On Shear-Banding in the Arrhenius Model

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Outline

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Principles

- Raised temperatures "softens" the material in some way.
- Temperature heterogeneities: localizing effect.
  Hot spots' local viscosity drop and material will flow.
- Heat dissipation smoothes heterogeneities: stabilizing effect.
  Homogeneizing the temperature homogeneizes the flow.

Mathematical Description

- Relevant quantities in simple shear:

deformation (shear)	: γ
shear-rate	: $\dot{\gamma} = u$
velocity field (lagrangian)	: v
scalar stress	: σ
temperature	: θ

- Equations (dimensionless)

$$\begin{aligned} \partial_t v &= \partial_x \sigma, \\ \dot{\gamma} &= \partial_x v, \\ \partial_t \theta &= \kappa \partial_{xx} \theta + \sigma \dot{\gamma} \\ \sigma &= f(\theta, \gamma, \dot{\gamma}) \end{aligned}$$

- In the following Arrhenius law: $\sigma = e^{-\alpha\theta} \dot{\gamma}^n$

Uniform Shear

Definition: solution whose shear-rate is independent of time and space:

$$\dot{\gamma}_s(t,x) = 1,$$
$$\gamma_s(t,x) = t,$$
$$v_s(t,x) = x,$$

θ_s and σ_s depend only on time and solve equations

$$\dot{\theta}_s = f(\theta_s, t, 1)$$
$$\sigma_s = \dot{\theta}_s$$

Questions

- Mathematical indication of the competition between dissipation and thermal softening?
- Mathematical description of the shear band velocity profile (if any)?

Outline

Shearing System

Shearing system is

$$\begin{cases} \partial_t v = \partial_x \sigma, \\ \partial_t \gamma = \partial_x v, \\ \partial_t \theta = \sigma \partial_t \gamma + \kappa \partial_{xx} \theta, \\ \sigma = e^{-\alpha\theta} (\partial_t \gamma)^n. \end{cases}$$

Setting $u = \dot{\gamma} = \partial_x v$ it reduces to

$$\begin{cases} \partial_t u = \partial_{xx} \sigma, \\ \partial_t \theta = \sigma u + \kappa \partial_{xx} \theta, \\ \sigma = e^{-\alpha\theta} u^n. \end{cases}$$

Rk: $\int_0^1 u(t,x)\mathrm{d}x = v(t,1) - v(t,0)$.

Uniform Shear

The uniform shear reads

$$u_s(t,x) = 1, \qquad \theta_s(t,x) = \frac{1}{\alpha}\log(\alpha t + k_0), \qquad \sigma_s(t,x) = \frac{1}{\alpha t + k_0},$$

Note that

$$\lim_{t\to+\infty} \theta_s(t,x) = +\infty$$

What is an unstable perturbation of $(u_s, \theta_s, \sigma_s)$?

Relative Perturbation

- Introduce $(\widetilde{\mathcal{U}}, \widetilde{\mathcal{T}}, \widetilde{\mathcal{S}})$ the relative perturbation to the uniform shear:

$$\widetilde{\mathcal{U}} = \frac{u}{u_s}, \qquad e^{-\alpha\widetilde{\mathcal{T}}} = \frac{e^{-\alpha\theta}}{e^{-\alpha\theta_s}}, \qquad \widetilde{\mathcal{S}} = \frac{\sigma}{\sigma_s}.$$

- $(\widetilde{\mathcal{U}}, \widetilde{\mathcal{T}}, \widetilde{\mathcal{S}})$ satisfies

$$\begin{cases} u_s \partial_t \widetilde{\mathcal{U}} = \sigma_s(t) \partial_{xx} \widetilde{\mathcal{S}}, \\ \partial_t \widetilde{\mathcal{T}} = \sigma_s(t) u_s \widetilde{\mathcal{S}} \widetilde{\mathcal{U}} - \partial_t \theta_s(t) + \kappa \partial_{xx} \widetilde{\mathcal{T}}, \\ \sigma_s(t) \widetilde{\mathcal{S}} = e^{-\alpha\theta_s(t)} u_s^n e^{-\alpha\widetilde{\mathcal{T}}} \widetilde{\mathcal{U}}^n \end{cases}$$

