Can stratified turbulence be a state of wave turbulence of internal waves ?

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European Research Council
Established by the European Commission

Wave turbulence ?

fluid turbulence:

large ensemble of vortices
with wide range of sizes and time scales

Image: van Dyke, *An album of fluid motion* 1982

wave turbulence:

- waves
- large number of degrees of freedom
- nonlinear
- out of equilibrium

applicable to internal waves ?

Weak turbulence

two major successive hypotheses:

- large system (no discrete wave vectors and frequencies)
- weak nonlinearity
  - ➤ keep the wave structure
  - ➤ scale separation $T_{NL} >> T_{linear}$

Weak turbulence

from Zakharov's 1992 book

wave equation
(normal variables)

$$i\frac{\partial c(\boldsymbol{k},t)}{\partial t} - \omega_k c(\boldsymbol{k},t) = \int \Big[\frac{1}{2}V_{k12}c_1c_2\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2) + V^*_{1k2}c_1c^*_2\delta(\boldsymbol{k}_1-\boldsymbol{k}-\boldsymbol{k}_2)\Big]\,d\boldsymbol{k}_1 d\boldsymbol{k}_2$$

towards a statistical equation $\langle c_k c^*_{k'}\rangle = n(\boldsymbol{k})\delta(\boldsymbol{k}-\boldsymbol{k}')$ evolution in time of $n(\mathbf{k})$

To calculate $\partial n(\boldsymbol{k},t)/\partial t$, we shall multiply (2.1.3) by $c^*(\boldsymbol{k},t)$, the complex conjugate equation by $c(\boldsymbol{k}',t)$, subtract the latter from the former and average. Setting $\boldsymbol{k}=\boldsymbol{k}'$, we obtain:

$$\frac{\partial n(\boldsymbol{k},t)}{\partial t} = \mathrm{Im}\int [V_{k12}J_{k12}\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2) - 2V_{1k2}J_{1k2}\delta(\boldsymbol{k}_1-\boldsymbol{k}-\boldsymbol{k}_2)]\,d\boldsymbol{k}_1 d\boldsymbol{k}_2\ . \qquad (2.1.9a)$$

Here

$$J_{123}\delta(\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3) = \langle c^*_1 c_2 c_3\rangle \qquad (2.1.9b)$$

is the triple correlation function. For a free field, $J_{123}=J^{(0)}_{123}=0$. This implies that in first order perturbation theory in $\mathcal{H}_{\mathrm{int}}$ we have $\partial n(\boldsymbol{k},t)/\partial t = 0$. In order to calculate $\partial n(\boldsymbol{k},t)/\partial t$ in second order perturbation theory in $\mathcal{H}_{\mathrm{int}}$, one should know J_{123} in the first order. Using definition (2.1.9b) and the equations of motion (2.1.3), we calculate $\partial J/\partial t$:

Weak turbulence

from Zakharov's 1992 book

(2.1.3), we calculate $\partial J/\partial t$:

$$\left[i\frac{\partial}{\partial t} + (\omega_1 - \omega_2 - \omega_3)\right] J(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3; t)$$
$$= \int \Big[-\frac{1}{2} V^*_{145} J_{4523} \delta(\boldsymbol{k}_1 - \boldsymbol{k}_4 - \boldsymbol{k}_5)$$
$$+ V^*_{425} J_{1534} \delta(\boldsymbol{k}_4 - \boldsymbol{k}_2 - \boldsymbol{k}_5)$$
$$+ V_{435} J_{1524} \delta(\boldsymbol{k}_4 - \boldsymbol{k}_3 - \boldsymbol{k}_5)\Big] d\boldsymbol{k}_4 d\boldsymbol{k}_5 \ .$$

Here

$$J_{1234} \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4) = \langle c_1^* c_2 c_3 c_4 \rangle$$

the evolution of J_{123} is ruled by J_{1234}:
hierarchy of equations ➤ closure ?

at the lowest order: J_{1234} comes for random waves

related to hypothesis of random phase and amplitude

$$\langle c_1^* c_2^* c_3 c_4 \rangle = n(\boldsymbol{k}_1) n(\boldsymbol{k}_2) [\delta(\boldsymbol{k}_1 - \boldsymbol{k}_3)\delta(\boldsymbol{k}_2 - \boldsymbol{k}_4) + \delta(\boldsymbol{k}_1 - \boldsymbol{k}_4)\delta(\boldsymbol{k}_2 - \boldsymbol{k}_3)]$$

Weak turbulence

from Zakharov's 1992 book

Remaining consistantly within second order perturbation theory, one should neglect the time dependence of $n(\boldsymbol{k},t)$ on the right-hand-side of (2.1.10c), i.e., set A =const. Then this equation may be solved

$$J(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_1;t) = B\exp(-i\Delta\omega t) + A_{123}/\Delta\omega, \quad \Delta\omega = \omega_1 - \omega_2 - \omega_3 \; . \tag{2.1.10d}$$

Substituting the first term into (2.1.9a), we get at $t \neq 0$ an integral of a fast oscillating function. Its contribution decreases with increasing t and becomes nonessential at times larger than $1/\omega(\boldsymbol{k})$. The second term of (2.1.10d) gives for J_1 an expression depending via $n(\boldsymbol{k}_j,t)$ parametrically on the slow time:

$$J_1(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3;t) = \frac{V^*_{123}(n_1n_2 + n_1n_3 - n_2n_3)}{\omega_1 - \omega_2 - \omega_3 + i\delta} \; . \tag{2.1.11}$$

