DYNAMICS AND GEOMETRY OF THE RAUZY-VEECH INDUCTION FOR QUADRATIC DIFFERENTIALS

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ABSTRACT. Interval exchange maps are related to geodesic flows on translation surfaces; they correspond to the first return maps of the vertical flow on a transverse segment. The Rauzy-Veech induction on the space of interval exchange maps provides a powerful tool to analyze the Teichmüller geodesic flow on the moduli space of Abelian differentials. Several major results have been proved using this renormalization.

Danthony and Nogueira introduced in 1988 a natural generalization of interval exchange transformations, namely the linear involutions. These maps are related to general measured foliations on surfaces (orientable or not). In this paper we are interested by such maps related to geodesic flow on (orientable) flat surfaces with $\mathbb{Z}/2\mathbb{Z}$ linear holonomy. We relate geometry and dynamics of such maps to the combinatorics of generalized permutations. We study an analogue of the Rauzy-Veech induction and give an efficient combinatorial characterization of its attractors. We establish a natural bijection between the extended Rauzy classes of generalized permutations and connected components of the strata of meromorphic quadratic differentials with at most simple poles, which allows, in particular, to classify the connected components of all exceptional strata.

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Introduction

A geodesic flow in a given direction on a translation surface induces on a transverse segment an interval exchange map. Dynamic of such transformations has been extensively studied during these last thirty years providing applications to billiards in rational polygons, to measured foliations on surfaces, to Teichmüller geometry and dynamics, etc.

Interval exchange transformations are closely related to Abelian differentials on Riemann surfaces. It is well known that the continued fractions encode cutting sequences of hyperbolic geodesics on the Poincaré upper half-plane. Similarly, the Rauzy-Veech induction (analogous to Euclidean algorithm) provides a discrete model for the Teichmüller geodesics flow ([Rau79, Vee82, Arn94]).

Using this relation H. Masur in [Mas82] and W. A. Veech in [Vee82] have independently proved the Keane's conjecture (unique ergodicity of almost all interval exchange transformations). Using combinatorics of Rauzy classes, Kontsevich and Zorich classified the connected components of strata of the moduli spaces of Abelian differentials [KZ03]. More recently, Avila, Gouëzel and Yoccoz proved the exponential decay of correlations for the Teichmüller geodesic flow also using a renormalization of the Rauzy-Veech induction (see [Zor96, AGY06]). Avila and Viana used combinatorics of Rauzy-Veech induction to prove the simplicity of the essential part of the Lyapunov spectrum of the Teichmüller geodesic flow on the strata of Abelian differentials (see [AV07]). Recently Bufetov and Gurevich proved the existence and uniqueness of the measure of maximal entropy for the Teichmüller geodesic flow on the moduli space of Abelian differentials [BG07]. Avila and Forni proved the weak mixing for almost all interval exchange transformations and translation flows [AF07].

These examples show that Rauzy-Veech induction which was initially elaborated to prove ergodicity of interval exchange transformations and ergodicity of the Teichmüller geodesic flow is, actually, very efficient far beyond these initial problems.

However, all the aforementioned results concern only the moduli space of Abelian differentials. The corresponding questions for strata of strict quadratic differentials (i.e. of those, which are not global squares of Abelian differentials) remain open.

Note that the (co)tangent bundle to the moduli space of curves is naturally identified with the moduli space of *quadratic* differentials. From this point of view, the strata of Abelian differentials represent special orbifolds of high codimension in the total space of the tangent bundle. Our interest in Teichmüller dynamics and geometry of the strata of strict quadratic differentials was one of the main motivations for developing Rauzy-Veech induction for quadratic differentials.

Natural generalizations of interval exchange transformations were introduced by Danthony and Nogueira in [DN88, DN90] (see also [Nog89]) as cross sections of measured foliations on surfaces. They introduced the notion of linear involutions, as well as the notion of Rauzy induction on these maps.

Studying Lyapunov spectrum of the Teichmüller geodesic flow Kontsevich and Zorich have performed series of computer experiments with linear involutions corresponding to quadratic differentials [KZ97]. These experiments indicated appearance of attractors for the Rauzy-Veech induction, as well as examples of generalized permutations such that the corresponding linear involutions are minimal for a domain of parameters of positive measure, and non minimal for a complementary domain of parameters also of positive measure (examples of this type are presented in Figure 14 and Figure 15 in Appendix A). But at this point, there was no combinatorial explanation.

Thus, in order to generalize technique of Rauzy-Veech induction to quadratic differentials in a consistent way it was necessary to find combinatorial criteria allowing to identify generalized permutations, which belong to attractors and those ones, which represents cross sections of vertical foliations of quadratic differentials. It was also necessary to distinguish those generalized permutation which give rise to minimal linear involution, and to specify the domains of appropriate parameters.

In this paper we establish corresponding combinatorial criteria, which enable us to develop technique of Rauzy-Veech induction for quadratic differentials. Partial results in this direction were obtained by the second author in [Lan04]. We also study relations between combinatorics, geometry and dynamics of linear involutions.

To compare similarities and differences between linear involutions corresponding to Abelian and to quadratic differentials let us first briefly review the situation in the classical case.

An interval exchange transformation is encoded by a combinatorial data (permutation π on d elements) and by a continuous data (lengths $\lambda_1, \dots, \lambda_d$ of the intervals). Recall that the Keane's property (see below) is a criterion of "irrationality" (which, in particular, implies minimality) of an interval exchange transformation. This property is satisfied for almost all parameters λ when the permutation π is irreducible (i.e. $\pi(\{1,\dots,k\}) \neq \{1,\dots,k\}, \quad 1 \leq k \leq d-1$), while when π is reducible, the corresponding interval exchange map is *never* minimal. On the other hand the irrational interval exchange maps are precisely those that arise as cross sections of minimal vertical flows on well chosen transverse intervals.

The Rauzy-Veech induction consists in taking the first return map of an interval exchange transformation to an appropriate smaller interval. This induction can be viewed as a dynamical system on a finite-dimensional space of interval exchange maps. The behavior of an orbit of the induction provides important information on dynamics of the interval exchange transformation representing the starting point. This information is especially useful when all iterates are well defined and when the length of the underlying subintervals tends to zero. An interval exchange transformation satisfying the latter conditions is said to have *Keane's property*. For a given irreducible permutation π, the subset of parameters λ which give rise to interval exchange transformations satisfying Keane's property contains all irrational parameters, and so it is a full Lebesgue measure subset. Moreover, for the space of interval exchange transformations with irreducible combinatorial data, the renormalized induction process is recurrent with respect to the Lebesgue measure (and even ergodic by a theorem of Veech). Note that the corresponding invariant measure has infinite total mass.

In this paper we use the definition of *linear involution* [1] proposed by Danthony and Nogueira (see [DN88, DN90]).

As above, a *linear involution* is encoded by a combinatorial data ("generalized permutation") and by continuous data. A generalized permutation of type (l, m) (with $l+m = 2d$) is a two-to-one map $\pi : \{1, \dots, 2d\} \to \mathcal{A}$ to an alphabet $\mathcal{A}$.

A generalized permutation is called *irreducible* if there exists a linear involution associated to this generalized permutation, which represents an appropriate cross section of the vertical foliation of some quadratic differential. A generalized permutation is called *dynamically irreducible* if there exists a minimal linear involution associated to this generalized permutation. It is easy to show that any irreducible generalized permutation is dynamically irreducible; the converse is not true in general as we will see.

Theorem A. *Irreducible and dynamically irreducible generalized permutations can be characterized by natural criteria expressed in elementary combinatorial terms.*

The corresponding criteria are stated as Definitions 3.1 and Definition 4.3 respectively.

Consider a dynamically irreducible generalized permutation π. The parameter space of normalized linear involutions associated to π is represented by a hyperplane section of a simplex. We describe an explicit procedure which associates to each generalized permutation π an open subset in the parameter space defined by a system of linear inequalities determined by π. This subset is called the set of *admissible parameters*. When π is irreducible, the set of admissible parameters coincides with entire parameter space; in general it is smaller. The next result gives a more precise statement than Theorem A in the dynamically irreducible case.

Theorem B.

(1) *If π is not dynamically irreducible, or if π is dynamically irreducible, but λ does not belong to the set of admissible parameters, the linear involution $T = (\pi, \lambda)$ is not minimal.*
(2) *If π is dynamically irreducible, then for almost all admissible parameters λ the linear involution $T = (\pi, \lambda)$ satisfies the Keane's property, and hence is minimal.*

Since the Rauzy-Veech induction commutes with dilatations, it projectivizes to a map $\mathcal{R}_r$ on the space of normalized linear involutions; we shall call this map the *renormalized Rauzy-Veech induction*.

Theorem C. *Let T be a linear involution on the unit interval and let us consider a sequence $\left(\mathcal{R}_r^{(n)}(T) = (\pi^{(n)}, \lambda^{(n)})\right)_{n\in\mathbb{N}}$ of iterates by the renormalized Rauzy-Veech induction $\mathcal{R}_r$.*

(1) *If T has the Keane's property, then there exists n_0 such that $\pi^{(n)}$ is irreducible for all $n \geq n_0$.*

[1] Let f be the involution of $X \times \{0, 1\}$ given by $f(x, \varepsilon) = (x, 1 - \varepsilon)$. A linear involution is a map T, from $X \times \{0, 1\}$ into itself, of the form $f \circ \tilde{T}$, where $\tilde{T}$ is an involution of $X \times \{0, 1\}$ without fixed point, continuous except in finitely many points, and which preserves the Lebesgue measure. In this paper we will only consider linear involutions with the additional condition. The derivative of $\tilde{T}$ is -1 at (x, ε) if (x, ε) and $T(x, \varepsilon)$ belong to the same connected component, and -1 otherwise; see also Convention 2.2.

(2) *The renormalized Rauzy-Veech induction, defined on the set* $\{(\pi, \lambda) \mid \pi$ *irreducible*$\}$, *is recurrent.*

Having a generalized permutation π we can define one or two other generalized permutations $\mathcal{R}_0(\pi)$ and $\mathcal{R}_1(\pi)$ reflecting the possibilities for the image of the Rauzy-Veech induction $\mathcal{R}(T)$. These combinatorial Rauzy operations define a partial order in the set of irreducible permutations represented by an oriented graph. A *Rauzy class* is a connected component of this graph.

Note that geometry of the Rauzy graphs is very different and more complicated than in the case of "true" permutations since for some irreducible generalized permutations one of the Rauzy operations might be not defined. From Theorem C we will deduce that a Rauzy class is an equivalence class for the equivalence relation given by these combinatorial operations (see Proposition 6.1).

In analogy with the case of the "true" permutations, we introduce one more combinatorial operation on generalized permutations and define *extended Rauzy classes* as minimal subsets of irreducible generalized permutations invariant under these corresponding three operations.

The moduli spaces of Abelian differentials and of quadratic differentials are stratified by multiplicities of the zeroes of the corresponding differentials. We denote a stratum of the moduli space of strict quadratic differentials (with at most simple poles) by $\mathcal{Q}(k_1, \dots, k_n)$, where $k_i \geq -1$ are the multiplicities of the zeroes ($k_i = -1$ corresponds to a pole).

Theorem D. *Extended Rauzy classes of irreducible generalized permutations are in one-to-one correspondence with connected components of strata in the moduli spaces of quadratic differentials.*

Historically, extended Rauzy classes where used to prove the non-connectedness of some strata of Abelian differentials. For permutations of a small number of elements, it is easy to construct explicitly the subset of irreducible permutations and then using the Rauzy operations to decompose it into a disjoint union of extended Rauzy classes. Using this approach Veech proved that the minimal stratum in genus 3 has two connected components and Arnoux proved that the minimal stratum in genus 4 has three connected components (for Abelian differentials).

Having established an explicit combinatorial criterion of irreducibility of a generalized permutation (namely Theorem A) one can apply Theorem D to classify the connected components of all strata of quadratic differentials of sufficiently small dimension. This justifies, in particular, the following experimental result of Zorich.

Theorem (Zorich)**.** *Each of the following four exceptional strata of quadratic differentials* $\mathcal{Q}(-1, 9)$, $\mathcal{Q}(-1, 3, 6)$, $\mathcal{Q}(-1, 3, 3, 3)$ *and* $\mathcal{Q}(12)$ *contains exactly two connected components.*

Note that a theorem of the second author [Lan04] classifies all connected components of all other strata of meromorphic quadratic differentials with at most simple poles. These strata are either connected, or contain exactly two connected components one of which being hyperelliptic. The same theorem [Lan04] proves that each of the remaining four exceptional strata might have at most two connected components. However, the only currently available proof of the fact they are disconnected is the one based on explicit calculation of the extended Rauzy classes and corresponds to the theorem of Zorich. It would be interesting to have an algebraic-geometrico proof of the last theorem; namely a topological invariant as in the Kontsevich-Zorich's classification [KZ03]. Note also that a paper of Zorich [Zor07] gives explicits representatives elements for each extended Rauzy class. See also [Zor06] for programs concerning calculations of these Rauzy classes.

Reader's guide. In Section 1 we recall basic properties of flat surfaces, moduli spaces and interval exchange maps. In particular we recall the Rauzy-Veech induction and its dynamical properties. We relate these properties to irreducibility.
In section 2 we recall the definition of a linear involution and give basics properties. Then in section 3 we define a combinatorial notion of irreducibility, and prove the first part of Theorem A. The main tool we use to prove this theorem is the presentation proposed by Marmi, Moussa and Yoccoz which appears in [MMY05]
In section 4 we introduce the Keane's property for the linear involutions and prove the second part of Theorem A, that is Theorem B. For that we prove that T satisfies the Keane's property if and only if the Rauzy-Veech induction is always well defined and the length parameters tends to zero. Then if T does not satisfy the Keane's property we show that there exists n_0 such that $\mathcal{R}^{n_0}(T)$ is dynamically reducible which then implies that T is also dynamically reducible.
In section 5, we study the dynamics of the renormalized Rauzy-Veech map on the space of the linear involutions, and prove Theorem C. For that we use the Teichmüller geometry and the finiteness of the volume of the strata proved by Masur and Veech (see [Mas82, Vee90]).
Section 6 is devoted to a proof of Theorem D on extended Rauzy classes; we present a result of Zorich based on an explicit calculation of these classes in low genera.
In the Appendix we present some explicit Rauzy classes as illustration of the problems which appear in the general case. We also give a property concerning the extended Rauzy classes.

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1. Background

In this section we review basic notions concerning flat surfaces, moduli spaces and interval exchange maps. For general references see say [Mas82, Rau79, Vee78, Vee82, Zor96] and [MT02]. In this paper we will mostly follows notations presented in the paper [MMY05], or equivalently [Yoc03].

1.1. **Flat surfaces.** A *flat surface* is a (real, compact, connected) genus g surface equipped with a flat metric (with isolated conical singularities) such that the holonomy group belongs to $\{\pm \mathrm{Id}\}$. Here holonomy means that the parallel transport of a vector along a long loop brings the vector back to itself or to its opposite. This implies that all cone angles are integer multiples of π. Equivalently a flat surface is a triple $(S,\mathcal{U},\Sigma)$ such that S is a topological compact connected surface, Σ is a finite subset of S (whose elements are called *singularities*) and $\mathcal{U}=\{(U_i,z_i)\}$ is an atlas of $S\setminus\Sigma$ such that the transition maps $z_j\circ z_i^{-1}: z_i(U_i\cap U_j)\rightarrow z_j(U_i\cap U_j)$ are translations or half-turns: $z_i=\pm z_j+const$, and for each $s\in\Sigma$, there is a neighborhood of s isometric to a Euclidean cone. Therefore we get a *quadratic differential* defined locally in the coordinates z_i by the formula $q=dz_i^2$. This form extends to the points of Σ to zeroes, simple poles or marked points (see [MT02]). We will sometimes use the notation (S,q) or simply S.

Observe that the holonomy is trivial if and only if there exists a sub-atlas such that all transition functions are translations or equivalently if the quadratic differentials q is the global square of an Abelian differential. We will then say that S is a translation surface.

1.2. **Moduli spaces.** For $g\geq 1$, we define the moduli space of Abelian differentials $\mathcal{H}_g$ as the set of pairs (S,ω) modulo the equivalence relation generated by: $(S,\omega)\sim(S',\omega')$ if there exists an analytic isomorphism $f:S\rightarrow S'$ such that $f^*\omega'=\omega$.
For $g\geq 0$, we also define the moduli space of quadratic differentials $\mathcal{Q}_g$ as the moduli space of pairs (S,q) (where q is not the global square of any Abelian differential) modulo the equivalence relation generated by: $(S,q)\sim(S',q')$ if there exists an analytic isomorphism $f:S\rightarrow S'$ such that $f^*q'=q$.

The moduli space of Abelian differentials (respectively quadratic differentials) is stratified by the multiplicities of the zeroes. We will denote by $\mathcal{H}(k_1,\dots,k_n)$ (respectively $\mathcal{Q}(k_1,\dots,k_n)$) the stratum consisting of holomorphic one-forms (respectively quadratic differentials) with n zeroes (or poles) of multiplicities $(k_1,\dots,k_n)$. These strata are non-connected in general (for a complete classification see [KZ03] in the Abelian differentials case and [Lan04] in the quadratic differentials case).

The linear action of the 1-parameter subgroup of diagonal matrices $g_t:=\mathrm{diag}(e^{t/2},e^{-t/2})$ on the flat surfaces presents a particular interest. It gives a measure-preserving flow with respect to a natural measure $\mu^{(1)}$, preserving each stratum of area one flat surfaces. This flow is known as the *Teichmüller geodesic flow.* Masur and Veech proved the following theorem.

Theorem (Masur; Veech)**.** *The Teichmüller geodesic flow acts ergodically on each connected component of each stratum of the moduli spaces of area one Abelian and quadratic differentials (with respect to a finite measure in the Lebesgue class).*

This theorem was proved by Masur [Mas82] and Veech [Vee82] for the $\mathcal{H}(k_1,\dots,k_n)$ case and for the $\mathcal{Q}(4g-4)$ case.
The ergodicity of the Teichmüller geodesic flow is proved in full generality in [Vee86], Theorem 0.2. The finiteness of the measure appears in two 1984 preprints of Veech: Dynamical systems on analytic manifolds of quadratic differentials I,II (see also [Vee86] p.445). These preprints have been published in 1990 [Vee90].

1.3. **Interval exchange maps.** In this section we recall briefly the theory of interval exchange maps. We will show that, under simple combinatorial conditions, such transformations arise naturally as Poincaré return maps of measured foliations and geodesic flows on translation surfaces. Moreover we will present the Rauzy-Veech induction and its geometric and dynamical properties (see [Vee82] for more details).

Let $I \subset \mathbb{R}$ be an open interval and let us choose a finite subset $\{sing\}$ of I. Its complement is a union of $d \geq 2$ open subintervals. An interval exchange map is a one-to-one map T from $I \backslash \{sing\}$ to a co-finite subset of I that is a translation on each subinterval of its definition domain. It is easy to see that T is precisely determined by the following data: a permutation $\overline{\pi}$ that encodes how the intervals are exchanged (expressing that the k-th interval, when numerated from the left to the right, is sent by T to the place $\overline{\pi}(k)$), and a vector with positive entries that encodes the lengths of the intervals.

