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VEECH GROUPS WITHOUT PARABOLIC ELEMENTS

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Abstract. We prove that a translation surface that has two transverse parabolic elements has a totally real trace field. As a corollary, nontrivial Veech groups which have no parabolic elements do exist.

The proof follows Veech's viewpoint on Thurston's construction of pseudo-Anosov diffeomorphisms.

1. Introduction

For a long time, it has been known that the ergodic properties of linear flows on a translation surface are strongly related to the behavior of its $SL_2(\mathbb{R})$-orbit in the moduli space of holomorphic one forms (see [MT], [Z] for surveys of the literature on this subject). The $SL_2(\mathbb{R})$-orbit of a translation surface is called its *Teichmüller disc*. Its stabilizer under the action of $SL_2(\mathbb{R})$ is a Fuchsian group called the *Veech group*.

In 1989, Veech proved that a translation surface whose stabilizer is a lattice has optimal dynamical properties: the directional flows are periodic or uniquely ergodic (see [Ve2]). Since then, much effort has gone into the study of the geometry of Teichmüller discs ([Ve3], [Vo], [W]). Hubert and Schmidt [HS1, HS2] found the first examples of infinitely generated Veech groups. Just after that, McMullen [Mc1, Mc2] proved that, in genus two, the existence of a pseudo-Anosov diffeomorphism in the affine group implies that the Veech group is a Fuchsian group of the first kind (which means that either it is a lattice or it is infinitely generated; moreover McMullen proved that both cases occur) (see also [C] for related results). McMullen's proof uses the existence of infinitely many parabolic elements. By contrast, we give examples with a very different behavior in genus $g \geq 3$.

The trace field is a natural invariant of the Veech group. Thurston proved that the trace of the derivative of any pseudo-Anosov diffeomorphisms is an algebraic integer over $\mathbb{Q}$ with degree less that the dimension of the Teichmüller space divided by 2 (see [T], p. 427). In [GJ] it was shown that a translation surface is a covering of the torus ramified over one point if and only if its trace field equals to $\mathbb{Q}$. Kenyon and Smillie [KS] gave a simple criterion ensuring this property: if the Veech group of a translation surface contains a hyperbolic element whose trace belongs to $\mathbb{Q}$, then this group is commensurable to $SL_2(\mathbb{Z})$. In fact they showed that the trace field is generated by the trace of the derivative of any pseudo-Anosov diffeomorphisms. Moreover, if K is the trace field of (X, ω), then the Veech group is commensurable to a subgroup of $SL_2(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of K.

An interesting problem is to determine which Fuchsian groups can occur as the affine group of some surface. Up to now, there are no general methods to compute a Veech group. To date there are two methods to produce pseudo-Anosov diffeomorphisms in the coordinates of the flat surface. In the first one, due to Thurston, a pseudo-Anosov diffeomorphism is obtained as a product of two parabolic elements (see [FLP] exp. 13, Construction de difféomorphismes pseudo-Anosov p. 243–250, [T], [Ve2]). Veech computed the first non trivial examples of affine groups by making calculations with a pair of parabolic elements (see [Ve2]). Independently, a very general construction of pseudo-Anosov diffeomorphisms was discovered by Veech [Ve1]. It is based on the Rauzy induction of interval exchange transformations (see also [AY] for specific examples of such pseudo-Anosov diffeomorphisms, for any genus $g \geq 3$). A simple consequence of our result is that some pseudo-Anosov diffeomorphisms are not given by Thurston's construction (see [L] for another proof). In fact, we prove a stronger result.

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Theorem 1.1. *Let (X,ω) be a translation surface. Let us assume that the Veech group $SL(X,\omega)$ contains two transverse parabolic elements* [1]*. Then the trace field*

$$\mathbb{Q}\,[\mathit{Trace}(A)\ ,\ A \in SL(X,\omega)]$$

is totally real.

When a pseudo-Anosov diffeomorphism ϕ acts linearly on a translation surface by the diagonal matrix $D\phi = \left[\begin{smallmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{smallmatrix}\right]$ (with $\lambda^{-1} < 1 < \lambda$), its *expansion factor* is $\lambda(\phi) = \lambda$ (and $1/\lambda$ is the contraction factor).

From Theorem 1.1, we draw the following results.

