TRACE FIELD DEGREES IN THE TORELLI GROUP

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Abstract. We show that for $g \geq 2$, all integers $1 \leq d \leq 3g-3$ arise as trace field degrees of pseudo-Anosov mapping classes in the Torelli group of the closed orientable surface of genus g. Our method uses the Thurston-Veech construction of pseudo-Anosov maps, and we provide examples where the stretch factor has algebraic degree any even number between two and $6g-6$. This validates a claim by Thurston from the 1980s.

1. Introduction

Let S_g be the closed orientable surface of genus $g \geq 2$. A homeomorphism f of S_g is *pseudo-Anosov* if there exists a pair of transverse singular measured foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ and a real number $\lambda > 1$ such that $f(\mathcal{F}^u) = \lambda\mathcal{F}^u$ and $f(\mathcal{F}^s) = \lambda^{-1}\mathcal{F}^s$. The number $\lambda > 1$ is the *stretch factor* of the pseudo-Anosov map, and is an algebraic integer. The degree of the field extension $\mathbb{Q}(\lambda) : \mathbb{Q}$ is the *stretch factor degree*, and is bounded from above by the dimension of the Teichmüller space of S_g, namely $6g-6$ [Thu88].

There is another field which plays a central role in this article: $\mathbb{Q}(\lambda + \lambda^{-1})$. This field is the *trace field* and is uniquely determined by the pair $(\mathcal{F}^u, \mathcal{F}^s)$ [KS00, GJ00]. The degree of the field extension $\mathbb{Q}(\lambda + \lambda^{-1}) : \mathbb{Q}$ is the *trace field degree*, and is bounded from above by $3g-3$.

Strenner showed that for S_g, all integers $1 \leq d \leq 3g-3$ arise as trace field degrees of pseudo-Anosov maps [Str17]. Furthermore, Strenner determined the set of integers which arise as stretch factor degrees: all even integers between 2 and $\leq 6g-6$, as well as all odd integers between 3 and $3g-3$ [Str17].

1.1. Torelli groups.

The *Torelli group* $\mathcal{I}(S_g)$ is the kernel of the symplectic representation of the mapping class group $\mathrm{Mod}(S_g)$. In [Mar19, Problem 10.6], Margalit asked which stretch factor degrees arise for pseudo-Anosov mapping classes in the Torelli group. Our first main result completely answers the question of trace field degrees arising in Torelli groups.

Theorem 1. *Let $g \geq 2$. Every integer $1 \leq d \leq 3g-3$ arises as the trace field degrees of a pseudo-Anosov mapping class in the Torelli group $\mathcal{I}(S_g)$.*

The stretch factor λ satisfies the quadratic equation $t^2 - (\lambda + \lambda^{-1})t + 1$ over the trace field $\mathbb{Q}(\lambda + \lambda^{-1})$. Hence, the field extension $\mathbb{Q}(\lambda) : \mathbb{Q}(\lambda + \lambda^{-1})$ has degree one or two. We obtain our second main result by showing that for all trace field degrees there exist instances where this field extension degree equals two.

Theorem 2. *Let $g \geq 2$. Every even integer $2 \leq 2d \leq 6g-6$ arises as the stretch factor degree of a pseudo-Anosov mapping class in the Torelli group $\mathcal{I}(S_g)$.*

1.2. **The Thurston–Veech construction.** We prove our main results using the Thurston–Veech construction. This construction of pseudo-Anosov maps appeared independently in two papers by Thurston and Veech [Thu88, Vee89].

A *multicurve* is a disjoint union of simple closed curves, and a pair of multicurves $\alpha, \beta \subset S_g$ *fills* the surface S_g if α and β intersect transversally and if the complement $S_g \setminus (\alpha \cup \beta)$ is a union of topological discs none of which is a bigon. This in particular implies that each pair α_i and β_j of components realises the minimal number of intersection points within their respective isotopy classes.

Given a pair of filling multicurves $\alpha, \beta \subset S_g$, the Thurston–Veech construction provides pseudo-Anosov mapping classes in the subgroup $\langle T_\alpha, T_\beta \rangle$ of $\mathrm{Mod}(S_g)$ generated by multitwists along the multicurves α and β. In his seminal 1988 Bulletin paper [Thu88], Thurston provides the upper bound of $6g-6$ on the algebraic degree of a pseudo-Anosov stretch factor $\lambda(f)$ and claims, without proof, that "*the examples of* [Thu88, Theorem 7] *show that this bound is sharp*". The referenced examples are exactly the pseudo-Anosov maps in $\langle T_\alpha, T_\beta \rangle$.

Margalit remarked in 2011 what Strenner wrote down in his article on stretch factor degrees [Str17], namely that no proof of Thurston's claim has ever been published. We are finally able to substantiate Thurston's claim. Furthermore, we can even do so for pseudo-Anosov maps in the Torelli group. Our precise statement for the Thurston–Veech construction is the following.

Theorem 3. *Let $g \geq 2$ and $1 \leq d \leq 3g-3$ be integers. Then there exists a pseudo-Anosov map on S_g arising from the Thurston–Veech construction with trace field degree d and stretch factor degree $2d$. For $g \geq 3$, the pseudo-Anosov maps can be chosen in the Torelli group $\mathcal{I}(S_g)$.*

Clearly, Theorem 3 implies Theorem 1 and Theorem 2 for $g \geq 3$. Our proof of the Torelli case of Theorem 3 does not work for $g = 2$, and for this situation we directly prove Theorem 1 and Theorem 2 by using ad hoc examples (see Section 1.4).

1.3. **Proof strategy for Theorem 3.** For a pair $\alpha, \beta \subset S_g$ of filling multicurves, let $X = (|\alpha_i \cap \beta_j|)_{ij}$ be the matrix encoding the number of intersections of the components of α and β.

The matrix $XX^\top$ is primitive, hence by Perron-Frobenius theory its spectral radius equals its largest eigenvalue and is therefore an algebraic integer. Let d be its algebraic degree. We call the number d the *multicurve intersection degree* of α and β.

Our proof is based on the following existence result.

Theorem 4. *Let $\alpha, \beta \subset S$ be a pair of filling multicurves having multicurve intersection degree d. For $\varepsilon \in \mathbb{Z} \setminus \{0\}$, there exists $n \in \mathbb{Z}_{>0}$ such that the mapping class $T_\alpha^n \circ T_\beta^{n\varepsilon}$ is pseudo-Anosov with stretch factor λ of degree $2d$.*

Assuming Theorem 4, what remains to be done in order to prove the first part of Theorem 3 is to construct all multicurve intersection degrees $1 \leq d \leq 3g-3$ on S_g for $g \geq 2$. By the Thurston–Veech construction, the trace field degree of the resulting pseudo-Anosov maps equals exactly the multicurve intersection degree of the multicurves α and β used in the construction [Thu88, Vee89]. Hence, we are done by setting $\varepsilon = 1$ in Theorem 4.

In order to prove the Torelli part of Theorem 3, we construct the multicurves α and β realising the multicurve intersection degrees $1 \leq d \leq 3g-3$ in such a way that $T_\alpha \circ T_\beta^{-1}$ is an element of $\mathcal{I}(S_g)$. We will ensure this by choosing multicurves α and β which consist of components that are separating or that come in bounding pairs, where for each bounding pair one of the curves is a component of α and the other is a component of β. We can then finish the proof of Theorem 3 by setting $\varepsilon = -1$ in Theorem 4. To see this, note that if $T_\alpha \circ T_\beta^{-1}$ is an element of $\mathcal{I}(S_g)$, then so is $T_\alpha^n \circ T_\beta^{-n}$.

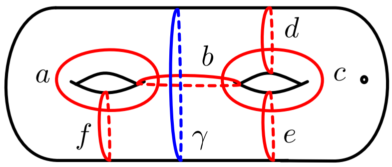
We note that our examples of multicurves α and β that we use in order to construct examples in the Torelli group $\mathcal{I}(S_g)$ cannot yield multicurve degrees greater than one on the surface S_2. Indeed, there exist no bounding pairs on S_2 and a multicurve can have at most one separating component.

To find the suitable multicurves α and β realising all possible multicurve intersection degrees $1 \leq d \leq 3g-3$ is the main technical contribution in this article.

1.4. **Proof of Theorem 1 and Theorem 2 for $g = 2$.** For the genus two surface we give ad hoc examples. For this purpose, we use the flipper software [Bel21] and start with the genus two surface with one puncture $S_{2,1}$, see the figure below. In flipper, mapping classes are defined via Dehn twists along the curves a, b, c, d, e, f.

Consider the separating curve γ depicted in blue. By the chain relation $T_\gamma = (T_a \circ T_f)^6$. We consider the following three conjugates of T_γ:

(1) $T_1 = (T_f \circ T_a \circ T_b) \circ T_\gamma \circ (T_f \circ T_a \circ T_b)^{-1}$,
(2) $T_2 = (T_c \circ T_b) \circ T_\gamma \circ (T_c \circ T_b)^{-1}$,
(3) $T_3 = (T_a \circ T_b) \circ T_\gamma \circ (T_a \circ T_b)^{-1}$.