To the denominator we have added the term $i\delta$ to circumvent the pole. It may be obtained in a consistent procedure by considering the free wave field at $t \to -\infty$ and adiabatically slowly including the interaction [for $t \gg 1/\omega(\boldsymbol{k})$]. The sign of δ can also be determined by accounting for the presence of small damping. Substituting (2.1.11) into (2.1.9a) and using $\mathrm{Im}\,\{\omega + i\delta\} = -\pi\delta(\omega)$, we obtain the three-wave kinetic equation

$$\boxed{\begin{aligned}\frac{\partial n(\boldsymbol{k},t)}{\partial t} = \pi \int \Big[&|V_{k12}|^2 f_{k12}\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\delta(\omega_k - \omega_1 - \omega_2) \\ &+ 2|V_{1k2}|^2 f_{1k2}\delta(\boldsymbol{k}_1-\boldsymbol{k}-\boldsymbol{k}_2)\delta(\omega_1-\omega_k-\omega_2)\Big]\, d\boldsymbol{k}_1 d\boldsymbol{k}_2,\end{aligned}} \tag{2.1.12a}$$

kinetic equation

$$f_{k12} = n_1n_2 - n_k(n_1+n_2), \quad n_j = n(\boldsymbol{k}_j,t). \tag{2.1.12b}$$

Weak turbulence

hypotheses: large system and weak nonlinearity

➤ scale separation $T_{NL} >> T_{linear}$

kinetic equation of the wave action spectrum:

$$n_{\mathbf{k}} = \langle a_{\mathbf{k}} a_{\mathbf{k}}^* \rangle$$

slow evolution of the wave spectrum:

$$\frac{\partial n_{\mathbf{k}_1}}{\partial t} = \epsilon^2 4\pi \int |V_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}|^2 \delta(\mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_1) n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} F(\omega_1, \omega_2, \omega_3) d\mathbf{k}_2 \mathbf{k}_3 \qquad N = 3$$

$$F(\omega_1,\omega_2,\omega_3) = \left(\frac{1}{n_{\mathbf{k}_1}} - \frac{1}{n_{\mathbf{k}_2}} - \frac{1}{n_{\mathbf{k}_3}}\right)\delta(\omega_1 - \omega_2 - \omega_3) + \left(\frac{1}{n_{\mathbf{k}_1}} - \frac{1}{n_{\mathbf{k}_2}} + \frac{1}{n_{\mathbf{k}_3}}\right)\delta(\omega_2 - \omega_1 - \omega_3) + \left(\frac{1}{n_{\mathbf{k}_1}} + \frac{1}{n_{\mathbf{k}_2}} - \frac{1}{n_{\mathbf{k}_3}}\right)\delta(\omega_3 - \omega_1 - \omega_2)$$

energy transfers only through resonant waves

$$\omega_1 = \omega_2 + \omega_3 \qquad\qquad \mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3$$

(energy conservation) (momentum conservation)

see S. Nazarenko, *Wave Turbulence* 2011

Weak turbulence

Kolmogorov-Zakharov spectrum: stationary solutions of out of equilibrium case

scale separation between forcing (usually at large scales) and dissipation (usually small scale)

- **capillary surface waves** (3 waves) $\omega = \sqrt{\frac{\gamma}{\rho} k^3}$ $E_k^{(1D)} \sim \sqrt{\varepsilon} \sigma^{1/4} k^{-7/4}$

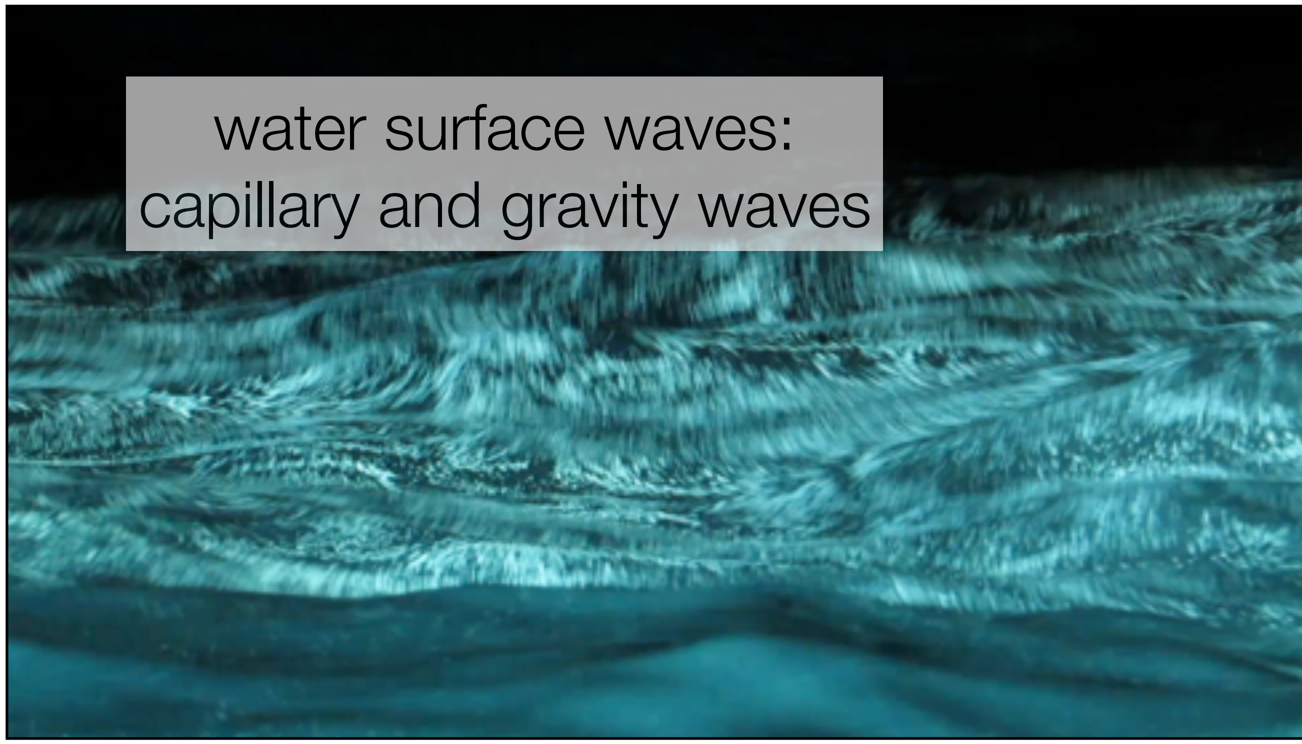
ε injected power

- **gravity surface waves** (4 waves) $\omega = \sqrt{gk}$ $E_k^{(1D)} \sim g^{1/2} \varepsilon^{1/3} k^{-5/2}$