Theorem 1.2. *Let (X,ω) be a translation surface endowed with a pseudo-Anosov diffeomorphism ϕ with expansion factor λ. Let us assume that the field $\mathbb{Q}[\lambda+\lambda^{-1}]$ is not totally real. Then $SL(X,\omega)$ does not contain any parabolic elements.*

Arnoux and Yoccoz [AY] discovered a family ϕ_n, $n \geq 3$, of pseudo-Anosov diffeomorphisms with expansion factor $\lambda_n = \lambda(\phi_n)$, where λ_n is the Pisot root of the irreducible polynomial P_n with

$$P_n(X) = X^n - X^{n-1} - \cdots - X - 1.$$

The pseudo-Anosov ϕ_n acts linearly on a genus n surface (the corresponding Abelian differential having two zeros of order $n-1$).

Corollary 1.3. *The Teichmüller disc stabilized by the Arnoux–Yoccoz pseudo-Anosov ϕ_n, $n \geq 3$ does not contain any parabolic direction.*
Therefore, for any genus $g \geq 3$, there exists a genus g translation surface such that its Veech group has (at least) one hyperbolic element and no parabolic elements.

Corollary 1.4. *The trace field of any Veech surface is totally real.*

Remark 1.1. Möller proved Corollary 1.4 by very different methods (see [Mö]).

Corollary 1.5. *There exists a Veech group which is commensurable to a Fuchsian group that only contains hyperbolic elements.*

2. Background

In order to establish notation and preparatory material, we review basic notions concerning translation surfaces, affine automorphisms groups, and trace fields. We end this section by recalling Veech's viewpoint on Thurston's construction. See say [KS], [MT], [Mc1, Mc2], [T], [Ve2] for more details; See also [Mc3, Mc4, Mc5], for recent related developments. For a general reference on Fuchsian groups, see [K].

2.1. Translation surfaces and affine diffeomorphisms group.

A *translation surface* is a (real) genus g surface with an atlas such that all transition functions are translations. As usual, we consider maximal atlases. These surfaces are precisely those given by a Riemann surface X and a holomorphic (nonnull) one-form $\omega \in \Omega(X)$; see [MT] for a general reference on translation surfaces and holomorphic one-forms.

We denote by X' the surface that arises from X by deleting the zeros of the form ω on X. The translation structure defines on X' a Riemannian structure; we therefore have notions of geodesic, length, angle, flow, measure... Orbits of the directional flows meeting singularities are called *separatrices*. Orbits of the flow going from a singularity to another one (possibly the same) are called *saddle connections*.

Given any matrix $A \in \mathrm{SL}_2(\mathbb{R})$, we can postcompose the coordinate functions of the charts of (X,ω) by A. One easily checks that this gives a new translation surface, denoted by $A \cdot (X,\omega)$. We therefore get an $\mathrm{SL}_2(\mathbb{R})$-action on these translation surfaces.

An *affine diffeomorphism* $f : X \to X$ is a homeomorphism of X such that f restricts to a diffeomorphism on X' of constant derivative. It is equivalent to say that f restricts to an isomorphism of X' which preserves the induced affine structure given by ω. Usually, one denotes by $\mathrm{Aff}(X,\omega)$ the group of orientation preserving affine diffeomorphisms. The function which takes

[1] The surface (X,ω) is then called a *prelattice surface* [HS1] or a "bouillabaisse surface" in honor of John Hubbard's Lecture (see [H]) at the Centre International de Rencontres Mathématiques, Marseille, in July 2003.

an affine diffeomorphism f to its derivative Df gives a homomorphism from $\mathrm{Aff}(X,\omega)$ into $\mathrm{SL}_2(\mathbb{R})$. The image of $\mathrm{Aff}(X,\omega)$ is the *Veech group* $\mathrm{SL}(X,\omega)$ of the surface (X,ω) – this is a discrete subgroup and, when X has genus greater than one, the kernel of the homomorphism is finite.