Obviously $T_\gamma, T_1, T_2, T_3 \in \mathcal{I}(S_{2,1})$. For each even degree $d \in \{2, 4, 6\}$, we exhibit a word in the above elements. We check that the mapping class is pseudo-Anosov with singularity pattern $(1,1,1,1;0)$, and we compute its stretch factor by using flipper [Bel21]. The vector $(1,1,1,1;0)$ means that the invariant foliations have four 3-pronged type singularities, and a 2-pronged one at the puncture.

pseudo–Anosov mapping class $[f] \in \mathrm{Mod}(S_{2,1})$	minimal polynomial of $\lambda(f)$
$T_\gamma \circ T_1 \circ T_2^{-1}$	$t^2 - 66t + 1$
$T_\gamma \circ T_1 \circ T_2$	$t^4 - 72t^3 + 110t^2 - 72t + 1$
$T_\gamma \circ T_1 \circ T_2 \circ T_3$	$t^6 - 266t^5 + 143t^4 - 204t^3 + 143t^2 - 266t + 1$

In order to obtain elements in $\mathrm{Mod}(S_2)$, we appeal to the forgetful map. Since our examples do not have 1-pronged singularity at the puncture, we can fill in it in order to get a pseudo–Anosov mapping class, with the same stretch factor, in $\mathrm{Mod}(S_2)$, see [HK06, Lemma 2.6] for details. This completes the proof for $g = 2$.

1.5. **Explicit examples.** In [Mar19, Problem 10.4], Margalit asks for explicit examples of pseudo-Anosov maps with specific stretch factor degrees. Our construction of multicurves allows us to do so for small genera.

More precisely, we provide an almost explicit construction of multicurves with intersection degrees $1 < d \leq 3g-3$ for which the pseudo-Anosov mapping class $T_\alpha \circ T_\beta$ has stretch factor degree $2d$. In the setting of Theorem 4, this means for $\varepsilon = 1$ one can choose $n = 1$. Our argument uses [LL24, Theorem 6] (see Section 6 for details). Unfortunately, for the multicurves we construct in the Torelli case, the criterion from [LL24] does not apply. Hence, our need to use Theorem 4 for the proof of Theorem 3, which provides a slightly less explicit result.

Aided by the computer, we can subsequently find completely explicit examples of multicurves and therefore entirely explicit pseudo-Anosov mapping classes on S_g arising from the Thurston–Veech construction realising the maximal stretch factor degree $6g-6$, for all genera $g \leq 201$.

1.6. **Odd degree stretch factors.** While Theorem 4 provides the existence of field extensions $\mathbb{Q}(\lambda) : \mathbb{Q}(\lambda + \lambda^{-1})$ of degree two for mapping classes in $\langle T_\alpha, T_\beta \rangle$, realising extensions of degree one seems to be more mysterious. For example, Veech [Vee82] discovered a family of Hecke groups $\langle T_\alpha, T_\beta \rangle = \langle \left(\begin{smallmatrix} 1 & \lambda_q \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ -\lambda_q & 1 \end{smallmatrix}\right) \rangle$, where $\lambda_q = 2\cos \pi/q$ for $q \geq 3$. The genus of the surface S_g is $(q-1)/2$ for odd q. For $q = 7, 9$ one can find stretch factors of degree one over the trace field $\mathbb{Q}(\lambda_q)$: for instance $T_\alpha \circ T_\beta^{-1}$ is an example for $q = 7$, and we refer to [Bou22] for $q = 9$. However, it is conjectured (see [HMTY08, Remark 9]) that stretch factors of degree one over $\mathbb{Q}(\lambda_q)$ do not exist for odd $q \geq 11$.

It remains an open problem to construct odd stretch factor degrees in the Torelli group $\mathcal{I}(S_g)$.

1.7. **On Thurston's upper bound.** One may impose restrictions on the foliations fixed by the pseudo-Anosov map, for example by prescribing the stratum, that is, the number of singularities as well as their orders as k-pronged singularities. It turns out that Thurston's upper bound for the stretch factor degree becomes

$$2g - 2 + \#\{\text{odd singularities}\} \leq 2g - 2 + 4g - 4 = 6g - 6.$$

In the context of the Thurston–Veech construction, the number and type of singularities can be read off directly from the geometry of the complement $S \setminus (\alpha \cup \beta)$: the number of k-pronged singularities coincides with the number of $2k$-gons in the complement.

It is a consequence of our proof of Theorem 3 that the upper bound

$$2g - 2 + \#\{\text{odd singularities}\}$$

for the stretch factor degree is sharp for every $g \geq 3$, and that examples realising this upper bound can be taken in the Torelli group.

1.8. **A natural field extension from the perspective of curves.** A pair of filling multicurves $\alpha, \beta \subset S_g$ naturally determines a bipartite graph whose vertices correspond to curve components and the number of edges between each pair of vertices equals the number of intersection points of the respective curve components. The adjacency matrix of this graph is $\Omega = \left(\begin{smallmatrix} 0 & X \\ X^\top & 0 \end{smallmatrix}\right)$. Clearly, the square root $\sqrt{\mu}$ of the spectral radius μ of $XX^\top$ equals the spectral radius of Ω. We call the algebraic degree of $\sqrt{\mu}$ the *multicurve bipartite degree* of α and β. Similarly to the field extension $\mathbb{Q}(\lambda) : \mathbb{Q}(\lambda + \lambda^{-1})$, also the field extension $\mathbb{Q}(\sqrt{\mu}) : \mathbb{Q}(\mu) = \mathbb{Q}(\lambda + \lambda^{-1})$ has degree one or two. We prove the following result.

Theorem 5. *Every even integer $2 \leq 2d \leq 6g-6$ is realised as a multicurve bipartite degree on S_g for $g \geq 2$.*

Organisation. In Section 2 we prove Theorem 4, which is the nonsplitting criterion used to reduce Theorem 3 to the construction of certain kinds of pairs of multicurves. In Section 3 we introduce an irreducibility criterion for the characteristic polynomial of matrices of the form $XX^\top$ which plays a central role throughout the rest of the article. Using this irreducibility criterion, we first give a proof of Thurston's claim as a warm-up in Section 4, before providing the multicurves needed to prove Theorem 3 in Section 5. Finally, we provide some explicit examples in Section 6.

Acknowledgments. The authors thank Dan Margalit and Jean-Claude Picaud for inspiring discussions, Mark Bell for his help with the software flipper [Bel21] and Curt McMullen for comments on an earlier version of the article. They are particularly grateful to Dan Margalit for pointing out a mistake in the formulation of the proof of Theorem 3 in an earlier version of the article. The first author has been partially supported by the LabEx PERSYVAL-Lab (ANR-11-LABX-0025-01) funded by the French program Investissement d'avenir. The first author would like to thank the University of Fribourg and the Center for Mathematical Modeling (CMM) at Universidad de Chile for excellent working conditions.

2. A nonsplitting criterion

In this section we prove Theorem 4, which is an algebraic criterion that allows us to deduce that the degree of the field extension $\mathbb{Q}(\lambda(f)) : \mathbb{Q}(\lambda(f)+\lambda(f)^{-1})$ equals two for certain f which are a product of multitwists. Compare with [LL24, Theorem 6]. For convenience, we repeat the statement of Theorem 4:

Theorem (Theorem 4)**.** *Let $\alpha, \beta \subset S$ be a pair of filling multicurves having multicurve intersection degree d. For every $\varepsilon \in \mathbb{Z}\setminus\{0\}$, there exists $n \in \mathbb{Z}_{>0}$ such that the mapping class $T_\alpha^n \circ T_\beta^{n\varepsilon}$ is pseudo-Anosov with stretch factor λ of degree $2d$.*

Proof of Theorem 4. By the Thurston–Veech construction, there exists a representation $\rho : \langle T_\alpha, T_\beta\rangle \to \mathrm{PSL}_2(\mathbb{R})$ mapping T_α to the matrix $\left(\begin{smallmatrix}1 & r\\ 0 & 1\end{smallmatrix}\right)$ and T_β to the matrix $\left(\begin{smallmatrix}1 & 0\\ -r & 1\end{smallmatrix}\right)$, where $r^2 = \mu$ is the spectral radius of the matrix $XX^\top$ for the multicurves α and β. Furthermore, the stretch factor $\lambda(f)$ of $f \in \langle T_\alpha, T_\beta\rangle$ equals the spectral radius of $\rho(f)$. Now, let us consider the product of multitwists $f = T_\alpha^{2n} \circ T_\beta^{2n\varepsilon}$. A direct computation provides that the trace of $\rho(f)$ equals $\mathrm{tr}(\rho(f)) = 2 - \varepsilon(2nr)^2$. Thus, $\lambda(f)+\lambda(f)^{-1} = |2-\varepsilon(2nr)^2|$ and hence $\mathbb{Q}(\lambda(f)+\lambda(f)^{-1}) = \mathbb{Q}(\mu) = K$. Note that by assumption, the degree of the field extension $K : \mathbb{Q}$ is d, the multicurve intersection degree of α and β.

Since $\lambda = \lambda(f)$ solves the quadratic equation $t^2 - (\lambda+\lambda^{-1})t + 1 = 0$, λ has degree 1 or 2 over K. All what we need to do is find $n \in \mathbb{Z}_{>0}$ such that $\lambda \notin K$, or equivalently such that the discriminant $D = (2-\varepsilon(2nr)^2)^2 - 4 = 16 \cdot n^2 \cdot ((n\varepsilon\mu)^2 - \varepsilon\mu)$ of the quadratic equation is not a square in K. We will proceed by contradiction. Let $\mu' = \varepsilon\mu$ and let us assume that $(n\mu')^2 - \mu'$ is a square in $K = \mathbb{Q}(\mu')$ for every $n > 0$. Since the expression is invariant under the transformation $n \mapsto -n$, we can assume the expression is a square for every $n \in \mathbb{Z}\setminus\{0\}$.

