Some mathematical issues in the modeling of dense suspensions

Charlotte Perrin (CNRS, Institut de Mathématiques de Marseille)

Groupe de travail MathsInFluids

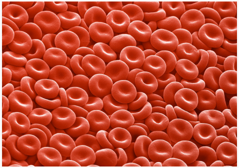
Lyon

September 22, 2023

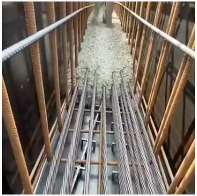
Dense suspensions

collection of non-brownian particles immersed in a fluid

geophysics



biology



industry

Dense suspensions

- suspensions and granular flows → complex macroscopic rheology

  ⚠ models in the literature are not always well-posed...

  ref: Schaeffer (Coulomb friction, 1987),

  Barker, Schaeffer, Bohorquez, Gray ($\mu(I)$-rheology 2015)

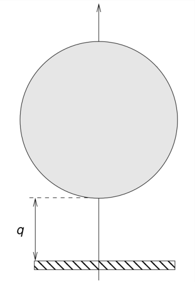
- dense regime
  → what physics for close interactions and contacts at the microscopic level ?
  → at the macroscopic level: phase transition in the compacted zones with complex phenomena emanating from the micro. structure

Outline

1. **microscopic level: handling contacts and lubrication forces**
   based on the works of B. Maury, A. Lefebvre-Lepot, M. Hillairet, ...
2. **macroscopic level: jamming**

Microscopic standpoint

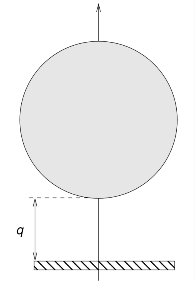
Dry granular media: Inelastic contact



inelastic contact → unilateral constraints (Signorini's conditions)

$$q \geq 0, \; F \geq 0, \; Fq = 0$$

Dry granular media: Inelastic contact



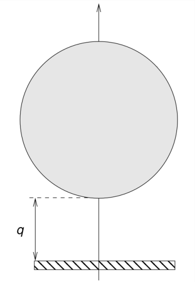
inelastic contact → unilateral constraints (Signorini's conditions)

$$q \geq 0, \ F \geq 0, \ Fq = 0$$

stiff problem → "regularizing methods"

- relaxation by allowing small negative values q: $F_\varepsilon(q) = \frac{q_-}{\varepsilon}$
- penalty method through a repulsive force $F_\varepsilon(q) \underset{q \to 0^+}{\longrightarrow} +\infty$

Dry granular media: Inelastic contact



inelastic contact → unilateral constraints (Signorini's conditions)

$$q \geq 0, \; F \geq 0, \; Fq = 0$$

stiff problem → "regularizing methods"

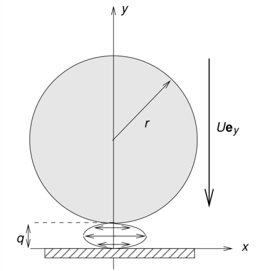
- relaxation by allowing small negative values q: $F_\varepsilon(q) = \frac{q_-}{\varepsilon}$
- penalty method through a repulsive force $F_\varepsilon(q) \underset{q \to 0^+}{\longrightarrow} +\infty$

Non-smooth convex analysis point of view → Contact dynamics Models

- collisions are not treated as events; def. of a set of admissible velocities
- typical model (ref: Moreau; ext. to multiple particles: Maury '06, Lefebvre)

$$\begin{cases} m\ddot{q} = mf + p \\ \text{supp } p \subset \{t, \; q(t) = 0\} \\ q \geq 0, \; p \geq 0 \\ \dot{q}(t+) = \mathrm{P}_{C_q} \dot{q}(t-) \end{cases} \quad \text{with} \quad C_q = \begin{cases} \mathbb{R}^+ & \text{if } q = 0 \\ \mathbb{R} & \text{otherwise} \end{cases}$$

Dense suspensions - lubrication effects



the particle is now immersed in a viscous fluid

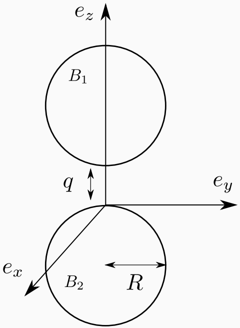
at first order, Newton's law:

$$m\ddot{q} = -6\pi r^2 \mu \frac{\dot{q}}{q} + m f_y$$

$\implies$ no contact in finite time

ref: Brenner, Cox '67
rigorous justification: Hillairet '07, Hesla '05

Lubrication - effects in the tangential direction



- two spheres of same radius almost in contact
- B_1 at rest, imposed motion for B_2

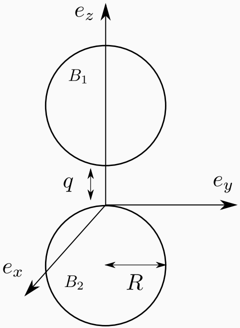
$\Longrightarrow$ calculation of the force on B_2 (Cox '73):

$$F_{x,y}^{\mathsf{lub}} = \pi R\, \mu \ln q\, U_{x,y}$$

$$F_z^{\mathsf{lub}} = -\frac{3\pi}{2} R^2 \frac{\mu}{q} U_z + O(\ln q)$$

rigorous justification in Hillairet, Kelaï '15

Lubrication - effects in the tangential direction



- two spheres of same radius almost in contact
- B_1 at rest, imposed motion for B_2

