Séminaire MathInFluid **5/03/21**

Séparation d'échelles en magnétohydrodynamique: l'effet alpha en dynamo

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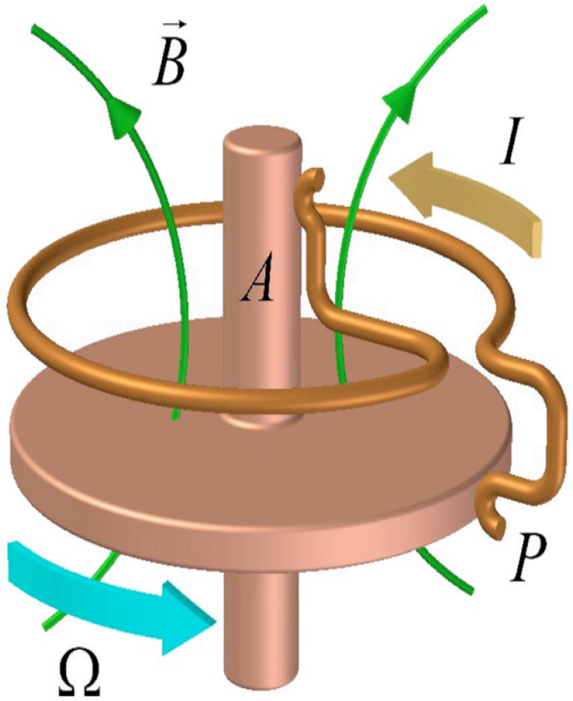
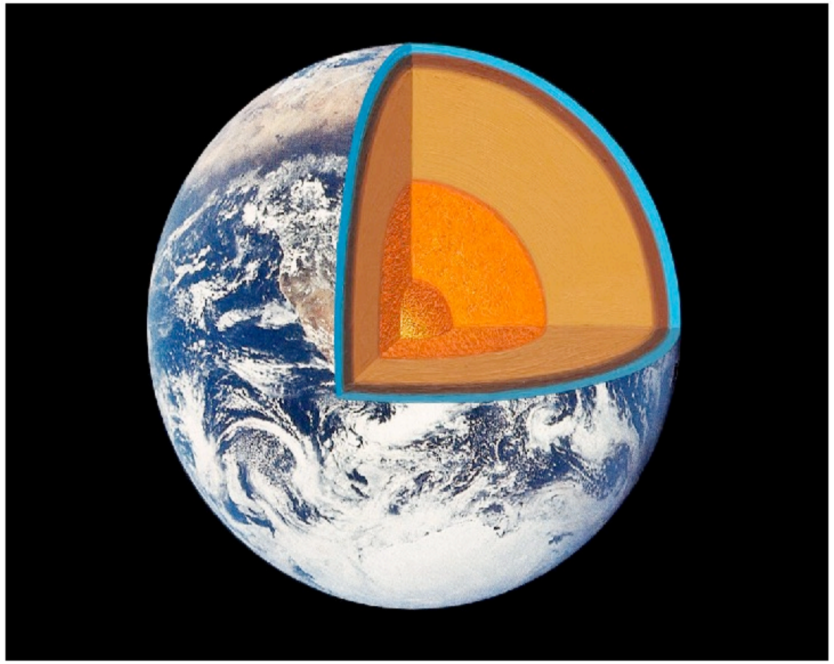
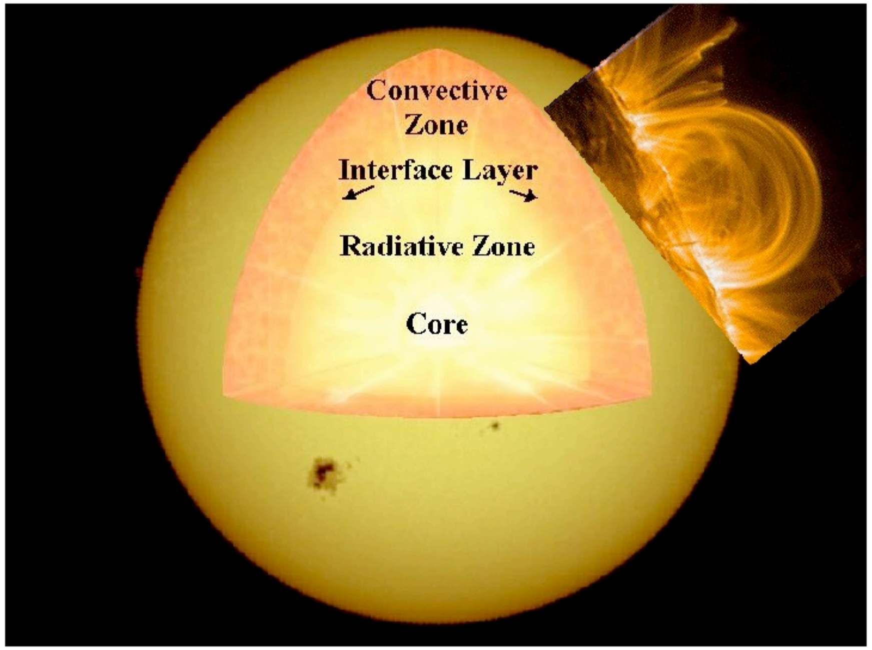
- Equations de la magnétohydrodynamique
- Amplification d'un champ magnétique par effet alpha: écoulement de Roberts
- Cas d'un écoulement turbulent: grande déviation et distribution du Lyapunov
- Extension à des effets hydrodynamiques (effet AKA).

