomorphism (3.43) this induces a real structure on the cohomology with complex coefficients.

We sum up this situation by the following statement.

Theorem 3.12 ([Nav87, § 8])**.** *The Dolbeaut complex computes the cohomology of Y over $\mathbb{R}$ and $\mathbb{C}$, with the induced filtrations W, F, via a fixed chain of canonical filtered quasi-isomorphisms.*

The second step is to construct functorial resolutions for these sheaves that will also have a structure of mixed Hodge diagram of algebras. Let $\mathscr{F}^\bullet$ be a sheaf of DG algebras on $\overline{Y}$ over the field $\mathbf{k}$ of characteristic zero. The *Thom-Whitney resolution* associates to $\mathscr{F}^\bullet$ a DG algebra over $\mathbf{k}$ whose cohomology computes the sheaf cohomology of $\mathscr{F}$ by the following sequence of operations:

1. First consider the Godement resolution $\mathscr{F}^\bullet \to \mathscr{G}$ as a strictly cosimplicial sheaf of DG algebras, see [God58, § II.4.3 and Appendice].
2. Then take the global sections of $\mathscr{G}$ over $\overline{Y}$. One gets a strictly cosimplicial DG algebra $\Gamma(\overline{Y}, \mathscr{G})$.
3. To $\Gamma(\overline{Y}, \mathscr{G})$ apply the *Thom-Whitney simple functor* [Nav87, § 2]. It is obtained by first tensoring with the simplicial algebra of polynomial differential forms on the simplex $\Omega_{\mathbf{k}}(\Delta)$ of Definition 1.62, which gives an algebra with components naturally indexed by four integers that is simplicial in one of them, cosimplicial in another, and a bidifferential bigraded algebra in the last two indices. Then consider the *end* of this simplicial-cosimplicial object, which is a bidifferential bigraded algebras (this eliminates the simplicial-cosimplicial dependency), and take the associated simple DG algebra.

The composition of these operations is denoted by $R_{\mathrm{TW}}\Gamma(\overline{Y}, \mathscr{F}^\bullet)$. If $\mathscr{F}^\bullet$ has filtrations then $R_{\mathrm{TW}}\Gamma(\overline{Y}, \mathscr{F}^\bullet)$ has induced filtrations and a filtered quasi-isomorphism between sheaves induces a filtered quasi-isomorphism between their Thom-Whitney resolutions. Furthermore the Thom-Whitney resolution commutes with the change of coefficients (at least for fields). So it can be used to construct mixed Hodge diagrams:

Theorem 3.13 ([Nav87, 8.15]). *The data*

$$R_{\mathrm{TW}}\Gamma\Big(\overline{Y}, (\mathscr{A}^\bullet_{\overline{Y}}(\log D)_{\mathbb{R}}, W)\Big), \quad R_{\mathrm{TW}}\Gamma\Big(\overline{Y}, (\mathscr{A}^\bullet_{\overline{Y}}(\log D)_{\mathbb{C}}, W, F)\Big), \tag{3.46}$$

defines a real mixed Hodge diagram of algebras that computes canonically the cohomology of Y with its mixed Hodge structure. It is functorial in the pair $(\overline{Y}, D)$. We denote it by $\mathrm{MHD}(\overline{Y}, D)_{\mathbb{R}}$.

Equivariant mixed Hodge diagram

Take back our variety X with a representation ρ of its fundamental group with finite image and the cover $\pi : Y \to X$ with Galois group Φ. Compactify $X \hookrightarrow \overline{X}$ by a divisor with normal crossings $D_X := \overline{X} \setminus X$ whose components are smooth, into a smooth projective variety $\overline{X}$, using the classical theorem of Hironaka. By the theorem of Sumihiro on equivariant completion [Sum74], it is possible to compactify $Y \hookrightarrow \overline{Y}'$ so that the action of Φ extends to $\overline{Y}'$. And then by the work of Bierstone-Milman on canonical resolutions of singularities [BM97, § 13] one can construct a resolution of singularities $\overline{Y} \to \overline{Y}'$ to which the action of Φ lifts. The smooth normal crossing divisor $D := \overline{Y} \setminus Y$ lives above D_X and π extends to a finite morphism

$$\pi : \overline{Y} \longrightarrow \overline{X} \tag{3.47}$$

ramified over D_X and invariant under the action of Φ. We sum up this situation in the diagram

$$\begin{array}{ccc} \Phi \curvearrowright Y & \hookrightarrow & \overline{Y} \curvearrowleft \Phi \\ \downarrow \pi & & \downarrow \pi \\ X & \hookrightarrow & \overline{X} \end{array} \tag{3.48}$$

and we call this an *equivariant compactification* of $\pi : Y \to X$. Actually, $\overline{X}$ plays no role to define $\overline{Y}$.

Then we can introduce the mixed Hodge diagram $\mathrm{MHD}(\overline{Y}, D)_{\mathbb{R}}$ of Theorem 3.13 that computes the cohomology of Y. To go from the cohomology of Y to that of X we use the following simple lemma. If a group Φ acts on a DG vector space V we denote by V^Φ the sub-DG vector space of invariants by Φ.

Lemma 3.14. *Let V be a DG vector space over a field $\mathbf{k}$ of characteristic zero and let Φ be a finite group that acts on V. Then on cohomology*

$$H(V^\Phi) = H(V)^\Phi. \tag{3.49}$$

As a consequence, if

$$\psi : V \xrightarrow{\approx} W$$

is a Φ-equivariant quasi-isomorphism between DG vector spaces on which Φ acts, then the induced morphism

$$\psi^\Phi : V^\Phi \xrightarrow{\approx} W^\Phi \tag{3.50}$$

is again a quasi-isomorphism.

Proof. An element in $H^n(V)^\Phi$ is given by a closed element $v \in V^n$ such that for all $g \in \Phi$ there exists an element $\beta_g \in V^{n-1}$ with $g.v = v + d(\beta_g)$. Of course there is a natural map

$$\varphi : H^n(V^\Phi) \longrightarrow H^n(V)^\Phi$$

induced by sending a closed element v of V^n such that $g.v = v$ for all $g \in \Phi$ to an element of $H^n(V)^\Phi$ (taking $\beta_g = 0$). And there is a cross-section to φ: if $v \in H^n(V)^\Phi$, take

$$w := \frac{1}{|\Phi|} \sum_{g\in\Phi} g.v = v + d\left(\frac{1}{|\Phi|} \sum_{g\in\Phi} d(\beta_g)\right)$$

then $g.w = w$ for all $g \in \Phi$, so w defines a cohomology class in $H^n(V^\Phi)$ that is sent by φ to v. This proves that φ is an isomorphism. □

Since the group Φ acts on $(\overline{Y}, D)$, it acts on all objects that are functorially attached to it. In particular it acts on the whole mixed Hodge diagram $\mathrm{MHD}(\overline{Y}, D)_\mathbb{R}$ (by morphisms of mixed Hodge diagrams) and one can define the invariant diagram $\mathrm{MHD}(\overline{Y}, D)_\mathbb{R}^\Phi$ by taking the invariants in each component.

Lemma 3.15. *The invariant diagram* $\mathrm{MHD}(\overline{Y}, D)_\mathbb{R}^\Phi$ *is again a mixed Hodge diagram and it computes canonically the cohomology of X with its mixed Hodge structure.*

Proof. Lemma 3.14 allows us to check all the axioms of mixed Hodge diagrams of Definition 2.27. Note that Φ automatically acts on the cohomology of such a diagram by morphisms of mixed Hodge structures, so the invariant cohomology is again a mixed Hodge structure.

Then Lemma 3.14 again, combined with the chain of canonical quasi-isomorphisms described for the Theorem 3.12 relating the Dolbeaut complex to other canonical and functorial complexes computing the cohomology of Y, shows that $\mathrm{MHD}(\overline{Y}, D)_\mathbb{R}^\Phi$ computes the cohomology of X: for example it is related canonically to the algebra $\mathscr{E}^\bullet(Y, \mathbb{C})$ via the maps (3.44) and (3.45) and the algebra of invariants under Φ of $\mathscr{E}^\bullet(Y, \mathbb{C})$ is $\mathscr{E}^\bullet(X, \mathbb{C})$. □

The controlling mixed Hodge diagram of augmented Lie algebras

Equation (3.33) and the related remarks explain how to get the controlling DG algebra of Goldman-Millson in this situation. Consider the diagram of DG Lie algebras

$$M := \left(\mathrm{MHD}(\overline{Y}, D)_\mathbb{R} \otimes \mathfrak{g}\right)^\Phi \tag{3.51}$$

(in the component over the field $\mathbf{k}$, we tensor by $\mathfrak{g}_\mathbf{k}$ over $\mathbf{k}$ then take the invariants where Φ acts on both the DG algebra over $\mathbf{k}$ and $\mathfrak{g}_\mathbf{k}$).

Lemma 3.16. *This M is a mixed Hodge diagram of Lie algebras that is canonically quasi-isomorphic to the DG Lie algebra L of Goldman-Millson controlling the deformation theory of ρ.*

Proof. First do it on Y, on which $\pi^* \operatorname{Ad}(P)$ is trivial and there is no action of Φ. Then the cohomology functors, as well as the functor Gr, commute with the operation of tensoring with $\mathfrak{g}$. So it is easy to see that M is a mixed Hodge diagram of Lie algebras. And by arguments similar to the proof of Lemma 3.15 this M is related by a canonical chain of quasi-isomorphisms to the DG Lie algebra of Goldman-Millson, which is simply here $\mathscr{E}^\bullet(Y, \mathfrak{g})$ (over $\mathbb{R}$ as well as over $\mathbb{C}$). For this we simply have to tensor by $\mathfrak{g}$ the whole chain of quasi-isomorphisms relating $\mathrm{MHD}(\overline{Y}, D)_\mathbb{R}$ to $\mathscr{E}^\bullet(Y)$.

Now on X, the controlling DG Lie algebra of Goldman-Millson is exactly

$$L = \mathscr{E}^\bullet(X, \operatorname{Ad}(P)) = (\mathscr{E}^\bullet(Y) \otimes \mathfrak{g})^\Phi. \tag{3.52}$$

So by arguments similar to the proof of Lemma 3.15, using the Lemma 3.14, we see that when we take the invariants by Φ, M is quasi-isomorphic to L. By the fundamental Theorem 1.19 one can use M to control the deformation theory of ρ. □

Remark 3.17. We see from the proof that a quasi-isomorphism φ between two such mixed Hodge diagrams associated with two equivariant compactifications $\overline{Y}, \overline{Y}'$, with φ equivariant with respect to Φ, induces a quasi-isomorphism $M \xrightarrow{\approx} M'$ between the corresponding mixed Hodge diagrams of Lie algebras.

The last step is to identify the augmentation ε_x, via its lift η_x to Y, but at the level of M. Roughly the argument is that on a manifold Y the evaluation of differential forms with values in $\mathbf{k}$ at a point y comes from the purely topological and sheaf-theoretical data of the canonical morphism of sheaves from the constant sheaf $\mathbf{k}_Y$ to the sheaf concentrated on y with stalk $\mathbf{k}$ denoted by $\mathbf{k}_y$. In some sense it evaluates sections of $\mathbf{k}_Y$ at y. Since different resolutions of $\mathbf{k}_Y$, that compute the cohomology of Y with coefficients in $\mathbf{k}$, are always quasi-isomorphic by usual arguments of homological algebra, and since $\mathbf{k}_y$ is already flasque so its resolution is essentially $\mathbf{k}$ itself, then to each such resolutions corresponds a canonical evaluation morphism to $\mathbf{k}$.