$\Longrightarrow$ calculation of the force on B_2 (Cox '73):

$$F_{x,y}^{\text{lub}} = \pi R\, \mu \ln q\, U_{x,y}$$
$$F_z^{\text{lub}} = -\frac{3\pi}{2} R^2\, \frac{\mu}{q}\, U_z + O(\ln q)$$

rigorous justification in Hillairet, Kelaï '15

no contact theoretically but numerical contacts may occur
due to the stiffness of the problem

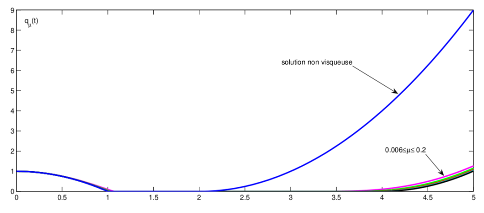
$\leadsto$ development of a "viscous contact model"

Gluey/viscous contact model - one particle over a plane

$$\begin{cases} \ddot{q}_\mu = -\mu \dfrac{\dot{q}_\mu}{q_\mu} + f, \\ \dot{q}_\mu(0) = 0, \quad q_\mu(0) = 1, \end{cases}$$

and compute the solution for $\mu \ll 1$

ext. forcing $f(t) = -2\mathbf{1}_{[0,2]}(t) + 2\mathbf{1}_{[2,+\infty[}(t)$



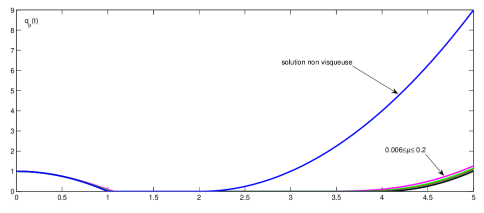
from A. Lefebvre's PhD thesis

Gluey/viscous contact model - one particle over a plane

$$\begin{cases} \ddot{q}_\mu = -\mu \dfrac{\dot{q}_\mu}{q_\mu} + f, \\ \dot{q}_\mu(0) = 0, \quad q_\mu(0) = 1, \end{cases}$$

and compute the solution for $\mu \ll 1$

ext. forcing $f(t) = -2\mathbf{1}_{[0,2]}(t) + 2\mathbf{1}_{[2,+\infty[}(t)$



from A. Lefebvre's PhD thesis

$\Longrightarrow$ non-negligible effect of the lubrication force as $\mu \to 0$

this effect results from the competition between $\mu \to 0$ and $q_\mu \to 0$

Gluey/viscous contact model - adhesion potential

$$\begin{cases} m\ddot{q}_\mu = -\mu \dfrac{\dot{q}_\mu}{q_\mu} + mf, \\ q(0) = q^0 > 0, \;\; \dot{q}(0) = u^0 \end{cases} \quad \rightsquigarrow \quad m\dot{q}_\mu(t) + \gamma_\mu(t) = mu^0 + \gamma_\mu(0) + m\int_0^t f(\tau)\, d\tau$$

where we have introduced the adhesion potential $\boxed{\gamma_\mu := \mu \ln(q_\mu)}$

Gluey/viscous contact model - adhesion potential

$$\begin{cases} m\ddot{q}_\mu = -\mu \dfrac{\dot{q}_\mu}{q_\mu} + mf, \\ q(0) = q^0 > 0, \;\; \dot{q}(0) = u^0 \end{cases} \quad \rightsquigarrow \quad m\dot{q}_\mu(t) + \gamma_\mu(t) = mu^0 + \gamma_\mu(0) + m\int_0^t f(\tau)\,d\tau$$

where we have introduced the adhesion potential $\boxed{\gamma_\mu := \mu \ln(q_\mu)}$

$\underline{\mu \to 0}$:

$$\begin{cases} m\dot{q} + \gamma(t) = mu^0 + m\int_0^t f(\tau)\,d\tau, \\ \gamma \le 0, \;\; q \ge 0, \;\; q\gamma = 0, \\ q(0) = q^0 > 0, \;\; \dot{q}(0) = u^0, \end{cases} \quad \Leftrightarrow$$

"$\gamma = m\,(u_{\text{free}} - u_{\text{real}})$"

Gluey/viscous contact model - adhesion potential

$$\begin{cases} m\ddot{q}_\mu = -\mu \dfrac{\dot{q}_\mu}{q_\mu} + mf, \\ q(0) = q^0 > 0,\ \dot{q}(0) = u^0 \end{cases} \quad \rightsquigarrow \quad m\dot{q}_\mu(t) + \gamma_\mu(t) = mu^0 + \gamma_\mu(0) + m\int_0^t f(\tau)\, d\tau$$

where we have introduced the adhesion potential $\boxed{\gamma_\mu := \mu \ln(q_\mu)}$

$\underline{\mu \to 0}$:

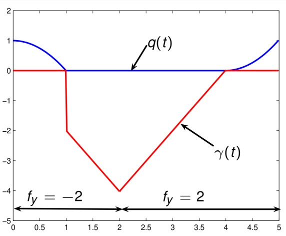
$$\begin{cases} m\dot{q} + \gamma(t) = mu^0 + m\int_0^t f(\tau)\, d\tau, \\ \gamma \le 0,\ q \ge 0,\ q\gamma = 0, \\ q(0) = q^0 > 0,\ \dot{q}(0) = u^0, \end{cases} \Leftrightarrow \begin{cases} \ddot{q} = f + p, \\ \text{supp}(p) \subset \{t, q(t) = 0\}, \\ \dot{q}^+ = \mathrm{P}_{C_{q,\gamma}} \dot{q}^-, \\ \dot{\gamma} = -p, \\ \gamma \le 0,\ q \ge 0,\ q\gamma = 0, \\ q(0) = q^0 > 0,\ \dot{q}(0) = u^0 \end{cases}$$

with $C_{q,\gamma} = \begin{cases} \{0\} & \text{if } \gamma^- < 0 \\ \mathbb{R}^+ & \text{if } q = 0, \gamma^- = 0 \\ \mathbb{R} & \text{otherwise} \end{cases}$