La magnétohydrodynamique

Effet Dynamo (Larmor 1919)







Applications astrophysiques et industrielles

Les équations: champ magnétique B et champ de vitesse v

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \cdot B = 0$$

$$B = \mu H$$

$$\nabla \times H = j + \frac{\partial D}{\partial t}$$

$$D = \epsilon E$$

$$\nabla \cdot D = \rho_e$$

$$\nabla \times B = \mu_o j + \frac{1}{c^2}\frac{\partial E}{\partial t}$$

On considère des écoulements à v<<c (non relativiste)

Equations de Maxwell sont simplifiées

$$\frac{E}{L} \sim \frac{B}{\tau} \rightarrow E \sim VB$$

$$\frac{\frac{1}{c^2}\frac{\partial E}{\partial t}}{\nabla \times B} \sim \frac{\frac{1}{c^2}\frac{E}{tau}}{\frac{B}{\tau}} \sim \frac{E}{B}\frac{v}{c^2} \sim \left(\frac{v}{c}\right)^2 << 1$$

Loi d'Ohm $j = \sigma(E + v \times B)$

Equation d'induction

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B) - \frac{1}{\mu_o \sigma} \nabla \times (\nabla \times B)$$

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B) + \frac{1}{\mu_o \sigma} \nabla^2 B$$

$$\partial_t B = \nabla \times (v \times B) + \eta \nabla^2 B \qquad \nabla \cdot B = 0$$

- Conditions aux limites (problème non local)

- Si v est donnée, c'est une équation linéaire

- B=0 est toujours solution. L'instabilité dynamo est la perte de stabilité de cette solution

Problème non linéaire en tenant compte de l'effet de B sur le champ de vitesse:

$$\rho\left(\frac{\partial v}{\partial t}+(v\cdot\nabla v)\right)=-\nabla p+\text{forcing}+\rho\nu\nabla^2 v+j\times B$$

$$\partial_t B=\nabla\times(v\times B)+\eta\nabla^2 B$$

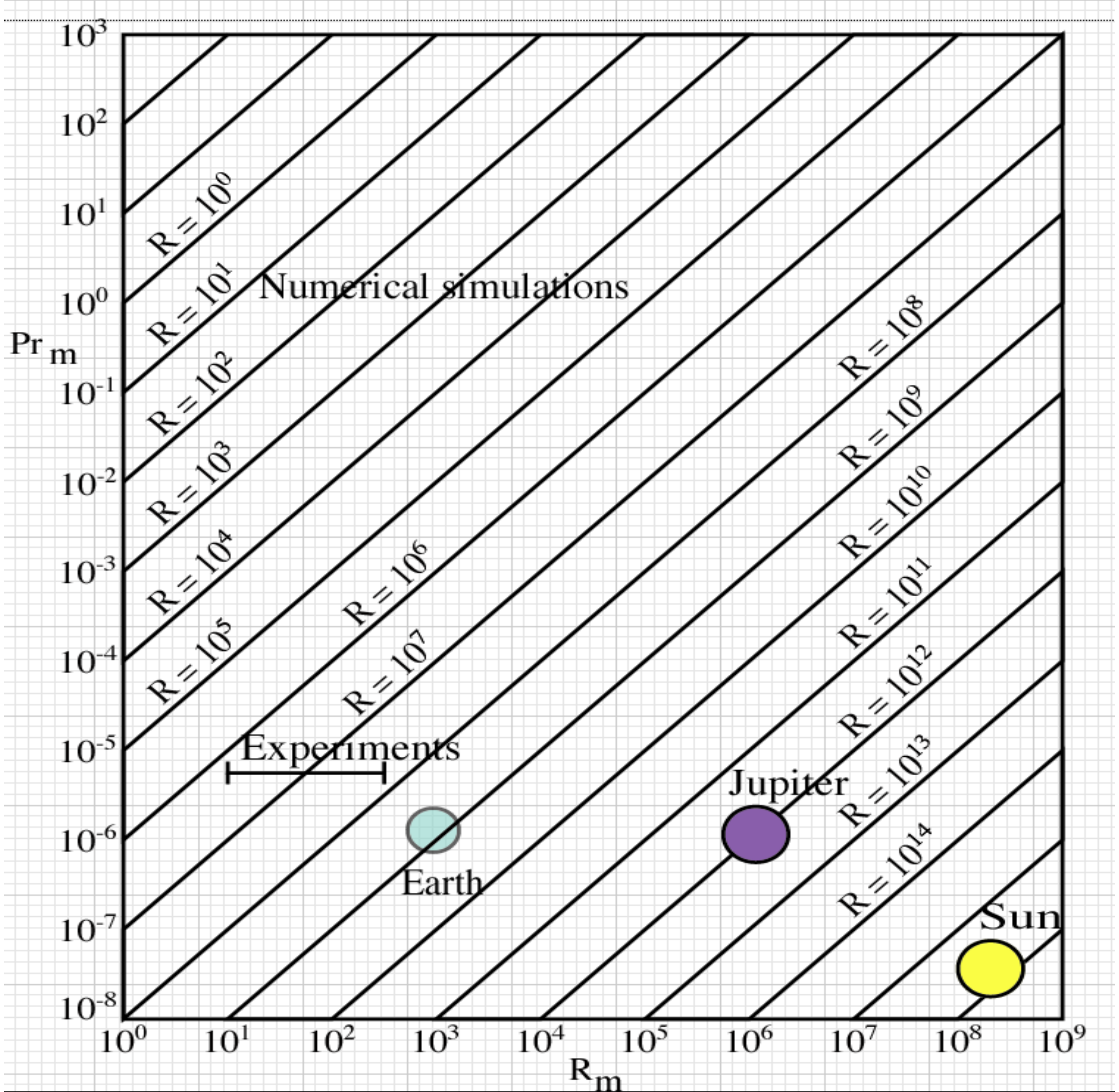
Si B est solution, -B est solution

Deux paramètres sans dimensions

$$Re = VL/\nu \qquad R_m = VL/\eta$$

ou bien Pm=Rm/Re= $\dfrac{\nu}{\eta}$ soit $P_m = \mu_0\sigma\nu$

Dimensionless parameters



Pm est très petit pour les métaux liquides Pm< 10^{-5}

Pourquoi?

