On "Full well-posedness of point vortex dynamics corresponding to stochastic 2D Euler equations" by Gubinelli, Flandoli, Priola

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1 Introduction: stochastic perturbation of fluid equations

Ergodicity for 2D and 3D incompressible Navier-Stokes equations: see the review [4].

Construction of a Feller semi-group for 3D incompressible Navier-Stokes equations: [3].

Stochastic perturbation of incompressible Euler equations: see [2] and references therein. In [2], the stochastic 2D Euler equations in vorticity form is considered. The equation is (with D the torus $\mathbb{T}^2$)

$$d\xi + u\cdot\nabla_x\xi dt + \sum_{k=1}^{N}\sigma_k(x)\nabla_x\xi\circ d\beta_k(t) = 0, \tag{1.1}$$

where $\beta_1(t), \beta_2(t), \dots$ are independent 1D Wiener processes on a filtered space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$, $\sigma_1, \sigma_2, \dots$ are divergence-free 2D vector fields and u is given by the Biot-Savart Law:

$$u = -\nabla^{\perp}\Delta^{-1}\xi, \quad u(x) = \int_D K_D(x,y)\xi(y)dy, \quad K_D(x,y) = \nabla_x^{\perp}G_D(x,y), \tag{1.2}$$

where D is the spatial domain, G_D the Green function of the Laplace operator with homogeneous Dirichlet boundary conditions on D. The main result of [2] is that (1.1)-(1.2) has unique solution when the initial vorticity is $L^\infty(\mathbb{T}^2)$.

In the paper [7], *Full well-posedness of point vortex dynamics corresponding to stochastic* 2*D Euler equations* by Gubinelli, Flandoli, Priola, the point vortex dynamics associated to (1.1)-(1.2) is considered: taking

$$\xi_{\mathrm{in}} = \sum_{j=1}^{n} a_j\delta_{x_j}, \quad x_1, x_2, \dots \in D, \tag{1.3}$$

as initial vorticity, the expected solution to (1.1)-(1.2) is $\xi_t = \sum_j a_j\delta_{x_j(t)}$, where the positions $x_j(t)$ of the center of the vortices evolve according to the following system of SDEs:

$$dx_j(t) = \sum_{i\neq j} a_i K_D(x_j(t), x_i(t))dt + \sum_k \sigma_k(x_j(t))\circ d\beta_k(t). \tag{1.4}$$

Last bibliographic references: in the paper [6], Flandoli studies the 2D Euler equation with (space) white noise initial condition and proves the convergence of the point vortex approximation. The result is extended to the stochastic 2D Euler equation (with the same structure of noise as in (1.1), - and $N=\infty$) in [8].

2 Point vortex deterministic dynamics

We refer to the book [10] by Marchioro and Pulvirenti (chapter 4). Contrary to [7], the spatial domain in [10, Chapter 4.] is the whole space $\mathbb{R}^2$ (the case of the $2D$-torus was considered in the anterior work [5]). For the deterministic version of (1.4) in the whole space:

$$\frac{dx_j(t)}{dt} = \sum_{i\neq j} a_i K(x_j(t), x_i(t)), \quad K(x,y) = \nabla_x^\perp G(x,y), \quad G(x,y) = -\frac{1}{2\pi}\ln|x-y|, \tag{2.1}$$

Marchioro and Pulvirenti show first that the vortices remain in a bounded domain under the condition

$$\forall \pi \subset \{1,\dots,n\}, \quad \sum_{i\in\pi} a_i \neq 0. \tag{2.2}$$

More precisely, consider a regularized version of (2.1):

$$\frac{dx_j^\varepsilon(t)}{dt} = \sum_{i\neq j} a_i K_\varepsilon(x_j^\varepsilon(t), x_i^\varepsilon(t)), \quad K_\varepsilon(x,y) = \nabla_x^\perp G_\varepsilon(x,y), \quad G_\varepsilon(x,y) = -\frac{1}{2\pi}\ln_\varepsilon|x-y|, \tag{2.3}$$

where $\varepsilon > 0$ and $\ln_\varepsilon \in C^\infty(\mathbb{R})$ satisfies:

$$\ln_\varepsilon(z) = \ln|z| \text{ if } |z| > \varepsilon, \quad |\ln_\varepsilon(z)| \leq |\ln|z|| \text{ and } \left|\frac{d}{dz}\ln_\varepsilon(z)\right| \leq |z|^{-1} \text{ for all } z\in\mathbb{R}. \tag{2.4}$$

For the dynamics defined by (2.3), we have global existence and the following result.