Let $P = a_d t^d + a_{d-1}t^{d-1} + \cdots + a_1 t + a_0 \in \mathbb{Z}[t]$ be the minimal polynomial of μ' over $\mathbb{Q}$. The Thurston–Veech construction implies that μ is an eigenvalue of a square matrix, so is μ' and $a_d = 1$. Thus, μ' and $n^2\mu' - 1$ are algebraic units. The norm of μ' equals $N(\mu') = (-1)^d a_0$. Similarly, the minimal polynomial of $n^2\mu' - 1$ is $n^{2d}P\left(\frac{t+1}{n^2}\right)$.

Inspecting the constant term, we have

$$N(n^2\mu' - 1) = (-1)^d \sum_{k=0}^{d} a_k n^{2d-2k}.$$

Altogether this gives $N((n\mu')^2 - \mu') = Q(n^2)$, where

$$Q(t) = a_0 \sum_{k=0}^{d} a_k t^{d-k}.$$

By assumption, $Q(n^2)$ is a square for every $n \in \mathbb{Z}\backslash\{0\}$. We show that $Q(0) = N(-\mu')$ is also a square. Indeed, for any prime integer p, the reduction modulo $n = p$ of $N((n\mu')^2 - \mu')$ gives that $N(-\mu')$ is a quadratic residue. Thus it is also a square in $\mathbb{Z}$. Hence $Q(t) \in \mathbb{Z}[t]$ is a polynomial taking integral square value at every integer specialisation. By a result of Murty [Mur08, Theorem 1], $Q(t^2)$ is the square of a polynomial.

Moreover, we observe that $Q(t) = a_0 \cdot t^d P\left(\frac{1}{t}\right) \in \mathbb{Q}[t]$. In particular $Q\left(\frac{1}{\mu'}\right) = 0$, and since μ' and $\frac{1}{\mu'}$ generate the field extension $K : \mathbb{Q}$, the polynomial Q is irreducible over $\mathbb{Q}$. It is in particular separable. Now each root $0 \neq a \in \mathbb{C}$ of Q gives rise to two distinct roots $\pm\sqrt{a}$ of $Q(t^2)$, and conversely. Thus $Q(t^2)$ is also separable, and cannot be a square. This concludes the proof of the theorem. $\square$

3. An irreducibility criterion

The goal of this section is to present an algebraic criterion that allows us to deduce that certain characteristic polynomials of matrices of the form $XX^\top$ are irreducible.

Proposition 6. *Let M be a square integer matrix, and let N be the principal submatrix of M obtained by deleting the first row and the first column. If M and N have no common eigenvalue, and if M has a simple eigenvalue ρ, then the characteristic polynomial of $\widetilde{M} = M + ay^pE_{11}$ is an irreducible element of $\mathbb{Z}[t, y]$, for all $p \geq 1$ and for all $0 \neq a \in \mathbb{Z}$.*

Proof. Our goal is to use Eisenstein's criterion on $\chi_{\widetilde{M}} \in \mathbb{Z}[t, y] \cong (\mathbb{Z}[t])[y]$, viewing it as a polynomial in the variable y and coefficients in $\mathbb{Z}[t]$. We calculate

$$\chi_{\widetilde{M}}(t, y) = \det(t \cdot \mathrm{Id} - \widetilde{M}) = -y^p a\chi_N(t) + \chi_M(t)$$

and notice that $a\chi_N$ and χ_M are relatively prime in $\mathbb{Z}[t]$. Indeed, χ_M has leading coefficient $+1$ and no root in common with χ_N by our assumption that M and N have no eigenvalue in common. In particular, they have no common factor, which shows that $\chi_{\widetilde{M}} \in (\mathbb{Z}[t])[y]$ is primitive. In order to apply Eisenstein's criterion, let $\mu_\rho \in \mathbb{Z}[t]$ be the minimal polynomial of the simple eigenvalue ρ of M. By assumption, μ_ρ divides χ_M exactly once, but it does not divide χ_N since χ_M and χ_N have no common root. In particular, Eisenstein's criterion applies to show that the polynomial $\chi_{\widetilde{M}} \in (\mathbb{Z}[t])[y] \cong \mathbb{Z}[t, y]$ is irreducible. $\square$

Remark 7. In the previous statement, one can easily replace $\chi_{\widetilde{M}}(t)$ by $\chi_{\widetilde{M}}(t^n)$ for any integer $n > 0$. Indeed $\chi_M(t^n)$ and $\chi_N(t^n)$ are still coprime and $\mu_\rho(t^n)$ divides $\chi_M(t^n)$ exactly once, so Eisenstein's criterion applies.

Remark 8. Oscillatory matrices satisfy a stronger version of Perron-Frobenius theory, namely *all* the eigenvalues are positive real, simple, and they strictly interlace when taking a principal submatrix [And87]. Hence, Proposition 6 applies very cleanly to this class of matrices.

We use Proposition 6 on the following two cases (Lemma 9 and Lemma 10).

Lemma 9. *For $n \geq 1$, let*

$$N = \left(\begin{array}{c|ccc} a_1 & a_2 & \dots & a_n \\ \hline a_2 & & & \\ \vdots & & * & \\ a_n & & & \end{array}\right), \quad M = \left(\begin{array}{c|ccc} 0 & \alpha a_1 & \dots & \alpha a_n \\ \hline \alpha a_1 & & & \\ \vdots & & N & \\ \alpha a_n & & & \end{array}\right)$$

be square integer matrices with $a_1 \geq 1$. If M is nonnegative and irreducible, and if $\chi_N \in \mathbb{Z}[t]$ is irreducible, then the characteristic polynomial of $\widetilde{M} = M + ay^2 E_{11}$ is irreducible in $\mathbb{Z}[t, y]$ for all $0 \neq a \in \mathbb{Z}$.

Proof. In order to use Proposition 6, we need to show that M has a simple eigenvalue and that M and N share no eigenvalue. The former holds since M is nonnegative and irreducible, and in particular has a Perron-Frobenius eigenvalue which is simple. For the latter, we compute

$$\chi_M(t) = t\chi_N(t) + q(t),$$

where $q(t) \in \mathbb{Z}[t]$ is of degree at most $n - 1$. We claim that it is not the zero polynomial either. Indeed, we directly verify

$$\begin{aligned} q(0) &= \det \left(\begin{array}{c|ccc} 0 & -\alpha a_1 & \dots & -\alpha a_n \\ \hline -\alpha a_1 & & & \\ \vdots & & -N & \\ -\alpha a_n & & & \end{array}\right) \\ &= \det \left(\begin{array}{c|ccc} \alpha^2 a_1 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & -N & \\ 0 & & & \end{array}\right) = \pm\alpha^2 a_1 \det N \neq 0. \end{aligned}$$

Now if there existed a common root $\lambda \in \mathbb{C}$ of χ_M and χ_N, then λ would also be a root of $q(t)$. But since χ_N is irreducible of degree n and $q(t)$ is a nonzero polynomial of degree at most $n - 1$, this is impossible. □

Lemma 10. *For $n, m \geq 1$, let*

$$A = \left(\begin{array}{c|ccc} a_1 & a_2 & \dots & a_n \\ \hline a_2 & & & \\ \vdots & & * & \\ a_n & & & \end{array}\right), \quad B = \left(\begin{array}{c|ccc} b_1 & b_2 & \dots & b_m \\ \hline b_2 & & & \\ \vdots & & * & \\ b_m & & & \end{array}\right)$$

be square integer matrices of dimension n and m, respectively, with $a_1, b_1 \geq 1$. Furthermore, let $\alpha, \beta \neq 0$ such that

$$M = \left(\begin{array}{c|ccc|ccc} 0 & \alpha a_1 & \dots & \alpha a_n & \beta b_1 & \dots & \beta b_m \\ \hline \alpha a_1 & & & & & & \\ \vdots & & A & & & & \\ \alpha a_n & & & & & & \\ \hline \beta b_1 & & & & & & \\ \vdots & & & & & B & \\ \beta b_m & & & & & & \end{array}\right)$$

is a matrix with integer coefficients. If M is nonnegative and irreducible, and if $\chi_A, \chi_B \in \mathbb{Z}[t]$ are irreducible and distinct, then the characteristic polynomial of $\widetilde{M} = M + ay^2 E_{11}$ is irreducible in $\mathbb{Z}[t, y]$ for all $0 \neq a \in \mathbb{Z}$.