$$P_m = \mu_0 \sigma \nu$$

$$\nu = v_i l_i \qquad \sigma = n e^2 \tau / m_e$$

Ces quantités sont déterminées par les propriétés microscopiques de la matière. En ordre de grandeur elles font intervenir $\hbar,\ q^2$ and m_e

On obtient

$$a_o = \hbar^2/(m_e q^2)$$

$$P_m = \alpha^2 \sqrt{\frac{m_e}{m_p}} \qquad \alpha = q^2/(\hbar c) \simeq (137)^{-1}$$

Pm est très petit pour les métaux liquides Pm< 10^{-5}

$$Re = R_m/P_m$$

Les écoulements sont donc turbulents au seuil dynamo

On ne peut résoudre numériquement les équations dans le bon régime de paramètres

C'est un exemple de problème de bruit multiplicatif

$$\partial_t B = \nabla \times (v \times B) + \eta \nabla^2 B$$

L'effet alpha.
Séparation d'échelle entre l'écoulement (l) et le champ magnétique (L)

Estimation optimiste:

$$\partial_t B = \nabla \times (v \times B) + \eta \nabla^2 B$$

B varie sur une échelle L, tandis que v varie sur une échelle l

Seuil dynamo quand v B/l= η B/L^2

Soit $R_{m1} = \mu_0 \sigma V L^2 / l$ assez grand

Séparation d'échelles

On écrit v=<v>+v'   où <> est une moyenne sur les petites échelles de l'écoulement

$$\vec{B} = \vec{B}_0 + \vec{b}, \qquad \langle\vec{b}\rangle = 0$$

On suppose <v>=0, on note v'=v

$$\partial_t B = \nabla \times (v \times B) + \eta \nabla^2 B$$

Equation pour champ moyen

$$\frac{\partial \vec{B}_0}{\partial t} = \vec{\nabla} \times \langle \vec{v} \times \vec{b} \rangle + \nu_m \Delta \vec{B}_0.$$

Equation pour fluctuations

$$\frac{\partial \vec{b}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{B}_0 + \vec{v} \times \vec{b} - \langle \vec{v} \times \vec{b} \rangle \right) + \nu_m \Delta \vec{b}.$$

b linéaire en Bo

On peut résoudre en « first order smoothing »

$$\langle \vec{v} \times \vec{b} \rangle_i = \alpha_{i,j} B_0\, j + \beta_{i,j,k} \frac{\partial B_0\, j}{\partial x_k} + \cdots$$

Exemple (écoulement ABC) $$\vec{v}(x,y,z) = V \begin{pmatrix} \sin ky + \cos kz \\ \sin kz + \cos kx \\ \sin kx + \cos ky \end{pmatrix}$$

Calcul b (first order smoothing) $$\langle \vec{v} \times \vec{b} \rangle \approx \frac{2V^2}{\nu_m k} \vec{B}_0$$

Role de l'hélicité

Eq. Champ grande échelle $$\frac{\partial \vec{B}_0}{\partial T} = \alpha \vec{\nabla} \times \vec{B}_0 + \nu_m \Delta \vec{B}_0.$$

$$\vec{B}_0 = \hat{B}_0 \exp(\mathrm{P}T \pm i\vec{K} \cdot \vec{R})$$

$$K < |\alpha| / \nu_m$$

$$\mathrm{P} = \pm|\alpha K| - \nu_m K^2,$$

Seuil pour

Soit 2 V^2/(η^2 k K) >1 donc Rm=V (l L)^(1/2)/η

Extension:

Au delà du first order smoothing

Effet d'un champ de vitesse moyen (dynamo alpha-omega)

Régime non linéaire de saturation de l'effet alpha

Ecoulement turbulent

Strange eigenmode et problème dynamo:

$$\partial_t B = \nabla \times (v \times B) + \eta \nabla^2 B$$

Turbulent dynamos:
which moment controls the dynamo onset?

Avec Kannabiran Seshasayanan

Multiplicative game

Start with 1 (euro, pound….)

Each time step:
With probability 1/100 multiply what you own by 10
With probability 99/100, divide it by 10

$$< X_n >=< X_{n-1} > (10\frac{1}{100} + 10^{-1}\frac{99}{100})$$

$$< X_n >=< X_{n-1} > \times 0.199$$

Multiplicative game

Start with 1 (euro, pounds....)

Each time step:

With probability 1/100 multiply what you own by 10

With probability 99/100, divide it by 10

For a moment of order p

$$< X_n^p > = < X_{n-1}^p > (10^p \frac{1}{100} + 10^{-p} \frac{99}{100})$$

$$< X_n^p > = < X_{n-1}^p > (10^{p-2} + 10^{-p} - 10^{-p-2})$$

For p>1 $$< X_n^p > \simeq < X_{n-1}^p > \times 10^{p-2}$$

The third moment grows while the first one decreases

Growth or decay depend on the considered moment

A canonical model of bifurcation

$$\dot{X} = (\mu + \zeta(t))X - X^3$$

A canonical model of bifurcation

Linearize $$\dot{X} = (\mu + \zeta(t))X$$

$$\frac{\partial \log X}{\partial t} = (\mu + \zeta(t))$$

$$X = X_0 \exp\left(\mu t + \int_0^T \zeta(t')dt'\right)$$

$$< X^n >= X_0^n \exp\left(nt(\mu + Dn/2)\right)$$

The onset depends on the moment $$\mu_c = \frac{-Dn}{2}$$

We can not forget the nonlinear term!