- **elastic waves in vibrating plate** (4 waves) $\omega = Ck^2$ $E_k^{(1D)} \sim \epsilon^{1/3} \frac{\log^{1/3}(k^\star/k)}{k^3}$

- **elastic waves in a membrane** (4 waves) $\omega = Tk$ $E_k^{(1D)} \sim \epsilon^{1/3} k^{-10/3}$

Systems that we study in LEGI

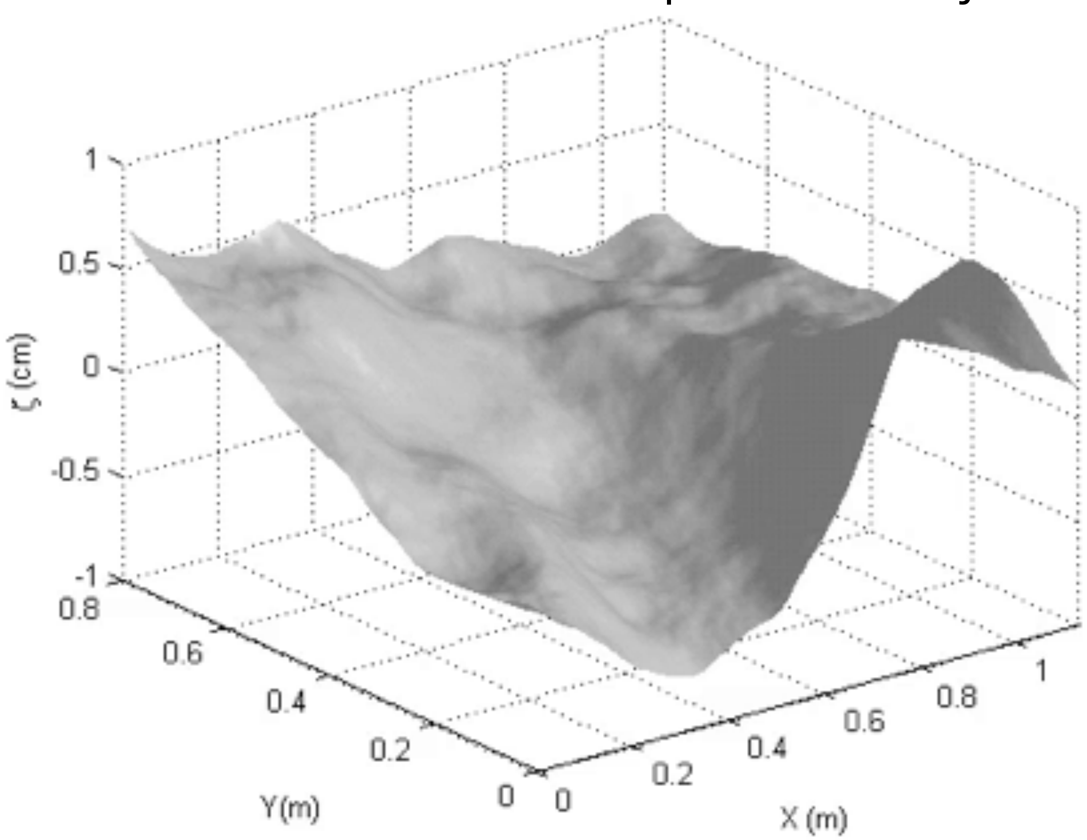


space and time resolved measurement