Theorem 2.1 (See [10] p. 142)**.** *Let $T > 0$, $\varepsilon \in (0,1)$. Under (2.2), there exists a constant $R \geq 0$ depending on $T, n, a_1, \dots, a_n$ only such that the solution to (2.3) satisfies $x_j^\varepsilon(t) \in \bar{B}(x_j^\varepsilon(0), R)$ for all $j \in \{1,\dots,n\}$, $t \in [0,T]$.*

Remark 2.1 (Condition (2.2) is necessary). If $n = 2$ and $a_1 = -a_2 = a > 0$ with $d = |x_1(0) - x_2(0)| > 0$, then the two vortices (solution to (2.1), but also (2.3) if $d > \varepsilon$) move in parallel, in the direction orthogonal to the line (x_1, x_2). Some explicit computations gives that, at time T, the vortices have travelled a distance $\frac{a}{\pi d}$, where d is their distance (constant in time thus). Taking $d = 2\varepsilon$, we see that the condition (2.2) is necessary in Theorem 2.1.

Once Theorem 2.1 is established, Marchioro and Pulvirenti prove that (2.1) is well posed for a.e. initial condition. The dynamics (2.1) and the regularized dynamics (2.3) coincide as long as the vortices remain at distance at most ε from each other, that is to say $d_\varepsilon(X) \geq \varepsilon$, where, for $X = (x_j)_{1,n} \in D^n$ ($D = \mathbb{R}^2$) the vector of initial positions,

$$d_\varepsilon(X) = \min_{i\neq j} \inf_{t\in[0,T]} |x_i^\varepsilon(t) - x_j^\varepsilon(t)|, \tag{2.5}$$

with $(x_j^\varepsilon(t))_{1,n}$ the solution to (2.3) starting from X. Therefore, in this approach to (2.1) by the regularization (2.3), we have to estimate the occurrence of the event $d_\varepsilon < \varepsilon$. Consider $X \in \bar{B}_{2n}(0,M)$ and let λ be the normalized Lebesgue measure on $\bar{B}_{2n}(0,M)$. We have the following result.

Theorem 2.2 (See [10] p. 144)**.** *Let $T > 0$. Under (2.2), the probability of ε-collisions between vortices vanishes when $\varepsilon \to 0$:*

$$\lim_{\varepsilon\to 0} \lambda(\{X \in \bar{B}_{2n}(0,M); d_\varepsilon(X) < \varepsilon\}) = 0.$$

Proof of Theorem 2.2. Let $\Phi_t^\varepsilon(X)$ denote the flow associated to (2.3): $X^\varepsilon(t) := \Phi_t^\varepsilon(X)$ is the solution to (2.3) starting from X. Let I_ε denote the functional

$$\mathrm{I}_\varepsilon(X) = -\pi \sum_{1\le i\neq j\le n} G_\varepsilon(x_i, x_j) = \frac{1}{2} \sum_{1\le i\neq j\le n} \ln_\varepsilon |x_i - x_j|. \tag{2.6}$$

The control of $\sup_{t\in[0,T]} |\mathrm{I}_\varepsilon \circ \Phi_t^\varepsilon|$ in $L^1(\lambda)$ gives the result. Indeed, for ε small enough, we have

$$d_\varepsilon(X) < \varepsilon \implies \sup_{t\in[0,T]} |\mathrm{I}_\varepsilon \circ \Phi_t^\varepsilon(X)| > \frac{1}{2} |\ln \varepsilon|, \tag{2.7}$$

and thus, by the Markov inequality,

$$\lambda(\{X \in \bar{B}_{2n}(0, M); d_\varepsilon(X) < \varepsilon\}) \le \frac{2}{|\ln \varepsilon|} \int_{\bar{B}_{2n}(0,M)} \sup_{t\in[0,T]} |\mathrm{I}_\varepsilon \circ \Phi_t^\varepsilon(X)| d\lambda(X). \tag{2.8}$$