A canonical model of bifurcation

$$\dot{X} = (\mu + \zeta(t))X - X^3$$

Solve the associated Fokker-Planck equation

$$P(X) = C|X|^{\frac{2\mu}{D}-1} e^{-\frac{X^2}{D}}$$

The onset of instability is $\mu=0$

It is the same as the one of <log(X)> calculated from the linear equation

To calculate <log(X)>, we can do <X^ε> = <e^(εlog(X))> ≈ <1+εlog(X)> for ε<<1

So <log(X)>=(<X^ε> -1)/ε

Why is that important for the dynamo instability?

Turbulent fluctuations act as a multiplicative noise in the induction equation

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B) + \eta \nabla^2 B = \nabla \times (< v > \times B) + \nabla \times (v_f \times B) + \eta \nabla^2 B$$

If we consider the linear problem (the kinematic dynamo problem),
then we should have the same behavior as for the canonical model

in particular, the moments of the field are unstable at different values of Rm

For the so-called Kazantsev dynamo, most analytical results consider the energy of the magnetic field, thus <B^2>, for the kinematic problem.

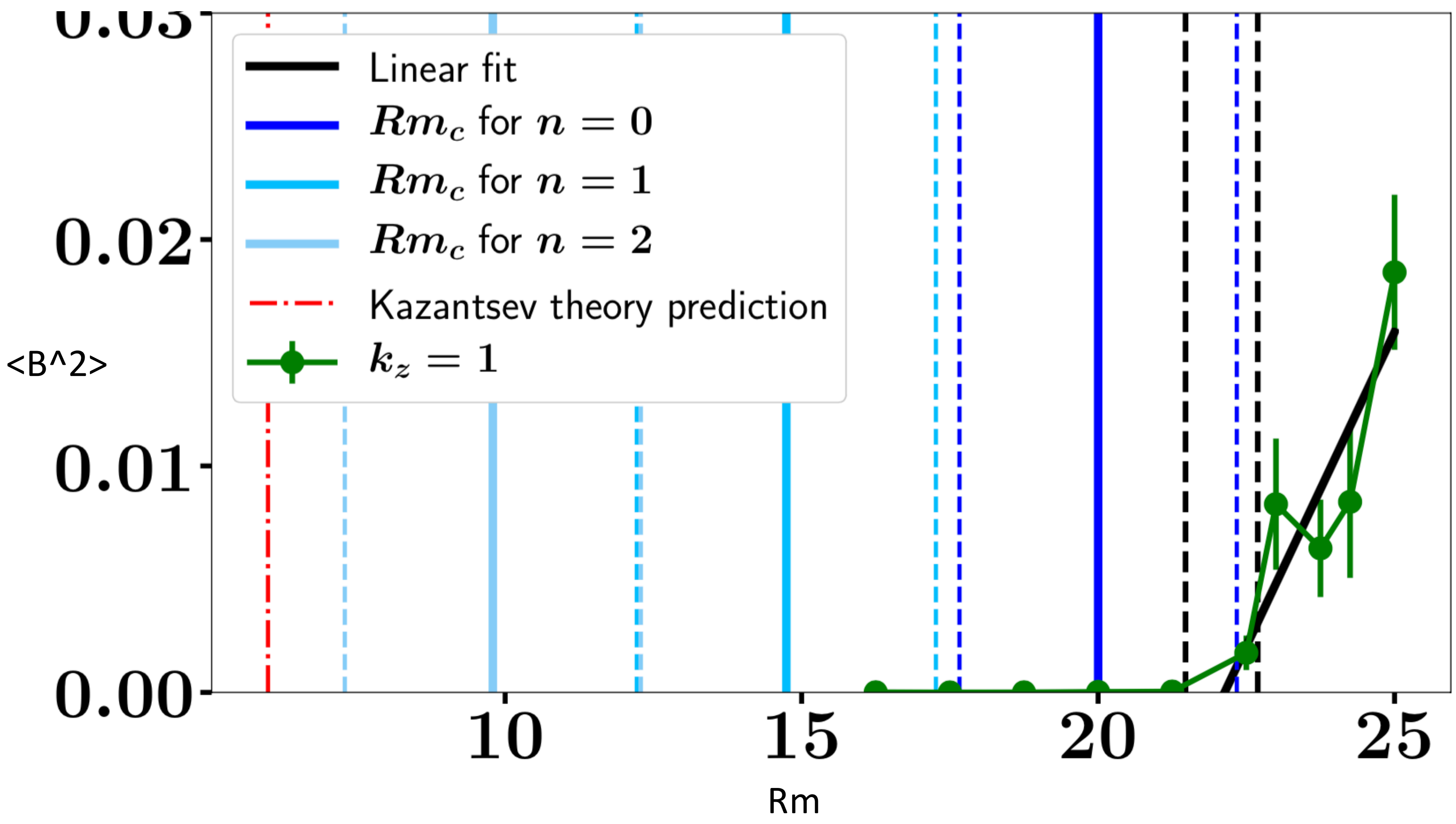
Are the predictions of the Kazantsev dynamo of any use?