One easily checks that the Veech group $\mathrm{SL}(X,\omega)$ is the $\mathrm{SL}_2(\mathbb{R})$-stabilizer of (X,ω). Thus, for any matrix $A \in \mathrm{SL}_2(\mathbb{R})$, the Veech group of (X,ω) and $A\cdot(X,\omega) = (Y,\alpha)$ are conjugate in $\mathrm{SL}_2(\mathbb{R})$:

$$\mathrm{SL}(Y,\alpha) \;=\; A\cdot \mathrm{SL}(X,\omega)\cdot A^{-1}$$

2.2. **Classification of affine diffeomorphisms.** There is a standard classification of the elements of $\mathrm{SL}_2(\mathbb{R})$ into three types: elliptic, parabolic and hyperbolic. This induces a classification of affine diffeomorphisms.

An affine diffeomorphism is parabolic, or elliptic, or pseudo-Anosov, respectively, if $|\mathrm{trace}(Df)| = 2$, $|\mathrm{trace}(Df)| < 2$, or $|\mathrm{trace}(Df)| > 2$, respectively.

Remark 2.1. If an elliptic element belongs to a Fuchsian group, its order is finite.

Remark 2.2. In a Fuchsian group, a parabolic direction (invariant direction of a parabolic element) is never fixed by a hyperbolic element. More precisely, if a hyperbolic element H fixes a parabolic direction of a parabolic element P then one can easily check that H^nPH^{-n} converges to Id as n tends to $+\infty$ (or $-\infty$), which is impossible in a discrete group.

2.3. **Cylinders decomposition and parabolic element.** A *cylinder* on (X,ω) is a maximal connected set of homotopic simple closed geodesics. If the genus of X is greater than one then every cylinder is bounded by saddle connections. A cylinder has a width (or circumference) x and a height y. The *modulus* of a cylinder is $\mu = y/x$. Veech proved the following (see [Ve2], prop. 2.4, p. 558).

Proposition (Veech)**.** *If a translation surface (X,ω) has a parabolic affine diffeomorphism f, then there is a decomposition of X into metric cylinders parallel to the fixed direction of Df. Furthermore, the moduli of the cylinders are commensurable (have rational ratios).*

Remark 2.3. In the above proposition, up to take a power of the affine diffeomorphism, we can assume that f acts as a power of the affine Dehn twist on each cylinder. Therefore the boundary of each cylinder is fixed by f.

Conversely, a cylinder decomposition into cylinders of commensurable moduli produces parabolic elements. Namely, the following holds.

Proposition (Veech)**.** *If (X,ω) has a decomposition into metric cylinders for the horizontal direction, with commensurable moduli, then the Veech group $SL(X,\omega)$ contains*

$$Df = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix},$$

where c is the least common multiple of the inverse of the moduli.

2.4. **Trace fields.** In this section we recall some general properties of the trace field of a group; see [GJ], [KS], [Mc1, Mc2].

The *trace field* of a group $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ is the subfield of $\mathbb{R}$ generated by $\mathrm{tr}(A)$, $A \in \Gamma$. One defines the trace field of a flat surface (X,ω) to be the trace field of its Veech group $\mathrm{SL}(X,\omega) \subset \mathrm{SL}_2(\mathbb{R})$.

Let (X,ω) be a genus g translation surface. Then the following holds (see [KS], appendix).

Theorem A. (Kenyon, Smillie). *The trace field of (X,ω) has degree at most g over $\mathbb{Q}$.*
Assume that the affine diffeomorphisms group of (X,ω) contains a pseudo-Anosov element f with expansion factor λ. Then the trace field of (X,ω) is $\mathbb{Q}[\lambda+\lambda^{-1}]$.

One defines the *holonomy vectors* to be the integrals of ω along the saddle connections. Let us denote by $\Lambda = \Lambda(\omega)$ the subgroup of $\mathbb{R}^2$ generated by holonomy vectors

$$\Lambda = \int_{H_1(X,\mathbb{Z})} \omega$$

Let $e_1, e_2 \in \Lambda$ be nonparallel vectors in $\mathbb{R}^2$. One defines the *holonomy field* k to be the smallest subfield of $\mathbb{R}$ such that every element of Λ may be written as $ae_1 + be_2$ with $a, b \in k$.

Theorem B. (Kenyon, Smillie) *The trace field of (X, ω) coincides with k. The space $\Lambda \otimes \mathbb{Q} \subset \mathbb{C}$ is a two dimensional vector space over k.*

See also [GJ] for a different approach of these notions. Note that these results have been reproved in [Mc1, Mc2].