observation of weak turbulence

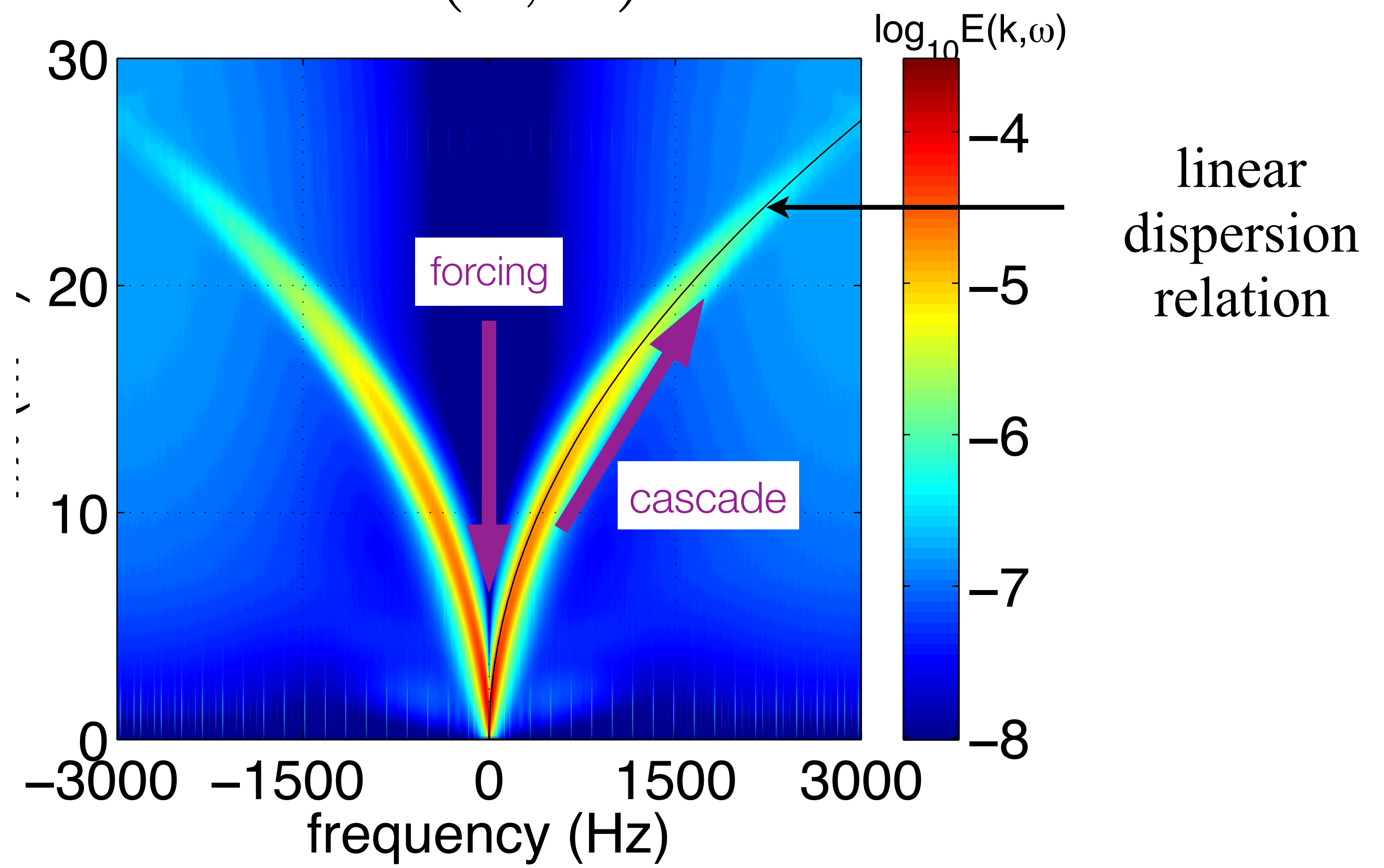
vibrating plate experiment

space and time resolved measurement:
Fourier transform profilometry



frequency-wavenumber
spectrum

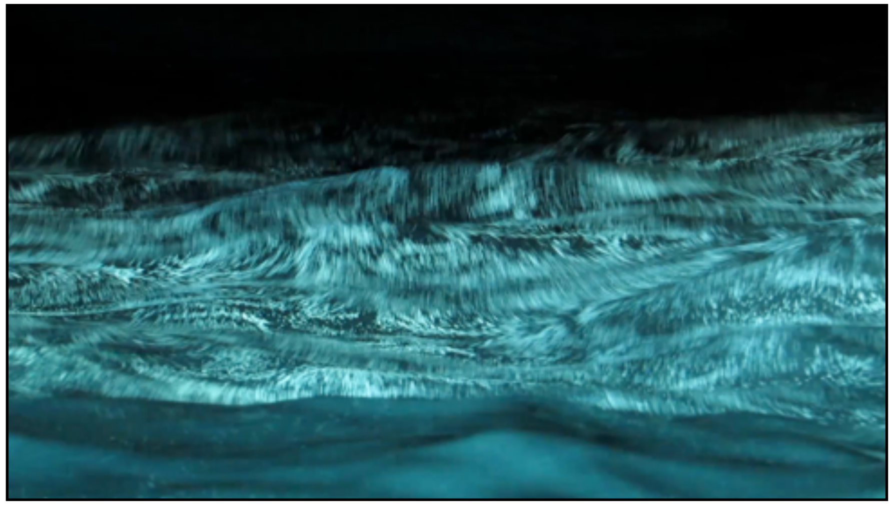
$$E(\mathbf{k},\omega)$$



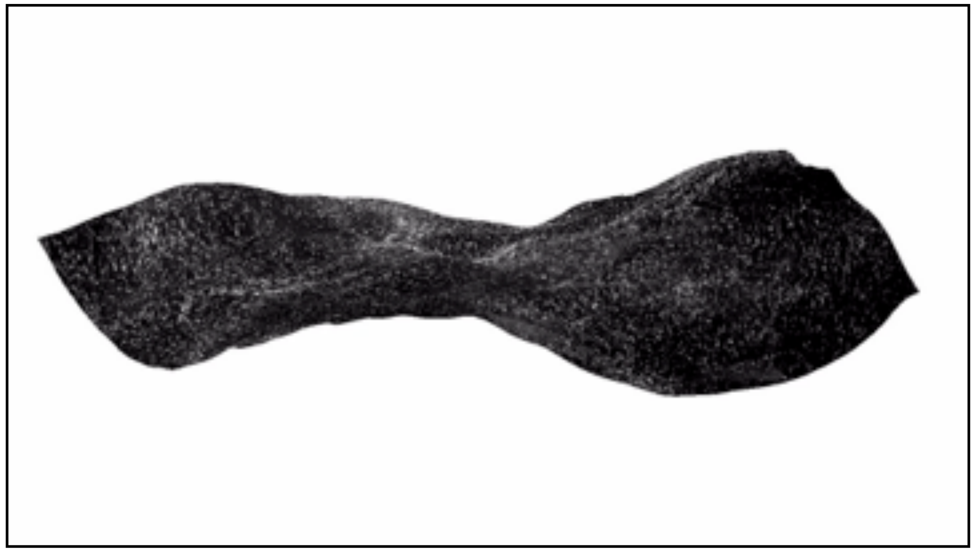