We test this numerically (A.Alexakis and K. Seshasayanan)

$$\mathbf{u} = \boldsymbol{\nabla} \times (\psi \mathbf{e}_z) + u_z \mathbf{e}_z,$$

$$\psi = U\zeta_1(t)\Big(\sin(\phi_1(t))\cos(k_f x + \phi_2(t)) + \cos(\phi_1(t))\sin(k_f y + \phi_2(t))\Big)/k_f,$$
$$u_z = U\zeta_2(t)\Big(\sin(\phi_1(t))\sin(k_f x + \phi_2(t)) + \cos(\phi_1(t))\cos(k_f y + \phi_2(t))\Big),$$

$$\partial_t \mathbf{B} = \boldsymbol{\nabla} \times \left(\mathbf{u} \times \mathbf{B} - |\mathbf{B}|^2 \mathbf{J}\right) + \eta \Delta \mathbf{B}$$



0.03
0.02
0.01
0.00
10
15
20
25
Rm
<B^2>
Linear fit
Rm_c for $n = 0$
Rm_c for $n = 1$
Rm_c for $n = 2$
Kazantsev theory prediction
$k_z = 1$

Using scale separation we can understand this in the dynamo context

We consider $\mathbf{v} = \zeta(t)\mathbf{u_r}$ with Ur a space-periodic flow

$$\langle \zeta(t)\zeta(0)\rangle = 2D\delta(t)$$

Write $\mathbf{B} = \langle \mathbf{B} \rangle + \mathbf{b}$

With < . > the spatial average over a flow period

We then have

$$\frac{\partial \mathbf{b}}{\partial t} - \eta \nabla^2 \mathbf{b} = \langle \mathbf{B} \rangle . \nabla \mathbf{v} = \zeta(t) \langle \mathbf{B} \rangle . \nabla \mathbf{u_r}$$

and

$$\frac{\partial \langle \mathbf{B} \rangle}{\partial t} = \nabla \times (\langle \mathbf{v} \times \mathbf{b} \rangle) + \eta \nabla^2 \langle \mathbf{B} \rangle$$

We Fourier transform and get

$$\frac{\partial \hat{\mathbf{b}}}{\partial t} + \eta k^2 \hat{\mathbf{b}} = \zeta(t) i \langle \mathbf{B} \rangle . \mathbf{k} \hat{\mathbf{u}}_\mathbf{r}$$

The amplitude of the unstable mode, $\langle \mathbf{B} \rangle = \bar{B} e^{-iKz}$, satisfies

$$\frac{dB_p}{dt} = Y_k(t)\zeta(t)\alpha\eta k^2 K B_p - \eta K^2 B_p$$

We can thus write the evolution of the field

$$B_p(t) = B_p(0)e^{\alpha\eta k^2 K I(t) - \eta K^2 t}$$

where $I(t) = \int_0^t Y_k(t')\zeta(t')dt'$

and $\dfrac{dY_k}{dt} + \eta k^2 Y_k = \zeta(t)$

We recognize the equation for the speed of a Brownian particle (Yk) and the injected energy by the fluctuating force (I(t))

This has been studied recently by J. Farago (J. Stat. Phys. 2002)

I is a random variable and satisfies a law of large deviation

$$P(I = t\varepsilon) \simeq e^{-tg(\varepsilon)} \qquad g(\varepsilon) = \frac{\eta k^2 D}{4\varepsilon}\left(\frac{\varepsilon}{D} - 1\right)^2$$

$$\langle \mathbf{B}^n \rangle_s \propto \int e^{n\eta k^2 \alpha K \varepsilon t - \eta n K^2 t} P(\varepsilon) d\varepsilon \simeq$$

$$\int e^{-t\left(g(\varepsilon) - n\eta k^2 \alpha K \varepsilon\right) - \eta n K^2 t} d\varepsilon$$

We thus have a formula for the growth rate of the n-th moment

$$\langle \mathbf{B}^n \rangle_s \propto e^{\lambda_n t}$$

$$\lambda_n = -n\eta K^2 + \frac{\eta k^2}{2}\left(1 - \sqrt{1 - 4DnK\alpha}\right)$$

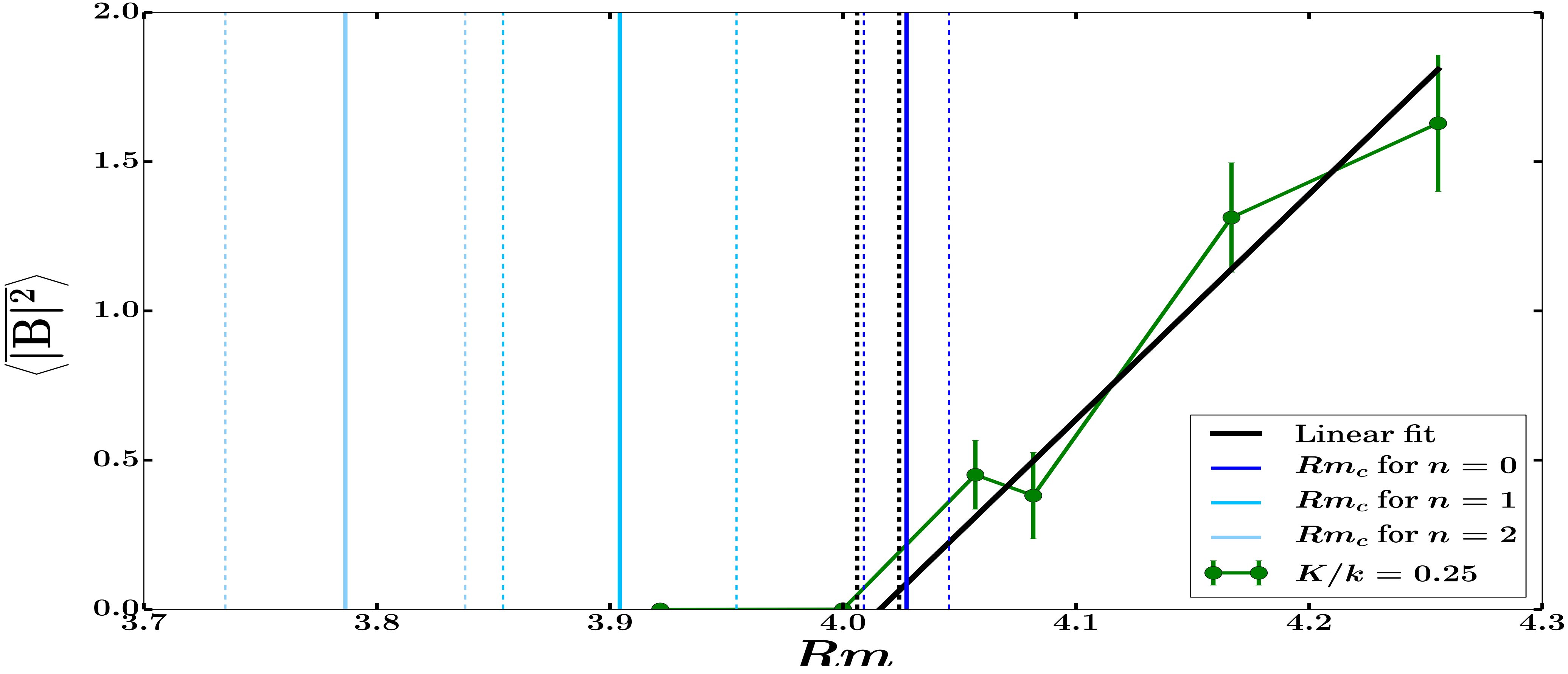
Multiscaling (growth rate not linear in n, larger n faster than exponential) due to intermittency,