Proof. The proof is similar to the proof of Lemma 9: the only thing to verify is that no eigenvalue of A or of B is also an eigenvalue of M. Again, we compute

$$\chi_M(t) = t\chi_A(t)\chi_B(t) \pm q_1(t)\chi_B(t) \pm q_2(t)\chi_A(t),$$

where $q_1(t) \in \mathbb{Z}[t]$ is of degree at most $n-1$ and $q_2(t) \in \mathbb{Z}[t]$ is of degree at most $m-1$. This is seen by developing the first column of the matrix $tI - M$. The first coefficient is responsible for the summand $t\chi_A(t)\chi_B(t)$, the next n coefficients are responsible for the summand $\pm q_1(t)\chi_B(t)$ and the final m coefficients are responsible for the summand $\pm q_2(t)\chi_A(t)$. We claim that neither among $q_1(t)$ and $q_2(t)$ is the zero polynomial. Indeed, by developing the first column of the matrix $t\mathrm{I} - M$, and evaluating at $t = 0$, we get

$$\begin{aligned} q_1(0) &= \det \left(\begin{array}{c|ccc} 0 & -\alpha a_1 & \dots & -\alpha a_n \\ \hline -\alpha a_1 & & & \\ \vdots & & -A & \\ -\alpha a_n & & & \end{array}\right) \\ &= \det \left(\begin{array}{c|ccc} \alpha^2 a_1 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & -A & \\ 0 & & & \end{array}\right) = \pm\alpha^2 a_1 \det A, \end{aligned}$$

which is not zero since χ_A is irreducible. Similarly, $q_2(0) \neq 0$. Now if there existed a common root $\lambda \in \mathbb{C}$ of χ_M and χ_A, then λ would also be a root of either χ_B or q_1. Since χ_A and χ_B are irreducible and distinct, we must have $q_1(\lambda) = 0$. But since χ_A is irreducible of degree n, and $q_1(t)$ is a nonzero polynomial of degree at most $n-1$, this is impossible. Similarly, no root of χ_B can be a root of χ_M, which concludes the proof. □

Remark 11. One could formulate Lemma 10 with $k \geq 2$ blocks $A_1, \dots, A_k$ of respective sizes $n_1, \dots, n_k$, instead of $k = 2$. In this case, all the k characteristic polynomials χ_{A_i} should be irreducible and pairwise distinct. The argument is identical by considering

$$\chi_M(t) = t\prod_{i=1}^{k} \chi_{A_i} + \sum_{i=1}^{k} \pm q_i(t) \prod_{j \neq i} \chi_{A_j},$$

where $q_i(t) \in \mathbb{Z}[t]$ is of degree at most $n_i - 1$ and nonzero.

4. Warm-up: proof of Thurston's claim

As a first illustration of our method, it is the goal of this section to provide a pair of filling multicurves α and β on S_g with multicurve intersection degree $3g-3$. By Theorem 4, this validates Thurston's claim that the product of two multitwists can realise the maximal possible algebraic degrees of stretch factors: $6g-6$.

Recall that the matrix encoding the number of intersections of the components of α and β is denoted by $X = (|\alpha_i \cap \beta_j|)_{ij}$ (see Section 1.3). In order to read off the matrix $XX^\top$ from our figures, we use the following formula for its coefficients, which is a direct consequence of the definition of matrix multiplication:

$$(XX^\top)_{ij} = \sum_k |\alpha_i \cap \beta_k| \cdot |\beta_k \cap \alpha_j|.$$

We start by realising, on the surface of genus $g \geq 1$ with 2 boundary components, a pair of filling multicurves α and β such that $\chi_{XX^\top} \in \mathbb{Z}[t]$ is irreducible and of degree $3g-1$. We proceed by induction on g.

For $g = 1$ with two boundary components. We consider the two multicurves α and β shown in Figure 1, where one of the components of β has $y-1$ parallel copies. Here, we think of y as a variable that we specify later on.

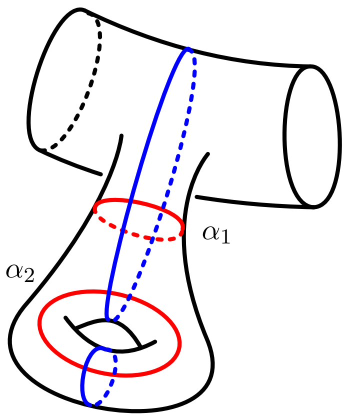


Figure 1. Two multicurves α (in red) and β (in blue) on the surface of genus one with two boundary components. The multicurve β contains $y-1$ parallel copies of one of its components.

We directly calculate

$$XX^\top = \begin{pmatrix} 4 & 2 \\ 2 & y \end{pmatrix}.$$

Observe that X is a matrix of size $2 \times y$ (the multicurve β has y components). We have $\chi_{XX^\top}(t) = t^2 - (4+y)t + 4(y-1)$ with discriminant $y^2 - 8y + 32$, which is not a square if $y \geq 12$. Indeed, in this case we have

$$(y-3)^2 = y^2 - 6y + 9 > y^2 - 8y + 32 > y^2 - 8y + 16 = (y-4)^2.$$

In particular, for $y \geq 12$ the polynomial $\chi_{XX^\top}$ is irreducible.

For $g > 1$ and two boundary components. For the inductive step, assume we have constructed on the surface of genus $g \geq 1$ with 2 boundary components a pair of multicurves α', β' such that the characteristic polynomial $\chi' = \chi_{XX^\top} \in \mathbb{Z}[t]$ is irreducible and of degree $3g-1$. Furthermore, assume that α'_1 is a simple closed curve that encircles all the handles of the surface, as illustrated in Figure 2. Take a surface of genus 1 and two boundary components, as in the case of genus $g = 1$, see Figure 1, and denote its multicurves by α'' and β''. Now glue its right boundary component to the left boundary component of the surface of genus g, and add two new curves α_0 and β_0 to the multicurves. The curve α_0 encircles all the handles of the newly formed surface, and the curve β_0 twice intersects α_0 but no other multicurve component. Again, see Figure 2 for an illustration.

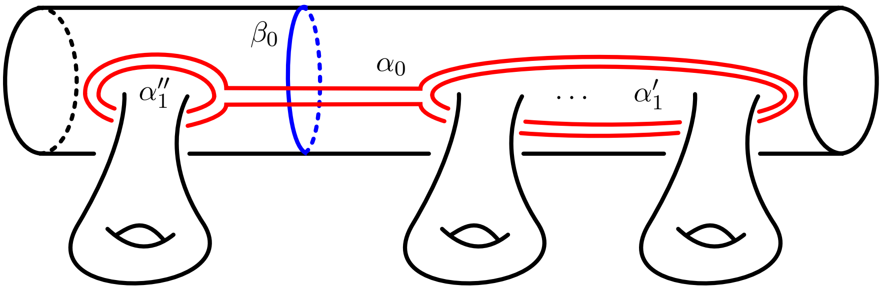


FIGURE 2. Two surfaces of genus g and 1, respectively, and two boundary components, glued together along one of their boundary components. The curves α'_1 and α''_1 are shown, each encircling all the handles of their respective surface. The new curve α_0 encircles all the handles of the newly formed surface, and the new curve β_0 runs along the glued boundary component.

Let A be the matrix $XX^\top$ for the pair of multicurves α', β', and let B be the matrix $XX^\top$ for the pair of multicurves α'', β''. We define the multicurves

$$\alpha = \alpha_0 \cup \alpha' \cup \alpha''$$
$$\beta = \beta_0 \cup \beta' \cup \beta''$$

A quick computation gives

$$A = \left(\begin{array}{c|ccc} a_1 & a_2 & \dots & a_n \\ \hline a_2 & & & \\ \vdots & & * & \\ a_n & & & \end{array}\right), \quad B = \begin{pmatrix} 4 & 2 \\ 2 & b \end{pmatrix}.$$

Let us choose b such that χ_B is irreducible and distinct from χ_A. We may assume inductively that $a_1 = 4a$. In the multicurve β, we take $y^2 - a - 1 \geq 1$ parallel copies of β_0, for $y > 0$ large enough. The matrix $XX^\top$ for the multicurves α and β takes the form

$$XX^\top = \left(\begin{array}{c|ccc|cc} 4y^2 & a_1 & \dots & a_n & 4 & 2 \\ \hline a_1 & & & & & \\ \vdots & & A & & & \\ a_n & & & & & \\ \hline 4 & & & & 4 & 2 \\ 2 & & & & 2 & b \end{array}\right).$$

By Lemma 10, $\chi_{XX^\top} \in \mathbb{Z}[t, y]$ is irreducible (recall that χ_A is irreducible). Hence, by Hilbert's irreducibility theorem, there exist infinitely many specifications of y (and in particular infinitely many specifications of y such that $y^2 - a - 1 > 0$) with $\chi_{XX^\top} \in \mathbb{Z}[t]$ irreducible. This polynomial is of degree

$$3g - 1 + 3 = 3(g + 1) - 1,$$

which is exactly what we wanted to show. Finally, to justify our inductive assumption on the top-left coefficient of the matrix A, note that the top-left coefficient of the matrix $XX^\top$ is again a multiple of 4.

The closed case for $g \geq 2$. Take any example of a pair of multicurves α' and β' we constructed on the surface of genus $g - 1 \geq 1$ with two boundary components in Section 4. Let

$$A = \left(\begin{array}{c|ccc} a_1 & a_2 & \dots & a_n \\ \hline a_2 & & & \\ \vdots & & * & \\ a_n & & & \end{array}\right)$$

be the matrix $XX^\top$ for the multicurves α' and β', where $a_1 = 4a$. We identify the two boundary components of the surface to increase the genus by one. Let α_0 be a longitude of the created handle, and let β_0 run along the glued boundary. Define the two new multicurves

$$\alpha = \alpha_0 \cup \alpha'$$
$$\beta = \beta_0 \cup \beta',$$

where we take $y^2 - a$ copies of β_0. Then the matrix $XX^\top$ for the multicurves α and β takes the form

$$XX^\top = \left(\begin{array}{c|ccc} y^2 & \frac{a_1}{2} & \dots & \frac{a_n}{2} \\ \hline \frac{a_1}{2} & & & \\ \vdots & & A & \\ \frac{a_n}{2} & & & \end{array}\right),$$

and $\chi_{XX^\top} \in \mathbb{Z}[t, y]$ is irreducible by Lemma 9. By Hilbert's irreducibility theorem, there exist infinitely many specifications of y (and in particular infinitely many specifications of y such that $y^2 - a > 0$) with $\chi_{XX^\top} \in \mathbb{Z}[t]$ irreducible. This polynomial is of degree $3(g - 1) - 1 + 1 = 3g - 3$.