Pisot numbers. An algebraic integer β is a Pisot number if $\beta \in \mathbb{R}$, $\beta > 1$ and all of its conjugates belong to the unit disc $\mathbb{D} = \{z \in \mathbb{C},\ |z| < 1\}$.

2.5. **Veech's viewpoint on Thurston's construction.** Let us recall the Thurston construction [T]. We follow the notations of the article of Veech [Ve2], section §9.

Let (Y, α) be a translation surface with vertical and horizontal parabolic directions. Up to taking a power of the parabolic elements, one can assume that the corresponding parabolic P_v (resp., P_h) is a multiple of the Dehn twist of each vertical (resp., horizontal) cylinder (see Remark 2.3).

In these coordinates our two parabolic elements are

$$P_h = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad P_v = \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix}$$

Without loss of the generality, we may assume that c and d are positive real numbers.

Claim 2.1. *Let $t = cd > 0$; then the trace field of $SL(Y, \alpha)$ is $\mathbb{Q}[t]$.*

Proof of Claim 2.1. The matrix $P_h P_v$ has trace $2 + t > 2$, thus this is a hyperbolic element and, following [KS] (see section 2.4, Theorem A), the trace field of $\mathrm{SL}(Y, \alpha)$ is $\mathbb{Q}[t]$. So the claim is proved. □

Let us denote by H_i, $1 \le i \le r$ and V_j, $1 \le j \le s$ the horizontal and vertical cylinders. Let us denote the width and heights of H_i and V_j respectively by (x_i, y_i) and (η_j, ξ_j). We insist that the first coordinate is the width and the second one is the height even for vertical cylinders.

With these notations, let E be the $r \times s$ integer matrix whose entry $E_{i,j}$ is the number of rectangles $(\xi_j \times y_i)$ in the intersection $H_i \cap V_j$. All of these rectangles have width y_j and heights ξ_i. In other words, $E_{i,j}$ is the intersection number of the core curves of the cylinders H_i and V_j.

Let us introduce the following notations of linear algebra: $\overrightarrow{x} = (x_1, \ldots, x_r)$, $\overrightarrow{y} = (y_1, \ldots, y_r)$, $\overrightarrow{\xi} = (\xi_1, \ldots, \xi_s)$ and $\overrightarrow{\eta} = (\eta_1, \ldots, \eta_s)$. Then one can summarize the above discussion by the matrix relations:

$$\left\{ \begin{array}{l} \overrightarrow{x} = E\overrightarrow{\xi} \\ \overrightarrow{\eta} = {}^{t}E\overrightarrow{y} \end{array} \right. \tag{1}$$

The moduli of the vertical cylinder V_j (resp horizontal cylinder H_i) is commensurable with d (resp with c). More precisely, there exist integers m_i, $1 \le i \le r$, and n_j, $1 \le j \le s$, such that

$$\left\{ \begin{array}{l} m_i x_i = c y_i \\ n_j \eta_j = d \xi_j \end{array} \right. \tag{2}$$

Let us denote by $D_m = \mathrm{Diag}(m_1, \ldots, m_r)$ and $D_n = \mathrm{Diag}(n_1, \ldots, n_s)$ the diagonal matrices. Then the above equation (2) becomes:

$$\left\{ \begin{array}{l} D_m \overrightarrow{x} = c\overrightarrow{y} \\ D_n \overrightarrow{\eta} = d\overrightarrow{\xi} \end{array} \right. \tag{3}$$

From equations (1) and (3) we get the following new one:

$$\left\{ \begin{array}{l} ED_n \overrightarrow{\eta} = d\overrightarrow{x} \\ {}^{t}ED_m \overrightarrow{x} = c\overrightarrow{\eta} \end{array} \right.$$

and therefore we deduce:

$$\left\{ \begin{array}{l} ED_n {}^{t}ED_m \overrightarrow{x} = cd\overrightarrow{x} \\ {}^{t}ED_m ED_n \overrightarrow{\eta} = cd\overrightarrow{\eta} \end{array} \right. \tag{4}$$

Now, in order to follow Veech's notations, let us introduce the two matrices $F_n = ED_n$ and $F_m = {}^{t}ED_m$. As remarked in [Ve2], the matrices $F_n F_m$ and $F_m F_n$ have a power with positive entries (see [HL] Appendix C for a proof). The vector $\overrightarrow{x}$ is a nonnegative eigenvector of the Perron–Frobenius matrix $F_n F_m$, therefore $t = cd > 0$ is the unique Perron–Frobenius eigenvalue of

$F_n F_m$ (the same is true for the Perron–Frobenius matrix $F_m F_n$). Thus, up to renormalization of the area of the surface, the coordinates of $\overrightarrow{x}$ and $\overrightarrow{\eta}$ belong to $\mathbb{Q}[t]$ (see section 2.4, Theorem B).