"$\gamma = m\,(u_{\text{free}} - u_{\text{real}})$"

ref: Maury '07, Lefebvre '09

Gluey/viscous contact model - adhesion potential



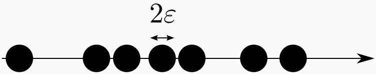
ref: Lefebvre '09

$$\begin{cases} \ddot{q} = f + p, \\ \mathrm{supp}(p) \subset \{t, q(t) = 0\}, \\ \dot{q}^+ = \mathrm{P}_{C_{q,\gamma}} \dot{q}^-, \\ \dot{\gamma} = -p, \\ \gamma \leq 0,\ \ q \geq 0,\ \ q\gamma = 0, \\ q(0) = q^0 > 0,\ \ \dot{q}(0) = u^0 \end{cases}$$

with $C_{q,\gamma} = \begin{cases} \{0\} & \textbf{if } \gamma^- < 0 \\ \mathbb{R}^+ & \textbf{if } q = 0, \gamma^- = 0 \\ \mathbb{R} & \textbf{otherwise} \end{cases}$

Macroscopic standpoint

1D micro-macro limit - effective viscosity of the suspension



neglecting inertia, the discrete system writes as

$$-A(q)\dot{q} + f(t,q) = 0$$

$$A(q) = \mu \begin{pmatrix} \frac{1}{d_1}+\frac{1}{d_2} & -\frac{1}{d_2} & & & \\ -\frac{1}{d_2} & \frac{1}{d_2}+\frac{1}{d_3} & -\frac{1}{d_3} & & \\ & \cdots & \cdots & \cdots & \\ & & & -\frac{1}{d_{N-1}} & \frac{1}{d_{N-1}}+\frac{1}{d_N} \end{pmatrix}$$

- define

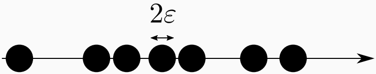
$$\phi^\varepsilon = \sum_i \phi_i^\varepsilon \mathbf{1}_{[q_{i-1}^\varepsilon, q_i^\varepsilon]} = \sum_i \left(1 - \frac{d_i^\varepsilon}{q_i^\varepsilon - q_{i-1}^\varepsilon}\right) \mathbf{1}_{[q_{i-1}^\varepsilon, q_i^\varepsilon]}$$

- $N = N^\varepsilon = \frac{1}{\varepsilon}$ and $\varepsilon \to 0$ $\rightsquigarrow$ 1D macroscopic system

$$\begin{cases} \partial_t \phi + \partial_x(\phi u) = 0, \\ -\partial_x \left(\frac{\mu}{1-\phi} \partial_x u \right) = \phi f \end{cases}$$

ref: Lefebvre-Lepot & Maury '11

1D micro-macro limit - effective viscosity of the suspension



with inertia, the discrete system writes as

$$-A(q)\dot{q} + f(t,q) = \ddot{q}$$

$$A(q) = \mu \begin{pmatrix} \frac{1}{d_1} + \frac{1}{d_2} & -\frac{1}{d_2} & & & \\ -\frac{1}{d_2} & \frac{1}{d_2} + \frac{1}{d_3} & -\frac{1}{d_3} & & \\ & \dots & \dots & \dots & \\ & & & -\frac{1}{d_{N-1}} & \frac{1}{d_{N-1}} + \frac{1}{d_N} \end{pmatrix}$$

- define

$$\phi^\varepsilon = \sum_i \phi_i^\varepsilon \mathbf{1}_{[q_{i-1}^\varepsilon, q_i^\varepsilon]} = \sum_i \left(1 - \frac{d_i^\varepsilon}{q_i^\varepsilon - q_{i-1}^\varepsilon}\right) \mathbf{1}_{[q_{i-1}^\varepsilon, q_i^\varepsilon]}$$

- $N = N^\varepsilon = \frac{1}{\varepsilon}$ and $\varepsilon \to 0$ $\rightsquigarrow$ 1D macroscopic system with inertia

$$\begin{cases} \partial_t \phi + \partial_x(\phi u) = 0, \\ \partial_t(\phi u) + \partial_x(\phi u^2) - \partial_x\left(\frac{\mu}{1-\phi}\partial_x u\right) = \phi f \end{cases}$$

ref: Lefebvre-Lepot & Maury '11

Towards a gluey macroscopic model

$$\begin{cases} \partial_t \phi + \partial_x(\phi u) = 0, \\ \partial_t(\phi u) + \partial_x(\phi u^2) - \partial_x\left(\frac{\mu}{1-\phi}\partial_x u\right) = \phi f \end{cases}$$

conjecture for $\mu \to 0$

$$\begin{cases} \partial_t \phi + \partial_x(\phi u) = 0, \\ \partial_t(\phi u) + \partial_x(\phi u^2) + \partial_x p = \phi f, \\ \partial_t \gamma + u \partial_x \gamma = -p, \\ \gamma \le 0, \ \phi \le 1, \ \gamma(1-\phi) = 0 \end{cases}$$