The argument for (2.7) is the following one: if $|x_i(t) - x_j(t)| < \varepsilon$ for some indices $i \neq j$ and some $t \in [0, T]$, then i and j both give a contribution at least $-\ln \varepsilon$ to $-P_t^\varepsilon \Phi_\varepsilon(X)$, which can not be compensated by some positive terms, due to Theorem 2.1, or at least not more than $\frac{1}{4}|\ln \varepsilon|$ if ε is small enough. Therefore the negative part of $\mathrm{I}_\varepsilon \circ \Phi_t^\varepsilon(X)$ is larger than $\frac{3}{4}|\ln \varepsilon|$. Similarly, the positive part of $\mathrm{I}_\varepsilon \circ \Phi_t^\varepsilon(X)$ is smaller than $\frac{1}{4}|\ln \varepsilon|$ for ε small enough, due to Theorem 2.1. This gives (2.7). In a second step, we try to get an estimate on the integral in (2.8), and for this, we take advantage of the special form of I_ε. Indeed,

$$\frac{d}{dt} \mathrm{I}_\varepsilon \circ \Phi_t^\varepsilon(X) = \frac{1}{2} \sum_i \sum_{j\neq i} (\nabla G_\varepsilon)(x_i^\varepsilon(t) - x_j^\varepsilon(t)) \frac{d}{dt}(x_i^\varepsilon(t) - x_j^\varepsilon(t))$$

because, with a slightly improper use of notations, we can write $G_\varepsilon(x, y) = G_\varepsilon(x - y)$. We use (2.3) then and see that the most singular terms

$$(\nabla G_\varepsilon)(x_i^\varepsilon(t) - x_j^\varepsilon(t)) \cdot (\nabla^\perp G_\varepsilon)(x_i^\varepsilon(t) - x_j^\varepsilon(t)),$$

which appear twice (with weight a_i and a_j respectively), vanish. More precisely, we obtain

$$\left| \frac{d}{dt} \mathrm{I}_\varepsilon \circ \Phi_t^\varepsilon(X) \right| \le \mathrm{J}_\varepsilon \circ \Phi_t^\varepsilon(X), \tag{2.9}$$

where

$$\mathrm{J}_\varepsilon(X) = a \sum_i \sum_{j\neq i} \sum_{k\notin\{i,j\}} |\nabla G_\varepsilon(x_i - x_j)||\nabla^\perp G_\varepsilon(x_i - x_k)|. \tag{2.10}$$

In (2.10), $a := \max_{1\le i\le n} |a_i|$. Using (2.4), we have $\mathrm{J}_\varepsilon(X) \le h_1(X)$, where the function

$$h_1 : X \mapsto a \sum_i \sum_{j\neq i} \sum_{k\notin\{i,j\}} \frac{1}{|x_i - x_j|} \frac{1}{|x_i - x_k|}$$

is in $L^1_{\mathrm{loc}}(\mathbb{R}^{2n})$. Indeed, the sum of h_1 over a n-cell $\bar{B}_2(0, R) \times \cdots \times \bar{B}_2(0, R)$ in $\mathbb{R}^{2n}$ is bounded by

$$a \sum_i \sum_{j\neq i} \sum_{k\notin\{i,j\}} |\bar{B}_2(0, R)|^{n-2} \int_{|x_j|\le 2R} \frac{dx}{|x_j|} \int_{|x_k|\le 2R} \frac{dx}{|x_k|},$$

which is finite. Similarly, using (2.4), we obtain $\mathrm{I}_\varepsilon(X) \leq h_0(X)$ where h_0 is in $L^1_{\mathrm{loc}}(\mathbb{R}^{2n})$. By (2.9), and the bounds by the functions h_j, we have

$$\int_{\bar{B}_{2n}(0,M)} \sup_{t\in[0,T]} |\mathrm{I}_\varepsilon \circ \Phi^\varepsilon_t(X)| d\lambda(X) \leq \int_{\bar{B}_{2n}(0,M)} h_0(X) d\lambda(X) + \int_0^T \int_{\bar{B}_{2n}(0,M)} h_1 \circ \Phi^\varepsilon_t(X) d\lambda(X). \tag{2.11}$$