Following Marmi, Moussa, Yoccoz [MMY05], we denote these intervals by $\{I_\alpha,\ \alpha \in \mathcal{A}\}$, with $\mathcal{A}$ a finite alphabet. The length of the intervals is a vector $\lambda = (\lambda_\alpha)_{\alpha \in \mathcal{A}}$, and the combinatorial data is a pair $\pi = (\pi_0, \pi_1)$ of one-to-one maps $\pi_\epsilon : \mathcal{A} \to \{1, \ldots, d\}$. Then $\overline{\pi}$ is a one-to-one map from $\{1, \ldots, d\}$ into itself given by $\overline{\pi} = \pi_1 \circ \pi_0^{-1}$. We will usually represent such a permutation by a table:

$$\overline{\pi} = \begin{pmatrix} 1 & 2 & \ldots & n \\ \overline{\pi}^{-1}(1) & \overline{\pi}^{-1}(2) & \ldots & \overline{\pi}^{-1}(n) \end{pmatrix} = \\ = \begin{pmatrix} \pi_0^{-1}(1) & \pi_0^{-1}(2) & \ldots & \pi_0^{-1}(n) \\ \pi_1^{-1}(1) & \pi_1^{-1}(2) & \ldots & \pi_1^{-1}(n) \end{pmatrix}.$$

Example 1.1. Let us consider the following alphabet $\mathcal{A} = \{A, B, C, D\}$ with $d = 4$. Then we define a permutation π as follows.

$$\overline{\pi} = \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}.$$

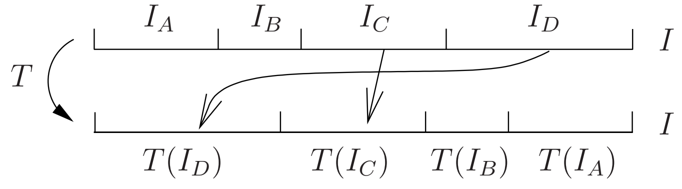


FIGURE 1. An interval exchange map.

1.3.1. *Rauzy-Veech induction.* In this section we introduce the notion of winner and loser, following the terminology of the paper of Avila, Gouëzel and Yoccoz [AGY06]. For $T = (\pi, \lambda)$ we define the *type* ε of T by $\lambda_{\pi_\varepsilon^{-1}(d)} > \lambda_{\pi_{1-\varepsilon}^{-1}(d)}$. We will then say that $I_{\pi_\varepsilon^{-1}(d)}$ is the winner and $I_{\pi_{1-\varepsilon}^{-1}(d)}$ is the loser. Then we define a subinterval J of I by removing the loser of I as follows.

$$\begin{cases} J = I \backslash T(I_{\pi_1^{-1}(d)}) & \text{if } T \text{ is of type } 0 \\ J = I \backslash I_{\pi_0^{-1}(d)} & \text{if } T \text{ is of type } 1. \end{cases}$$

The Rauzy-Veech induction $\mathcal{R}(T)$ of T is defined as the first return map of T to the subinterval J. It is easy to see that this is again an interval exchange transformation, defined on d letters (see e.g. [Rau79]). We now see how to compute the data of the new map.

There are two cases to distinguish depending on whether T is of type 0 or 1; the combinatorial data of $\mathcal{R}(T)$ only depends on π and on the type of T. This defines two maps $\mathcal{R}_0$ and $\mathcal{R}_1$ by $\mathcal{R}(T) = (\mathcal{R}_\varepsilon(\pi), \lambda')$, with ε the type of T.

(1) T has type 0; equivalently the winner is $I_{\pi_0^{-1}(d)}$.
In that case, we define k by $\pi_1^{-1}(k) = \pi_0^{-1}(d)$ where $k \leq d-1$. In an equivalent way $k = \pi_1 \circ \pi_0^{-1}(d) = \overline{\pi}(d)$. Then $\mathcal{R}_0(\pi_0, \pi_1) = (\pi_0', \pi_1')$ where $\pi_0 = \pi_0'$ and

$$\pi_1'^{-1}(j) = \begin{cases} \pi_1^{-1}(j) & \text{if } j \leq k \\ \pi_1^{-1}(d) & \text{if } j = k+1 \\ \pi_1^{-1}(j-1) & \text{otherwise.} \end{cases}$$

We have $\lambda'_\alpha = \lambda_\alpha$ if $\alpha \neq \pi_0^{-1}(d)$ and $\lambda'_{\pi_0^{-1}(d)} = \lambda_{\pi_0^{-1}(d)} - \lambda_{\pi_1^{-1}(d)}$.

(2) T has type 1; equivalently the winner is $I_{\pi_1^{-1}(d)}$.
In that case, we define k by $\pi_0^{-1}(k) = \pi_1^{-1}(d)$ where $k \leq d-1$. In an equivalent way $k = \pi_0 \circ \pi_1^{-1}(d) = \overline{\pi}^{-1}(d)$. Then $\mathcal{R}_1(\pi_0, \pi_1) = (\pi_0', \pi_1')$ where $\pi_1 = \pi_1'$ and

$$\pi_0'^{-1}(j) = \begin{cases} \pi_0^{-1}(j) & \text{if } j \leq k \\ \pi_0^{-1}(d) & \text{if } j = k+1 \\ \pi_0^{-1}(j-1) & \text{otherwise.} \end{cases}$$

We have $\lambda'_\alpha = \lambda_\alpha$ if $\alpha \neq \pi_1^{-1}(d)$ and $\lambda'_{\pi_1^{-1}(d)} = \lambda_{\pi_1^{-1}(d)} - \lambda_{\pi_0^{-1}(d)}$.

Example 1.2. Let $\mathcal{A} = \{A, B, C, D\}$ be an alphabet. Let us consider the permutation π of Example 1.1. Then

$$\mathcal{R}_0\pi = \begin{pmatrix} A & B & C & D \\ D & A & C & B \end{pmatrix} \qquad \text{and} \qquad \mathcal{R}_1\pi = \begin{pmatrix} A & D & B & C \\ D & C & B & A \end{pmatrix}.$$

We stress that the Rauzy-Veech induction is well defined if and only if the two rightmost intervals do not have the same length i.e. $\lambda_{\pi_0^{-1}(d)} \neq \lambda_{\pi_1^{-1}(d)}$. In the next, we want to study the Rauzy-Veech induction as a dynamical system defined on the space of interval exchange transformations. Thus we want the iterates of the Rauzy-Veech induction on T to be always well defined. We also want this induction to be a good renormalization process, in the sense that the iterates correspond to inductions on subintervals that tend to zero. This leads to the definition of reducibility and to the Keane's property.

1.3.2. *Rauzy-Veech induction and Keane's property.* We will say that $\pi = (\pi_0, \pi_1)$ is reducible if there exists $1 \leq k \leq d-1$ such that $\{1, \ldots, k\}$ is invariant under $\overline{\pi} = \pi_1 \circ \pi_0^{-1}$. This means exactly that T splits into two interval exchange transformations.

We will say that T satisfies the Keane's property (also called the infinite distinct orbit condition or i.d.o.c. property), if the orbits of the singularities of T^{-1} by T are infinite. This ensures that π is irreducible and the iterates of the Rauzy-Veech induction are always well defined.

If the λ_α are rationally independent vectors, that is $\sum r_\alpha \lambda_\alpha \neq 0$ for all nonzero integer vectors (r_α), then T satisfies the Keane's property (see [Kea75]). However the converse is not true. Note that if T satisfies the Keane's property then T is minimal.

Let $T = (\pi, \lambda)$ be an interval exchange map. Let us denote by $\lambda_\alpha^{(n)}$ the length of the interval associated to the symbol $\alpha \in \mathcal{A}$ for the n-th iterate of T by $\mathcal{R}$; we denote $\mathcal{R}^n(T) =: (\pi^{(n)}, \lambda^{(n)})$ if it is well defined.

Proposition. *The following are equivalent.*

1. *T satisfies the Keane's property.*
2. *The Rauzy-Veech induction $\mathcal{R}$ is always well-defined and for any $\alpha \in \mathcal{A}$, the length of the intervals $\lambda_\alpha^{(n)}$ goes to zero as n tends to infinity.*

As we will see this situation is very similar in the case of linear involutions.

If we want to study the Rauzy-Veech induction as a dynamical system on the space of interval exchange maps, it is useful to consider the Rauzy-Veech renormalisation on the projective space of lengths parameters space. The natural associated object is the renormalized Rauzy-Veech induction defined on length one intervals:

$$\text{if } \mathcal{R}(\pi, \lambda) = (\pi', \lambda') \text{ then } \mathcal{R}_r(\pi, \lambda) := (\pi', \lambda'/|\lambda'|).$$

1.3.3. *Rauzy classes.* Given a permutation π, we can define two other permutations $\mathcal{R}_\varepsilon(\pi)$ with $\varepsilon = 0, 1$. Conversely, any permutation π' has exactly two predecessors: there exist exactly two permutations π^0 and π^1 such that $\mathcal{R}_\varepsilon(\pi^\varepsilon) = \pi'$. Note that π is irreducible if and only if $\mathcal{R}_\varepsilon(\pi)$ is irreducible. Thus the relation generated by $\pi \sim \mathcal{R}_\varepsilon(\pi)$ is a partial order on the set of irreducible permutations; we represent it as a directed graph G. We call Rauzy classes the connected components of this graph.

Proposition (Rauzy)**.** *The above relation is an equivalence relation on the set of permutations. In particular, the equivalent class of a permutation is the Rauzy class.*

Proof. The key remark is the following: if $\pi' = \mathcal{R}_\varepsilon(\pi)$ then there exists $n > 0$ such that $\pi = \mathcal{R}_\varepsilon^n(\pi')$. Now assume there exists an oriented path in G joining π and π', *i.e.* there exist $\varepsilon_1, \ldots, \varepsilon_r$ such that $\pi' = \mathcal{R}_{\varepsilon_1} \circ \cdots \circ \mathcal{R}_{\varepsilon_r}(\pi)$. Then there exists n_1 such that $\mathcal{R}_{\varepsilon_1}^{n_1}(\pi') = \mathcal{R}_{\varepsilon_2} \circ \cdots \circ \mathcal{R}_{\varepsilon_r}(\pi)$. Iterating this argument, there exist $n_1, \ldots, n_r$ such that $\pi = \mathcal{R}_{\varepsilon_r}^{n_r} \circ \cdots \circ \mathcal{R}_{\varepsilon_1}^{n_1}(\pi')$. Thus there is an oriented path in G that joins π' and π. □

We will see that there is an analogous proposition in the case of generalized permutations although the situation is much more complicated.

1.3.4. *Suspension data over an interval exchange transformation.* Here we describe the construction of a *suspension* over an interval exchange map T, that is a flat surface for which T is the first return map of the vertical flow on a well chosen segment.

Let $T = (\pi, \lambda)$ be an interval exchange transformation. A *suspension data* for T is a collection of vectors $(\zeta_\alpha)_{\alpha \in \mathcal{A}}$ such that:

(1) $\forall \alpha \in \mathcal{A},\ Re(\zeta_\alpha) = \lambda_\alpha$.
(2) $\forall 1 \leq k \leq d-1,\ Im(\sum_{\pi_0(\alpha) \leq k} \zeta_\alpha) > 0$.
(3) $\forall 1 \leq k \leq d-1,\ Im(\sum_{\pi_1(\alpha) \leq k} \zeta_\alpha) < 0$.

Given a suspension datum ζ, we consider the broken line L_0 on $\mathbb{C} = \mathbb{R}^2$ defined by concatenation of the vectors $\zeta_{\pi_0^{-1}(j)}$ (in this order) for $j = 1, \dots, d$ with starting point at the origin (see Figure 2). Similarly, we consider the broken line L_1 defined by concatenation of the vectors $\zeta_{\pi_1^{-1}(j)}$ (in this order) for $j = 1, \dots, d$ with starting point at the origin. If the lines L_0 and L_1 have no intersections other than the endpoints, then we can construct a translation surface S as follows: we can identify each side ζ_α on L_0 with the side ζ_α on L_1 by a translation (in the general case, we must use the Veech zippered rectangle construction, see section 1.3.5). Let $I \subset S$ be the horizontal interval defined by $I = (0, \sum_{\alpha \in \mathcal{A}} \lambda_\alpha) \times \{0\}$. Then the interval exchange map T is precisely the one defined by the first return map to I of the vertical flow on S.

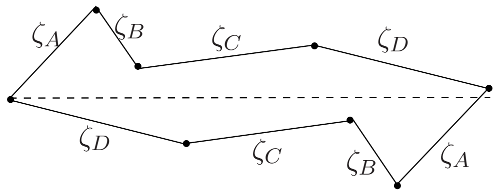


FIGURE 2. Suspension over an interval exchange transformation.

We have not yet discussed the existence of such a suspension datum for a general interval exchange map. A necessary condition for T to have suspension data is that π is irreducible. Indeed, if we have $1 \leq k \leq d-1$ such that $\pi_1 \circ \pi_0^{-1}(\{1, \dots, k\}) = \{1, \dots, k\}$, and let $\zeta = (\zeta_\alpha)_\alpha$ be a collection of complex numbers, then:

$$\sum_{\pi_0(\alpha) \leq k} \zeta_\alpha = \sum_{\pi_1(\alpha) \leq k} \zeta_\alpha.$$

So the imaginary part of this number cannot be both positive and negative, and ζ is not a suspension data for T. If π is irreducible, the existence of a suspension data is given by Masur and Veech independently (see [Mas82] page 174 and [Vee82] formula 3.7 page 207). We explain the construction here.

First, let us remark that $\pi = (\pi_0, \pi_1)$ is irreducible if and only if

$$\sum_{i=1}^{k} \pi_1 \circ \pi_0^{-1}(i) - i > 0 \qquad \text{for any } 1 \leq k \leq d-1. \tag{1}$$

Of course if π is irreducible, then so is π^{-1}, therefore

$$\sum_{i=1}^{k} \pi_0 \circ \pi_1^{-1}(i) - i > 0 \qquad \text{for any } 1 \leq k \leq d-1. \tag{2}$$

Let us define a collection of complex number $\zeta = (\zeta_\alpha)_\alpha$ as follows:

$$\zeta_\alpha = \lambda_\alpha + i(\pi_1(\alpha) - \pi_0(\alpha)) \qquad \text{for any } \alpha \in \mathcal{A}.$$

Then following (1) and (2), the collection $(\zeta_\alpha)_{\alpha\in\mathcal{A}}$ is a suspension data over T.

1.3.5. *Zippered rectangles.* Here we describe an alternative construction of the suspension over an interval exchange transformation that works for *any* suspension data, namely the so called zippered rectangles construction due to Veech [Vee82]. Let $T = (\pi, \lambda)$ be an interval exchange map, and let us assume that π is irreducible. Let ζ be any suspension over T. Then we define $h = (h_\alpha)_{\alpha\in\mathcal{A}}$ by

$$h_\alpha = \sum_{\pi_0(\beta)<\pi_0(\alpha)} Im(\zeta_\beta) - \sum_{\pi_1(\beta)<\pi_1(\alpha)} Im(\zeta_\beta) > 0.$$

For each $\alpha \in \mathcal{A}$ let us consider a rectangle R_α of width $Re(\zeta_\alpha)$ and of height h_α based on $I_{\pi_0(\alpha)} \subset I$. The zippered rectangle construction is the translation surface $\bigcup_{\alpha\in\mathcal{A}} R_\alpha / \sim$ where $\sim$ is the following equivalence relation: we identify the top and the bottom of these rectangles by $(x, h_\alpha) \sim (T(x), 0)$ for $x \in I_{\pi_0(\alpha)}$. Then we "zip" the vertical boundaries of these rectangles that are adjacent (see figure 3; see also [Vee82] for a more precise description).

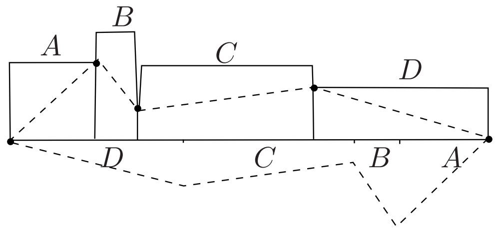


FIGURE 3. Zippered rectangles construction.

1.3.6. *Rauzy-Veech induction on suspensions.* We can define the Rauzy-Veech induction on the space of suspensions, as well as on the space of zippered rectangles. Let $T = (\pi, \lambda)$ be an interval exchange map and let ζ be a suspension over T. Then we define $\mathcal{R}(\pi, \zeta) = (\pi', \zeta')$ as follows.

We define $(\pi', Re(\zeta')) = \mathcal{R}(\pi, Re(\zeta))$ (the standard Rauzy-Veech induction). If $I_{\pi_\varepsilon^{-1}(d)}$ is the winner for $T = (\pi, Re(\zeta))$ then

$$\begin{cases} \zeta'_{\pi_\varepsilon^{-1}(d)} = \zeta_{\pi_\varepsilon^{-1}(d)} - \zeta_{\pi_{1-\varepsilon}^{-1}(d)} \\ \zeta'_\alpha = \zeta_\alpha \qquad \text{otherwise.} \end{cases}$$

Remark 1.3. Since (π', ζ') is obtained from (π, ζ) by "cutting" and "gluing", these two surfaces differ by an element of the mapping class group, hence they define the same point in the moduli space (see Figure 4 for an example).

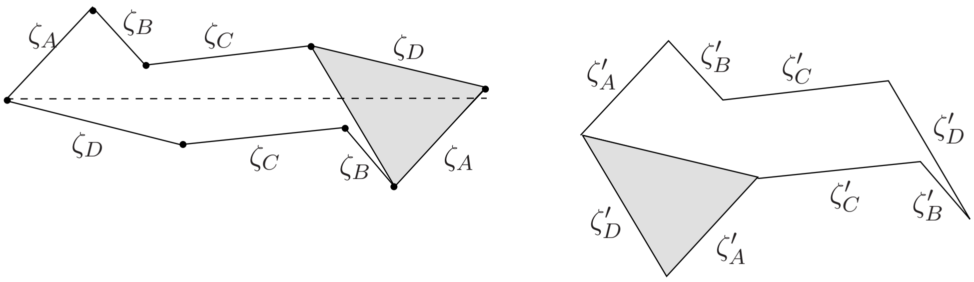


FIGURE 4. Rauzy-Veech induction on a suspension over an interval exchange transformation. The corresponding map is of type 0 hence the new suspension data are $\zeta'_A = \zeta_A$, $\zeta'_B = \zeta_B$, $\zeta'_C = \zeta_C$ and $\zeta'_D = \zeta_D - \zeta_A$.

If C is a Rauzy class, we define

$$\mathcal{T}_C = \{(\pi, \zeta),\ \pi \in C, \zeta \text{ is a suspension data for } \pi\}.$$

We have thus defined the Rauzy-Veech map on the space $\mathcal{T}_C$. It is easy to check that it defines an almost everywhere invertible map: If $\sum Im(\zeta'_\alpha) \neq 0$ then every (π', ζ') has exactly one preimage for $\mathcal{R}$.

1.3.7. *Moduli spaces and Rauzy-Veech induction.* We define the quotient $\mathcal{H}_C = \mathcal{T}_C / \sim$ of $\mathcal{T}_C$ by the equivalence relation generated by $(\pi, \zeta) \sim \mathcal{R}(\pi, \zeta)$. The zippered rectangle construction, provides a mapping p from $\mathcal{H}_C$ to a stratum $\mathcal{H}(k_1, \ldots, k_n)$ of the moduli space of Abelian differentials (see Remark 1.3). Observe that $(k_1, \ldots, k_n)$ can be calculated in terms of $C \ni \pi$. One can also show that $\mathcal{H}_C$ is connected and so the image belongs to a connected component of a stratum.

We will denote by m the natural Lebesgue measure on $\mathcal{T}_C$ i.e. $m = d\pi d\zeta$, were $d\pi$ is the counting measure on C and $d\zeta$ is the Lebesgue measure. The mapping $\mathcal{R}$ preserves m, so it induces a measure, denoted again by m on $\mathcal{H}_C$.