Towards a gluey macroscopic model

$$\begin{cases} \partial_t \phi + \partial_x(\phi u) = 0, \\ \partial_t(\phi u) + \partial_x(\phi u^2) - \partial_x \left(\frac{\mu}{1-\phi} \partial_x u \right) = \phi f \end{cases}$$

conjecture for $\mu \to 0$

$$\begin{cases} \partial_t \phi + \partial_x(\phi u) = 0, \\ \partial_t(\phi u) + \partial_x(\phi u^2) + \partial_x p = \phi f, \\ \partial_t \gamma + u \partial_x \gamma = -p, \\ \gamma \le 0, \ \phi \le 1, \ \gamma(1-\phi) = 0 \end{cases}$$

recall the micro. model

$$\begin{cases} \ddot{q} = f + p, \\ \dot{\gamma} = -p, \\ \gamma \le 0, \ q \ge 0, \ q\gamma = 0 \end{cases}$$

ref: Lefebvre-Lepot & Maury '11

in the following:

- existence of solutions to the limit system
- justification of the limit $\mu \to 0$ (alternative proof of existence for the limit system)
- what possible extension in multi-d ?

Existence of solutions to the limit system

remove the pressure from the momentum eq. by using the effective velocity v

$$\begin{cases} \partial_t \phi + \partial_x(\phi u) = 0 \\ \partial_t(\phi u) + \partial_x(\phi u^2) + \partial_x p = \phi f \\ \partial_t \gamma + u \partial_x \gamma = -p \\ \phi \le 1, \ (1-\phi)\gamma = 0, \ \gamma \le 0 \end{cases} \quad \rightsquigarrow \quad \begin{cases} \partial_t \phi + \partial_x(\phi u) = 0 \\ \partial_t(\phi v) + \partial_x(\phi u v) = \phi f \\ v = u - \partial_x \gamma \\ \phi \le 1, \ (1-\phi)\gamma = 0, \ \gamma \le 0 \end{cases}$$

Existence of solutions - Lagrangian point of view

$$\begin{cases} \partial_t \phi + \partial_x(\phi u) = 0 \\ \partial_t(\phi v) + \partial_x(\phi u v) = \phi f \\ v = u - \partial_x \gamma \\ \phi \leq 1, \ (1-\phi)\gamma = 0, \ \gamma \leq 0 \end{cases}$$

"Lagrangian" point of view:

$\phi_t(x) \to X_t(y)$ monotone rearrangement,

$$u_t(x) \to U_t(y) = u_t(X_t(y)) = \frac{dX_t}{dt}$$

$$\rightsquigarrow \quad \phi_t(x) = \frac{1}{\partial_y X_t\left(X_t^{-1}(x)\right)} \leq 1$$

$$\Longrightarrow \mathrm{Adm} = \{X = \mathrm{Id} + S, \ S \nearrow\}$$

Existence of solutions - Lagrangian point of view

$$\begin{cases} \partial_t \phi + \partial_x(\phi u) = 0 \\ \partial_t(\phi v) + \partial_x(\phi u v) = \phi f \\ v = u - \partial_x \gamma \\ \phi \le 1, \ (1-\phi)\gamma = 0, \ \gamma \le 0 \end{cases}$$

"Lagrangian" point of view:

$\phi_t(x) \to X_t(y)$ monotone rearrangement,

$$u_t(x) \to U_t(y) = u_t(X_t(y)) = \frac{dX_t}{dt}$$

$$\rightsquigarrow \quad \phi_t(x) = \frac{1}{\partial_y X_t\left(X_t^{-1}(x)\right)} \le 1$$

$$\Longrightarrow \mathrm{Adm} = \{X = \mathrm{Id} + S, \ S \nearrow\}$$

we define

$$\begin{cases} X_t = \mathrm{P}_{\mathrm{Adm}}\left(X^0 + \int_0^t V^{\text{free}}(\tau)\,d\tau\right), \\ V^{\text{free}}(t) = U^0 + \int_0^t f(\tau, X_\tau)\,d\tau \end{cases}$$

Existence of solutions - Lagrangian point of view

$$\begin{cases} \partial_t \phi + \partial_x(\phi u) = 0 \\ \partial_t(\phi v) + \partial_x(\phi u v) = \phi f \\ v = u - \partial_x \gamma \\ \phi \le 1, \; (1-\phi)\gamma = 0, \; \gamma \le 0 \end{cases}$$

we define

$$\begin{cases} X_t = \mathrm{P}_{\mathrm{Adm}}\left(X^0 + \int_0^t V^{\text{free}}(\tau)d\tau\right), \\ V^{\text{free}}(t) = U^0 + \int_0^t f(\tau, X_\tau)\, d\tau \end{cases}$$

"Lagrangian" point of view:

$\phi_t(x) \to X_t(y)$ monotone rearrangement,

$$u_t(x) \to U_t(y) = u_t(X_t(y)) = \frac{dX_t}{dt}$$

$$\rightsquigarrow \quad \phi_t(x) = \frac{1}{\partial_y X_t\big(X_t^{-1}(x)\big)} \le 1$$

$$\Longrightarrow \mathrm{Adm} = \{X = \mathrm{Id} + S, \; S \nearrow\}$$

$$\Longrightarrow \; U_t := \frac{dX_t}{dt} = \mathrm{P}_{C_{X_t}}(V^{\text{free}})$$

C_{X_t}: set of admissible velocities (constant in the congested domain)

define Γ_t such that $\partial_y \Gamma_t = U_t - V^{\text{free}}$

Existence of solutions - Lagrangian point of view

$$\begin{cases} \partial_t \phi + \partial_x(\phi u) = 0 \\ \partial_t(\phi v) + \partial_x(\phi u v) = \phi f \\ v = u - \partial_x \gamma \\ \phi \le 1, \ (1-\phi)\gamma = 0, \ \gamma \le 0 \end{cases}$$