We use the fact that the flow Φ^ε_t preserves the $2n$-dimensional Lebesgue measure (a consequence of the identity $\mathrm{div}\, \nabla^\perp = 0$ in 2D) and sends the ball $\bar{B}_{2n}(0,M)$ into the ball $\bar{B}_{2n}(0,M+R)$ (Theorem 2.1) to deduce from (2.11) the bound

$$\int_{\bar{B}_{2n}(0,M)} \sup_{t\in[0,T]} |\mathrm{I}_\varepsilon \circ \Phi^\varepsilon_t(X)| d\lambda(X) \leq \int_{\bar{B}_{2n}(0,M)} h_0(X) d\lambda(X) + T \int_{\bar{B}_{2n}(0,M+R)} h_1(X) d\lambda(X), \tag{2.12}$$

which is the desired estimate. □

3 Point vortex stochastic dynamics

Let us consider now the full stochastic system (1.4). In [7], Gubinelli, Flandoli, Priola study (1.4) and the regularized version

$$\frac{dx^\varepsilon_j(t)}{dt} = \sum_{i\neq j} a_i K^\sharp_\varepsilon(x^\varepsilon_j(t), x^\varepsilon_i(t)) + \sum_k \sigma_k(x^\varepsilon_j(t)) \circ d\beta_k(t), \tag{3.1}$$

on the Torus $\mathbb{T}^2$. Since $\mathbb{T}^2$ is compact, there is no need to establish some bounds on X^ε, solution to (3.1), as in Theorem 2.1. Gubinelli, Flandoli, Priola show that (1.4) is well posed not only a.e., but *for all* starting point $X_{\mathrm{in}} \in \Gamma^c$, where

$$\Gamma = \bigcup_{1\leq i\neq j\leq n} \{X \in (\mathbb{T}^2)^n; x_i = x_j\} \tag{3.2}$$

is the set where collisions of vortices take place. To get such a result, Gubinelli, Flandoli, Priola prove first that (1.4) is well posed a.e., a.s. (with a proof analogous to the proof of Theorem 2.2 in the deterministic case), and then use the fact that the law of X_t solution to (3.1) is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}^2$, as soon as $t > 0$, to obtain

$$\mathbb{P}(X_t \in \Gamma^c \text{ for all } t \in [0,T]) = 1. \tag{3.3}$$

Actually, one has to consider not exactly X_t, but the process X_t killed at the time where collision possibly occurs to establish (3.3) (see Section 3.2). A more important remark is that the absolute continuity with respect to the Lebesgue measure on $\mathbb{T}^2$ of the law of X_t ($t > 0$) is not always true. If X_t starts from X at time $t = 0$, then the absolute continuity is ensured if the vector fields

$$A_k(X) := (\sigma_k(x_1), \dots, \sigma_k(x_n)) \tag{3.4}$$

satisfy the following bracket generating condition:

$$\mathcal{V}(X) = \mathbb{R}^{2n}, \tag{3.5}$$

where

$$V_0 = \{A_k; 1 \leq k \leq n\}, \quad V_{m+1} = V_m \cup \{[B, A_k]; B \in V_m, 1 \leq k \leq n\}, \quad V = \cup_{m\geq 0} V_m, \tag{3.6}$$

and $\mathcal{V}(X)$ is the vector space generated by all the elements of V evaluated at X. The precise statement of the main result in [7] is the following one.

Theorem 3.1. *Let $\sigma_k \in C^\infty(\mathbb{T}^2)$ be divergence-free vector fields. Assume that the vector fields A_k defined by (3.4) satisfy the bracket generating condition (3.5) at every point $X \in \Gamma^c$. Then, for all $X \in \Gamma^c$, the system (1.4) has a unique global strong solution starting from X.*

We have to specify the last sentence in Theorem 3.1.

Definition 3.1. A strong solution on $[0,T]$ to (1.4) starting from $X \in \Gamma^c$ is a continuous adapted process $(X(t))_{t\in[0,T]}$ satisfying (3.3) and the identity

$$x_j(t) = x_j + \int_0^t \sum_{i\neq j} a_i K^\sharp(x_j(s), x_i(s))ds + \sum_k \int_0^t \sigma_k(x_j(s)) \circ d\beta_k(s), \tag{3.7}$$

for all $t \in [0,T]$.

The uniqueness assertion in Theorem 3.1 can be proved as follows. Given two solutions $(X(t))$, $(\tilde{X}(t))$, show that for all $\delta > 0$, a.s., $X(t) = \tilde{X}(t)$ for all $t \in [0, \tau_\delta]$, where τ_δ is the stopping time defined as the minimal time at which $X(t)$ or $\tilde{X}(t)$ leaves the compact $\overline{\Lambda}_\delta$, where

$$\Lambda_\delta = \{X \in (\mathbb{T}^2)^n; d(X,\Gamma) > \delta\}. \tag{3.8}$$

Indeed, the drift part in (1.4) is Lipschitz continuous on $\overline{\Lambda}_\delta$. Then we use (3.3), which gives $\tau_\delta \to T$ when $\delta \to 0$, to conclude that $X = \tilde{X}$. The sketch of the proof of the existence part of Theorem 3.1 is given in Section 3.1 and Section 3.2 below.