Time Change

Introduce a new time variable τ such that

$$\begin{cases} \partial_t \tau(t) = \sigma_s(t), \\ \tau(0) = 0. \end{cases}$$

- For Arrhenius' Law

$$\tau(t) = \frac{1}{\alpha} \log\left(\frac{\alpha}{k_0} t + 1\right)$$

- Obtain the system

$$\begin{cases} \partial_\tau \mathcal{U} = \partial_{xx} \mathcal{S}, \\ \partial_\tau \mathcal{T} = \mathcal{S}\mathcal{U} - 1 + \kappa k_0 e^{\alpha\tau} \partial_{xx} \mathcal{T}, \\ \mathcal{S} = e^{-\alpha \mathcal{T}} \mathcal{U}^n, \end{cases}$$

Outline

Linearization around Uniform Shear

- Investigate the linear stability of the uniform shear $(\mathcal{U}, \mathcal{T}, \mathcal{S}) = (1, 0, 1)$
- Linearized system:

$$\begin{cases} \partial_\tau \mathcal{U} = \partial_{xx} \mathcal{S} \\ \partial_\tau \mathcal{T} = \mathcal{U} + \mathcal{S} + \widetilde{\kappa} \partial_{xx} \mathcal{T} \\ \mathcal{S}_1 = n\mathcal{U} - \alpha \mathcal{T} \end{cases}$$

- Study the differential system

$$\begin{cases} \partial_\tau \mathcal{U} = n \partial_{xx} \mathcal{U} - \alpha \partial_{xx} \mathcal{T} \\ \partial_\tau \mathcal{T} = (n+1)\mathcal{U} - \alpha \mathcal{T} + \widetilde{\kappa} \partial_{xx} \mathcal{T} \end{cases}$$

Fourier Analysis

- Apply Fourier transform in x:

$$\begin{cases} \partial_\tau \widehat{\mathcal{U}}_j = -j^2 n \widehat{\mathcal{U}}_j + j^2 \alpha \widehat{\mathcal{T}}_j \\ \partial_\tau \widehat{\mathcal{T}}_j = (n+1)\widehat{\mathcal{U}}_j - \alpha \widehat{\mathcal{T}}_j - j^2 \widetilde{\kappa} \widehat{\mathcal{T}}_j \end{cases}$$

- In the following, either $\kappa = 0$ or $\kappa e^{\alpha\tau}$ is frozen to a constant value
- Always diagonalizable
- Sum of eigenvalues < 0. Product of eigenvalues:

$$j^2 \left(n \widetilde{\kappa} j^2 - \alpha \right)$$

If $\kappa = 0$ all modes unstable;

- if $n = 0$ Hadamard instability
- if $n > 0$ Turing instability

if $\widetilde{\kappa} > 0$ a few modes linearly unstable

A word of caution

- Frozen coefficient analysis can be misleading

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 + \frac{3}{2}\cos(t)^2 & 1 - \frac{3}{2}\cos(t)\sin(t) \\ -1 - \frac{3}{2}\cos(t)\sin(t) & -1 + \frac{3}{2}\sin(t)^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

- $\forall t$ fixed, eigenvalues are $-\frac{1}{4} \pm i\frac{\sqrt{7}}{4}$
- $x(t) = -e^{t/2}\cos(t), y(t)e^{t/2}\sin(t)$ is a solution.

Linear stability with time

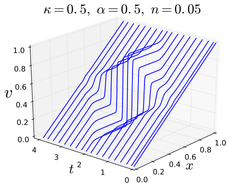
When $\kappa > 0$:

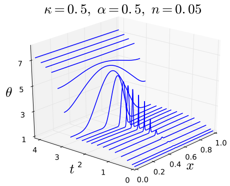
Theorem

- *Linearized system is* L^2 *stable in the sense that* $\forall \varepsilon > 0, \exists \delta > 0$,

$$\|(\mathcal{U}(0), \mathcal{T}(0))\|_{\mathrm{L}^2} \leq \delta \implies \sup_{\tau > 0} \|(\mathcal{U}(\tau), \mathcal{T}(\tau)\|_{\mathrm{L}^2} \leq \varepsilon$$

- (u_s, θ_s) *is linearly asymptotically stable.*





Ideas of proof

Take appropriate boundary conditions.