There is natural action of the matrix

$$g_t = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}$$

on $\mathcal{T}_C$ by $g_t(\pi, \zeta) = (\pi, (g_t(\zeta_\alpha))_\alpha)$, where g_t acts on $\zeta_\alpha \in \mathbb{C} = \mathbb{R}^2$ linearly. This action preserves the measure m on $\mathcal{T}_C$ and commutes with $\mathcal{R}$, so it descends to a 1-parameter action on $\mathcal{H}_C$ called the *Teichmüller flow*. Since the action of g_t on $\mathcal{H}_C$ preserves the area of the corresponding flat surface, the Teichmüller flow also acts on the subset $\mathcal{H}^1_C$ corresponding to area one surfaces, and preserves the measure $m^{(1)}$ induced by the measure m on that subset. Note also that

$$\Big\{(\pi, \zeta) \in \mathcal{T}_C;\ 1 \leq |Re(\zeta)| \leq 1 + \min\big(Re(\zeta_{\pi_0^{-1}(d)}), Re(\zeta_{\pi_1^{-1}(d)})\big)\Big\}$$

is a fundamental domain of $\mathcal{T}_C$ for the relation $\sim$ and the Poincaré map of the Teichmüller flow on

$$\mathcal{S} = \{(\pi,\zeta);\ \pi \text{ irreducible},\ |Re(\zeta)| = 1\}/\sim$$

is precisely the renormalized Rauzy-Veech induction on suspensions.

One can show (see [Vee82]) that the mapping p is a finite covering from $\mathcal{H}^1_C$ onto a subset of full measure in a connected component of a stratum and the measure m projects to the measure $\mu^{(1)}$ defined in section 1.2. Moreover the action of g_t is equivariant with respect to p, that is $p \circ g_t(\pi,\zeta) = g_t \circ p(\pi,\zeta)$. Hence if we restrict to area one surfaces, the result of Masur and Veech (finiteness of the measure) implies that the measure $m^{(1)}$ is finite on $\mathcal{H}^1_C$.

Corollary 1.4. *The renormalized Rauzy-Veech induction is recurrent on $\mathcal{S}$.*

Remark 1.5. Veech proved a stronger result, that is the ergodicity of g_t (on the level of $\mathcal{H}_C$ for any Rauzy class C), which implies the ergodicity of the Teichmüller flow for Abelian differentials (see [Vee82]). He also proved that the induced measure on $\mathcal{S}$ is always infinite.

2. Linear involutions

2.1. **linear involutions and generalized permutations.** Let S be a (compact, connected, oriented) flat surface with $\mathbb{Z}/2\mathbb{Z}$ linear holonomy and let X be a horizontal segment with a choice of a positive vertical direction (or equivalently, a choice of left and right ends). We consider the first return map $T_0 : X \to X$ of vertical geodesics starting from X in the positive direction. Any vertical geodesic which start from X and doesn't hit a singularity will intersect X again. Therefore, the map T_0 is well defined outside a finite number of points $\{sing\}$ (called singular points) that correspond to vertical geodesics that stop at a singularity before intersecting again the interval X. The set $X\backslash\{sing\}$ is a finite union of open intervals (X_i) and the restriction of T_0 on each of these intervals is of the kind $x \mapsto \pm x + c_i$.

The map T_0 alone does not properly correspond to the dynamics of vertical geodesics since when $T_0(x) = -x + c_i$ on the interval X_i, then $T_0^2(x) = x$, and $(x, T_0(x), T_0^2(x))$ does not correspond to the successive intersections of a vertical geodesic with X starting from x. To fix this problem, we have to consider T_1 the first return map of the vertical geodesics starting from X in the negative direction. Now if $T_0(x) = -x + c_i$ then the successive intersections with X of the vertical geodesic starting from x will be $x, T_0(x), T_1(T_0(x))\dots$

We get a dynamical system on $X\times\{0,1\}$. Following Danthony and Nogueira (see [Nog89, DN88, DN90]) we will call such a dynamical system a linear involution. We recall here the definition that we have restricted to our purpose.

Definition 2.1. Let X be an open interval and let $\widehat{X} = X \times \{0,1\}$ be two disjoint copies of X. A linear involution on X is a map $T := f \circ \tilde{T}$, where:

- $\tilde{T}$ is a smooth involution without fixed point defined on $\widehat{X}\backslash\{sing\}$, where $\{sing\}$ is a finite subset of $\widehat{X}$.

- If $p = (x,\varepsilon)$ and $T(p)$ belong to the same connected component of $\widehat{X}$ then the derivative of $\tilde{T}$ at p is -1 otherwise the derivative of $\tilde{T}$ at p is 1.
- f is the involution $(x,\varepsilon) \mapsto (x, 1-\varepsilon)$.

Convention 2.2. *In this paper, we are interested with non oriented measured foliations defined on oriented surfaces. Observe that the orientability of the surface S forces the second condition on the derivative of T in Definition 2.1.*

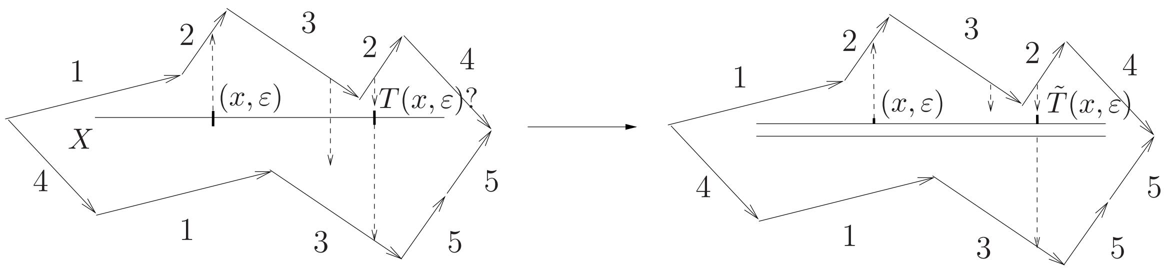


FIGURE 5. Linear involution defined by the vertical foliation of a flat surface.

The previous definition is motivated by the following remark.

Remark 2.3. The first return map of the vertical geodesic foliation on a horizontal segment X in a flat surface S defines a linear involution in the following way. Choose a positive vertical direction in a neighborhood of X, and replace X be two copies of X as in Figure 5. We denote by $X \times \{0\}$ the one on the top and by $X \times \{1\}$ the one on the bottom. Then we consider the first return map on $X \times \{0,1\}$ of vertical geodesics, where a geodesic starting from $X \times \{0\}$ is taken in the positive vertical direction, and a geodesic starting from $X \times \{1\}$ is taken in the negative direction. We obtain a map $\tilde{T}$ and it is easy to check that $\tilde{T}$ satisfies the condition of Definition 2.1. Then it is clear that the map $T = f \circ \tilde{T}$ encodes the successive intersections of a vertical geodesic with X.

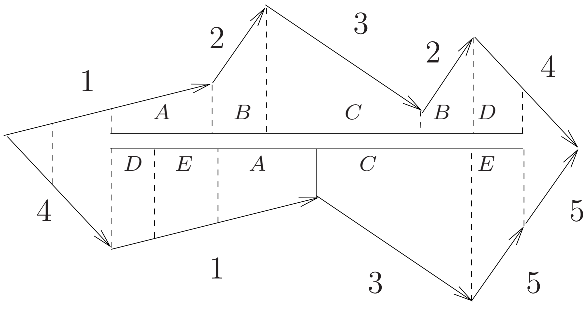
Recall that interval exchange maps are encoded by combinatorial and metric data: these are a permutation and a vector with positive entries. We define an analogous object for linear involutions.

Definition 2.4. Let $\mathcal{A}$ be an alphabet of d letters. A *generalized permutation* of type (l,m), with $l+m = 2d$, is a two-to-one map $\pi : \{1,\dots,2d\} \to \mathcal{A}$. We will usually represent such generalized permutation by the table:

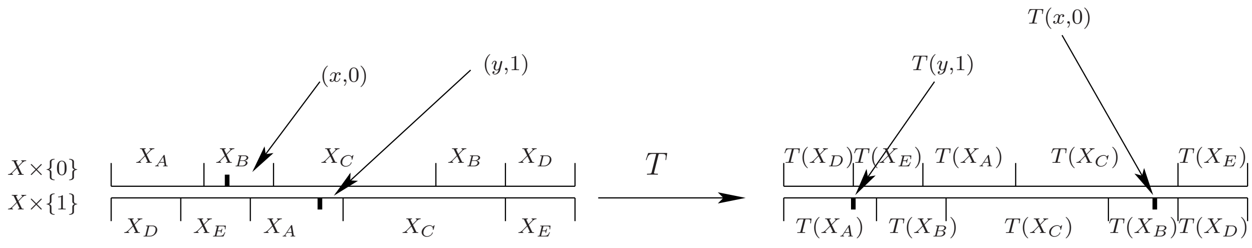
$$\begin{pmatrix} \pi(1) & \dots & \pi(l) \\ \pi(l+1) & \dots & \pi(l+m) \end{pmatrix}.$$

A generalized permutation π defines an involution σ without fixed points by the following way

$$\pi^{-1}(\{\pi(i)\}) = \{i, \sigma(i)\}.$$



(a)



(b)

FIGURE 6. A linear involution associated to a measured foliation on a flat surface.

Note that a permutation defines in a natural way a generalized permutation.

We now describe how a linear involution naturally defines a generalized permutation. Let T be a linear involution and let $\tilde{T}$ be the corresponding involution as in Definition 2.1. The domain of definition of $\tilde{T}$ is a finite union $X_1, \dots, X_{l+m}$ of open intervals, where $X_1, \dots, X_l$ are subintervals of $X \times \{0\}$ and $X_{l+1}, \dots, X_{l+m}$ are subintervals of $X \times \{1\}$. Since $\tilde{T}$ is an isometric involution without fixed point, each X_i is mapped isometrically to a X_j, with $j \neq i$, hence $\tilde{T}$ induces an involution without fixed point σ_T on $\{1, \dots, l+m\}$. As in section 1, we choose a name $\alpha_i \in \mathcal{A}$ to each pair $\{i, \sigma_T(i)\}$ and we get a generalized permutation in the sense of the above definition which is defined up to a one-to-one map of $\mathcal{A}$.

Example 2.5. In view of Figure 6, let us consider the following alphabet $\mathcal{A} = \{A, B, C, D\}$ with $d = 5$. Then we define a generalized permutation π as follows.

$$l = m = 5, \quad \pi(1) = \pi(8) = A,\ \pi(2) = \pi(4) = B,$$
$$\pi(3) = \pi(9) = C,\ \pi(5) = \pi(6) = D,\ \pi(7) = \pi(10) = E.$$

In an equivalent way, we can define an involution without fixed point in order to define π.

$$\sigma(1) = 8,\ \sigma(2) = 4,\ \sigma(3) = 9,\ \sigma(5) = 6,\ \sigma(7) = 10.$$

We represent π by the following table

$$\pi = \begin{pmatrix} A & B & C & B & D \\ D & E & A & C & E \end{pmatrix}.$$

One can check that the discrete datum associated to the linear involution described in Figure 6 is the generalized permutation π.

Example 2.6. Note that π is a "true" permutation on d letters if and only if $l = m = d$ and for any $i \leq l$, $\sigma(i) > l$. In this case (if $\mathcal{A} = \{1, \dots, d\}$):

$$\pi = \begin{pmatrix} 1 & 2 & \dots & d \\ \sigma(1) - d & \sigma(2) - d & \dots & \sigma(d) - d \end{pmatrix}.$$

Conversely, let π be a generalized permutation of type (l, m) and let σ be the associated involution. If π is not a "true" permutation, then an obvious necessary and sufficient condition for π to come from a linear involution is that there exist at least two indices $i \leq l$ and $j > l$ such that $\sigma(i) \leq l$ and $\sigma(j) > l$.

Convention 2.7. *From now, unless explicitly stated (in particular in section 3.2), we will always assume that generalized permutations will satisfy the following convention. There exist at least two indices $i \leq l$ and $j > l$ such that $\sigma(i) \leq l$ and $\sigma(j) > l$.*

Let $(\lambda_\alpha)_{\alpha \in \mathcal{A}}$ be a collection of positive real numbers such that

$$L := \sum_{i=1}^{l} \lambda_{\pi(i)} = \sum_{i=l+1}^{l+m} \lambda_{\pi(i)}. \tag{3}$$

It is easy to construct a linear involution on the interval $X = (0, L)$ with combinatorial data (π, λ). As in section 1, we will denote by $T = (\pi, \lambda)$ a linear involution.

2.2. **Rauzy-Veech induction on linear involutions.** We recall the *Rauzy-Veech induction* on linear involutions introduced by Danthony and Nogueira (see [DN90] p. 473).

Let $T = (\pi, \lambda)$ be a linear involution on $X = (0, L)$, with π of type (l, m). If $\lambda_{\pi(l)} \neq \lambda_{\pi(l+m)}$, then the Rauzy-Veech induction $\mathcal{R}(T)$ of T is the linear involution obtained by the first return map of T to

$$\big(0, \max(L - \lambda_{\pi(l)}, L - \lambda_{\pi(l+m)})\big) \times \{0, 1\}.$$

As in the case of interval exchange maps, the combinatorial data of the new linear involution depends only on the combinatorial data of T and whether $\lambda_{\pi(l)} > \lambda_{\pi(l+m)}$ or $\lambda_{\pi(l)} < \lambda_{\pi(l+m)}$. As before, we say that T has type 0 or type 1 respectively. The corresponding combinatorial operations are denoted by $\mathcal{R}_\varepsilon$ for $\varepsilon = 0, 1$ respectively. Note that if π is a given generalized permutation, the subsets $\{T = (\pi, \lambda),\ \lambda_{\pi(l)} > \lambda_{\pi(l+m)}\}$ and $\{T = (\pi, \lambda), \lambda_{\pi(l)} < \lambda_{\pi(l+m)}\}$ can be empty because $\pi(l) = \pi(l+m)$ or because of the linear relation on the λ_i that must be satisfied.

We first describe the combinatorial Rauzy operations $\mathcal{R}_\varepsilon$. Let σ be the associated involution to π.

(1) map $\mathcal{R}_0$.
• If $\sigma(l) > l$ and if $\pi(l) \neq \pi(l+m)$ then we define $\mathcal{R}_0\pi$ to be of type (l,m) and such that:

$$\mathcal{R}_0\pi(i) = \begin{cases} \pi(i) & \text{if } i \leq \sigma(l) \\ \pi(l+m) & \text{if } i = \sigma(l)+1 \\ \pi(i-1) & \text{otherwise.} \end{cases}$$

• If $\sigma(l) \leq l$, and if there exists a pair $\{x, \sigma(x)\}$ included in $\{l+1, \dots, l+m-1\}$ then we define $\mathcal{R}_0\pi$ to be of type $(l+1, m-1)$ and such that:

$$\mathcal{R}_0\pi(i) = \begin{cases} \pi(i) & \text{if } i < \sigma(l) \\ \pi(l+m) & \text{if } i = \sigma(l) \\ \pi(i-1) & \text{otherwise.} \end{cases}$$

• Otherwise $R_0\pi$ is not defined.

(2) map $\mathcal{R}_1$.
• If $\sigma(l+m) \leq l$ and if $\pi(l) \neq \pi(l+m)$ then we define $\mathcal{R}_1\pi$ to be of type (l,m) such that:

$$\mathcal{R}_1\pi(i) = \begin{cases} \pi(l) & \text{if } i = \sigma(l+m)+1 \\ \pi(i-1) & \text{If } \sigma(l+m)+1 < i \leq l \\ \pi(i) & \text{otherwise.} \end{cases}$$

• If $\sigma(l+m) > l$ and if there exists a pair $\{x, \sigma(x)\}$ included in $\{1, \dots, l-1\}$ then $\mathcal{R}_1\pi$ is of type $(l-1, m+1)$ and:

$$\mathcal{R}_1\pi(i) = \begin{cases} \pi(i+1) & \text{if } l \leq i < \sigma(l+m)-1 \\ \pi(l) & \text{If } i = \sigma(l+m)-1 \\ \pi(i) & \text{otherwise.} \end{cases}$$

• Otherwise $R_1\pi$ is not defined.

We now describe the Rauzy-Veech induction $\mathcal{R}(T)$ of T:

- If $T = (\pi, \lambda)$ has type 0, then $\mathcal{R}(T) = (\mathcal{R}_0\pi, \lambda')$, with $\lambda'_\alpha = \lambda_\alpha$ if $\alpha \neq \pi(l)$ and $\lambda'_{\pi(l)} = \lambda_{\pi(l)} - \lambda_{\pi(l+m)}$.
- If $T = (\pi, \lambda)$ has type 1, then $\mathcal{R}(T) = (\mathcal{R}_1\pi, \lambda')$, with $\lambda'_\alpha = \lambda_\alpha$ if $\alpha \neq \pi(l+m)$ and $\lambda'_{\pi(l+m)} = \lambda_{\pi(l+m)} - \lambda_{\pi(l)}$.

Example 2.8. Let us consider the permutation of Example 2.5, namely $\pi = \left(\begin{smallmatrix} A & B & C & B & D \\ D & E & A & C & E \end{smallmatrix}\right)$. Then

$$\mathcal{R}_0(\pi) = \begin{pmatrix} A & B & C & B & D \\ D & E & E & A & C \end{pmatrix} \text{ and } \mathcal{R}_1(\pi) = \begin{pmatrix} A & B & C & B & & \\ D & D & E & A & C & E \end{pmatrix}.$$

Example 2.9. Let us consider the permutation π defined on the alphabet $\mathcal{A} = \{A, B, C, D\}$ by $\pi = \left(\begin{smallmatrix} A & B & A & & \\ B & D & C & C & D \end{smallmatrix}\right)$. Then

$$\mathcal{R}_0(\pi) = \begin{pmatrix} D & A & B & A \\ B & D & C & C \end{pmatrix}$$

and $\mathcal{R}_1(\pi)$ is not defined. Indeed, consider any linear involution with π as combinatorial data. Then we must have

$$2\lambda_A + \lambda_B = \lambda_B + 2\lambda_C + 2\lambda_D.$$

Therefore we necessarily have $\lambda_D < \lambda_A$ and $\lambda_D > \lambda_A$ never happens.

Example 2.10. Consider the permutation π defined on the alphabet $\mathcal{A} = \{A, B, C\}$ by $\pi = \left(\begin{smallmatrix} A & B & A \\ B & C & C \end{smallmatrix}\right)$. Then $\mathcal{R}_\varepsilon(\pi)$ is not defined for any ε. Indeed, consider any linear involution with π as combinatorial data. Then we must have $\lambda_A = \lambda_C$, hence the Rauzy-Veech induction of T is not defined for any parameters.

In the case of interval exchange maps, one usually define the Rauzy-Veech induction only for irreducible combinatorial data. Here we have not yet defined irreducibility. However, it will appear in section 3 that some interesting phenomena with respect to Rauzy-Veech induction appear also in the reducible case.

In the next section we will define a notion of irreducibility which is equivalent to have a suspension data. It is easy to see that a generalized permutation π such that neither $\mathcal{R}_0(\pi)$ nor $\mathcal{R}_1(\pi)$ is defined is necessarily reducible. However, the permutation π of Example 2.9 is irreducible (see Definition 3.1 and Theorem 3.2) while $\mathcal{R}_1(\pi)$ is not defined.

2.3. **Suspension data and zippered rectangles construction.** Starting from a linear involution T, we want to construct a flat surface and a horizontal segment whose corresponding first return maps (T_0, T_1) of the vertical foliation give T. Such surface will be called a *suspension* over T, and the parameters encoding this construction will be called *suspension data*.