Now we have all necessary tools to prove the announced results.

3. Proofs

We first prove Theorem 1.2 assuming Theorem 1.1.

Proof of Theorem 1.2. Let us assume that there is a parabolic element P in $\mathrm{SL}(X,\omega)$. Let us denote by H the derivative of the pseudo-Anosov ϕ. Then the conjugate HPH^{-1} is another parabolic element in $\mathrm{SL}(X,\omega)$. Let $x \in \partial\mathbb{H}$ be the fixed point of P. Thus, $H(x)$ is a fixed point of HPH^{-1}. But by Remark 2.2, $H(x) \neq x$, then $HPH^{-1} \in \mathrm{SL}(X,\omega)$ is certainly a parabolic element transverse to the parabolic P. Therefore Theorem 1.1 applies. □

Proof of Theorem 1.1. Let us assume that the surface (X,ω) has two transverse parabolic elements. By a standard argument, one can find a matrix $A \in \mathrm{SL}_2(\mathbb{R})$ which sends the two invariant directions of our parabolic elements into horizontal and vertical direction. The Veech group

$$\mathrm{SL}(Y,\alpha) = A \cdot \mathrm{SL}(X,\omega) \cdot A^{-1}$$

possesses the same trace field as $\mathrm{SL}(X,\omega)$.

Now up to taking a power of the parabolic element, one can assume that there is a multiple of the Dehn twist on each vertical (resp., horizontal) cylinder (see Remark 2.3). Thus one can apply Veech's viewpoint on Thurston's construction, section 2.5. In particular we follow the notations introduced in that section.

Recall that the trace field of $\mathrm{SL}(Y,\alpha)$ is $\mathbb{Q}[t]$ (see Claim 2.1). Now let us prove that $\mathbb{Q}[t]$ is totally real.

Let σ be an embedding of $\mathbb{Q}[t]$ into $\mathbb{C}$ and $t' = \sigma(t) \in \mathbb{C}$ be a conjugate of t. Applying σ to the first part of equation (4): $F_n F_m \overrightarrow{x} = t\overrightarrow{x}$ and recalling that $F_n F_m$ is an *integer matrix*, one gets

$$F_n F_m \sigma(\overrightarrow{x}) = t'\sigma(\overrightarrow{x}) \tag{5}$$

Now, let us denote by $D_{\sqrt{m}} = \mathrm{Diag}(\sqrt{m_1},\dots,\sqrt{m_r})$ and $D_{\sqrt{n}} = \mathrm{Diag}(\sqrt{n_1},\dots,\sqrt{n_s})$ the diagonal matrices. Then

$$F_n F_m = ED_n{}^tED_m = ED_{\sqrt{n}}\, D_{\sqrt{n}}{}^tE\, D_{\sqrt{m}}D_{\sqrt{m}} \;=\; ED_{\sqrt{n}}\, {}^t(ED_{\sqrt{n}})\, D_{\sqrt{m}}D_{\sqrt{m}}$$

Let us set $A = ED_{\sqrt{n}}$. Substituting this into the last equation yields

$$F_n F_m = A^tAD_{\sqrt{m}}D_{\sqrt{m}}$$

Letting $M = D_{\sqrt{m}}\, A$, it becomes:

$$\begin{aligned} F_n F_m \;&=\; D_{\sqrt{m}}^{-1}D_{\sqrt{m}}\, A^tAD_{\sqrt{m}}D_{\sqrt{m}} \;= \\ &=\; D_{\sqrt{m}}^{-1}D_{\sqrt{m}}\, A\, {}^t(D_{\sqrt{m}}\, A)\, D_{\sqrt{m}} \;= \\ &=\; D_{\sqrt{m}}^{-1}M^tM\, D_{\sqrt{m}} \end{aligned} \tag{6}$$

Now equation (5) and the fact that $\sigma(\overrightarrow{x}) \neq \overrightarrow{0}$ imply that t' is an eigenvalue of $F_n F_m$. But by equation (6), the two matrices $F_n F_m$ and M^tM are similar, they thus have the same eigenvalues. But M^tM is symmetric, thus all of its eigenvalues are real, and so $t' \in \mathbb{R}$.