➥ **weak wave turbulence...**

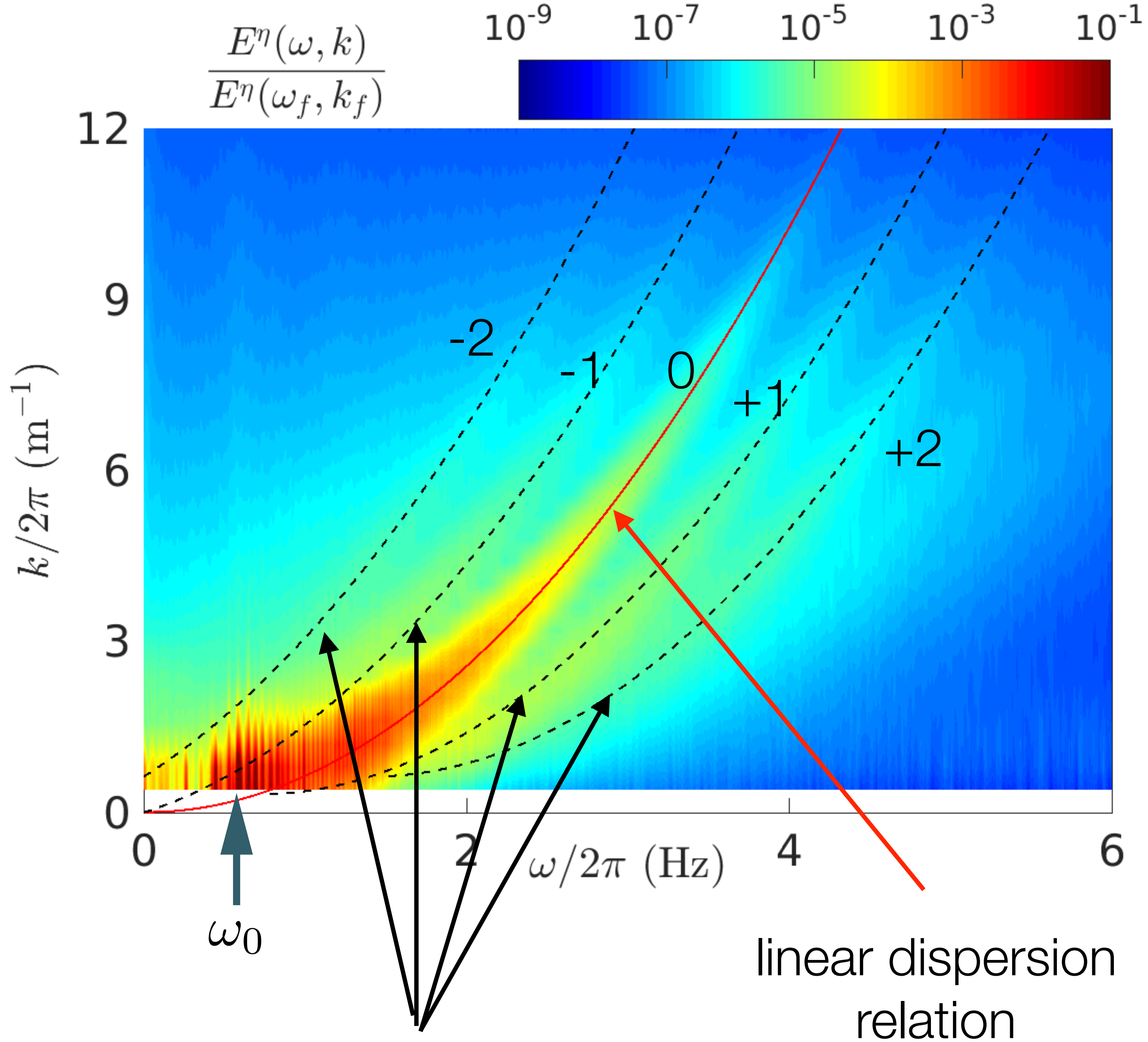
Cobelli *et al. PRL* 2009
Mordant *EPJB* 2010

surface gravity waves: bound waves



space and time resolved measurement:
stereoscopic imaging





bound waves

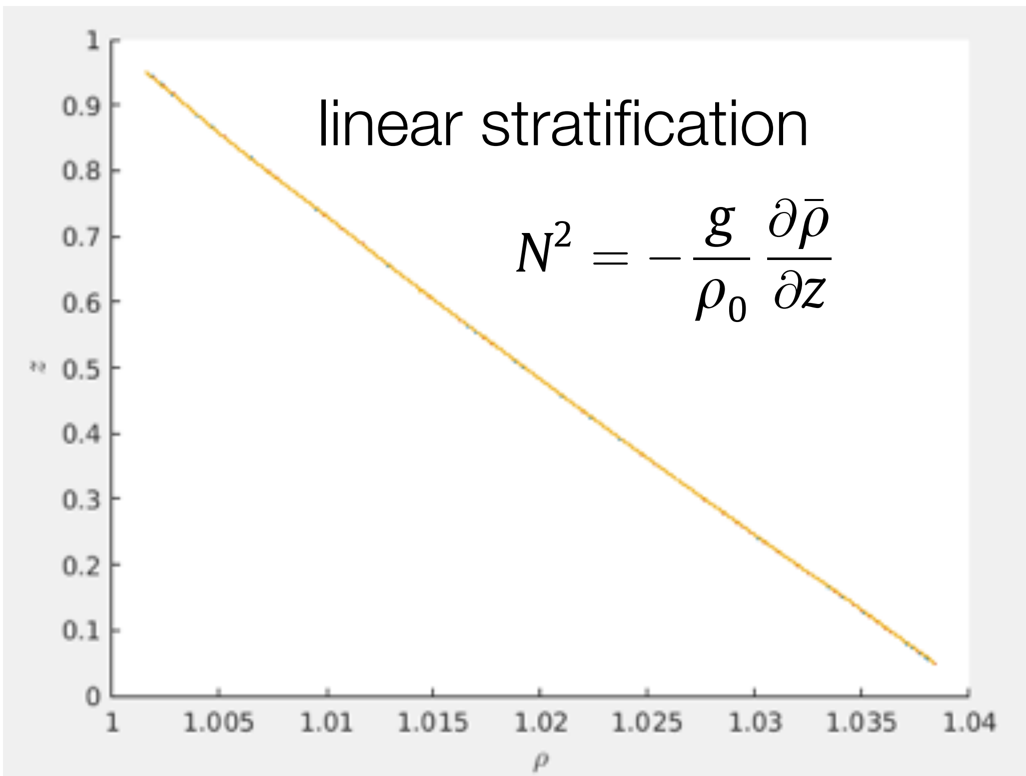
linear dispersion relation
$$\omega^2 = g|\mathbf{k}|$$