Definition 2.11. Let T be a linear involution and let $(\lambda_\alpha)_{\alpha\in\mathcal{A}}$ be the lengths of the corresponding intervals. Let $\{\zeta_\alpha\}_{\alpha\in\mathcal{A}}$ be a collection of complex numbers such that:

(1) $\forall \alpha \in \mathcal{A} \quad Re(\zeta_\alpha) = \lambda_\alpha$.
(2) $\forall 1 \leq i \leq l-1 \quad Im(\sum_{j\leq i} \zeta_{\pi(j)}) > 0$
(3) $\forall 1 \leq i \leq m-1 \quad Im(\sum_{1\leq j\leq i} \zeta_{\pi(l+j)}) < 0$
(4) $\sum_{1\leq i\leq l} \zeta_{\pi(i)} = \sum_{1\leq j\leq m} \zeta_{\pi(l+j)}$.

The collection $\zeta = \{\zeta_\alpha\}_{\alpha\in\mathcal{A}}$ is called a *suspension data* over T.

We will also speak in an obvious manner of a suspension data for a generalized permutation.

Let L_0 be a broken line (with a finite number of edges) on the plane such that the edge number i is represented by the complex number $\zeta_{\pi(i)}$, for $1 \leq i \leq l$, and L_1 be a broken line that starts on the same point as L_0, and whose edge number j is represented by the complex number $\zeta_{\pi(l+j)}$ for $1 \leq j \leq m$ (Figure 7).

If L_0 and L_1 only intersect on their endpoints, then L_0 and L_1 define a polygon whose sides comes by pairs and for each pair the corresponding sides are parallel and have the same length. Then identifying these sides together, one gets a flat surface. It is easy to check that the first return map of the vertical foliation on the segment corresponding to

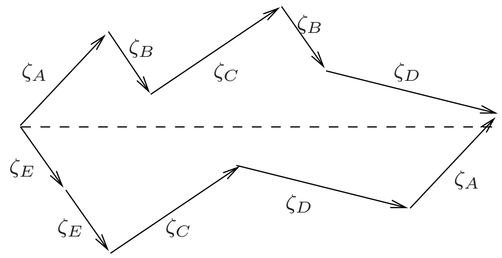


FIGURE 7. A suspension over a linear involution.

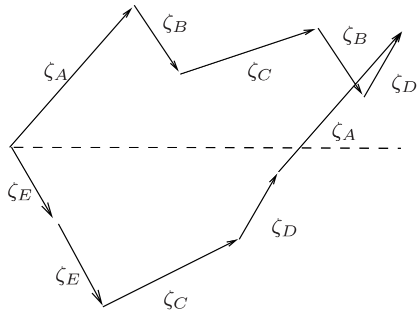


FIGURE 8. Suspension data that does not give a suitable polygon.

X in S defines the same linear involution as T, so we have constructed a suspension over T. We will say in this case that ζ defines a *suitable polygon*.

The broken lines L_0 and L_1 might intersect at other points (see Figure 8). However, we can still define a flat surface by using an analogous construction as the zippered rectangles construction. We now give a sketch of this construction (see e.g. [Vee82, Yoc03] for the case of interval exchange maps, or section 1.3.5). This construction is very similar to the usual one, although its precise description is very technical. Still, for completeness, we give an equivalent but rather implicit formulation.

We first consider the previous case when L_0 and L_1 define a suitable polygon. For each pair of interval $X_i, X_{\sigma(i)}$ on X, the return time $h_{\pi(i)}$ of the vertical foliation starting from $x \in X_i$ and returning in $y \in X_{\sigma(i)}$ is constant. This value depends only on the generalized permutation and on the imaginary part of the suspension data ζ. There is a natural embedding of the open rectangle $R_{\pi(i)} = (0, \lambda_i) \times (0, h_{\pi(i)})$ into the flat surface S and this surface is obtained from $\sqcup_\alpha R_\alpha$ by identifications on the boundaries of the R_α. Identifications for the horizontal sides $[0, \lambda_\alpha]$ are given by the linear involution and identifications for the vertical sides only depend on the generalized permutation and of $\{Im(\zeta_\alpha)\}_{\alpha\in\mathcal{A}}$.

For the general case, we construct the rectangles R_α using the same formulas. Identifications for the horizontal sides are straightforward. Identifications for the vertical sides,

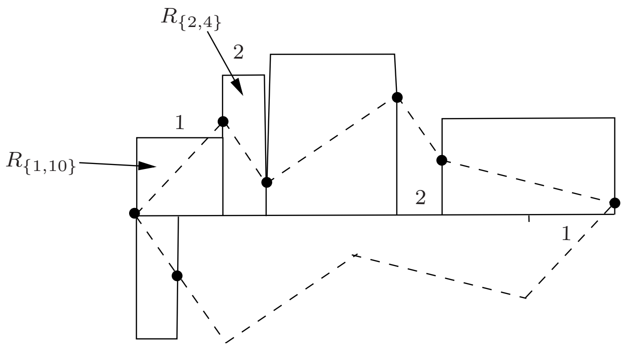


FIGURE 9. Zippered rectangle construction of the flat surface of Figure 7.

that do not depends on the horizontal parameters, will be well defined after the following lemma.

Lemma 2.12. *Let ζ be a suspension data for a linear involution T, and let π be the corresponding generalized permutation. There exists a linear involution T' and a suspension data ζ' for T' such that:*

- *The generalized permutation associated to T' is π.*
- *For any α the complex numbers ζ_α and ζ'_α have the same imaginary part.*
- *The suspension data ζ' defines a suitable polygon.*

Proof. We can assume that $\sum_{k=1}^{l} Im(\zeta_{\pi(k)}) > 0$ (the negative case is analogous and there is nothing to prove when the sum is zero). It is clear that $\sigma(l+m) \neq l$ otherwise there would be no possible suspension data. If $\sigma(l+m) < l$, then we can shorten the real part of $\zeta_{\pi(l+m)}$, keeping conditions (1)–(4) satisfied, and get a suspension data ζ' with the same imaginary part as ζ, and such that $Re(\zeta'_{\pi(l+m)}) < Re(\zeta'_{\pi(l)})$. This last condition implies that ζ' defines a suitable polygon.

If $\sigma(l+m) > l$, then condition (4) implies that $Re(\zeta_{\pi(l+m)})$ is necessary bigger than $Re(\zeta_{\pi(l)})$. However, we can still change ζ into a suspension data ζ', with same imaginary part, and such that $Re(\zeta'_{\pi(l+m)})$ is very close to $Re(\zeta'_{\pi(l)})$. In that case, ζ' also defines a suitable polygon. See [Boi07], Lemma 2.1 for more details. □

We have therefore defined the zippered rectangle construction for any suspension data. Note that we have not yet discussed the existence of a suspension data. This will be done in the upcoming section. This notion is natural. See [Vee82] and the following Proposition.

Proposition 2.13. *Let S be a flat surface with no vertical saddle connections and let X be a horizontal interval attached to a singularity on the left. Let γ be the vertical leaf passing through the right endpoint of X, we assume that γ meets a singularity before returning to X, in positive or negative direction. Let $T = (\pi, \lambda)$ be the linear involution given by the cross section on X of the vertical flow. There exists a suspension data ζ such that (π, ζ) defines a surface isometric to S.*

Proof. See the construction given in the proof of Proposition 2.2 in [Boi07]. □

We can define the Rauzy-Veech induction on the space of suspensions, as well as on the space of zippered rectangles. Let $T = (\pi, \lambda)$ be a linear involution and let ζ be a suspension over T. Then we define $\mathcal{R}(\pi, \zeta) = (\pi', \zeta')$ as follows.

- If $T = (\pi, \lambda)$ has type 0, then $\mathcal{R}(\pi, \zeta) = (\mathcal{R}_0\pi, \zeta')$, with $\zeta'_\alpha = \zeta_\alpha$ if $\alpha \neq \pi(l)$ and $\zeta'_{\pi(l)} = \zeta_{\pi(l)} - \zeta_{\pi(l+m)}$.
- If $T = (\pi, \lambda)$ has type 1, then $\mathcal{R}(\pi, \zeta) = (\mathcal{R}_1\pi, \zeta')$, with $\zeta'_\alpha = \zeta_\alpha$ if $\alpha \neq \pi(l+m)$ and $\zeta'_{\pi(l+m)} = \zeta_{\pi(l+m)} - \zeta_{\pi(l)}$.

We can show that (π', ζ') is a suspension over $\mathcal{R}(T)$ and defines a surface isometric to the one corresponding to (π, ζ).

As in the case of interval exchange maps we consider the renormalized Rauzy-Veech induction defined on lengths one intervals:

$$\text{if } \mathcal{R}(\pi, \lambda) = (\pi', \lambda') \text{ then } \mathcal{R}_r(\pi, \lambda) := (\pi', \lambda'/|\lambda'|).$$

One can define obviously the corresponding renormalized Rauzy-Veech induction on the suspensions data by contracting the imaginary parts by a factor $|\lambda'|$ which preserves the area of the corresponding flat surface.

3. Geometry of generalized permutations

In this section we give a necessary and sufficient condition for a generalized permutation to admit a suspension; this will prove the first part of Theorem A. Let us first introduce some notations to make clear the definition.

Notation: If $\mathcal{A} = \{\alpha_1, \dots, \alpha_d\}$ is an alphabet, we will denote by $\mathcal{A} \sqcup \mathcal{A}$ the set with multiplicities $\{\alpha_1, \alpha_1, \dots, \alpha_d, \alpha_d\}$ of cardinal $2d$, and we will use analogous notations for subsets of $\mathcal{A}$.

We will also call *top* (respectively *bottom*) the restriction of a generalized permutation π to $\{1, \dots, l\}$ (respectively $\{l+1, \dots, l+m\}$) where (l, m) is the type of π.

Notation: Let F_1, F_2, F_3, F_4 be (possibly empty) unordered subsets of $\mathcal{A}$ or $\mathcal{A} \sqcup \mathcal{A}$. We say that a generalized permutation π of type (l, m) is decomposed if

$$\pi = \left(\begin{array}{c|c|c} F_1 & *** & F_2 \\ \hline F_3 & *** & F_4 \end{array}\right),$$

and there exist $0 \leq i_1 \leq i_2 \leq l$ and $l \leq i_3 \leq i_4 \leq l+m = 2d$ such that

- $\{\pi(1), \dots, \pi(i_1)\} = F_1$
- $\{\pi(i_2), \dots, \pi(l)\} = F_2$
- $\{\pi(l+1), \dots, \pi(i_3)\} = F_3$
- $\{\pi(i_4), \dots, \pi(2d)\} = F_4$.

The sets F_1, F_2, F_3, and F_4 will be referred as top-left, top-right, bottom-left and bottom-right corners respectively.

We do not assume that $card(F_1) = card(F_3)$, or $card(F_2) = card(F_4)$.

Definition 3.1. We will say that π is *reducible* if π admits a decomposition

$$(*)\qquad \left(\begin{array}{c|c|c} A\cup B & *** & D\cup B \\ \hline A\cup C & *** & D\cup C \end{array}\right),\ A,B,C,D \text{ disjoint subsets of } \mathcal{A},$$

where the subsets A,B,C,D are not all empty and one of the following statements holds

i- No corner is empty

ii- Exactly one corner is empty and it is on the left.

iii- Exactly two corners are empty and they are either both on the left, either both on the right.

A permutation that is not reducible is *irreducible.*

The main result of this section is the next theorem which, being combined with Proposition 2.13 implies first part of Theorem A. We make clear that in this section, we only speak of suspensions given by the construction of section 2.3.

Theorem 3.2. *Let $T=(\pi,\lambda)$ be a linear involution. Then T admits a suspension ζ if and only if the underlying generalized permutation π is irreducible.*

Remark 3.3. Note that the existence or not of a suspension is independent of the length data λ.

Remark 3.4. One can see that this reducibility notion is not symmetric with respect to the left/right, contrary to the case of interval exchange maps. Therefore, the choice of attaching a singularity on the left end of the segment in the construction of section 2.3 is a real choice. This will have an important consequence in terms of extended Rauzy classes.

Remark 3.5. In the usual case of interval exchange maps, one can always choose ζ in such a way that $Im(\sum_{i=1}^{l}\zeta_{\pi(i)})=0$ (*i.e.* there is a singularity on the left and on the right of the interval). Here it is not always possible. More precisely one can show that T admits such a suspension with this extra condition if and only if for any decomposition of π as in equation $(*)$ above, all the corners are empty.

3.1. **Necessary condition.**

Proposition 3.6. *A reducible generalized permutation does not admit any suspension data.*

Proof of the Proposition. Let us consider π a reducible generalized permutation. It is convenient to introduce some notations. Let us assume that there exists a suspension ζ over π. Then we define a the real number $a=\sum_{j\in A} Im(\zeta_{\pi(j)})$; we define $a=0$ if the set A is empty. Finally we define b, c and d in an analogous manner for B,C and D. We also define $t=\sum_{i=1}^{l} Im(\zeta_{\pi(i)})$. We distinguish three cases following Definition 3.1.

i- No corner is empty.
Then the following inequalities hold

$$\left\{\begin{array}{lcl} a+b & > & 0 \\ a+c & < & 0 \\ t-d-b & > & 0 \\ t-d-c & < & 0 \end{array}\right.$$

Subtracting the second one from the first one, and the fourth one from the third one, we get:

$$\begin{cases} b-c & > & 0 \\ c-b & > & 0 \end{cases}$$

which is a contradiction.

ii- Exactly one corner is empty, and it is on the left.
We can assume without loss of generality that it is the top-left one. That means that A, B are empty, and C, D are nonempty. Therefore the following inequalities holds:

$$\begin{cases} c & < & 0 \\ t-d & > & 0 \\ t-d-c & < & 0 \end{cases}$$

Subtracting the third inequality from the second one, we get $c>0$, which contradicts the first one.

iii- Exactly two corners are empty.
If they are both on the left side, then we have B and C empty and D non empty. This implies that $t-d$ is both positive and negative, which is impossible.
If they are both on the right side, it is similar. If the two corners forming a diagonal were empty, then it is easy to see that all the corners would be empty, hence this case doesn't occur by assumption. The proposition is proven. □

3.2. **Sufficient condition.** In this section, we will not necessarily assume that generalized permutations satisfy Convention 2.7, since for technical reasons, some intermediary results of this section must be stated for an arbitrary generalized permutation.

We will have to work only on the imaginary part of the ζ_i in order to built a suspension. Hence, in order to simplify the notations we will use the following ones. We will use this vocabulary only in this section.

Definition 3.7. A *pseudo-suspension* is a collection of real numbers $\{\tau_i\}_{i\in\mathcal{A}}$ such that:

- For all $k\in\{1,\dots,l\}$ $\quad\sum_{i\leq k}\tau_{\pi(i)}\geq 0$.
- For all $k\in\{1,\dots,m\}$ $\quad\sum_{l<i\leq l+k}\tau_{\pi(i)}\leq 0$.
- $\sum_{i\leq l}\tau_{\pi(i)}=\sum_{l<i\leq l+m}\tau_{\pi(i)}=0$

A pseudo-suspension is *strict* if all the previous inequalities are strict except for the extremal ones.

A *vanishing index* on the top (respectively bottom) of a pseudo-suspension is an integer $k_0<l$ (respectively $k_0<m$) such that $\sum_{i\leq k_0}\tau_{\pi(i)}=0$ (respectively $\sum_{l<i\leq l+k_0}\tau_{\pi(i)}=0$).

A pseudo-suspension τ' is *better* than τ if the set of vanishing indices of τ' is strictly included into the set of vanishing indices of τ.

We will say that π is *strongly irreducible* if for any decomposition of π as in $(*)$ of Definition 3.1, all the corners are empty. Of course strong irreducibility implies irreducibility.

The following lemma is obvious and left to the reader.

Lemma 3.8. *Let π be generalized permutation satisfying Convention 2.7 that admits a strict pseudo-suspension. Then π admits a suspension ζ with $Im(\sum_{1\leq i\leq l}\zeta_{\pi(i)})=0$.*

Let us assume that π is any irreducible permutation. One has to find a suspension ζ over π. We will first assume that π is strongly irreducible and we will show that π admits such a suspension with the extra equality $Im(\sum_{1\leq i\leq l}\zeta_{\pi(i)})=0$. This corresponds to a special case of Proposition 3.16. We will then relax the condition on the irreducibility of π and prove our main result. Note that one can extend the proof of Proposition 3.6 to show that if ζ is a suspension data such that $Im(\sum_{1\leq i\leq l}\zeta_{\pi(i)})=0$, then π is strongly irreducible.

From Lemma 3.8 we have reduced the problem to the construction of a strict pseudo-suspension. As we have seen in section 1, in the case of true permutations, there is an explicit formula, due to Masur and Veech, that gives a suspension when the permutation is irreducible. We will first build a pseudo-suspension τ_{MV} by extending this formula to generalized permutations. This will not give in general a strict pseudo-suspension.

Let $\pi:\{1,\dots l+m\}\to\mathcal{A}$ be a generalized permutation. We can decompose $\mathcal{A}$ into three disjoint subsets

- The subset $\mathcal{A}_{01}$ of elements $\alpha\in\mathcal{A}$ such that $\pi^{-1}(\{\alpha\})$ contains exactly one element in $\{1,\dots,l\}$ and one element in $\{l+1,\dots,l+m\}$. The restriction of π on $\pi^{-1}(\mathcal{A}_{01})$ defines a true permutation.
- The subset $\mathcal{A}_0$ of elements $\alpha\in\mathcal{A}$ such that $\pi^{-1}(\{\alpha\})$ contains exactly two elements in $\{1,\dots,l\}$ (and hence no elements in $\{l+1,\dots,l+m\}$).
- The subset $\mathcal{A}_1$ of elements $\alpha\in\mathcal{A}$ such that $\pi^{-1}(\{\alpha\})$ contains exactly two elements in $\{l+1,\dots,l+m\}$ (and hence no elements in $\{1,\dots,l\}$).

The next lemma is just a reformulation of the construction of a suspension data in section 1.3.4

Lemma 3.9 (Masur; Veech). *Let π be a true permutation defined on $\{1,\dots,d\}$, then the integers $\tau_i=\pi(i)-i$ for $1\leq i\leq d$ define a pseudo-suspension over π. Furthermore, we have:*
$$\sum_{i\leq i_0}\tau_i=0\Leftrightarrow\sum_{i\leq i_0}\tau_{\pi^{-1}(i)}=0\Leftrightarrow\pi(\{1,\dots,i_0\})=\{1,\dots,i_0\}.$$

Recall that we do not assume any more that a generalized permutation satisfies Convention 2.7.

Lemma 3.10. *Let π be a generalized permutation of type $(l,m)=(2d,0)$ and σ the associated involution. There exists a collection of real numbers $(\tau_1,\dots,\tau_{2d})$ with $\sum_{i\leq i_0}\tau_i\geq 0$ for all i_0 and such that*
$$\sum_{i\leq i_0}\tau_i=0\Leftrightarrow\sigma(\{1,\dots,i_0\})=\{2d,\dots,2d-i_0+1\}).$$

Proof. We will construct from $\pi_0:=\pi$ a new permutation $\tilde{\pi}$ on d symbols. Let us consider the "mirror symmetry" π_1 of π_0 as follows. In tabular representation π_0 is $(\tau(1),\dots,\tau(2d))$; π_1 is of type $(0,2d)$ and its tabular representation is $(\tau(2d),\dots,\tau(1))$.