the onset depends on the considered moment

Test for a Roberts flow with random amplitude

$$\mathbf{v} = \zeta(t)U\left(cos(ky), sin(kx), cos(kx) + sin(ky)\right)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\mathbf{v} \times \mathbf{B} - |\mathbf{B}|^2 \mathbf{B}\right) + \eta \nabla^2 \mathbf{B}$$



Conclusion

For a turbulent dynamo:

* the growth rate of the moment of order n calculated from the kinematic equation depends non linearly on n

So that the onset then depends on the considered moment.

This is due to the multiplicative effect of the fluctuations

and

To considering the linear induction equation only

Indeed, there are rare events where the fluctuations act coherently and increase the field to huge values.

These rare but large events contribute to the moments of high order
but they would be erased if we were to consider the full nonlinear equations

* the onset of instability is correctly predict by the behavior of the log of the field (moment of 0th order)

* Predictions on the Kazantsev dynamo, using the kinematic equation and based on the energy of the field (2^{nd} moment, or other moments of $n>0$) are at best approximations

K. Seshasayanan and F. Pétrélis, EPL 122, 64004, 2018

Effet AKA (Frish, She et Sulem)

Problème hydro $$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla\Pi + \nu\nabla^2\mathbf{v} + \mathbf{f}\,.$$

Séparation échelle, Eq. Petites échelles $$\frac{\partial \tilde{\mathbf{v}}}{\partial t} + \mathbf{V}\cdot\nabla\tilde{\mathbf{v}} - \nu\nabla^2\tilde{\mathbf{v}} = -\nabla\tilde{\Pi} + \mathbf{f}\,.$$

Pour les grandes échelles, tenseur de Reynolds $\partial_j R_{ij}$ où $R_{ij} = V_i V_j + \langle \tilde{\mathbf{v}}_i \tilde{\mathbf{v}}_j \rangle$.

Exemple
$$\begin{aligned} \mathbf{f}_x &= A\cos(ky + \nu k^2 t) \\ \mathbf{f}_y &= A\cos(kx - \nu k^2 t) \\ \mathbf{f}_z &= \mathbf{f}_x + \mathbf{f}_y \end{aligned}$$
crée une instabilité grande échelle