So our augmentation of M is obtained by several elementary steps. Recall that x is the base point of X and y is the base point of Y over x. Let $\mathbf{k}$ be the field $\mathbb{R}$ or $\mathbb{C}$. Let $\mathbf{k}_y$ be the sheaf on $\overline{Y}$ supported on y with stalk $\mathbf{k}$ at y. There is a natural augmentation seen as a morphism of sheaves

$$\mu_y : \mathscr{A}^\bullet_{\overline{Y}}(\log D)_\mathbf{k} \longrightarrow \mathbf{k}_y \tag{3.53}$$

that, as usual, evaluates forms of degree zero at y. By the functoriality of the Thom-Whitney construction, this induces a morphism

$$R_{TW}(\mu_y) : R_{\mathrm{TW}}\Gamma\Big(\overline{Y}, \mathscr{A}^\bullet_{\overline{Y}}(\log D)_\mathbf{k}\Big) \longrightarrow R_{\mathrm{TW}}\Gamma(\overline{Y}, \mathbf{k}_y). \tag{3.54}$$

The left-hand side is a component of the mixed Hodge diagram $\mathrm{MHD}(\overline{Y}, D)_\mathbb{R}$. The right-hand side is simply the field $\mathbf{k}$. Namely the sheaf $\mathbf{k}_y$ is flasque so its Godement resolution is simply $\mathbf{k}_y$. Then the set of global sections is the field $\mathbf{k}$, seen as a cosimplicial DG algebra. And the Thom-Whitney construction just gives $\mathbf{k}$.

From this one defines an augmentation

$$\nu_x : R_{\mathrm{TW}}\Gamma\Big(\overline{Y}, \mathscr{A}^\bullet_{\overline{Y}}(\log D)_\mathbf{k}\Big) \otimes \mathfrak{g} \longrightarrow \mathfrak{g} \tag{3.55}$$

as (compare with (3.36))

$$\nu_x(\omega \otimes u) := \frac{1}{|\Phi|} \sum_{g \in \Phi} (R_{\mathrm{TW}}(\mu_y) \otimes \mathrm{id}_{\mathfrak{g}})(g.(\omega \otimes u)). \tag{3.56}$$

When restricted to the invariants by Φ, this induces an augmentation (still denoted by ν_x)

$$\nu_x : M = \Big(\mathrm{MHD}(\overline{Y}, D)_{\mathbb{R}} \otimes \mathfrak{g} \Big)^{\Phi} \longrightarrow \mathfrak{g}. \tag{3.57}$$

Lemma 3.18. *Via the canonical chain of quasi-isomorphisms relating M to L, this ν_x corresponds to the augmentation ε_x of* (3.34)*.*

By this we mean that the whole canonical chain of quasi-isomorphisms relating M to L has augmentations to $\mathfrak{g}$, commuting with the quasi-isomorphisms, and relating ν_x to ε_x.

Proof. First do it on Y, without the action of Φ. Then ε_y is also induced by the augmentation at the level of sheaves

$$\mathscr{E}^{\bullet}_{Y,\mathbf{k}} \longrightarrow \mathbf{k}_y$$

then by taking global sections and tensoring with $\mathfrak{g}$. This augmentation commutes with the chain of quasi-isomorphisms relating $\mathscr{E}^{\bullet}_Y$ to $\mathscr{A}^{\bullet}_{\overline{Y}}(\log D)$, via the intermediate augmentation of $\Omega^{\bullet}_{\overline{Y}}(\log D)$ which is defined by the same obvious way, evaluating degree zero holomorphic forms at y (important is the fact that y is in Y and not on D). This is enough to prove the claim on Y, since then it is easy to tensor all the chain of augmented quasi-isomorphisms by $\mathfrak{g}$.

One goes from Y to X by simply comparing the formulas (3.56) and (3.36), from which we see by construction that ν_x corresponds to the augmentation that we denoted by η_x, and then by going to the invariants under Φ we see that ν_x on M corresponds to ε_x on L. □

Conclusion

Putting everything together, we get a mixed Hodge structure on $\widehat{\mathcal{O}}_\rho$ which is a priori functorial in the whole data of $X, x, \rho, Y, y, \overline{Y}$. But Y, y are obtained functorially from X, x, ρ and different choices for $\overline{Y}$ will lead to quasi-isomorphic mixed Hodge diagrams, as in the classical case where the mixed Hodge structure of Deligne on the cohomology of a variety is independent of the choice of a compactification. Such quasi-isomorphic augmented mixed Hodge diagrams of Lie algebras will lead to the same mixed Hodge structure on $\widehat{\mathcal{O}}_\rho$.

So our main theorem in the non-compact case is:

Theorem 3.19. *The data of M in* (3.51) *and ν_x in* (3.57) *forms an augmented mixed Hodge diagram of Lie algebras that controls the deformation theory of ρ. From this there is a mixed Hodge structure on $\widehat{\mathcal{O}}_\rho$ that is functorial in X, x, ρ.*

Proof. Combining our Lemma 3.16 with our Lemma 3.18 shows the first part of the statement. Then the mixed Hodge structure on $\widehat{\mathcal{O}}_\rho$ is constructed as in our model Theorem 3.3. By construction it is functorial in the data of $X, x, \rho, Y, y, \overline{Y}$.

Now check the functoriality. As we said Y, y are obtained functorially from the data of X, x, ρ. This is part of the usual Galois correspondence between covering spaces of (X, x) and subgroups of $\pi_1(X, x)$. Since the augmentation depends only on these on not on the compactification, then by combining Lemma 2.39, Lemma 2.45 and Corollary 2.46 we have to show the independence of the controlling mixed Hodge diagram of Lie algebras M on the compactifications, up to quasi-isomorphisms, and the functoriality of this. And for this, by Remark 3.17, we are reduced to studying the independence of $\mathrm{MHD}(\overline{Y}, D)_{\mathbb{R}}$ on the compactifications and the functoriality of this up to quasi-isomorphism. This is well-known already in the work of Navarro Aznar but we have to be careful with the equivariance condition on compactifications.

So as a first step we prove the independence on $\overline{Y}$. We start with X, x, Y, y fixed. One can also fix a compactification $\overline{X}$ that plays anyway no role in the construction. Let $\overline{Y}', \overline{Y}''$ be two equivariant compactifications of Y. As in [Del71b, 3.2.II.C] we look for a third compactification $\overline{Y}$ which dominates both, i.e. with two morphisms of pairs

$$\begin{array}{ccccc} & & (\overline{Y}, D) & & \\ & \swarrow^{j'} & & \searrow^{j''} & \\ (\overline{Y}', D') & & & & (\overline{Y}'', D''). \end{array} \tag{3.58}$$

This $\overline{Y}$ can be obtained as a resolution of singularities of the closure of the image of the diagonal embedding of Y into $\overline{Y}' \times \overline{Y}''$. But by invoking again the combination of the theorem of Sumihiro on equivariant completion combined with the theorem of Bierstone-Milman on canonical resolutions of singularities, one can find such an $\overline{Y}$ to which the action of Φ lifts. So in the diagram (3.58) the compactification $\overline{Y}$ dominates the two others as equivariant compactifications. Then j' and j'' both induce quasi-isomorphisms of mixed Hodge diagrams.

As we said, by tensoring with $\mathfrak{g}$ and taking the invariants, they also induce quasi-isomorphisms of the corresponding controlling mixed Hodge diagrams of Lie algebras M, M', M'', augmented over $\mathfrak{g}$, and then lead to the same mixed Hodge structure on $\widehat{\mathcal{O}}_\rho$.

Then we prove the functoriality. A morphism

$$f : (X_1, x_1, \rho_1) \longrightarrow (X_2, x_2, \rho_2),$$

meaning that we require $f(x_1) = x_2$ and the commutativity of

$$\begin{array}{ccccc} \pi_1(X_1, x_1) & & \xrightarrow{\quad f_* \quad} & & \pi_1(X_2, x_2) \\ & \searrow^{\rho_1} & & \swarrow^{\rho_2} & \\ & & G(\mathbb{R}), & & \end{array}$$

induces a morphism

$$\begin{array}{ccc} \Phi_1 \curvearrowright Y_1 & \xrightarrow{\ h\ } & Y_2 \curvearrowleft \Phi_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ X_1 & \xrightarrow[f]{} & X_2 \end{array}$$

compatibly with the base points and equivariant with respect to Φ_1, that acts on Y_1 and on Y_2 via the induced morphism of groups

$$\varphi : \Phi_1 \to \Phi_2.$$

Then we can upgrade this to a morphism of equivariant compactifications: start with such compactifications $\overline{Y}_1$, $\overline{Y}_2$ and consider the graph Γ_h of h seen as a subset

$$\Gamma_h \subset Y_1 \times Y_2 \subset \overline{Y}_1 \times \overline{Y}_2.$$

By construction, Γ_h is Φ_1-invariant, where Φ_1 acts diagonally. So is its Zariski closure. This defines a morphism $\overline{Y}_1 \to \overline{Y_2}$ which is φ-equivariant. Such a morphism induces a morphism of mixed Hodge diagrams

$$\mathrm{MHD}(\overline{Y_2}, D_2)_{\mathbb{R}} \longrightarrow \mathrm{MHD}(\overline{Y_1}, D_1)_{\mathbb{R}}$$

and then a morphism of mixed Hodge diagrams of Lie algebras $M_2 \to M_1$ augmented over $\mathfrak{g}$. So the canonically induced morphism

$$\widehat{\mathcal{O}}_{\rho_1} \longrightarrow \widehat{\mathcal{O}}_{\rho_2}$$

is a morphism of mixed Hodge structures. □

Description of this mixed Hodge structure

Again it is clear that the mixed Hodge structure we constructed on $\widehat{\mathcal{O}}_\rho$ has non-positive weights and we can describe its cotangent space via the spectral sequence for the bar-weight filtration. The proof of Proposition 3.4 applies in the same way to show that that the graded piece of weight zero of $\widehat{\mathcal{O}}_\rho$ is isomorphic to the algebra of formal power series on $\mathfrak{g}/\varepsilon(H^0(L))$ at 0 and there are two additional pieces, of weight -1 and -2. However in the case of finite representations ε is surjective so the only weights are $-1, 2$. So via the functorial Deligne splitting (Definition 2.18) $\widehat{\mathcal{O}}_\rho$ has a grading (as complete local ring) with generators of weight $1, 2$.

Proposition 3.20. *The mixed Hodge structure on $\widehat{\mathcal{O}}_\rho$ has weights only -1 and -2 on its cotangent space. Thus the grading of $\widehat{\mathcal{O}}_\rho$ obtained via the Deligne splitting has generators of weight* $1, 2$.

Proof. Take back the proof of Proposition 3.4. Denote simply by $\varepsilon : L \to \mathfrak{g}$ a controlling augmented mixed Hodge diagram of Lie algebras and by C its desuspended mapping cone. Re-write the spectral sequence for the cotangent space ($s = 1$ in (3.16))

$$_{\mathscr{C}W}E_1^{-k,q}(\mathscr{C}_1(C)) = H^{-k+1+q}(\mathrm{Gr}^W_{k-1} C). \tag{3.59}$$

By definition L^n has weights only between 0 and n. The page E_1 of the spectral sequence is now

$$\begin{array}{c|cccc} q & \vdots & \vdots & \vdots & \\ 2 & H^1(\mathrm{Gr}^W_1 L) & H^2(\mathrm{Gr}^W_0 L) & 0 & \\ 1 & 0 & H^1(\mathrm{Gr}^W_0 L) & 0 & \\ 0 & 0 & H^0(L) & \mathfrak{g} & \\ \hline & -2 & -1 & 0 & -k \end{array} \tag{3.60}$$

from which we see that on $H^0(\mathscr{C}_1(C))$ there are a priori three graded pieces, of weight $0, 1, 2$. The part of weight zero is as in the compact case

$$\mathrm{Gr}_0^{\mathscr{C}W} H^0(\mathscr{C}_1(C)) = \mathfrak{g}/\varepsilon(H^0(L)).$$

But for representations with finite image ε is surjective on $H^0(L)$. Namely it is trivially surjective at the level of the finite cover Y and an element ω in $\mathfrak{g}$ can be lifted to Y and moreover, by an averaging procedure as in Lemma 3.14, can be lifted as an equivariant element and then descends to X. So there is no graded part of weight zero.

By passing to the dual, on the cotangent space the only weights of the mixed Hodge structure are $-1, -2$. And then by the Deligne splitting on can see the grading of $\widehat{\mathcal{O}}_\rho$ over $\mathbb{C}$ to be generated in weights $1, 2$. □

Via this spectral sequence we understand well where these weights come from. We can also recover some form of Theorem A.2.