"Lagrangian" point of view:

$\phi_t(x) \to X_t(y)$ monotone rearrangement,

$$u_t(x) \to U_t(y) = u_t(X_t(y)) = \frac{dX_t}{dt}$$

$$\rightsquigarrow \quad \phi_t(x) = \frac{1}{\partial_y X_t\big(X_t^{-1}(x)\big)} \le 1$$

$$\Longrightarrow \mathrm{Adm} = \{X = \mathrm{Id} + S, \ S \nearrow\}$$

we define

$$\begin{cases} X_t = \mathrm{P}_{\mathrm{Adm}}\left(X^0 + \int_0^t V^{\text{free}}(\tau)\, d\tau\right), \\ V^{\text{free}}(t) = U^0 + \int_0^t f(\tau, X_\tau)\, d\tau \end{cases}$$

$$\Longrightarrow \ U_t := \frac{dX_t}{dt} = \mathrm{P}_{C_{X_t}}(V^{\text{free}})$$

C_{X_t}: set of admissible velocities (constant in the congested domain)

define Γ_t such that $\partial_y \Gamma_t = U_t - V^{\text{free}}$

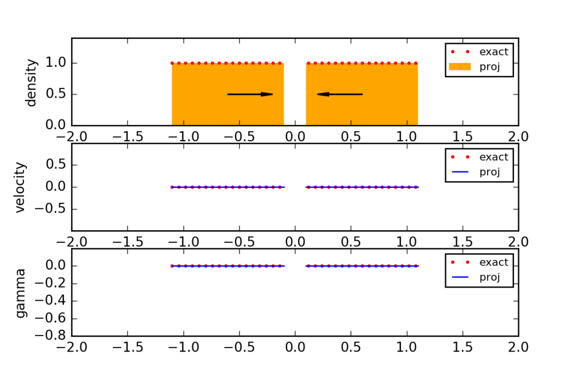
one can check that the associated "Eulerian" variables (ϕ, u, γ) form a weak solution to the system

ref: Perrin, Westdickenberg '18

Lagrangian point of view - convex optimization scheme

→ minimization at each t_n of $\left\| X_0 + \int_0^{t_n} V_\tau^{\text{free}}\, d\tau - X \right\|_2^2$ under the constraint $X \in \text{Adm}$

→ Python software for convex optimization CVXOPT



ref: Perrin, Westdickenberg '18

Asymptotics $\mu = \varepsilon \to 0$ - Notion of effective velocity

$$\begin{cases} \partial_t \phi + \partial_x(\phi u) = 0, \\ \partial_t(\phi u) + \partial_x(\phi u^2) - \underbrace{\partial_x\left(\varepsilon \frac{\phi^\alpha}{(1-\phi)^\beta}\partial_x u\right)}_{=\partial_x(\lambda_\varepsilon(\phi)\partial_x u)} = 0 \end{cases} \quad \xrightarrow[\varepsilon\to 0]{??} \quad \begin{cases} \partial_t \phi + \partial_x(\phi u) = 0, \\ \partial_t(\phi u) + \partial_x(\phi u^2) + \partial_x p = 0, \\ \partial_t \gamma + u\partial_x \gamma = -p, \\ \phi \le 1, \ \gamma \le 0, \ (1-\phi)\gamma = 0 \end{cases}$$

Asymptotics $\mu = \varepsilon \to 0$ - Notion of effective velocity

$$\begin{cases} \partial_t \phi + \partial_x(\phi u) = 0, \\ \partial_t(\phi u) + \partial_x(\phi u^2) - \underbrace{\partial_x\left(\varepsilon \frac{\phi^\alpha}{(1-\phi)^\beta}\partial_x u\right)}_{=\partial_x(\lambda_\varepsilon(\phi)\partial_x u)} = 0 \end{cases} \quad \overset{??}{\underset{\varepsilon\to 0}{\longrightarrow}} \quad \begin{cases} \partial_t \phi + \partial_x(\phi u) = 0, \\ \partial_t(\phi u) + \partial_x(\phi u^2) + \partial_x p = 0, \\ \partial_t \gamma + u\partial_x \gamma = -p, \\ \phi \le 1,\ \gamma \le 0,\ (1-\phi)\gamma = 0 \end{cases}$$

observe that

$$\partial_t \phi + \partial_x(\phi u) = 0 \implies \partial_t b(\phi) + u\partial_x(b(\phi)) = -b'(\phi)\phi\partial_x u$$

Asymptotics $\mu = \varepsilon \to 0$ - Notion of effective velocity

$$\begin{cases} \partial_t \phi + \partial_x(\phi u) = 0, \\ \partial_t(\phi u) + \partial_x(\phi u^2) - \underbrace{\partial_x \left(\varepsilon \frac{\phi^\alpha}{(1-\phi)^\beta} \partial_x u \right)}_{= \partial_x(\lambda_\varepsilon(\phi) \partial_x u)} = 0 \\ \end{cases} \quad \underset{\varepsilon \to 0}{\overset{??}{\longrightarrow}} \quad \begin{cases} \partial_t \phi + \partial_x(\phi u) = 0, \\ \partial_t(\phi u) + \partial_x(\phi u^2) + \partial_x p = 0, \\ \partial_t \gamma + u \partial_x \gamma = -p, \\ \phi \leq 1, \ \gamma \leq 0, \ (1-\phi)\gamma = 0 \end{cases}$$

observe that

$$\partial_t \phi + \partial_x(\phi u) = 0 \implies \partial_t b(\phi) + u \partial_x (b(\phi)) = -b'(\phi) \phi \partial_x u$$

for $\gamma'_\varepsilon(\phi) = \frac{b'(\phi)}{\phi} = \frac{\lambda_\varepsilon(\phi)}{\phi^2}$:

$$\boxed{\partial_t \big(\phi \partial_x \gamma_\varepsilon(\phi) \big) + \partial_x \big(\phi \partial_x \gamma_\varepsilon(\phi) u \big) = -\partial_x \big(\lambda_\varepsilon(\phi) \partial_x u \big)}$$