3.1 Argument 1

Theorem 3.2. *Let $\sigma_k \in C^2(\mathbb{T}^2)$ be divergence-free vector fields. Then, for almost all $X \in \Gamma^c$, the system (1.4) has a unique global strong solution starting from X.*

Proof of Theorem 3.2. It is sufficient to prove the result for $X \in \Lambda_\delta$, where δ is arbitrary (Λ_δ, defined in (3.8)), is the complement of the closed δ-neighbourhood of Γ). In Section 2, the proof of Theorem 2.2 on ε-collisions in the deterministic case uses the estimate

$$\lambda(\{X \in \bar{B}_{2n}(0,M); d_\varepsilon(X) < \varepsilon\}) \leq \frac{C}{|\ln \varepsilon|}, \tag{3.9}$$

where C is twice the right-hand side of (2.12) (we combine (2.8) and (2.12)). Assume we have an equivalent result in our stochastic context: for $\varepsilon < \delta$,

$$(\lambda \otimes \mathbb{P})(\{X \in \Lambda_\delta; d_\varepsilon(X) < \varepsilon\}) \leq \frac{C}{|\ln \varepsilon|}. \tag{3.10}$$

Let (ε_k) be a decreasing sequence such that the series of general term $|\ln \varepsilon_k|^{-1}$ is convergent. By the Borel-Cantelli Lemma, we deduce from (3.10) that, for $\lambda \otimes \mathbb{P}$-almost all $(X,\omega) \in \Lambda_\delta \times \Omega$, there is a $k_0(X,\omega)$ such that $d_{\varepsilon_k}(X^\omega) \geq \varepsilon_k$ for all $k \geq k_0(X,\omega)$. This can be rephrased as

$$\tau_X(\omega) = T, \tag{3.11}$$

for $\lambda \otimes \mathbb{P}$-almost all $(X,\omega) \in \Lambda_\delta \times \Omega$, where τ_X is the stopping time defined by

$$\tau_X = \lim_{\varepsilon\to 0} \tau_X^\varepsilon = \sup_{0<\varepsilon<1} \tau_X^\varepsilon, \quad \tau_X^\varepsilon = \inf\{t \in [0,T]; X^\varepsilon(t) \notin \Lambda_\varepsilon\}. \tag{3.12}$$

By the Fubini Theorem, we obtain: for λ-almost all $X \in \Lambda_\delta$, $\mathbb{P}(\tau_X = T) = 1$. This gives the result. To establish (3.10), the procedure is the same as in (2.7)-(2.8): we need an estimate

$$\int_{(\mathbb{T}^2)^n} \mathbb{E}\left[\sup_{t\in[0,T]} |\mathrm{I}_\varepsilon \circ \Phi_t^\varepsilon(X)|\right] d\lambda(X) \leq C, \tag{3.13}$$

where Φ_t^ε is the (stochastic) flow for (3.1), I_ε is the functional defined by (2.6), and the constant C is independent on ε. The argument is simply a perturbation of the argument used in the deterministic case. We rewrite (3.1) in Itô form:

$$dx_j^\varepsilon(t) = \sum_{i\neq j} a_i K_\varepsilon^\sharp(x_j^\varepsilon(t), x_i^\varepsilon(t))dt + \sum_k \frac{1}{2}(\nabla\sigma_k \cdot \sigma_k)(x_j^\varepsilon(t))dt + \sigma_k(x_j^\varepsilon(t))d\beta_k(t). \tag{3.14}$$