Proposition 3.21. *If the mixed Hodge structure on $H^1(Y)$ is pure of weight 2 (equivalently, if $b_1(\overline{Y}) = 0$) then the mixed Hodge structure on the tangent space to $\widehat{\mathcal{O}}_\rho$ is pure of weight -2. The induced grading on $\widehat{\mathcal{O}}_\rho$ has generators of pure weight 2.*

Proof. As is explained through chapter A, under this condition $H^1(L)$ also has pure weight 2. In the above spectral sequence (3.60) the term $H^1(\mathrm{Gr}_0^W L)$ vanishes. And then the only weight on $H^0(\mathscr{C}_1(C))$, which is dual to the cotangent space, is 2. □

3.2.2 Variations

As in the compact case, there are several variations we can work out from the main Theorem 3.19. We keep the same notations and hypothesis as in the preceding section.

The first one is that when we drop the part over $\mathbb{R}$ of the mixed Hodge diagram $\mathrm{MHD}(\overline{Y}, D)_\mathbb{R}$ one gets a complex mixed Hodge diagram and we denote it by $\mathrm{MHD}(\overline{Y}, D)_\mathbb{C}$.

Proposition 3.22. *Assume that G and ρ are defined over $\mathbb{C}$. Then $\widehat{\mathcal{O}}_\rho$ has a functorial complex mixed Hodge structure.*

Proof. In this case $\mathfrak{g}$ is defined over $\mathbb{C}$. Re-write everything with the complex mixed Hodge diagram $\mathrm{MHD}(\overline{Y}, D)_\mathbb{C}$. The controlling mixed Hodge diagram of Lie algebras is

$$M := \Big(\mathrm{MHD}(\overline{Y}, D)_\mathbb{C} \otimes \mathfrak{g}\Big)^\Phi$$

where the tensor product is over $\mathbb{C}$. □

The second one, that until now we didn't do in the compact case, is to show that the mixed Hodge structure of $\widehat{\mathcal{O}}_\rho$ is defined over a subfield $\mathbf{k} \subset \mathbb{R}$ if G and ρ are. For this we review a little bit more of the theory of Navarro Aznar, who constructs a mixed Hodge diagram $\mathrm{MHD}(\overline{Y}, D)_\mathbf{k}$ to compute the mixed Hodge structure over $\mathbf{k}$ on the cohomology of Y.

For any sheaf of DG algebras $\mathscr{F}^\bullet$ over $\mathbf{k}$ on Y and for a continuous map $j : Y \to Z$ of topological spaces, the *Thom-Whitney direct image* is the sheaf of DG algebras on Z defined by the successive operations:

1. Consider the Godement resolution $\mathscr{F}^\bullet \to \mathscr{G}$.
2. Apply the functor j_* to $\mathscr{G}$. This is a strictly cosimplicial sheaf of DG algebras on Z.
3. Then apply the Thom-Whitney simple functor for sheaves to $f_*\mathscr{G}$.

The resulting sheaf is denoted by $R_{\mathrm{TW}} j_* \mathscr{F}^\bullet$. Its cohomology computes canonically the derived direct image.

If we apply this to $j : Y \hookrightarrow \overline{Y}$, and if $\mathbf{k}_Y$ is the constant sheaf $\mathbf{k}$ on Y, then

$$R_{\mathrm{TW}}\Gamma(\overline{Y}, R_{\mathrm{TW}} j_* \mathbf{k}_Y) \tag{3.61}$$

is the part over $\mathbf{k}$ of the mixed Hodge diagram $\mathrm{MHD}(\overline{Y}, D)_{\mathbf{k}}$. It is equipped with the *canonical filtration* [Nav87, § 6.16]. Its tensor product with $\mathbb{R}$, which is

$$R_{\mathrm{TW}}\Gamma(\overline{Y}, R_{\mathrm{TW}} j_* \mathbb{R}_Y), \tag{3.62}$$

is related by a fixed canonical chain of filtered quasi-isomorphisms to the real Dolbeaut complex. So it induces the structure over $\mathbf{k}$ on the cohomology of Y and defines the filtration W over $\mathbf{k}$. All this is of course defined at least over $\mathbb{Q}$ and obtained over $\mathbf{k}$ by change of coefficients.

Proposition 3.23. *Assume that G and ρ are defined over a subfield $\mathbf{k} \subset \mathbb{R}$. Then the mixed Hodge structure of $\widehat{\mathcal{O}}_\rho$ is defined over $\mathbf{k}$.*

Proof. Re-write everything with the mixed Hodge diagram $\mathrm{MHD}(\overline{Y}, D)_{\mathbf{k}}$. The controlling mixed Hodge diagram of Lie algebras is

$$M := \left(\mathrm{MHD}(\overline{Y}, D)_{\mathbf{k}} \otimes \mathfrak{g} \right)^{\Phi}$$

where, since $\mathfrak{g}$ is defined over $\mathbf{k}$, the tensor product is taken over $\mathbf{k}$ in the component over $\mathbf{k}$.

The augmentation is obtained from the natural map $\mathbf{k}_Y \to \mathbf{k}_y$ and thus is also defined over $\mathbf{k}$. So this defines an augmented mixed Hodge diagram of Lie algebras over $\mathbf{k}$ and constructs on $\widehat{\mathcal{O}}_\rho$ a mixed Hodge structure over $\mathbf{k}$. □

With these ideas we can come back to the compact case and define mixed Hodge structures over $\mathbf{k}$. Let X be a compact Kähler manifold. Then in [Nav87, § 6–7] Navarro Aznar proves that a mixed Hodge diagram for computing the cohomology and the Hodge structure of X over $\mathbf{k}$ is given by the algebra $R_{\mathrm{TW}}\Gamma(X, \mathbf{k}_X)$, in addition to the usual algebras of differential forms $\mathscr{E}^\bullet(X, \mathbb{R})$ and $\mathscr{E}^\bullet(X, \mathbb{C})$.

Then we can complete the proof of Proposition 3.11. Recall that in this case ρ is the monodromy of a variation of Hodge structure defined over $\mathbf{k}$, so that V is a local system of finite-dimensional vector spaces over $\mathbf{k}$.

Proof of Proposition 3.11. Let $\mathcal{L}$ be the sheaf of sections of $\mathrm{End}(V)$. It is a local system of finite-dimensional Lie algebras over $\mathbf{k}$. Then $R_{\mathrm{TW}}\Gamma(X, \mathcal{L})$ is a DG Lie algebra over $\mathbf{k}$ (the construction described in [Nav87, § 3] for commutative algebras works as well for Lie algebras). So we use it as the part over $\mathbf{k}$ of the mixed Hodge diagram of Lie algebras L of Theorem 3.2. The augmentation at x comes from the canonical morphism of sheaves

$$\mathcal{L} \longrightarrow \mathcal{L}_x$$

where the right-hand side is the sheaf supported at x with stalk the Lie algebra $\mathrm{End}(V_x)$, and with $R_{\mathrm{TW}}\Gamma(X, \mathcal{L}_x) = \mathrm{End}(V_x)$. □

Appendix A

A criterion for quadraticity of a representation of the fundamental group of an algebraic variety

This is a reproduction of the published article [Lef17], with only few typesetting and notational changes for coherence.

Important however is the reference [Eys16], which is an essential part in the proof of Theorem A.14 and was cited as “personal communication” and “work in preparation”, that has now appeared.

Let us comment briefly: in section A.3.2 we claimed that we constructed a *minimal model in the sense of Lie algebras*. Since then we worked out this theory and found that this $\mathscr{G}$ is *not* minimal; namely a minimal model should at least be free as graded Lie algebra and obtained in a similar way to Sullivan's construction of minimal models [Sul77] for commutative DG algebras by an inductive limit of elementary extensions. Such a minimal model is then a cofibrant replacement for an appropriate model structure on the category $\mathbf{DG{-}Lie_k}$ and this is very well-known to homotopy theorists, though not written in such a concrete way in the literature. The most recent work in this direction is [CR16] which treats many other operads at the same time.

Also, our main result could be related to a series of statements *purity implies formality* (see [Dup15]), that we were not aware of.

A.1 Introduction

Many restrictions are known on the question of whether a given finitely presented group Γ can be obtained as the fundamental group of a compact Kähler manifold, and some restrictions are known for smooth complex algebraic varieties. See [ABC$^+$96] for an introduction to these questions. One way to study these groups is via their representations into a linear algebraic group G over $\mathbb{R}$: there exists a scheme $\mathrm{Hom}(\Gamma, G)$ parametrizing such representations ρ (see section A.2.1) and it is sometimes possible to describe ρ as a singularity in $\mathrm{Hom}(\Gamma, G)$, up to analytic isomorphism (see section A.2.1).

The first known theorem in this direction is obtained by Goldman and Millson in [GM88].

Theorem A.1 (Goldman-Millson)**.** *Let X be a compact Kähler manifold and Γ its fundamental group. Let $\rho : \Gamma \to G(\mathbb{R})$ be a representation with image contained in a compact subgroup of $G(\mathbb{R})$. Then* $(\mathrm{Hom}(\Gamma, G), \rho)$ *is a quadratic singularity.*

We will need to review this theory in section A.2.1. We exhibit a criterion under which the same conclusion holds that is inspired by the case of arrangements of hyperplanes. It is known that when ρ is the trivial representation of the fundamental group of the complement of an arrangement of hyperplanes, or of the complement of a projective algebraic curve, then ρ is a quadratic singularity in $\mathrm{Hom}(\Gamma, G)$. This is related to the notion of 1-formality developed in the work of Dimca, Papadima and Suciu, see [DPS09], [PS09]. Our main result proved in section A.3.2 is:

Theorem A.2. *Let X be a smooth complex quasi-projective variety and Γ its fundamental group. Let $\rho : \Gamma \to G(\mathbb{R})$ be a representation with finite image. Corresponding to* $\mathrm{Ker}(\rho)$ *there is a finite étale Galois cover $Y \to X$. Assume that Y has a smooth compactification $\overline{Y}$ with first Betti number $b_1(\overline{Y}) = 0$. Then* $(\mathrm{Hom}(\Gamma, G), \rho)$ *is a quadratic singularity.*

To pass from the case of Kähler manifolds to that of algebraic varieties (we restrict ourself to quasi-projective ones), we must study mixed Hodge theory and use the older results of Deligne [Del71b] and Morgan [Mor78], reviewed in section A.2.2. Our theorem is obtained by reviewing the ideas of Kapovich and Millson in [KM98], who proved:

Theorem A.3 (Kapovich-Millson)**.** *Let X be a smooth complex algebraic variety and Γ its fundamental group. Let $\rho : \Gamma \to G(\mathbb{R})$ be a representation with finite image. Then* $(\mathrm{Hom}(\Gamma, G), \rho)$ *is a weighted homogeneous singularity with generators of weight* $1, 2$ *and relations of weight* $2, 3, 4$.

In section A.4 we look for new examples to apply our theorem. We are first motivated by the case of arrangements of lines and some special representations. One case that has been studied, starting from the work of Hironaka [Hir93], is the abelian covers of the complex projective plane branched over an arrangement of lines, whose compactifications are called Hirzebruch surfaces. There exists some inequalities on their first Betti number and we use them in section A.4.1 to find the cases where our theorem applies.

We next study other classes of examples where we can apply our criterion with respect to *all* representations with finite image. We say a smooth quasi-projective variety X has property (P) (see Definition A.15) if for all finite Galois cover $Y \to X$ there is a smooth compactification $\overline{Y}$ of Y such that $b_1(\overline{Y}) = 0$; and then any representation of $\pi_1(X)$ with finite image satisfies the hypothesis of our main theorem. We show that this class of varieties is stable by taking products, by taking a $\mathbb{C}^*$-principal bundle over a base that satisfies (P) (Theorem A.18), and by taking some families of abelian varieties over a base that satisfies (P) (section A.4.2). Finally, related to famous rigidity results, we treat the case of hermitian locally symmetric spaces in section A.4.2.

A.2 Preliminaries

A.2.1 Review of Goldman-Millson theory

We first give a review of [GM88]. We fix a field $\mathbf{k}$ of characteristic zero, usually $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$. Our schemes will always be of finite type over $\mathbf{k}$.