Lemma

There exists $A, c > 0$, and $T > 0$ such that for all $\tau > T$:

$$\int_0^1 \frac{A}{2}\mathcal{U}^2(\tau) + \frac{1}{2}\mathcal{T}^2(\tau) \leq \left(\int_0^1 \frac{A}{2}\mathcal{U}^2(T) + \frac{1}{2}\mathcal{T}^2(T)\right) e^{-c(\tau - T)}.$$

Lemma

There exists $B, C_B > 0$ such that for all $\tau > 0$:

$$\int_0^1 \frac{1}{2}\mathcal{U}^2(\tau) + \frac{B}{2}\mathcal{T}^2(\tau) \leq \left(\int_0^1 \frac{1}{2}\mathcal{U}^2(0) + \frac{B}{2}\mathcal{T}^2(0)\right) e^{C_B \tau}.$$

Outline

Framework

- $\kappa = 0$
- Introduce a large time scale T and $s = \tau/T$, $y = x/\sqrt{T}$.
- Rescaled equation:

$$\begin{cases} \partial_s \mathcal{U} = \partial_{yy} \mathcal{S} \\ \dfrac{1}{T} \partial_s \mathcal{T} = \mathcal{U}\mathcal{S} - 1 \\ \mathcal{S} = e^{-\alpha \mathcal{T}} \mathcal{U}^n \end{cases}$$

- and expand in $1/T$

Main Order

- Main order system:

$$\begin{cases} \partial_s \mathcal{U}_0 = \partial_{yy} \mathcal{S}_0 \\ 0 = \mathcal{U}_0 \mathcal{S}_0 - 1 \\ \mathcal{S}_0 = e^{-\alpha \mathcal{T}_0} \mathcal{U}_0^n \end{cases}$$

- How to solve:

$$\mathcal{S}_0 = \frac{1}{\mathcal{U}_0}, \qquad \mathcal{T}_0 = \frac{n+1}{\alpha} \log(\mathcal{U}_0), \qquad \partial_s \mathcal{U}_0 = \partial_{yy} \left(\frac{1}{\mathcal{U}_0} \right)$$

Linearized Order

- System:

$$\begin{cases} \partial_s \mathcal{U}_1 = \partial_{yy} \mathcal{S}_1 \\ \partial_s \mathcal{T}_0 = \mathcal{U}_0 \mathcal{S}_1 + \mathcal{S}_0 \mathcal{U}_1 \\ \mathcal{S}_1 = e^{-\alpha \mathcal{T}_0} \left(n \mathcal{U}_0^{n-1} \mathcal{U}_1 - \alpha \mathcal{U}_0^n \mathcal{T}_1 \right) \end{cases}$$

- Only $\mathcal{U}_1$ needed for the sequel. From 2$^{\text{nd}}$ equation:

$$\mathcal{S}_1 = -\frac{\mathcal{U}_1}{\mathcal{U}_0^2} + \frac{1}{\mathcal{U}_0} \partial_s \left(\frac{n+1}{\alpha} \log \mathcal{U}_0 \right)$$

- Conclusion:

$$\partial_s \mathcal{U}_1 = \partial_{yy} \left(-\frac{\mathcal{U}_1}{\mathcal{U}_0^2} + \frac{1}{\mathcal{U}_0} \partial_s \left(\frac{n+1}{\alpha} \log \mathcal{U}_0 \right) \right)$$

Chapman-Enskog

- Goal: find an effective equation approximating the system at order $1/T^2$.
- Define $\widetilde{u} = \mathcal{U}_0 + \frac{1}{T}\mathcal{U}_1$
- Combine equations for $\mathcal{U}_0$ and $\mathcal{U}_1$:

$$\partial_s \widetilde{u} = \partial_{yy}\left(\frac{1}{\widetilde{u}}\right) + \frac{1}{T}\frac{n+1}{\alpha}\partial_{yy}\left(\frac{1}{\widetilde{u}^2}\partial_{yy}\left(\frac{1}{\widetilde{u}}\right)\right) + O\left(\frac{1}{T^2}\right)$$

- Red part: backward parabolic; unstable
- Green part: fourth order; stabilizing as $\frac{n+1}{\alpha} > 0$