5. Proof of Theorem 3

The goal of this section is to realise every positive integer $d \leq 3g-3$ as the multicurve intersection degree of a pair of multicurves $\alpha, \beta \subset S$ on S_g for $g \geq 3$ in such a way that the multicurves α and β consist of components that are separating or that come in bounding pairs, where for each bounding pair one of the curves is a component of α and the other is a component of β. Theorem 4 then provides Theorem 3 in the case $g \geq 3$. For the case $g = 2$ we note that the statement is proved for $d = 3g-3 = 3$ in Section 4, and for $d \leq 2 = g$ it is proved in [LL24].

We start with the maximal degree $3g - 3$ and then discuss how to adapt the construction in order to realise smaller degrees.

5.1. **Multicurve intersection degree $3g-3$.** We start by realising, on the surface of genus $g \geq 2$ with one boundary component, a pair of filling multicurves α and β such that $\chi_{XX^\top} \in \mathbb{Z}[t]$ is irreducible and of degree $3g-2$, in such a way that the multicurves α and β consist of components that are separating or that come in bounding pairs, where for each bounding pair one of the curves is a component of α and the other is a component of β. The construction is done by induction on the genus $g \geq 2$.

5.1.1. *For $g=2$ with one boundary component.* We consider the two multicurves α and β shown in Figure 3. We first note that the components α_1 and α_3 are sep-

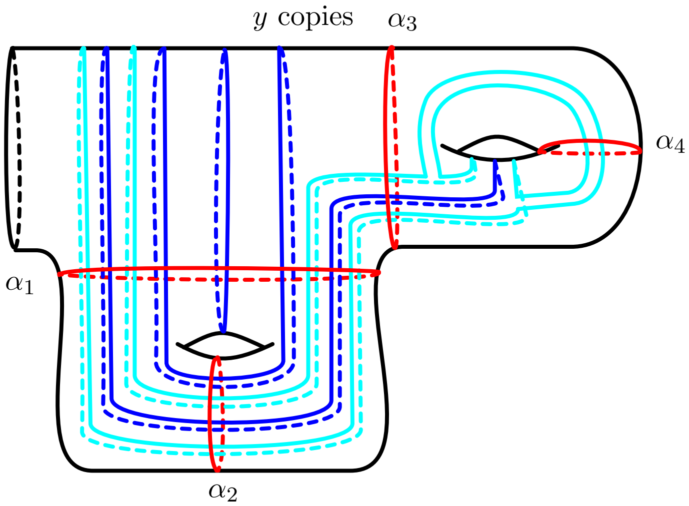


FIGURE 3. Two multicurves α and β on the surface of genus two with one boundary component. One component of β has y parallel copies.

arating. Furthermore, the components α_2 and α_4 have their counterparts in the multicurve β with which they each form a bounding pair. Finally, the component of β of which there are y parallel copies and the component of β drawn in light blue in Figure 3 are separating.

We directly calculate

$$XX^\top = \begin{pmatrix} 84+16y & 40+8y & 40 & 16 \\ 40+8y & 20+4y & 20 & 8 \\ 40 & 20 & 20 & 8 \\ 16 & 8 & 8 & 4 \end{pmatrix},$$

and it is a direct check (by the computer) that the characteristic polynomial of $XX^\top$ is irreducible if $y=2$ or $y=3$. This finishes the case $g=2$ with one boundary component.

5.1.2. *For $g>2$ and one boundary component.* In order to increase the genus by one, we glue a surface of genus one with two boundary components as follows. On this surface, we consider the two multicurves α and β shown in Figure 4. We directly

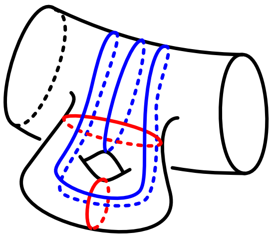


FIGURE 4. Two multicurves α (in red) and β (in blue) on the surface of genus one with two boundary components. The multicurve β has y parallel copies of its separating component.

calculate

$$XX^\top = \begin{pmatrix} 16y+4 & 8y \\ 8y & 4y \end{pmatrix} =: C_y,$$

and $\chi_{XX^\top}(t) = t^2 - (20y+4)t + 16y$ with discriminant $16 \cdot (25y^2 + 6y + 1)$, which is never a square. Indeed, we have

$$(5y)^2 = 25y^2 < 25y^2 + 6y + 1 < 25y^2 + 10y + 1 = (5y+1)^2.$$

In particular, the polynomial $\chi_{XX^\top}$ is irreducible for every positive integer y.

For the inductive step, let $g \geq 2$. Assume we have constructed on the surface of genus g with one boundary component a pair of multicurves α', β' such that the characteristic polynomial $\chi_{XX^\top} \in \mathbb{Z}[t]$ is irreducible and of degree $3g-2$, in such a way that the multicurves α and β consist of components that are separating or that come in bounding pairs, where for each bounding pair one of the curves is a component of α and the other is a component of β. Further, assume that α_1' is a simple closed curve that encircles all the handles of the surface, except for the rightmost one. Then, we take such a model surface and glue to its boundary a surface of genus one with two boundary components, as shown in Figure 4, and add two new curves α_0 and β_0 to the multicurves. The curve α_0 encircles all the handles of the newly formed surface, except for the rightmost one, and the curve β_0 runs along the glued boundary components, and twice intersects α_0 but no other component of α, see Figure 5.

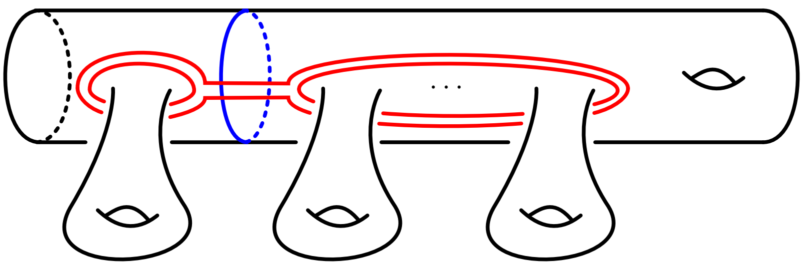


FIGURE 5

The proof of irreducibility is now exactly the same as in the non-Torelli case in Section 4. The only thing we need to check is that the multicurves α and β consist of components that are separating or that come in bounding pairs, where for each bounding pair one of the curves is a component of α and the other is a component of β. But this is clearly the case, since all the curves we add in the inductive step are separating or come as a bounding pair.

5.1.3. *The closed case for $g \geq 4$.* The last step is to make the surfaces closed. We simply glue together two pieces of genera g', g'', where $g'+g''=g$, and one boundary component together along their boundaries. The same argument as in the inductive step provides irreducible characteristic polynomials of degree

$$3g' - 2 + 3g'' - 2 + 1 = 3g - 3.$$

5.1.4. *The closed case for $g = 3$.* We need a different argument. In this case, we start with the surface of genus two and one boundary component depicted in Figure 3, and close it off to the left by glueing a surface of genus one with one boundary component, see Figure 6. First add the curves α_5 and β_5 with $y^2 - 29$ parallel

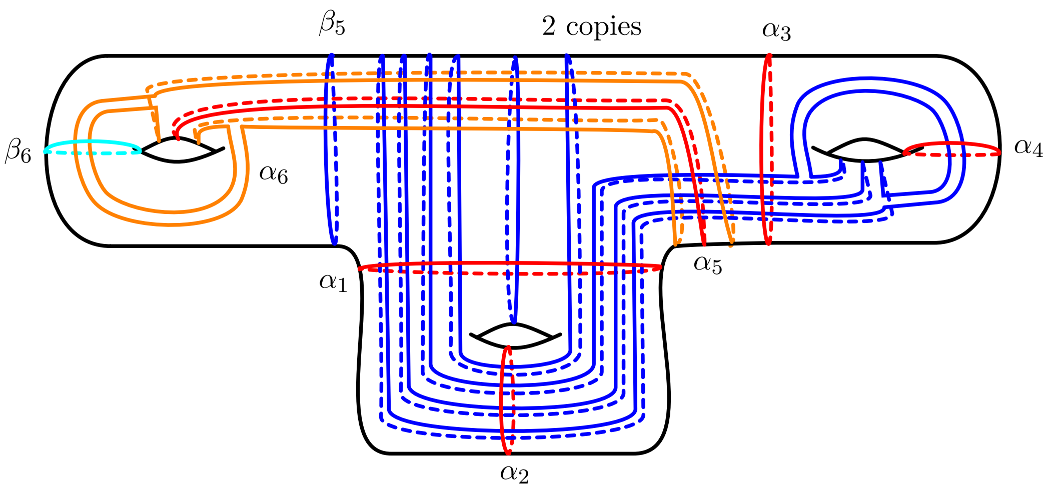


FIGURE 6. Two multicurves α and β on the surface of genus three. There are to new components of α when compared to Figure 3: a nonseparating component (red) that we call α_5 and a separating component (orange) that we call α_6. Similarly, there are two new components of β: a separating component (blue) that we call β_5 and a nonseparating component (light blue) that we call β_6.

copies. The resulting characteristic polynomials is irreducible for infinitely many choices of y by Lemma 9. Repeat the same process with α_6 and β_6 (adjusting the number of parallel copies of β_6 suitably), and we are done.

5.2. **Multicurve intersection degrees $d < 3g - 3$.** We now show how to modify our construction from Section 5.1 in order to realise multicurve intersection degrees smaller than the maximal multicurve intersection degree $3g - 3$. We need new building blocks to construct our surfaces.

Block 1. Our first block is obtained from the surface depicted in Figure 3, simply by dropping the component α_3. A direct verification yields that for $y = 1, 2$ the characteristic polynomial of $XX^\top$ is irreducible and of degree 3.