Asymptotics $\mu = \varepsilon \to 0$ - Notion of effective velocity

$$\begin{cases} \partial_t \phi + \partial_x(\phi u) = 0, \\ \partial_t(\phi u) + \partial_x(\phi u^2) - \underbrace{\partial_x\left(\varepsilon \frac{\phi^\alpha}{(1-\phi)^\beta}\partial_x u\right)}_{=\partial_x(\lambda_\varepsilon(\phi)\partial_x u)} = 0 \end{cases} \quad \xrightarrow[\varepsilon\to 0]{??} \quad \begin{cases} \partial_t \phi + \partial_x(\phi u) = 0, \\ \partial_t(\phi u) + \partial_x(\phi u^2) + \partial_x p = 0, \\ \partial_t \gamma + u \partial_x \gamma = -p, \\ \phi \le 1,\ \gamma \le 0,\ (1-\phi)\gamma = 0 \end{cases}$$

observe that

$$\partial_t \phi + \partial_x(\phi u) = 0 \implies \partial_t b(\phi) + u\partial_x(b(\phi)) = -b'(\phi)\phi\partial_x u$$

for $\gamma_\varepsilon'(\phi) = \frac{b'(\phi)}{\phi} = \frac{\lambda_\varepsilon(\phi)}{\phi^2}$:

$$\boxed{\partial_t\big(\phi\partial_x\gamma_\varepsilon(\phi)\big) + \partial_x\big(\phi\partial_x\gamma_\varepsilon(\phi)u\big) = -\partial_x\big(\lambda_\varepsilon(\phi)\partial_x u\big)}$$

the system at $\varepsilon > 0$ fixed rewrites

$$\begin{cases} \partial_t \phi_\varepsilon + \partial_x(\phi_\varepsilon u_\varepsilon) = 0, \\ \partial_t(\phi_\varepsilon v_\varepsilon) + \partial_x(\phi_\varepsilon u_\varepsilon v_\varepsilon) = 0, \\ v_\varepsilon = u_\varepsilon + \partial_x \gamma_\varepsilon(\phi_\varepsilon) \end{cases}$$

Asymptotics $\mu = \varepsilon \to 0$ - Notion of effective velocity

$$\begin{cases} \partial_t \phi + \partial_x(\phi u) = 0, \\ \partial_t(\phi u) + \partial_x(\phi u^2) - \underbrace{\partial_x\left(\varepsilon \dfrac{\phi^\alpha}{(1-\phi)^\beta} \partial_x u\right)}_{=\partial_x(\lambda_\varepsilon(\phi)\partial_x u)} = 0 \end{cases} \quad \overset{??}{\underset{\varepsilon\to 0}{\longrightarrow}} \quad \begin{cases} \partial_t \phi + \partial_x(\phi u) = 0, \\ \partial_t(\phi u) + \partial_x(\phi u^2) + \partial_x p = 0, \\ \partial_t \gamma + u \partial_x \gamma = -p, \\ \phi \le 1,\ \gamma \le 0,\ (1-\phi)\gamma = 0 \end{cases}$$

observe that

$$\partial_t \phi + \partial_x(\phi u) = 0 \implies \partial_t b(\phi) + u \partial_x(b(\phi)) = -b'(\phi)\phi \partial_x u$$

for $\gamma'_\varepsilon(\phi) = \dfrac{b'(\phi)}{\phi} = \dfrac{\lambda_\varepsilon(\phi)}{\phi^2}$:

$$\boxed{\partial_t\big(\phi \partial_x \gamma_\varepsilon(\phi)\big) + \partial_x\big(\phi \partial_x \gamma_\varepsilon(\phi) u\big) = -\partial_x\big(\lambda_\varepsilon(\phi)\partial_x u\big)}$$

the system at $\varepsilon > 0$ fixed rewrites

$$\begin{cases} \partial_t \phi_\varepsilon + \partial_x(\phi_\varepsilon u_\varepsilon) = 0, \\ \partial_t(\phi_\varepsilon v_\varepsilon) + \partial_x(\phi_\varepsilon u_\varepsilon v_\varepsilon) = 0, \\ v_\varepsilon = u_\varepsilon + \partial_x \gamma_\varepsilon(\phi_\varepsilon) \end{cases}$$

key estimates to pass to the limit $\varepsilon \to 0$

- uniform controls on γ_ε
- one-sided Lipschitz control: $\partial_x u_\varepsilon \le C$

ref: Chaudhuri, Navoret, Perrin, Zatorska '23

Multi-d extension

Toy model

$$\begin{cases} \partial_t \phi + \operatorname{div}(\phi u) = 0, \\ \partial_t(\phi u) + \operatorname{div}(\phi u \otimes u) - 2\operatorname{div}\big(\mu(\phi)\,\mathrm{D}(u)\big) - \nabla\big(\lambda(\phi)\operatorname{div} u\big) = 0 \end{cases}$$