Linearization+Fourier

- Linearization around $\tilde{u} = 1$:

$$\partial_s h = -\partial_{yy} h - \frac{n+1}{T\alpha} \partial_{yyyy} h.$$

- Fourier:

$$\partial_s \hat{h} = \xi^2 \left(1 - \frac{n+1}{\alpha T} \xi^2\right) \hat{h}.$$

Outline

Localization

- Go back to the perturbation system ($\kappa = 0$)

$$\begin{cases} \partial_\tau \mathcal{U} = \partial_{xx} \mathcal{S}, \\ \partial_\tau \mathcal{T} = \mathcal{S}\mathcal{U} - 1, \\ \mathcal{S} = e^{-\alpha \mathcal{T}} \mathcal{U}^n, \end{cases}$$

- Goal: examine the possibility for some solutions to concentrate.

Self-similarity

Looking for concentrating solution set

$$\xi(\tau, x) = \frac{x}{R(\tau)}$$

- Find R such that $\lim_{\tau \to \infty} R(\tau) = 0$
- Find profile solutions satisfying:

$$\mathcal{U}(\tau, x) = \frac{1}{R(\tau)} U\left(\frac{x}{R(\tau)}\right),$$

$$\mathcal{T}(\tau, x) = \Theta\left(\frac{x}{R(\tau)}\right) - \frac{n+1}{\alpha} \log R(\tau),$$

$$\mathcal{S}(\tau, x) = R(\tau) \Sigma\left(\frac{x}{R(\tau)}\right),$$

Profile Equations

- Plugging the ansaetze for $(\mathcal{U}, \mathcal{T}, \mathcal{S})$ yields

$$
\begin{cases}
-\dfrac{\partial_\tau R}{R}\left(U + \dfrac{x}{R}\partial_\xi U\right) = \partial_{\xi\xi}\Sigma, \\
-\dfrac{\partial_\tau R}{R}\dfrac{x}{R}\partial_\xi \Theta - \dfrac{n+1}{\alpha}\dfrac{\partial_\tau R}{R} = \Sigma U - 1, \\
\Sigma = e^{-\alpha\Theta} U^n
\end{cases}
$$

- Set $-\frac{\partial_\tau R}{R} = \varepsilon$ ($R(\tau) = e^{-\varepsilon\tau}$)
- Define $c_\varepsilon = 1 + \frac{n+1}{\alpha}\varepsilon$
- Assume U, Θ, Σ even

Outline

Goal

We aim to find solutions to

$$\begin{cases} \Sigma' = \xi U, \\ \varepsilon\xi\Theta' = \Sigma U - \underbrace{\left(1 + \varepsilon\frac{n+1}{\alpha}\right)}_{c_\varepsilon}, \\ \Sigma = e^{-\alpha\Theta} U^n. \end{cases}$$

- Singular when $\xi \to 0$
- Singular when $\varepsilon \to 0$
- We want $\mathcal{C}^1$ solutions on $[0, +\infty[$.
  Consequence: $\Sigma(0)$ and $U(0)$ are positive

“Algebraicity”

- Set $\Psi = e^{-\frac{\alpha}{n}\Theta}$.
- Obtain the system

$$\begin{cases} \Sigma' = \xi \dfrac{\Sigma^{1/n}}{\Psi}, \\ \varepsilon \xi \Psi' = \dfrac{\alpha}{n} \left(c_\varepsilon \Psi - \Sigma^{1+1/n} \right), \\ U = \dfrac{\Sigma^{1/n}}{\Psi}. \end{cases}$$

- Same singularity, no exponential

Far-field

- For ξ away from 0: exact solution

$$U(\xi) = \frac{1}{\xi}, \qquad \Psi(\xi) = \xi^{1+1/n}, \qquad \Sigma(\xi) = \xi;$$

  corresponds physically to $\mathcal{U}(t,x) = \frac{1}{|x|}$
- Correct features but unbounded at $x = 0$.
- Build on that solution: introduce

$$U(\xi) = \frac{1}{\xi}\overline{U}(\xi), \qquad \Psi(\xi) = \xi^{1+1/n}\overline{\Psi}(\xi), \qquad \Sigma(\xi) = \xi\overline{\Sigma}(\xi).$$