Block 2. Our second block is obtained from the surface depicted in Figure 7. The characteristic polynomial of the matrix $XX^\top$ for the multicurves α and β is irreducible and of degree 1. Versions of this block with distinct characteristic polynomial can be obtained by taking y parallel copies of β.

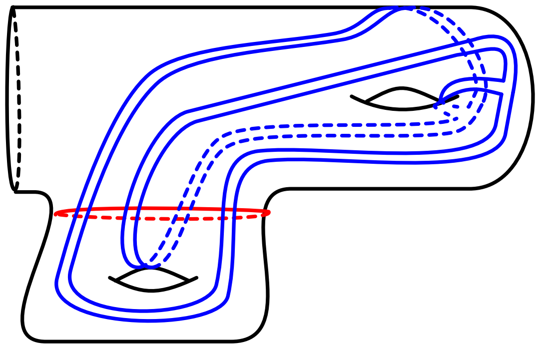


FIGURE 7. Two separating and filling curves α and β on the surface of genus two with one boundary component.

Block 3. Take a surface as depicted in Figure 8. We denote the red multicurve by α and the blue multicurve by β. The multicurve α has $k+1$ separating components: one for each of the handles that separates the handle, and one that separates all the handles. We denote the component of α that separates all the handles of the surface in Figure 8 by α_1, and we denote the other separating components of α by $\alpha_2, \alpha_4, \dots, \alpha_{2g-2}$ from left to right. Finally, the remaining nonseparating components of α are $\alpha_3, \alpha_5, \dots, \alpha_{2g-1}$ from left to right. In this situation, we have

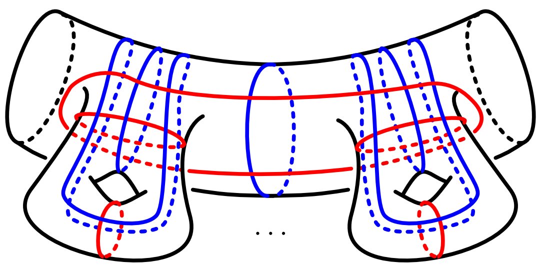


FIGURE 8. A surface of genus k with two boundary components, as well as two multicurves α (in red) and β (in blue). The separating components of β can have several parallel copies: the ones separating the handles have $y_1, \dots, y_k$ copies, and the separating component in the middle has $y^2 - 4k - y_1 - \dots - y_k$ copies.

$$XX^\top = \left(\begin{array}{c|cccc} 4y^2 & v_{y_1}^\top & v_{y_2}^\top & \cdots & v_{y_k}^\top \\ \hline v_{y_1} & C_{y_1} & 0 & & \\ v_{y_2} & 0 & C_{y_2} & & \\ \vdots & & & \ddots & \\ v_{y_k} & & & & C_{y_k} \end{array}\right), \quad C_{y_i} = \begin{pmatrix} 16y_i + 4 & 8y_i \\ 8y_i & 4y_i \end{pmatrix}, \quad v_{y_i} = \begin{pmatrix} 16y_i + 4 \\ 8y_i \end{pmatrix}.$$

By Remark 11, $\chi_{XX^\top}(t, y) \in \mathbb{Z}[t][y]$ is irreducible. By Hilbert's irreducibility theorem, there are infinitely many specifications of y such that $y^2 - 4k - y_1 - \dots - y_k > 0$

and such that $\chi_{XX^\top}(t) \in \mathbb{Z}[t]$ is irreducible and of degree $2k+1$. Note that we can drop the separating components of α winding around one handle one by one in order to decrease the degree, reducing a 2-by-2 block to a 1-by-1 block, consisting of the coefficient $4y_i$, for each component dropped in this way. If all the y_i are chosen pairwise distinct, Remark 11 guarantees that the polynomial $\chi_{XX^\top}(t, y) \in \mathbb{Z}[t][y]$ is irreducible. We can in this way construct all degrees $k+1 \leq d \leq 2k+1$ for the surface of genus k and 2 boundary components.

5.2.1. *Realising multicurve intersection degrees* $3g-6 \leq d < 3g-3$. Using a block of type 1 or 2 instead of our standard starting surface depicted in Figure 3, we can reduce the multicurve intersection degree by 1 or 3, respectively. Since we use such block on both sides of the surface in our construction, this gives the possibility to reduce the degree by any among the numbers 1,2,3,4 or 6. In particular, we can clearly realise the multicurve intersection degrees $3g-3, 3g-4$ and $3g-5$. This argument works for $g \geq 4$.

In case of $g = 3$, we need a separate argument. The idea is to copy our example of maximal degree from Figure 6, but leave out first α_3 and then also α_1. We start from the multicurves depicted in Figure 3 and drop α_3. Letting $y = 2$, we then get

$$XX^\top = \begin{pmatrix} 116 & 56 & 16 \\ 56 & 28 & 8 \\ 16 & 8 & 4 \end{pmatrix},$$

which has irreducible characteristic polynomial. We can now close off the surface by glueing a torus with one boundary component and add more components to α and β, in the same way as in Figure 6. The exact same argument we used to realise degree 6 now yields degree 5 instead.

In order to realise degree 4 for $g = 3$, we note that if we start from the multicurves depicted in Figure 6 and drop the components $\alpha_1, \alpha_3, \alpha_6$ as well as β_5, β_6, then the matrix $XX^\top$ for $\alpha_5, \alpha_2, \alpha_4$ is exactly the matrix as above:

$$XX^\top = \begin{pmatrix} 116 & 56 & 16 \\ 56 & 28 & 8 \\ 16 & 8 & 4 \end{pmatrix},$$

with irreducible characteristic polynomial. If we add back α_6 and β_6 with $y^2 - 116$ parallel copies, the resulting characteristic polynomials is irreducible for infinitely many choices of y by Lemma 9, realising degree 4. Note that all in all, we have dropped the components α_1, α_3 and β_6, which are all separating. Therefore, we have not changed the fact that the multicurves α and β consist of components that are separating or that come in bounding pairs, where for each bounding pair one of the curves is a component of α and the other is a component of β.

5.2.2. *Realising multicurve intersection degrees* $g \leq d \leq 3g-6$. We start by constructing a surface of genus $g-2$ with two boundary components, which we then close off in a second step.

Using surfaces of the type depicted in Figure 4 and applying the inductive step procedure, we can construct a surface of genus $g-2 \geq 1$ and two boundary components, as well as filling multicurves α and β with intersection degree $3(g-2)-1 = 3g-7$. Using at some point in the inductive procedure a block of type 3 of genus $k \leq g-2$, as depicted in Figure 8, we can reduce the degree by up to $2k-2 \leq 2g-6$, realising multicurve intersection degrees from $g-1$ to $3g-7$ on the surface of genus $g-2$ with two boundary components. Now we close the surface, as depicted in Figure 9, adding the new components α_0 and β_0 to the multicurves α and β, respectively.

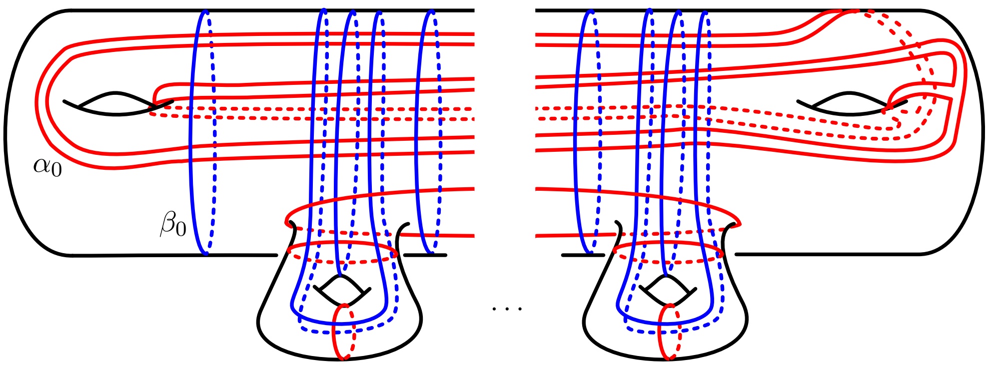


FIGURE 9. Two separating curves α_0 and β_0. There are ρ parallel copies of β_0.

We obtain the matrix

$$XX^\top = \left(\begin{array}{c|ccc} 64\rho + 16a_1 & 4a_1 & \dots & 4a_n \\ \hline 4a_1 & & & \\ \vdots & & A & \\ 4a_n & & & \end{array}\right),$$

where A is the matrix $XX^\top$ before adding the curves α_0 and β_0. Since $a_1 = 4a$, we can set $\rho = y^2 - a$ to have the top-left coefficient $64y^2$, which is exactly the form of the matrix in Lemma 9. Finishing the argument as usual, we can realise the multicurve intersection degrees $g \le d \le 3g-6$ for $g \ge 3$.

5.2.3. *Realising multicurve intersection degrees* $1 \le d < g$. Realising multicurve intersection degree one is clearly achieved by taking a pair of separating filling curves on the surface S_g.