Representation variety

Let Γ be a finitely presented group and let G be a linear algebraic group over $\mathbf{k}$. There exists an affine scheme over $\mathbf{k}$, denoted by $\mathrm{Hom}(\Gamma, G)$, that represents the functor from $\mathbf{k}$-algebras to sets

$$\begin{array}{rcl} A & \longmapsto & \mathrm{Hom}(\Gamma, G(A)) \\ (f : A \to B) & \longmapsto & \Big(f_* : \mathrm{Hom}(\Gamma, G(A)) \to \mathrm{Hom}(\Gamma, G(B))\Big). \end{array} \tag{A.1}$$

It is called the *representation variety* (it is in general not a variety, but a scheme). Thus, giving a representation $\rho : \Gamma \to G(\mathbf{k})$ is the same as giving a $\mathbf{k}$-point of $\mathrm{Hom}(\Gamma, G)$. When doing topology we just write G for $G(\mathbb{R})$.

Analytic germs

Given a scheme S and a $\mathbf{k}$-rational point s, the isomorphism class of the complete local ring $\widehat{O}_{S,s}$ is referred to as the *analytic germ* of S at s. That is, two germs (S_1, s_1) and (S_2, s_2) are said to be *analytically isomorphic* if their complete local rings are isomorphic.

A *weighted homogeneous cone* is an affine scheme defined by equations of the form $P_j(X_1, \ldots, X_n) = 0$ in $\mathbf{k}^n$ where the variables X_i have given *weights* w_i and the polynomials P_j are homogeneous of degree $d_j > 0$ with respect to the weights w_i (a monomial $X_1^{\alpha_1} \ldots X_n^{\alpha_n}$ is of weighted degree $w_1\alpha_1 + \cdots + w_n\alpha_n$). We say that w_i are the *weights of the generators* and d_j are the *weights of the relations*. The cone is said to be *quadratic* if the X_i have weight 1 and all the relations have weight 2. We say that an analytic germ (S, s) is a *weighted homogeneous singularity* with given weights (for example, is a quadratic singularity) if it is analytically isomorphic to a weighted homogeneous cone with these given weights.

Lemma A.4. *A germ (S, s) is a weighted homogeneous singularity over $\mathbb{R}$ if and only if it is over $\mathbb{C}$, with the same weights.*

Proof. Of course a weighted homogeneous cone over $\mathbb{R}$ can be complexified to a cone over $\mathbb{C}$ with the same weights. In the other direction, given the equations $P_j(X_1, \ldots, X_n)$ over $\mathbb{C}$, replace the variables X_i by their real and imaginary parts x_i, y_i and give the same weight w_i to the two new variables. Then expand the relations $P_j(x_1 + iy_1, \ldots, x_n + iy_n) = 0$, separate real and imaginary part, and this gives two equations both with the same weighted homogeneous degree d_j. $\square$

We denote by $\mathbf{Art_k}$ the category of Artin local $\mathbf{k}$-algebras. An element A in $\mathbf{Art_k}$ has a unique maximal ideal which we always denote by $\mathfrak{m}$, has residue field $\mathbf{k}$ and is of finite dimension over $\mathbf{k}$. This implies that $\mathfrak{m}$ is a nilpotent ideal, and this gives a natural map $A \to \mathbf{k}$ which is reduction modulo $\mathfrak{m}$. An analytic germ (S, s) defines a functor of Artin rings

$$\begin{array}{rcl} F_{S,s} : \mathbf{Art_k} & \longrightarrow & \mathbf{Set} \\ A & \longmapsto & \mathrm{Hom}(\widehat{O}_{S,s}, A). \end{array} \tag{A.2}$$

Such a functor is called *pro-representable*.

Theorem A.5 ([GM88, 3.1])**.** *Two germs (S_1, s_1) and (S_2, s_2) are analytically isomorphic if and only if the associated pro-representable functors F_{S_1,s_1}, F_{S_2,s_2} are isomorphic.*

Thus, in order to study the analytic germ of a representation ρ in the representation variety we only have to study its pro-representable functor, which is also the functor

$$\begin{array}{rcl} \mathbf{Art_k} & \longrightarrow & \mathbf{Set} \\ A & \longmapsto & \left\{ \tilde{\rho} \in \operatorname{Hom}(\Gamma, G(A)) \;\middle|\; \tilde{\rho} = \rho \bmod \mathfrak{m} \right\} \end{array} \tag{A.3}$$

interpreted as the functor of deformations of ρ over $\mathbf{Art_k}$; and the type of analytic singularity corresponds to the obstruction theory for deformations of ρ.

Differential graded Lie algebras

Let L be a differential graded Lie algebra. It has a grading $L = \oplus_{i \geq 0} L^i$, a bracket $[-,-]$ with $[L^i, L^j] \subset L^{i+j}$, and a derivation d of degree 1 satisfying the usual identities in the graded sense, see [GM88] and see also [Man99]. The basic example is: take a differential graded algebra A commutative in the graded sense (for example the De Rham algebra of a smooth manifold) and a Lie algebra $\mathfrak{g}$ and consider the tensor product $A \otimes \mathfrak{g}$ with bracket

$$[\alpha \otimes u, \beta \otimes v] := (\alpha \wedge \beta) \otimes [u, v] \tag{A.4}$$

and differential

$$d(\alpha \otimes u) := (d\alpha) \otimes u. \tag{A.5}$$

For such an L, L^0 is a usual Lie algebra and for an Artin local $\mathbf{k}$-algebra A, $L^0 \otimes \mathfrak{m}$ is a nilpotent Lie algebra on which we can define a group structure via the Baker-Campbell-Hausdorff formula. This group is denoted by $\exp(L^0 \otimes \mathfrak{m})$.

Recall that a *groupoid* is a small category $\mathscr{C}$ in which all arrows are invertible. An example is provided by the so-called *action groupoid*: let a group H act on a set E, take the set of objects to be $\operatorname{Obj} \mathscr{C} := E$ and the arrows $x \to y$ are the $h \in H$ such that $h.x = y$. We denote by $\operatorname{Iso} \mathscr{C}$ the set of isomorphism classes of $\mathscr{C}$; in the case of an action groupoid this is just the usual quotient E/H.

We define a functor $A \mapsto \mathscr{C}(L, A)$ from $\mathbf{Art_k}$ to groupoids, called the *Deligne-Goldman-Millson functor*: the set of objects is

$$\operatorname{Obj} \mathscr{C}(L, A) := \left\{ \eta \in L^1 \otimes \mathfrak{m} \;\middle|\; d\eta + \frac{1}{2}[\eta, \eta] = 0 \right\} \tag{A.6}$$

(this one is called the *Maurer-Cartan equation*) and the arrows of the groupoid are given by the action of the group $\exp(L^0 \otimes \mathfrak{m})$ by

$$\exp(\alpha).\eta := \eta + \sum_{n=0}^{\infty} \frac{(\operatorname{ad}(\alpha))^n}{(n+1)!}([\alpha, \eta] - d\alpha). \tag{A.7}$$

We then have a functor $A \mapsto \operatorname{Iso} \mathscr{C}(L, A)$ from $\mathbf{Art_k}$ to sets.

Given a scheme S over $\mathbf{k}$ and a $\mathbf{k}$-rational point s, we say that L *controls* the germ (S, s) if the functor $A \mapsto \mathscr{C}(L, A)$ is isomorphic to the functor $F_{S,s}$.

Recall that if L and L' are two differential graded Lie algebras, a morphism $\varphi : L \to L'$ is said to be a 1-*quasi-isomorphism* if it induces an isomorphism on the cohomology groups H^0, H^1 and an injection on H^2. The algebras L, L' are said to be 1-*quasi-isomorphic* if there is a sequence of 1-quasi-isomorphisms connecting them.

Theorem A.6 ([GM88, 2.4])**.** *If L and L' are two differential graded Lie algebras and $\varphi : L \to L'$ is a 1-quasi-isomorphism, then the germs controlled by L and L' are analytically isomorphic.*

Thus to understand an analytic germ (S, s) it suffices to understand the functor $A \mapsto \operatorname{Iso} \mathscr{C}(L, A)$ for some controlling Lie algebra L up to 1-quasi-isomorphism.

Remark A.7. If $L^0 = 0$ then this functor is equal to $A \mapsto \operatorname{Obj} \mathscr{C}(L, A)$. If furthermore L^1 is finite-dimensional this is exactly the pro-representable functor associated to the analytic germ at 0 of the Maurer-Cartan equation $d\eta + \frac{1}{2}[\eta, \eta] = 0$ for $\eta \in L^1$.

Main construction

We explain the main construction to relate theses objects. Let X be a real manifold, x a base point, Γ its fundamental group and G a linear algebraic group over $\mathbb{R}$. Let $\mathfrak{g}$ be the Lie algebra of G. Let $\rho : \Gamma \to G(\mathbb{R})$ be a representation. Let P be the principal bundle obtained by the left monodromy action of Γ on G via ρ. If $\widetilde{X}$ is a universal covering space for X, on which we make Γ act on the left, then

$$P := \widetilde{X} \times_\Gamma G = (\widetilde{X} \times G)/\Gamma \tag{A.8}$$

where Γ acts diagonally. The group G acts on its Lie algebra via the adjoint representation Ad and Γ acts on $\mathfrak{g}$ by $\operatorname{Ad} \circ \rho$. We denote by $\operatorname{Ad}(P)$ the adjoint bundle

$$\operatorname{Ad}(P) := P \times_G \mathfrak{g} = \widetilde{X} \times_\Gamma \mathfrak{g} \tag{A.9}$$

and it comes with a flat connection such that the algebra of differential forms with value in $\operatorname{Ad}(P)$, denoted by $\mathscr{E}^\bullet(X, \operatorname{Ad}(P))$, has the structure of a differential graded Lie algebra.

Given the base point x we define an augmentation $\varepsilon : \mathscr{E}^\bullet(X, \operatorname{Ad}(P)) \to \mathfrak{g}$ by evaluating degree 0 forms at x and sending the others to zero. We put $\mathscr{E}^\bullet(X, \operatorname{Ad}(P))_0 := \operatorname{Ker}(\varepsilon)$.

Theorem A.8 ([GM88, 6.8])**.** *The differential graded Lie algebra $\mathscr{E}^\bullet(X, \operatorname{Ad}(P))_0$ controls the analytic germ* $(\operatorname{Hom}(\Gamma, G), \rho)$.

It is then proven in [GM88, § 7] that when X is a compact Kähler manifold, $\mathscr{E}^\bullet(X, \operatorname{Ad}(P))_0$ is quasi-isomorphic (over $\mathbb{C}$) to a differential graded Lie algebra with zero differential and this implies that the analytic germ controlled by it is quadratic.

A.2.2 Mixed Hodge theory and rational homotopy

Next we give a short review of [Del71b] and [Mor78]. See also [PS08].

Hodge structures

Given a finite-dimensional vector space V over $\mathbb{R}$, a *Hodge structure* of weight n over V is the data of a decreasing (finite) filtration F of $V_\mathbb{C} := V \otimes \mathbb{C}$ and a decomposition in bigraded parts $V_\mathbb{C} = \bigoplus_{i+j=n} V^{i,j}$, where $F^p(V_\mathbb{C}) = \bigoplus_{i \geq p} V^{i,j}$, with $\overline{V^{j,i}} = V^{i,j}$. A *mixed Hodge structure* on V is the data of an increasing (finite) filtration W, a decreasing filtration F of $V_\mathbb{C}$, such that F induces on $\operatorname{Gr}_n^W(V)$ a Hodge structure of weight n, for all

n; so $\mathrm{Gr}_n^W(V_{\mathbb{C}}) = \bigoplus_{i+j=n} V^{i,j}$ with $\overline{V^{j,i}} = V^{i,j}$ modulo $W_{n-1}(V_{\mathbb{C}})$. The category of mixed Hodge structures is abelian (this is not trivial, see [Del71b, 1.2.10]).