This gives the differential equation

$$d\mathbb{X}^\varepsilon(t) = (H^\sharp(\mathbb{X}^\varepsilon(t)) + R(X^\varepsilon(t)))dt + Q(X^\varepsilon(t))dW(t), \tag{3.15}$$

where $\mathbb{X}^\varepsilon(t)$ is the matrix with ij-component $x_j^\varepsilon(t) - x_i^\varepsilon(t)$, $H^\sharp(\mathbb{X}^\varepsilon(t))$ is the deterministic part (the one that remains when all the functions $\sigma_k \equiv 0$), $W(t)$ is the vector with components $\beta_1(k), \ldots, \beta_N(t)$ (an N-dimensional Wiener process) and

$$R(X)_{ij} = \frac{1}{2}\sum_k \left[(\nabla\sigma_k \cdot \sigma_k)(x_j) - (\nabla\sigma_k \cdot \sigma_k)(x_i)\right], \quad |R(X)_{ij}| \lesssim |\mathbb{X}_{ij}|, \tag{3.16}$$

$$Q(X)_{ij}^k = \sigma_k(x_j) - \sigma_k(x_i), \quad |Q(X)_{ij}^k| \lesssim |\mathbb{X}_{ij}|. \tag{3.17}$$

To write the deterministic part in (3.15) as a function of $\mathbb{X}$, we use the fact that, like[1] the Green function of the Laplace operator on the whole space, the Green function $G^\sharp(x,y)$ on the torus is a function of $x-y$: $G^\sharp(x,y) = G^\sharp(x-y)$ where

$$G^\sharp(x) = \sum_{m\in\mathbb{Z}^2\setminus\{0\}} \frac{e^{im\cdot x}}{|m|^2} \tag{3.18}$$

is an even function. (Note however that only the fact that $(x,y) \mapsto G^\sharp(x,y)$ is symmetric does matter). For the same reason, we have $\mathrm{I}_\varepsilon(X) = \hat{\mathrm{I}}_\varepsilon(\mathbb{X})$, where

$$\hat{\mathrm{I}}_\varepsilon(\mathbb{X}) = -\pi \sum_{1\leq i\neq j\leq n} G_\varepsilon^\sharp(\mathbb{X}_{ij}). \tag{3.19}$$

By Itô's Formula, we deduce from (3.15) that

$$\begin{aligned}\mathrm{I}_\varepsilon \circ \Phi_t^\varepsilon(X) = \mathrm{I}_\varepsilon(X) &+ \int_0^t D\hat{\mathrm{I}}_\varepsilon(\mathbb{X}^\varepsilon(s)) \cdot (H^\sharp(\mathbb{X}^\varepsilon(s)) + R(X^\varepsilon(s)))ds \\ &+ \frac{1}{2}\sum_k \int_0^t D^2\hat{\mathrm{I}}_\varepsilon(\mathbb{X}^\varepsilon(s)) \cdot (Q^k(X^\varepsilon(s)), Q^k(X^\varepsilon(s)))ds + M_\varepsilon(t),\end{aligned} \tag{3.20}$$

where $M_\varepsilon(t)$ is the martingale

$$M_\varepsilon(t) = \int_0^t D\hat{\mathrm{I}}_\varepsilon(\mathbb{X}^\varepsilon(s)) \cdot Q(X^\varepsilon(s))dW(s).$$

[1] we also use the fact that $G^\sharp$ defined by (3.18) has the same singularity that G at 0 and can be regularized in a similar manner

Since (3.1) is in Stratonovitch form and the σ_k are divergence free, the stochastic flow Φ_t^ε preserves the Lebesgue measure λ on $(\mathbb{T}^2)^n$. We can use the results of the deterministic case to get an estimate

$$\int_{(\mathbb{T}^2)^n} |\mathrm{I}_\varepsilon(X)| d\lambda(X) + \int_0^T \int_{(\mathbb{T}^2)^n} \mathbb{E}|D\hat{\mathrm{I}}_\varepsilon(\mathbb{X}^\varepsilon(t)) \cdot H^\sharp(\mathbb{X}^\varepsilon(t))| d\lambda(X) dt \leq C_1. \tag{3.21}$$

For the third and fourth term in the right-hand side of (3.20), we have, thanks to (3.16) and (3.17),

$$\int_0^T \int_{(\mathbb{T}^2)^n} \mathbb{E}|D\hat{\mathrm{I}}_\varepsilon(\mathbb{X}^\varepsilon(t)) \cdot R(X^\varepsilon(t))| d\lambda(X) dt \leq C_2, \tag{3.22}$$

and

$$\sum_k \int_0^T \int_{(\mathbb{T}^2)^n} \mathbb{E}|D^2\hat{\mathrm{I}}_\varepsilon(\mathbb{X}^\varepsilon(t)) \cdot (Q^k(X^\varepsilon(t)), Q^k(X^\varepsilon(t)))| d\lambda(X) dt \leq C_3. \tag{3.23}$$