- $\lim_{\xi\to 0} \overline{\Sigma}(\xi) = +\infty$,
  $\lim_{\xi\to 0} \xi^{1+1/n}(c_\varepsilon \overline{\Psi}(\xi) - \overline{\Sigma}(\xi)^{1+1/n}) = 0$

Near-field

- The system on $(\overline{U}, \overline{\Psi}, \overline{\Sigma})$:

$$\begin{cases} \xi \overline{\Sigma}' = \dfrac{\overline{\Sigma}^{1/n}}{\overline{\Psi}} - \overline{\Sigma}, \\ \xi \overline{\Psi}' = \dfrac{\alpha}{n\varepsilon} \left(\overline{\Psi} - \overline{\Sigma}^{1+1/n} \right), \\ \overline{U} = \dfrac{\overline{\Sigma}^{1/n}}{\overline{\Psi}}. \end{cases}$$

- Singularity at $\xi = 0$: now removable!

Near-field

- The system on $(\overline{U}, \overline{\Psi}, \overline{\Sigma})$:

$$\begin{cases} \xi \overline{\Sigma}' = \dfrac{\overline{\Sigma}^{1/n}}{\overline{\Psi}} - \overline{\Sigma}, \\ \xi \overline{\Psi}' = \dfrac{\alpha}{n\varepsilon}\left(\overline{\Psi} - \overline{\Sigma}^{1+1/n}\right), \\ \overline{U} = \dfrac{\overline{\Sigma}^{1/n}}{\overline{\Psi}}. \end{cases}$$

- Singularity at $\xi = 0$: now removable!
- Set $\eta = \log \xi$ and $\overline{f}(\xi) = \widehat{f}(\eta)$:

$$\xi \partial_\xi \overline{f}(\xi) = \partial_\eta \widehat{f}(\eta)$$

- Resulting system is autonomous

From Infinite to Finite

- $(\widehat{\Sigma}, \widehat{\Psi})$ now satisfies:

$$\begin{cases} \widehat{\Sigma}' = \dfrac{\widehat{\Sigma}^{1/n}}{\widehat{\Psi}} - \widehat{\Sigma} \\ \widehat{\Psi}' = \dfrac{\alpha}{n\varepsilon}\left(\widehat{\Psi} - \widehat{\Sigma}^{1+1/n}\right) \end{cases}$$

- Its asymptotic properties:

$$\lim_{\eta \to -\infty} \widehat{\Sigma}(\eta) = +\infty \qquad \lim_{\eta \to +\infty} \widehat{\Sigma}(\eta) = 1$$

$$\lim_{\eta \to -\infty} \widehat{\Psi}(\eta) = +\infty \qquad \lim_{\eta \to +\infty} \widehat{\Psi}(\eta) = 1$$

- Bring infinite to finite:

$$\widehat{B} = \frac{1}{\widehat{\Sigma}}, \qquad \widehat{\Lambda} = \frac{\widehat{\Psi}}{\widehat{\Sigma}^{1+1/n}}$$

Final System

- $(\widehat{B}, \widehat{\Lambda})$ satisfies:

$$\begin{cases} \widehat{B}' = \widehat{B}\left(1 - \dfrac{\widehat{B}^2}{\widehat{\Lambda}}\right) \\ \widehat{\Lambda}' = \dfrac{\alpha}{n\varepsilon}\left(\left(1 + \varepsilon\dfrac{n+1}{\alpha}\right)\widehat{\Lambda} - 1 - \varepsilon\dfrac{n+1}{\alpha}\widehat{B}^2\right) \end{cases}$$

- We look for solutions satisfying:

$$\lim_{\eta\to-\infty} \widehat{B}(\eta) = 0, \qquad \lim_{\eta\to+\infty} \widehat{B}(\eta) = 1,$$

$$\lim_{\eta\to-\infty} \widehat{\Lambda}(\eta) < +\infty, \qquad \lim_{\eta\to-\infty} \widehat{\Lambda}(\eta) = 1$$