For $2 \le d < g$, let us define $f = g - 1 - d$. We start with a surface block of type 3 of genus $g-2$, where we deleted all the components of α that are separating. We also remove the component of β in the middle of Figure 8. Furthermore, we let the $f+1 \le g-2$ first of the parameters y_i be equal to 1. Then we close off the surface as in the previous case, adding one component α_0 to α and one component β_0 to β, compare with Figure 9. Assume there are ρ parallel copies of β_0. We get

$$XX^\top = \begin{pmatrix} 64(\rho - g + 2) + 256\delta & 32y_1 & 32y_2 & \cdots & 32y_{g-2} \\ 32y_1 & 4y_1 & & & \\ 32y_2 & & 4y_2 & & \\ \vdots & & & \ddots & \\ 32y_{g-2} & & & & 4y_{g-2} \end{pmatrix},$$

where $\delta = y_1 + \cdots + y_{g-2}$. We choose ρ such that $64(\rho - g + 2) + 256\delta = 64y^2$. To simplify the calculations, we let $z_i = 4y_i$ for $i = 1, \ldots, g-2$. The matrix becomes

$$XX^\top = \begin{pmatrix} 64y^2 & 8z_1 & 8z_2 & \cdots & 8z_{g-2} \\ 8z_1 & z_1 & & & \\ 8z_2 & & z_2 & & \\ \vdots & & & \ddots & \\ 8z_{g-2} & & & & z_{g-2} \end{pmatrix}.$$

By Lemma 9 in [LL24], the characteristic polynomial of $XX^\top$ equals

$$p(t,y,\mathbf{z}) = -64y^2 \prod_{i=1}^{g-2}(t - z_i) + t \prod_{i=1}^{g-2}(t - z_i) - \sum_{i=1}^{g-2} 64 z_i^2 \prod_{j \neq i}(t - z_j).$$

If all the z_i are pairwise distinct, this polynomial is irreducible as a polynomial in t, y by Lemma 10 in [LL24]. However, we chose that the first $f + 1$ coefficients $y_1, \ldots, y_{f+1}$ are equal to 1 and the other $y_i \neq 1$ and pairwise distinct. In particular, the polynomial $p(t,y)$ factors as $(t-4)^f \tilde{p}(t,y)$, where $\tilde{p}(t,y)$ is of degree $g - 1 - f = d$ in the variable t and with pairwise distinct z_i. In particular, Lemma 10 in [LL24] implies that $\tilde{p}(t,y) \in \mathbb{Z}[t,y]$ is irreducible. Hilbert's irreducibility theorem guarantees the existence of infinitely many positive specifications of y such that the resulting polynomial is irreducible in $\mathbb{Z}[t]$.

This finishes the proof of Theorem 3.

Finally, we end this section we a proof of Theorem 5.

Proof of Theorem 5. For every $g \geq 3$ and every integer $1 \leq d \leq 3g-3$, we have constructed a pair of filling multicurves α and β, with a parameter y, such that $\chi_{XX^\top}(t,y) \in \mathbb{Z}[t,y]$ is irreducible. By Remark 7, we may run the same argument to show that also the polynomial $\chi_{XX^\top}(t^2,y) \in \mathbb{Z}[t,y]$ is irreducible. By Hilbert's irreducibility theorem, we find infinitely many specifications of y such that $\chi_{XX^\top}(t^2) \in \mathbb{Z}[t]$ is irreducible of degree $2d$. The leading eigenvalue μ of $XX^\top$ is a root of a characteristic polynomial $\chi_{XX^\top}(t)$, so $\chi_{XX^\top}(t^2) \in \mathbb{Z}[t]$ is the minimal polynomial of $\sqrt{\mu}$. Hence, the multicurve bipartite degree of α and β equals $2d$.

For $g = 2$, we use the example constructed in Section 4 for $d = 3$ and the examples in [LL24] for $d = 1, 2$. Similarly to Remark 7, one can run the same proof as [LL24, Lemma 10] to show that $\chi_{XX^\top}(t^2,y) \in \mathbb{Z}[t,y]$ is irreducible. □

6. Explicit pseudo-Anosov maps

The goal of this section is to construct, as explicitly as possible, pseudo-Anosov maps whose stretch factors have prescribed algebraic degrees. In a first step, we build upon our examples in Section 4 to realise also all trace field degrees $d < 3g-3$ on S_g for $g \geq 2$.

6.1. Multicurve intersection degrees $d < 3g - 3$.

We need a new building block for our surfaces, see Figure 10.

We denote the red multicurve by α and the blue multicurve by β. The multicurve α has $k+1$ separating components: one for each of the handles that separates the handle, and one that separates all the handles. We denote the component of α that separates all the handles of the surface in Figure 10 by α_1, and we denote the other

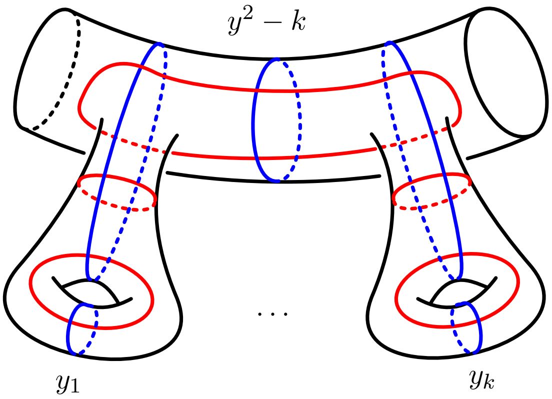


FIGURE 10. A surface of genus k with two boundary components, as well as two multicurves α (in red) and β (in blue). Some components of β have several parallel copies, as indicated by $y_1, \ldots, y_k$ and $y^2 - k$.

separating components of α by $\alpha_2, \alpha_4, \ldots, \alpha_{2g-2}$ from left to right. Finally, the remaining nonseparating components of α are $\alpha_3, \alpha_5, \ldots, \alpha_{2g-1}$ from left to right. In this situation, we have

$$XX^\top = \left(\begin{array}{c|cccc} 4y^2 & v^\top & v^\top & \cdots & v^\top \\ \hline v & B_{y_1} & 0 & & \\ v & 0 & B_{y_2} & & \\ \vdots & & & \ddots & \\ v & & & & B_{y_k} \end{array}\right), \qquad B_{y_i} = \begin{pmatrix} 4 & 2 \\ 2 & y_i \end{pmatrix}, \qquad v = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

Let $p_{y_i}(t) = t^2 - (4 + y_i)t + 4(y_i - 1)$ be the characteristic polynomial of B_{y_i}. We know from Section 4 that p_{y_i} is irreducible if $y \geq 12$. So, choosing all $y_i \geq 12$ pairwise distinct, Remark 11 guarantees that the polynomial $\chi_{XX^\top}(t, y) \in \mathbb{Z}[t][y]$ is irreducible. By Hilbert's irreducibility theorem, there are infinitely many specifications of y such that $y^2 - k > 0$ and such that $\chi_{XX^\top}(t) \in \mathbb{Z}[t]$ is irreducible and of degree $2k + 1$.

Case 1: $2g \leq d < 3g - 3$. Assume we want to realise the multicurve intersection degree $3g - 3 - f$ for $0 < f \leq g - 3$. Let $k = f + 2 \leq g - 1$. Start the inductive procedure as in Section 4 with the surface from Figure 10 as a starting point, adding $g - 1 - k$ more handles. The exact same argument yields a surface of genus $g - 1$ with two boundary components, and a characteristic polynomial $\chi_{XX^\top} \in \mathbb{Z}[t]$ that is irreducible and of degree $2k + 1 + 3(g - 1 - k) = 3g - 3 - k + 1$. Closing up the surface exactly as in Section 4 yields $3g - 3 - k + 2 = 3g - 3 - f$ as a multicurve intersection degree on the closed orientable surface of genus g.

Case 2: $g < d < 2g$. Assume we want to realise the multicurve intersection degree $2g - f$ for $0 < f \leq g - 1$. Take the surface depicted in Figure 10 for $k = g - 1$. Now, remove f of the separating curve $\alpha_2, \ldots, \alpha_{2g-2}$. This slightly modifies the matrix $XX^\top$: f of the 2-by-2 blocks on the diagonal are now 1-by-1 blocks, with the single coefficient y_i. Nevertheless, since all the y_i are chosen pairwise distinct, Remark 11 guarantees that the polynomial $\chi_{XX^\top}(t, y) \in \mathbb{Z}[t][y]$ is irreducible. We note that for the coefficients y_i in the 1-by-1 blocks, any positive integer can be chosen.

By Hilbert's irreducibility theorem, there are infinitely many specifications of y such that $\chi_{XX^\top}(t) \in \mathbb{Z}[t]$ is irreducible and of degree $2g-1-f$. Closing up the surface as in Section 4 yields the multicurve intersection degree $2g-f$ on the closed orientable surface of genus g.

Case 3: $1 \le d \le g$. This is the case we have already dealt with in [LL24].

6.2. **Even degree stretch factors.** In this section, we show that in our construction of multicurves in Section 6.1, the degree of the stretch factor of $T_\alpha \circ T_\beta$ equals two over the trace field. It uses the nonsplitting criterion of [LL24, Theorem 6] that we recall below.

Theorem 12 ([LL24], Theorem 6). *Let $\alpha, \beta \subset S$ be a pair of filling multicurves. Let X be their geometric intersection matrix, let d be their multicurve intersection degree and let $\Omega = \left(\begin{smallmatrix} 0 & X \\ X^\top & 0 \end{smallmatrix}\right)$. If $\dim(\Omega) > \sigma(\Omega + 2I) + \mathrm{null}(\Omega + 2I) > \dim(\Omega) - 2d$, then the mapping class $T_\alpha \circ T_\beta$ is pseudo-Anosov with stretch factor λ of degree $2d$.*

Here, $\sigma(A)$ and $\mathrm{null}(A)$ denote the signature and the nullity, respectively, of the matrix A.

Theorem 13. *Let α and β be an example of a pair of multicurves described in Section 6.1, realising a multicurve intersection degree $1 \le d \le 3g-3$. Then the mapping class $T_\alpha \circ T_\beta$ is pseudo-Anosov with stretch factor λ of degree $2d$.*

For the case $1 \le d \le g$, this is shown in [LL24].

