Inertial effects in the Lubrication Limit.

(1) Landau-Levich drag-out problem, (2) Bubble rise in Hele-Shaw cells

J John Soundar Jerome

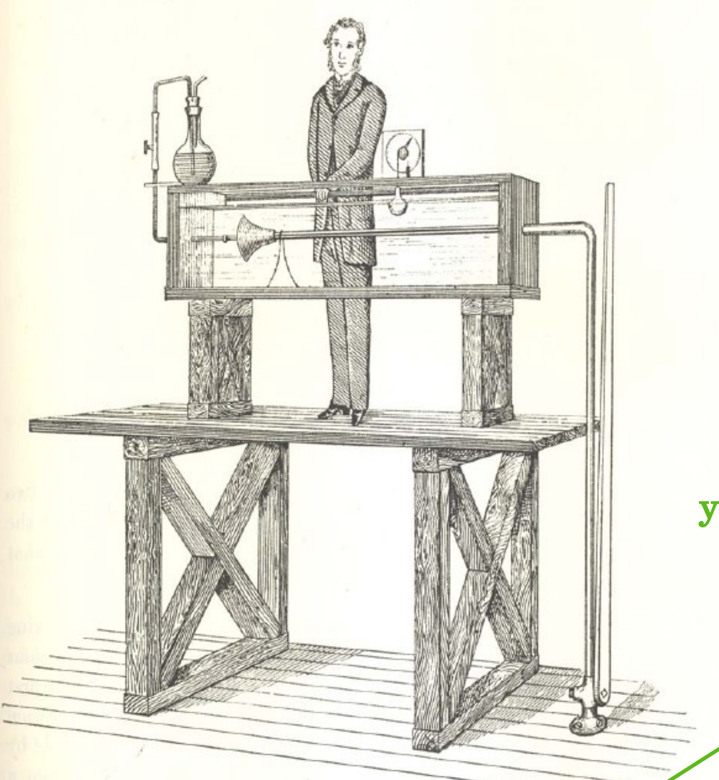
LMFA CNRS UMR – 5509

Université Claude-Bernard Lyon1 (France)

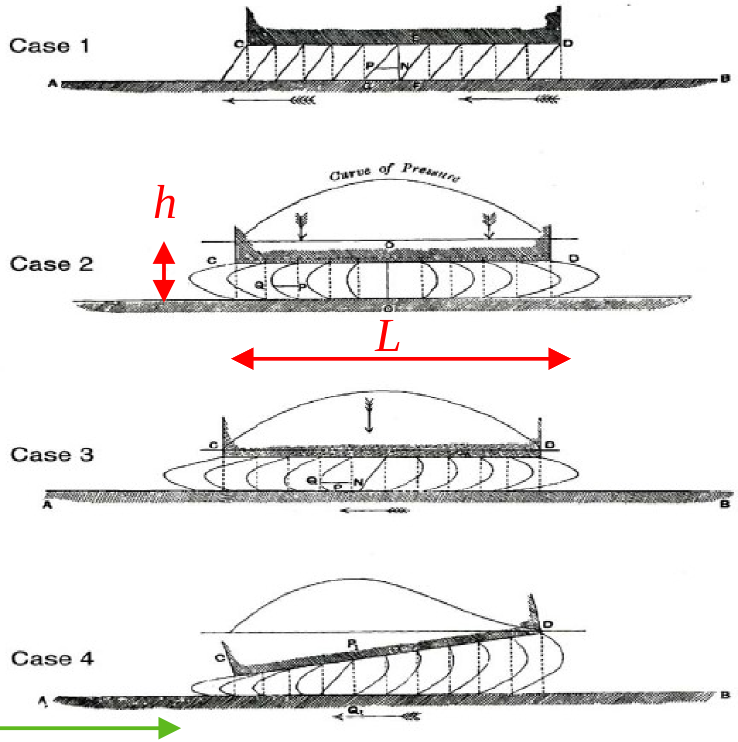
Reynolds' lubrication theory

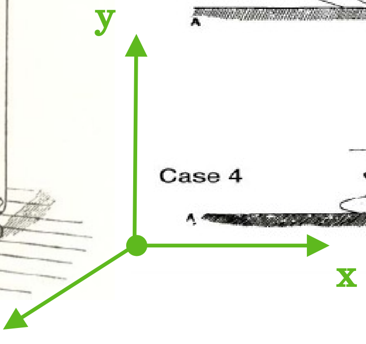
Historical background

(Art. 11). Also that the forces arising from weight and inertia are altogether small compared with the stresses arising from viscosity.



Reynolds (1883)





Reynolds (1886)

$$L \gg h \quad \Rightarrow v \ll u, w$$

$$Re = \frac{\rho U L}{\mu}\left(\frac{h}{L}\right)^2 \ll 1$$

$$\left.\begin{aligned} \frac{dp}{dx} &= \mu \frac{d^2u}{dy^2} \\ \frac{dp}{dy} &= 0 \\ \frac{dp}{dz} &= \mu \frac{d^2w}{dy^2} \\ 0 &= \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \end{aligned}\right\}$$

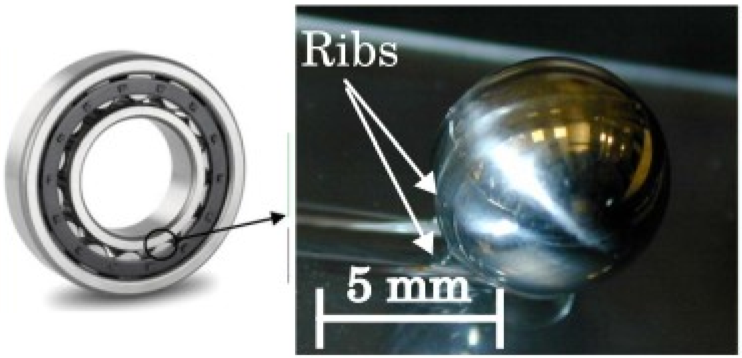
$$\left.\begin{aligned} u &= \frac{1}{2\mu}\frac{dp}{dx}(y-h)y + U_0\frac{h-y}{h} + U_1\frac{y}{h} \\ w &= \frac{1}{2\mu}\frac{dp}{dz}(y-h)y \end{aligned}\right\}$$

The life and work of Osborne Reynolds

J. D. Jackson (2006)

Liquid entrainment problem
Introduction

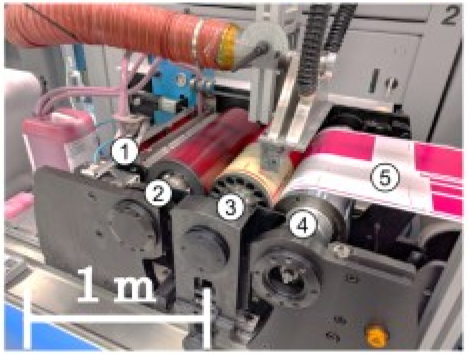
(a) Journal & bearing lubrication



Bico et al. (PoF 2009) ;

(b) Wheels on wet tracks

(c) Off-set printing

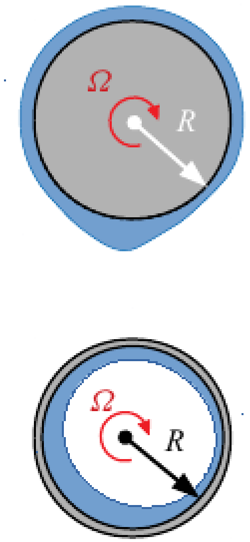


Brumm et al. (Coll. Int. 2009) ;

Coating flows in scientific literature

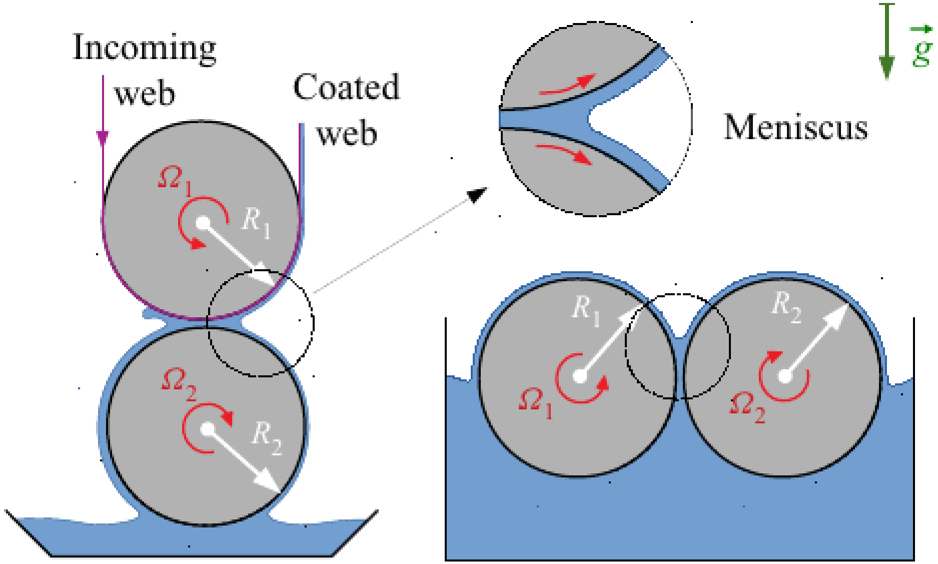
Introduction

(a) Rimming flows



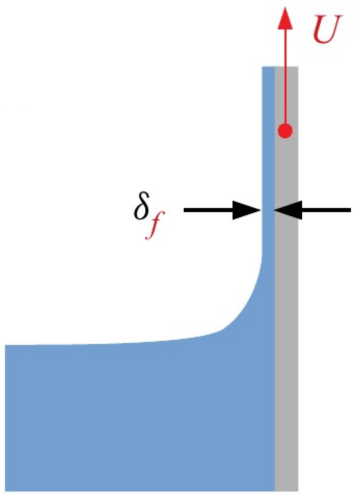
Moffatt (1976) ;
Hosoi & Mahadevan (PoF 1999)

(b) Film-splitting flows



Taylor (JFM 1963);
Ruschak (Ann. Rev. F. Mech. 1985) ;

(c) Self-metering film flows

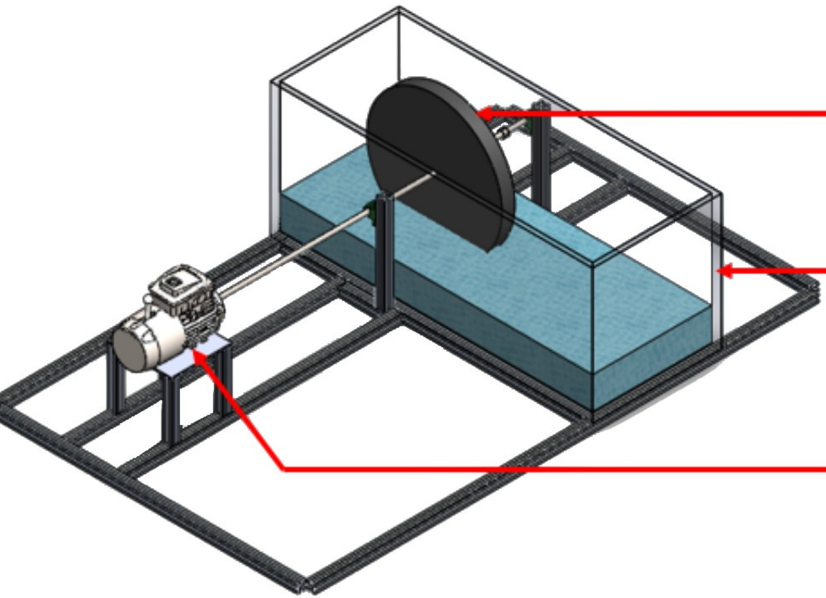


Landau & Levich (1942) ;
Deryaguin & Levi (1965)

Speeds less than a few cm/s .. & Thin films –> Lubrication approximation in thin gaps

Rotary drag-out on a disk (4 cm wide)

Landau-Levich flow at $Re_{lub} >> 1$



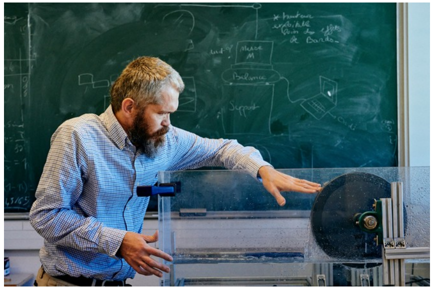
Rotating disc
Diameter = 0.2, 0.27, 0.42 m
Width =4.5 , 13.5 cm

Plexiglass tank
Height = 0.415 m
Width = 0.4 m
Length = 1 m

Asynchronous Motor (IP55) coupled with a Parker AC10 frequency controller (10 – 400 rpm)

Sébastien Thévenin @CEA/Paris-Saclay

Mickaël Bourgoin @LPENS/ENS

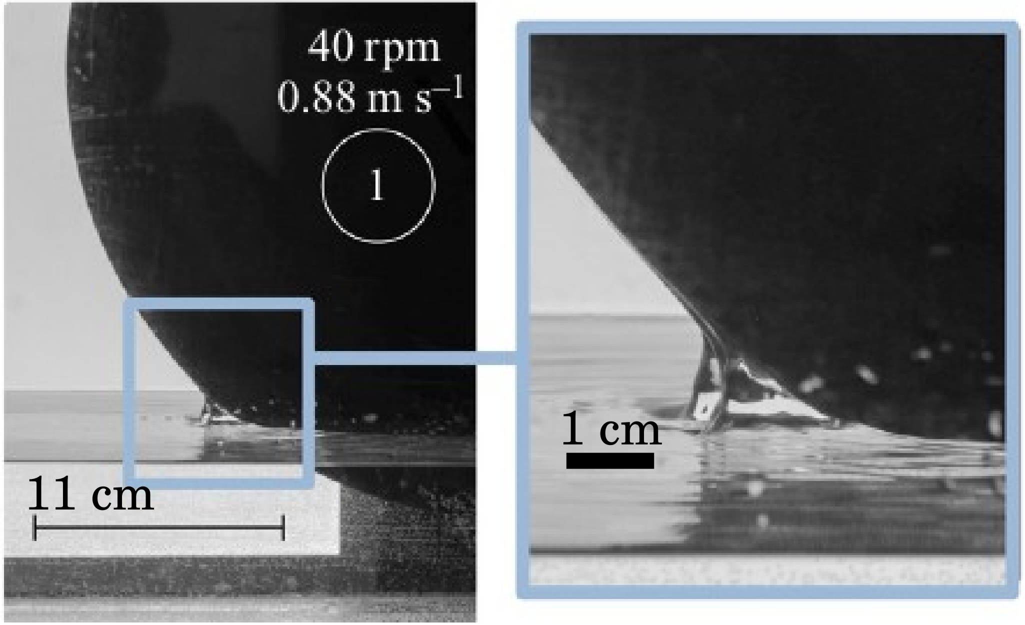


Jean-Philippe Matas @LMFA/UCBL

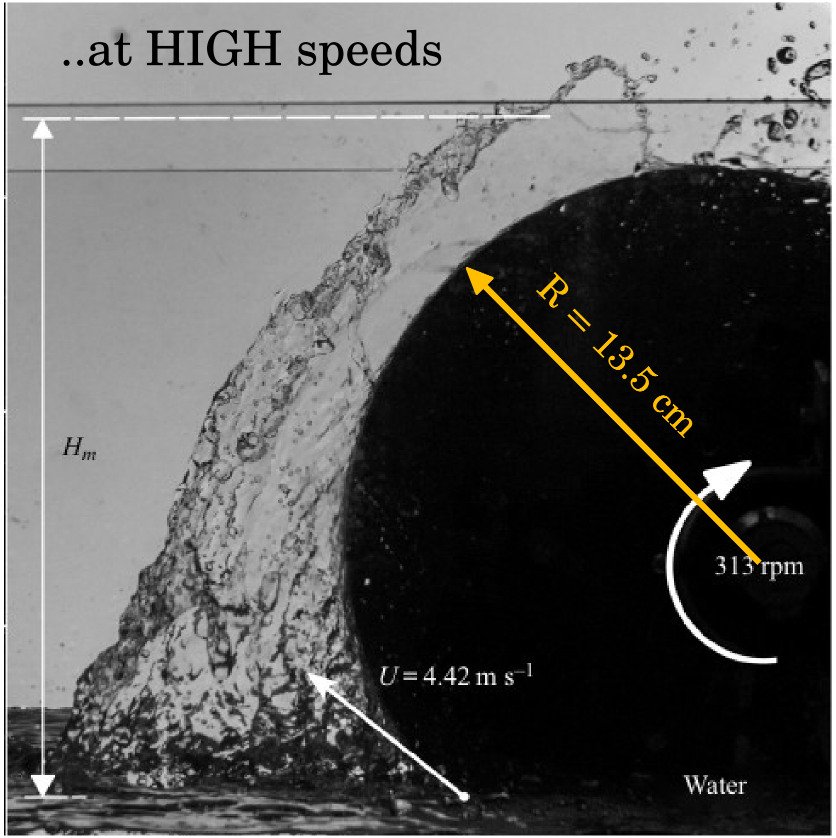
Life of a liquid sheet on a rotating disc

Water drag-out on a rotating disc

...at moderately LOW speeds

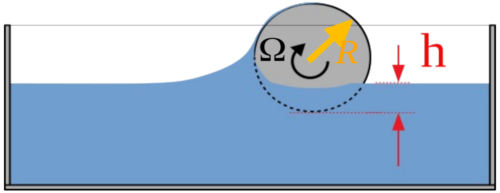


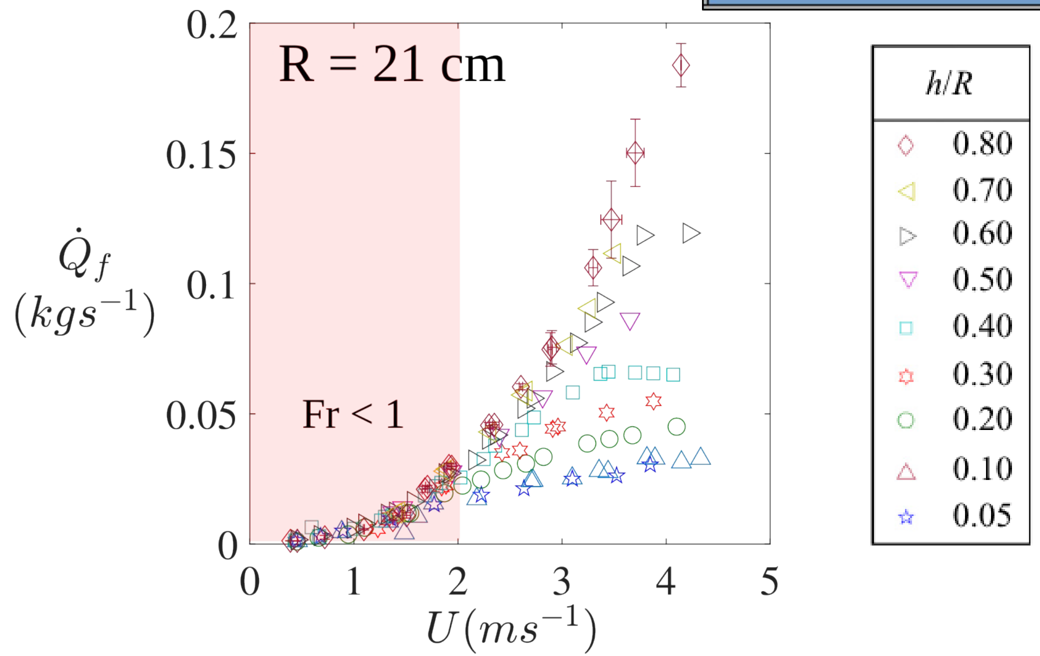
..at HIGH speeds

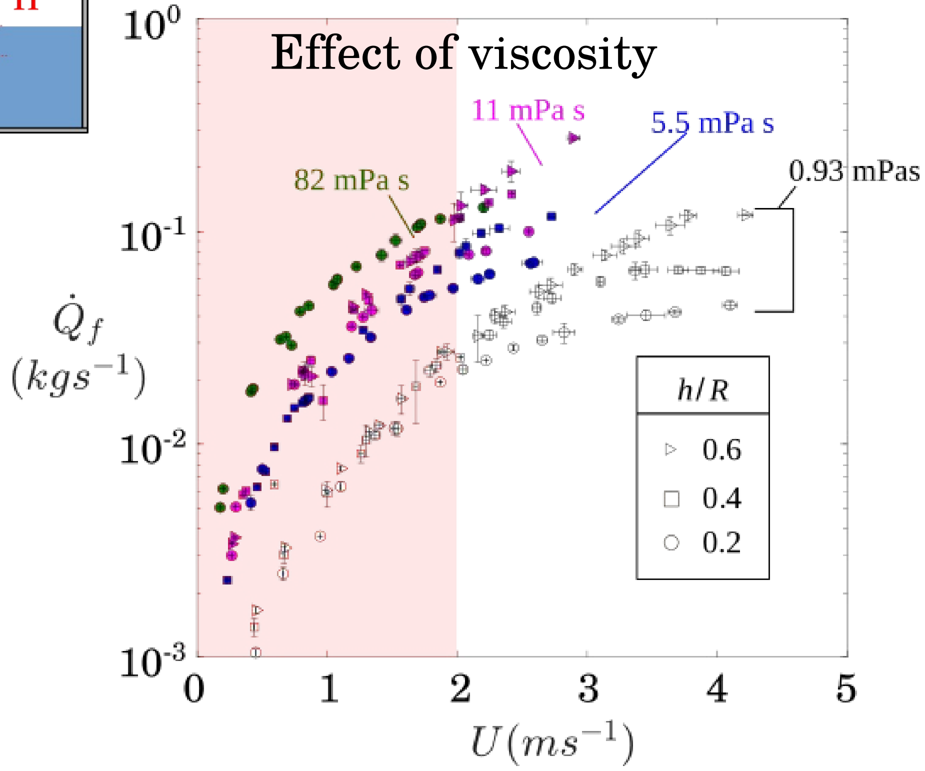


Drag-out flow rate (time-average)

Effect of speed and depth



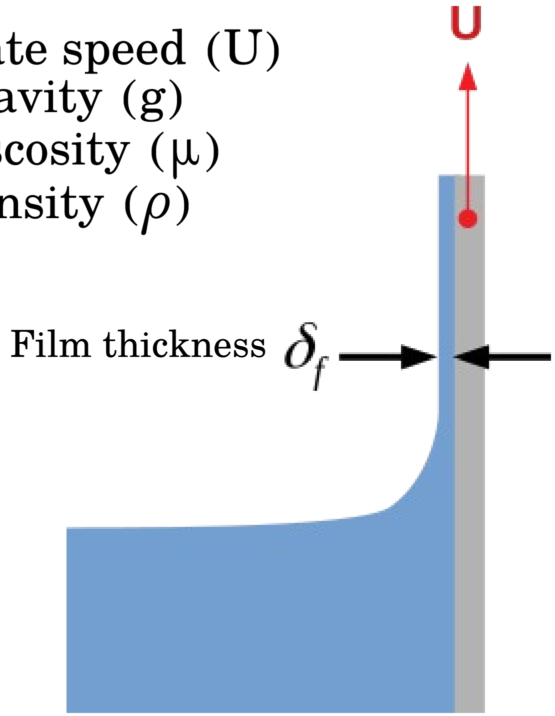




The drag-out, or the entrainment problem (at Re_f << 1)

Deryaguin's model

- Plate speed (U)
- Gravity (g)
- Viscosity (μ)
- Density (ρ)



Deryaguin's drag out problem

Viscosity-Gravity regime

$$\rho g \sim \mu U / \delta_f^2$$

(grad(p) – hydrostatic) *(plate drag/unit volume)*

$$\Rightarrow \delta_f \sim \sqrt{\frac{\mu U}{\rho g}}$$

Jeffreys (1930) ; Deryaguin (1945) ;

The drag-out, or the entrainment problem (at $Re_f << 1$)

Morey's experiments for the bureau of standards to check Jeffrey's drainage law.. *Morey (1940);*

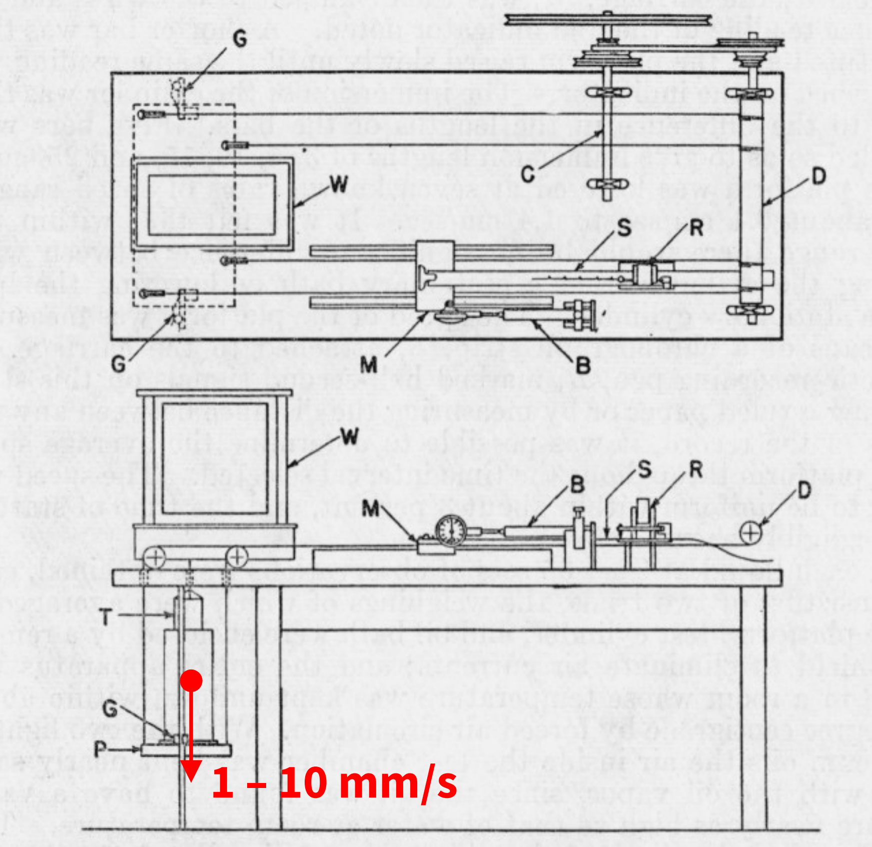


FIGURE 1.—*Schematic diagram of apparatus.*

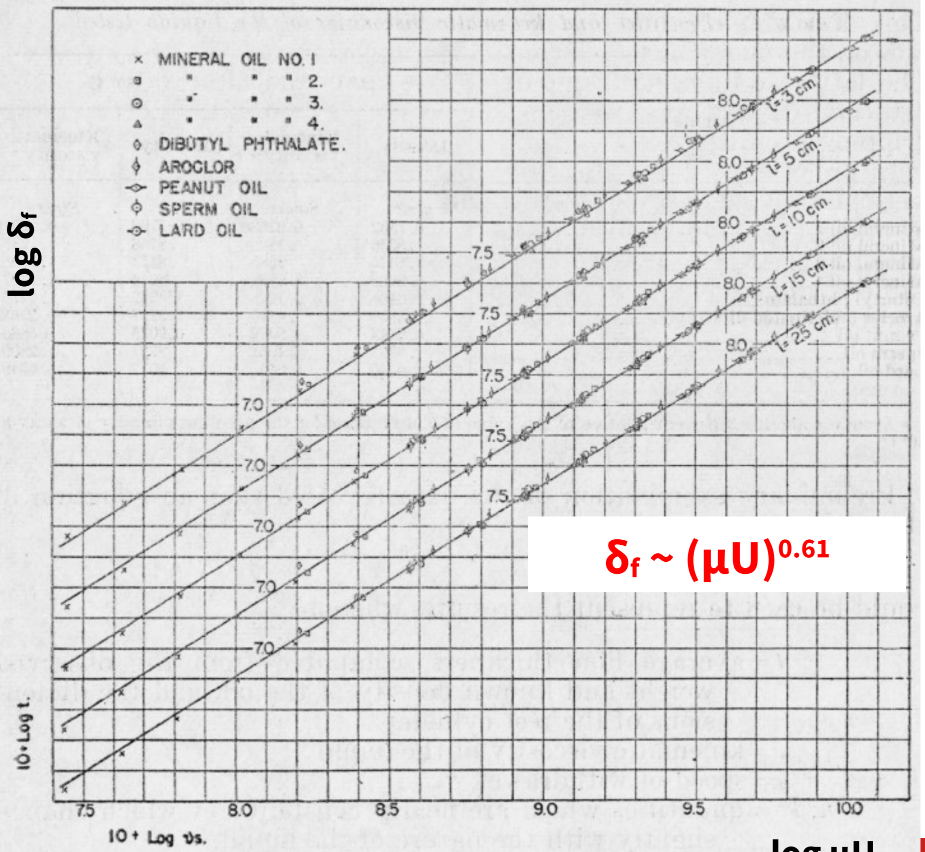
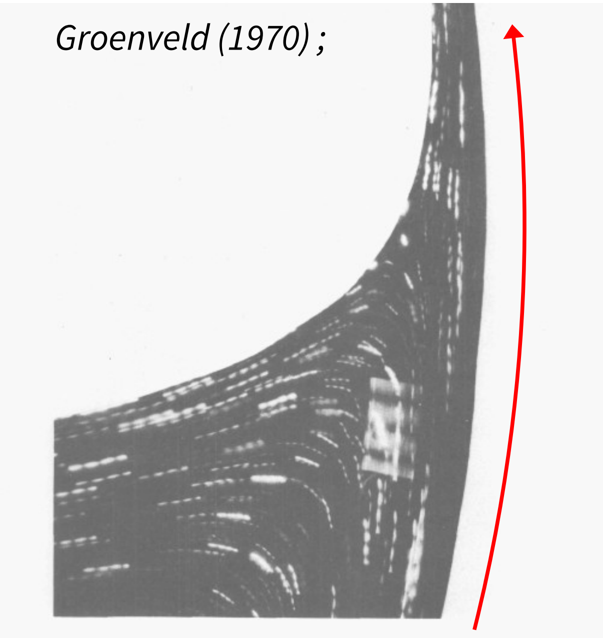


FIGURE 2.—*Graph of observed film thicknesses.*

The drag-out, or the entrainment problem (at $Re_f \ll 1$ and $Ca \ll 1$)