Finally the trace field $\mathbb{Q}[t]$ of (Y,α), and that of (X,ω), is totally real. Theorem 1.1 is proved. □

Proof of Corollary 1.3. Let $n \geq 3$ be any odd integer. We denote by (X_n,ω_n) a flat surface in the Teichmüller disc stabilized by the Arnoux–Yoccoz pseudo-Anosov ϕ_n. By Theorem A (see section 2.4), the trace field of (X_n,ω_n) is $\mathbb{Q}[\lambda_n + \lambda_n^{-1}]$.

Now the polynomial $X^n - X^{n-1} - \cdots - 1$ has two real roots if n is even and one if n is odd. Indeed, as in [AY], let Q_n be the polynomial

$$Q_n(X) = (X^n - X^{n-1} - \cdots - 1)(X-1) = X^{n+1} - 2X^n + 1.$$

One can directly check, by calculating Q_n', that $Q_n(X)$ has two real roots if n is odd and three if n is even.

Therefore this shows that $\mathbb{Q}[\lambda_n]$ is not totally real. Recall that λ_n is a Pisot number (see [AY]). Applying the next Lemma 3.1 we get that $\mathbb{Q}[\lambda_n + \lambda_n^{-1}]$ is not totally real.

Thus Corollary 1.3 follows from Theorem 1.2. □

Lemma 3.1. *Let β be any Pisot number. Let us assume that $\mathbb{Q}[\beta]$ is not totally real. Then the field $\mathbb{Q}[\beta + \beta^{-1}]$ is not totally real.*

Proof of Lemma 3.1. Let δ be a conjugate of β which is not real. Galois theory ensures that there is a field homomorphism $\chi : \mathbb{Q}[\beta] \to \mathbb{Q}[\delta]$. The complex number $\chi(\beta+\beta^{-1}) = \delta+\delta^{-1}$ is a conjugate of $\beta + \beta^{-1}$. It is enough to show that $\delta + \delta^{-1}$ is not real to prove that $\mathbb{Q}[\beta + \beta^{-1}]$ is not totally real. Writing $\delta = \rho e^{i\theta}$ (with $\sin(\theta) \neq 0$), we have $\Im m(\delta + \delta^{-1}) = (\rho - \rho^{-1})\sin(\theta)$. As β is a Pisot number, $\rho = |\delta| < 1$. Therefore $\delta + \delta^{-1}$ is not real. So Lemma 3.1 is proved. □

Proof of Corollary 1.4. On a Veech surface, the direction of every saddle connection is a parabolic direction. There are thus at least two transverse parabolic elements in the Veech group and Theorem 1.1 applies. □

Proof of Corollary 1.5. Let (X, ω) be any genus $g \geq 3$ translation surface whose Veech group only contains hyperbolic and elliptic elements. Any elliptic element in $\mathrm{SL}(X, \omega)$ is conjugate in $\mathrm{SL}_2(\mathbb{R})$ to a rotation. As a rotation preserves the underlying complex structure of the Riemann surface X, it is an automorphism of a genus g Riemann surface. Therefore, by a theorem of Hurwitz, the order of any elliptic element belonging to $\mathrm{SL}(X, \omega)$ is bounded by $84(g - 1)$ (see, say, [FK] §5 p. 242).

Now we recall a Theorem of Purzitsky on Fuchsian groups (see [P], Theorem 7, p. 241).

Theorem (Purzitsky)**.** *Let Γ be a Fuchsian group. Then Γ contains a finite index subgroup without elliptic elements if and only if there exists a constant N such that the order of any elliptic element of Γ is less than N.*

Now recalling that any elliptic element belonging to a Fuchsian group has finite order (see Remark 2.1), Corollary 1.5 follows from Purzitsky's Theorem taking $\Gamma = \mathrm{SL}(X, \omega)$. □

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