To estimate the contribution of martingale term in (3.20), we use the Burkholder - Davis - Gundy inequality [1, Theorem 5.70]: $\mathbb{E}\left[\sup_{t\in[0,T]} |M^\varepsilon(t)|\right] \leq C\mathbb{E}\left[\langle M^\varepsilon, M^\varepsilon\rangle_T^{1/2}\right]$, where

$$\langle M^\varepsilon, M^\varepsilon\rangle_t = \sum_k \int_0^t |D\hat{\mathrm{I}}_\varepsilon(\mathbb{X}^\varepsilon(s)) \cdot Q^k(X^\varepsilon(s))|^2 ds$$

is the quadratic variation of $(M^\varepsilon(t))$. Using again (3.17), we get

$$\int_{(\mathbb{T}^2)^n} \mathbb{E}\left[\sup_{t\in[0,T]} |M^\varepsilon(t)|\right] d\lambda(X) \leq C_4. \tag{3.24}$$

Combining (3.21), (3.22), (3.23), (3.24), we get (3.13). □

3.2 Argument 2

A corollary of Theorem 3.2 is the following result: consider the resolution of (1.4) starting from an initial state ξ which is a $\mathcal{F}_0$-measurable random variable such that $\xi \in \Gamma^c$ a.s. In that context, we use the integral formulation (3.7) with ξ_j in place of x_j and the approximation by regularization of the singular kernel $K^\sharp$. We also define τ_ξ as in (3.11). Assume that the law μ_ξ of ξ is absolutely continuous with respect to the Lebesgue measure. Then, a.s., $(X(t))$ is defined globally in time. To prove this result, we introduce Δ, an isolated point, and let $(\hat{X}(t))$ be the killed process which coincides with $(X(t))$ if $t < \tau_\xi$ and is equal to Δ if $t \geq \tau_\xi$. Then $(\hat{X}(t))$ is a Markov process. In particular, we have

$$\mathbb{P}(\hat{X}(t_1;\xi) \in B_1, \ldots, \hat{X}(t_m;\xi) \in B_m) = \int_{(\mathbb{T}^2)^n} \mathbb{P}(\hat{X}(t_1;X) \in B_1, \ldots, \hat{X}(t_m;X) \in B_m) d\mu_\xi(X), \tag{3.25}$$

for all $0 \leq t_1 \leq \ldots \leq t_m$, for all Borel subsets $B_1, \ldots, B_m$ of $(\mathbb{T}^2)^n \cup \{\Delta\}$. Take

$$B_1 = \ldots = B_m = \overline{\Lambda}_\delta$$

and let $t_1, \ldots, t_m$ be the elements of a dense countable subset (t_k) of $[0,T]$. By continuity of the trajectories, (3.25) gives at the limit $[m \to +\infty]$,

$$\mathbb{P}(\hat{X}([0,T];\xi) \in \overline{\Lambda}_\delta) = \int_{(\mathbb{T}^2)^n} \mathbb{P}(\hat{X}([0,T];X) \in \overline{\Lambda}_\delta) d\mu_\xi(X), \tag{3.26}$$

where

$$\{\hat{X}([0,T];\xi) \in \overline{\Lambda}_\delta\} = \{\hat{X}(\cdot;\xi)([0,T]) \subset \overline{\Lambda}_\delta\}.$$

Taking then the limit $\delta \to 0$ gives

$$\mathbb{P}(\hat{X}([0,T];\xi) \in \Gamma^c) = \int_{(\mathbb{T}^2)^n} \mathbb{P}(\hat{X}([0,T];X) \in \Gamma^c) d\mu_\xi(X). \tag{3.27}$$

By Theorem 3.2, under the hypothesis that μ_ξ is absolutely continuous with respect to λ, we obtain

$$\mathbb{P}(\hat{X}([0,T];\xi) \in \Gamma^c) = 1. \tag{3.28}$$

This proves our corollary. Eventually, we will complete the proof Theorem 3.1 by means of (3.28). Let $X \in \Gamma^c$. There exists $\delta > 0$ such that $X \in \Lambda_{2\delta}$. Let $\eta \in (0,T)$. We make the distinction between the case $\tau_X^\delta \leq \eta$ and $\tau_X^\delta > \eta$:

$$\begin{aligned}
\mathbb{P}(\hat{X}([0,T];X) \in \Gamma^c) =& \mathbb{P}(\hat{X}([0,T];X) \in \Gamma^c; \tau_X^\delta \leq \eta) + \mathbb{P}(\hat{X}([0,T];X) \in \Gamma^c; \tau_X^\delta > \eta) \\
=& \mathbb{P}(\hat{X}([0,T];X) \in \Gamma^c; \tau_X^\delta \leq \eta) + \mathbb{P}(\hat{X}([\eta,T];X^\delta(\eta)) \in \Gamma^c; \tau_X^\delta > \eta) \quad (3.29) \\
=& \mathbb{P}(\hat{X}([0,T];X) \in \Gamma^c; \tau_X^\delta \leq \eta) - \mathbb{P}(\hat{X}([\eta,T];X^\delta(\eta)) \in \Gamma^c; \tau_X^\delta \leq \eta) \\
&+ \mathbb{P}(\hat{X}([\eta,T];X^\delta(\eta)) \in \Gamma^c) \quad (3.30)
\end{aligned}$$

The identity (3.29) is clear: if $\tau_X^\delta > \eta$, then X and X^δ coincide on $[0,\eta]$, and the occurrence of collision for $X(t)$, $t \in [0,T]$ is equivalent to the occurrence of collision for the trajectory $X(t)$, $t \in [\eta,T]$, which starts from $X(\eta) = X^\delta(\eta)$. Identity (3.30) is clear also (formula of total probability). By (3.28), we have $\mathbb{P}(\hat{X}([\eta,T];X^\delta(\eta)) \in \Gamma^c) = 1$. Indeed, under (3.5), the law of $X^\delta(\eta)$ is absolutely continuous with respect to the Lebesgue measure. This is a non-trivial result of course, see the discussion in Section 4. It follows from (3.30) that

$$\mathbb{P}(\hat{X}([0,T];X) \in \Gamma^c) \geq 1 - \mathbb{P}(\tau_X^\delta \leq \eta).$$

By continuity of the trajectories, we have $\lim_{\eta\to 0} \mathbb{P}(\tau_X^\delta \leq \eta) = 0$, hence $\mathbb{P}(\hat{X}([0,T];X) \in \Gamma^c) = 1$, which concludes the proof.

4 Hörmander's condition

The condition (3.5) is not exactly Hörmander's condition (the latter involves the vector field which appears in the drift part of the equation). We will not discuss (3.5) however, but the more simple condition

$$\forall X \in \Gamma^c, \quad \mathrm{span}(A_k(X))_{1\leq k\leq N} = (\mathbb{R}^2)^n. \tag{4.1}$$

Let μ_t denote the law of the solution X_t to the SDE (Itô form)

$$dX_t = A_0(X_t)dt + \sum_{k=1}^N A_k(X_t)d\beta_k(t), \quad X_t \in \mathbb{R}^{2n}. \tag{4.2}$$

The evolution equation for μ_t is the Fokker-Planck equation $\partial_t \mu_t + \mathcal{L}\mu_t = 0$, where

$$\mathcal{L}\mu = \nabla_X \cdot (A_0(X)\mu) - D_X^2 : (K(X)\mu). \tag{4.3}$$

The diffusion matrix in (4.3) is $K_{ij}(X) = \frac{1}{2}\sum_k A_k^i(X)A_k^j(X)$. For $Y \in \mathbb{R}^{2n}$, we have

$$K(X)Y \cdot Y = \frac{1}{2}\sum_k |A_k(X) \cdot Y|^2,$$

hence (4.1) implies that $\det(K(X)) > 0$, which in turn implies that μ_t is absolutely continuous with respect to the Lebesgue measure for $t > 0$ (we use the representation of μ_t thanks to the Green function of $\partial_t + \mathcal{L}$, see [9, Chapter IV-11]).

Theorem 3.1 holds true under the hypotheses that the vector fields $\sigma_k \in C^2(\mathbb{T}^2)$ are divergence free and (4.1) is satisfied. In [7, Section 4], the authors show that (4.1) is a generic condition.

Theorem 4.1. *Let $N > n^2$. There exists a residual subset Q of*

$$\left\{(\sigma_k)_{1,N}; \sigma_k \in C^\infty(\mathbb{T}^2); \operatorname{div}\sigma_k = 0\right\},$$

such that, for every $(\sigma_k)_{1,N} \in Q$, the vector fields A_k defined by (3.4) satisfy (4.1).

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