Landau-Levich's model

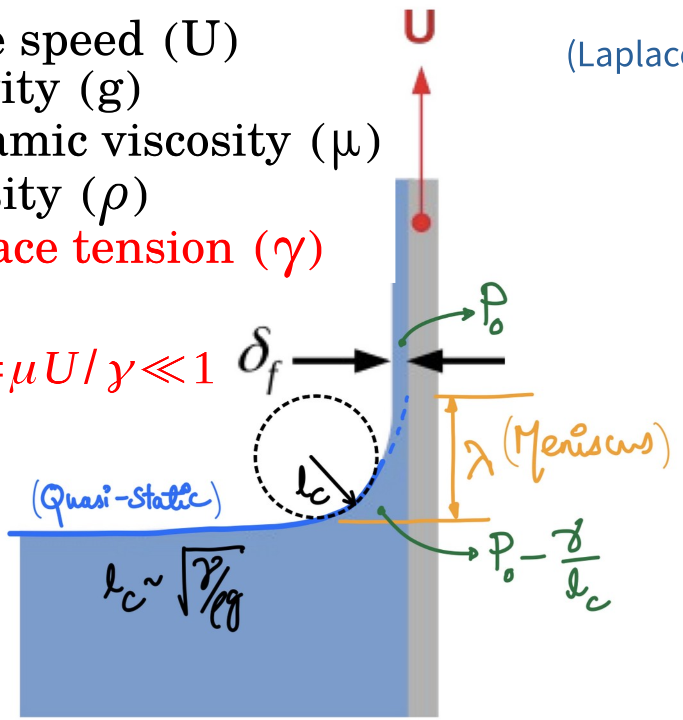
Groenveld (1970) ;



Photograph of the streamlines during withdrawal of a viscous liquid

- Plate speed (U)
- Gravity (g)
- Dynamic viscosity (μ)
- Density (ρ)
- Surface tension (γ)

$Ca = \mu U / \gamma \ll 1$



Landau-Levich flow

(Laplace pressure drop) (Plate drag/volume)

$$\frac{(\gamma/l_c)}{\lambda} \sim \frac{\mu U}{\delta_f^2}$$

\+

(Curvature matching)

$$\frac{\delta_f}{\lambda^2} \sim \frac{1}{l_c}$$

$$\Rightarrow \delta_f \simeq \sqrt{\frac{\mu U}{\rho g}}\, Ca^{1/6} = l_c\, Ca^{2/3}$$

$$\Rightarrow \lambda \simeq l_c\, Ca^{1/3}$$

Landau & Levich (1942) ;
Wilson (1982) ;

$$Re_{lub} = U^2 / g\, l_c$$

Capillary number, Ca = μU/γ

The drag-out, or the entrainment problem (at $Re_f << 1$)

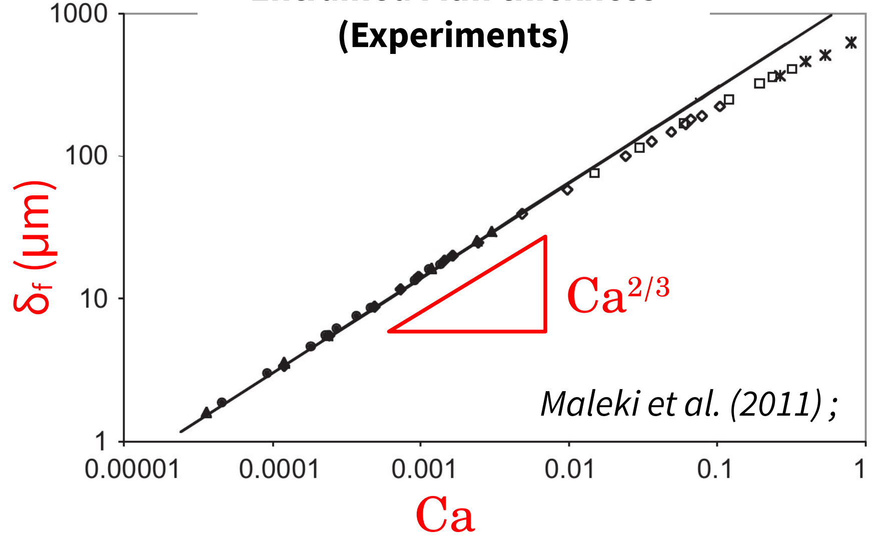
Overview

$$\delta_f \simeq \sqrt{\frac{\mu U}{\rho g}} Ca^{1/6} = l_c Ca^{2/3}$$ *Landau & Levich (1942) ; Wilson (1982) ;*

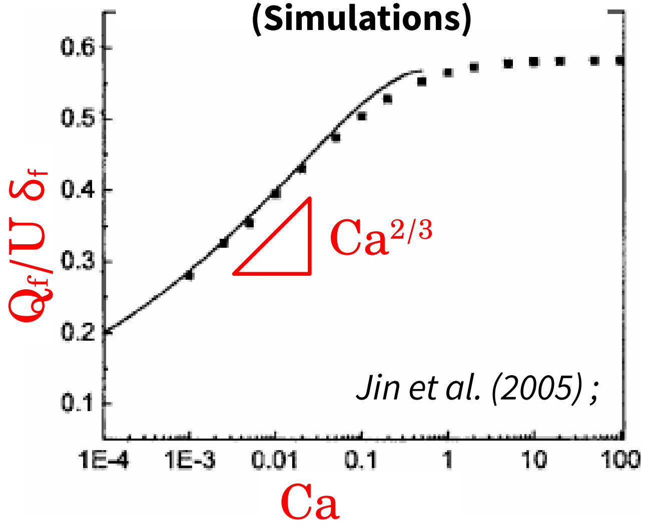
$$\delta_f \sim \sqrt{\frac{\mu U}{\rho g}}$$ *Jeffreys (1930) ; Deryaguin (1945) ;*

$$Ca = \mu U / \gamma$$

Entrained Film thickness (Experiments)



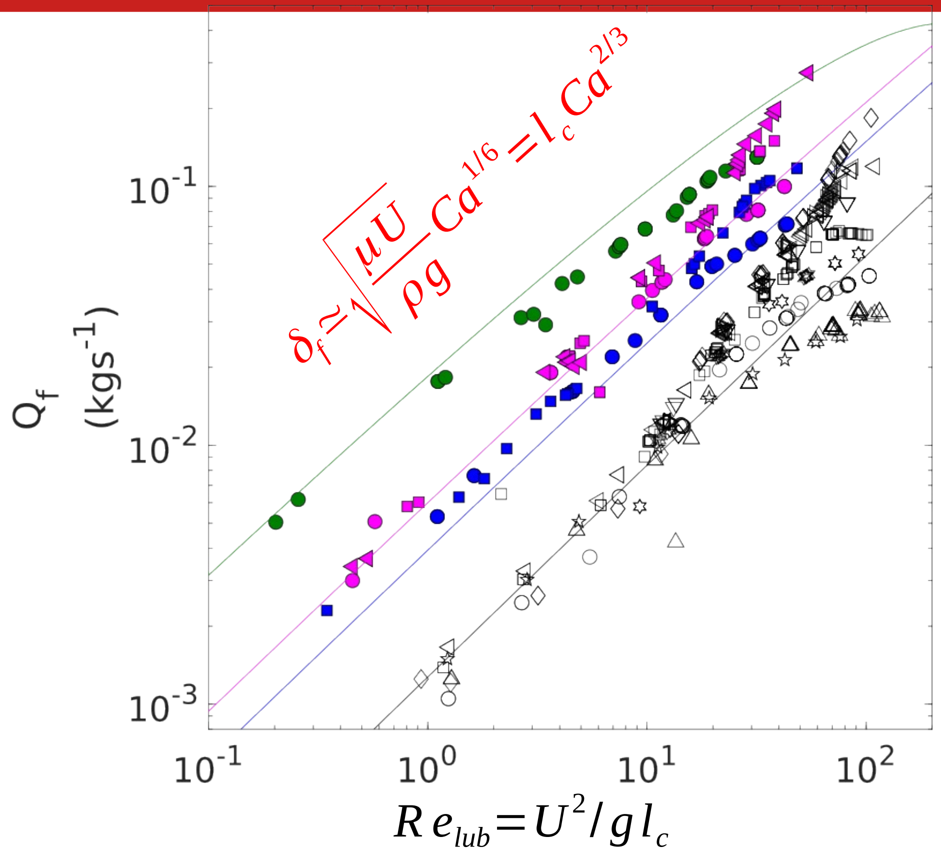
Non-dimensional Mass flux (Simulations)

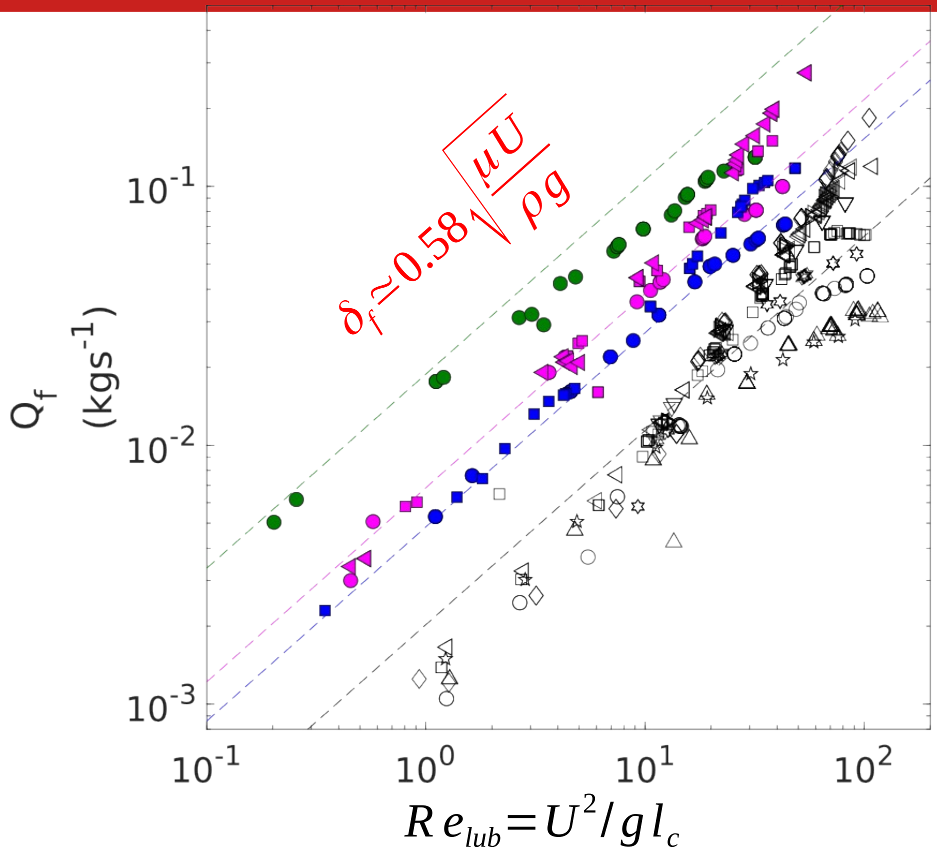


Capillary number, $Ca = \mu U/\gamma$

Comparison with Landau-Levich-Deryaguin flow rate scaling

Our experiments

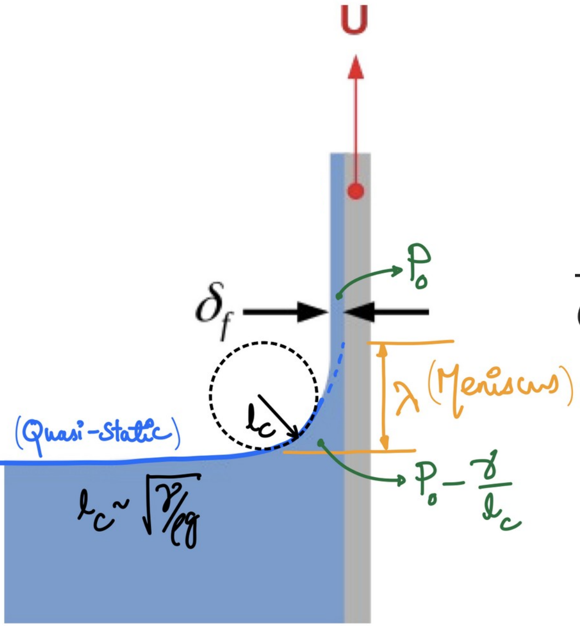




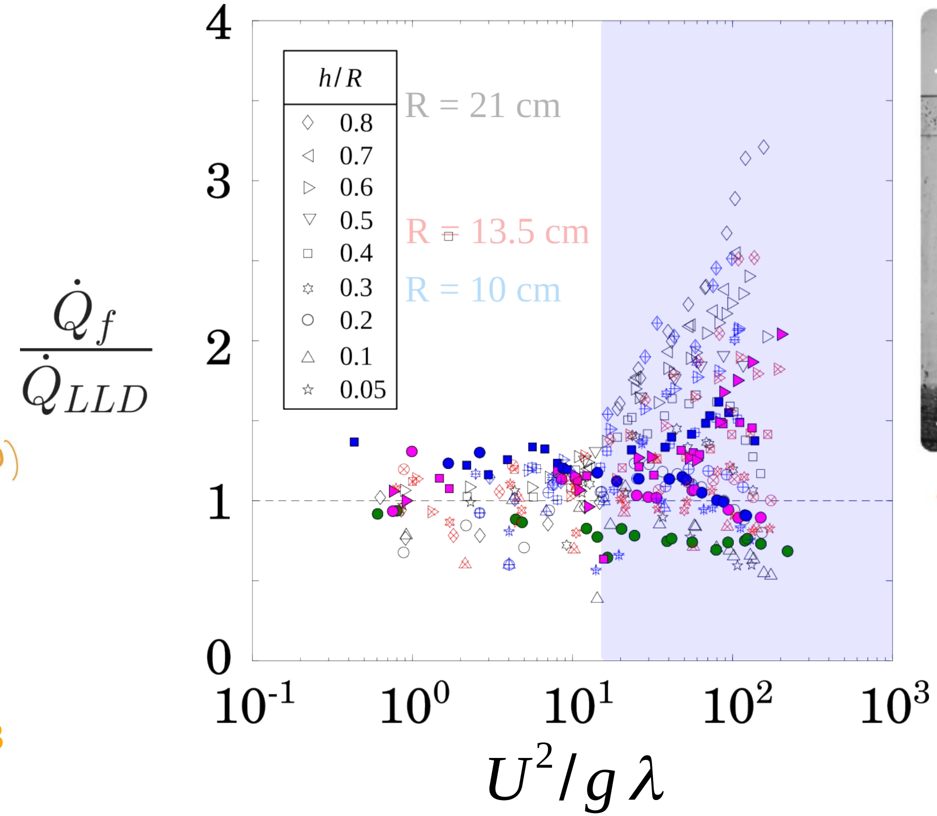
[Wilson (1982)'s 2nd order Ca - corrections where used for viscous UCON/Water mixtures]

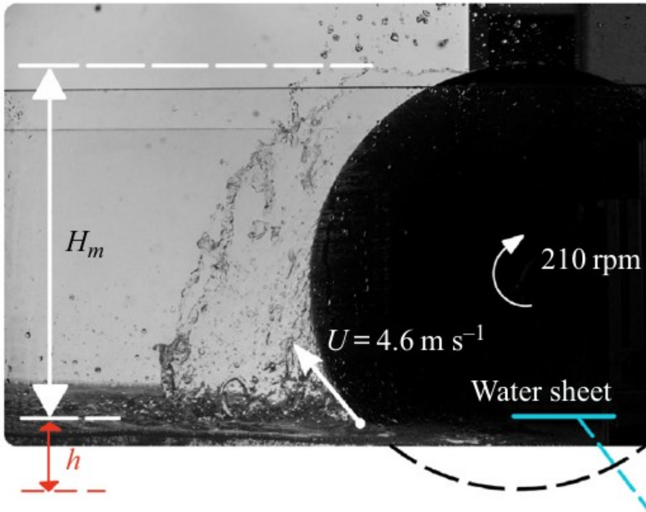
Comparison with Landau-Levich flow rate scaling

Critical Weber for inertial effects



$$\lambda \sim l_c Ca^{1/3}$$





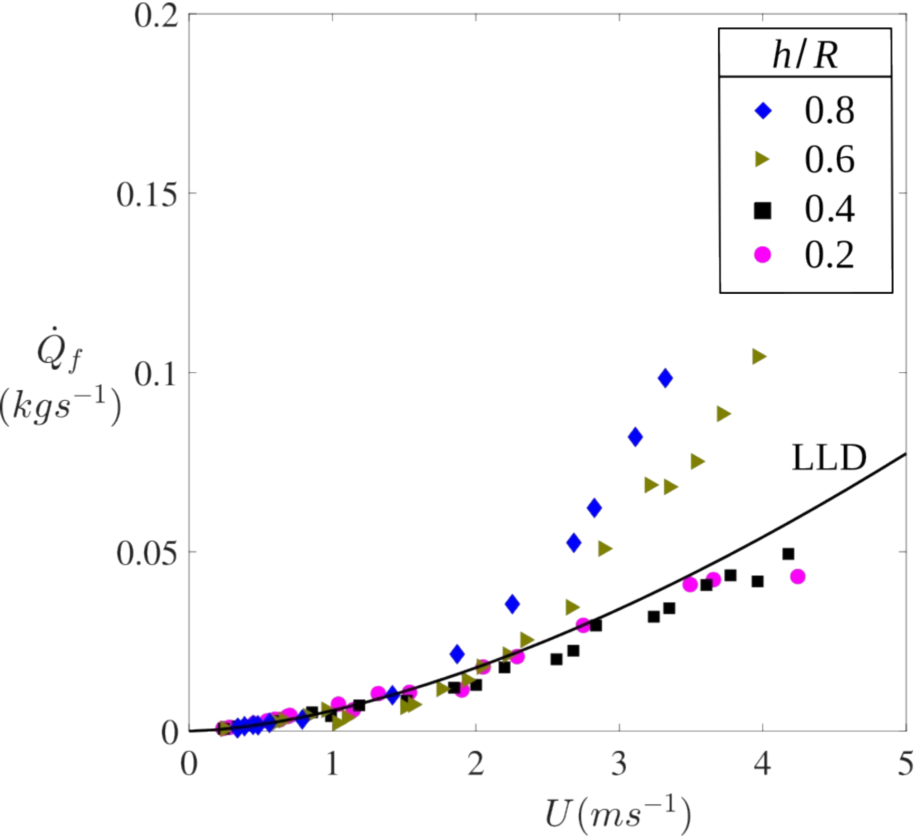
Inertia overwhelms surface tension effect, when $U^2 \gg g\lambda$

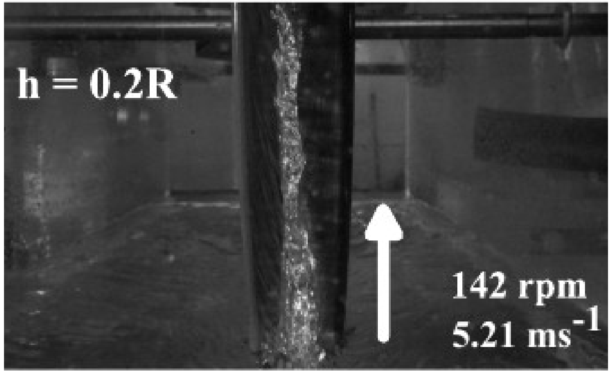
[Wilson (1982)'s 2nd order Ca - corrections where used for viscous UCON/Water mixtures]

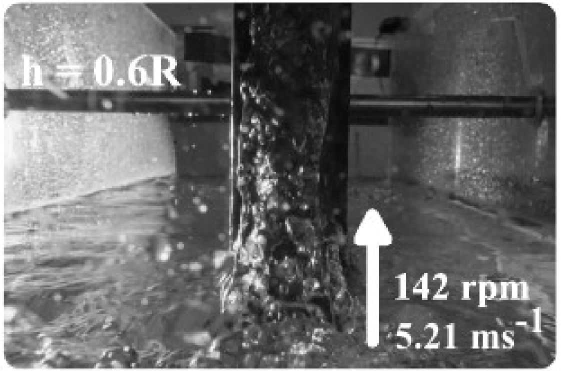
..as in fiber coating,
de Ryck & Quéré (JFM 1996)

Inertial effect in rotating drum

Revisiting the Landau-Levich curvature matching







Capillary number, $Ca = \mu U/\gamma$; Morton number, $Mo = \mu^4 g/\rho\gamma^3$; $l_c^2 = \gamma/\rho g$;

Jerome et al. (JFM 2021)

Inertial effect in rotating drum

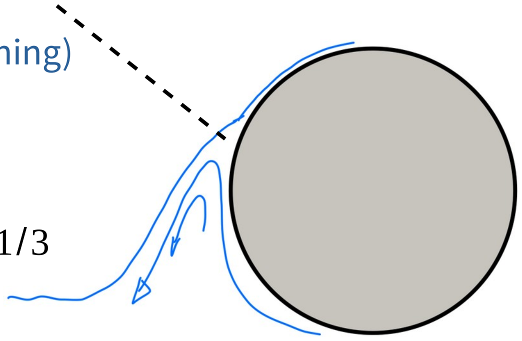
Revisiting the Landau-Levich curvature matching

$$\frac{(\gamma/l_c)}{\lambda_I} \sim \frac{\mu U}{\delta_f^2}$$

(Laplace pressure drop) (plate drag/volume)

$$\frac{\delta_f}{\lambda_I^2} \sim \frac{g}{U^2}$$

(Curvature matching)

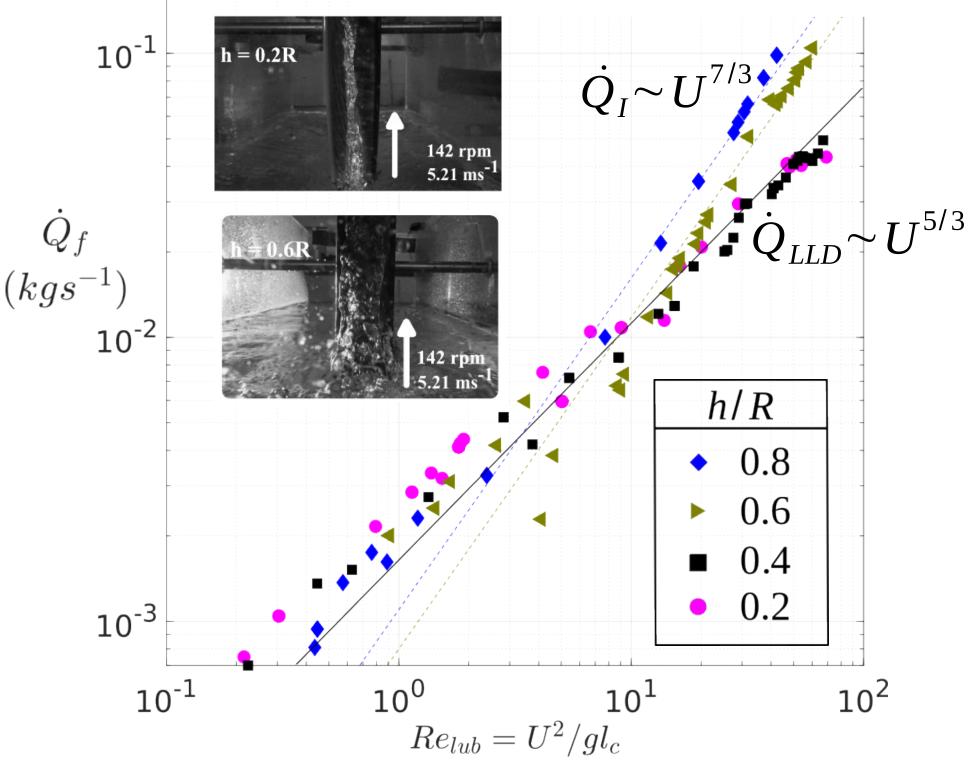


Film-thickness

$$\delta_f \sim \delta_f^{LLD}\left(\frac{U^2}{g\,l_c}\right)^{1/3}$$

..for $U^2 \gg g\lambda$

Entrained flow rate thickness



Capillary number, Ca = $\mu U/\gamma$; $\delta_{LLD} = l_c Ca^{2/3}$; $\dot{Q}_f = \rho U\,\delta_f$;

Jerome et al. (JFM 2021)

Inertial effect in rotating drum

Revisiting the Landau-Levich curvature matching

$$\frac{(\gamma/l_c)}{\lambda_I} \sim \frac{\mu U}{\delta_f^2}$$

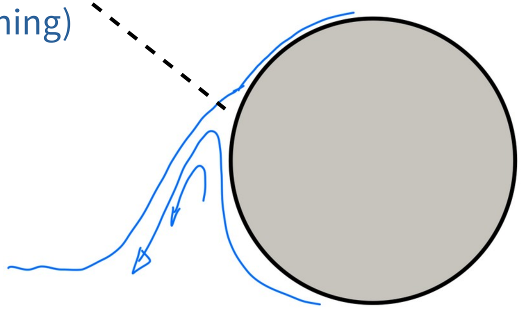
(Laplace pressure drop) (plate drag/volume)

$$\frac{\delta_f}{\lambda_I^2} \sim \frac{g}{U^2}$$

(Curvature matching)

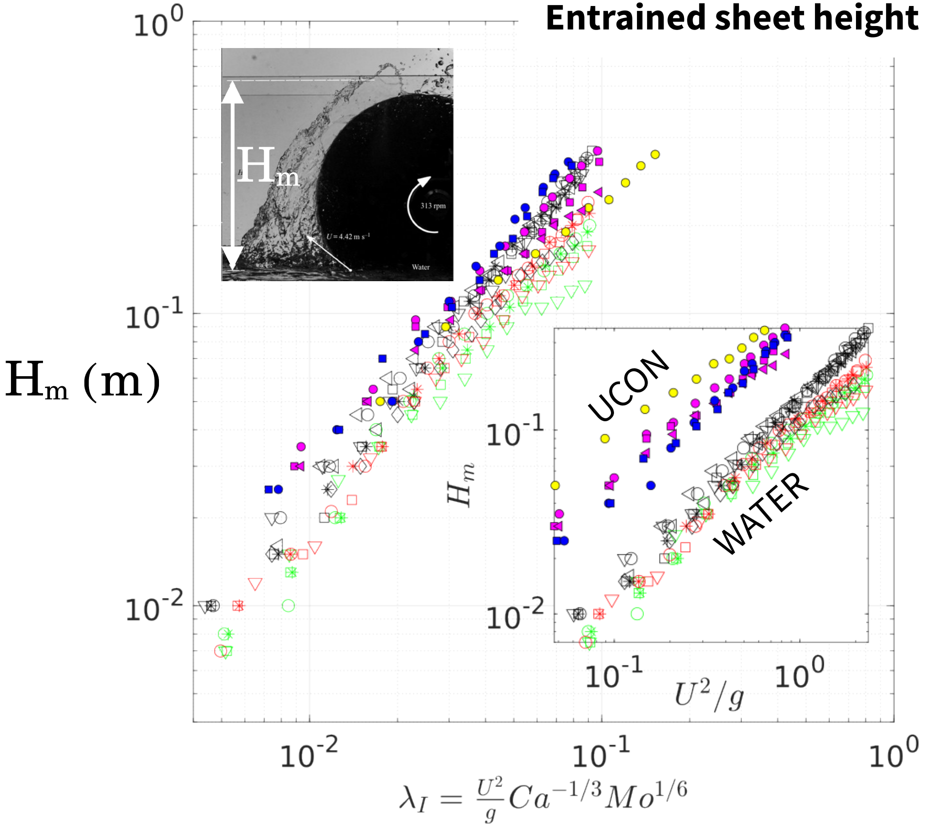
Meniscus height

$$\lambda_I \sim \frac{U^2}{g} Ca^{-1/3} Mo^{1/6}$$



..for $U^2 \gg g\lambda$

Entrained sheet height



Capillary number, $Ca = \mu U/\gamma$; Morton number, $Mo = \mu^4 g/\rho\gamma^3$; $l_c^2 = \gamma/\rho g$;

Jerome et al. (JFM 2021)

Conclusions

J John Soundar Jerome et al. (JFM 2021)

- Inertia effect in rotary flow manifests in an unsteady and non-uniform flow with strong 3D effects (liquid sheet formation)

- Surprisingly, the classical Landau-Levich (LL) model "holds on an average" despite of these inertial effects if the sheet width is small ($h/R < 0.5$) !