Proof of Theorem 13. According to Theorem 12, all there is to show is

$$\dim(\Omega) > \sigma(\Omega + 2I) + \mathrm{null}(\Omega + 2I) > \dim(\Omega) - 2d. \tag{1}$$

We now make a case distinction depending on d.

Case 1: $2g \le d \le 3g-3$. We consider the submatrix Ω' of Ω that is obtained by deleting all the rows and columns corresponding to components of the multicurve α that have been added during the inductive step or closing up of the surface. Furthermore, if $d < 3g-3$, we also remove the component of α encircling multiple handles of the starting surface, that is, the surface depicted in Figure 10.

A base change by a permutation matrix brings $\Omega' + 2I$ into block diagonal form with $g-1$ blocks corresponding to genus one surface pieces as depicted in Figure 1, and a block of the form $2I$. For a block of the former type, and for $y > 4$, we directly calculate that the nullity is zero and the signature equals the dimension of the block minus two. Already, this implies that certainly the signature of $\Omega + 2I$ is not equal to its dimension, and it only remains to verify the lower bound in Equation (1).

By construction, if the genus equals $g \ge 2$, we have $g-1$ surface pieces as in Figure 1. This in particular implies that $\sigma(\Omega') = \dim(\Omega') - 2g + 2$. Furthermore, we have $\dim(\Omega) - \dim(\Omega') = d - 2g + 2$. The latter equality follows from that fact that the number of components of α in our construction is exactly d, and there are two components per surface pieces as in Figure 1. We now calculate

$$\begin{aligned}
\sigma(\Omega + 2I) &\ge \sigma(\Omega') - (\dim(\Omega) - \dim(\Omega')) \\
&= (\dim(\Omega') - 2g + 2) - (d - 2g + 2) \\
&= \dim(\Omega') - d \\
&> \dim(\Omega) - 2d,
\end{aligned}$$

which implies Equation (1), so we are done for this case.

Case 2: $g < d < 2g$. We consider the submatrix Ω' of Ω that is obtained by deleting two rows and two columns corresponding to components of the multicurve α: the one corresponding to the component encircling multiple handles in Figure 10 and the one obtained from closing the surface. Recall that we have removed $f = 2g - d$ separating curves $\alpha_2, \ldots, \alpha_{2g-2}$.

A base change by a permutation matrix brings $\Omega' + 2I$ into block diagonal form with $g - 1 - f$ blocks corresponding to surface pieces as in Figure 1, f blocks corresponding to surface pieces as in Figure 1 but with the separating component of α removed, and a block of the form $2I$.

For a block of the first type, and for $y > 4$, recall from the previous case that the nullity is zero and the signature equals the dimension of the block minus two. For a block of the second type, the sum of the nullity and the signature equals the dimension of the block if $y \leq 3$, and it equals the dimension of the block minus two if $y > 3$. We may assume that for at least one block of the second type, we have $y = 3$. This is enough to ensure that $\dim(\Omega) > \sigma(\Omega + 2I) + \mathrm{null}(\Omega + 2I)$, so again we only need to verify the lower bound in Equation (1).

By construction, if the genus equals $g \geq 2$, we have $g-1$ surface pieces as in Figure 1. Having at least one piece with $y \leq 3$, this implies that $\sigma(\Omega') > \dim(\Omega') - 2g + 2$. Furthermore, we have $\dim(\Omega) - \dim(\Omega') = 2$. We now calculate

$$\begin{aligned}\sigma(\Omega + 2I) &\geq \sigma(\Omega') - (\dim(\Omega) - \dim(\Omega')) \\ &> (\dim(\Omega') - 2g + 2) - 2 \\ &= \dim(\Omega') - 2g \\ &= \dim(\Omega) - 2g + 2 \geq \dim(\Omega) - 2d,\end{aligned}$$

which implies Equation (1) also in the case $g < d < 2g$, so we are done. □

6.3. **Explicit examples with stretch factor degree $6g - 6$.** We conclude this section by giving explicit computations supporting a conjecture on the irreducibility of the characteristic polynomials constructed in Section 4 for specific values of y. In the inductive step of Section 4, one uses a map

$$\phi_k : \begin{array}{ccc} M_k(\mathbb{Z}) \times \mathbb{Z} & \longrightarrow & M_{k+3}(\mathbb{Z}) \\ (C, y) & \mapsto & \left(\begin{array}{c|c|c} 4y^2 & * & * \\ \hline * & C & \\ \hline * & & A \end{array}\right) \end{array}, \qquad \text{with } A = \left(\begin{smallmatrix} 4 & 2 \\ 2 & 12 \end{smallmatrix}\right).$$

For $g > 1$ we inductively construct the $(3g - 1) \times (3g - 1)$ matrix M_g with the maps ϕ_{3i-1} for $i = 1, \ldots, g - 1$:

$$M_g = \phi_{3(g-1)-1}(\phi_{3(g-2)-1}(\ldots \phi_{3\cdot 2-1}(\phi_{3\cdot 1-1}(B, y^{(1)}), y^{(2)}), \ldots, y^{(g-2)}), y^{(g-1)}),$$

with $B = \left(\begin{smallmatrix} 4 & 2 \\ 2 & 13 \end{smallmatrix}\right)$ and suitable parameters $y^{(i)}$ given by Hilbert's irreducibility theorem. The condition $y^2 > \frac{1}{4}c_{11} + 1$ appearing in the construction is obviously equivalent to $(y^{(i)})^2 > (y^{(i-1)})^2 + 1$. Finally, following Section 4 the matrix $XX^\top$ for the multicurves α and β on the closed surface of genus $g + 1$ takes the form

$$N_g = \left(\begin{array}{c|c} y^2 & * \\ \hline * & M_g \end{array}\right)$$

with the condition $y^2 > \frac{1}{4}(M_g)_{11} = (y^{(g)})^2$.

By computer assistance, one immediately checks the following proposition.

Proposition 14. *For any $1 < g \leq 200$, if $y^{(i)} = i + 1$ for $i = 1, \dots, g - 1$, then the characteristic polynomial χ_{M_g} is irreducible over $\mathbb{Q}$. Moreover for $y = g + 1$, χ_{N_g} is irreducible over $\mathbb{Q}$.*

Together with Theorem 13, this gives explicit examples of pseudo-Anosov maps realizing the upper bound $6g - 6$ in Theorem 3 for every $1 < g \leq 201$. We don't know whether χ_{M_g} and χ_{N_g} are actually irreducible for every $g > 200$ with the parameters $y^{(i)} = i + 1$ chosen as in Proposition 14.

References

[And87] Tsuyoshi Ando. Totally positive matrices. *Linear Algebra Appl.*, 90:165–219, 1987. doi:10.1016/0024-3795(87)90313-2.

[Bel21] Mark Bell. flipper (computer software). https://pypi.org/project/flipper/, 2013–2021. Version 2 July 2021.

[Bou22] Julien Boulanger. Central points of the double heptagon translation surface are not connection points. *Bull. Soc. Math. Fr.*, 150(2):459–472, 2022. doi:10.24033/bsmf.2851.

[GJ00] Eugene Gutkin and Chris Judge. Affine mappings of translation surfaces: Geometry and arithmetic. *Duke Math. J.*, 103(2):191–213, 2000. doi:10.1215/S0012-7094-00-10321-3.

[HK06] Eriko Hironaka and Eiko Kin. A family of pseudo-Anosov braids with small dilatation. *Algebraic & Geometric Topology*, 6(2):699 – 738, 2006. doi:10.2140/agt.2006.6.699.

[HMTY08] E. H. Hanson, Adam Benjamin Merberg, Christopher Towse, and Elena Yudovina. Generalized continued fractions and orbits under the action of hecke triangle groups. *Acta Arithmetica*, 134:337–348, 2008. URL: https://api.semanticscholar.org/CorpusID:58923865.

[KS00] Richard Kenyon and John Smillie. Billiards on rational-angled triangles. *Comment. Math. Helv.*, 75(1):65–108, 2000. doi:10.1007/s000140050113.

[LL24] Erwan Lanneau and Livio Liechti. Trace field degrees of abelian differentials. *Comment. Math. Helv.*, 99(1):81–110, 2024. doi:10.4171/CMH/565.

[Mar19] D. Margalit. Problems, questions, and conjectures about mapping class groups. In *Breadth in contemporary topology*, volume 102 of *Proc. Sympos. Pure Math.*, pages 157–186. Amer. Math. Soc., Providence, RI, 2019.

[Mur08] M. Ram Murty. Polynomials assuming square values. In *Number theory and discrete geometry. Proceedings of the international conference in honour of Professor R. P. Bambah, Chandigarh, India, November 30–December 3, 2005*, pages 155–163. Mysore: Ramanujan Mathematical Society, 2008.

[Str17] Balázs Strenner. Algebraic degrees of pseudo-Anosov stretch factors. *Geom. Funct. Anal.*, 27(6):1497–1539, 2017. doi:10.1007/s00039-017-0429-4.

[Thu88] William P. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Am. Math. Soc., New Ser.*, 19(2):417–431, 1988. doi:10.1090/S0273-0979-1988-15685-6.

[Vee82] William A. Veech. Gauss measures for transformations on the space of interval exchange maps. *Ann. Math. (2)*, 115:201–242, 1982. doi:10.2307/1971391.

[Vee89] William A. Veech. Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards. *Invent. Math.*, 97(3):553–583, 1989. doi:10.1007/BF01388890.

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