Then $\tilde{\pi}$ is in tabular representation $\left(\begin{smallmatrix} L_0 \\ L_1 \end{smallmatrix}\right)$ with L_i is obtained from π_i by removing the second occurrence of each letter. For instance, if $\pi_0 = (A\ B\ C\ C\ D\ D\ A\ B)$ then $\pi_1 = (B\ A\ D\ D\ C\ C\ B\ A)$ and $\tilde{\pi} = \left(\begin{smallmatrix} A & B & C & D \\ B & A & D & C \end{smallmatrix}\right)$. It is easy to check that $\tilde{\pi}$ is reducible if and only if there exists i_0 such that $\sigma(\{1,\dots,i_0\}) = \{2d - i_0 + 1, \dots, 2d\}$. Moreover the solution of Lemma 3.9 gives the desired collection of numbers τ_i. □

Definition/Lemma 3.11. We define the pseudo-suspension τ_{MV} over π by the collection of real numbers given by

- The solutions given by Lemma 3.9 and Lemma 3.10 for the restrictions of π on $\pi^{-1}(\mathcal{A}_{01})$ and on $\pi^{-1}(\mathcal{A}_0)$.
- The solution of Lemma 3.10 for the restriction of π on $\pi^{-1}(\mathcal{A}_1)$, taken with opposite sign.

Lemma 3.12. *Let $k \in \{1,\dots,l\}$ be any vanishing index on the top of τ_{MV}. Setting $A = \pi(\{1,\dots,k\}) \cap \mathcal{A}_{01}$ and $B = \pi(\{1,\dots,k\}) \cap (\mathcal{A}_0 \sqcup \mathcal{A}_0)$, there exists $C \subset \mathcal{A}_1 \sqcup \mathcal{A}_1$ and $D \subset \mathcal{A}_{01}$ such that the generalized permutation π decomposes as*

$$\left(\begin{array}{c|c|c} A \cup B & *** & D \cup B' \\ \hline A \cup C & *** & *** \end{array}\right)$$

with $A \cup B \neq \emptyset$ and with one of the following properties: either $B = B' \subset \mathcal{A}$ or there exist $i_1, i_2 \le k$ such that $\pi(i_1) = \pi(i_2) \in B$ and $B' \subset B$.

There is an analogous decomposition for vanishing indices in $\{l+1,\dots,l+m\}$ but with different subsets A' B', C' and D' a priory.

Proof. It follows from Lemmas 3.9 and 3.10. □

Remark 3.13. If τ is a pseudo-suspension of $\pi = \left(\begin{smallmatrix} \alpha_1 & \alpha_2 & *** & \alpha_l \\ \alpha_{l+1} & \alpha_{l+2} & *** & \alpha_{l+m} \end{smallmatrix}\right)$ then $\tau' = -\tau$ is a pseudo-suspension of $\pi' = \left(\begin{smallmatrix} \alpha_{l+1} & \alpha_{l+2} & *** & \alpha_{l+m} \\ \alpha_1 & \alpha_2 & *** & \alpha_l \end{smallmatrix}\right)$, and τ is a pseudo-suspension of the generalized permutation $\pi'' = \left(\begin{smallmatrix} \alpha_l & \alpha_{l-1} & *** & \alpha_1 \\ \alpha_{l+m} & \alpha_{l+m-1} & *** & \alpha_{l+1} \end{smallmatrix}\right)$.
Hence we can "flip" the generalized permutation π by top/bottom or left/right without loss of generality.

In the next two lemmas, we denote by τ a pseudo-suspension that is better than τ_{MV} and maximal (i.e. there is no better pseudo-suspensions).

Lemma 3.14. *Let i_1 and i_2 be the two first top and bottom vanishing indices for τ (possibly $i_1 = l, i_2 = m$). Let $A = \pi(\{1,\dots,i_1\}) \cap \mathcal{A}_{01}$ and $A' = \pi(\{l+1,\dots,l+i_2\}) \cap \mathcal{A}_{01}$. Then either $A = A'$ or $A = \emptyset$ or $A' = \emptyset$.*

Proof. We assume that neither A nor A' is empty. Lemma 3.12 implies that one of this set is a subset of the other one.

Without loss of generality, we can assume that $A \subseteq A'$. Let us assume $A \neq A'$; we will get a contradiction. So there exist j_1, j_2 in $\pi^{-1}(\mathcal{A}_{01})$ such that $1 \le j_1 \le i_1 < j_2 \le l$. But by definition of A and A', we also have $\sigma(j_1) < \sigma(j_2)$.

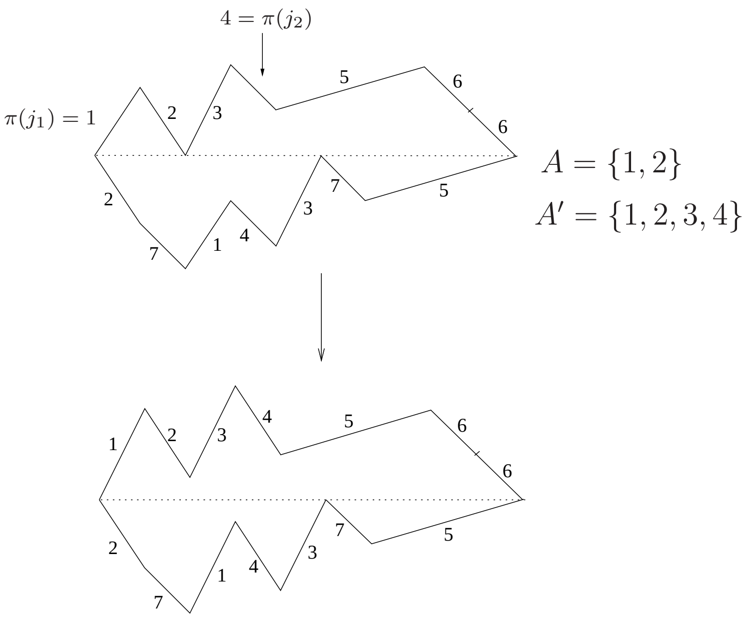


FIGURE 10. Construction of a pseudo-suspension τ' better than τ.

The definition of i_2 implies that there exists $c < 0$ such that, for $l + 1 \leq k < i_2$, the following inequality holds:

$$\sum_{1+l\leq i\leq l+k} \tau_{\pi(i)} \leq c < 0.$$

Now we replace $\tau_{\pi(j_1)}$ (respectively $\tau_{\pi(j_2)}$) by $\tau_{\pi(j_1)} - \frac{c}{2}$ (respectively $\tau_{\pi(j_2)} + \frac{c}{2}$) and get a vector τ', see Figure 10. We have

- $\sum_{1\leq i\leq k} \tau'_{\pi(i)} > 0$ for $k < j_2$.
- $\sum_{1\leq i\leq k} \tau'_{\pi(i)} = \sum_{1\leq i\leq k} \tau_{\pi(i)}$ for $k \geq j_2$.
- $\sum_{1+l\leq i\leq k} \tau'_{\pi(i)} \leq c/2 < 0$ for $l + 1 \leq k < \sigma(j_2)$.
- $\sum_{1+l\leq i\leq k} \tau'_{\pi(i)} = \sum_{1+l\leq i\leq k} \tau_{\pi(i)}$ for $k \geq \sigma(j_2)$ (since $\sigma(j_1) < \sigma(j_2)$).

Hence, τ' is a pseudo-suspension better than τ, contradicting its maximality. Therefore $A = A'$ and the lemma is proven. □

Lemma 3.15. *Let i_1 and i_2 be the first and last top vanishing indices of π. Let $B = \pi(\{1, \dots, i_1\}) \cap (\mathcal{A}_0 \sqcup \mathcal{A}_0)$ and $B' = \pi(\{i_2+1, \dots, l\}) \cap (\mathcal{A}_0 \sqcup \mathcal{A}_0)$. Then either $B' = B \subset \mathcal{A}_0$ or $B = \emptyset$ or $B' = \emptyset$. Moreover if there exist $i_{b_1} \neq i_{b_2}$ in $\{1, \dots, i_1\}$ such that $\pi(i_{b_1}) = \pi(i_{b_2})$ then $B = \mathcal{A}_0 \sqcup \mathcal{A}_0$.*

Proof. We sketch the proof here. We assume that there exist i_{b_1} and i_{b_2} in $\{1, \dots, i_1\}$ such that $\pi(i_{b_1}) = \pi(i_{b_2})$. If there exists $i_{b_3} > i_1$ such that $\pi(i_{b_3}) \in B$, then we set:

$$\tau'_{\pi(i_{b_1})} = \tau_{\pi(i_{b_1})} + \varepsilon$$
$$\tau'_{\pi(i_{b_3})} = \tau_{\pi(i_{b_3})} - \varepsilon.$$

Then is is easy to see that, for ε small enough, τ' is a pseudo-suspension and is better than τ, contradicting its maximality. Remark 3.13 implies that the same statement is true for B'; hence, we can assume that $B, B' \subset \mathcal{A}_0$. We conclude using the same argument as the one of the proof of the previous Lemma 3.14. $\square$

Proposition 3.16. *Let π be a strongly irreducible generalized permutation. Let τ be any pseudo-suspension which is better than τ_{MV} and maximal. Then τ is a strict pseudo-suspension.*

Proof of Proposition 3.16. Let us assume that τ is not strict. From Lemmas 3.14 and 3.15 and Remark 3.13 we have the following decomposition of π.

$$\left(\begin{array}{c|c|c} A\cup B & *** & D\cup B' \\ \hline A'\cup C & *** & D'\cup C' \end{array}\right)$$

with $A, A', D, D' \subset \mathcal{A}_{01}$, $B, B' \subset \mathcal{A}_0$ and $C, C' \subset \mathcal{A}_1$ by assumption, and with the condition that either A, A' are equal, or at least one of them is empty (and similar statement for the pair (D, D')); and the condition that if $B, B' \subset \mathcal{A}_0$ then they are either equal, or at least one of them is empty, otherwise one of them is $\mathcal{A}_0 \sqcup \mathcal{A}_0$ (and similar statements for C, C'). By convention from now on, we will keep the notation B or C only when they are not equal to $\mathcal{A}_0 \sqcup \mathcal{A}_0$ or $\mathcal{A}_1 \sqcup \mathcal{A}_1$, and therefore subsets of $\mathcal{A}_0$ or $\mathcal{A}_1$.

Let us note that if there is no vanishing index in $\{1, \dots, l-1\}$ or in $\{l+1, \dots, l+m-1\}$, the corresponding right corner is just empty. But if τ is not strict, then there exists at least a pair of nonempty corners in the top or in the bottom.

If there is a vanishing index on the top, then the two corresponding corners are non-empty. Then it is easy to see that either there is a corner with only A, B or D, or the corners are respectively $A \cup B$ or $D \cup B$, with A, B, D nonempty. In this case Lemma 3.14 implies that there must be a vanishing index in $\{l+1, \dots, l+m\}$.

Since there must be a vanishing index in the top, or in the bottom, the previous argument implies that either π is not strongly irreducible, or there is one corner that only consists of one set A, B, C or D. Thanks to Remark 3.13, we assume that this is the top-left corner; this leads to the two next cases.

The general idea of the next part of the proof is first to remove the cases that correspond to not strongly irreducible permutations, and then show that the other cases correspond to a non-maximal pseudo-solution.

First case: The top-left corner is B.
There is necessary a vanishing index in $\{1, \dots, l-1\}$, and hence the top-right corner is not empty. It also does not contains all $\mathcal{A}_0 \sqcup \mathcal{A}_0$, hence it is necessary B, D or $D \cup B$. Recall that π is assumed to be strongly irreducible, so the top-right corner is not B. If the bottom-right corner were D, the generalized permutation π would decompose as

$$\left(\begin{array}{c|c|c} B & *** & D\cup B \\ \hline & *** & D \end{array}\right),$$

or

$$\left(\begin{array}{c|c|c} B & *** & D \\ \hline & *** & D \end{array}\right)$$

which are not strongly irreducible. Hence the bottom-right corner is not D. This also implies that $\mathcal{A}_1$ cannot be empty.

Let us assume that there are no vanishing indices in the bottom line. We choose any element $b \in B$, $c \in \mathcal{A}_1$, and $d \in D$ and change τ_b by $\tau_b+\varepsilon$, τ_c by $\tau_c+\varepsilon$ and τ_d by $\tau_d-2\varepsilon$. If ε is small enough, then the new vector τ' is better than τ, which contradicts its maximality.

So, the bottom admits vanishing indices; then the bottom-left corner can be $C, \mathcal{A}_1 \sqcup \mathcal{A}_1, \mathcal{A}_1 \sqcup \mathcal{A}_1 \cup A, A$ or $A \cup C$. Let us discuss these cases in details.

- C: the bottom-right corner is C, D or $C \cup D$. In the first and second cases, π is not strongly irreducible. If for instance, the top-right is $D \cup B$, then π decomposes as
$$\left(\begin{array}{c|c|c} B & *** & D\cup B \\ \hline C & *** & D \cup C \end{array}\right),$$
and therefore π is not strongly irreducible. The other case is similar.
- $\mathcal{A}_1 \sqcup \mathcal{A}_1$ or $\mathcal{A}_1 \sqcup \mathcal{A}_1 \cup A$: in that case, the bottom-right corner is necessary D and we have already proved that π is not strongly irreducible in this situation.
- A or $A \cup C$: We construct a better pseudo-suspension τ'.
Let $j_1 \leq l$ be the smallest index such that $\sigma(j_1) > l$ and let $j_2 \leq l$ be the largest one. Let i_1 be the first vanishing index. There exists $j_b \in \{1, \dots, i_1\}$ such that $\sigma(j_b) < j_2$ otherwise the top-line would have a decomposition as $B| *** |B$, and π would be not strongly irreducible. Let j_c be the first index in $\pi^{-1}(\mathcal{A}_1)$ (see Figure 11).
Now we define τ' in the following way:
$$\begin{aligned} \tau'_{\pi(j_1)} &= \tau_{\pi(j_1)} - \varepsilon \\ \tau'_{\pi(j_2)} &= \tau_{\pi(j_2)} - \varepsilon \\ \tau'_{\pi(j_b)} &= \tau_{\pi(j_b)} + \varepsilon \\ \tau'_{\pi(j_c)} &= \tau_{\pi(j_c)} + \varepsilon \\ \forall \alpha \notin \pi(\{j_1, j_2, j_b, j_c\}) \quad \tau'_\alpha &= \tau_\alpha. \end{aligned}$$
In the extremal case $j_1 = j_2$, the following arguments will work similarly if we define $\tau'_{\pi(j_1)}$ by $\tau_{\pi(j_1)} - 2\varepsilon$. We have
$$\forall k \in \{1, \dots, l\} \quad \sum_{i=1}^{k} \tau'_{\pi(i)} = \sum_{i=1}^{k} \tau_{\pi(i)} + n_k \varepsilon$$
$$\forall k \in \{1, \dots, m\} \quad \sum_{i=l+1}^{l+k} \tau'_{\pi(i)} = \sum_{i=l+1}^{l+k} \tau_{\pi(i)} + m_k \varepsilon$$

Here n_k is the difference between the number of indices in $\{j_b, \sigma(j_b)\}$ smaller than or equal to k, and number of indices in $\{j_1, j_2\}$ smaller than or equal to k. This value is *always* greater than or equal to zero for $k \in \{1, \dots, l\}$, and is strictly greater than zero when k is the first vanishing index.

Similarly m_k is the difference between the number of indices in $\{j_c, \sigma(j_c)\}$ that are in $\{l+1, \dots, k\}$, and number of indices in $\{\sigma(j_1), \sigma(j_2)\}$ that are in $\{l+1, \dots, k\}$. This value might be positive. Let $i_3 \leq i_4 < l+m$ be respectively the first and last bottom vanishing indices. We have the following facts:

– $\sigma(j_1) \leq i_3$ otherwise the bottom-left corner is C.
– $\sigma(j_c) > i_4$ otherwise the bottom-right corner is D.

Hence it is easy to check that m_k can be strictly positive only for $l < k < i_3$ or $i_4 < k < l+m$.

Then if ε is small enough, τ' is a pseudo-suspension, and is better than τ (see Figure 11), which contradicts the maximality of τ.

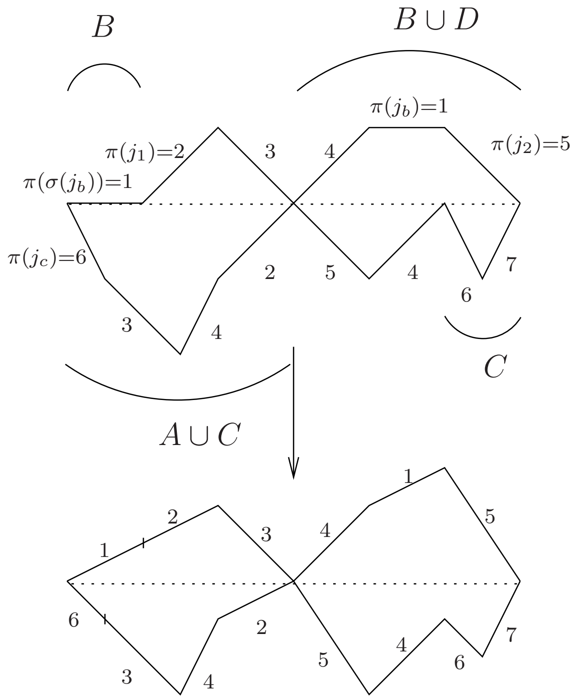


FIGURE 11. Construction a pseudo-solution τ' better than τ.

Second case: The top-left corner is A.
We assume that there are no corners B or C, since this case has already been discussed.

Let us assume that there is no vanishing index in the bottom line. Then, according to Lemma 3.14, $A = \mathcal{A}_{01}$; therefore the top-right corner is $\mathcal{A}_0 \sqcup \mathcal{A}_0$ or B. If $\mathcal{A}_1$ is empty, then

π decomposes as

$$\left(\begin{array}{c|c} A & \mathcal{A}_0 \sqcup \mathcal{A}_0 \\ \hline & A \end{array}\right)$$

so π is not strongly irreducible. If $\mathcal{A}_1$ is not empty, we choose any element $a \in A$, $b \in \mathcal{A}_0$, $c \in \mathcal{A}_1$, and replace τ_a by $\tau_a + 2\varepsilon$, τ_b by $\tau_b - \varepsilon$, and τ_c by $\tau_c - \varepsilon$. This new pseudo-suspension we have constructed is better that the old one for ε small enough.

If there are vanishing indices in the bottom, then the bottom-left corner belongs to the list: $A, A \cup C, A \cup \mathcal{A}_1 \sqcup \mathcal{A}_1$ or $\mathcal{A}_1 \sqcup \mathcal{A}_1$.

- A: the permutation π is then obviously not strongly irreducible.
- $A \cup C$: the bottom-right corner is necessary D or $D \cup C$. If the top-right corner where D, then π would be not strongly irreducible. In particular, that means $\mathcal{A}_0$ is not empty. Hence there exists $j_d < j_1 \leq l$ such that $d = \pi(j_d) \in D$ and $b = \pi(j_1) \in \mathcal{A}_0$. Then we choose any index $a \in A$ and any index $c \in C$, and set:
$$\begin{aligned} \tau'_a &= \tau_a + \varepsilon \\ \tau'_d &= \tau_d + \varepsilon \\ \tau'_b &= \tau_b - \varepsilon \\ \tau'_c &= \tau_c - \varepsilon \\ \forall i \notin \{a,b,c,d\} \quad \tau'_i &= \tau_i. \end{aligned}$$
Then τ' is better than τ for $\varepsilon > 0$ small enough, which contradicts its maximality.
- $A \cup \mathcal{A}_1 \sqcup \mathcal{A}_1$, or $\mathcal{A}_1 \sqcup \mathcal{A}_1$. The bottom-right corner is necessary D. If $\mathcal{A}_0$ is empty, then the top-right corner is also D, and therefore π is not strongly irreducible. If $\mathcal{A}_0$ is not empty, then we choose $a \in A$, $b \in \mathcal{A}_0$ and $c \in \mathcal{A}_1$, and set:
$$\begin{aligned} \tau'_a &= \tau_a + 2\varepsilon \\ \tau'_b &= \tau_b - \varepsilon \\ \tau'_c &= \tau_c - \varepsilon \\ \forall i \notin \{a,b,c\} \quad \tau'_i &= \tau_i \end{aligned}$$
And τ' is better than τ.