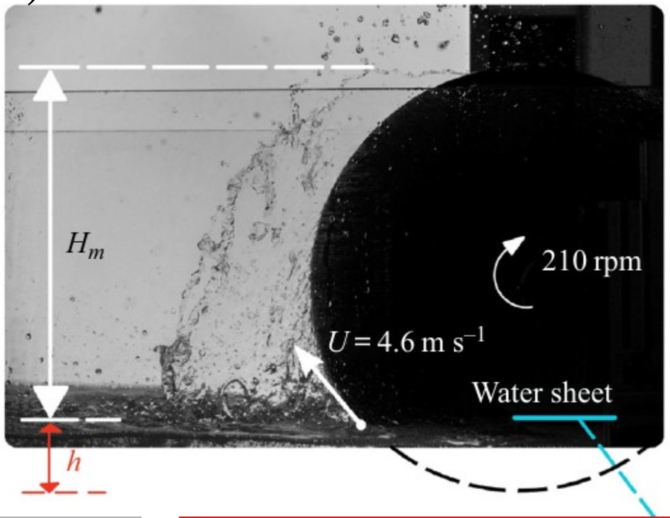
  § Strong impact of inertia when $U^2/g\lambda > 10$

  § A modified LL provides an order of magnitude :

$$\Rightarrow \dot{Q}_f \sim \dot{Q}_{LLD}\, Re_{lub}^{1/3}$$

$$\Rightarrow H_m \propto \frac{U^2}{g} Ca^{-1/3} Mo^{1/6}$$

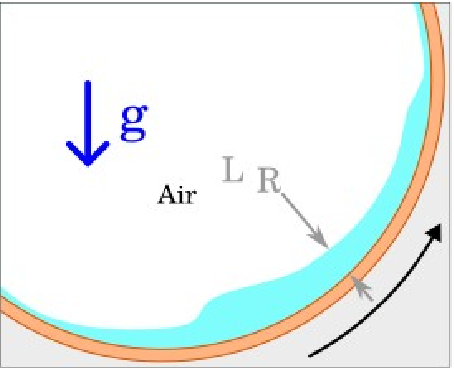
Reynolds number:
$Re_{lub} = U^2/gl_c$

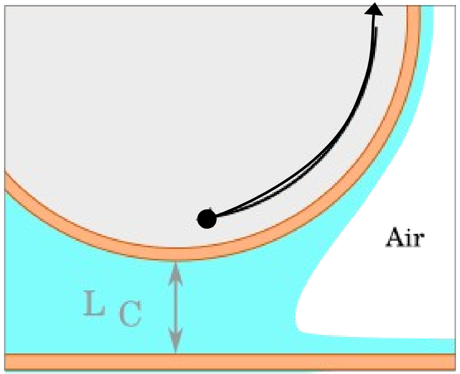


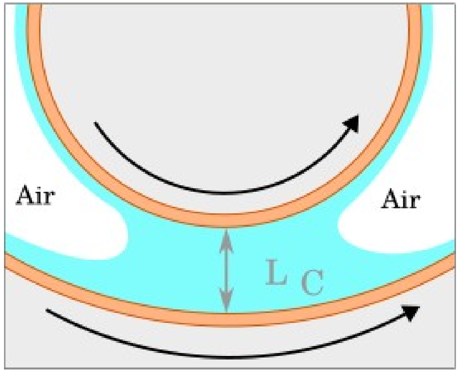
Capillary number, $Ca = \mu U/\gamma$; Morton number, $Mo = \mu^4 g/\rho\gamma^3$; $l_c^2 = \gamma/\rho g$;

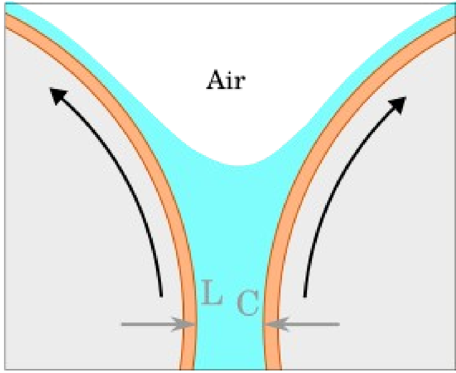
Axial-patterns in coating flows

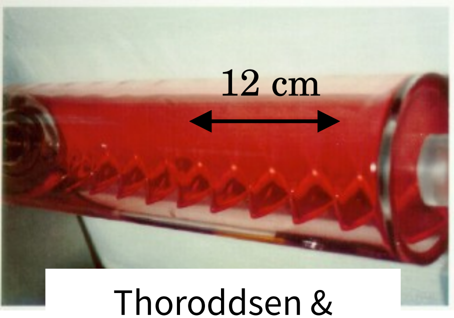
Film-splitting flows at Re << 1 and thin gaps



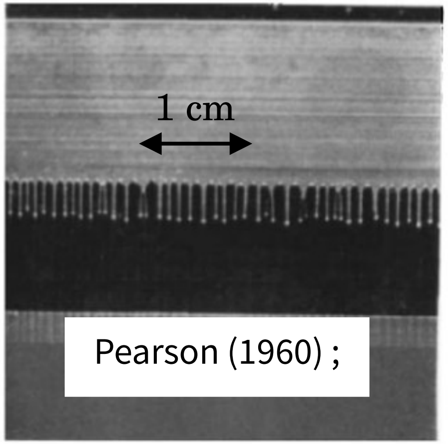




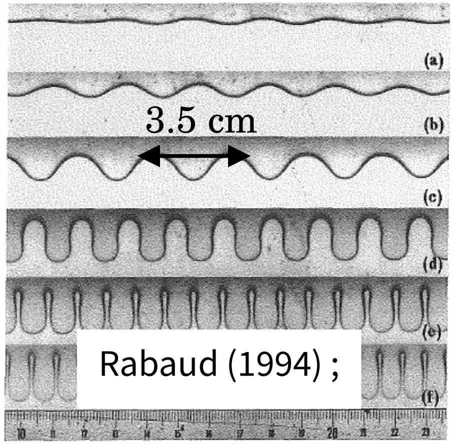




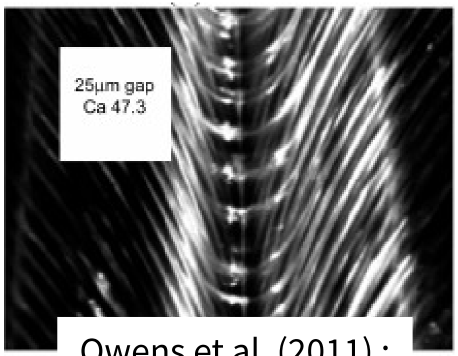
Thoroddsen & Mahadevan (1997) ;



Pearson (1960) ;



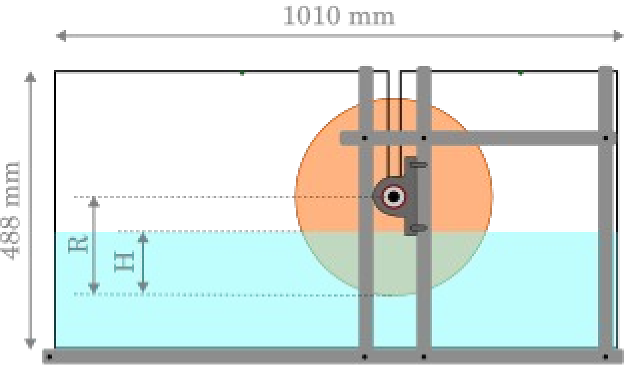
Rabaud (1994) ;

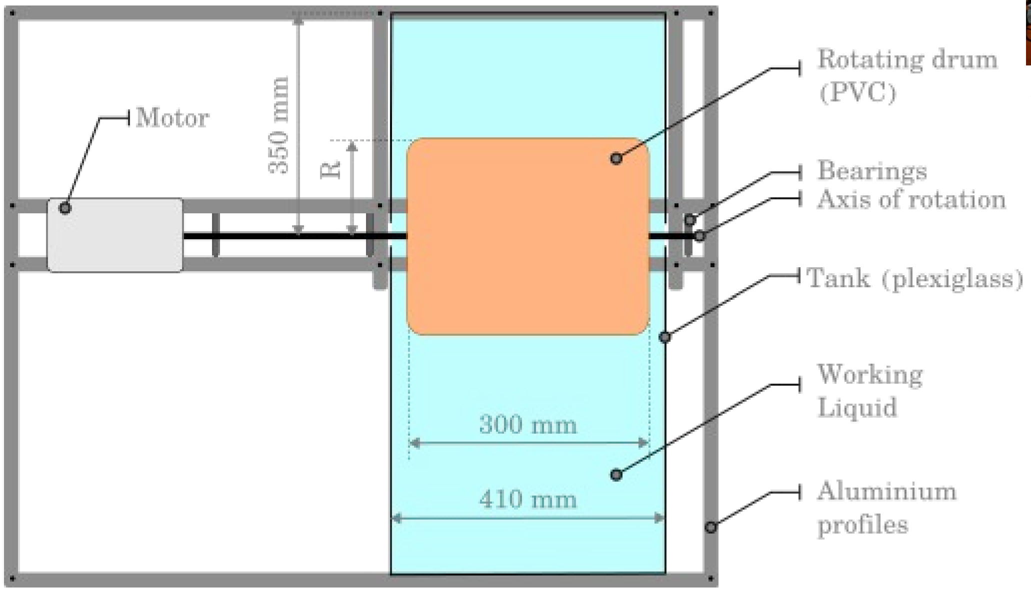


Owens et al. (2011) ;

Rotary drag-out on a drum (30 cm wide)

Landau-Levich flow at $Re_{lub} > 1$



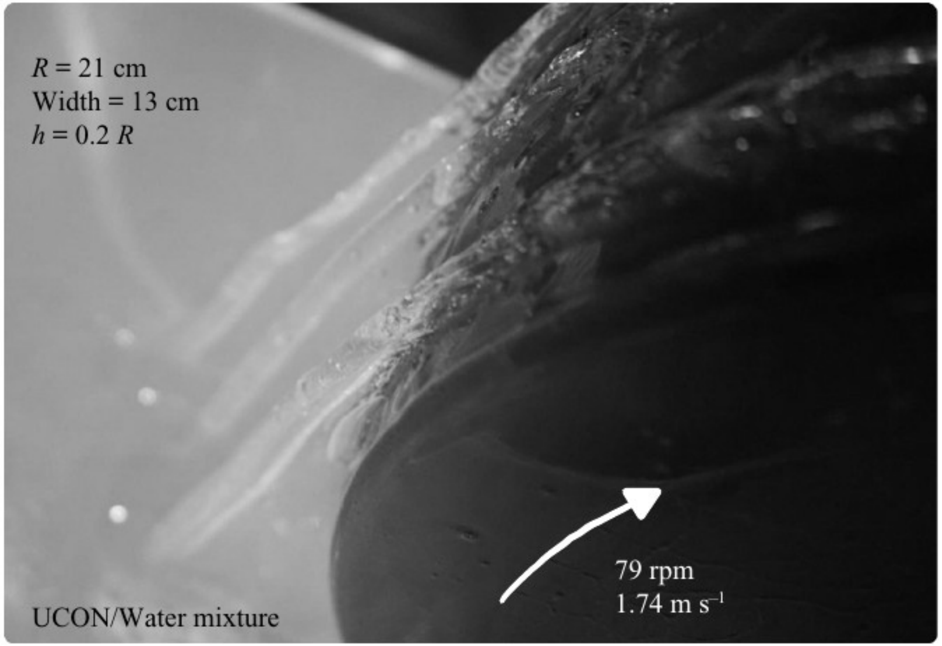


Jean-Philippe Matas
@ LMFA/UCBL

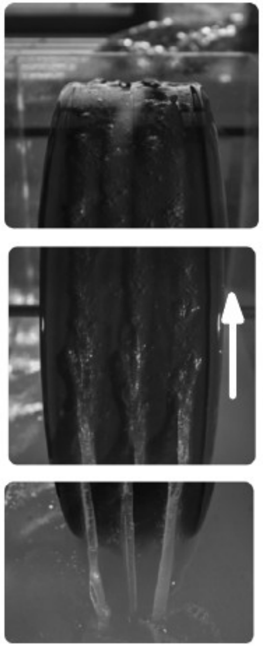
Pierre Trontin
@ LMFA/UCBL

Wider discs = Multiple ribs occur!!

UCON/Water



(front view)

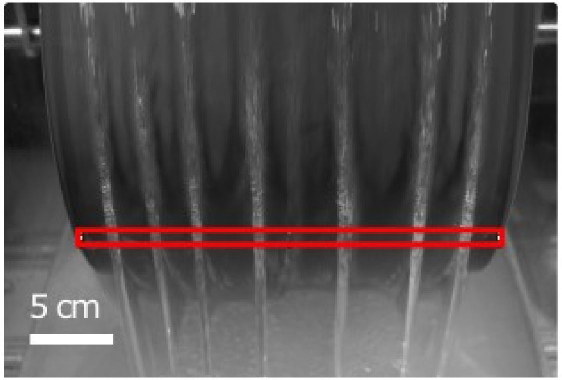


Water
(front view)

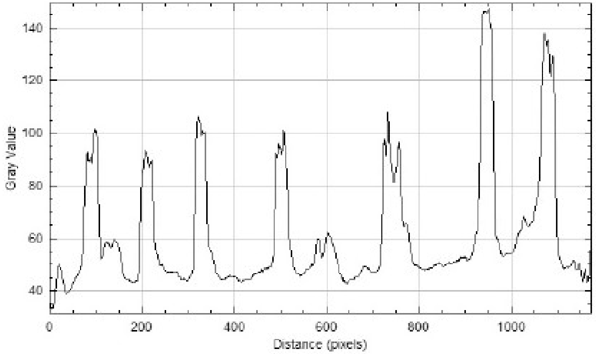
Rotary LLD flow at $Re_{lub} > 1$ is 3D (strong axial flow) !!!

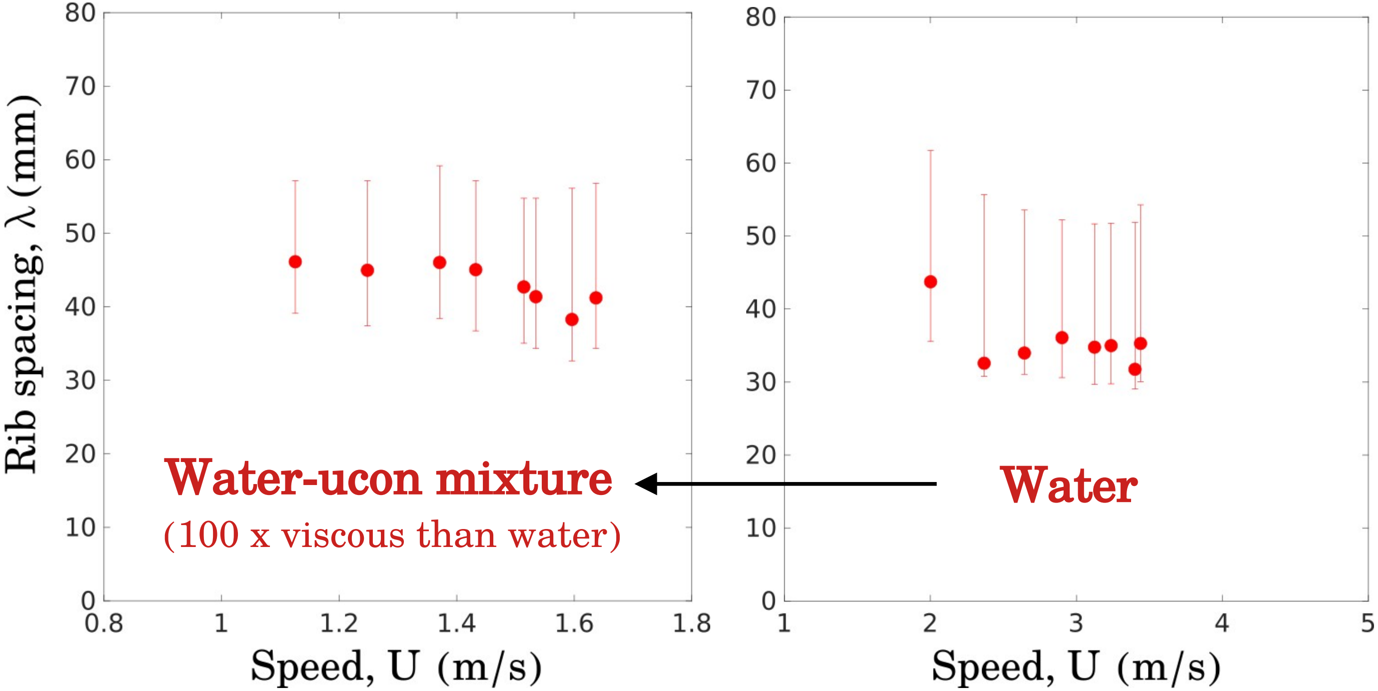
Rotating drum – rib detection

(a) UCON oil

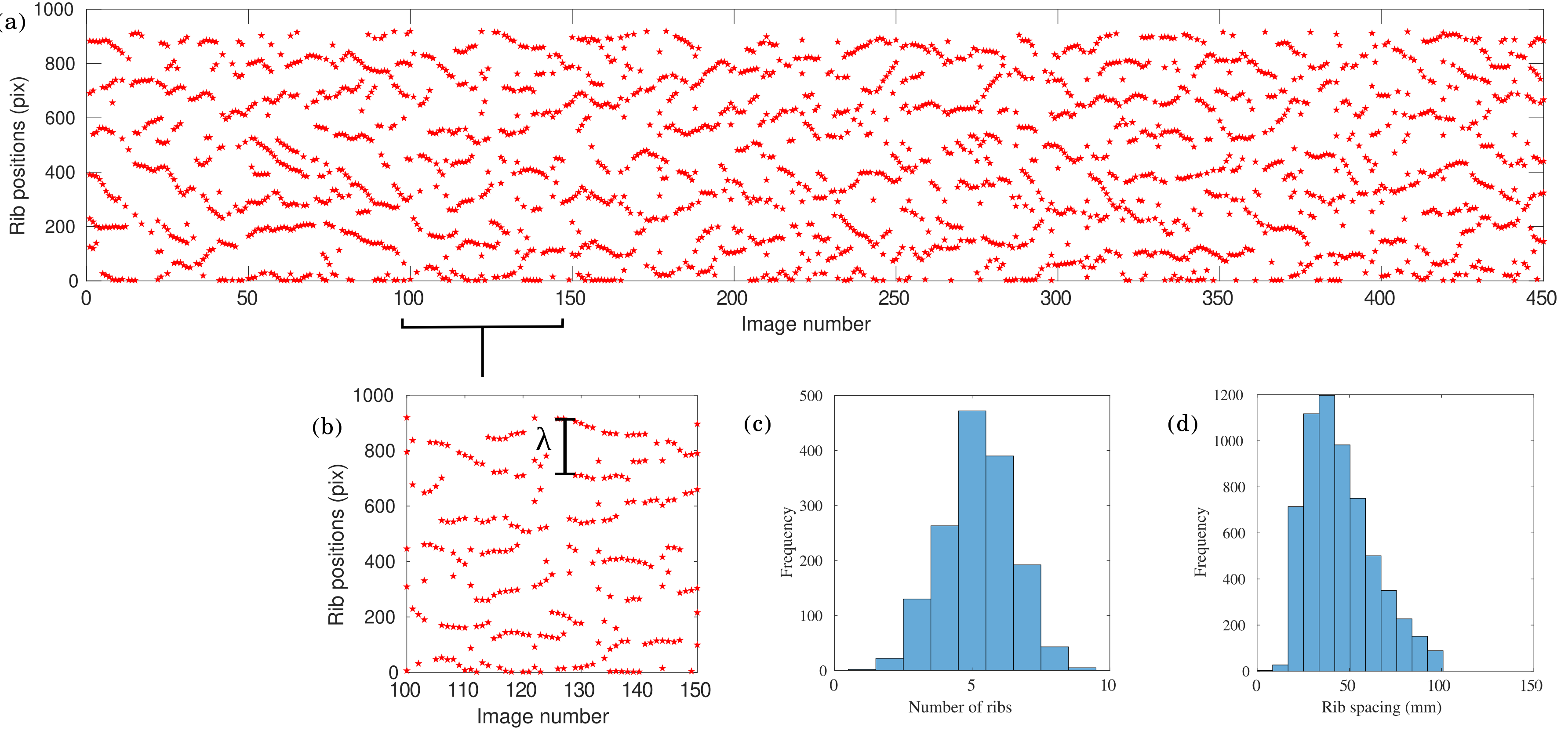


(b) Instensity profile





Rotating drum – rib spacing evolution

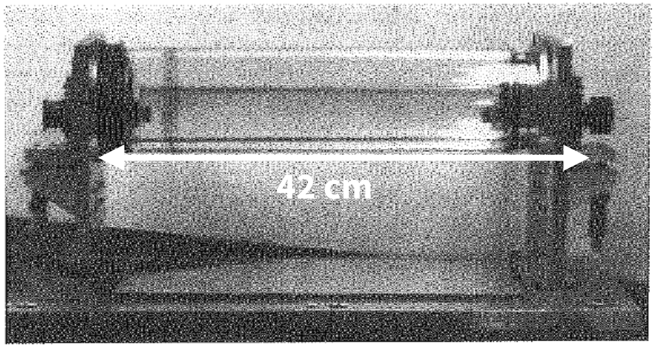


25 – 30 images per second ..

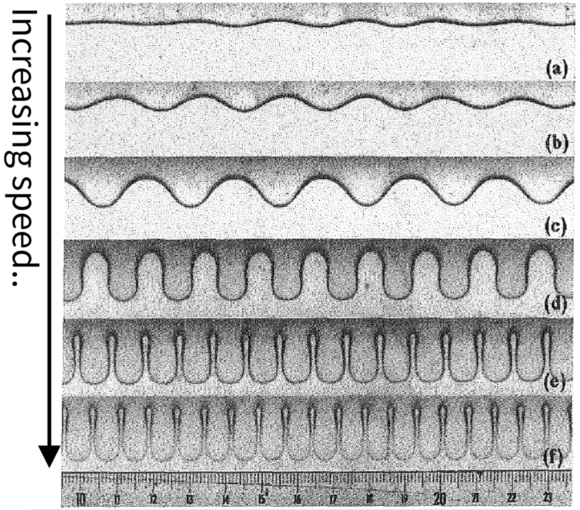
Axial pattern formation mechanism

Film-splitting flows

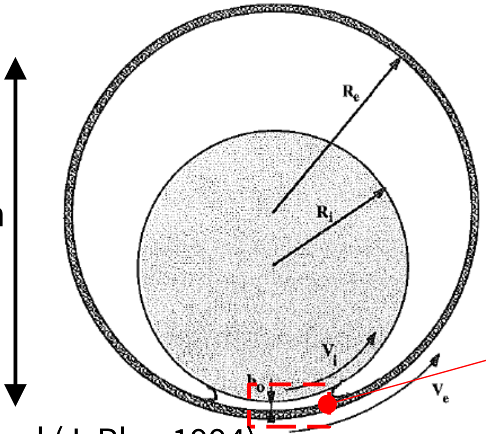
Rabaud's printer

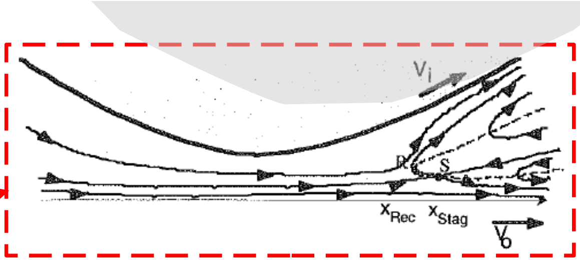


Bottom view



10 cm

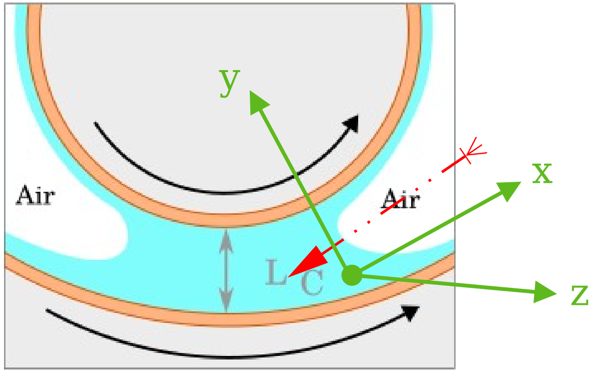


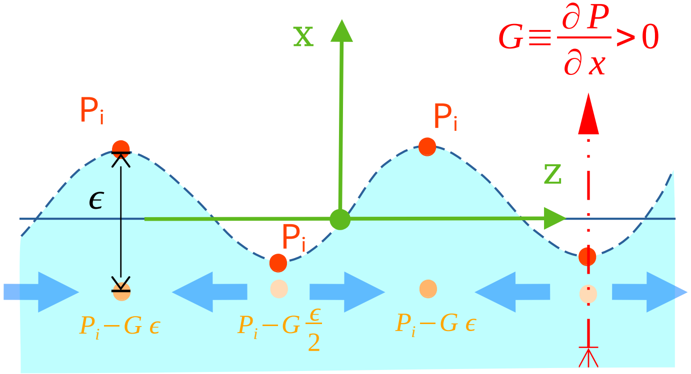


Rabaud (J. Phy. 1994)

Figure 1 : Le montage expérimental à deux cylindres (a) Photographie de l'expérience. (b) Schéma d'une section du montage.

$$\lambda_i = 2\pi\sqrt{3\gamma/(\partial p/\partial s)^{-1}}$$

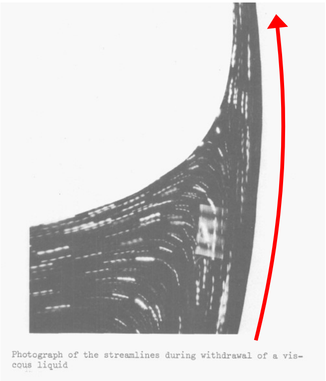




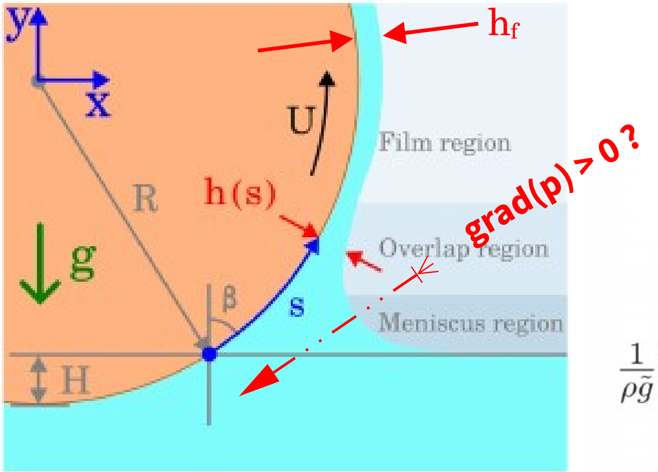
Schweizer & Kistler. (2012)

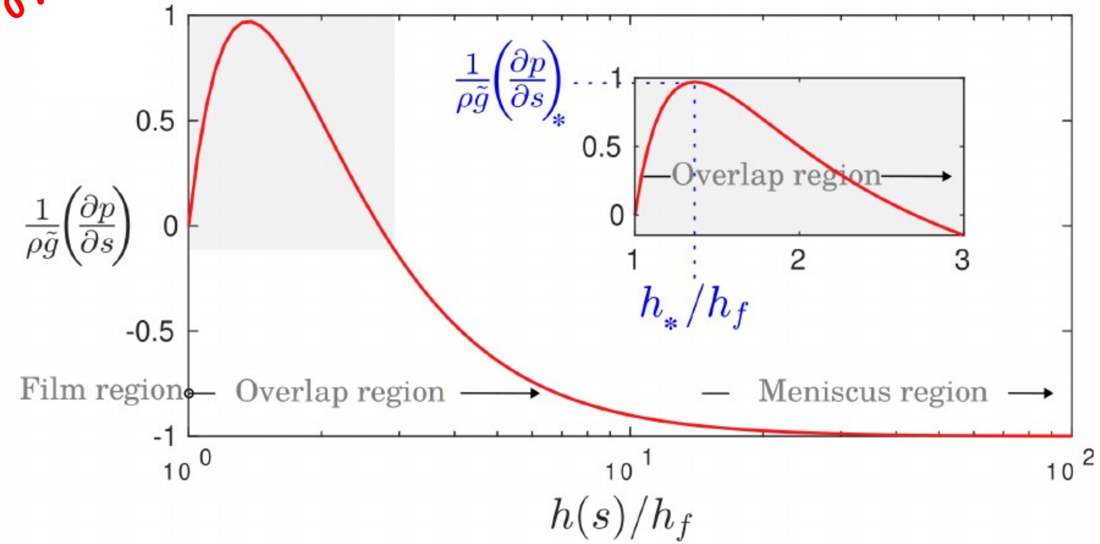
Saffman & Taylor (1956)

Adverse pressure gradient from lubrication model



Groenveld (1970) ;





Reynolds' Lubrication approximation and
Mass conservation between *Film* region and *Overlap* region...

$$\frac{\partial p}{\partial s} = -\rho\tilde{g}\left(1 - \frac{h_f}{h}\right)\left(1 + \frac{h_f}{h} - \left(a^2 - 1\right)\frac{h_f^2}{h^2}\right)$$