Given a mixed Hodge structure on V, there is one preferred way of splitting $V_{\mathbb{C}} = \bigoplus A^{i,j}$ such that $W_n(V_{\mathbb{C}}) = \bigoplus_{i+j\leq n} A^{i,j}$, $F^p(V_{\mathbb{C}}) = \bigoplus_{i\geq p} A^{i,j}$ and then $V^{i,j}$ becomes naturally isomorphic with $A^{i,j}$. We call it the *Deligne splitting*, see [Mor78, 1.9]

Hodge structures on cohomology groups

Suppose that X is a smooth complex quasi-projective variety. We denote by $\overline{X}$ a smooth compactification such that $D := \overline{X} \setminus X$ is a divisor with normal crossings. By Deligne [Del71b], the cohomology groups of X are equipped with a mixed Hodge structure which is independent of the choice of $\overline{X}$. On $H^n(X)$ the nonzero graded parts for W are of weight between n and $2n$ and $\mathrm{Gr}_n^W H^n(X) = H^n(\overline{X})$.

In our case of interest, we will have a finite étale Galois cover $Y \to X$ and we are interested in the condition $b_1(\overline{Y}) = 0$. Remark that since b_1 is a birational invariant, it is enough to have for $\overline{Y}$ a smooth compactification, not necessarily by a divisor with normal crossings.

The condition $b_1(\overline{Y}) = 0$ is equivalent to one of the following:

1. $H^1(\overline{Y}, \mathbb{Q}) = 0$.
2. $\pi_1(\overline{Y})^{\mathrm{ab}}$ if finite.
3. $q(\overline{Y}) = 0$, where $q := \dim H^0(\overline{Y}, \Omega^1_{\overline{Y}})$ is the irregularity.
4. The mixed Hodge structure on $H^1(Y)$ is pure of weight 2.

There is also one characterization that will motivate our section A.4.2.

Lemma A.9. *Let X be a smooth quasi-projective variety. Then $b_1(\overline{X}) = 0$ if and only if there is no nonconstant holomorphic map $f : X \to A$ to an abelian variety A.*

Proof. Recall that $\overline{X}$ has an Albanese variety, which is an abelian variety Alb with a map $\mathrm{alb} : \overline{X} \to \mathrm{Alb}$ through which every map to an abelian variety factors. Its dimension is exactly $q(\overline{X}) = b_1(\overline{X})/2$ and as soon as this is strictly positive then f is nonconstant and its image generates Alb.

If $b_1(\overline{X}) > 0$ then alb restricts to a map $f : X \to \mathrm{Alb}$. Conversely given a nonconstant map $f : X \to A$ then first f extends to $\overline{X}$ (see [BL04, § 4, 9.4]), then factors through alb and so Alb must be of positive dimension. □

Rational homotopy theory

We explain very briefly the ideas we need. We refer to [GM81] and [Mor78].

Let $A^\bullet$ be a (commutative) differential graded algebra over a field $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$: this means that A has a grading $A = \oplus_{i\geq 0} A^i$, a multiplication $\wedge : A^i \otimes A^j \to A^{i+j}$ with $\alpha \wedge \beta = (-1)^{ij} \beta \wedge \alpha$ if $\alpha \in A^i$, $\beta \in A^j$, and a derivation of degree 1 with $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^i \alpha \wedge d\beta$. We denote by $A^+ := \bigoplus_{i>0} A^i$.

There is a notion of 1-*minimal algebra* but we won't need the details. Briefly it means that A is obtained as an increasing union of elementary extensions, with in addition $A^0 = \mathbf{k}$ and d is *decomposable* which means $d(A) \subset A^+ \wedge A^+$.

A 1-*minimal model* for A is a 1-minimal differential algebra $\mathscr{N}$ with a morphism $\nu : \mathscr{N} \to A$ which is a 1-quasi-isomorphism. See [GM81] and [Mor78] for more statements about existence and unicity. Remark that if $\nu : \mathscr{N} \to A$ is a 1-minimal model over $\mathbb{R}$ then $\nu_{\mathbb{C}} : \mathscr{N}_{\mathbb{C}} \to A_{\mathbb{C}}$ is a 1-minimal model over $\mathbb{C}$.

Hodge structures on minimal models

We recall the work of Morgan [Mor78]. The goal is to put mixed Hodge structures on several rational homotopy invariants of algebraic varieties, however we only need to study the 1-minimal model.

Let X be a smooth complex quasi-projective variety admitting a compactification $\overline{X}$ by a divisor with normal crossings D. We denote as always by $\mathscr{E}^\bullet(X, \mathbb{R})$ the algebra of real-valued differential forms on X, and by $\mathscr{E}^\bullet_{\overline{X}}(\log D)$ the algebra of $\mathcal{C}^\infty$ complex-valued forms on X with logarithmic poles along D, with its filtration W by the order of poles. There is a canonical map $\mathscr{E}^\bullet_{\overline{X}}(\log D) \hookrightarrow \mathscr{E}^\bullet(X, \mathbb{C})$ that is a quasi-isomorphism.

In addition, Morgan constructs a real algebra $E^\bullet_{\mathcal{C}^\infty}(X)$ (see [Mor78, § 2] for details), with a filtration W similar to the weight filtration on $\mathscr{E}_{\overline{X}}(\log D)$, and constructs a quasi-isomorphism $E_{\mathcal{C}^\infty}(X) \otimes \mathbb{C} \to \mathscr{E}_{\overline{X}}(\log D)$ that respects W. Now let $A := E_{\mathcal{C}^\infty}(X)$. Recall the Dec-weight filtration defined by

$$\operatorname{Dec} W_i(A^n) = \{x \in W_{i-n}(A) \mid dx \in W_{i-n-1}(A^{n+1})\} \tag{A.10}$$

such that for the spectral sequence

$${}_W E_r^{p,q} = {}_{\operatorname{Dec} W} E_{r-1}^{-q, p+2q} \tag{A.11}$$

and on cohomology

$$\operatorname{Dec} W_i H^n(A) = W_{i-n} H^n(A). \tag{A.12}$$

Morgan proves that A has a real minimal model $\nu : \mathscr{N} \to A$ with a filtration W such that ν respects the filtrations $(\mathscr{N}, W) \to (A, \operatorname{Dec} W)$; of course by transitivity $(\mathscr{N}_{\mathbb{C}}, W)$ is a also a filtered (complex) minimal model for $(\mathscr{E}_{\overline{X}}(\log D), W)$.

Over $\mathbb{C}$, the weight filtration on $\mathscr{N}_{\mathbb{C}}$ splits and we will denote by lower indices the grading by the weight, that is compatible with the grading of $\mathscr{N}$ as differential graded algebra. So $\mathscr{N}_{\mathbb{C}} = \bigoplus_{i \geq 0} \mathscr{N}_i$, with $d : \mathscr{N}_i \to \mathscr{N}_i$ and $\mathscr{N}_i \wedge \mathscr{N}_j \subset \mathscr{N}_{i+j}$. The grading has the following properties:

1. Via the 1-quasi-isomorphism ν, the grading by weight on $H^n(\mathscr{N}_{\mathbb{C}})$ coincide with the grading induced on $H^n(X, \mathbb{C})$ by the Deligne splitting of its mixed Hodge structure.
2. For $n = 1$ the only possible weights induced are $1, 2$ and for $n = 2$ they are $2, 3, 4$.
3. Each component $\mathscr{N}_i^j$ is of finite dimension.
4. $\mathscr{N}_0 = \mathbb{C}$ and $\mathscr{N}^0 = \mathbb{C}$.
5. $d(\mathscr{N}_1^1) = 0$ (if $x \in \mathscr{N}_1^1$ then as d is decomposable, $dx = \sum \alpha_i \wedge \beta_i \in \mathscr{N}_1^2$, so for each i we must have $\alpha_i \in \mathscr{N}_1^1$ and $\beta_i \in \mathscr{N}_0^1$ or the other way around; but $\mathscr{N}_0 = \mathbb{C}$ concentrated in degree 0 so $\mathscr{N}_0^1 = 0$).

A.3 Equivariant constructions and proof of the main theorem

Now we rewrite section A.2.1, taking into account a finite covering space and equivariance, as needed in [KM98, § 15]. From now on we fix the objects we introduce: X is a smooth complex quasi-projective variety with a base point x and fundamental group Γ. We fix a linear algebraic group G over $\mathbb{R}$ with Lie algebra $\mathfrak{g}$ and a representation $\rho : \Gamma \to G(\mathbb{R})$ with finite image. We introduce the finite group $\Phi := \Gamma / \operatorname{Ker}(\rho) \simeq \rho(\Gamma)$.

A.3.1 Covering spaces

Covering spaces and compactifications

To $\operatorname{Ker}(\rho)$ corresponds a finite étale Galois cover $\pi : Y \to X$ with automorphism group Φ that acts simply transitively and with a base-point y over x. The cover Y can be taken to be algebraic by [Gro03, XII.5.1] and it will be quasi-projective by [Gro61, 5.3.4]. By [Sum74, 3], it is possible to compactify Y into a variety (possibly singular) $\overline{Y}'$ such that the action of Φ extends to $\overline{Y}'$, which is there called an *equivariant completion.* In [BM97] there is a construction of a canonical resolution of singularities $\overline{Y} \to \overline{Y}'$ on which the action of Φ lifts (section 13 therein). We also compactify smoothly X to $\overline{X}$.

Now, π is a finite cover $\overline{Y} \to \overline{X}$ ramified over $\overline{X} \setminus X$ which is equivariant. We call this construction an *equivariant compactification* and this is summarized in the following diagram:

$$\begin{array}{ccc} \Phi \curvearrowright Y & \hookrightarrow & \overline{Y} \curvearrowleft \Phi \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ X & \hookrightarrow & \overline{X}. \end{array} \tag{A.13}$$

Bundles and augmentations

We construct the bundle $\operatorname{Ad}(P)$ taking into account the augmentation.

First fix some notations: if E is a flat bundle, there is a twisted algebra of differential forms with values in E denoted by $\mathscr{E}^\bullet(X, E)$. If E is globally trivial this is just $\mathscr{E}^\bullet(X) \otimes E$ where we write $\mathscr{E}^\bullet(X)$ for $\mathscr{E}^\bullet(X, \mathbb{R})$. Given a group Φ acting on an algebra A (which can be graded, commutative, Lie, etc, and the action must respect this structure) we always denote by A^Φ the sub-algebra of invariants by Φ. Given an augmentation of A, we always denote by A_0 the kernel of the augmentation.

Introduce the trivial bundle $Q := Y \times G$ and its adjoint bundle $\operatorname{Ad}(Q) := Y \times \mathfrak{g}$. Recall that Φ acts on Y with $Y/\Phi = X$; on G (via ρ) and on $\mathfrak{g}$ (via $\operatorname{Ad} \circ \rho$); and also on $\mathscr{E}^\bullet(Y)$ with $\mathscr{E}^\bullet(Y)^\Phi = \mathscr{E}^\bullet(X)$. It also acts naturally on products and tensor products.

So: we have $P = Q/\Phi$ and $\operatorname{Ad}(P) = \operatorname{Ad}(Q)/\Phi$, and for the twisted version

$$\mathscr{E}^\bullet(X, \operatorname{Ad}(P)) = (\mathscr{E}^\bullet(Y, \operatorname{Ad}(P')))^\Phi = (\mathscr{E}^\bullet(Y) \otimes \mathfrak{g})^\Phi. \tag{A.14}$$

We want to lift the augmentation $\beta : \mathscr{E}^\bullet(X) \to \mathbb{R}$, which is the evaluation of 0-forms at x, to Y. We let

$$\beta_Y(f) := \frac{1}{|\Phi|} \sum_{\gamma \in \Phi} (\gamma . f)(y) \tag{A.15}$$

for $f \in \mathscr{E}^0(Y)$. Thus we sum over all of $\pi^{-1}(x)$. Then naturally

$$\mathscr{E}^\bullet(X)_0 = (\mathscr{E}^\bullet(Y)^\Phi)_0 = (\mathscr{E}^\bullet(Y)_0)^\Phi \tag{A.16}$$

(the first two augmentations are with respect to β, the last one to β_Y) and we can write it $\mathscr{E}^\bullet(Y)_0^\Phi$.