Utilisation de détecteurs CO2 pour améliorer l’aération dans les locaux confinés

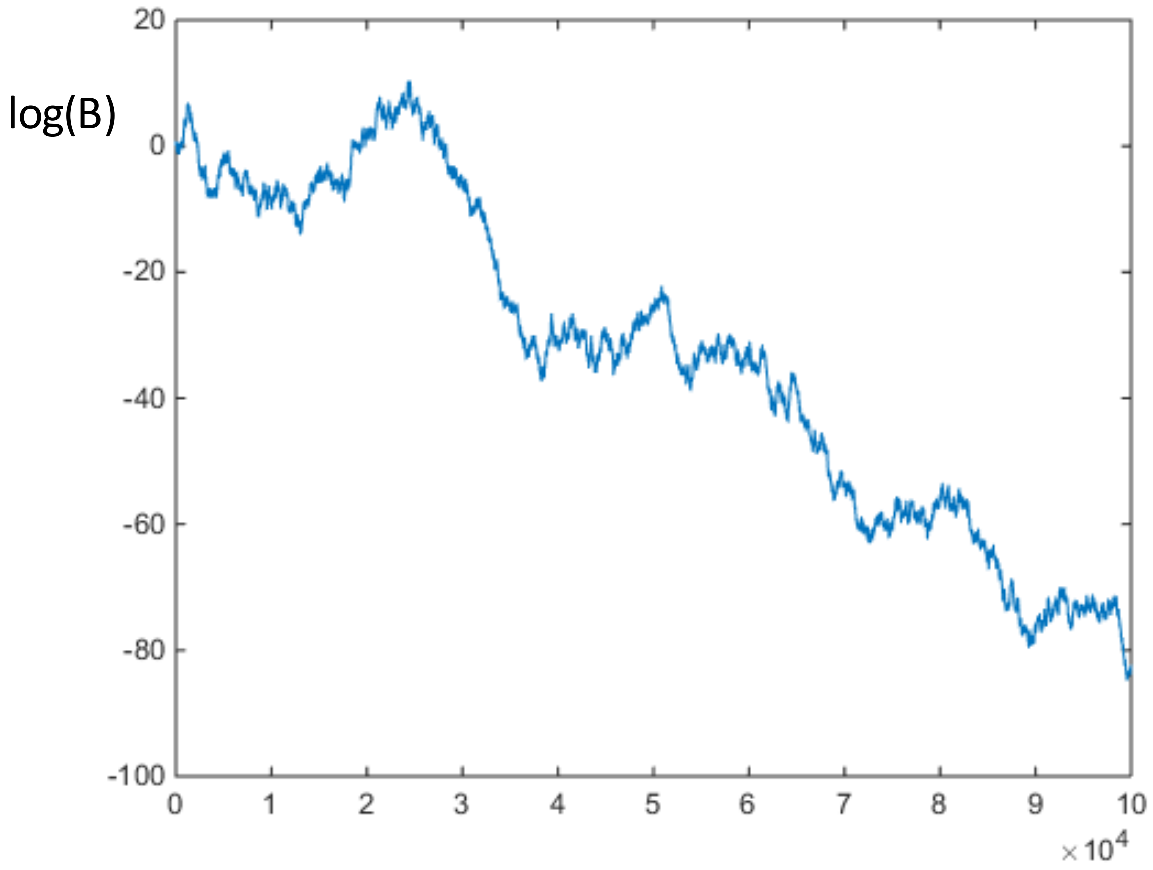
http://projetco2.fr/

Webinaires (chaine youtube du département de physique de l’ENS)

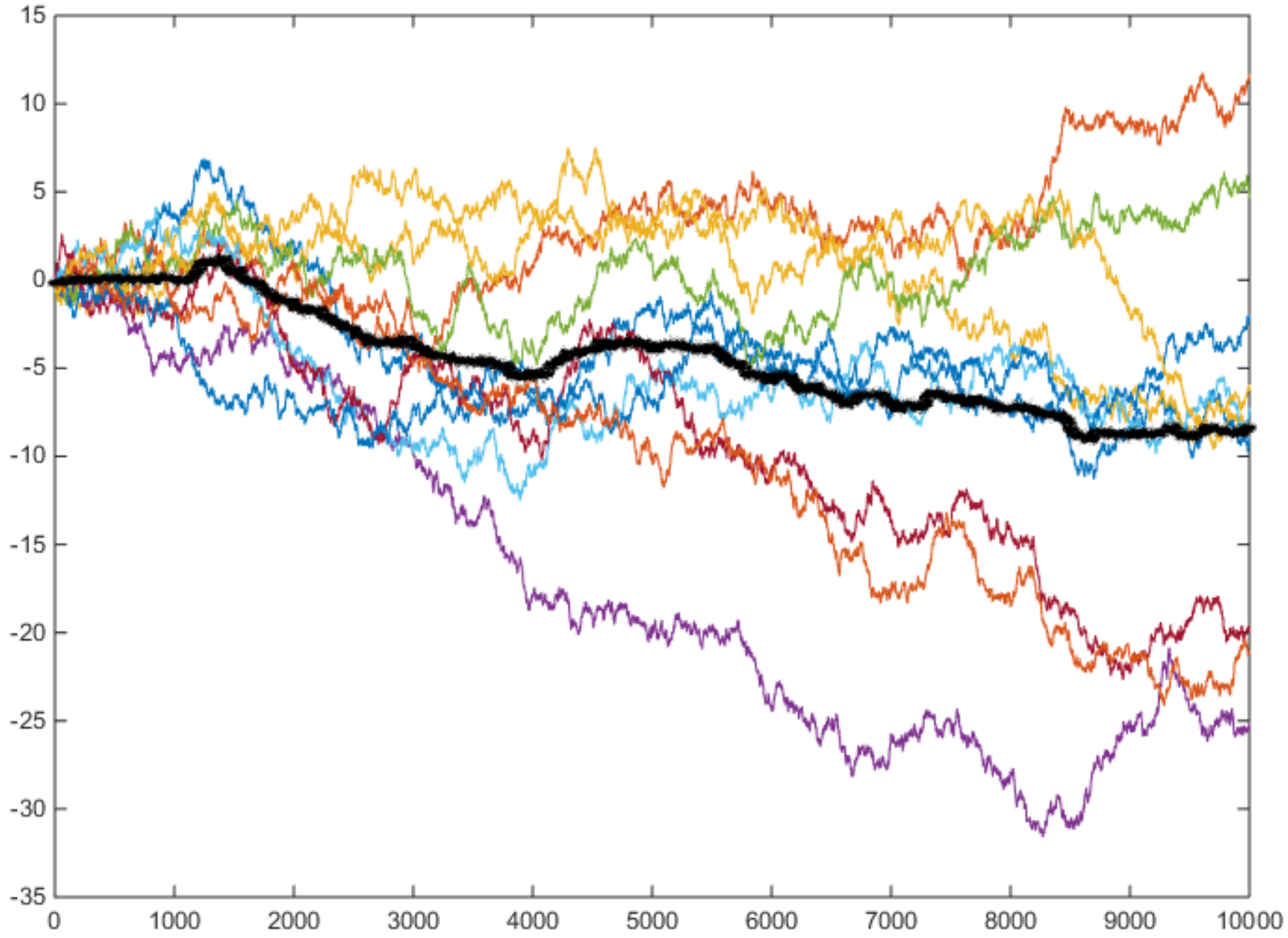
https://www.youtube.com/watch?v=XAFM7LRgZ2k