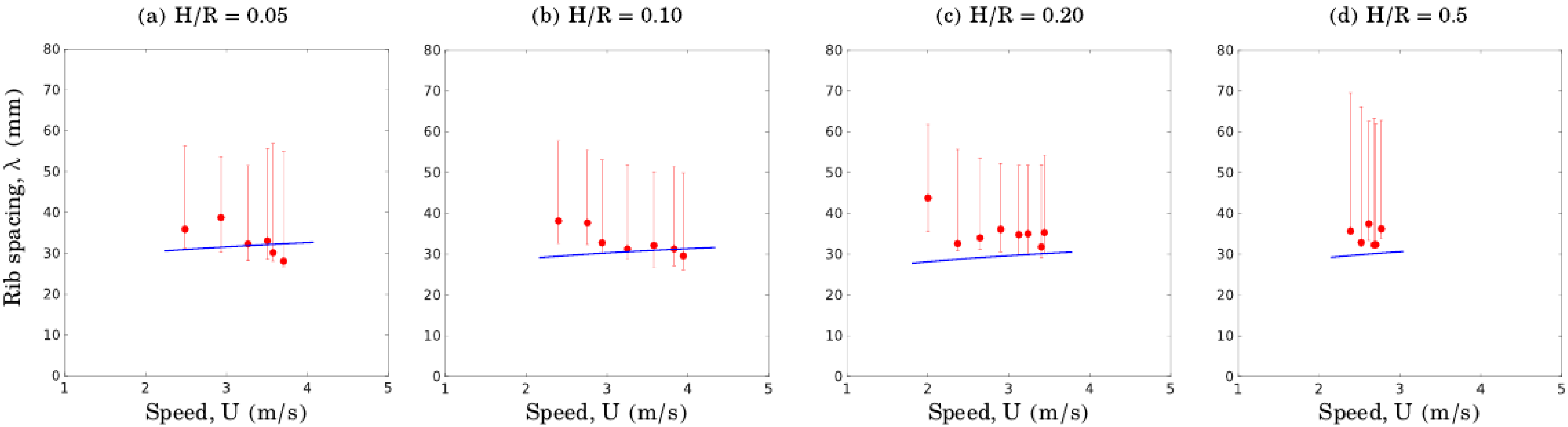
$$a = \sqrt{3}Ca^{1/2}l_c/h_f$$

Rotating drum – Comparaison with Saffman-Taylor wavelength

$$\lambda_{¿} = 2\pi l_c \sqrt{3} f(Ca)$$

WATER

$l_c^2 = \gamma/\rho g;$

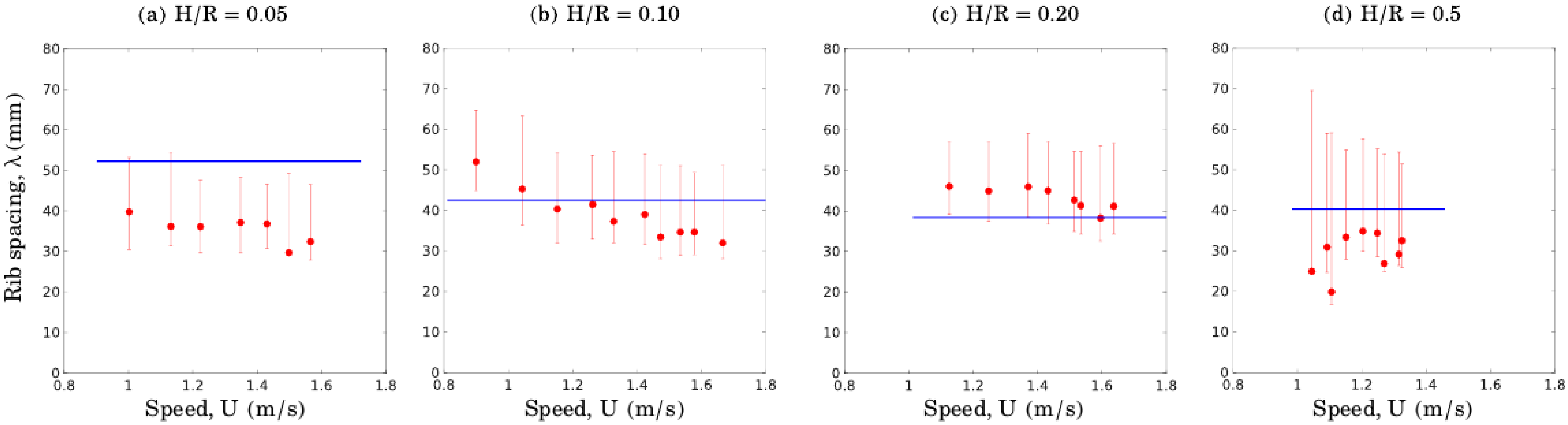


Rotating drum – Comparaison with Saffman-Taylor wavelength

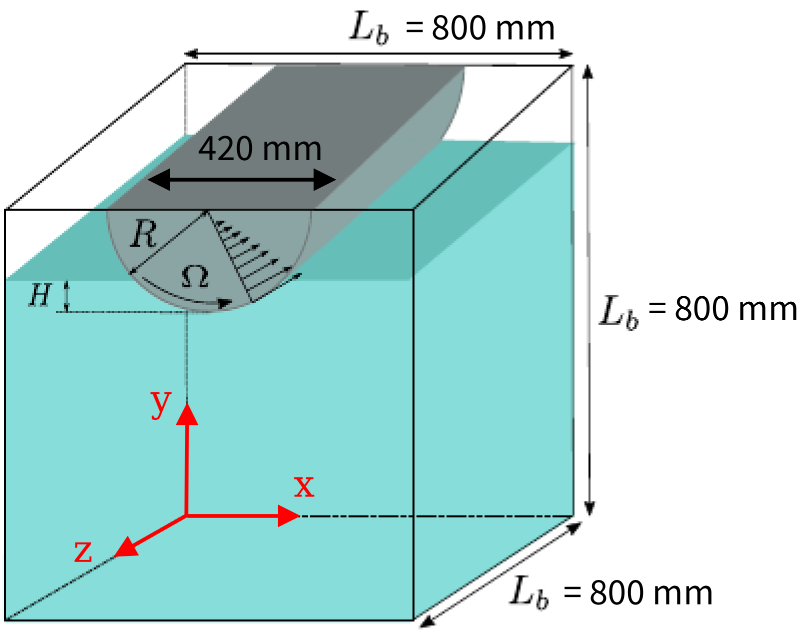
$$\lambda_i = 2\pi l_c \sqrt{3} f(Ca)$$

$l_c^2 = \gamma/\rho g;$

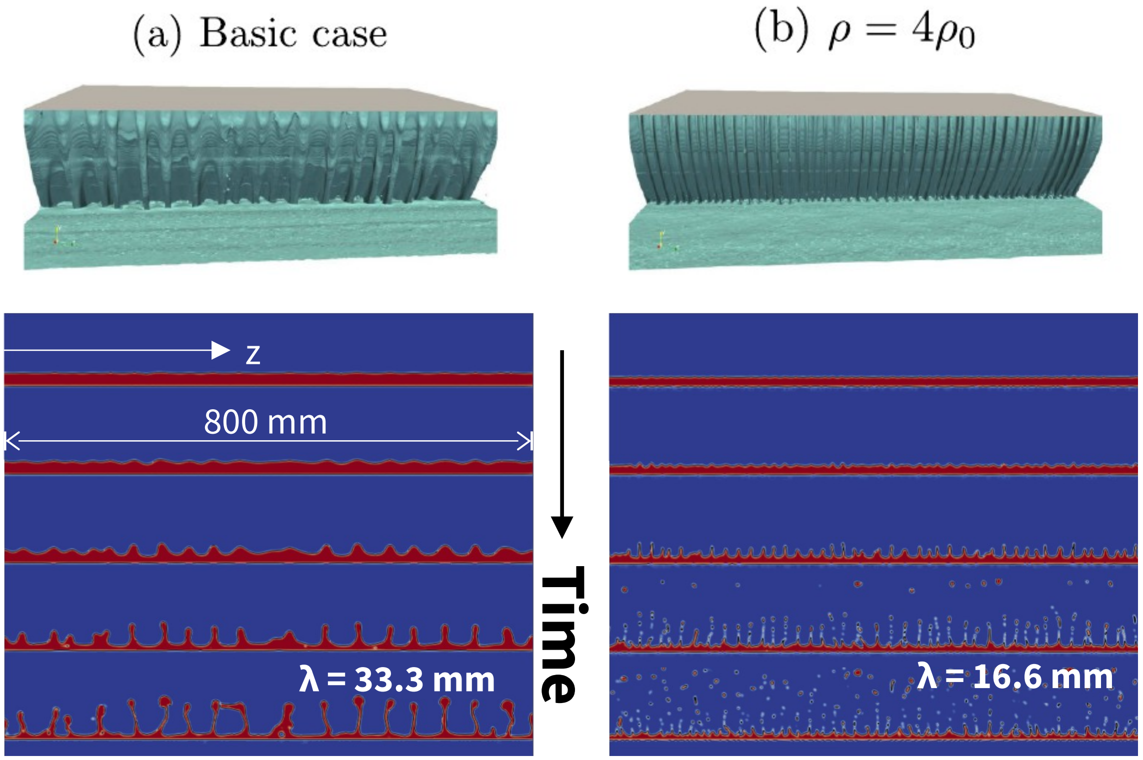
UCON/WATER
(~100 times more viscous)



Rotating drum – numerical experiments (basilisk)



$$\lambda_{¿} = 2\pi l_c \sqrt{3} f(Ca)$$



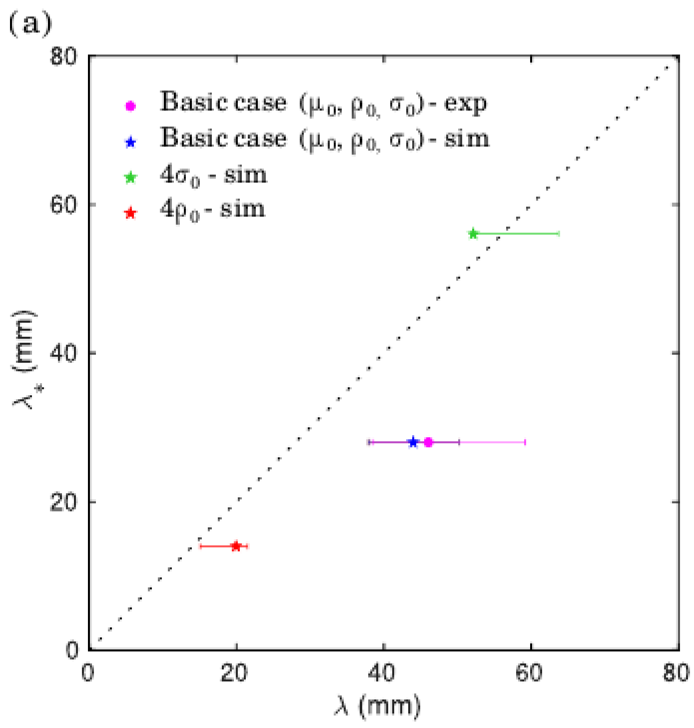
Model : λ = 28 mm

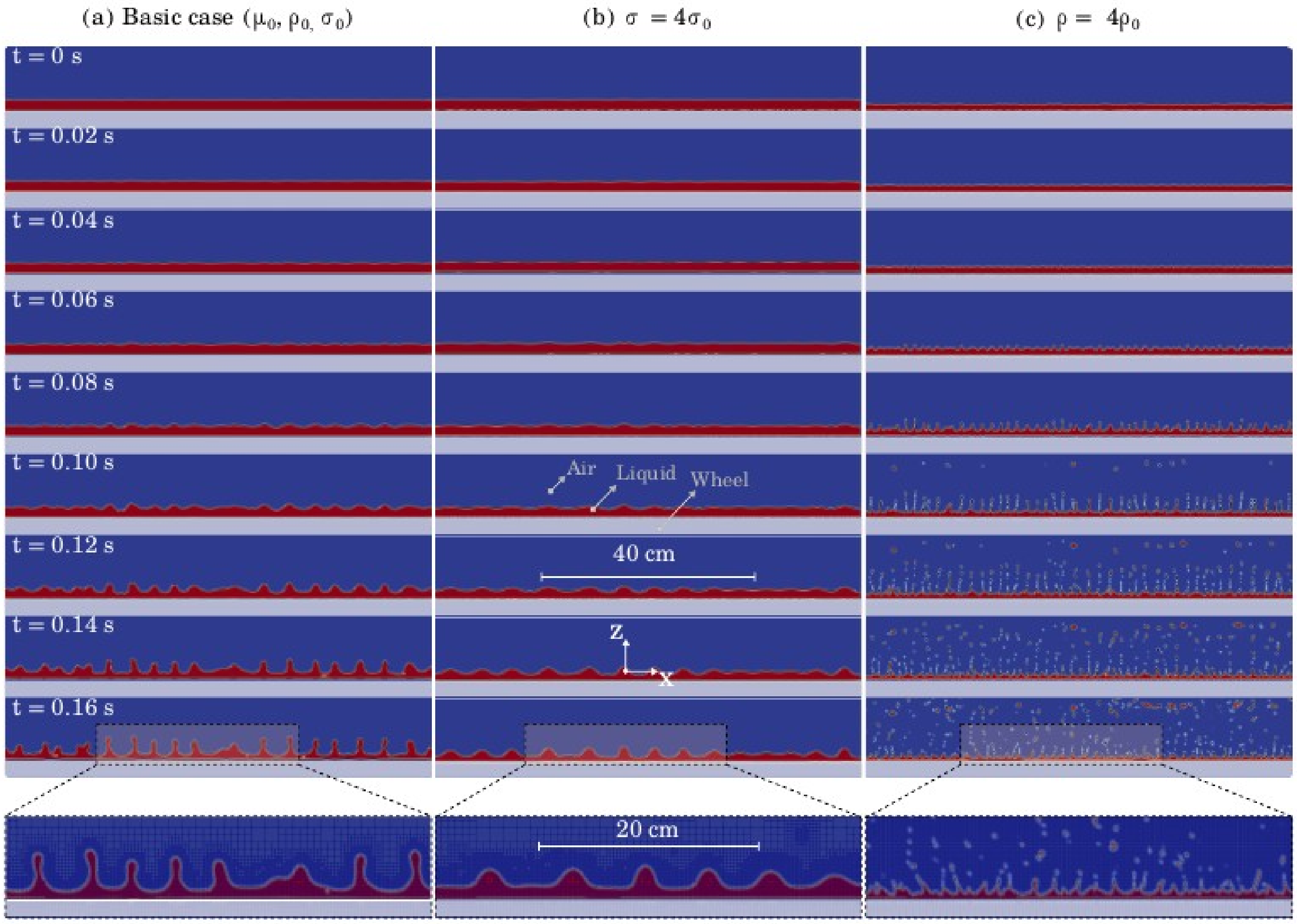
Model : λ =14.01 mm

$l_c^2 = \gamma/\rho g$;

…by Pierre Trontin (LMFA, UCBL)

Rotating drum – numerical experiments (basilisk)





…by Pierre Trontin (LMFA, UCBL)

Conclusions: Ribbing patterns on a rotating drum

- Inertia effect in rotary flow leads to axial flow patterns resulting in multiple liquid sheets with a rim

- This can be related to the diverging flow near the meniscus in the classical Landau-Levich (LL) flow – Adverse pressure gradient

  § Estimations based on free-surface flow with lubrication approximation

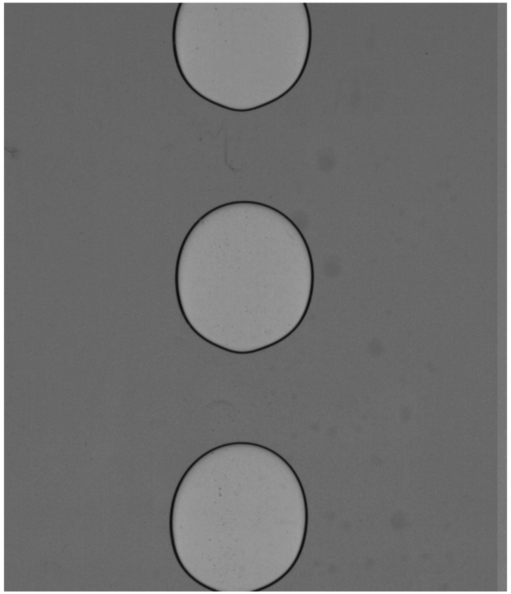
$$\Rightarrow \left(\frac{\partial p}{\partial s}\right)_* = \rho\tilde{g}\mathcal{F}\left(\frac{\mu U}{\gamma}\right) > 0$$

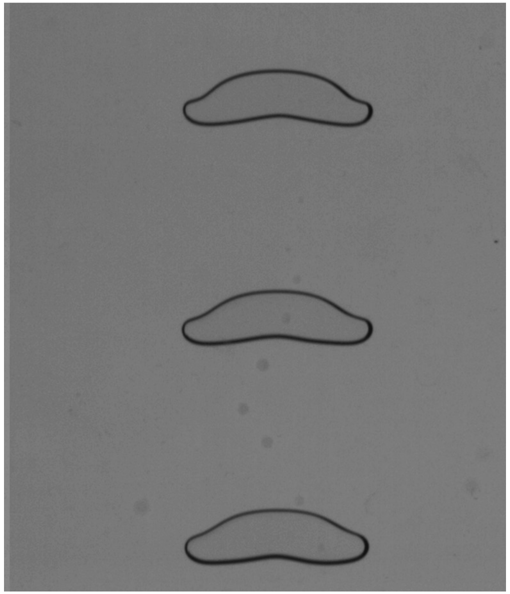
  § Wavelength order of magnitude from Saffman-Taylor instability

$$\Rightarrow \lambda_{¿} = 2\pi\sqrt{3\gamma(\partial p/\partial s)^{-1}}$$

- Numerical simulations using BASILISK for a longer cylinder : good match with experiments, confirms scaling with density and surface tension

Bubble rise in Hele-Shaw





Christopher Madec
(Thèse 2021)

Benjamin Monnet
(Thèse 2024)

Sylvain Joubaud
@ LPENS/ENS

Valérie Vidal
@ LPENS/ENS

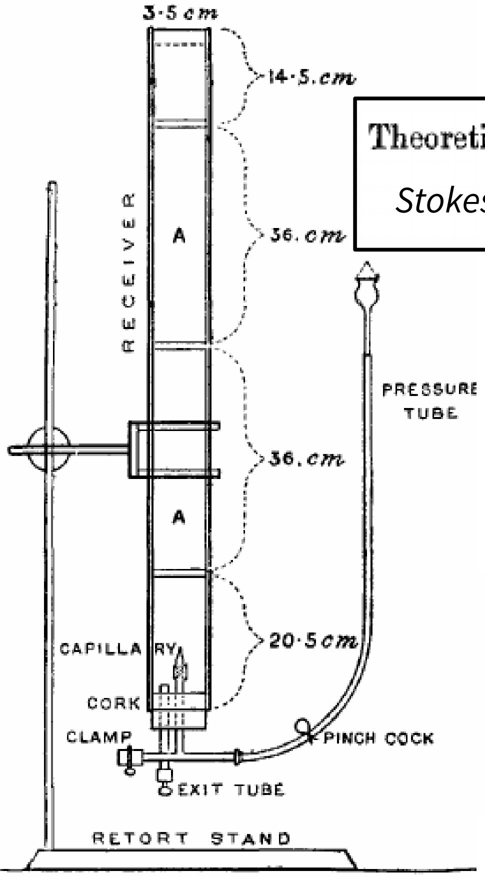
Speed of a freely-rising single bubble

...in an infintely large tank (historical backgound)

Reynolds number, $Re = U_b l_b/\nu$

Re << 1

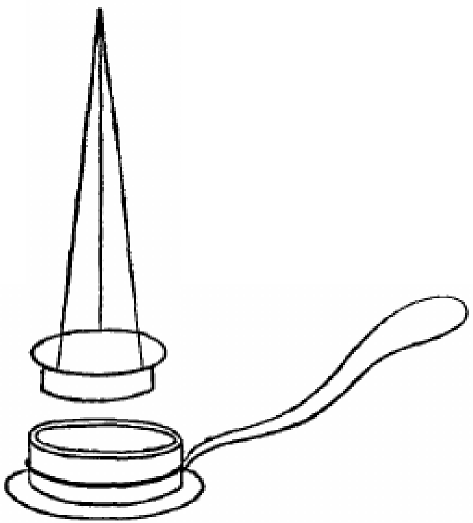
Fig. 1.



Theoretical Formula of Stokes,

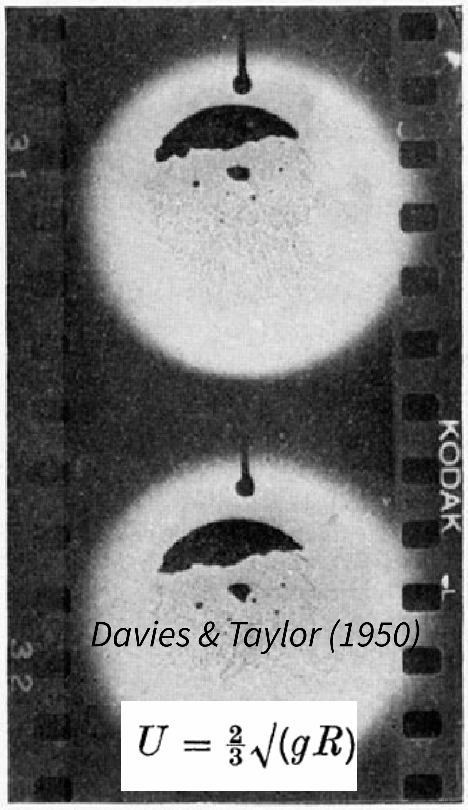
Stokes (1850)

$$V = \frac{2}{9} \frac{\sigma - \rho \, . \, g}{\rho} \frac{a^2}{\nu}.$$



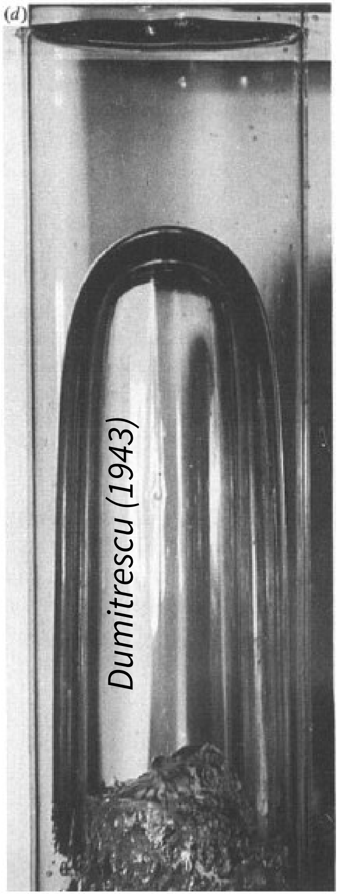
Allen (1900)

"Re >> 1"

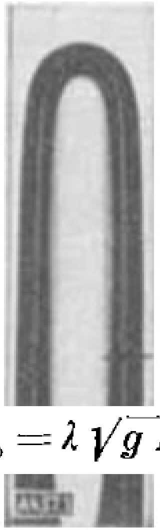


Davies & Taylor (1950)

$$U = \tfrac{2}{3}\sqrt{(gR)}$$



Dumitrescu (1943)



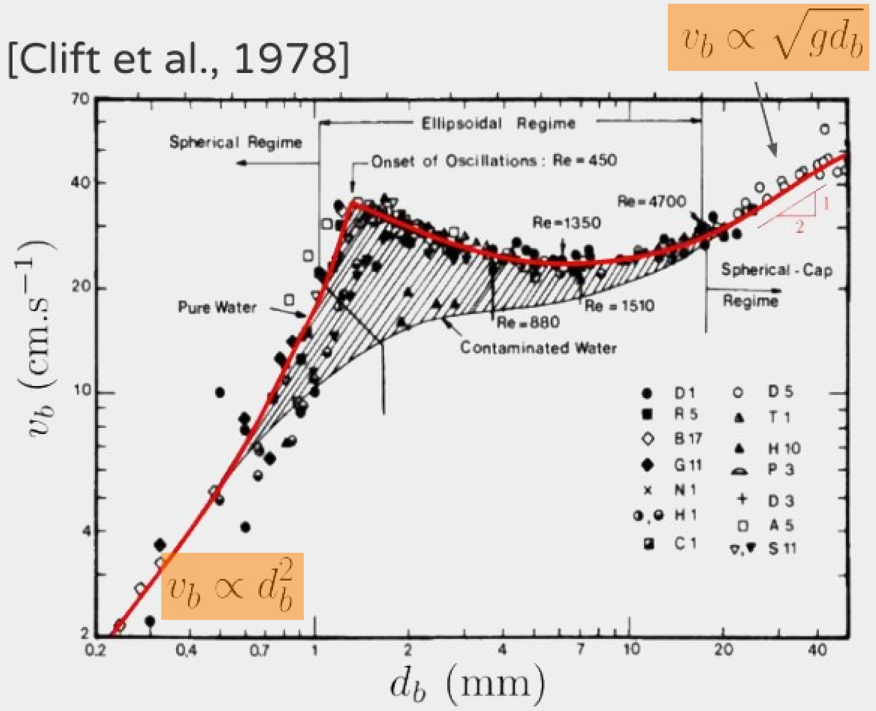
$$w_\infty = \lambda \sqrt{g R}$$

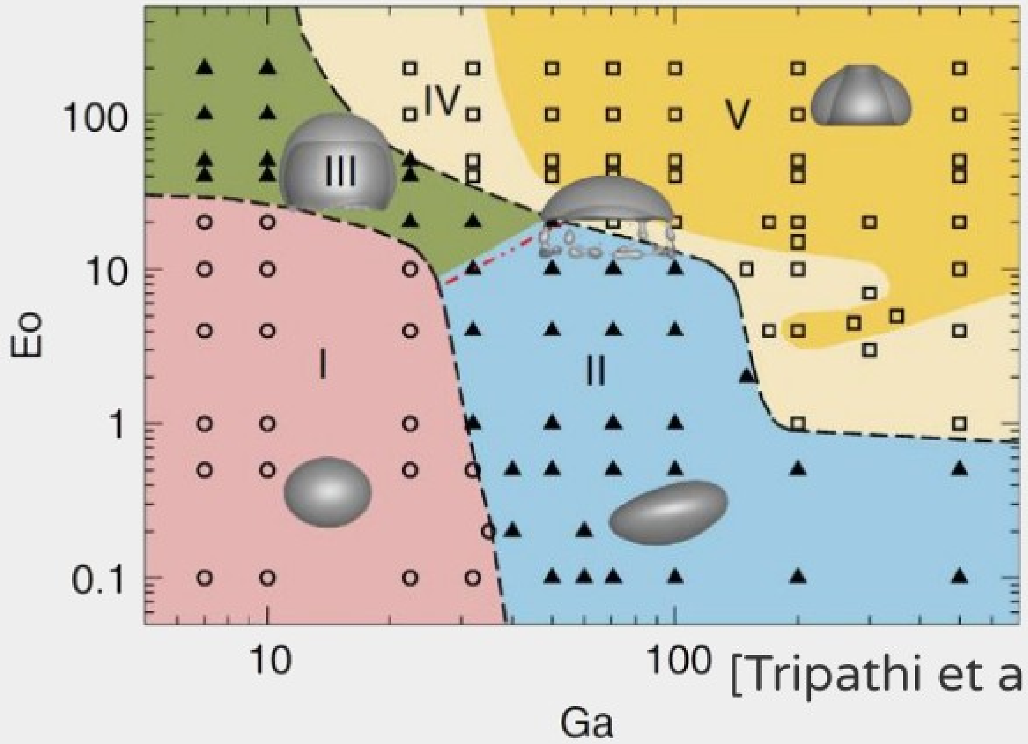
Bild 1. In einem mit Wasser gefüllten Rohr aufsteigende Luftblase.

Speed of a freely-rising single bubble

The Objective ..

[Clift et al., 1978]





[Tripathi et al., 2015

Speed of a freely-rising single bubble (small Re ?)

Previous works – Inclinced Hele-Shaw cell

Eck & Siekmann (Ing. Arv 1978)

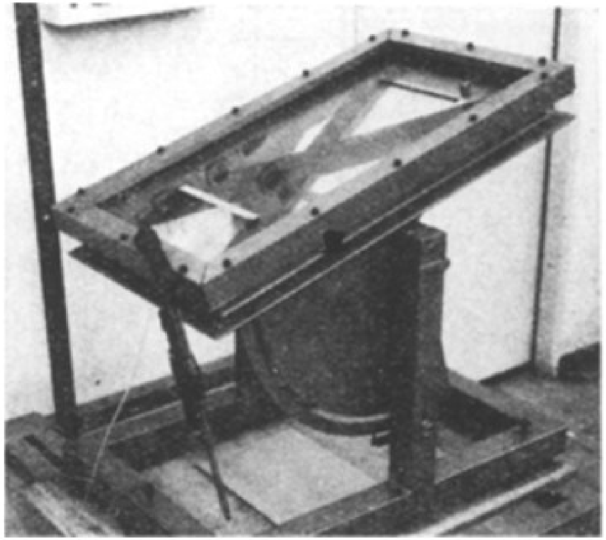
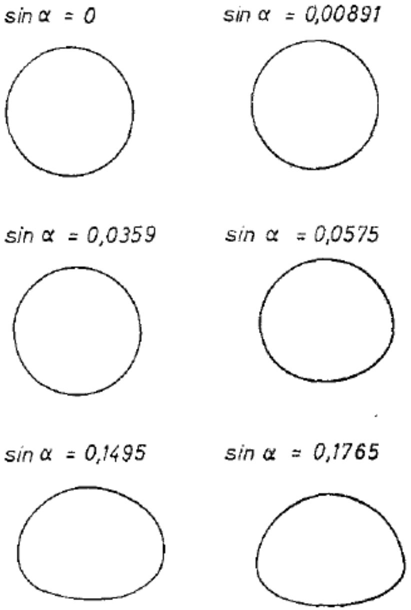
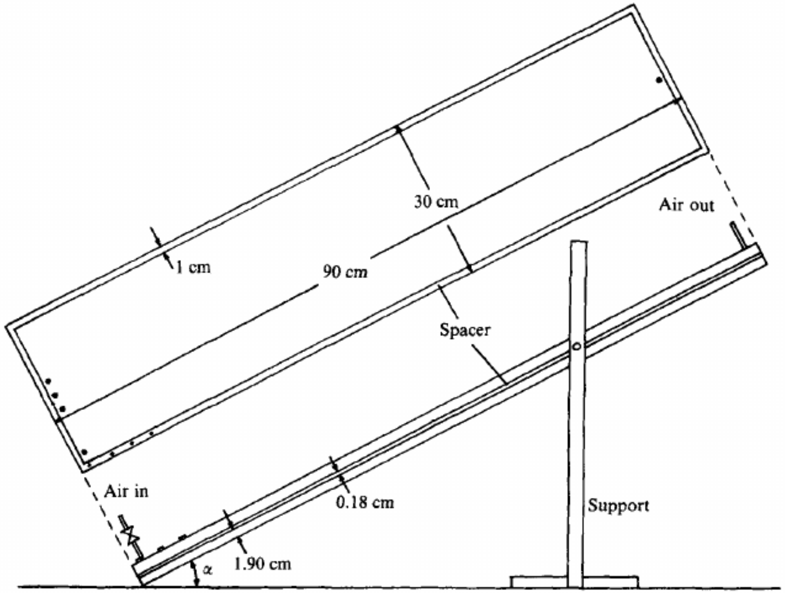
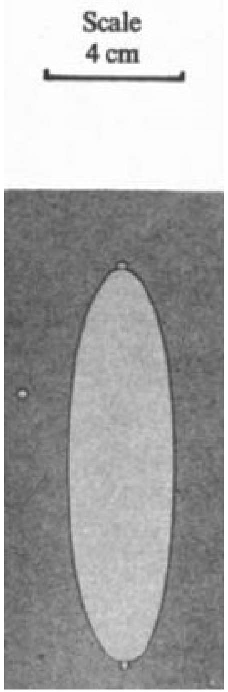


Fig. 7. Reduced gravity simulator



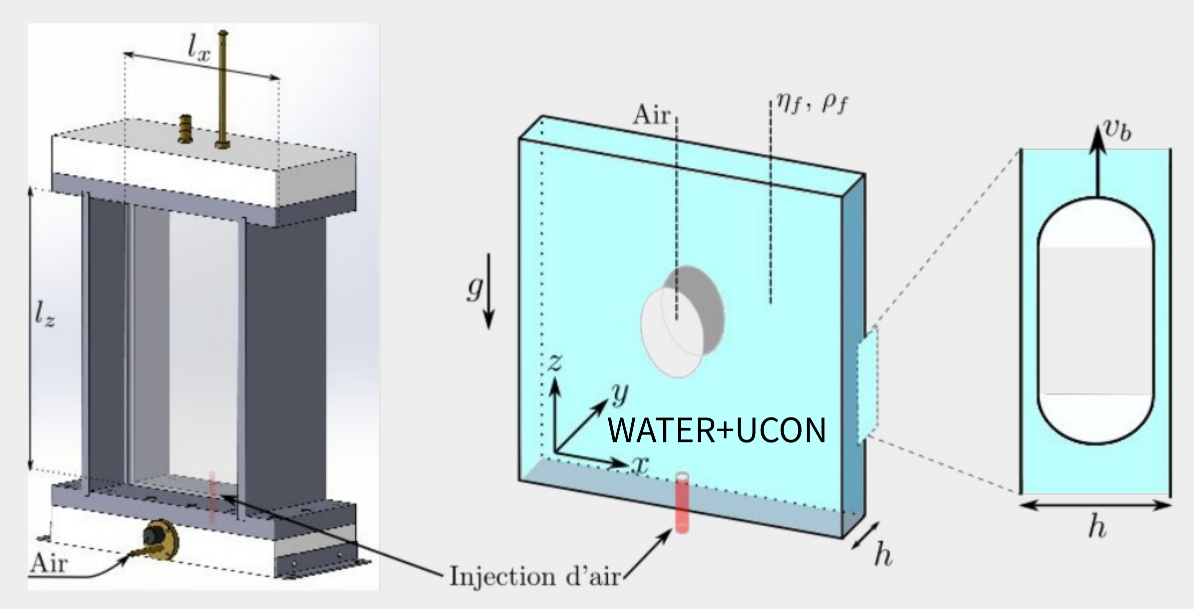
Maxworthy (JFM 1986)