In the same way we want to lift $\varepsilon : \mathscr{E}^\bullet(X, \mathrm{Ad}(P)) \to \mathfrak{g}$ to $\mathscr{E}^\bullet(Y) \otimes \mathfrak{g}$. Put

$$\varepsilon_Y(f \otimes u) := \frac{1}{|\Phi|} \sum_{\gamma \in \Phi} (\gamma.(f \otimes u))(y). \tag{A.17}$$

Then naturally

$$\mathscr{E}^\bullet(X, \mathrm{Ad}(P))_0 = ((\mathscr{E}^\bullet(Y) \otimes \mathfrak{g})^\Phi)_0 = ((\mathscr{E}^\bullet(Y) \otimes \mathfrak{g})_0)^\Phi = (\mathscr{E}^\bullet(Y)_0 \otimes \mathfrak{g})^\Phi \tag{A.18}$$

and we can denote all this by $(\mathscr{E}^\bullet(Y) \otimes \mathfrak{g})_0^\Phi$.

Observe that all theses constructions extend naturally to $\mathbb{C}$.

Lemma A.10 (See [GM88, 5.12])**.** *The four augmentations defined above are surjective. This implies that $H^0(\mathscr{E}^\bullet(X, \mathbb{R})_0) = 0$ and*

$$H^0(\mathscr{E}^\bullet(X, \mathrm{Ad}(P))_0) = 0. \tag{A.19}$$

Cohomology

We just give an elementary property relating cohomology and the action of a finite group, that will be used often.

Lemma A.11. *Let $A^\bullet$ be a differential graded commutative algebra (or Lie algebra) over a field of characteristic zero. Let Φ be a finite group acting on $A^\bullet$. Then on cohomology*

$$H^\bullet(A^\Phi) = (H^\bullet(A))^\Phi. \tag{A.20}$$

As a consequence, if A and A' are differential graded algebras (or Lie algebras) with a finite group Φ acting on both, if $\psi : A \to A'$ a 1-quasi-isomorphism commuting with the actions, then ψ induces a 1-quasi-isomorphism $\psi^\Phi : A^\Phi \to A'^\Phi$.

Equivariant minimal model

We refer to [KM98, § 15] for this technical part. We denote by $A^\bullet$ the algebra $E^\bullet_{\mathcal{C}^\infty}(Y)$ with its filtration W. It is shown that Φ acts on $(A^\bullet, W)$ and $(A^\bullet)^\Phi$ is then quasi-isomorphic to $E^\bullet_{\mathcal{C}^\infty}(X)$. It is shown how to construct a 1-minimal model $\nu : \mathscr{N} \to A$ with a filtration W and an action of Φ which commutes with ν; and ν respects the filtration $\mathrm{Dec}\, W$ on A. So (Lemma A.11) $\mathscr{N}^\Phi$ is a 1-minimal model for $A^\Phi = E^\bullet_{\mathcal{C}^\infty}(X)$.

Also, by transitivity and over $\mathbb{C}$, $\mathscr{N}_\mathbb{C}$ is a 1-minimal model for $\mathscr{E}(Y, \mathbb{C})$ and $\mathscr{N}_\mathbb{C}^\Phi$ is a 1-minimal model for $\mathscr{E}(X, \mathbb{C})$. The filtration W on $\mathscr{N}_\mathbb{C}$ splits and Φ commutes with the splitting.

A.3.2 Minimal model for a Lie algebra

We are now able to describe an explicit differential graded Lie algebra which controls the germ of ρ in $\mathrm{Hom}(\Gamma, G)$, and which comes with a grading by weight. Everything is done in [KM98, § 15].

The minimal model

Consider $\mathscr{G} := \mathscr{N} \otimes \mathfrak{g}$ and put $\nu \otimes \mathrm{id} : \mathscr{G} \to A \otimes \mathfrak{g}$. Then $\mathscr{G}$ is a differential graded Lie algebra with a differential d decomposable, and $\nu \otimes \mathrm{id}$ is a 1-quasi-isomorphism. We call $\mathscr{G}$ a 1-*minimal model in the sense of Lie algebras.* We do not give a precise definition for this notion: it seems possible to rework all the theory of Sullivan minimal models in the context of differential graded Lie algebras. However what matters here are the properties of this algebra $\mathscr{G}$.

The grading by weight on $\mathscr{N}_{\mathbb{C}}$ induces one on $\mathscr{G}_{\mathbb{C}}$ with the properties that:

1. $\mathscr{G}_{\mathbb{C}} = \bigoplus_{i \geq 0} \mathscr{G}_i$.
2. Each $\mathscr{G}_i^j$ is finite dimensional.
3. $[\mathscr{G}_i, \mathscr{G}_j] \subset \mathscr{G}_{i+j}$.
4. $d(\mathscr{G}_i) \subset \mathscr{G}_i$, so the cohomology is also graded.
5. $\mathscr{G}_0 = \mathfrak{g}_{\mathbb{C}}$ and $\mathscr{G}_{\mathbb{C}}^0 = \mathfrak{g}_{\mathbb{C}}$.
6. The only non-zero induced weights on $H^n(\mathscr{G}_{\mathbb{C}})$ are $1, 2$ for $n = 1$ and $2, 3, 4$ for $n = 2$.

The group Φ acts on both factors of $\mathscr{G}$ and preserve the grading on $\mathscr{G}_{\mathbb{C}}$. Put $\mathscr{M} := \mathscr{G}^{\Phi}$, so that $\mathscr{M}_{\mathbb{C}}$ has a bigrading with the same properties; by transitivity $\mathscr{M}_{\mathbb{C}}$ is 1-quasi-isomorphic to $\mathscr{E}^{\bullet}(X, \mathrm{Ad}(P)_{\mathbb{C}})$.

Augmentation

Recall the augmentations $\beta, \varepsilon, \beta_Y, \varepsilon_Y$, extend them over $\mathbb{C}$. It is easy to pull them back respectively to $A_{\mathbb{C}}$, $A_{\mathbb{C}} \otimes \mathfrak{g}$, $(A_{\mathbb{C}})^{\Phi}$, $(A_{\mathbb{C}} \otimes \mathfrak{g})^{\Phi}$ and then to $\mathscr{N}_{\mathbb{C}}$, $\mathscr{G}_{\mathbb{C}}$, $\mathscr{N}_{\mathbb{C}}^{\Phi}$, $\mathscr{G}_{\mathbb{C}}^{\Phi}$ all in a compatible way: $\mathscr{N}_{\mathbb{C},0}$ (warning with the notations, this is the kernel of the augmentation and $\mathscr{N}_0$ is the degree zero graded part by weight on $\mathscr{N}_{\mathbb{C}}$) is then 1-quasi-isomorphic to $\mathscr{E}(Y, \mathbb{C})_0$ such that $(\mathscr{N}_{\mathbb{C},0})^{\Phi} = (\mathscr{N}_{\mathbb{C}}^{\Phi})_0$ and $\mathscr{G}_{\mathbb{C},0}$ is 1-quasi-isomorphic to $\mathscr{E}(X, \mathrm{Ad}(P)_{\mathbb{C}})_0$ such that $(\mathscr{G}_{\mathbb{C},0})^{\Phi} = (\mathscr{G}_{\mathbb{C}}^{\Phi})_0$; and we denote this last one by $\mathscr{L}$.

Recall that when constructing a minimal model $\mathscr{N} \to A$, $\mathscr{N}^0$ is sent isomorphically to $H^0(A)$. Combining this with Lemma A.10, and with the various compatibilities of the augmentations, we see that $\mathscr{L}^0 = 0$. Furthermore $\mathscr{L}_0 = 0$.

A.3.3 The controlling Lie algebra and proof of the main theorem

The controlling Lie algebra

By Theorem A.6 and Remark A.7, the analytic germ of ρ in $\mathrm{Hom}(\Gamma, G)$ is isomorphic (over $\mathbb{C}$) to the one at 0 in $\mathscr{L}^1$ of the equation $d\eta + \frac{1}{2}[\eta, \eta] = 0$. So $\mathscr{L}$ is the controlling Lie algebra to our problem.

We can simplify it as in [KM98, § 15]. Put $\mathscr{I} := \mathscr{L}_4^1 \oplus d(\mathscr{L}_4^1) \oplus \bigoplus_{i\geq 5} \mathscr{L}_i$, observe that it is and ideal in $\mathscr{L}$ (in the sense of differential graded Lie algebras with an additional grading by weight, that is: bigraded homogeneous, stable by d, and stable by Lie bracket with $\mathscr{L}$) such that the projection $\mathscr{L} \to Q := \mathscr{L}/\mathscr{I}$ is a 1-quasi-isomorphism. This Q is simpler to study because it has all the properties of $\mathscr{G}_\mathbb{C}$ above and in addition $Q_i = 0$ for $i \geq 5$ and $Q_4^1 = 0$.

Proof of the main theorem

Theorem A.12. *If we assume that the compactification $\overline{Y}$ has $b_1(\overline{Y})$=0, then the differential graded Lie algebra Q above controls a quadratic germ.*

Proof. By hypothesis $H^1(Y)$ is a pure Hodge structure of weight 2. Looking carefully at our 1-quasi-isomorphisms that preserve filtrations by weight and gradings over $\mathbb{C}$, we see that on $H^1(\mathscr{N}_i)$ the only nonzero induced weight is for $i = 2$. This special property is also true for $\mathscr{G}_\mathbb{C}$ because

$$H^\bullet(\mathscr{G}_i) = H^\bullet(\mathscr{N}_i) \otimes \mathfrak{g}$$

and also for $\mathscr{G}_{\mathbb{C},0}$ because

$$H^\bullet(\mathscr{G}_{i,0}) = H^\bullet((\mathscr{N}_i \otimes \mathfrak{g})_0) = H^\bullet(\mathscr{N}_{i,0} \otimes \mathfrak{g}) = H^\bullet(\mathscr{N}_{i,0}) \otimes \mathfrak{g}$$

and $\mathscr{N}_{\mathbb{C},0}$ is a subcomplex of $\mathscr{N}_\mathbb{C}$, so all restrictions apply to it.

This restriction on weights holds for $\mathscr{L}$ because the action of Φ preserves the grading and

$$H^\bullet(\mathscr{L}_i) = H^\bullet((\mathscr{G}_{i,0})^\Phi) = (H^\bullet(\mathscr{G}_{\mathbb{C},0}))^\Phi,$$

and by the 1-quasi-isomorphism it holds for Q.

Now look at the equation $d\eta + \frac{1}{2}[\eta,\eta] = 0$ in Q^2, for $\eta \in Q^1$. By construction $Q^1 = Q_1^1 \oplus Q_2^1 \oplus Q_3^1$ (and $Q_0^1 = 0$). By our hypothesis $H^1(Q_1) = 0$, which is $\mathrm{Ker}(d : Q_1^1 \to Q_1^2)/d(Q_1^0)$. Combined with the fact that $d(Q_1^1) = 0$ and $Q_0^1 = 0$ we have $Q_1^1 = 0$.

So, decompose $\eta = \eta_2 + \eta_3$ where η_i is of weight i. The equation on η becomes (we truncate parts of weight ≥ 5)

$$\begin{aligned} d\eta_2 &= 0 \\ d\eta_3 &= 0 \\ \frac{1}{2}[\eta_2,\eta_2] &= 0. \end{aligned}$$

Since we have $H^1(Q_3) = 0$ and $d\eta_3 = 0$, η_3 must be exact. But a primitive must be in Q_3^0, which is 0. So we can eliminate the equation $d\eta_3 = 0$. Since $d\eta_2 = 0$, we can just assume η_2 is in the linear subspace $Z_2^1 := \mathrm{Ker}(d) \cap Q_2^1$ and it remains only the equation

$$\frac{1}{2}[\eta_2,\eta_2] = 0, \quad \eta_2 \in Z_2^1 \tag{A.21}$$

which is weighted homogeneous, with the generator η_2 of weight 2 and the relation of weight 4. But we can divide the degrees by two and this is isomorphic to a weighted homogeneous cone with generators of weight 1 and relations of weight 2, that is, a quadratic cone. □

A.4 Examples

We now investigate several situations where we can apply our main theorem.

A.4.1 Abelian coverings of line arrangements

Motivated by the case of the trivial representation, the first example is to take for X the complement of an arrangement of hyperplanes in some complex projective space. We reduce to the case of the projective plane, thus we denote by $\mathcal{L}$ a finite union of lines in $\mathbb{P}^2(\mathbb{C})$ and by X its complement. A smooth compactification of X is obtained as a blow-up of $\mathbb{P}^2(\mathbb{C})$ at the points of intersection with multiplicity at least 3 so has clearly $b_1(\overline{X}) = 0$.