The proposition is proved. □

We now have all necessary tools for proving our main result.

Proof of Theorem 3.2. We only have to prove the sufficient condition. We consider a pseudo-suspension τ better than τ_{MV} and maximal for this property. We can assume that $\pi : \{1, \ldots, l+m\} \to \mathcal{A}$ is not strongly irreducible (i.e. at least one corner is non empty in the decomposition) otherwise the theorem follows from Lemma 3.8 and Proposition 3.16. Let us consider a decomposition of π as

$$\left(\begin{array}{c|c|c} A \cup B & U & D \cup B \\ \hline A \cup C & V & D \cup C \end{array}\right).$$

where $A\cup B\cup C\cup D$ is maximal. Note that $\pi' = \left(\begin{smallmatrix} U \\ V \end{smallmatrix}\right)$ defines a generalized permutation which is not strongly irreducible by assumption. Note also that π' does not necessary satisfy Convention 2.7, even if π satisfies that convention.

We define $\mathcal{A}' = \mathcal{A}\backslash\big(A\cup B\cup C\cup D\big)$; from Proposition 3.16, the restriction of τ to $\mathcal{A}'$ is strict for π'.

Since π is irreducible, there is one or two empty corners in the decomposition.

- If only one corner is empty, then it is on the right. So we can assume that π decomposes as:

$$\left(\begin{array}{c|c|c} A\cup B & U & B \\ \hline A & V & \end{array}\right)$$

with $\pi' = \left(\begin{smallmatrix} U \\ V \end{smallmatrix}\right)$ strongly irreducible.

Let i_1 be the first vanishing index in the top line of π and i_2 be the first vanishing index of the second line. Consider i_b the first index such that $b = \pi(i_b) \in B$. Then $i_b \leq i_1$ otherwise there would be a subdecomposition of π as

$$\left(\begin{array}{c|c} A' & *** \\ \hline A' & *** \end{array}\right)$$

and π would be reducible. Now let $a = \pi(l+1) \in A$ and let $c \in \mathcal{A}_1$. We set:

$$\begin{aligned} \tau'_b &= \tau_b + 2\varepsilon \\ \tau'_c &= \tau_c + 2\varepsilon \\ \tau'_a &= \tau_a - \varepsilon \\ \forall j \notin \{a,b,c\} \quad \tau'_j &= \tau_j \end{aligned}$$

If ε is small enough, then τ' satisfies:
  - For all $k \in \{1,\dots,l\}$ $\quad \sum_{i\leq k} \tau'_{\pi(i)} > 0$.
  - For all $k \in \{1,\dots,m-1\}$ $\quad \sum_{l<i\leq l+k} \tau'_{\pi(i)} < 0$.

And then, we can deduce from τ' a suspension over π.
- If two corner are empty, then we can assume that π decomposes as:

$$\left(\begin{array}{c|c|c} B & U & B \\ \hline & V & \end{array}\right)$$

with $\pi' = \left(\begin{smallmatrix} U \\ V \end{smallmatrix}\right)$ irreducible. Now we choose $b \in B$ and $c \in \mathcal{A}_1$, and then set:

$$\begin{aligned} \tau'_b &= \tau_b + 2\varepsilon \\ \tau'_c &= \tau_c + 2\varepsilon \\ \forall j \notin \{b,c\} \quad \tau'_j &= \tau_j. \end{aligned}$$

Then τ' defines a suspension over π for ε small enough. The theorem is proven. $\square$

4. Irrationality of linear involutions

For an interval exchange map $T = (\pi, \lambda)$ either the underlying permutation is reducible and then the transformation is never minimal or π is irreducible and T has the Keane's property (and hence is minimal) for almost every λ (see section 1). Furthermore T admits a suspension if and only if π is irreducible.
Hence the combinatorial set for which the dynamics of T is good coincides with the one for which the geometry is good. As we will see, the situation is more complicated in the general case. In this section we prove Theorem B and the second half of Theorem A.

4.1. Keane's property.

Definition 4.1. A linear involution has a connection (of length r) if there exist $(x, \varepsilon) \in X \times \{0, 1\}$ and $r \geq 0$ such that

- (x, ε) is a singularity for T^{-1}.
- $T^r(x, \varepsilon)$ is a singularity for T.

A linear involution with no connection is said to have the Keane's property.

Note that, by definition of a singularity, if we have a connection of length r starting from (x, ε), then $\forall r' < r$, $T^{r'}(x, \varepsilon)$ is not a singularity for T.
We first prove the following proposition:

Proposition 4.2. *Let T be a linear involution. The following statements are equivalent.*

1. *T satisfies the Keane's property.*
2. *$\mathcal{R}^n(T)$ is well defined for any $n \geq 0$ and the lengths of the intervals $\lambda^{(n)}$ tends to 0 as n tends to infinity.*

Moreover in the above situation the transformation T is minimal.

Proof of Proposition 4.2. We denote by $\lambda^{(n)}$ the length parameters of the map $\mathcal{R}^{(n)}(T)$, by $\pi^{(n)}$, $\sigma^{(n)}$, $(l^{(n)}, m^{(n)})$ the combinatorial data, and by $X^{(n)}$ the subinterval of X corresponding to $\mathcal{R}^{(n)}(T)$. Let us assume that T has no connection. Then all the iterates of T by the Rauzy-Veech induction are well defined. Indeed it is easy to see that T has the Keane's property if and only if its image $\mathcal{R}(T)$ by the Rauzy-Veech induction is well defined and has the Keane's property. Hence if T has the Keane's property, then by induction, all its iterates by $\mathcal{R}$ are well defined and have the Keane's property.

Now we have to prove that $\lambda^{(n)}$ goes to zero as n tends to infinity. Let $\mathcal{A}'$ be the subset of elements $\alpha \in \mathcal{A}$ such that $(\lambda_\alpha^{(n)})_n$ decreases an infinite number of time in the sequence $\{\mathcal{R}^n(T))\}_n$, and let $\mathcal{A}''$ be its complement.
Repeating the arguments for the Proposition and Corollary 1 and 2 of section 4.3 in [Yoc03], we have that:

- For n large enough, the permutation $\pi^{(n)}$ can be written as:
$$\left(\begin{array}{ccc|c} \alpha_1 & \dots & \alpha_{i_0} & *** \\ \hline \beta_1 & \dots & \beta_{j_0} & *** \end{array}\right),$$
with $\{\alpha_1, \dots, \beta_{j_0}\} = \mathcal{A}'' \sqcup \mathcal{A}''$

- For all $\alpha \in \mathcal{A}'$, $\lambda_\alpha^{(n)}$ tends to zero.

If $\mathcal{A}' = \mathcal{A}$, then the proposition is proven. So we can assume that $\mathcal{A}'$ is a strict subset of $\mathcal{A}$. Note that $\mathcal{A}'$ cannot be empty. Therefore, we must have

$$\sum_{i=1}^{i_0} \lambda_{\alpha_i} = \sum_{j=1}^{j_0} \lambda_{\beta_j},$$

for some $1 \leq i_0 \leq l^{(n)} - 1$ and $1 \leq j_0 \leq m^{(n)} - 1$. This means that $\mathcal{R}^n(T)$ has a connection of length zero, hence T has a connection. This contradicts the hypothesis. So we have proven that if T has no connections, then the sequence $\{\mathcal{R}^n(T))\}_n$ of iterates of T by the Rauzy-Veech induction is infinite and all length parameters of $\mathcal{R}^n(T)$ tend to zero when n tends to infinity.

Now we assume that T has a connection. So, there exists $u_0 = (x, \varepsilon)$ in $X \times \{0, 1\}$ which is a singularity of T^{-1}, and such that its sequence $u_1, \dots, u_m$ of iterates by T is finite, with u_m a singularity of T. We denote by $\overline{u_1}, \dots, \overline{u_m}$ the projections of $u_0, \dots, u_m$ on X. Let u_{min} be the element of $\{u_0, \dots, u_m\}$ whose corresponding projection to X is minimal. We have $u_{min} > 0$. If for all $n \geq 0$, the map $\mathcal{R}^n(T)$ is well defined and $u_{min} \in X^{(n)}$, then $X^{(n)}$ does not tend to zero, and hence there exists $\alpha \in \mathcal{A}$ such that $\lambda_\alpha^{(n)}$ does not tend to zero. Hence we can assume that there exists a maximal n_0 such that $\mathcal{R}^{n_0}(T)$ is well defined, and $X^{(n_0)}$ contains $\overline{u_{min}}$. We want to show that $\mathcal{R}^{n_0+1}(T)$ is not defined.

Assume that $\mathcal{R}^{n_0+1}(T)$ is defined, then $\overline{u_{min}} \notin X^{(n_0+1)}$. Since $\mathcal{R}^{n_0}(T)$ is an acceleration of T, there must exists an iterate of u_{min} by T, say u_k which is a singularity for $\mathcal{R}^{n_0}(T)$. Either $\overline{u_k}$ is in $X^{(n_0+1)}$, either it is its right end. However, $X^{(n_0+1)}$ does not contain $\overline{u_{min}}$, and $\overline{u_{min}} \leq \overline{u_k}$. Therefore, we must have $u_{min} = u_k$, so u_{min} is a singularity for $\mathcal{R}^{n_0}(T)$.

We prove in the same way that u_{min} is also a singularity for $\mathcal{R}^{n_0}(T)^{-1}$. This implies that we are precisely in the case when the Rauzy-Veech induction is not defined. Hence we have proven that if T has a connection, then either the sequence $(\mathcal{R}^n(T))_n$ is finite, either the length parameters do not all tend to zero.

This proves the first part of the proposition. Now let T be a linear involution on X satisfying the Keane's property. Recall that T is defined on the set $X \times \{0, 1\}$. Let us consider the first return map T_0 on $X \times \{0\}$. By definition of T_0, one has for each (x, ε) some return time $k = k(x, \varepsilon) > 0$ such that $T_0(x, \varepsilon) = T^k(x, \varepsilon)$. But T is piecewise linear thus for any (y, ε) in a small neighborhood of (x, ε), the return time $k(y, \varepsilon) = k(x, \varepsilon)$. Since the derivative of T is 1 if (x, ε) and $T(x, \varepsilon)$ belong to the same connected component and -1 otherwise, the derivative of T_0 is necessary 1. Hence T_0 is an *interval exchange* map. Obviously T_0 has no connexion since it is an acceleration of T, hence T_0 is minimal. Similarly, the first return map T_1 of T on $X \times \{1\}$ is also minimal. Since T satisfies Convention 2.7, any orbit of T is dense on $X \times \{0\}$ and $X \times \{1\}$ therefore T is minimal. The proof is complete. □

4.2. **Dynamical irreducibility.**

Definition 4.3. Let $T=(\pi,\lambda)$ be a linear involution. We will say that λ is *admissible* for π (or T has admissible parameters) if none of the following assertions holds:

(1) π decomposes as $\left(\frac{A|\ ***}{A|\ ***}\right)$, $\left(\frac{***\ |D}{***\ |D}\right)$ or $\left(\frac{A\cup B|\ D\cup B}{A\cup C|\ D\cup C}\right)$ with $A,D\subset\mathcal{A}_{01}$ and $B=\mathcal{A}_0$, $C=\mathcal{A}_1$ and A,D non empty in the two first cases.

(2) There is a decomposition of π as $\left(\frac{A\cup B|\qquad ***\qquad |B\cup D}{A\cup C|\ \alpha_0\ ***\ \alpha_0\ |C\cup D}\right)$, with (up to switching the top and the bottom of π) $A,D\subset\mathcal{A}_{01}$ and $\emptyset\neq B\subset\mathcal{A}_0$, $C\subset\mathcal{A}_1$ and the length parameters λ satisfy the following inequality

$$\sum_{\alpha\in C}\lambda_\alpha\leq\sum_{\alpha\in B}\lambda_\alpha\leq\lambda_{\alpha_0}+\sum_{\alpha\in C}\lambda_\alpha.$$

A generalized permutation π will be called *dynamically irreducible* if the corresponding set of admissible parameters is nonempty.

The set of admissible parameters of a generalized permutation is always open.

Remark 4.4. These two combinatorial notions of reducibility were introduced by the second author (see [Lan04]). Observe that if λ is not admissible for π, then $T=(\pi,\lambda)$ have a connection of length 0 or 1 depending on cases (1) or (2) of Definition 4.3, and is never minimal. More precisely there exist two invariant sets of positive measure.
One can also note that if π is irreducible then π is dynamically irreducible (the set of admissible parameters being the entire parameters space).

The length parameters for T cannot be linearly independent over $\mathbb{Q}$ since they must satisfy a nontrivial relation with integer coefficients. A linear involution $T=(\lambda,\pi)$ is said to have *irrational parameters* if $\{\lambda_\alpha\}$ generates a $\mathbb{Q}$-vector space of dimension $\#\mathcal{A}-1$. Almost all linear involutions have irrational parameters, and this property is preserved by the Rauzy-Veech induction.

Proof of Theorem B. If π is dynamical reducible, the non minimality comes from Remark 4.4. Conversely let us assume that π is a dynamical irreducible permutation and let $T=(\pi,\lambda)$ be a linear involution with irrational parameters and λ admissible for π.

We still denote by $\lambda^{(n)}$ the length parameters of $\mathcal{R}^{(n)}$ and by $\pi^{(n)}$, $\sigma^{(n)}$, $(l^{(n)},m^{(n)})$ the combinatorial data.

The proof has two steps: first we show using Proposition 4.2 that if T does not have the Keane's property, then there exists n_0 such that $\mathcal{R}^{n_0}(T)$ does not have admissible parameter (case (1) of Definition 4.3). Then we show that in this case λ is not admissible for π. This will imply the theorem.

First step: We assume that the sequence is finite. Then there exists $\mathcal{R}^{n_0}(T)$ that admits no Rauzy-Veech induction. Since $\lambda^{(n_0)}$ is irrational then either $\sigma^{(n_0)}(l^{(n_0)})=l^{(n_0)}+m^{(n_0)}$, or $l^{(n_0)}$ belongs to the only pair $\{i,\sigma^{(n_0)}(i)\}$ on the top of the permutation and $l^{(n_0)}+m^{(n_0)}$ belongs to the only pair $\{j,\sigma^{(n_0)}(j)\}$ on the bottom of the permutation. In each case, $\mathcal{R}^{n_0}(T)$ does not have admissible parameter (case (1)).

Now we assume that the lengths parameters do not all tend to zero. As in the proof of Proposition 4.2, for n large enough, the generalized permutation $\pi^{(n)}$ decomposes as:

$$\left(\begin{array}{ccc|c} a_1 & \dots & a_{i_0} & *** \\ \hline b_1 & \dots & b_{j_0} & *** \end{array}\right),$$

with $\{a_1,\dots,b_{j_0}\} = \mathcal{A}'' \sqcup \mathcal{A}''$, for some $\emptyset \neq \mathcal{A}'' \subset \mathcal{A}$ and some $1 \leq i_0 < l^{(n)}$ and $1 \leq j_0 < m^{(n)}$. Recall that

$$\sum_{i=1}^{i_0} \lambda_{\pi^{(n)}(i)} = \sum_{j=1}^{j_0} \lambda_{\pi^{(n)}(j)}.$$

The map $\mathcal{R}^n(T)$ has irrational parameters, therefore $\pi^{(n)}$ must decompose as:

$$\left(\begin{array}{c|c} A & *** \\ \hline A & *** \end{array}\right), \text{ or } \left(\begin{array}{c|c} *** & D \\ \hline *** & D \end{array}\right),$$

so $\mathcal{R}^n(T)$ does not have admissible parameter (case (1)).

Second step: It is enough to prove that if $T' = \mathcal{R}(T)$ does not have admissible parameter, then so is T. We can assume without loss of generality that the combinatorial Rauzy-Veech transformation is $\mathcal{R}_0$. We denote by π,σ,λ the data of T and by π',σ',λ' the data of T'. If π' decomposes as:

$$\left(\begin{array}{c|c} *** & D \\ \hline *** & D \end{array}\right),$$

let us consider l' the last element of the top line. Its twin $\sigma'(l')$ is on the bottom-right corner, but is not $l'+m'$. We denote by $\beta = \pi'(\sigma'(l')+1)$. Then it is clear that we obtain π by removing β from that place and putting it at the right-end of the bottom line. Then T does not have admissible parameter (case (1)).

Now we assume that π' decomposes as:

$$\left(\begin{array}{c|c} A & *** \\ \hline A & *** \end{array}\right).$$

If $\sigma'(l')$ is on the bottom line, the situation is analogous to the previous case. If not, then we denote by $\beta = \pi'(\sigma'(l')-1)$ and $\alpha = \pi'(l')$, and we get π by removing β from $\sigma'(l')-1$ and putting it on the right-end of the bottom line. If this place is in the top-right corner, then clearly, T does not have admissible parameter (case (1)). However, it might be the last element of the top-left corner. In that case, setting $A = A' \cup \{\beta\}$, the generalized permutation π decomposes as:

$$\left(\begin{array}{cc|c|c} A' & \alpha & *** & \alpha \\ \hline A'\cup\{\beta\} & & *** & \beta \end{array}\right),$$

with $\lambda_\beta = \lambda'_\beta > 0$ and $\lambda_\alpha = \lambda'_\alpha + \lambda'_\beta > \lambda_\beta$, hence T does not have admissible parameter (case (2)).

Now we assume that π' decomposes as

$$\left(\begin{array}{c|c} A\cup B & B\cup D \\ \hline A\cup C & C\cup D \end{array}\right).$$

Then we obtain π from π' by removing an element on the top-left corner or on the bottom-right corner, and putting it at the right-end of the bottom line. Then T does not have admissible parameter (case (1)). The other cases are similar. $\square$

5. Dynamics of the renormalized Rauzy-Veech induction

As we have seen previously, there are two notions of irreducibility for a linear involution.

- "Geometrical irreducibility" as stated in section 3, that we just called irreducibility.
- Dynamical irreducibility as stated in section 4.

In this section, we first prove that the set of irreducible linear involutions in an attractor for the renormalized Rauzy-Veech induction. Then we show that (analogously to the case of interval exchange transformations) the renormalized Rauzy-Veech induction is recurrent for almost all irreducible linear involutions.

5.1. **An attraction domain.**

Proof of the first part of Theorem C. We can find a non-zero pseudo-suspension $(\tau_\alpha)_{\alpha\in\mathcal{A}}$ (see Definition 3.7) otherwise it is easy to show that T does not have admissible parameter (case (1)). For all α, we denote by ζ_α the complex number $\zeta_\alpha = \lambda_\alpha + i\tau_\alpha$. Then, as in section 4.2, we consider a broken line L_0 which starts at 0, and whose edge number i is represented by the complex number $\zeta_{\pi(i)}$, for $1 \leq i \leq l$. Then we consider a broken line L_1, which starts on the same point as L_0, and whose edge number j is represented by the complex number $\zeta_{\pi(l+j)}$ for $1 \leq j \leq m$.