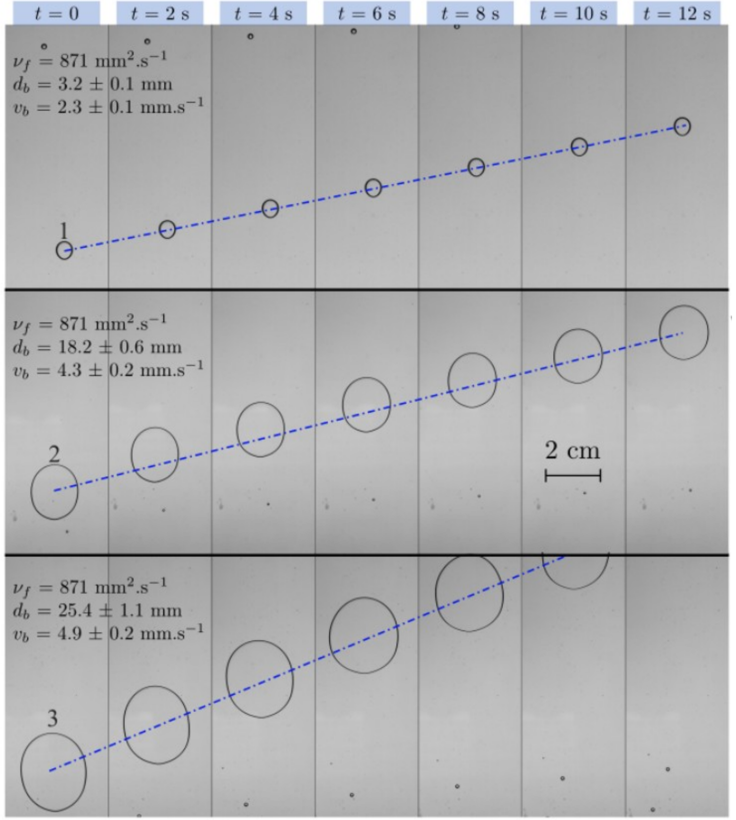




Hele-Shaw cell & Lubrication approximation

Taylor-Saffman bubble





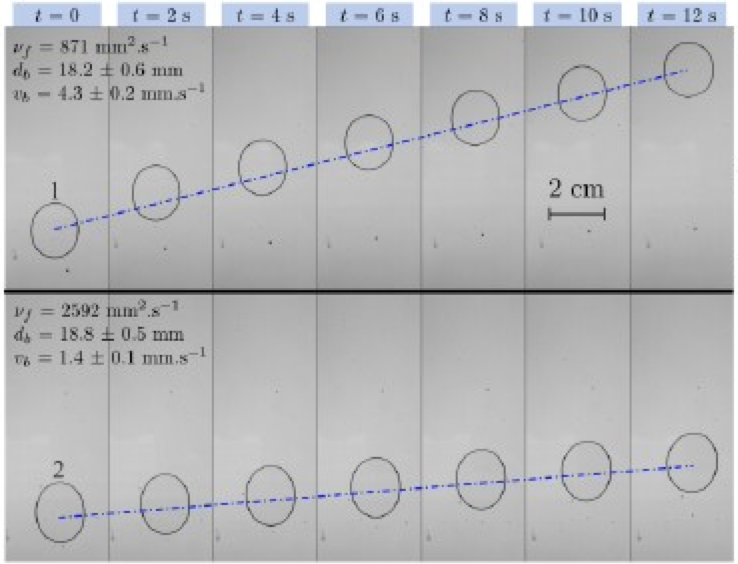
Gap: h = 2.3 ± 0.05 mm
Size: $L_x = L_z \approx 20$ cm

Dynamic viscosity:
~ 100 to 2500 times water

Madec (Thèse, ENS de Lyon 2021)

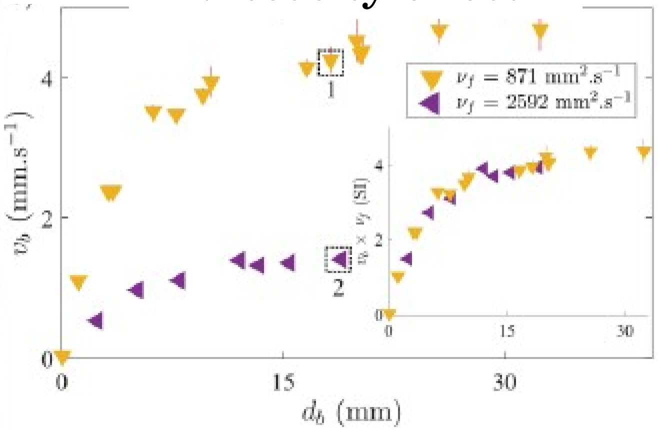
Hele-Shaw cell & Lubrication approximation

Taylor-Saffman flow

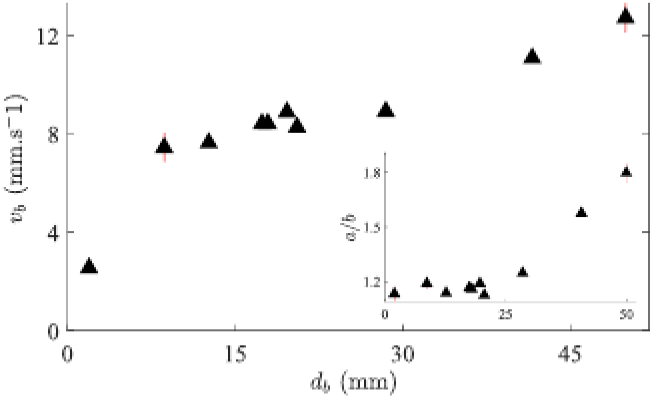


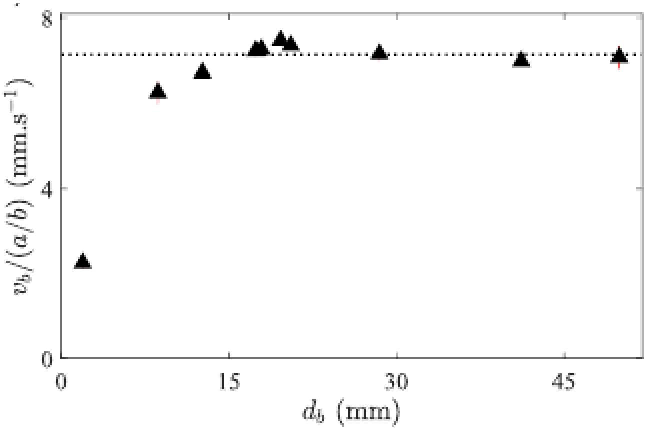
$$v_M = \frac{\Delta\rho g h^2}{12\eta}\frac{a}{b}$$

Viscosity effect



a) Diameter & Aspect ratio





Madec (Thèse, ENS de Lyon 2021)

Bubble rise in Hele-Shaw

...lubrication approximation in thin gaps (historical backgound)

IV. *On the Theory of Lubrication and its Application to Mr.* BEAUCHAMP TOWER'S *Experiments, including an Experimental Determination of the Viscosity of Olive Oil.*

By Professor OSBORNE REYNOLDS, *LL.D., F.R.S.*

Received December 29, 1885,—Read February 11, 1886.

$$u=\frac{1}{2\mu}\frac{dp}{dx}(y-h)y+U_0\frac{h-y}{h}+U_1\frac{y}{h}$$

$$w=\frac{1}{2\mu}\frac{dp}{dz}(y-h)y$$

Reynolds (1886)

REPORT

OF THE

SIXTY-EIGHTH MEETING

OF THE

BRITISH ASSOCIATION

FOR THE

ADVANCEMENT OF SCIENCE

HELD AT

BRISTOL IN SEPTEMBER 1898.

(II.) *Mathematical Proof of the Identity of the Stream Lines obtained by Means of a Viscous Film with those of a Perfect Fluid moving in Two Dimensions. By* Sir G. G. STOKES, *F.R.S.*

Stokes (1898)

The beautiful photographs obtained by Professor Hele-Shaw of the stream lines in a liquid flowing between two close parallel walls are of very great interest, because they afford a complete graphical solution, experimentally obtained, of a problem which, from its complexity, baffles the mathematician, except in a few simple cases.

Depth-averaged velocity field is irrotational !!

$$\frac{dp}{dx}=-\frac{3\mu}{c^2}u^1,\quad \frac{dp}{dy}=-\frac{3\mu}{c^2}v^1,\quad \frac{du^1}{dx}+\frac{dv^1}{dy}=0.$$

Bubble rise in Hele-Shaw

...lubrication approximation in thin gaps (historical backgound)

Taylor & Saffman (1958)

A NOTE ON THE MOTION OF BUBBLES IN A HELE-SHAW CELL AND POROUS MEDIUM

By SIR GEOFFREY TAYLOR and P. G. SAFFMAN
(*Cavendish Laboratory, Cambridge*)

[Received 12 August 1958. Revise received 13 November 1958]

$$\tilde{u} = -\frac{h^2}{12\mu}\frac{\partial p}{\partial x} = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad \tilde{v} = -\frac{h^2}{12\mu}\frac{\partial p}{\partial y} = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

- Depth-averaged velocity field is irrotational,
- Constant pressure at the bubble interface,
- Absence, or weak, Surface Tension,

+ a lot of calculus, analysis complex-plane...

Rise speed of a free isolated bubble
(free = > no channel flow)

$$v_M = \frac{\Delta \rho g h^2}{12\eta}\frac{a}{b}$$

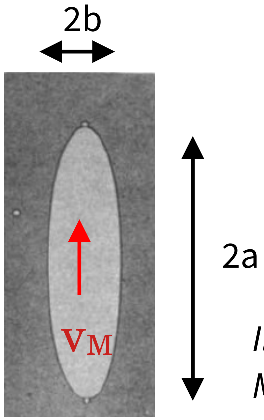
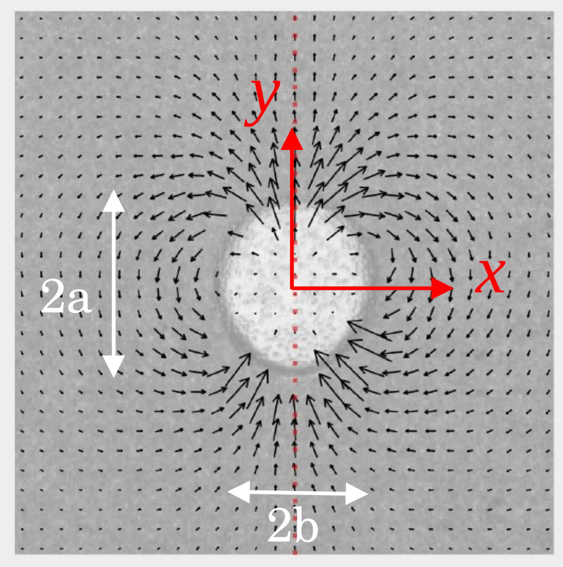


Illustration: Maxworthy (JFM 1986)

Hele-Shaw cell & Lubrication approximation

Taylor-Saffman speed as rederived by Maxworthy (JFM 1986)

Remember that the depth-averaged velocity field is irrotational ....



$$\boldsymbol{v_f} = \frac{3v_b}{2}\left(\frac{d_b/2}{r}\right)^2\left[1-\left(\frac{y}{h/2}\right)^2\right](\sin(\theta)\boldsymbol{e_r}-\cos(\theta)\boldsymbol{e_\theta})\ \forall r > d_b/2.$$ [..circular !]

P_η : Dissipation rate

$$\mathcal{P}_\eta = \int_{d_b/2}^{+\infty}\int_{-\pi}^{\pi}\int_{-h/2}^{h/2}\eta_f\left|\frac{\partial \boldsymbol{v_f}(r,\theta,y)}{\partial y}\right|^2 r\mathrm{d}r\mathrm{d}\theta\mathrm{d}y = \frac{3\pi\eta_f v_b^2 d_b^2}{h}$$

P_B : Injected power to generate the flow ..

$$\mathcal{P}_B = \Delta\rho g\left(\frac{\pi d_b^2 h}{4}\right)v_b.$$

Taylor-Saffman bubble speed

$$v_M = \frac{\Delta\rho g h^2}{12\eta}\frac{a}{b}$$

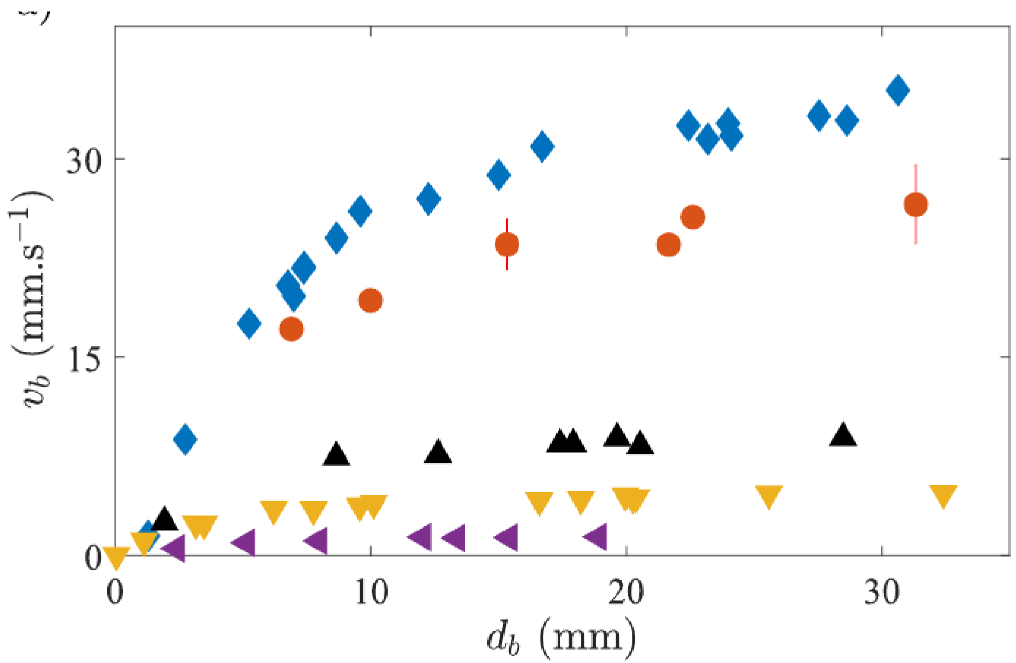
‡elliptical

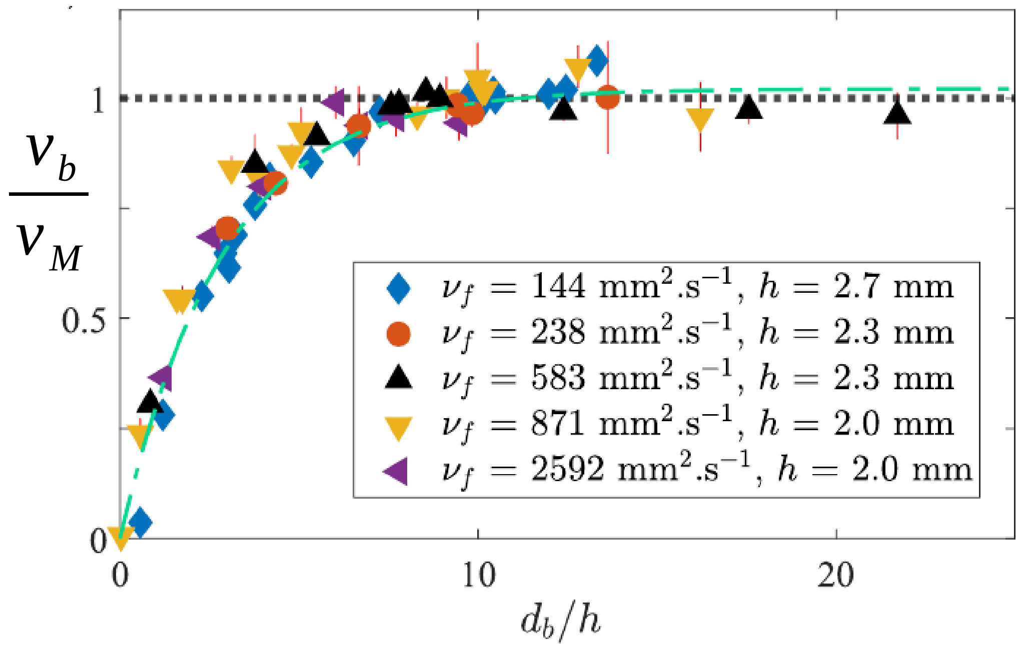
‡a little more elaborate for an elliptical bubble..

Hele-Shaw cell & Lubrication approximation

Taylor-Saffman bubble

$$v_M = \frac{\Delta\rho g h^2}{12\eta}\frac{a}{b}$$



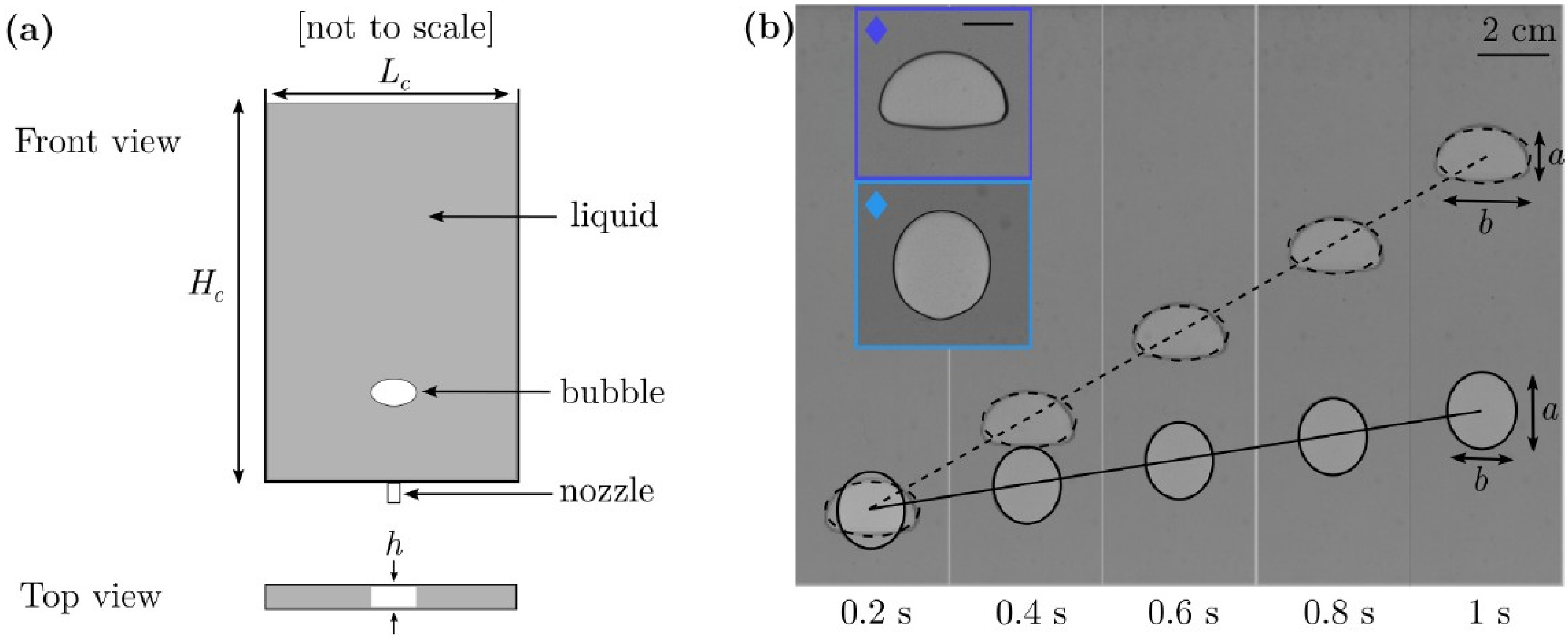


Madec et al. (PRL 2020)

Speed of a freely-rising single bubble (increasing Re_{lub})

..from Taylor-Saffman bubble to Taylor bubbles ?

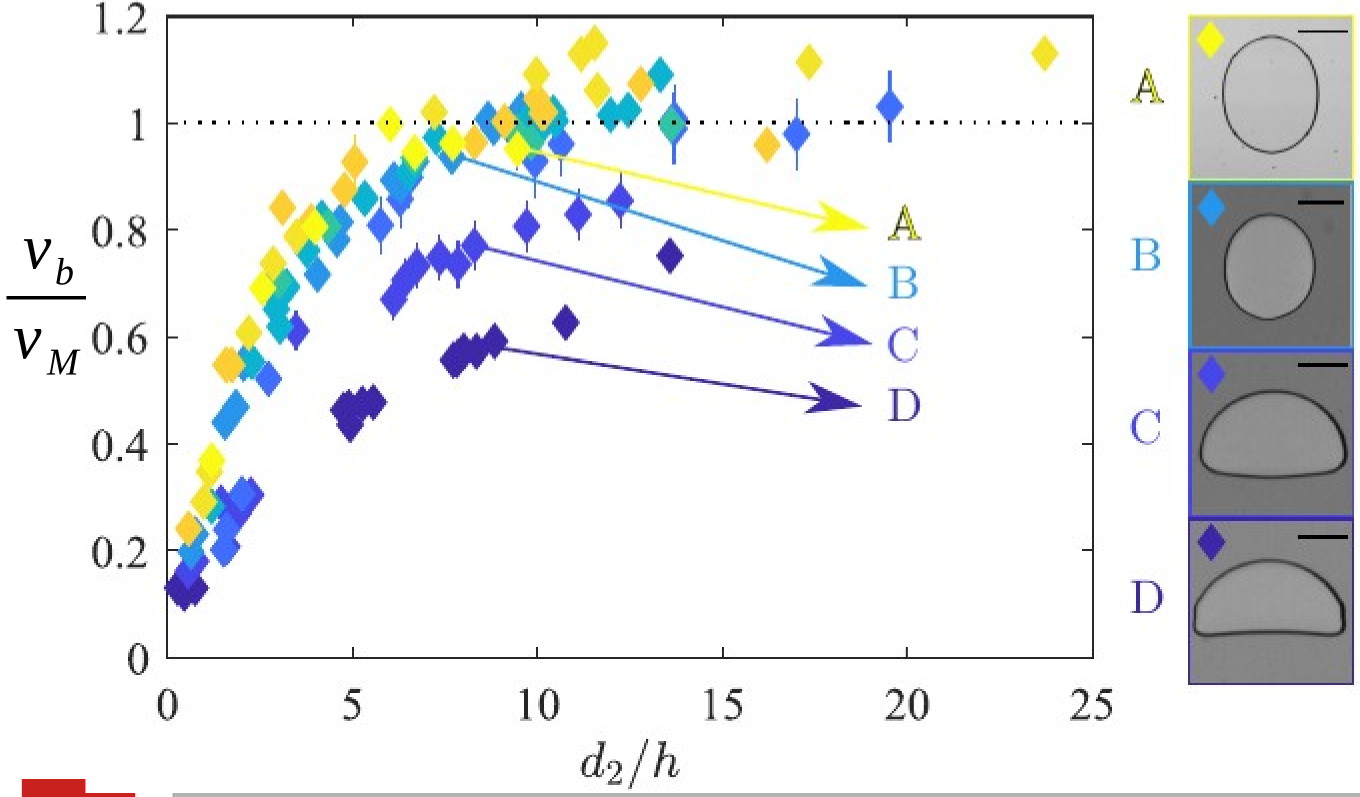
Monnet (Thèse ENS Lyon – en cours)



Speed of a freely-rising single bubble (increasing Re_{lub})

..from Taylor-Saffman bubble to Taylor bubbles ?

$$v_M = \frac{\Delta\rho g h^2}{12\eta}\frac{a}{b}$$

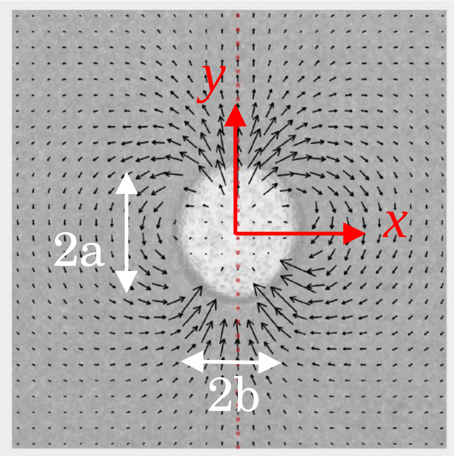


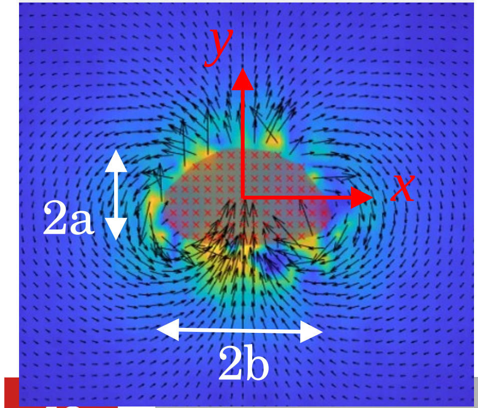
	η (mPa.s)	v_z (mm/s)	Re_{2h}
A	2890	1.4	1.1×10^{-4}
B	140	32	7.2×10^{-2}
C	17	120	2.0
D	8	150	5.5

Hele-Shaw cell & Lubrication approximation

...in the inertial limit ?

$$v_M = \frac{\Delta\rho g h^2}{12\eta}\frac{a}{b}$$





P : Viscous dissipation rate to push the flow in the thin gap..

$$\mathcal{P}_\eta = \int_{d_b/2}^{+\infty}\int_{-\pi}^{\pi}\int_{-h/2}^{h/2}\eta_f\left|\frac{\partial \boldsymbol{v_f}(r,\theta,y)}{\partial y}\right|^2 r\mathrm{d}r\mathrm{d}\theta\mathrm{d}y = \frac{3\pi\eta_f v_b^2 d_b^2}{h}$$

P_I : Energy required to move liquid per unit time..

$$\mathcal{P}_I = \frac{1}{2}\rho v_b^2(\pi d_b^2 h)\left(\beta\frac{v_b}{d_3}\right)$$

P_B : Injected power ..

$$\mathcal{P}_B = \Delta\rho g\left(\frac{\pi d_b^2 h}{4}\right)v_b$$

$$\mathcal{P}_B = \mathcal{P}_\eta + \mathcal{P}_I$$

$$v_b = \frac{2v_M}{1+\sqrt{1+2\beta\left(\frac{v_M}{\sqrt{gd_3}}\right)^2}}$$

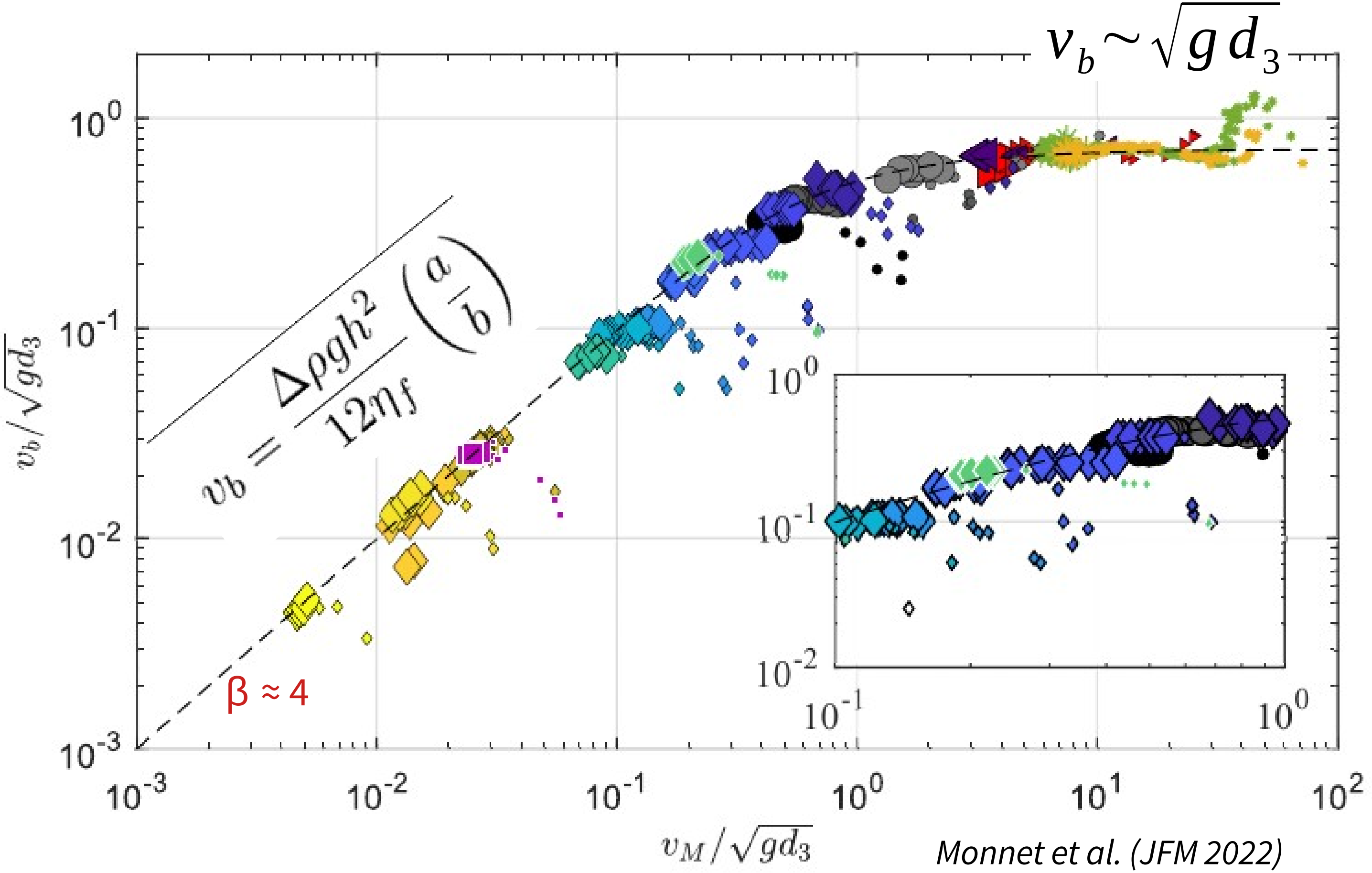
Speed of a freely-rising single bubble (large Re)