There is a special class of coverings of $\mathbb{P}^2(\mathbb{C})$ branched over $\mathcal{L}$ that has already been studied: the *Hirzebruch surfaces.* The definition appears first in [Hir83], and a study of the Betti numbers was done in [Hir93]. See also [Suc01] for a survey of these results.

For each integer $N > 0$, we define $X_N(\mathcal{L})$ to be the covering of X corresponding to the morphism $\pi_1(X) \to H_1(X, \mathbb{Z}) \to H_1(X, \mathbb{Z}/N\mathbb{Z})$. It is known that if n is the number of lines of the arrangement then $H_1(X, \mathbb{Z})$ is free of rank $n-1$, so $X_N(\mathcal{L})$ is a Galois cover of degree N^{n-1}. It extends to a branched covering $\widehat{X}_N(\mathcal{L})$ over $\mathcal{L}$; $\widehat{X}_N(\mathcal{L})$ is a normal algebraic surface. We define the Hirzebruch surface associated to $\mathcal{L}$, which we denote $M_N(\mathcal{L})$, to be a minimal desingularization of $\widehat{X}_N$ (see [Hir93] for more details). There are various formulas for computing the Betti number $b_1(M_N(\mathcal{L}))$ and we will refer to Tayama [Tay00].

Theorem A.13 ([Tay00, 1.2])**.** *Define the function*

$$b(N, n) := (N-1)\left((n-2)N^{n-2} - 2\sum_{k=0}^{n-3} N^k\right). \tag{A.22}$$

It is $b_1(M_N(\mathcal{O}_n))$ where $\mathcal{O}_n$ is an arrangement made of n lines passing through a common point. Let m_r be the number of points of multiplicity r of $\mathcal{L}$. Let $\beta(\mathcal{L})$ be the number of braid sub-arrangements of $\mathcal{L}$. Then

$$b_1(M_N(\mathcal{L})) \geq \sum_{r\geq 3} m_r b(N, r) + \beta(\mathcal{L}) b(N, 3). \tag{A.23}$$

Furthermore the following conditions are equivalent:

1. *$\mathcal{L}$ is a general position line arrangement.*
2. *$b_1(M_N(\mathcal{L})) = 0$ for any $N \geq 2$.*
3. *$b_1(M_N(\mathcal{L})) = 0$ for some $N \geq 3$.*

From this we deduce easily one necessary condition to $b_1(M_N(\mathcal{L})) = 0$ and in fact there is a converse.

Theorem A.14. *We have $b_1(M_N(\mathcal{L})) = 0$ if and only if*
— *Either $\mathcal{L}$ is a general position line arrangement, and N is any integer.*
— *Either $N = 2$ and $\mathcal{L}$ has at most triple points.*

Proof. The case of general position is already treated in Tayama's theorem. Now suppose that $\mathcal{L}$ is not in general position and $b_1(M_N(\mathcal{L})) = 0$. Then $N = 2$. As $b(3, 2) = 0$ and $b(3, r) > 0$ if $r > 3$ it follows from the inequality (A.23) that we must have $m_r = 0$ for $r > 3$, that is $\mathcal{L}$ contains at most triple points.

For the converse, one can show ([Eys16, 2.5]) that if $\mathcal{L}$ has at most triple points and $N = 2$ then $M_N(\mathcal{L})$ is simply connected. So $b_1(M_N(\mathcal{L})) = 0$. □

A.4.2 Criterion with respect to all finite representations

As we have seen the case of arrangements is quite limited. But we have other sources of interesting examples where we can apply our theorem with respect to all representations with finite image.

Definition A.15. A smooth complex quasi-projective variety X is said to have property (P) if for all normal subgroup of finite index $H \subset \pi_1(X)$, the corresponding finite étale Galois cover $\pi : Y \to X$ has a (smooth) compactification $\overline{Y}$ with $b_1(\overline{Y}) = 0$.

We will study this property and give two interesting classes of examples.

Construction of varieties with property (P)

It is easy to see that if X is a smooth projective variety with $\pi_1(X)$ finite, then X has property (P). Indeed a finite cover Y a smooth projective variety corresponding to $H \subset \pi_1(H)$ is automatically a smooth projective variety and recall that $\pi_1(Y) = H$. Conversely it is known (by work of J.-P. Serre) that every finite group is the fundamental group of some smooth projective variety. It is also clear that if X satisfies (P), then any finite étale Galois cover of X satisfies (P).

Example A.16. The variety $\mathbb{C}^*$ has property (P).

Proof. It is known that every n-sheeted cover of $\mathbb{C}^*$ is of the form $\mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto z^n$. This can be compactified in a ramified cover $\mathbb{P}^1 \to \mathbb{P}^1$ over $\{0, \infty\}$ and $b_1(\mathbb{P}^1) = 0$. □

So with the next theorem we will easily see that for every finitely generated abelian group G, there is a smooth quasi-projective variety X with $\pi_1(X) \simeq G$ which has property (P).

Theorem A.17. *If X_1 and X_2 satisfy (P), then $X_1 \times X_2$ satisfies (P).*

Proof. Denote by $\Gamma_i := \pi_1(X_i)$ $(i = 1, 2)$. Let $H \subset \Gamma_1 \times \Gamma_2$ be a normal subgroup of finite index. Put $U_i := H \cap \Gamma_i$. Then U_i is a normal subgroup of finite index in Γ_i. So the Galois cover Y for H lies under the finite Galois cover corresponding to $U_1 \times U_2$, which is obtained as a product cover $Y_1 \times Y_2$, and we have a sequence of finite étale Galois covers

$$Y_1 \times Y_2 \longrightarrow Y \longrightarrow X_1 \times X_2. \tag{A.24}$$

Taking compactifications this gives two ramified covers $\overline{Y_1 \times Y_2} \to \overline{Y}$ and $\overline{Y} \to \overline{X_1 \times X_2}$ where we take for each the corresponding equivariant compactifications. Since we are only interested in b_1, that does not depend on the choice of the smooth compactification, we can compute $b_1(\overline{Y_1 \times Y_2}) = b_1(\overline{Y_1} \times \overline{Y_2})$ (both are a smooth compactification of $Y_1 \times Y_2$)

but by property (P) we have $b_1(\overline{Y}_1) = b_1(\overline{Y}_2) = 0$ and so $b_1(\overline{Y_1 \times Y_2}) = 0$. Now if we had $b_1(\overline{Y}) > 0$, there would be holomorphic one-forms on $\overline{Y}$ which could be pulled-back injectively to $\overline{Y_1 \times Y_2}$, which is not possible. So $b_1(\overline{Y}) = 0$. □

We also have a twisted version, which may allow to construct new examples.

Theorem A.18. *If X satisfies (P), then any (algebraic) principal $\mathbb{C}^*$-bundle P over X satisfies (P).*

Proof. First we show that a finite étale Galois cover of P is a bundle Q with fiber $\mathbb{C}^*$ over some finite étale Galois cover $\pi : Y \to X$. Let $\tau : Q \to P$ be such a cover, with $H := \tau_* \pi_1(Q) \subset \pi_1(P)$ a normal subgroup of finite index. Since $p : P \to X$ is a bundle, the induced map $p_* : \pi_1(P) \to \pi_1(X)$ is surjective, so $p_*(H)$ is a normal subgroup of finite index in $\pi_1(X)$. This corresponds to a finite étale Galois cover $\pi : Y \to X$ with $\pi_* \pi_1(Y) = p_* H$:

$$\begin{array}{ccc} Q & \xrightarrow{\tau} & P \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[\pi]{} & X. \end{array} \tag{A.25}$$

Now by construction $(p \circ \tau)_* \pi_1(Q) = \pi_* \pi_1(Y)$ which means there is a lifting of $p \circ \tau : Q \to X$ to a map $q : Q \to Y$. All theses spaces and maps can be taken to be algebraic and Q is a bundle whose general fiber is a finite cover of $\mathbb{C}^*$, that is $\mathbb{C}^*$: we can see this by first looking at $\pi^* P$, which is a principal $\mathbb{C}^*$-bundle over Y, then the induced map $Q \to \pi^* P$ over Y which is finite étale.

Now we want to apply Lemma A.9. Let A be an abelian variety and $f : Q \to A$. Restricted to each fiber, f is a map $\mathbb{C}^* \to A$. By rigidity it extends to $\mathbb{P}^1(\mathbb{C})$, but a map $\mathbb{P}^1(\mathbb{C}) \to A$ is constant. So f is constant on each fiber and thus is determined by its restriction to Y. But since X satisfies (P) this one is constant so f is globally constant, which proves $b_1(\overline{Q}) = 0$. □

Families of complex tori

Our motivation for studying complex tori is the use of Lemma A.9 and the various rigidity lemmas for abelian varieties and families.

Let X be a smooth quasi-projective variety. We would like to prove, using these lemmas, that if $E \to X$ is a family of abelian varieties over a base that satisfies (P), then E satisfies (P). To deduce that $b_1(\overline{E}) = 0$ it will be enough to show that a map $E \to A$ to an abelian variety A is constant along fibers, so descends to X, and by hypothesis $b_1(\overline{X}) = 0$. Of course the family must not contain a constant factor, else the fundamental group would contain a $\mathbb{Z}^{2r}$ summand. However there is some technical difficulty coming from the fact that a finite cover of a family of abelian varieties may not be a family of abelian varieties, because of the lack of a zero section. Thus we have to work with complex tori, by which we mean a projective variety isomorphic as variety to a complex torus but without specifying an origin.

Definition A.19. A *family of complex tori* is a smooth quasi-projective variety E with a smooth projective morphism $T \to X$ such that all fibers are isomorphic to complex tori, without specifying an origin. By a *family of abelian varieties* we mean the data of a family of complex tori $E \to X$ with a section, called the *zero section*.

When $E \to X$ is a family of abelian varieties, then there are global maps $E \times E \to E$ (addition) and $E \to E$ (inverse) over X. Many definitions and results from abelian varieties carry over directly to families. A *morphism* of families of abelian varieties must preserve the zero section. An *isogeny* is a surjective morphism with finite fibers. A *factor* of $E \to X$ is a sub-family F such that there is another sub-family G with addition $F \times G \to E$ being an isomorphism.

Definition A.20. A family of abelian varieties $E \to X$ is called *almost constant* if it becomes constant after a finite étale base change.

To each family $T \to X$ of complex tori, we can attach a family of abelian varieties $E \to X$ such that for the associated sheaves of holomorphic sections, E acts on T (in each fiber this is the action by translation) and T is a torsor under E. An idea to construct it is that from a complex torus T_x without origin, one can recover an abelian variety E_x as $\mathrm{Aut}(T_x)^0$ which acts on T_x making it a torsor. Wee see then that a family of complex tori is isomorphic (over X) to its associated family of abelian varieties if and only if it has a section. See [Cam85, Lemme 2] for related questions.

Theorem A.21. *Suppose that X satisfies (P) and let $T \to X$ be a family of complex tori. Let $E \to X$ be the associated family of abelian varieties. Assume that, up to isogeny, E has no non-trivial almost constant factor. Then T has property (P).*

Proof. First, exactly as in the case of principal $\mathbb{C}^*$-bundles, a finite étale Galois cover of T is a projective and smooth morphism $R \to Y$ over some finite étale Galois cover Y of X, whose general fiber is a finite cover of a complex torus, that is R is a family of complex tori:

$$\begin{array}{ccc} R & \xrightarrow{\tau} & T \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[\pi]{} & X. \end{array} \tag{A.26}$$

We want to apply Lemma A.9. Let A be an abelian variety and $f : R \to A$. Let F be the associated family of abelian varieties to R. On each fiber we have a map of varieties $f : R_y \to A$, which, by rigidity, is the composition of a translation in R_y and a morphism of abelian varieties. In this way f induces a map on F and a morphism of families of abelian varieties $g : F \to A \times Y$, then an injective morphism $F/\operatorname{Ker}(g) \hookrightarrow A \times Y$. By rigidity (see [Mil08, 16.3]) this implies that $F/\operatorname{Ker}(g)$ is a constant family. But by Poincaré's reducibility theorem for families, it is possible to find a family G over Y such that addition $\operatorname{Ker}(g) \times G \to F$ is an isogeny, and $F/\operatorname{Ker}(g)$ is isogenous to G. There is an induced map $F \to \pi^*E$ over Y such that G projects to a constant factor of π^*E. This corresponds to an almost constant factor of E over X. But by our hypothesis there is no non-trivial such factor, thus G is trivial which means g is constant (as morphism of families), therefore f is constant so $b_1(\overline{R}) = 0$. □

Symmetric spaces

Our motivation comes now from rigidity theorems for hermitian locally symmetric spaces and for lattices in Lie groups, which translate into the vanishing of the first Betti number.