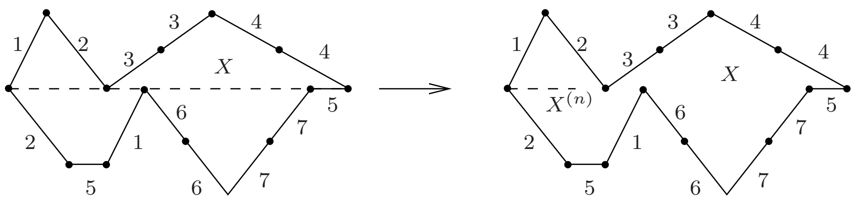


Figure 12. The transformation T is the first return map of the vertical foliation on a union of saddle connections.

Special case: We assume that L_0 and L_1 only intersect on their endpoints. Then they define a flat surface S, and T appears as a first return map of the vertical foliation on a segment X which is a union of horizontal saddle connections (see Figure 12). After n steps of the Rauzy-Veech induction, the resulting linear involution $\mathcal{R}^n(T)$ is the first return map of the vertical flow of S on a shorter segment $X^{(n)}$, which is adjacent to the same singularity as X. Since T has no connection, then the length of $X^{(n)}$ tends to zero when n tends to infinity by the first part of Proposition 4.2. Hence for n large enough, $\mathcal{R}^n(T)$ is the first return map of the vertical flow of S on a segment, adjacent to a singularity, and with no singularities in its interior. With our construction of S, it is clear that any vertical saddle connection would intersect X and would give a connexion on S. Since T has no connection, the surface S has no vertical saddle connection (note that this is not true in general for a first return map on a transverse segment). According to Proposition 2.13, $(\pi^{(n)}, \lambda^{(n)})$ admits a suspension and hence Theorem 3.2 implies that $\pi^{(n)}$ is irreducible. The theorem is proven for that case.

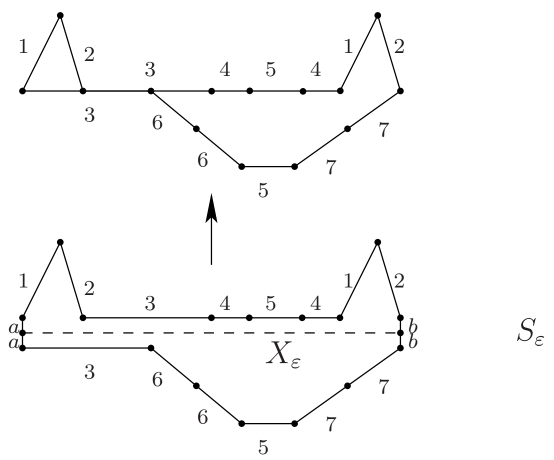


FIGURE 13. Constructing T as a first return map on a regular segment of a surface S_ε.

General case: The two broken lines L_0 and L_1 might have other intersection points. We first show this still defines a flat surface. We consider the line L_0^ε that starts at the complex number $2i\varepsilon$. Then we join the first points of L_0^ε and L_1 by a vertical segment, and do the same for their ends points (see Figure 13). This defines a polygon and the non vertical sides come by pairs, so we can glue them as previously. There are two vertical segments left. We decompose each vertical segment into a pair of vertical segments of the same length and glue them together (see the figure). This creates a pole for each initial segment. We denote by S_ε the resulting flat surface. The first return map of the vertical flow on the horizontal segment X_ε joining the two poles is T. The surface S_ε has two vertical saddle connections of length ε starting from the poles, but there is no other vertical saddle connection on S_ε since T has no connections. When ε tends to zero, the two vertical saddle connections are the only ones that shrink to zero. Hence there is no loop that shrink to zero. Furthermore, the initial pseudo-suspension is nonzero, so the area of S_ε is bounded from below. Hence, the surface S_ε does not degenerate when ε tends to zero and so there exists a sequence (ε_k) that tends to zero such that (S_{ε_k}) tends to a surface S.

The segment $X \subset S$ corresponding to the limit of X_{ε_k}, as k tends to infinity, might be very complicated and the first return map on X is not well defined.

The transformation $\mathcal{R}^n(T)$ is the first return map of the vertical flow of S_{ε_k} on a shortest horizontal segment $X_\varepsilon^{(n)}$, adjacent to one of the poles. If n is large enough, then the segment $X^{(n)} \subset S$ corresponding to the limit of $X_{\varepsilon_k}^{(n)}$ has no singularity on its interior. Since the surgery corresponding to contracting ε does not change the the vertical foliation, the first return map of the vertical foliation of S on $X^{(n)}$ is precisely $\mathcal{R}^n(T)$.

As in the special case, the surface S does not have any vertical saddle connection, so the generalized permutation corresponding to $\mathcal{R}^n(T)$ is irreducible and the proposition is proven.

□

5.2. **Recurrence.** The following lemma is analogous to Proposition 9.1 in [Vee82].

Lemma 5.1. *Let T be a linear involution on $X = (0, L)$ with no connection and let $(x, \varepsilon) \in X \times \{0, 1\}$ be a singularity for T. Let $X^{(n)} \subset X$ be the subinterval corresponding to the linear involution $\mathcal{R}^n(T)$. There exists $n > 0$ such that $X^{(n)} = (0, x)$.*

Proof. Since T has no connection, there exists a first $n > 0$ such that $x \notin X^{(n)}$. So $x \in X^{(n-1)}$, and (x, ε) is still a singularity for $\mathcal{R}^{n-1}(T)$. We obtain $\mathcal{R}^n(T)$ from $\mathcal{R}^{n-1}(T)$ by considering the first return map on the largest subinterval $X^{(n)} \subset X^{(n-1)}$ whose right endpoint corresponds to a singularity of $\mathcal{R}^{n-1}(T)$. So $X^{(n)} = (0, x)$. □

Let π_0 be an irreducible generalized permutation, and let C be the set of generalized permutations that can be obtained by iterations of the maps $\mathcal{R}_0$ and $\mathcal{R}_1$ (when possible).

We define $\mathcal{T}_C = \{(\pi, \zeta),\ \pi \in C, \zeta \text{ is a suspension data for } \pi\}$. We have defined the Rauzy-Veech map on the space $\mathcal{T}_C$. It defines an almost everywhere invertible map: If $\sum_{i=1}^{l} Im(\zeta_{\pi(i)}) \neq 0$ then (π, ζ) has exactly one preimage for $\mathcal{R}$.

We define the quotient $\mathcal{Q}_C$ of $\mathcal{T}_C$ by the equivalence relation generated by $(\pi, \zeta) \sim \mathcal{R}(\pi, \zeta)$.

One will denote by m the natural Lebesgue measure on $\mathcal{T}_C$ i.e. $m = d\pi d\zeta$, where $d\zeta$ is the natural Lebesgue measure on the hyperplane $\sum_{i=1}^{l} \zeta_{\pi(i)} = \sum_{j=l+1}^{2d} \zeta_{\pi(j)}$, and $d\pi$ is the counting measure. The mapping $\mathcal{R}$ preserves m, so it induces a measure, again denoted by m on $\mathcal{Q}_C$.

The matrix g_t acts on $\mathcal{T}_C$ by $g_t(\pi, \zeta) = (\pi, (g_t(\zeta_\alpha))_\alpha)$, where g_t acts on $\zeta_\alpha \in \mathbb{C} = \mathbb{R}^2$ linearly. This action preserves the measure m on $\mathcal{T}_C$ and commutes with $\mathcal{R}$, so it descends to a measure preserving flow on $\mathcal{Q}_C$ called the Teichmüller flow.

If (π, ζ) is a suspension data, we denote by $|Re(\zeta)|_\pi$ the length of the corresponding interval, *i.e.* $\sum_{i=1}^{l} Re(\zeta_{\pi(i)})$. The subset

$$\Big\{(\pi, \zeta) \in \mathcal{T}_C;\ 1 \leq |Re(\zeta)|_\pi \leq 1 + \min\big(Re(\zeta_{\pi(l)}), Re(\zeta_{\pi(2d)})\big)\Big\}$$

is a fundamental domain of $\mathcal{T}_C$ for the relation $\sim$ and the first return map of the Teichmüller flow on

$$\mathcal{S} = \{(\pi, \zeta);\ \pi \in C,\ |Re(\zeta)|_\pi = 1\}/\sim$$

is the renormalized Rauzy-Veech induction on suspensions.

Proposition 5.2. *The zippered rectangle construction provides a finite covering Z from $\mathcal{Q}_C$ to a subset of full measure in a connected component of a stratum $\mathcal{Q}(k_1, \dots, k_n)$ of the moduli space of quadratic differentials. The degree of this cover is $h!$ where h is the dimension of the stratum. Moreover $h = 2g + n - 2$ (g is the genus of the surfaces).*

Proof. Let S be a (generic) flat surface in $\mathcal{Q}(k_1,\dots,k_n)$ with no vertical and no horizontal saddle connection. Consider a horizontal separatrix l adjacent to a given singularity P. We call *admissible* a segment X adjacent to P, such that the vertical geodesic passing through the right endpoint of X meets a singularity before returning to X, in positive or negative direction. Then Proposition 2.13 implies that there exists a corresponding suspension datum ζ such that $S = Z(\pi,\zeta)$. Conversely, any ζ such that $S = Z(\pi,\zeta)$ is obtained by this construction.

Now let X_0, X_1 be two admissible segments, and let ζ_0, ζ_1 be the corresponding suspension data. One can assume without loss of generality that $X_0 \subset X_1$ and their left endpoint is the singularity P. Let T_0, T_1 be the linear involutions corresponding to X_0, X_1. The right endpoint of X_0 corresponds to a singularity of T_1. Hence there exists $n \geq 0$ such that $\mathcal{R}^n(T_1) = T_0$, and therefore $\mathcal{R}^n(\zeta_1) = \zeta_0$.

So we have proven that for each separatrix l adjacent to a singularity, there is only one preimage of S by the mapping Z. So Z is a finite covering. The degree of Z is obvious by construction. If $2h$ is the number of possible choices of horizontal separatrices then the degree of Z is $h!$ (choices of labels and the choice of the intervals $X \times \{0,1\}$).

For each singularity, one has $k_i + 2$ separatrices. Thus

$$2h = \sum_{i=1}^{n}(k_i+2) = 4g-4+2n = 2(2g+n-2).$$

The proposition is proven. □

Proof of the second part of Theorem C. The subset $\mathcal{Q}_C^1$ corresponding to surfaces of area 1 is a finite ramified cover of a connected component of a stratum of quadratic differentials, and the corresponding Lebesgue measures are proportional.

By Theorem 0.2 in [Vee90] the volume of the moduli space of quadratic differentials is finite, and so, $\mathcal{Q}_C^1$ has finite measure. Hence the Teichmüller geodesic flow on $\mathcal{Q}_C$ is recurrent for the Lebesgue measure. Recall that the Rauzy-Veech renormalization for suspensions $\mathcal{R}_r$ is the cross section of the Teichmüller geodesic flow on $\mathcal{S}$; therefore the Rauzy-Veech renormalization for suspension is recurrent.

We have $d\zeta = d\lambda d\tau$, and the Rauzy-Veech induction commutes with the projection $(\pi,\zeta) \mapsto (\pi,\lambda)$. So, for almost all parameters λ, the sequence $(\mathcal{R}_r^n(\pi,\lambda))_n$ is recurrent. □

Remark 5.3. Note that the proof of theorem C does not use the fact that a linear involution satisfying the Keane's property is minimal. We can use this theorem to give an alternative proof of the minimality of such map. let T be a linear involution with the Keane's property. From Theorem C, there exists $n \geq 0$ such that $\mathcal{R}^n(T) = (\pi,\lambda)$ is the cross section of the vertical foliation on a flat surface with no vertical saddle connection. Any infinite vertical geodesic on such a surface is dense (see e.g. [MT02]). Thus $\mathcal{R}^n(T)$ is minimal and so is T.

6. Rauzy classes

As we have seen previously, the irreducible generalized permutations are an attractor for the Rauzy-Veech induction. In this section, we prove that there is no smaller attractor. We also prove Theorem D.

We first define the Rauzy classes and then the extended Rauzy classes.

Given a permutation π, we can define *at most* two other permutations $\mathcal{R}_\varepsilon(\pi)$ with $\varepsilon = 0, 1$ when $\mathcal{R}_\varepsilon$ is well defined. The relation $\pi \sim \mathcal{R}_\varepsilon(\pi)$ generates a partial order on the set of generalized permutations; we represent it as a directed graph G, and as for permutations, we will call Rauzy classes the connected components of this graph.

In the case of interval exchanges, the periodicity of the maps $\mathcal{R}_0$ and $\mathcal{R}_1$ gives an easy proof of the fact that the above relation is an equivalence relation (proposition of section 1.3.3). Here the argument fails because these maps are not always defined, and it may happen that $\mathcal{R}_0(\pi)$ is well defined, but not $\mathcal{R}_0^2(\pi)$. However, the corresponding statement is still true.

Proposition 6.1. *The above partial order is an equivalence relation on the set of irreducible generalized permutations.*

Proof. Let π and π' be two generalized permutations. Assume that there is a sequence of maps $\mathcal{R}_0$ and $\mathcal{R}_1$ that sends π to π'. If $\pi' = \mathcal{R}_\varepsilon(\pi'')$, then for any parameters λ', there exist parameters λ'' such that $\mathcal{R}(\pi'', \lambda'') = (\pi', \lambda')$. Iterating this argument, there exists (π, λ^0) and n_0 such that $\mathcal{R}^{n_0}(\pi, \lambda^0) = (\pi', \lambda')$. But for any λ in a sufficiently small neighborhood U of λ^0, the generalized permutation corresponding to $\mathcal{R}^{n_0}(\pi, \lambda^0)$ is π'.

Recall that renormalized Rauzy-Veech induction map is recurrent (Theorem C) thus one can find $\lambda \in U$ such that the sequence $(\mathcal{R}_r^n(\pi, \lambda))_n$ come back in a neighborhood of (π, λ) infinitely many time. Furthermore, $\mathcal{R}_r^{n_0}(\pi, \lambda) = (\pi', \lambda^{(n_0)})$. Thus $(\mathcal{R}_r^n(\pi, \lambda))_n$ gives a sequence of generalized permutations that reach π' and then reach π. So, it gives a combination of the maps $\mathcal{R}_0$ and $\mathcal{R}_1$ that sends π' to π. This proves the proposition. $\square$

Definition 6.2. Let $2d = l + m$. We define the symmetric permutation s of $\{1, \ldots, 2d\}$ by $s(i) = 2d + 1 - i$, $\forall i = 1, \ldots, 2d$. If π is a generalized permutation of type (l, m) defined over an alphabet $\mathcal{A}$ of d letters, we define the generalized permutation $s\pi$ to be of type (m, l) by

$$(s\pi)(k) := \pi \circ s(k).$$

We start from an irreducible generalized permutation π and we construct the subset of irreducible generalized permutation that can be obtained from π by some composition of the maps $\mathcal{R}_0$, $\mathcal{R}_1$, and s. The quotient of this set by the equivalence relation generated by $\pi \sim f \circ \pi$ for any bijective map f from $\mathcal{A}$ onto $\mathcal{A}$ is called the *extended Rauzy class* of π.

Remark 6.3. The quotient by the equivalence relation generated by $\pi \sim f \circ \pi$ means that we look at generalized permutations defined up to renumbering. This is needed for technical reasons in the proof of Theorem D.

Remark 6.4. In opposite to the case of interval exchange maps, the definition of irreducibility we gave in section 3 is not invariant by the map s: for instance, the generalized permutation $\pi = \left(\begin{smallmatrix} 1 & 2 & 1 & & \\ 2 & 3 & 3 & 4 & 4 \end{smallmatrix}\right)$ is irreducible while $s\pi = \left(\begin{smallmatrix} 4 & 4 & 3 & 3 & 2 \\ 1 & 2 & 1 & & \end{smallmatrix}\right)$ is reducible.

So an extended Rauzy class is obtained after considering the set of generalized permutations obtained from π by the extended Rauzy operations, and intersecting this set by irreducible generalized permutations. The results from the previous section shows that our definition of irreducibility is the good one with respect to the Rauzy-Veech induction, but we see that the convention of the "left-end singularity" is a real choice.

Remark 6.5. Let T be a linear involution defined on an interval $X = (0, L)$. Recall that Rauzy-Veech induction applied on T consists in considering the first return map on $(0, L')$, where L' is the maximal element of $(0, L)$ that corresponds to a singularity of T. In terms of generalized permutation, this corresponds to the $\mathcal{R}_\varepsilon$ mapping.

One can consider the first return map of T on the interval (L'', L), where L'' is the minimal element of $(0, L)$ that corresponds to a singularity of T. In terms of generalized permutations, this corresponds to the the conjugaison of $s \circ \mathcal{R}_\varepsilon \circ s$ map. We will call this the "Rauzy-Veech induction of T by cutting on the left of X", while the usual Rauzy-Veech induction will on the opposite called the "Rauzy-Veech induction of T by cutting on the right of X".

Proof of Theorem D. Let π_1 be an irreducible generalized permutation. The corresponding set of suspension data is connected (even convex), so the set of surfaces constructed from a suspension data, using the zippered rectangle construction, belongs to a connected component of the moduli space of quadratic differentials.

It is also open and invariant by the action of the Teichmüller geodesic flow, hence it is a subset of full measure by ergodicity.

Let π_2 be a generalized permutation that corresponds to the same connected component of the moduli space. Then there exists a surface S and two segments X_1 and X_2, each one being adjacent to a singularity x_1 and x_2, such that for each i, the linear involution T_i given by the first return maps on X_i has combinatorial data π_i. We can assume that S has no vertical saddle connection.

We recall that each X_i has an orientation so that the corresponding singularity x_i is in its left end. Consider the vertical separatrix l starting from x_2, in the positive direction and let y_1 be its first intersection point with $X_1 \cup \{x_1\}$.

Applying the usual Rauzy-Veech induction for T_2, the map $\mathcal{R}^n(T_2)$ is a first return map of the vertical flow on a subinterval $X_2^{(n)} \subset X_2$, adjacent to x_2. If n is large enough, then $\mathcal{R}^n(T_2)$ is isomorphic to the first return map on the subinterval $(y_1, y_2) \subset X_1$, of the same length as $X_2^{(n)}$. We assume first that $y_1 < y_2$, hence this first return map is consistent with the positive direction on X_1.

We now have to apply Rauzy-Veech inductions (on the right and on the left) on T_1 until we get a first return map on (y_1, y_2) with corresponding generalized permutation π_3. Since π_3 is by construction, up to renumbering the alphabet, in the same Rauzy class as π_2, we will therefore find some composition of the maps $\mathcal{R}_\varepsilon$, $s \circ \mathcal{R}_\varepsilon \circ s$ that send π_1 to π_2.

Note that y_2 might not correspond a priory to some singularities of T_1, so naive Rauzy-Veech induction on X_1 might miss the interval (y_1, y_2). But $(y_1, 0)$ or $(y_1, 1)$ is a singularity, so we can cut the interval on the left until y_1 is the left end, this will eventually occurs because of Lemma 5.1. Then after cutting on the right y_2 will become the right end of the corresponding interval.

If $y_2 < y_1$, then similarly, by cutting on the right and then on the left, we get two linear involutions corresponding to first returns maps that only differ by a different choice of orientation. Hence we have found some composition of the maps $\mathcal{R}_\varepsilon$, $s \circ \mathcal{R}_\varepsilon \circ s$ that send π_1 to some π_3, such that $s\pi_3$ is in the same Rauzy class as π_2.