Viscous to Inertial regimes

$$v_M = \frac{\Delta\rho g h^2}{12\eta}\frac{a}{b}$$

$$v_b = \frac{2v_M}{1+\sqrt{1+2\beta\left(\frac{v_M}{\sqrt{gd_3}}\right)^2}}$$

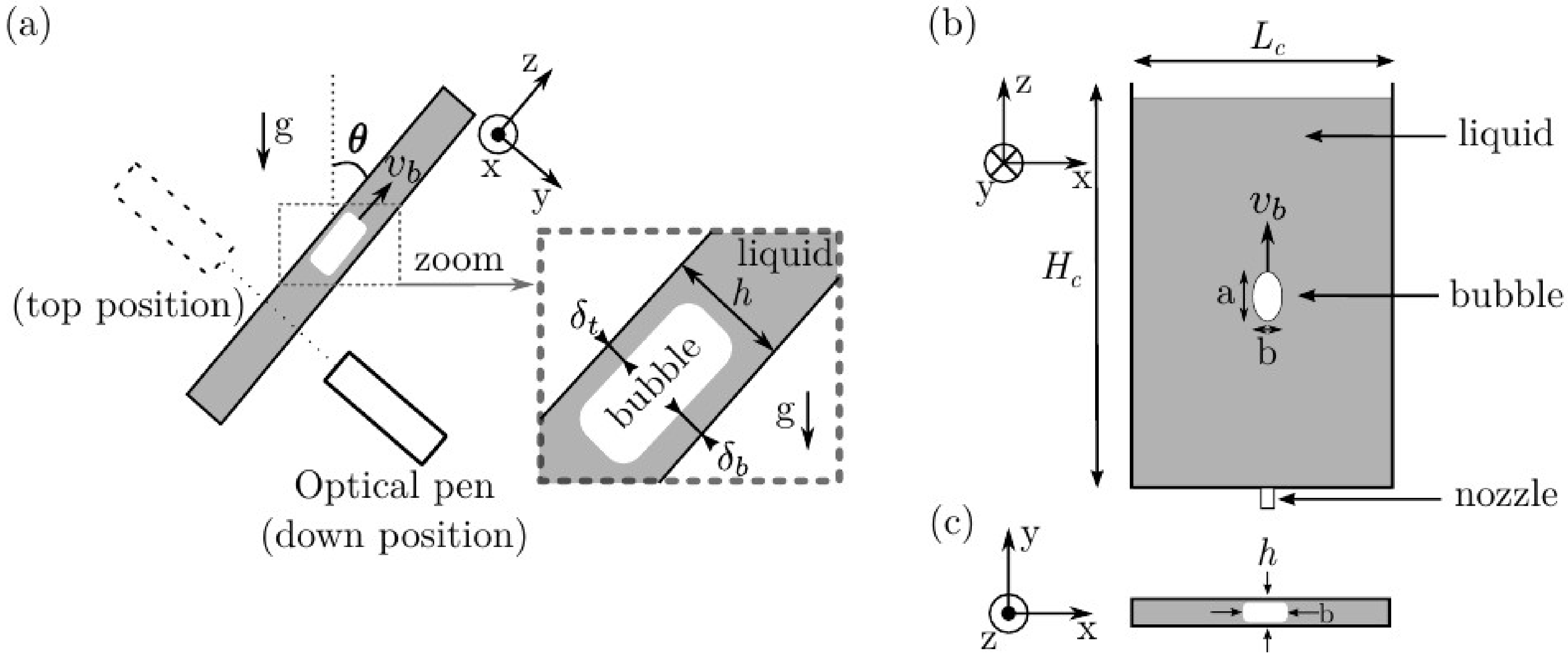
Solution	Viscosity η (mPa s)	Density ρ (kg m^{-3})	Surface Tension γ (mN m^{-1})	gap h (mm)	Symbol
water	0.94	997	72	2.3	▶
ethanol (95 %)	1.2	789	22	2.3	◀
WG3	3.0	1100	67 ± 3	2.3	●
WG13	13	1187	67 ± 3	2.3	●
WG24	24	1208	67 ± 3	2.3	●
WU8	8	1011	53 ± 1	2.3	◆
WU17	17	1020	52 ± 1	2.3	◆
WU42	42	1032	52 ± 1	2.3	◆
WU80	80	1041	52 ± 1	2.3	◆
WU140	140	1048	51 ± 1	2.3	◆
WU152	152	1051	51 ± 1	2.3	◆
WU210	210	1057	50 ± 1	2.3	◆
WU260	260	1058	49 ± 1	5.2	◆
WU620	620	1066	47 ± 1	2.3	◆
WU930	930	1074	47 ± 1	2.0	◆
WU1120	1120	1075	46 ± 1	2.3	◆
WU2890	2890	1085	45 ± 1	2.0	◆
WT2700	2700	1187	32 ± 1	5.2	■



Monnet et al. (JFM 2022)

Speed of a freely-rising single bubble (inclined cell)

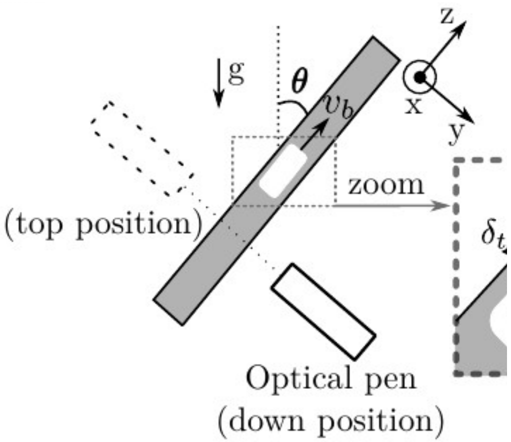
..from Taylor-Saffman bubble to Taylor bubbles ?



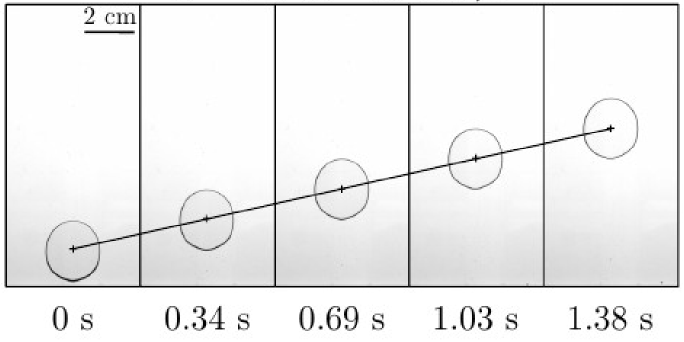
Monnet et al. (PoF 2024)

Speed of a freely-rising single bubble (inclined cell)

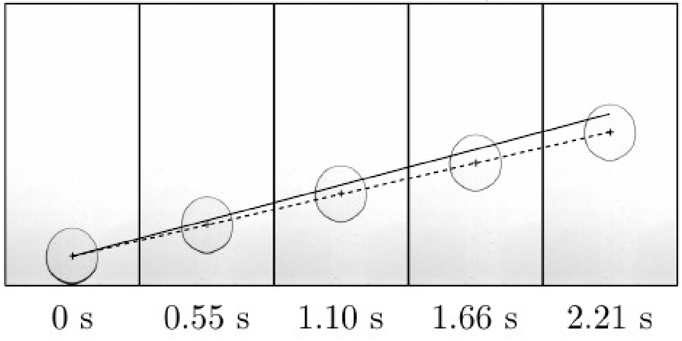
..from Taylor-Saffman bubble to Taylor bubbles ?



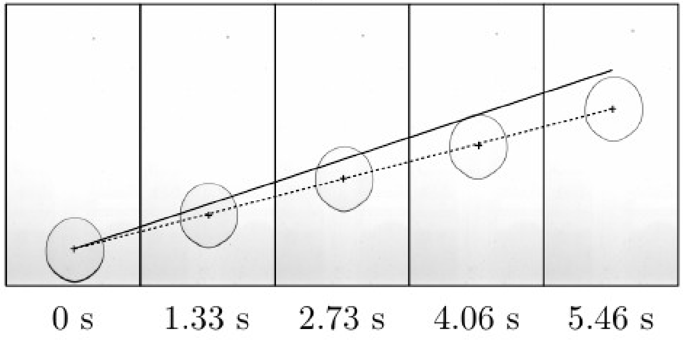
$\theta = 0°,\ v_b = 35$ mm/s



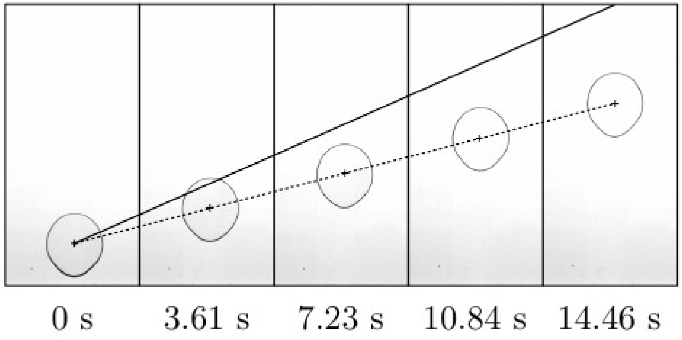
$\theta = 45°,\ v_b = 22$ mm/s



$\theta = 68°,\ v_b = 10.2$ mm/s



$\theta = 80°,\ v_b = 3.85$ mm/s



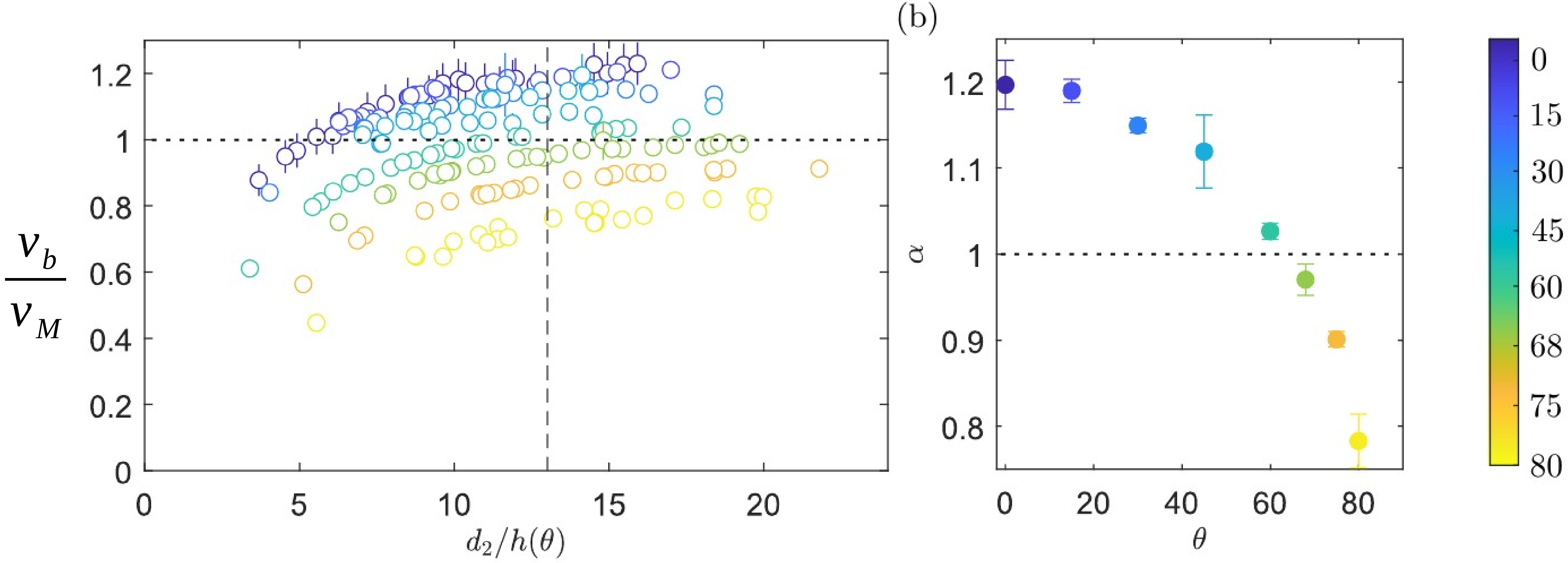
Monnet et al. (PoF 2024)

Speed of a freely-rising single bubble (inclined cell)

..from Taylor-Saffman bubble to Taylor bubbles ?

$$v_M = \frac{\Delta\rho\tilde{g}h^2}{12\eta}\frac{a}{b}$$

$$\tilde{g} = g\cos\theta$$



Monnet et al. (PoF 2024)

Speed of a freely-rising single bubble (inclined cell)

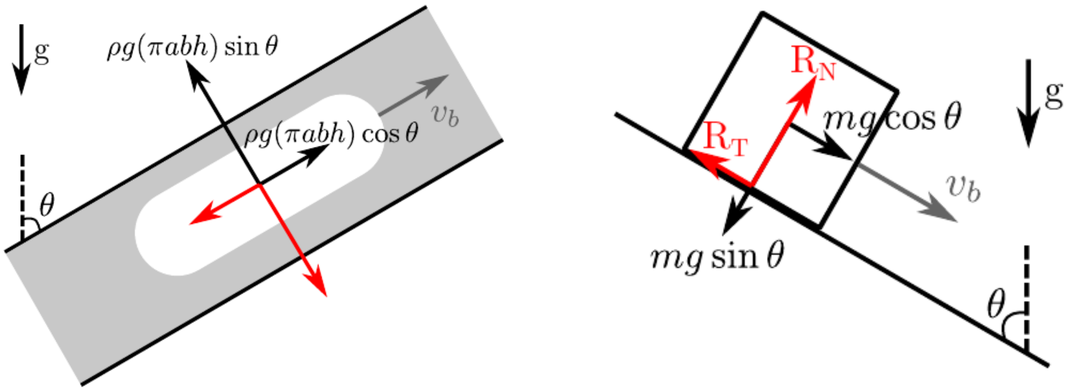
..from Taylor-Saffman bubble to Taylor bubbles ?

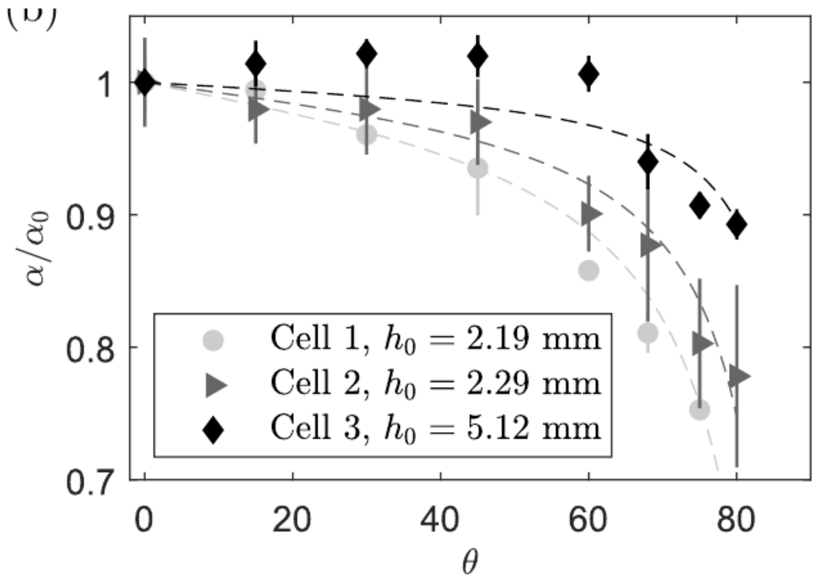
$$v_M = \frac{\Delta\rho\tilde{g}h^2}{12\eta}\frac{a}{b}$$

$$\tilde{g} = g\cos\theta$$

$$\rho\tilde{g}\cos\theta(\pi abh)v_b = \frac{12\eta v_b^2\pi b^2}{h} + \kappa\rho\tilde{g}\sin\theta(\pi abh)v_b,$$

$$v_b = v_M(1-\kappa\tan\theta) \quad \Rightarrow \alpha(\theta) = 1-\kappa\tan\theta.$$





Monnet et al. (PoF 2024)

Conclusions : bubble rise speed in thin gaps

- Taylor-Saffman bubble speed is verified for vertical bubble rise in thin gap cells

$$v_M = \frac{\Delta\rho g h^2}{12\eta}\frac{a}{b}$$

- Bubble aspect ratio is crucial and inertia flattens the bubbles ! (but why?)

- Bubble speed between viscous and inertial regime is modeled by an power balance argument

$$v_b = \frac{2v_M}{1+\sqrt{1+2\beta\left(\frac{v_M}{\sqrt{gd_3}}\right)^2}}$$

- Bubble speed in inclined cells depends non-trivially on the inclination angle due to symmetry loss in lubrication films

$$v_M = \frac{\Delta\rho g h^2}{12\eta}(1-\kappa\tan\theta)\frac{a}{b}$$

Perspectives :

- Bubble pair interactions

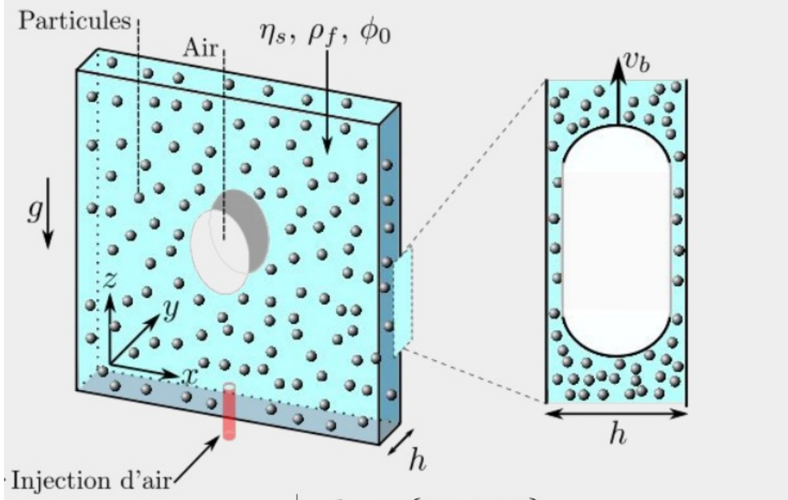
- Lubrication films ? Taylor-Saffman analysis with symmetry loss near bubble ?

Monnet et al. (JFM 2022, PoF 2024)

$\beta \approx 4$

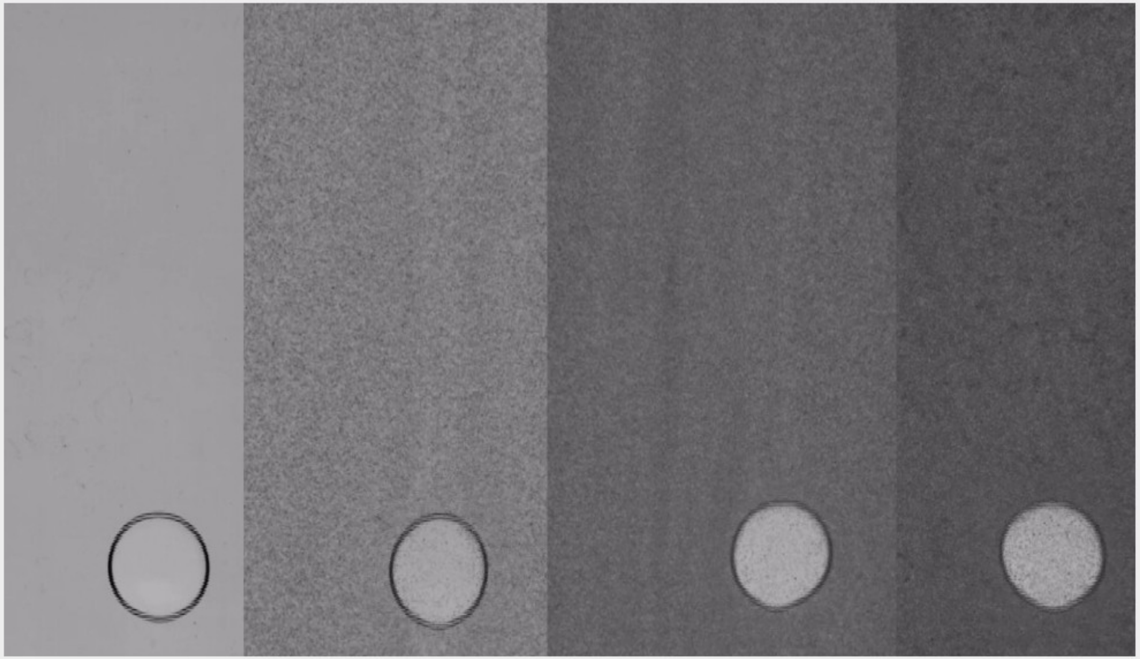
Hele-Shaw cell & Lubrication approximation

Taylor-Saffman bubble in suspensions



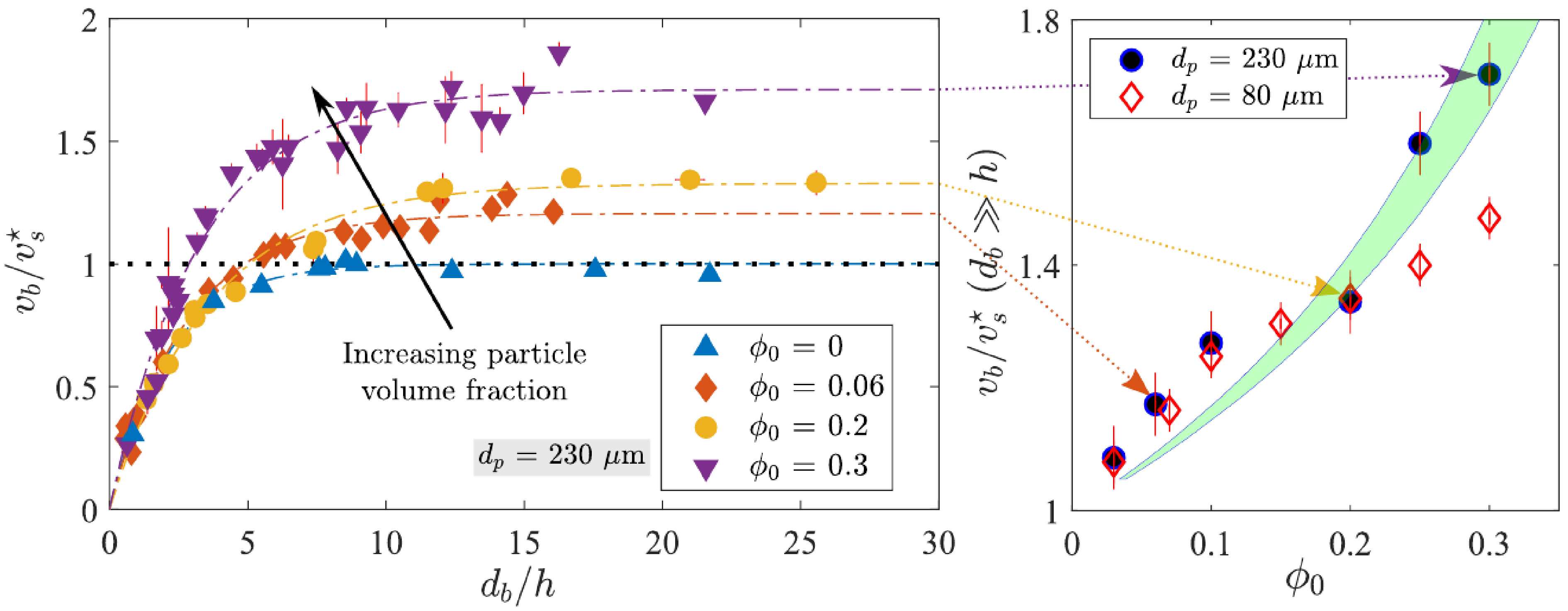
$d_p \in \{80, 230\}\ \mu\text{m}$

$\rho_p = 1057 \pm 3\ \text{kg.m}^{-3} = \rho_f$



Hele-Shaw cell & Lubrication approximation

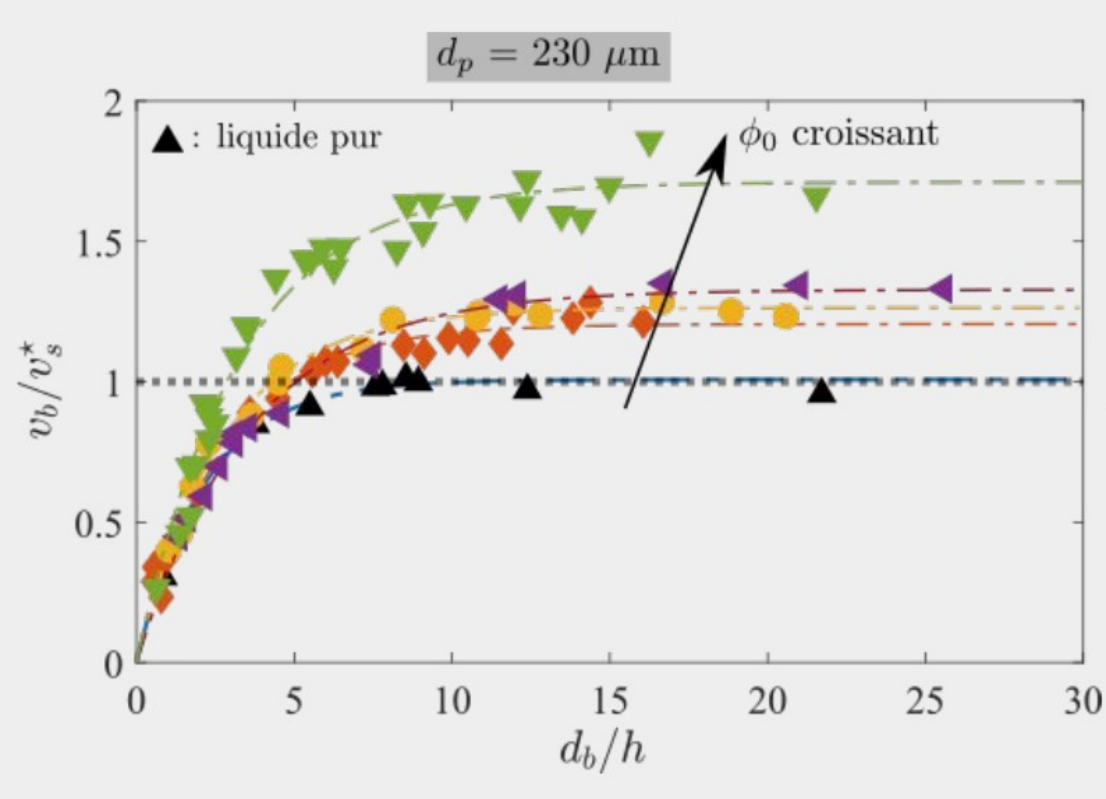
Faster than a TS bubble without particles?

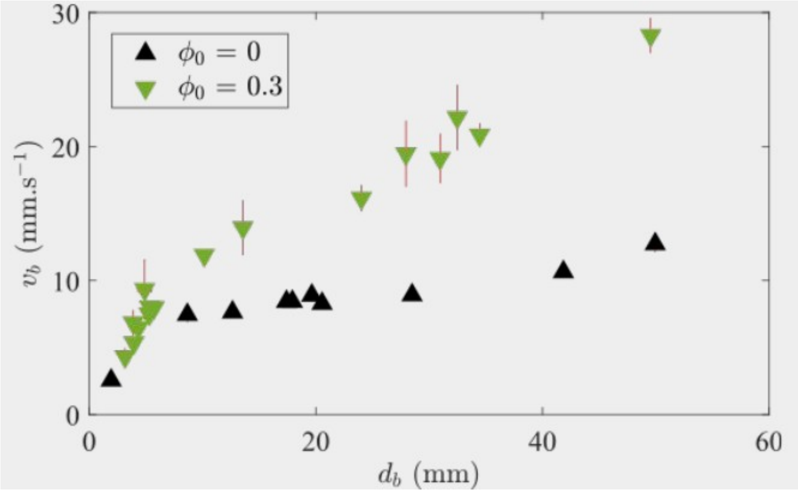


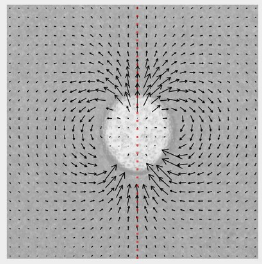
Madec et al. (PRL 2021)

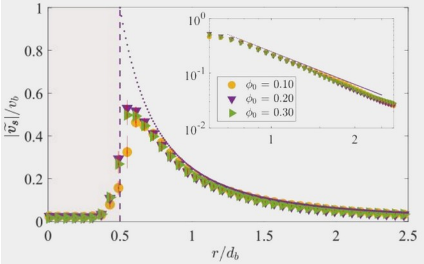
Hele-Shaw cell & Lubrication approximation

Taylor-Saffman bubble in suspensions



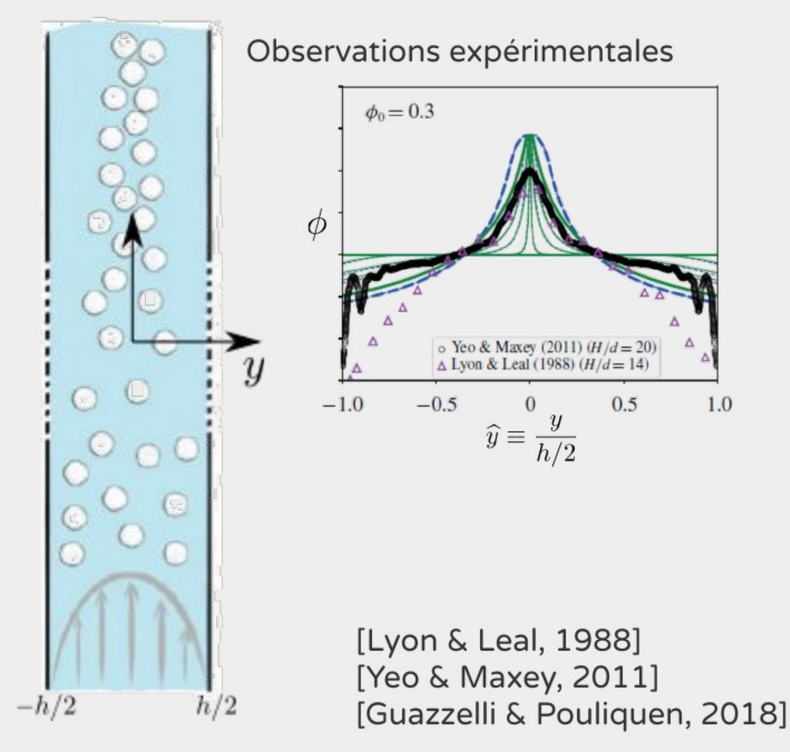




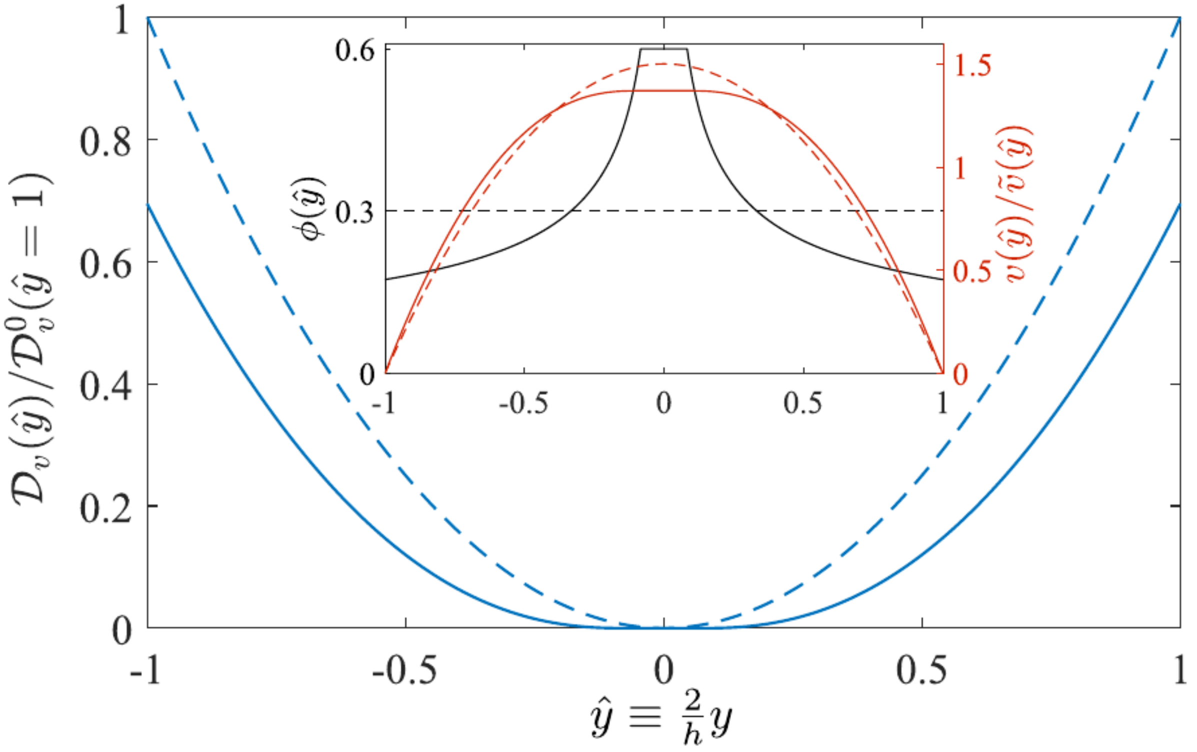


Hele-Shaw cell & Lubrication approximation

Shear-induced migration => less dissipation in the gap !