- $\mathrm{D}(u) = \dfrac{\nabla u + (\nabla u)^T}{2}$
- $\mu(\phi) \geq 0$: shear viscosity, $\mu(0) = 0$
- $\lambda(\phi)$: bulk viscosity

we assume that $\mu(\phi)$, $\lambda(\phi) \underset{\phi \to \phi^*}{\longrightarrow} +\infty$

Multi-d extension - BD relation

Toy model

$$\begin{cases} \partial_t \phi + \operatorname{div}(\phi u) = 0, \\ \partial_t (\phi u) + \operatorname{div}(\phi u \otimes u) - 2 \operatorname{div}\left(\mu(\phi) \mathrm{D}(u)\right) - \nabla\left(\lambda(\phi) \operatorname{div} u\right) = 0 \end{cases}$$

- "Navier-Stokes eq with density-dependent viscosities"

→ theory of weak solutions developed by Bresch, Desjardins and co-authors

additional entropy estimate by imposing

$$\lambda(\phi) = 2\big(\mu'(\phi)\phi - \mu(\phi)\big)$$

$$\rightsquigarrow \quad \partial_t \mu(\phi) + \operatorname{div}(\mu(\phi) u) = -\frac{1}{2} \lambda(\phi) \operatorname{div}(u)$$

Multi-d extension - BD relation

Toy model

$$\begin{cases} \partial_t \phi + \operatorname{div}(\phi u) = 0, \\ \partial_t(\phi u) + \operatorname{div}(\phi u \otimes u) - 2\operatorname{div}\big(\mu(\phi)\,\mathrm{D}(u)\big) - \nabla\big(\lambda(\phi)\operatorname{div} u\big) = 0 \end{cases}$$

- "Navier-Stokes eq with density-dependent viscosities"

→ theory of weak solutions developed by Bresch, Desjardins and co-authors

additional entropy estimate by imposing

$$\boxed{\lambda(\phi) = 2\big(\mu'(\phi)\phi - \mu(\phi)\big)} \quad \rightsquigarrow \quad \partial_t \mu(\phi) + \operatorname{div}(\mu(\phi) u) = -\frac{1}{2}\lambda(\phi)\operatorname{div}(u)$$

introducing the effective velocity $\boxed{v = u + 2\dfrac{\nabla \mu(\phi)}{\phi}}$, we get

$$\partial_t(\phi v) + \operatorname{div}(\phi v \otimes u) - 2\operatorname{div}(\mu(\phi) A(u)) = 0$$

$$\Longrightarrow \quad \frac{d}{dt}\int \phi \frac{|v|^2}{2}\,dx + 2\int \mu(\phi)|A(u)|^2\,dx = 0$$

Multi-d extension - Asymptotic gluey/granular model

- take $\mu_\varepsilon(\phi) = \begin{cases} -\varepsilon\phi^\alpha \ln(\phi^* - \phi) \\ \varepsilon\phi^\alpha(\phi^* - \phi)^{-\beta} \end{cases} \quad \rightsquigarrow \quad \lambda_\varepsilon(\phi) \underset{\phi\to\phi^*}{\sim} \begin{cases} \varepsilon(\phi^* - \phi)^{-1} \\ \varepsilon(\phi^* - \phi)^{-(\beta+1)} \end{cases}$

  rmk: in the literature $\beta = 2$

Multi-d extension - Asymptotic gluey/granular model

- take $\mu_\varepsilon(\phi) = \begin{cases} -\varepsilon\phi^\alpha \ln(\phi^* - \phi) \\ \varepsilon\phi^\alpha (\phi^* - \phi)^{-\beta} \end{cases} \rightsquigarrow \lambda_\varepsilon(\phi) \underset{\phi\to\phi^*}{\sim} \begin{cases} \varepsilon(\phi^* - \phi)^{-1} \\ \varepsilon(\phi^* - \phi)^{-(\beta+1)} \end{cases}$

  rmk: in the literature $\beta = 2$

$$\begin{cases} \partial_t\phi_\varepsilon + \mathrm{div}(\phi_\varepsilon u_\varepsilon) = 0, \\ \partial_t(\phi_\varepsilon u_\varepsilon) + \mathrm{div}(\phi_\varepsilon u_\varepsilon \otimes u_\varepsilon) - 2\,\mathrm{div}(\mu_\varepsilon(\phi_\varepsilon)\,\mathrm{D}(u_\varepsilon)) \\ \qquad -\nabla\big(\lambda_\varepsilon(\phi_\varepsilon)\,\mathrm{div}\,u_\varepsilon\big) = 0, \\ \partial_t\mu_\varepsilon(\phi_\varepsilon) + \mathrm{div}(\mu_\varepsilon(\phi_\varepsilon)u_\varepsilon) = -\frac{1}{2}\lambda_\varepsilon(\phi_\varepsilon)\,\mathrm{div}(u_\varepsilon) \end{cases} \underset{\varepsilon\to 0}{\longrightarrow} \begin{cases} \partial_t\phi + \mathrm{div}(\phi u) = 0, \\ \partial_t(\phi u) + \mathrm{div}(\phi u \otimes u) \\ \qquad -2\,\mathrm{div}\big((-\gamma)\,\mathrm{D}(u)\big) + \nabla p = 0, \\ \partial_t\gamma + u\cdot\nabla\gamma = -p, \\ \gamma \le 0,\ (\phi^* - \phi)\gamma = 0,\ 0 \le \phi \le \phi^* \end{cases}$$