$$\omega^{(\pm n)} = \omega(k \mp nk_0) \pm n\omega_0$$

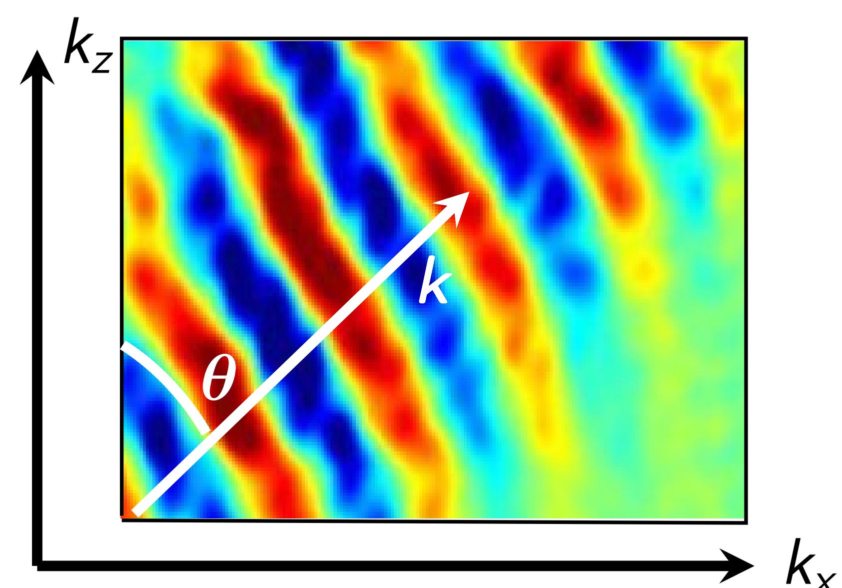
n=0: free waves

what about internal waves ?

linear internal waves



$$\omega^2 = N^2 \sin^2\theta$$

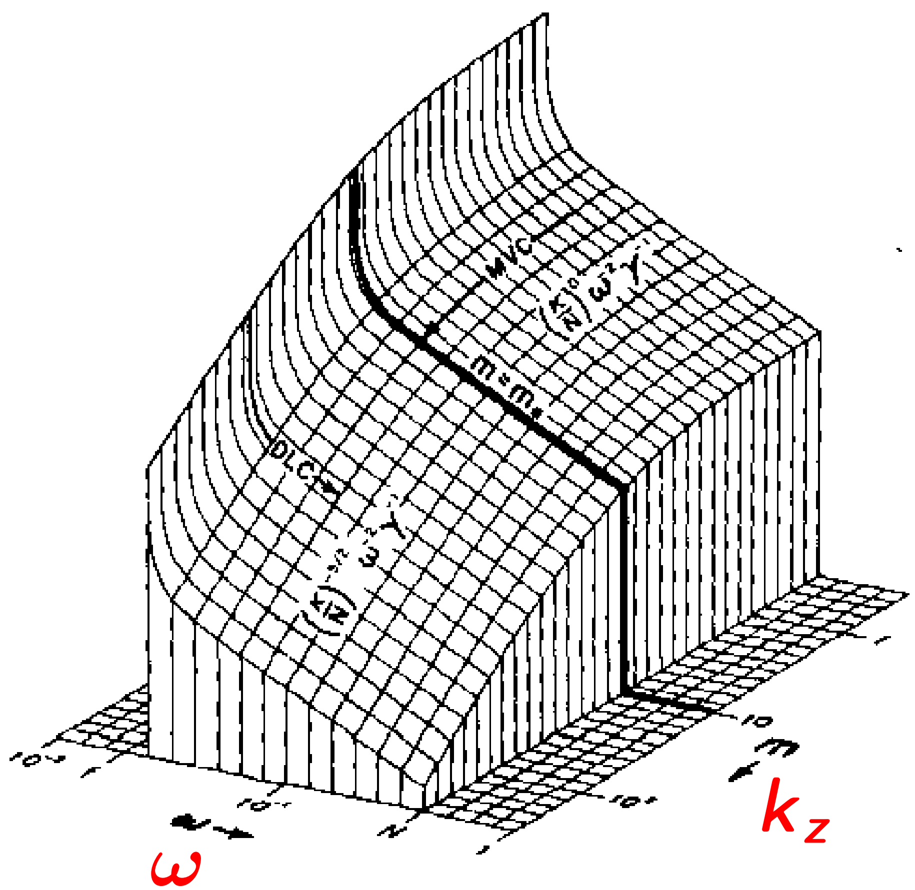


Background field courtesy of Lérisson, Chomaz and Ortiz (LadHyX, France)