[Lyon & Leal, 1988]
[Yeo & Maxey, 2011]
[Guazzelli & Pouliquen, 2018]



Conclusions : bubble rise speed in suspensions within thin gaps

- Taylor-Saffman bubble speed is verified for vertical bubble rise in thin gap cells, if the dissipation is appropriately modified !
  - § Shear-Induced-Migration increases particle fraction at the center of the cell gap
  - § This in turn decreases local viscosity and hence the shear near the wall
  - § Overall dissipation is smaller than in a Newtonian fluid of same bulk viscosity
  - § Bubble rise is thus faster !
- Bubble aspect ratio remains unchanged (but why ?)

<u>Perspectives :</u>

- Shear Induced Migration during bubble ? How much ?
- Effect of inclination and inertia ?

For questions ...?

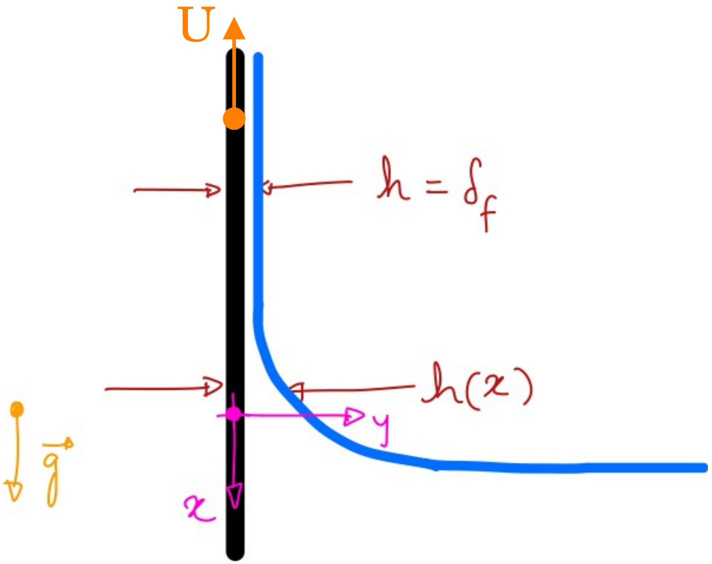
J John Soundar Jerome

LMFA CNRS UMR – 5509

Université Claude-Bernard Lyon1 (France)

The drag-out, or the entrainment problem (at $Re_f << 1$ and $Ca << 1$)

Landau-Levich's asymptotics



- Plate speed (U)
- Gravity (g)
- Dynamic viscosity (μ)
- Density ()
- Surface tension ()

- Incompressible: $\vec{\nabla} \cdot \vec{u} = 0$
- Parallel flow: $\vec{u} = u(y)\,\vec{e}_x$ ← to first-order
- Stokes flow: $(\vec{u} \cdot \vec{\nabla})\,\vec{u} << \nu \nabla^2 \vec{u}$
- Steady flow: $\frac{\partial \vec{u}}{\partial t} = \vec{0}$

$$0 = g - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial y^2}\right) \longrightarrow (1)$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \longrightarrow (2)$$

Landau-Levich flow

with the boundary conditions,

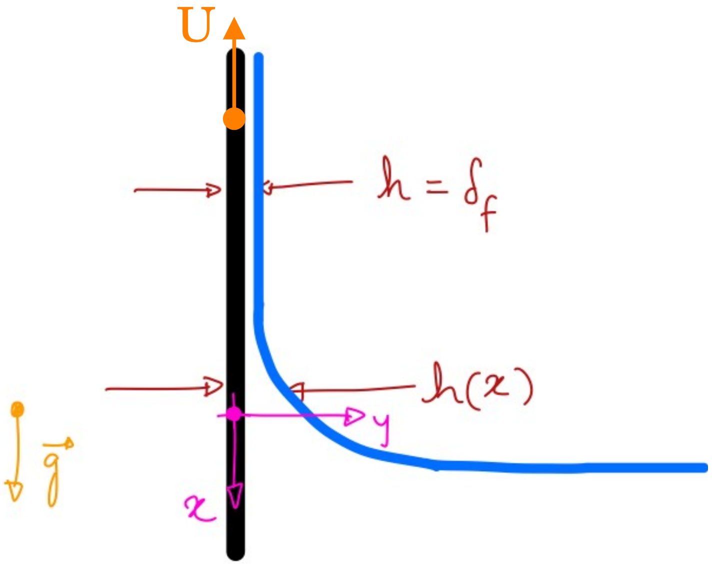
$y = 0: u = U$

$y = h(x): \begin{cases} \frac{\partial u}{\partial y} = 0 \\ p = -\sigma \frac{\frac{d^2 h}{dx^2}}{\left(1 + \left(\frac{dh}{dx}\right)^2\right)^{3/2}} + p_0 \end{cases}$

Laplace pressure

The drag-out, or the entrainment problem (at $Re_f << 1$ and $Ca << 1$)

Landau-Levich's asymptotics



- Plate speed (U)
- Gravity (g)
- Dynamic viscosity (μ)
- Density ()
- Surface tension ()

So,

$$u(x,y) = U + \frac{1}{2}k(x)y(y-2h) \longrightarrow (4)$$

$$k(x) = -\frac{\rho g}{\mu} - \left(\frac{\sigma h''}{(1+h'^2)^{3/2}}\right)'$$

⚠

Note that the film flow rate per unit width is given by

$$Q(x) = \int_0^{h(x)} u(x,y)\,dy \Leftrightarrow Uh + \frac{1}{3}k(x)h^3,$$

where $k(x) = -\frac{\rho g}{\mu}$ and $h(x) = \delta_f$ when $x \to -\infty$,

$$Uh + \frac{1}{3}k(x)h^3 = U\delta_f - \frac{\rho g}{3\mu}\delta_f^3$$

⚠

$$k(x) = -\frac{\rho g}{\mu} - \left(\frac{\sigma h''}{(1+h'^2)^{3/2}}\right)'$$

Capillary number, Ca = μU/

The drag-out, or the entrainment problem (at $Re_f << 1$ and $Ca << 1$)

Landau-Levich's asymptotics

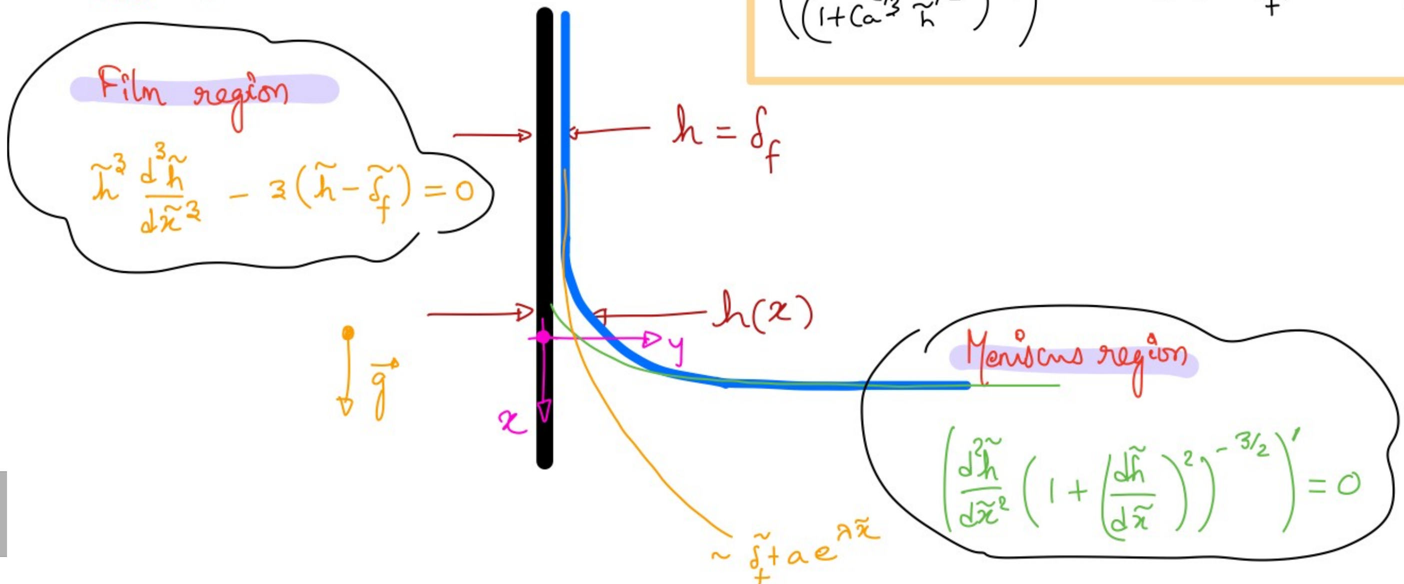
$$\left(\frac{\sigma h''}{(1+h'^2)^{3/2}}\right)' - 3 Ca \frac{l_c^2}{h^2}\left(1-\frac{\delta_f}{h}\right) + \left(1-\frac{\delta_f^3}{h^3}\right) = 0$$

where $(\cdot)' \equiv \frac{d}{dx}$, $l_c = \sqrt{\frac{\sigma}{\rho g}}$, $Ca = \frac{\mu U}{\sigma}$

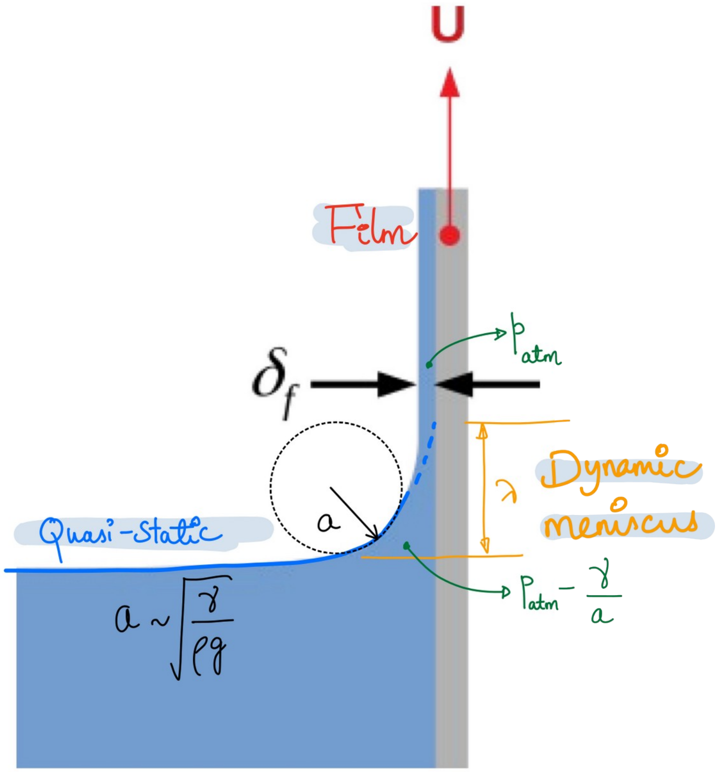
$h = l_c Ca^{2/3} \tilde{h}$ and $x = l_c Ca^{1/3} \tilde{x}$:

$$\left(\frac{\tilde{h}''}{(1+Ca^{2/3}\tilde{h}'^2)^{3/2}}\right)' \tilde{h}^3 - 3(\tilde{h}-\tilde{\delta}_f) + Ca^{1/3}(\tilde{h}^3 - \tilde{\delta}_f^3) = 0$$

$Ca^{2/3} << 1$:



Inertial effect in rotating drum: depth effect



$$\frac{(\gamma / l_c)}{\lambda} \sim \frac{\mu U}{\delta_f^2}$$

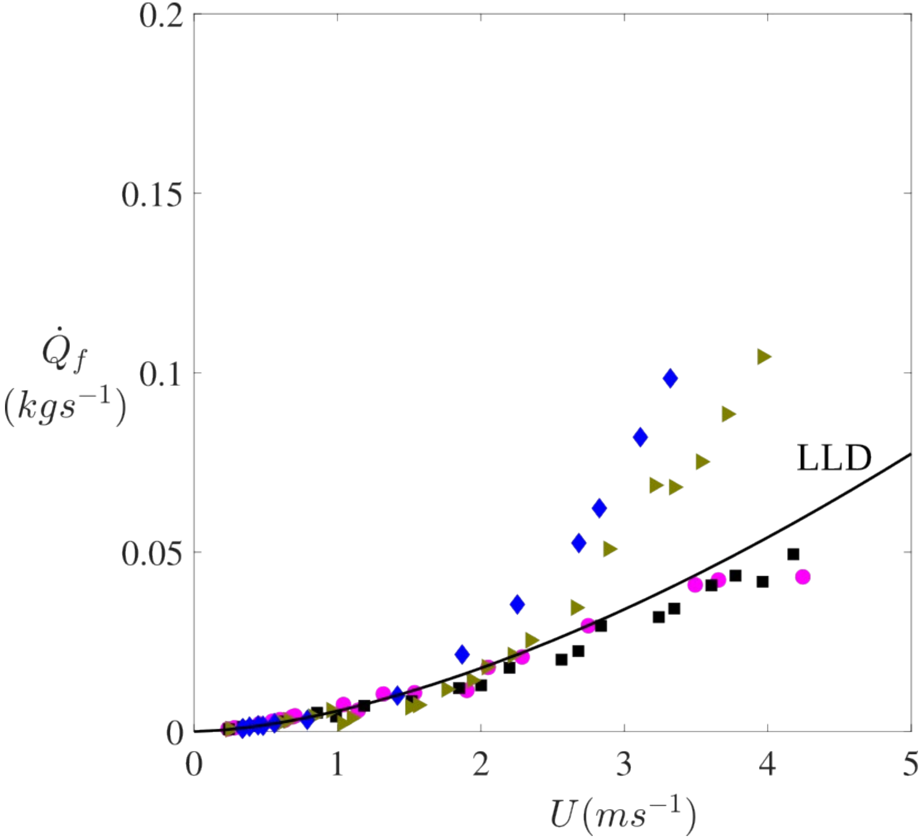
(Laplace pressure drop) (plate drag)

$$\frac{\lambda}{\delta_f^2} \sim \frac{g}{U^2}$$

(Curvature matching)

$$\delta_f \sim \delta_f^{LLD} \left(\frac{U^2}{g\, l_c} \right)^{1/3} \qquad \Rightarrow \dot{Q}_f \sim We^{1/3}$$

Inertial effect in rotating drum: depth effect



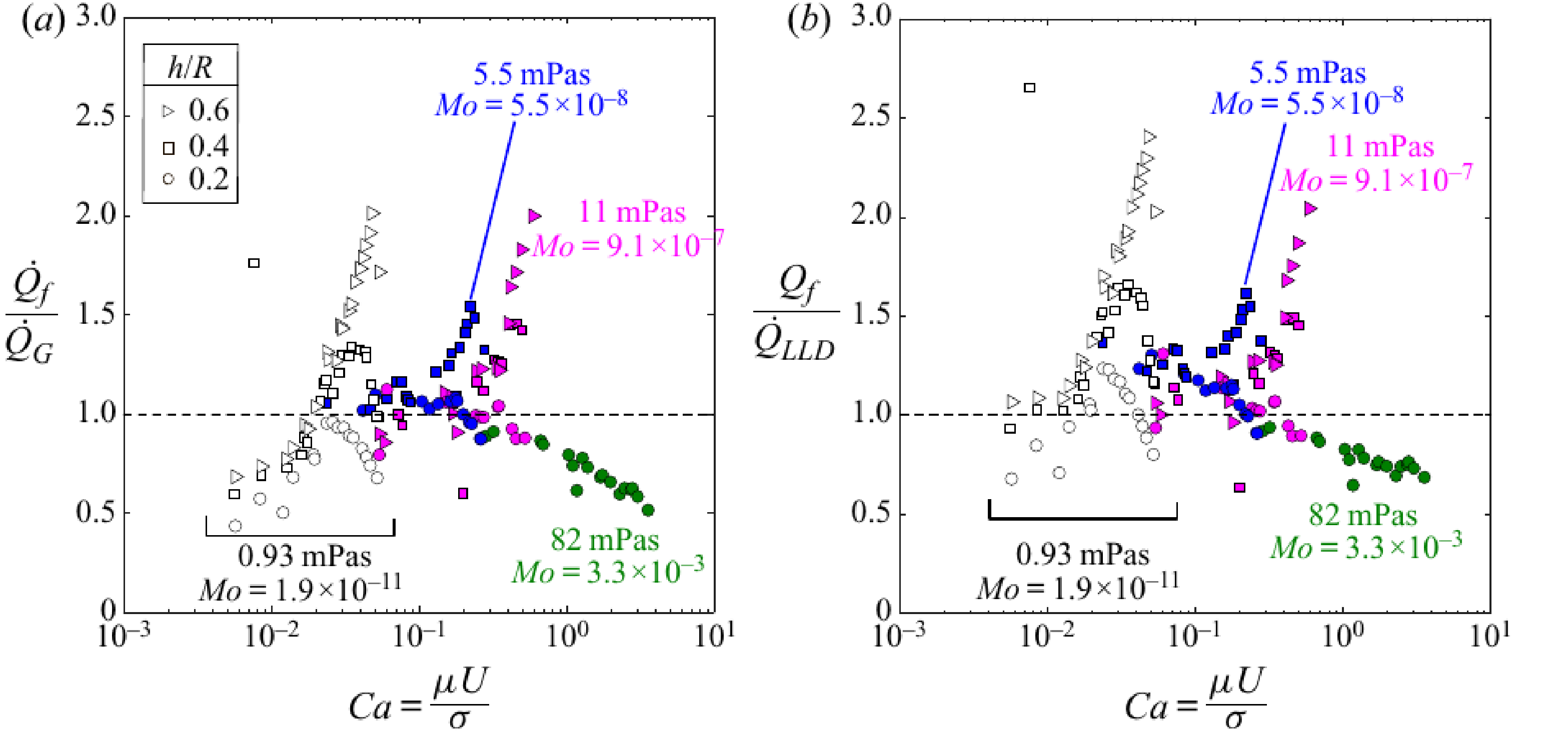
$$\frac{(\gamma / l_c)}{\lambda} \sim \frac{\mu U}{\delta_f^2}$$

(Laplace pressure drop) (plate drag)

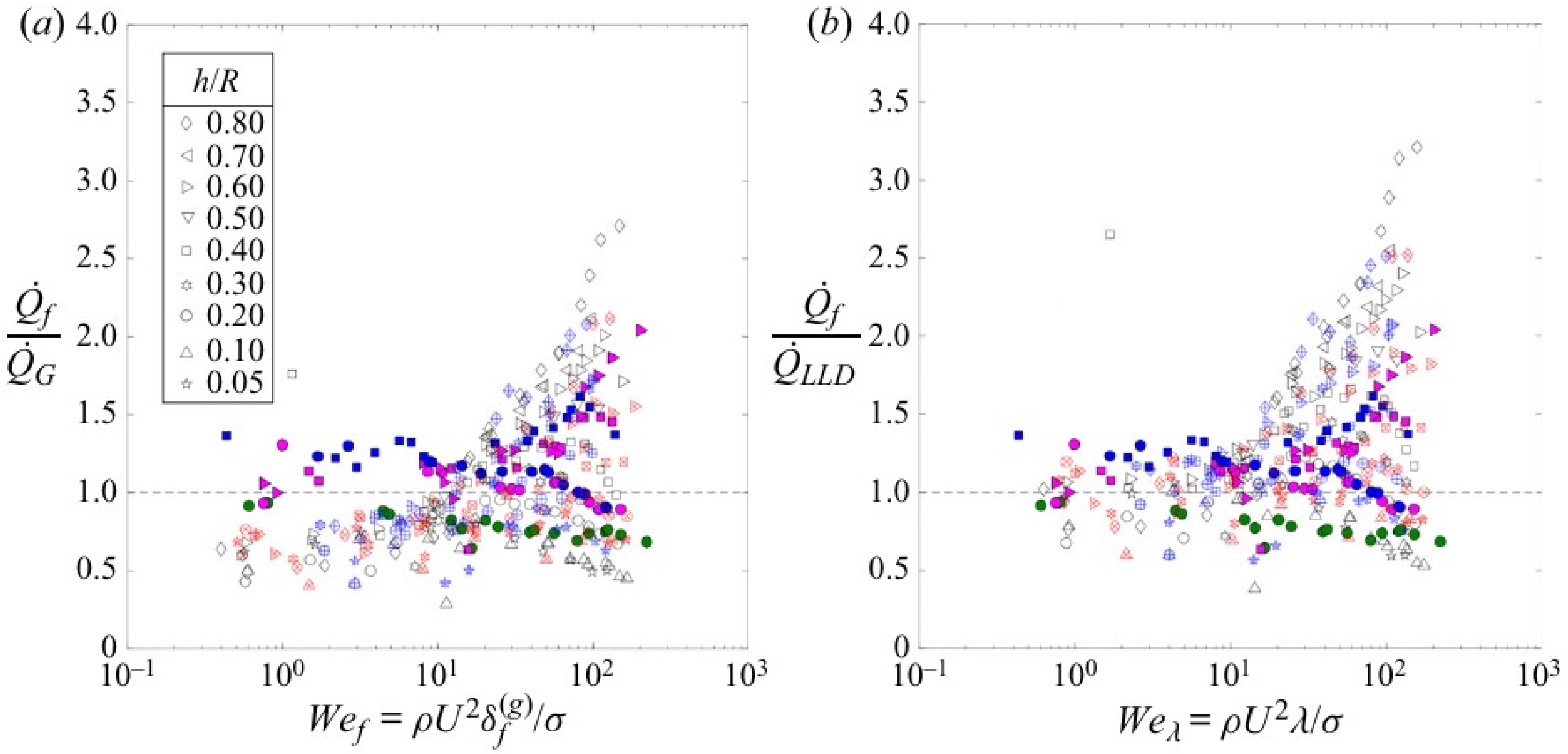
$$\frac{\lambda}{\delta_f^2} \sim \frac{g}{U^2}$$

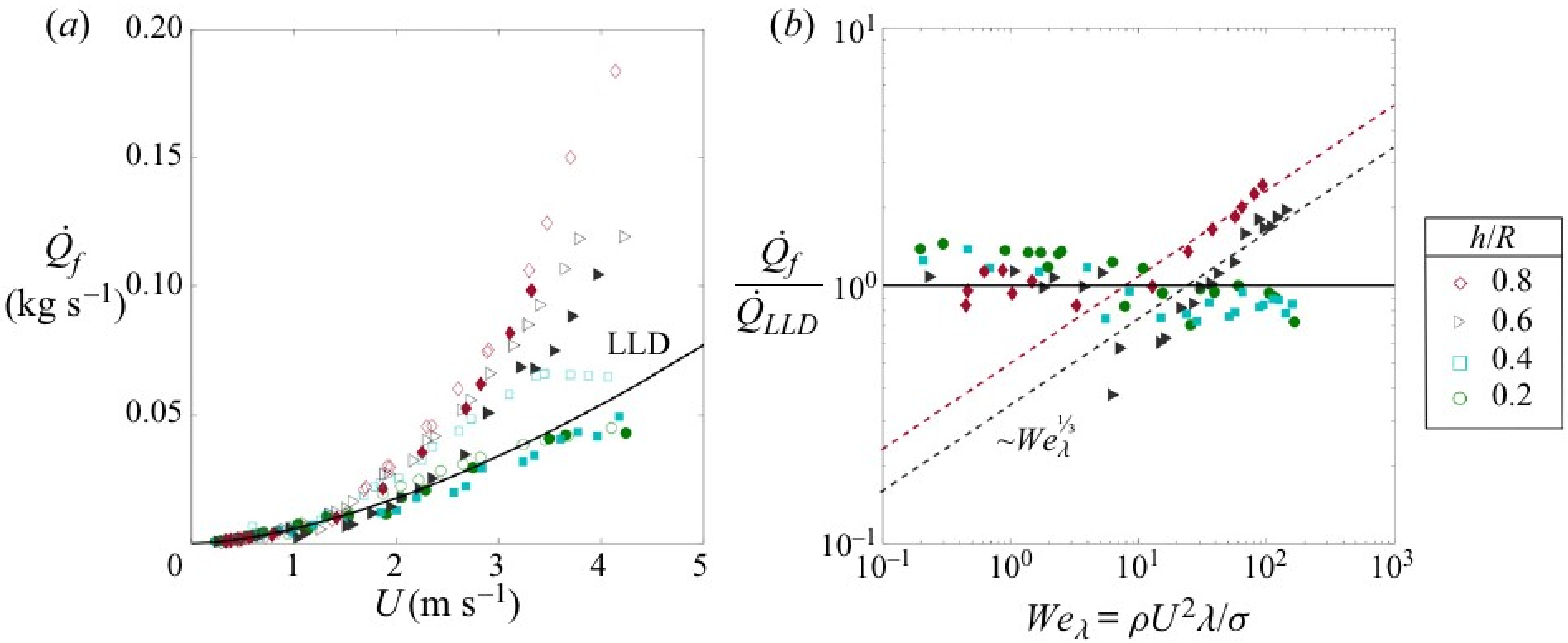
(Curvature matching)

$$\delta_f \sim \delta_f^{LLD} \left(\frac{U^2}{g\, l_c} \right)^{1/3} \qquad \Rightarrow \dot{Q}_f \sim We^{1/3}$$



(a)
h/R
0.6
0.4
0.2
5.5 mPas
Mo = 5.5×10^−8
11 mPas
Mo = 9.1×10^−7
0.93 mPas
Mo = 1.9×10^−11
82 mPas
Mo = 3.3×10^−3
Q̇f / Q̇G
Ca = μU/σ
(b)
Qf / Q̇LLD


(a)
h/R
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.05
$\dot{Q}_f / \dot{Q}_G$
$We_f = \rho U^2 \delta_f^{(g)} / \sigma$
(b)
$\dot{Q}_f / \dot{Q}_{LLD}$
$We_\lambda = \rho U^2 \lambda / \sigma$



(a)
0.20
0.15
0.10
0.05
$\dot{Q}_f$ (kg s^{-1})
LLD
0
1
2
3
4
5
U (m s^{-1})
(b)
10^1
10^0
10^{-1}
$\dfrac{\dot{Q}_f}{\dot{Q}_{LLD}}$
$\sim We_\lambda^{1/3}$
10^{-1}
10^0
10^1
10^2
10^3
$We_\lambda = \rho U^2 \lambda / \sigma$
h/R
0.8
0.6
0.4
0.2

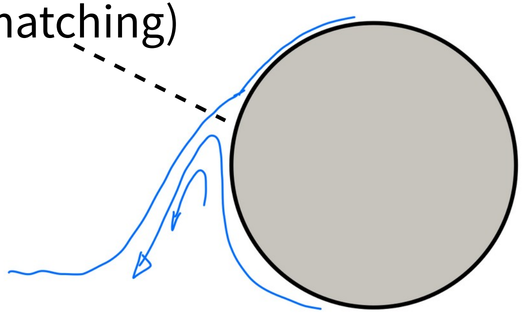
Inertial effect in rotating drum: We ≫ 1

$$\frac{(\gamma/l_c)}{\lambda} \sim \frac{\mu U}{\delta_f^2}$$

(Laplace pressure drop) (plate drag/volume)

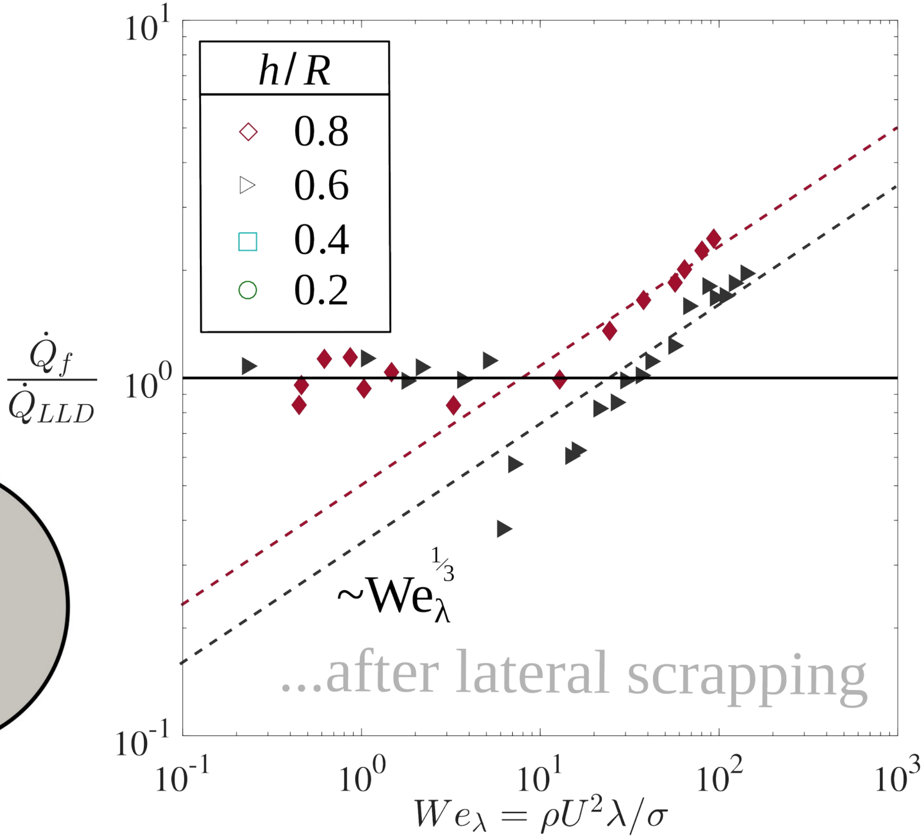
$$\frac{\lambda}{\delta_f^2} \sim \frac{g}{U^2}$$