Theorem A.22. *Let $\Omega = G/K$ be an irreducible hermitian symmetric space of noncompact type, where G is a simple Lie group of rank greater than 2, K is a maximal compact subgroup, and let $\Gamma \subset G$ be a torsion-free lattice. Then $X := \Gamma \setminus \Omega$ has property (P).*

Proof. First it is known that Ω is simply connected and that X is a smooth quasi-projective variety, see the Baily-Borel compactification (for example [BJ06]). A finite étale Galois cover Y of X is a quotient $\Gamma' \setminus \Omega$ where $\Gamma' \subset \Gamma$ has finite index, so Γ' is still a torsion-free lattice in G and is the fundamental group of Y. Under our hypothesis it is known by the results of Kazhdan (see [BdV08, p. 12]) that G has property (T), and so do Γ, Γ'. This implies that $b_1(\Gamma') = 0$ and this is also $b_1(Y)$.

In case X (and Y) is compact, we are done. Else we take any smooth compactification $\overline{Y}$ (which may not be the Baily-Borel compactification since this one is usually not smooth) and the natural morphism $\Gamma' = \pi_1(Y) \to \pi_1(\overline{Y})$ is surjective. This implies that $\pi_1(\overline{Y})^{\mathrm{ab}}$ is finite and so $b_1(\overline{Y}) = 0$. $\square$

Bibliographie

[ABC$^+$96] J. Amorós, M. Burger, K. Corlette, D. Kotschick, and D. Toledo, *Fundamental groups of compact Kähler manifolds*, Mathematical Surveys and Monographs, no. 44, American Mathematical Society, 1996.

[AM69] M. F. Atiyah and I. G. MacDonald, *Introduction to commutative algebra*, Addison-Wesley, 1969.

[BdV08] B. Bekka, P. de la Harpe, and A. Valette, *Kazhdan's property (T)*, New Mathematical Monographs, vol. 11, Cambridge University Press, 2008.

[BG76] A. K. Bousfield and V. K. A. M. Gugenheim, *On PL De Rham theory and rational homotopy type*, Memoirs of the American Mathematical Society **179** (1976).

[BJ06] A. Borel and L. Ji, *Compactifications of symmetric and locally symmetric spaces*, Mathematics: Theory & Applications, Birkhäuser, 2006.

[BL04] C. Birkenhake and H. Lange, *Complex abelian varieties*, Grundlehren der mathematischen Wissenschaften, vol. 302, Springer, 2004.

[BM97] E. Bierstone and P. D. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Inventiones Mathematicae **128** (1997), no. 2, 207–302.

[Cam85] F. Campana, *Réduction d'Albanèse d'un morphisme propre et faiblement kählérien. I*, Compositio Mathematica **54** (1985), no. 3, 373–398.

[CG16] J. Cirici and F. Guillén, *Homotopy theory of mixed Hodge complexes*, Tohoku Mathematical Journal **68** (2016), no. 3, 349–375.

[Cir15] J. Cirici, *Cofibrant models of diagrams: mixed Hodge structures in rational homotopy*, Transactions of the American Mathematical Society **367** (2015), no. 8, 5935–5970.

[CR16] J. Cirici and A. Roig, *Sullivan minimal models of operad algebras*, arXiv:1612.03862, 2016.

[Del71a] P. Deligne, *Théorie de Hodge, I*, Actes du Congrès International des Mathématiciens **1** (1971), 425–430.

[Del71b] ———, *Théorie de Hodge, II*, Publications Mathématiques de l'IHES **40** (1971), 5–57.

[Del74] ———, *Théorie de Hodge, III*, Publications Mathématiques de l'IHES **44** (1974), 5–77.

[Del86] ———, *Letter to Millson*, 1986.

[DGMS75] P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan, *Real homotopy theory of Kähler manifolds*, Inventiones Mathematicae **29** (1975), 245–274.

[DPS09] A. Dimca, S. Papadima, and A. I. Suciu, *Topology and geometry of cohomology jump loci*, Duke Mathematical Journal **148** (2009), no. 3, 405–457.

[Dri88] V. Drinfeld, *Letter to V. Schechtman*, 1988.

[Dup15] C. Dupont, *Purity, formality, and arrangement complements*, International Mathematics Research Notices **2016** (2015), no. 13, 4132–4144.

[EKPR12] P. Eyssidieux, L. Katzarkov, T. Pantev, and M. Ramachandran, *Linear Shafarevich conjecture*, Annals of Mathematics **176** (2012), no. 3, 1545–1581.

[ES11] P. Eyssidieux and C. Simpson, *Variations of mixed Hodge structure attached to the deformation theory of a complex variation of Hodge structures*, Journal of the European Mathematical Society **13** (2011), no. 6, 1769–1798.

[Eys16] P. Eyssidieux, *Orbifold Kähler groups and the Shafarevich conjecture for Hirzebruch's covering surfaces with equal weights*, arXiv:1611.09178, to appear in Asian Journal of Mathematics, 2016.

[FM07] D. Fiorenza and M. Manetti, *L_∞ structures on mapping cones*, Algebra & Number Theory **1** (2007), no. 3, 301–330.

[GJ99] P. G. Goerss and J. F. Jardine, *Simplicial homotopy theory*, Progress in Mathematics, no. 174, Birkhäuser, 1999.

[GM81] P. A. Griffiths and J. W. Morgan, *Rational homotopy theory and differential forms*, Progress in Mathematics, no. 16, Birkhäuser, 1981.

[GM88] W. M. Goldman and J. J. Millson, *The deformation theory of representations of fundamental groups of compact Kähler manifolds*, Publications Mathématiques de l'IHES **67** (1988), 43–96.

[GM90] ______, *The homotopy invariance of the Kuranishi space*, Illinois Journal of Mathematics **34** (1990), no. 2, 337–367.

[God58] R. Godement, *Topologie algébrique et théorie des faisceaux*, Hermann, 1958.

[Gro61] A. Grothendieck, *Éléments de géométrie algébrique: II. étude globale élémentaire de quelques classes de morphismes*, Publications Mathématiques de l'IHES **8** (1961), 5–222.

[Gro03] A. Grothendieck (ed.), *Séminaire de géométrie algébrique du Bois Marie 1960-1961. Revêtements étales et groupe fondamental (SGA 1)*, Société Mathématique de France, 2003.

[Hai87] R. M. Hain, *The de Rham homotopy theory of complex algebraic varieties I*, *K*-Theory **1** (1987), no. 3, 271–324.

[Hai98] ______, *The Hodge de Rham theory of relative Malcev completion*, Annales Scientifiques de l'École Normale Supérieure **31** (1998), no. 1, 47–92.

[Hin01] V. Hinich, *DG coalgebras as formal stacks*, Journal of Pure and Applied Algebra **162** (2001), no. 2-3, 209–250.

[Hir83] F. Hirzebruch, *Arrangements of lines and algebraic surfaces*, Arithmetic and geometry, Vol. II, Progress in Mathematics, vol. 36, Birkhäuser, 1983, pp. 113–140.

[Hir93] E. Hironaka, *Abelian coverings of the complex projective plane branched along configurations of real lines*, Memoirs of the American Mathematical Society **105** (1993), no. 502.

[Hov99] M. Hovey, *Model categories*, Mathematical Surveys and Monographs, no. 63, American Mathematical Society, 1999.

[KM98] M. Kapovich and J. J. Millson, *On representation varieties of Artin groups, projective arrangements and the fundamental groups of smooth complex algebraic varieties*, Publications Mathématiques de l'IHES **88** (1998), 5–95.

[Kon94] M. Kontsevich, *Topics in algebra, deformation theory*, Lectures notes, 1994.

[Lef17] L.-C. Lefèvre, *A criterion for quadraticity of a representation of the fundamental group of an algebraic variety*, Manuscripta Mathematica **152** (2017), no. 3-4, 381–397.

[LM85] A. Lubotzky and A. R. Magid, *Varieties of representations of finitely generated groups*, Memoirs of the American Mathematical Society **336** (1985).

[Lur11] J. Lurie, *Formal moduli problems*, `http://www.math.harvard.edu/~lurie/`, 2011.

[LV12] J.-L. Loday and B. Vallette, *Algebraic operads*, Grundlehren der mathematischen Wissenschaften, no. 346, Springer, 2012.

[Mac88] S. MacLane, *Categories for the working mathematician*, Graduate Texts in Mathematics, no. 5, Springer, 1988.

[Man99] M. Manetti, *Deformation theory via differential graded Lie algebras*, Seminari di Geometria Algebrica 1998-1999, Scuola Normale Superiore (1999).

[Man02] ______, *Extended deformation functors*, International Mathematics Research Notices **2002** (2002), no. 14, 719–756.

[Man04] ______, *Lectures on deformations of complex manifolds. Deformations from differential graded viewpoint*, Rendiconti di Matematica **24** (2004), no. 1, 1–183.

[Mil08] J. S. Milne, *Abelian Varieties*, `www.jmilne.org/math/`, 2008.

[Mor78] J. W. Morgan, *The algebraic topology of smooth algebraic varieties*, Publications Mathématiques de l'IHES **48** (1978), 137–204.

[Mor86] ______, *Correction to: "The algebraic topology of smooth algebraic varieties"*, Publications Mathématiques de l'IHES **64** (1986), 185.

[Nav87] V. Navarro Aznar, *Sur la théorie de Hodge-Deligne*, Inventiones Mathematicae **90** (1987), 11–76.

[Pri10] J. P. Pridham, *Unifying derived deformation theories*, Advances in Mathematics **224** (2010), no. 3, 772–826.

[PS08] C. A. M. Peters and J. H. M. Steenbrink, *Mixed Hodge structures*, Ergebnisse der Mathematik und ihrer Grenzgebiete, no. 52, Springer, 2008.

[PS09] S. Papadima and A.I. Suciu, *Geometric and algebraic aspects of 1-formality*, Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie **52(100)** (2009), no. 3.

[Qui69] D. Quillen, *Rational homotopy theory*, Annals of Mathematics **90** (1969), 205–295.

[Sch68] M. Schlessinger, *Functors of Artin rings*, Transactions of the American Mathematical Society **130** (1968), 208–222.

[Sim92] C. T. Simpson, *Higgs bundles and local systems*, Publications Mathématiques de l'IHES **75** (1992), 5–95.

[Suc01] A. I. Suciu, *Fundamental groups of line arrangements: enumerative aspects*, Advances in algebraic geometry motivated by physics (Lowell, MA, 2000), Contemporary Mathematics, vol. 276, American Mathematical Society, 2001, pp. 43–79.

[Sul77] D. Sullivan, *Infinitesimal computations in topology*, Publications Mathématiques de l'IHES **47** (1977), 269–331.

[Sum74] H. Sumihiro, *Equivariant completion*, Journal of Mathematics of Kyoto University **14** (1974), 1–28.

[Swe69] M. E. Sweedler, *Hopf algebras*, Mathematics Lecture Notes Series, W. A. Benjamin, Inc., 1969.

[Tay00] I. Tayama, *First Betti numbers of abelian coverings of the complex projective plane branched over line configurations*, Journal of Knot Theory and its Ramifications **9** (2000), no. 2, 271–284.

[Toë17] B. Toën, *Problèmes de modules formels. Exposé 1111*, Astérisque. Séminaire Bourbaki, volume 2015/2016 **390** (2017), 199–244.

[Voi02] C. Voisin, *Théorie de Hodge et géométrie algébrique complexe*, Cours spécialisés, Société Mathématique de France, 2002.

[Zuc79] S. Zucker, *Hodge theory with degenerating coefficients: L_2 cohomology in the Poincaré metric*, Annals of Mathematics **109** (1979), 415–476.