Illustration on a typical turbulent dynamo simulation

Averaged spatially, a moment of the field (should be fine if not too spatially intermittent)

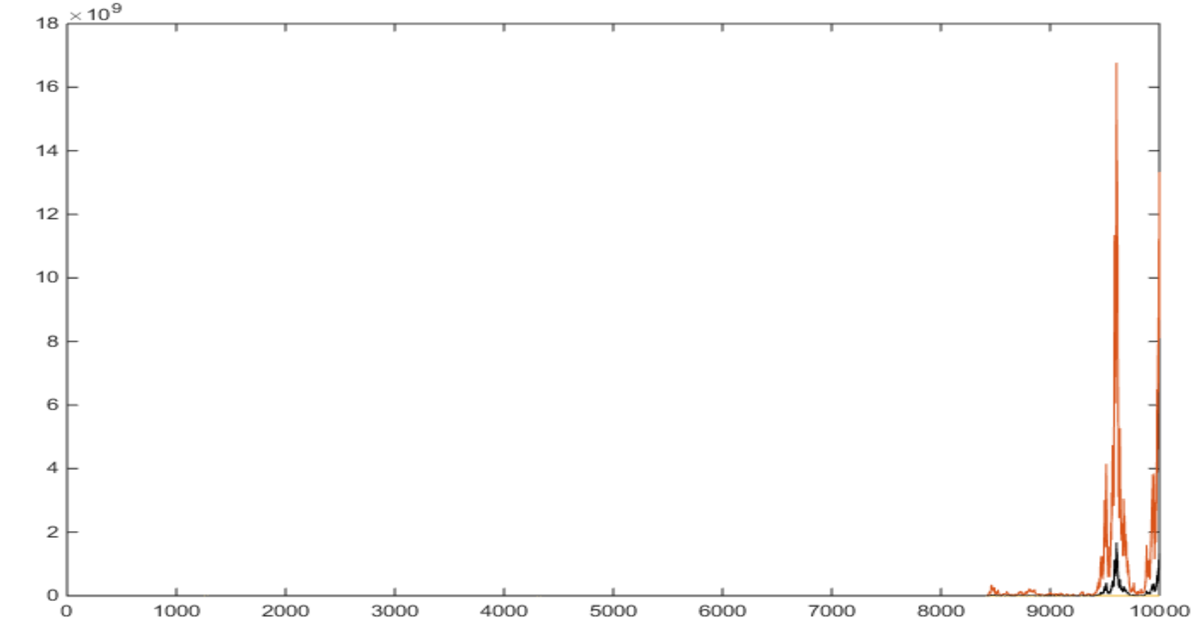


<log(B)>



15
10
5
0
-5
-10
-15
-20
-25
-30
-35
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000

<B^2>



×10^9
18
16
14
12
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8
6
4
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1000
2000
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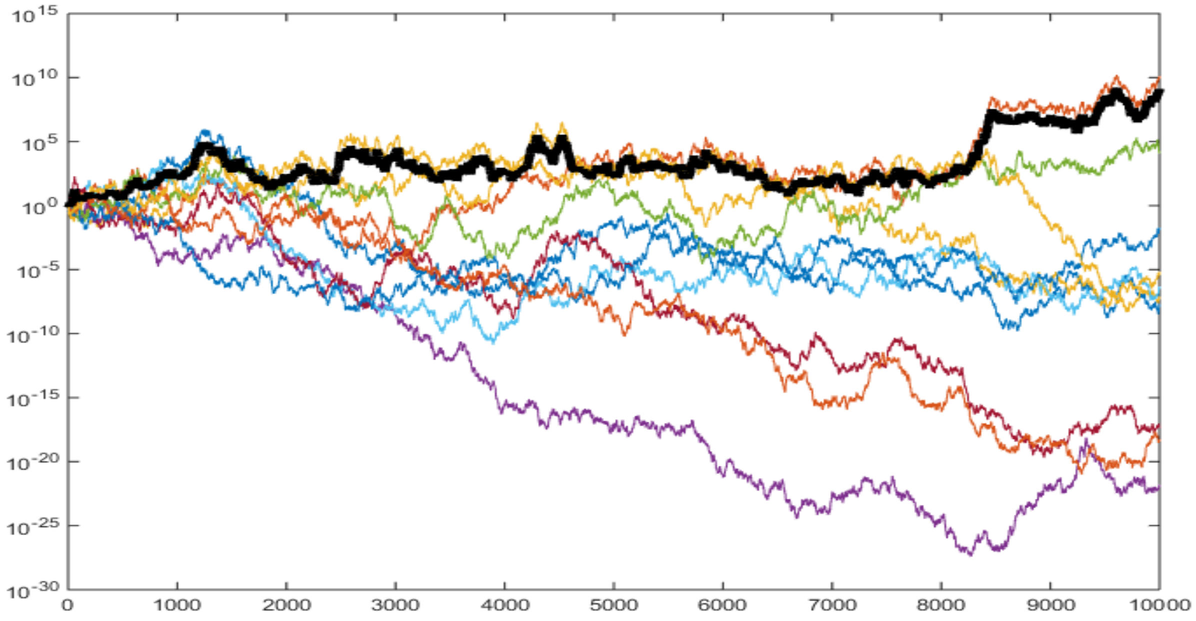
4000
5000
6000
7000
8000
9000
10000



10^15
10^10
10^5
10^0
10^-5
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10^-20
10^-25
10^-30
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000