- at the limit → pressure-dependent viscosity model

Multi-d extension - Asymptotic gluey/granular model

- take $\mu_\varepsilon(\phi) = \begin{cases} -\varepsilon\phi^\alpha \ln(\phi^* - \phi) \\ \varepsilon\phi^\alpha(\phi^* - \phi)^{-\beta} \end{cases} \rightsquigarrow \lambda_\varepsilon(\phi) \underset{\phi\to\phi^*}{\sim} \begin{cases} \varepsilon(\phi^* - \phi)^{-1} \\ \varepsilon(\phi^* - \phi)^{-(\beta+1)} \end{cases}$

  rmk: in the literature $\beta = 2$

$$\begin{cases} \partial_t \phi_\varepsilon + \operatorname{div}(\phi_\varepsilon u_\varepsilon) = 0, \\ \partial_t(\phi_\varepsilon u_\varepsilon) + \operatorname{div}(\phi_\varepsilon u_\varepsilon \otimes u_\varepsilon) - 2\operatorname{div}(\mu_\varepsilon(\phi_\varepsilon)\mathrm{D}(u_\varepsilon)) \\ \qquad -\nabla\big(\lambda_\varepsilon(\phi_\varepsilon)\operatorname{div} u_\varepsilon\big) = 0, \\ \partial_t \mu_\varepsilon(\phi_\varepsilon) + \operatorname{div}(\mu_\varepsilon(\phi_\varepsilon) u_\varepsilon) = -\frac{1}{2}\lambda_\varepsilon(\phi_\varepsilon)\operatorname{div}(u_\varepsilon) \end{cases} \underset{\varepsilon\to 0}{\longrightarrow} \begin{cases} \partial_t \phi + \operatorname{div}(\phi u) = 0, \\ \partial_t(\phi u) + \operatorname{div}(\phi u \otimes u) \\ \qquad -2\operatorname{div}\big((-\gamma)\mathrm{D}(u)\big) + \nabla p = 0, \\ \partial_t \gamma + u\cdot\nabla\gamma = -p, \\ \gamma \le 0,\ (\phi^* - \phi)\gamma = 0,\ 0 \le \phi \le \phi^* \end{cases}$$

- at the limit → pressure-dependent viscosity model

justification of the limit with additional (regularizing) terms: Perrin '16

case $\mu(\phi) \equiv$ cst, $\lambda_\varepsilon(\phi) \underset{\phi\to\phi^*}{\longrightarrow} +\infty$ and neglecting inertia: Bresch, Necasova, Perrin '18

Remark on the BD relation between viscosities

- we have imposed the algebraic relation

$$\boxed{\lambda_\varepsilon(\phi) = 2\big(\mu_\varepsilon'(\phi)\phi - \mu_\varepsilon(\phi)\big)} \quad \rightsquigarrow \quad \partial_t \mu_\varepsilon(\phi) + \mathrm{div}\,(\mu_\varepsilon(\phi)\,u) = -\frac{1}{2}\lambda_\varepsilon(\phi)\,\mathrm{div}\,(u)$$

$\Longrightarrow$ $-\mu_\varepsilon(\phi)$ plays the role of the adhesion potential

Remark on the BD relation between viscosities

- we have imposed the algebraic relation

$$\boxed{\lambda_\varepsilon(\phi) = 2\big(\mu_\varepsilon'(\phi)\phi - \mu_\varepsilon(\phi)\big)} \qquad \rightsquigarrow \quad \partial_t \mu_\varepsilon(\phi) + \mathrm{div}\,(\mu_\varepsilon(\phi)\,u) = -\frac{1}{2}\lambda_\varepsilon(\phi)\,\mathrm{div}\,(u)$$

$\Longrightarrow$ $-\mu_\varepsilon(\phi)$ plays the role of the adhesion potential

- recall the microscopic case (Cox '74, Hillairet, Kelai '15, Maury '07)

$$\begin{cases} F_{x,y}^{\text{lub}} = \pi R\, \varepsilon \ln q\, U_{x,y} \\ F_z^{\text{lub}} = -\dfrac{3\pi}{2} R^2\, \dfrac{\varepsilon}{q}\, U_z + O(\ln q) \end{cases} \quad \text{and} \quad \gamma_{\text{micro}} = \varepsilon \ln q \tag{1}$$

→ the tangential lubrication term scales at the same order as the adhesion potential

the algebraic relation between $\mu(\phi), \lambda(\phi)$ can be understood in other contexts
→ Euler-Korteweg eq. (see Bresch, Couderc, Noble, Vila '16)

Some on-going projects, perspectives

Some on-going projects, perspectives

- IMPT / INRAE project
  team: G. Chambon, E. Deléage (PhD), T. Faug, S. Gavrilyuk, C.P, G. Richard
  - derivation of nonlocal models via variational principles (analogy with capillarity fluids) + theoretical and numerical analysis
  - derivation of depth-averaged models for compressible granular rheologies with a reference profile which accounts for the shear
    + simulation and comparison/validation with experimental data
  - addition of nonlocal effects in the depth-averaged models and coupling of the granular regime with a viscoplastic regime

Some on-going projects, perspectives

- include more physics in the models
    - friction
    - yield stress / rate-dependent rheology

Some on-going projects, perspectives

- include more physics in the models
    - friction
    - yield stress / rate-dependent rheology
- toy models for jamming/congestion
    - dynamics/stability of interfaces
    - development of jammed regions
    - link with other constrained models (dynamics of floating objects in particular)