Observations

Garrett & Munk spectrum:
observation of oceanic data
empirical spectrum

C. J. R. Garrett and W. H. Munk, Geophys. Fluid Dyn. **2**, 225 (1972).



high frequency part:

$$E(m, \omega) \simeq N m^{-2} \omega^{-2}$$

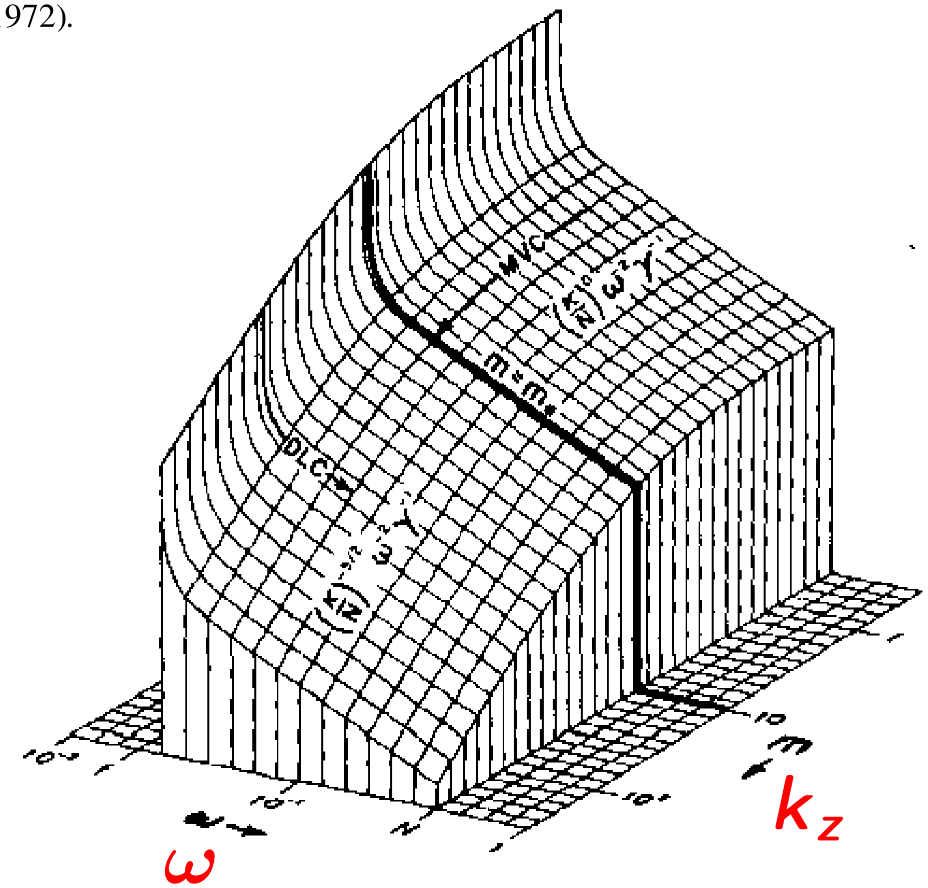
m: vertical wavenumber

interpreted as a spectrum
of nonlinear internal waves

Observations

Garrett & Munk spectrum:
observation of oceanic data
empirical spectrum

C. J. R. Garrett and W. H. Munk, Geophys. Fluid Dyn. **2**, 225 (1972).



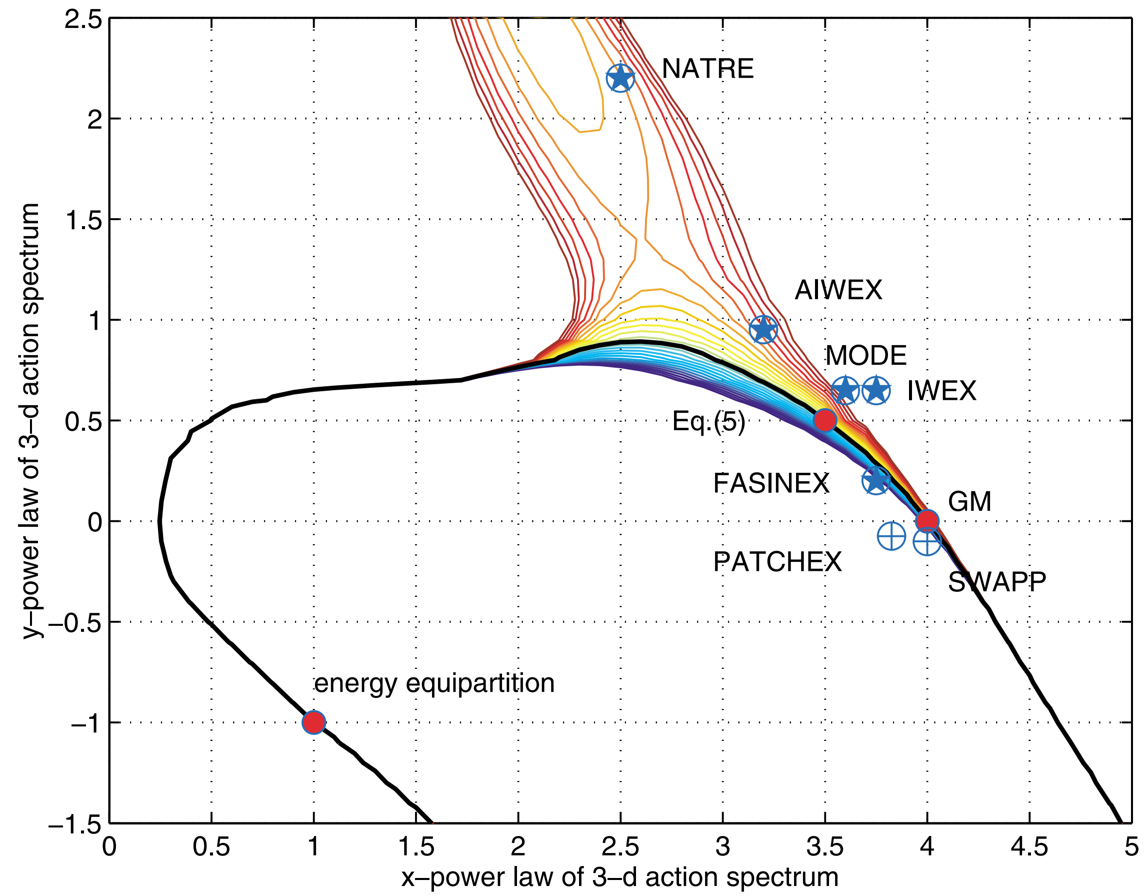
high frequency part:

$$E(m,\omega) \simeq N m^{-2}\omega^{-2}$$

m: vertical wavenumber

other datasets:

$$E(m,\omega) \propto \omega^{2-x} m^{2-x-y}$$



L'vov, Polzin & Tabak, *PRL* 2004

universality of the GM spectrum ?