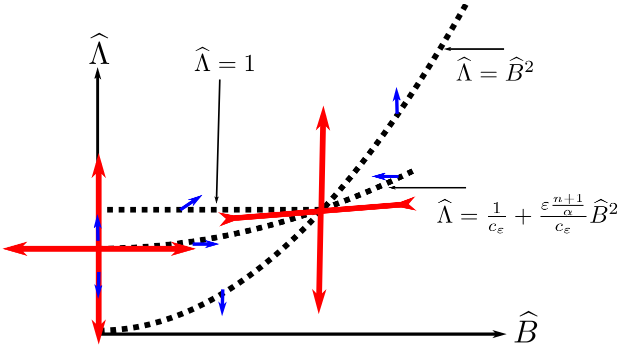
Theoretical Result

Theorem

- *The* $(\widehat{B}, \widehat{\Lambda})$ *system has the following properties:*
  1. *Exactly two stationary hyperbolic points* $P = (0, 1/c_\varepsilon)$ $Q = (1, 1)$. *P is a pure node while Q is a saddle point.*
  2. *For all* $\varepsilon > 0$, *there exists an heteroclinic orbit that connects P to Q.*
- *Let* $(\widehat{B}_0, \widehat{\Lambda}_0)$ *generate the heteroclinic orbit and let* $\widehat{B}_{\eta_0}(\eta) = \widehat{B}_0(\eta + \eta_0)$ *and* $\widehat{\Lambda}_{\eta_0}(\eta) = \widehat{\Lambda}_0(\eta + \eta_0)$.
  We have
  $$\lim_{\eta \to -\infty} \frac{e^\eta}{\widehat{B}_{\eta_0}(\eta)} = \Sigma_{\eta_0} > 0$$
  and the function $\eta_0 \mapsto \Sigma_{\eta_0}$ *is a bijection from* $\mathbf{R}$ *to* $\mathbf{R}_+^*$.

Proof

- Phase portrait



- Global estimation:

$$0 \leq \widehat{B}' \leq \widehat{B}(1 - \widehat{B}^2) \text{ implies } \frac{e^\eta}{\widehat{B}(\eta)} \geq \frac{\sqrt{1 + C_0^2 e^{2\eta}}}{C_0}$$

Existence of Shear Band Profiles

Theorem

The family (depending on $r > 0$)

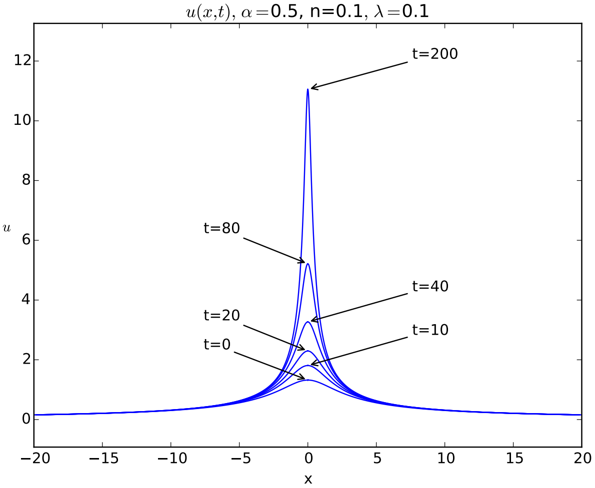
$$\mathcal{U}(t,x) = \left(\frac{\alpha}{k_0}t + 1\right)^{1/r} \frac{\widehat{B}_{\eta_0}(\log \xi)}{\xi \widehat{\Lambda}_{\eta_0}(\log \xi)},$$

$$\mathcal{T}(t,x) = \frac{n+1}{\alpha r} \log\left(\frac{\alpha}{k_0}t + 1\right) - \frac{n}{\alpha} \log\left(\xi^{1+1/n} \frac{\widehat{\Lambda}_{\eta_0}(\log \xi)}{\widehat{B}_{\eta_0}(\log \xi)^{1+1/n}}\right)$$

$$\mathcal{S}(t,x) = \frac{1}{\left(\frac{\alpha}{k_0}t + 1\right)^{1/r}} \frac{\xi}{\widehat{B}_{\eta_0}(\log \xi)},$$

where $\xi(t,x) = \left(\frac{\alpha}{k_0}t + 1\right)^{\frac{1}{\alpha r}} x$, describes a shear-band.

Numerical illustration



Summary

- In adiabatic regime, perturbations of the uniform shear are Hadamard unstable for fluids following Arrhenius' Law.
- In the long time asymptotics, an effective equation recovers some stability.
- There exist a solution describing a shear band.