Hence we have proved that if two irreducible generalized permutations correspond to the same connected component, then they are in the same extended Rauzy class. To prove the converse, we must consider a slightly more general kind of suspensions that do not necessary corresponds to a singularity on the left. The corresponding "extended" suspension data satisfy

(1) $\forall \alpha \in \mathcal{A} \quad Re(\zeta_\alpha) > 0$.
(2) $\forall 1 \leq i \leq l-1 \quad t + Im(\sum_{j \leq i} \zeta_{\pi(j)}) > 0$
(3) $\forall 1 \leq i \leq m-1 \quad t + Im(\sum_{1 \leq j \leq i} \zeta_{\pi(l+j)}) < 0$
(4) $\sum_{1 \leq i \leq l} \zeta_{\pi(i)} = \sum_{1 \leq j \leq m} \zeta_{\pi(l+j)}$

for some $t \in \mathbb{R}$ (the case $t = 0$ corresponds to suspension data as seen previously).

Then we can extend the zippered rectangle construction to these extended suspension data. As in the usual case, the space of extended suspension data corresponding to a generalized permutation is convex, so the set of surfaces corresponding to a given generalized permutation belong to a connected component of stratum. Then it is easy to see that if π' is obtained from π by the map $\mathcal{R}_0$, $\mathcal{R}_1$ or s, then the corresponding connected component is the same. $\square$

Historically, extended Rauzy classes have been used to prove the non connectedness of some stratum of Abelian differentials (see for instance [Vee90]). For this case, some topological invariants were found by Kontsevich and Zorich [KZ03] (hyperellipticity and spin structure). For the case of quadratic differentials, all non-connected components (except four special cases) are distinguished by hyperellipticity [Lan04]. For the four "exceptional ones", the only known proof up to now is an explicit computation of the corresponding extended Rauzy classes. Theorem D, which is now formally proven complete the proof of the following

Theorem (Zorich)**.** *The strata $\mathcal{Q}(-1, 9)$, $\mathcal{Q}(-1, 3, 6)$, $\mathcal{Q}(-1, 3, 3, 3)$ and $\mathcal{Q}(12)$ are non connected.*

Proof. The generalized permutations $\left(\begin{smallmatrix} 1 & 1 & 2 & 3 & 2 & 3 & 4 \\ 5 & 4 & 5 & 6 & 7 & 6 & 7 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 7 & 5 & 7 & 6 & 4 \end{smallmatrix}\right)$ are irreducible. The corresponding suspension surfaces belong to the stratum $\mathcal{Q}(-1, 9)$. According to Zorich's computation, these two permutations do not belong to the same extended Rauzy classes (see Table 1 in the Appendix). Hence the stratum $\mathcal{Q}(-1, 9)$ is not connected. In fact this stratum has precisely two connected components corresponding to the two extended Rauzy classes.

We have similar conclusions for other strata with the following generalized permutations.
For the stratum $\mathcal{Q}(-1,3,6)$ one can consider the generalized permutations

$$\left(\begin{smallmatrix} 1 & 1 & 2 & 3 & 2 & 3 & 4 & 5 \\ 4 & 6 & 5 & 6 & 7 & 8 & 7 & 8 \end{smallmatrix}\right) \qquad \text{and} \qquad \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 2 & 3 \\ 7 & 1 & 7 & 6 & 5 & 4 & 8 & 8 \end{smallmatrix}\right).$$

For the stratum $\mathcal{Q}(-1,3,3,3)$ one can consider the generalized permutations

$$\left(\begin{smallmatrix} 1 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 6 \\ 7 & 8 & 5 & 8 & 2 & 4 & 9 & 3 & 9 \end{smallmatrix}\right) \qquad \text{and} \qquad \left(\begin{smallmatrix} 1 & 1 & 2 & 3 & 2 & 3 & 4 & 5 & 6 \\ 4 & 7 & 8 & 9 & 7 & 8 & 6 & 5 & 9 \end{smallmatrix}\right).$$

For the stratum $\mathcal{Q}(12)$ one can consider the generalized permutations

$$\left(\begin{smallmatrix} 1 & 2 & 1 & 2 & 3 & 4 & 5 & 3 \\ 6 & 7 & 6 & 7 & 5 & 8 & 4 & 8 \end{smallmatrix}\right) \qquad \text{and} \qquad \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 6 \\ 8 & 7 & 5 & 8 & 4 & 3 & 2 & 1 \end{smallmatrix}\right).$$

The theorem is proven. $\square$

Appendix A. Computation of the Rauzy classes

Here we give explicit examples of *reduced* Rauzy classes (*i.e.* up to the equivalence $\pi \sim f \circ \pi$, for any permutation f of $\mathcal{A}$).

It is easy to see that there is only one Rauzy class filled by (irreducible) generalized permutations defined over 3 letters. In that case the Rauzy class contains 4 generalized permutations and a permutation is irreducible if and only if it is dynamically irreducible. Thus there is no interesting phenomenon in this "simple" case.

If we consider a slightly more complicated case, for instance permutations defined over 4 letters we get some interesting phenomenon. Figure 14 illustrates such a Rauzy class. It corresponds to the stratum $\mathcal{Q}(2,-1,-1)$. The generalized permutations $\left(\begin{smallmatrix} 1&1&2&2&3 \\ 4&3&4&& \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1&2&1&& \\ 3&3&4&4&2 \end{smallmatrix}\right)$ are not formally in the Rauzy class since they are reducible, but we can see there is concretely the "attraction" phenomenon. As we can see the (reduced) Rauzy classes for generalized permutations are in general much more complicated than the one for usual permutation since the vertex are either of valence one or of valence two. In Figure 15 we present a more complicated case with an "unstable" set of permutations.

We end this section with an explicit calculation of the cardinality of the Rauzy classes of the four exceptional strata (performed with Anton Zorich's software [Zor06]).

connected components	representatives elements	cardinality of extended Rauzy classes
$\mathcal{Q}(-1,9)^{adj}$	$\left(\begin{smallmatrix} 1&1&2&3&2&3&4 \\ 5&4&5&6&7&6&7 \end{smallmatrix}\right)$	95944
$\mathcal{Q}(-1,9)^{irr}$	$\left(\begin{smallmatrix} 1&1&2&3&4&5&6 \\ 3&2&7&5&7&6&4 \end{smallmatrix}\right)$	12366
$\mathcal{Q}(-1,3,6)^{adj}$	$\left(\begin{smallmatrix} 1&1&2&3&2&3&4&5 \\ 4&6&5&6&7&8&7&8 \end{smallmatrix}\right)$	531674
$\mathcal{Q}(-1,3,6)^{irr}$	$\left(\begin{smallmatrix} 1&2&3&4&5&6&2&3 \\ 7&1&7&6&5&4&8&8 \end{smallmatrix}\right)$	72172
$\mathcal{Q}(-1,3,3,3)^{adj}$	$\left(\begin{smallmatrix} 1&1&2&3&4&5&6&7&6 \\ 7&8&5&8&2&4&9&3&9 \end{smallmatrix}\right)$	612838
$\mathcal{Q}(-1,3,3,3)^{irr}$	$\left(\begin{smallmatrix} 1&1&2&3&2&3&4&5&6 \\ 4&7&8&9&7&8&6&5&9 \end{smallmatrix}\right)$	88374
$\mathcal{Q}(12)^{adj}$	$\left(\begin{smallmatrix} 1&2&1&2&3&4&5&3 \\ 6&7&6&7&5&8&4&8 \end{smallmatrix}\right)$	881599
$\mathcal{Q}(12)^{irr}$	$\left(\begin{smallmatrix} 1&2&3&4&5&6&7&6 \\ 8&7&5&8&4&3&2&1 \end{smallmatrix}\right)$	146049

Table 1. Representatives elements for the special strata.

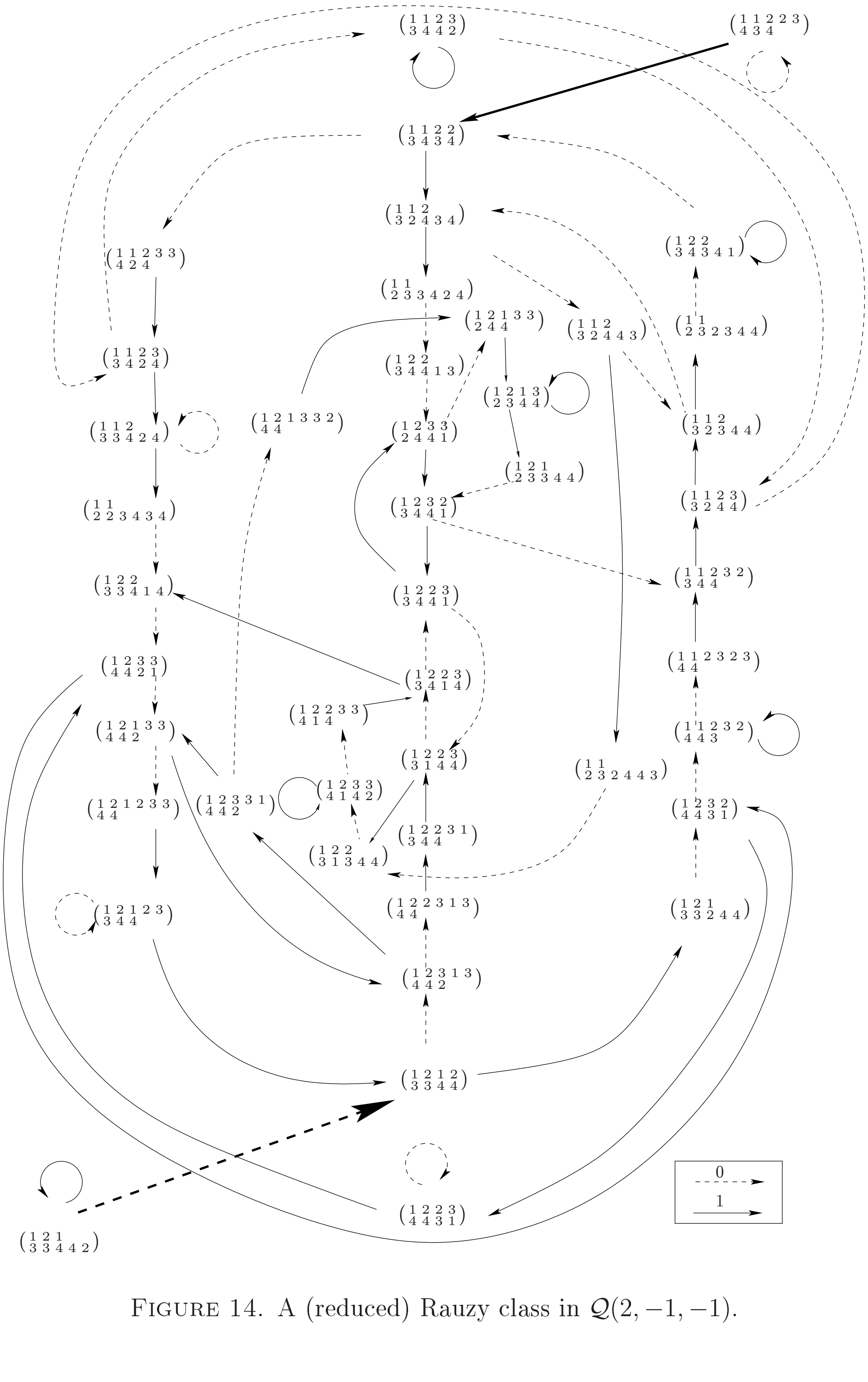


FIGURE 14. A (reduced) Rauzy class in $\mathcal{Q}(2,-1,-1)$.

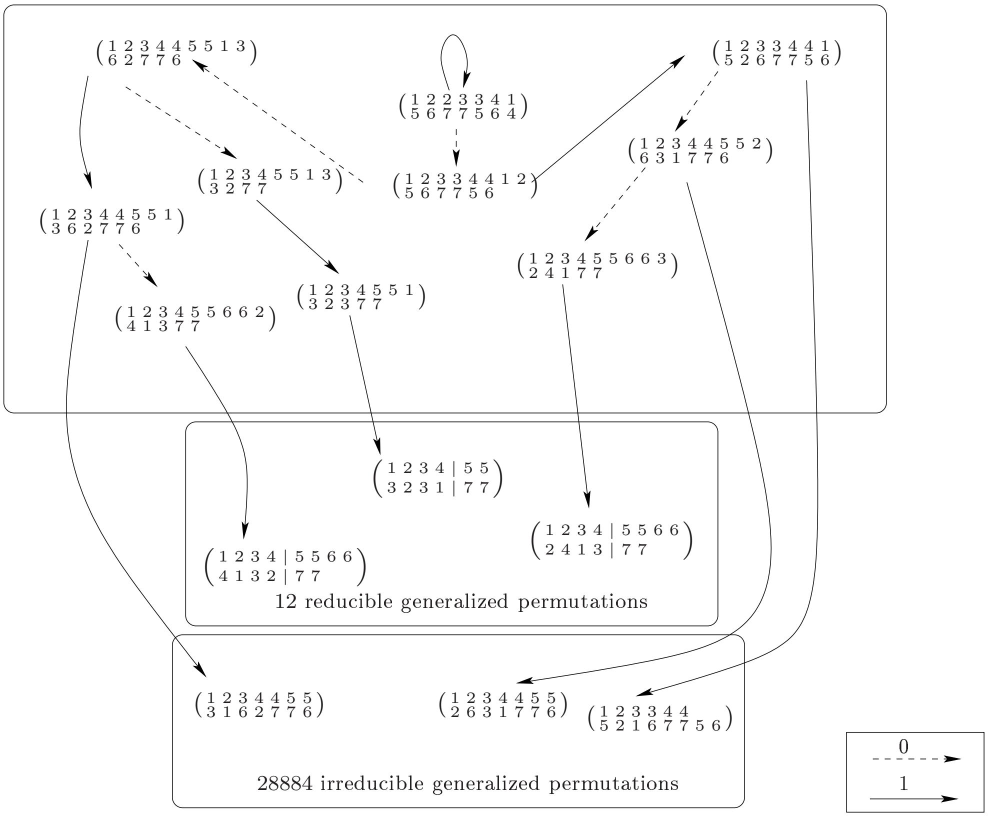


FIGURE 15. An example of a Rauzy class. The corresponding stratum is $\mathcal{Q}(-1,-1,-1,7)$. There are 28906 permutations in the whole "class" and 28884 permutations in the "good" Rauzy class. The $28906 - 28884 = 22$ remaining permutations belong to the reducible part (12 permutations) and the "unstable" part (10 permutations). Note that there is no smaller attractor set: the three irreducible permutations belong to the same Rauzy class. Let us also note that the extended Rauzy class has 38456 elements.

Appendix B. An other definition of the extended Rauzy class

In section 6, we have defined an extended Rauzy class by considering the set of generalized permutations obtained from an irreducible permutation π by the extended Rauzy operations. This set is not in general a subset of the irreducible generalized permutations, therefore we must intersect it with the set of irreducible generalized permutations to get an extended Rauzy class.

One could also define an extended Rauzy class in the following way: it is a minimal subset of the irreducible generalized permutations stable by the operations $\mathcal{R}_0$, $\mathcal{R}_1$, and s. It is equivalent to say that we forbid the operation s for π' such that $s\pi'$ is reducible. For the purpose of this section, let us call this new class a *weakly extended Rauzy class*.

A priory, an extended Rauzy class is a union of weakly extended Rauzy classes. We will prove:

Proposition B.1. *The extended Rauzy classes and the weakly extended Rauzy classes coincide.*

Proof. All we have to prove is that if two irreducible generalized permutations π_1 and π_2 correspond to the same connected component of a stratum of quadratic differentials, then we can join them (up to relabelling) by a combination of the maps $\mathcal{R}_0$, $\mathcal{R}_1$, and s, such that *all* the corresponding intermediary generalized permutations are irreducible. Recall that if π is irreducible, then so are $\mathcal{R}_0(\pi)$ and $\mathcal{R}_1(\pi)$ (when defined).

The idea is now to modify the proof of Theorem D, by using the three following elementary remarks. Let ζ be a suspension datum over an irreducible generalized permutation π (of type (l,m)).

(1) In Remark 3.5 we gave a condition in order to have $Im(\sum_{i=1}^{l}\zeta_{\pi(i)})=0$. Equivalently if a decomposition of π holds then there is no empty corner. It is obvious to check that, under this condition, $s\pi$ is irreducible.
(2) Let us assume that the two lines joining the end points of L_0 and the end points of L_1 do not have any other intersection point with L_0 and L_1. Then applying to ζ_i the matrix $\left(\begin{smallmatrix}1&0\\t&1\end{smallmatrix}\right)$ for a suitable t, we get a new suspension data ζ' over T with $Im(\sum_{i=1}^{l}\zeta'_{\pi(i)})=0$. Hence $s\pi$ is irreducible.
(3) Let $k\leq l$ minimize the value $Im(\sum_{i\leq k}\zeta_{\pi(i)})$. Lemma 5.1 implies that there exists $n>0$ such that $\mathcal{R}^n(T)$ is the first return map of T to the subinterval $\Big(0, Re(\sum_{i=1}^{k}\lambda_i)\Big)$. Let us consider $(\pi^{(n)},\zeta^{(n)})=\mathcal{R}^n(\pi,\zeta)$. By construction $\zeta^{(n)}$ satisfies the previous condition, hence $s\pi^{(n)}$ is irreducible.

Let us now prove the proposition. Let π_1 and π_2 be two generalized permutations in the same extended Rauzy class. The proof of Theorem D asserts that there exists a surface S and two segments X_1 and X_2, each one being adjacent to a singularity x_1 and x_2, such that for each i, the linear involution T_i given by the first return maps on X_i has combinatorial datum π_i. We can assume that S has no vertical saddle connection.

The previous remark implies that, up to replacing T_1 by some $\mathcal{R}^{n_0}(T_1)$ for some well chosen n_0, one can and do assume that $s\pi_1$ is irreducible. Let (π_1,ζ) be the suspension over T_1 that corresponds to the surface S, then up to applying to ζ the matrix $\left(\begin{smallmatrix}1&0\\t&1\end{smallmatrix}\right)$ for a suitable t (which does not change the vertical foliation), we can assume that $Im(\sum_{i=1}^{l}\zeta_{\pi(i)})=0$.

For n large enough, $\mathcal{R}^n(T_2)$ is isomorphic to the first return map on a subinterval (y_1,y_2) of X_1, with $(y_1,0)$ or $(y_1,1)$ a singularity of T_1. Let $k\leq l$ that minimizes the value $Im(\sum_{i\leq k}\zeta_{\pi(i)})$ and let $x\in X_1$ be the corresponding point. If $y_1<x$ then we also have $y_2<x$ (since y_2 can be chosen arbitrarily close to y_1). We then apply the Rauzy-Veech induction to T_1 until we get a first return map on (x_1,x). If $y_1>x$ then we also have $y_2>x$. By definition ζ is a suspension data over $(\lambda,s\pi_1)$ (i.e. we are “rotating by 180°” the polygon and the linear involution T_1). We apply the Rauzy-Veech induction on $(\lambda,s\pi_1)$ until we get a a first return map on (x'_1,x) that contains y_1,y_2. The result is a linear involution $T_1' = (\lambda', \pi')$ such that $s\pi'$ is irreducible, and a suspension ζ' over T_1'. As before we can assume that $Im(\sum_{i=1}^{l} \zeta'_{\pi'(i)}) = 0$ and then $(\zeta', s\pi_1')$ is a suspension over $(\lambda', s\pi_1')$ that corresponds to a first return map of T_1 on the subinterval $X_1' = (x, x_1')$. Moreover the sequence of generalized permutations joining π_1 to $s\pi_1'$ corresponding to our description consists entirely of irreducible elements.

Iterating this argument, there will be a step where the point x'' minimizing the value $Im(\sum_{i\leq k} \zeta''_{\pi''(i)})$ is precisely y_1 (because the surface admits a finite number of vertical separatrices starting from the singularities). The same argument produces a sequence of irreducible generalized permutations joining π'' to π_2.

This proves the equivalence of the two definitions of the extended Rauzy classes. □

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