(Curvature matching)



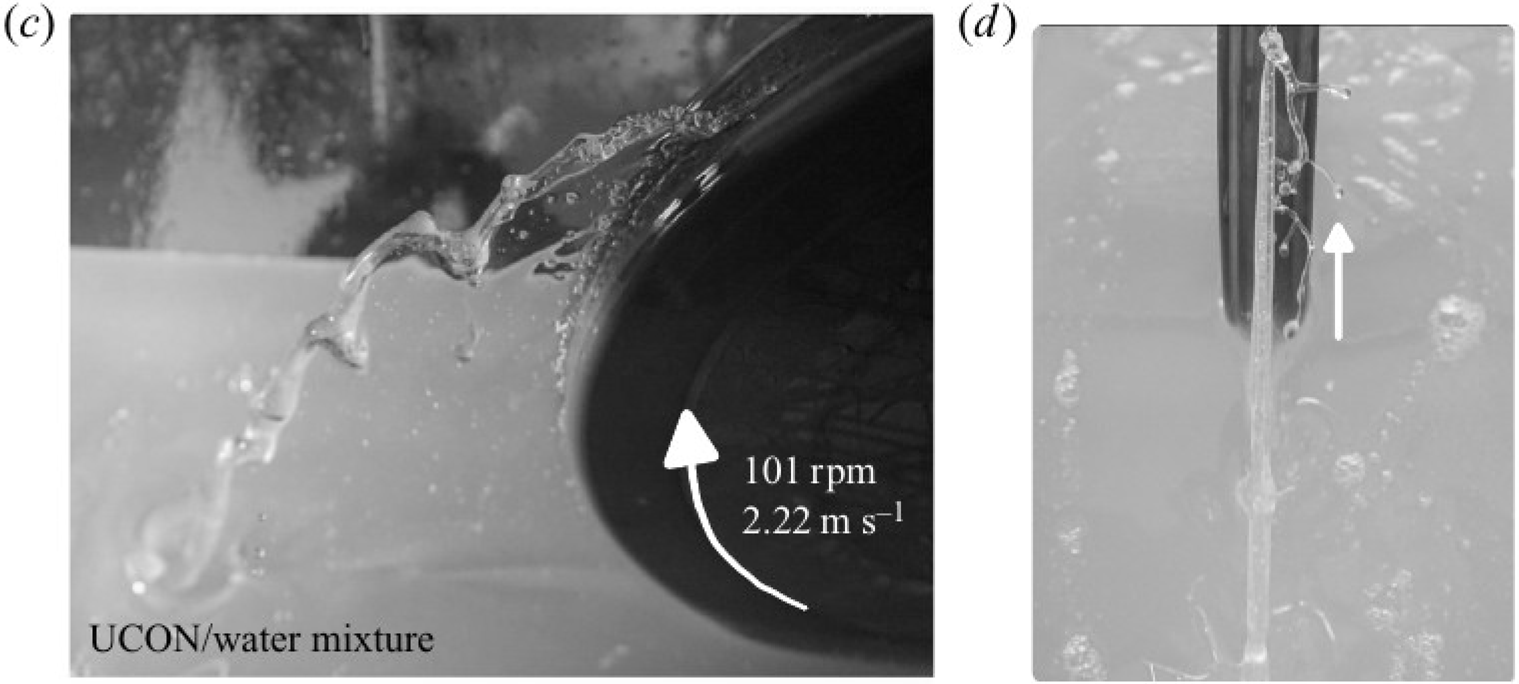
$$\Rightarrow \delta_f \sim \delta_f^{LLD}\left(U^2/g\,l_c\right)^{1/3}$$

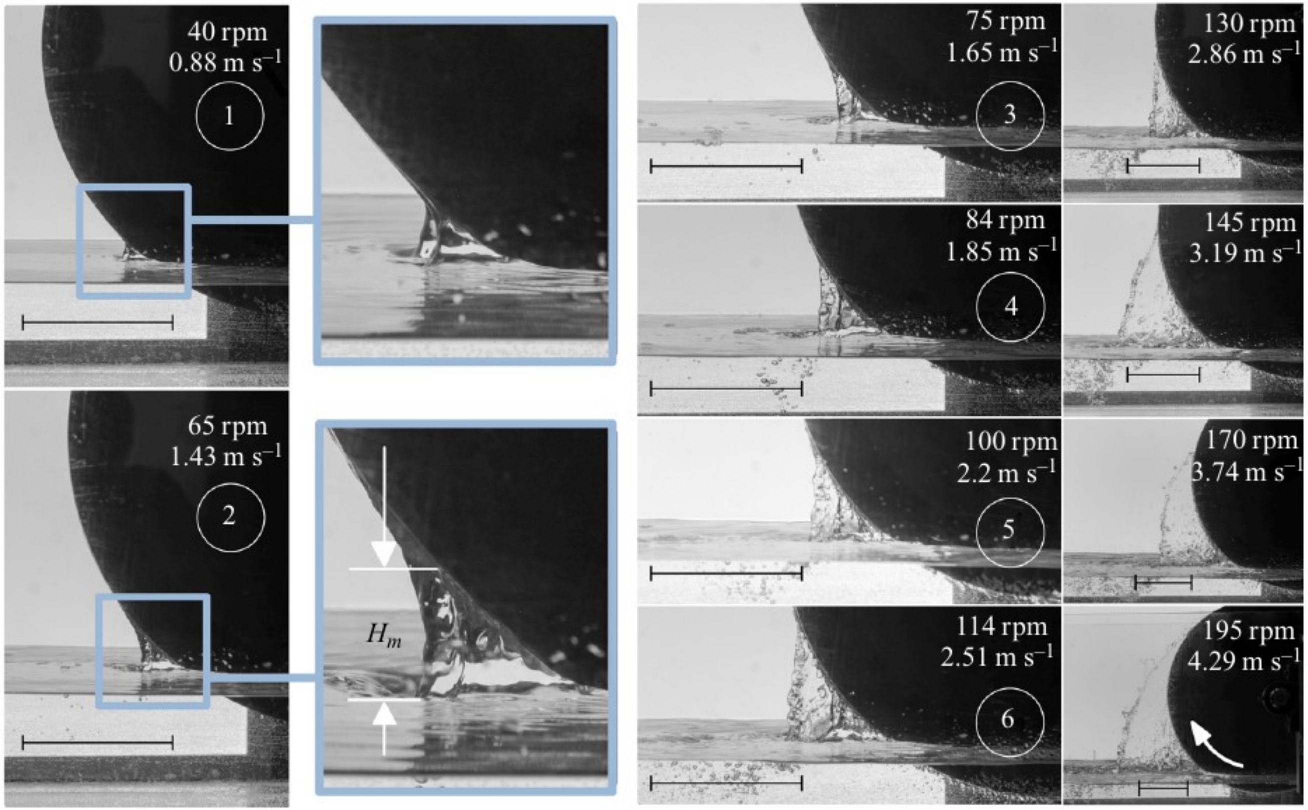
$$\Rightarrow \dot{Q}_f \sim \dot{Q}_{LLD}\, We_\lambda^{1/3} \quad \text{[for We} \gg 1\text{]}$$



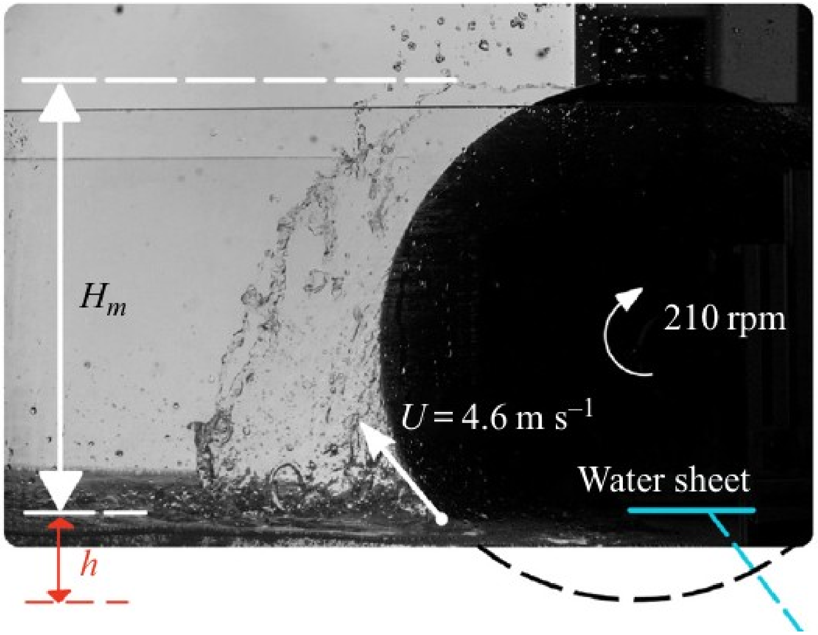
Jerome et al. (JFM 2020)

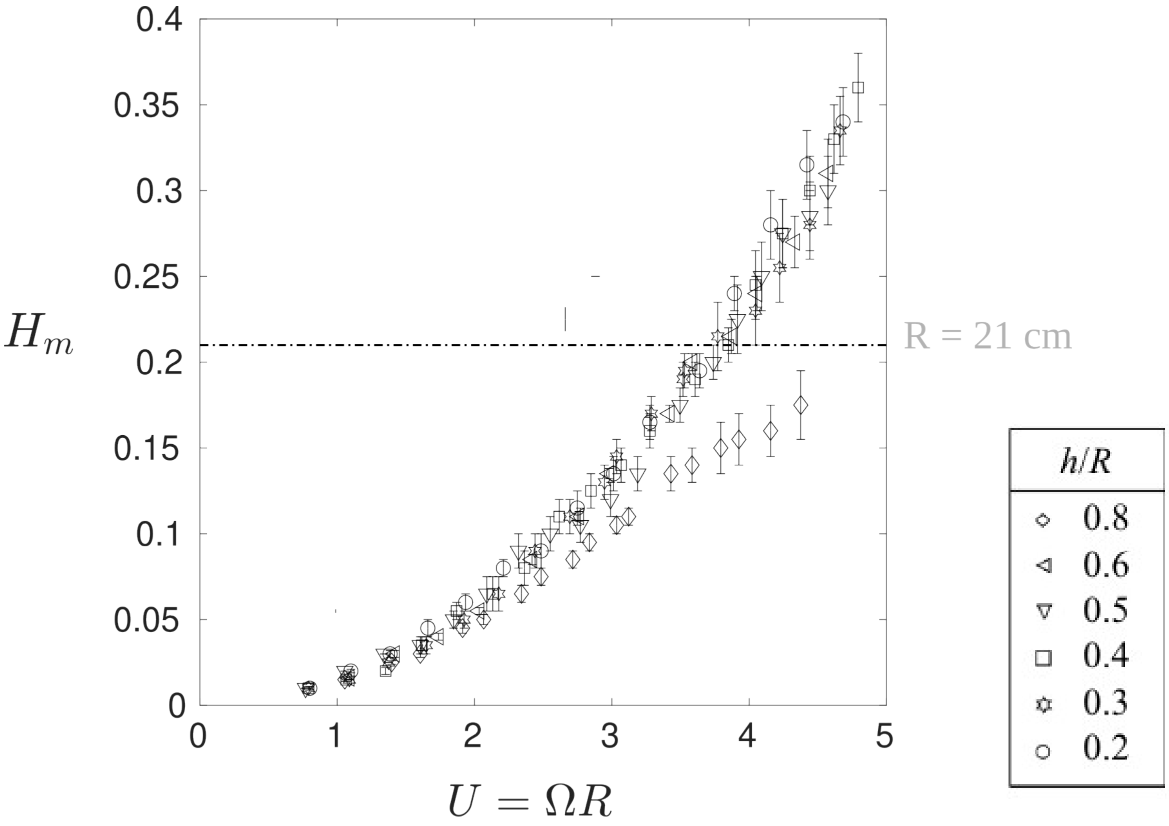
A wiggling rib (sheet)




40 rpm
0.88 m s⁻¹
1
65 rpm
1.43 m s⁻¹
2
H_m
75 rpm
1.65 m s⁻¹
3
84 rpm
1.85 m s⁻¹
4
100 rpm
2.2 m s⁻¹
5
114 rpm
2.51 m s⁻¹
6
130 rpm
2.86 m s⁻¹
145 rpm
3.19 m s⁻¹
170 rpm
3.74 m s⁻¹
195 rpm
4.29 m s⁻¹

Water sheet height Vs Linear speed & Depth



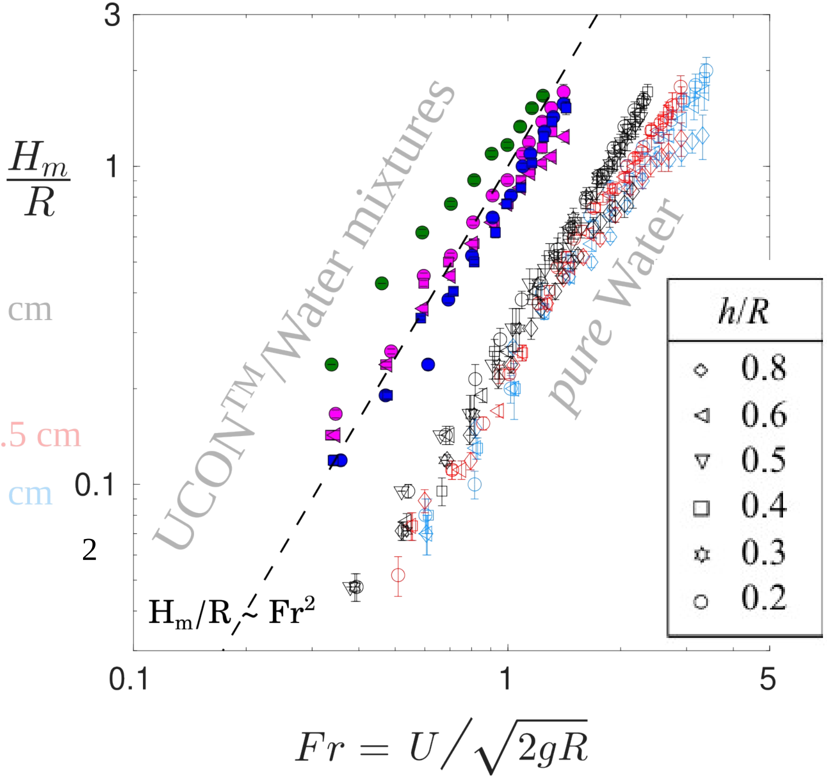


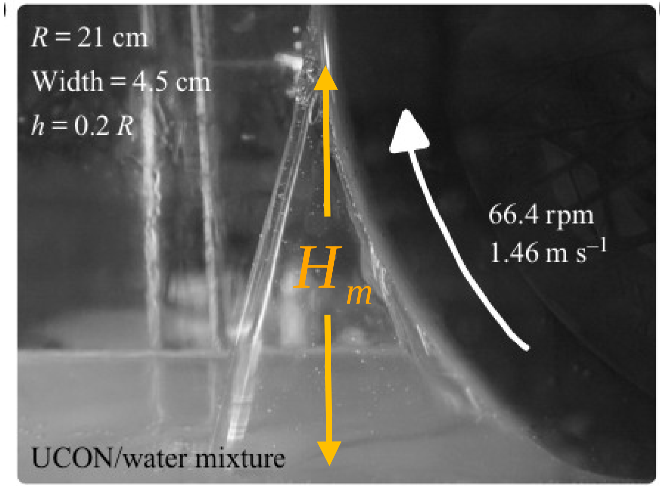
Rib height: ballistic mechanism

R = 21 cm

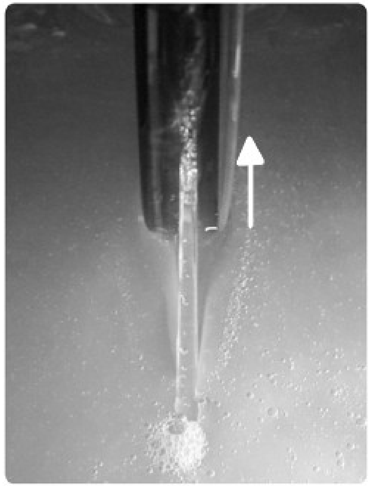
R = 13.5 cm

R = 10 cm





(side view)

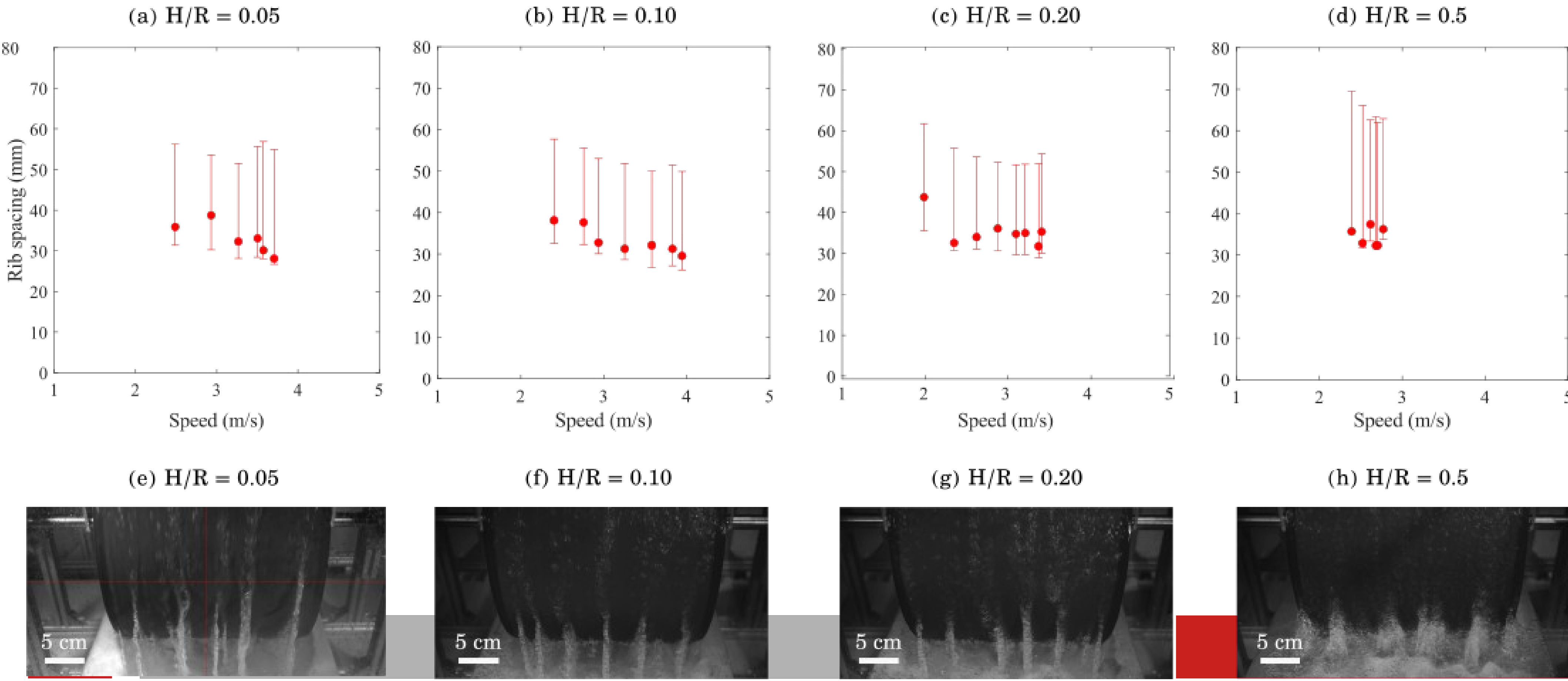


(front view)

$$H_m \simeq \frac{U^2}{g} f(Ca)$$

J John Soundar Jerome et al. (2020)

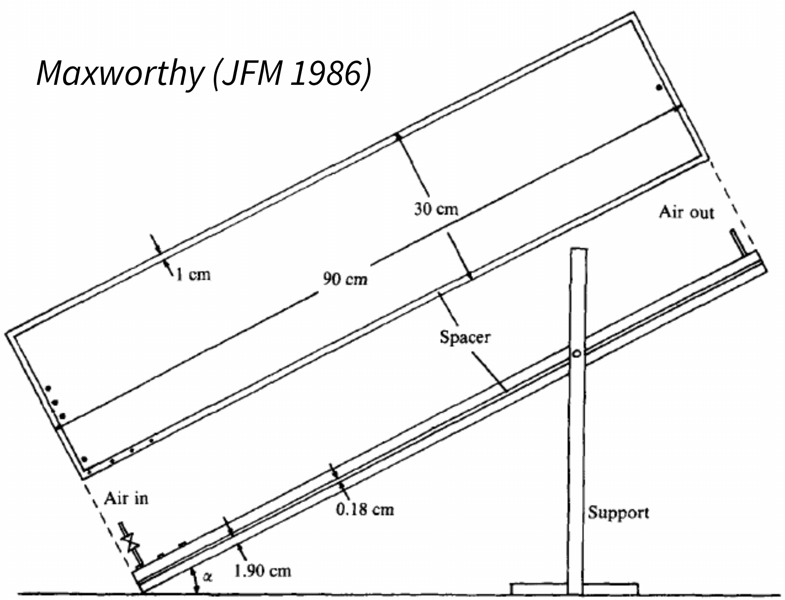
Rotating drum – Maximum rib spacing

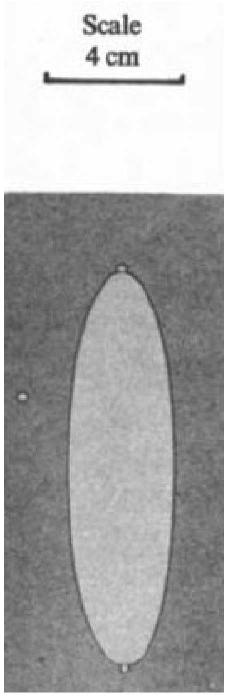


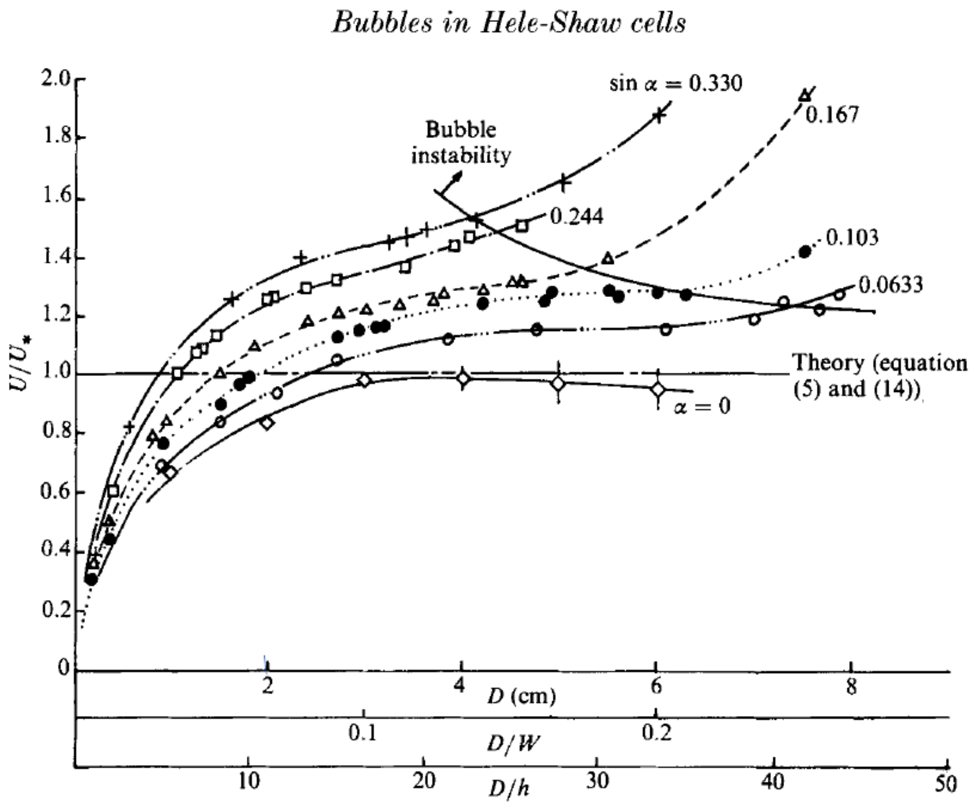
Speed of a freely-rising single bubble (small Re ?)

…in large Hele-Shaw cell

Maxworthy (JFM 1986)

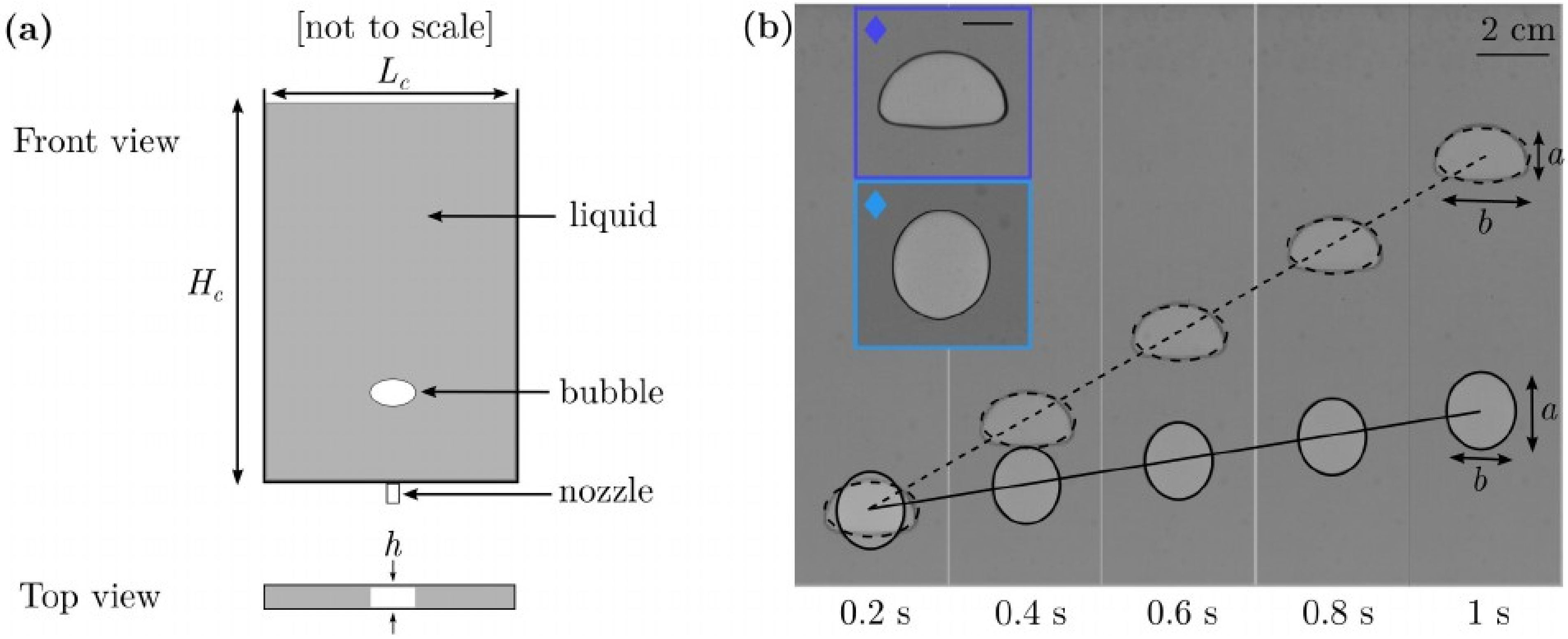






Speed of a freely-rising single bubble (large Re)

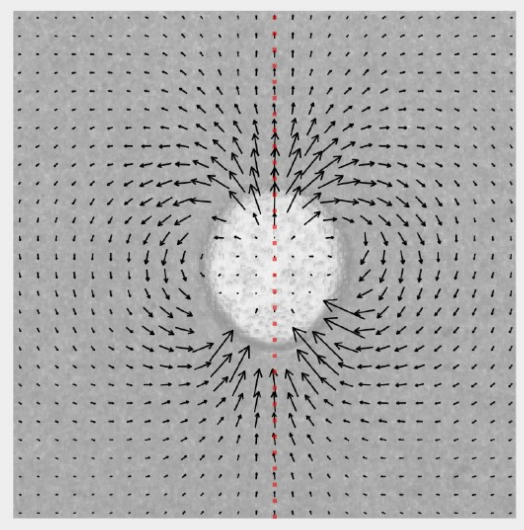
...in large Hele-Shaw cell

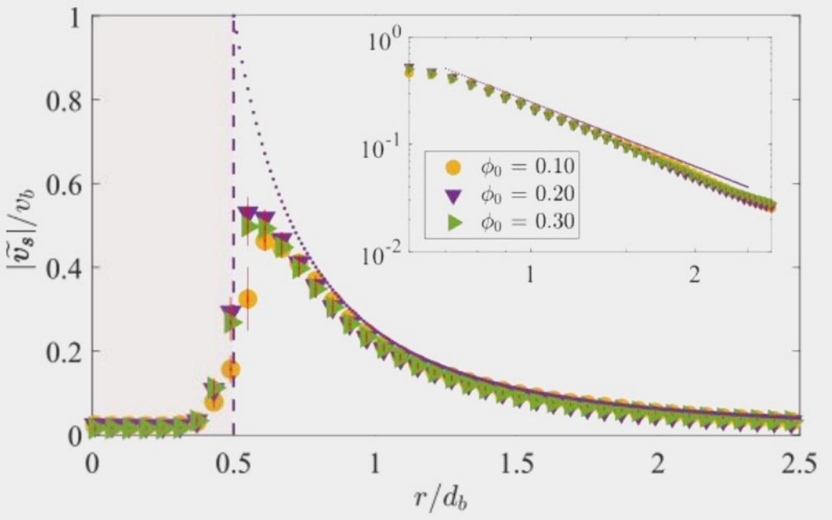


Monnet et al. (JFM 2022)

Hele-Shaw cell & Lubrication approximation

Taylor-Saffman flow





$$\frac{v_b}{4}\left(\frac{d_b}{r}\right)^2 [\sin(\theta)\boldsymbol{e}_r - \cos(\theta)\boldsymbol{e}_\theta]$$