Weak turbulence theory applied to internal waves

L'vov, Polzin & Tabak, *PRL* 2004

derivation of the kinetic equation
for internal gravity waves

$$\frac{dn_{k,m}}{dt} = \frac{1}{k}\int (R^p_{p_1p_2} - R^{p_1}_{pp_2} - R^{p_2}_{p_1p})\, dp_1 dp_2/\Delta^k_{k_1k_2},$$

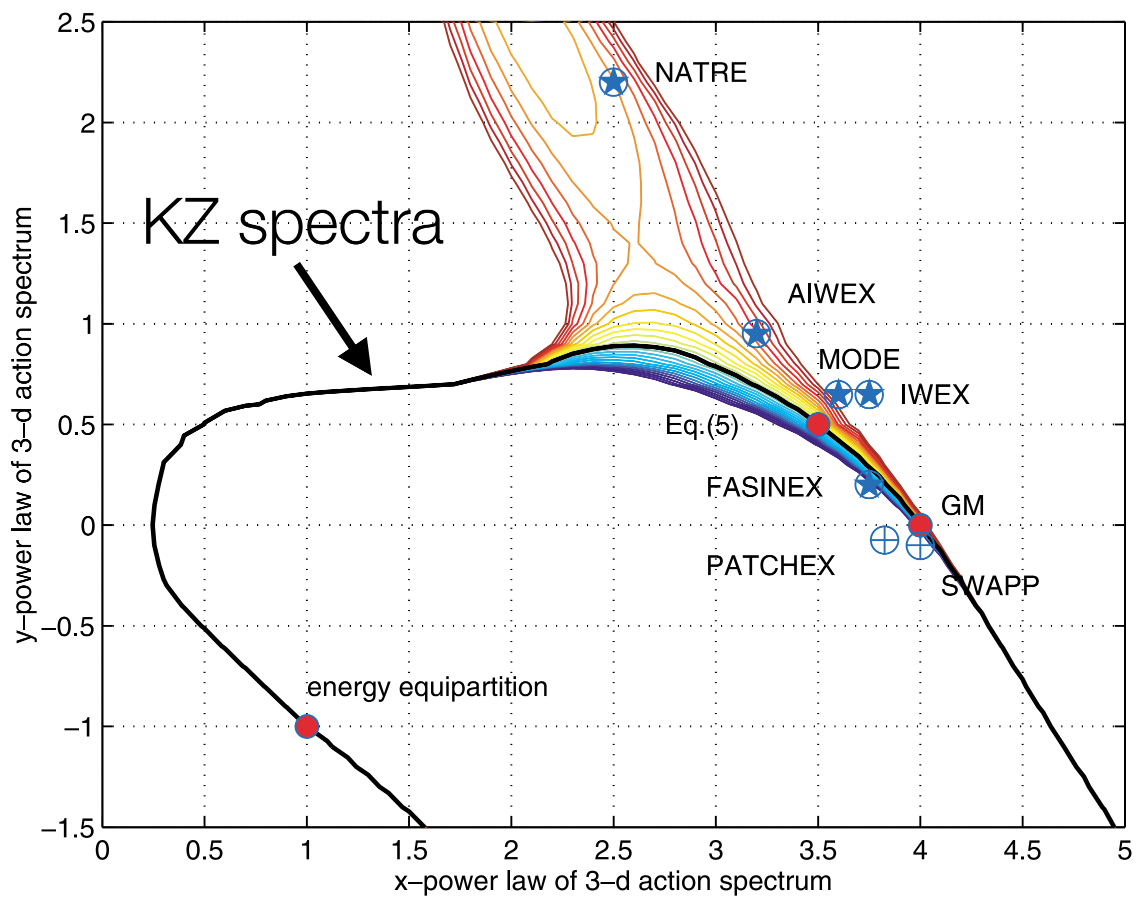
$$R^p_{p_1p_2} = \delta_{\omega_p-\omega_{p_1}-\omega_{p_2}} f^p_{p_1p_2} |V^p_{p_1p_2}|^2 \delta_{m-m_1-m_2 kk_1k_2}, \qquad (4)$$

where $f^p_{p_1p_2} = n_{p_1}n_{p_2} - n_p(n_{p_1} + n_{p_2})$ and $\Delta^k_{k_1k_2} = \{2[(kk_1)^2 + (kk_2)^2 + (k_1k_2)^2] - k^4 - k_1^4 - k_2^4\}^{1/2}/2.$

$$V^{\mathbf{p}}_{\mathbf{p}_1\mathbf{p}_2} = U^{\mathbf{p}}_{\mathbf{p}_1\mathbf{p}_2} + U^{\mathbf{p}_1}_{\mathbf{p}\mathbf{p}_2} + U^{\mathbf{p}_2}_{\mathbf{p}\mathbf{p}_1} \text{ with } U^{\mathbf{p}}_{\mathbf{p}_1\mathbf{p}_2} = -\frac{N}{4\sqrt{2g}}\frac{\mathbf{k}_2\cdot\mathbf{k}_3}{k_2k_3}\sqrt{\frac{\omega_{\mathbf{p}_2}\omega_{\mathbf{p}_3}}{\omega_{\mathbf{p}_1}}}k_1.$$

a family of solutions

$$E(m,\omega) \propto \omega^{2-x} m^{2-x-y}$$



includes the GM spectrum

BUT: issues with the KZ spectra

Oceanic Internal-Wave Field: Theory of Scale-Invariant Spectra

J. Phys. Oceanogr. 2010

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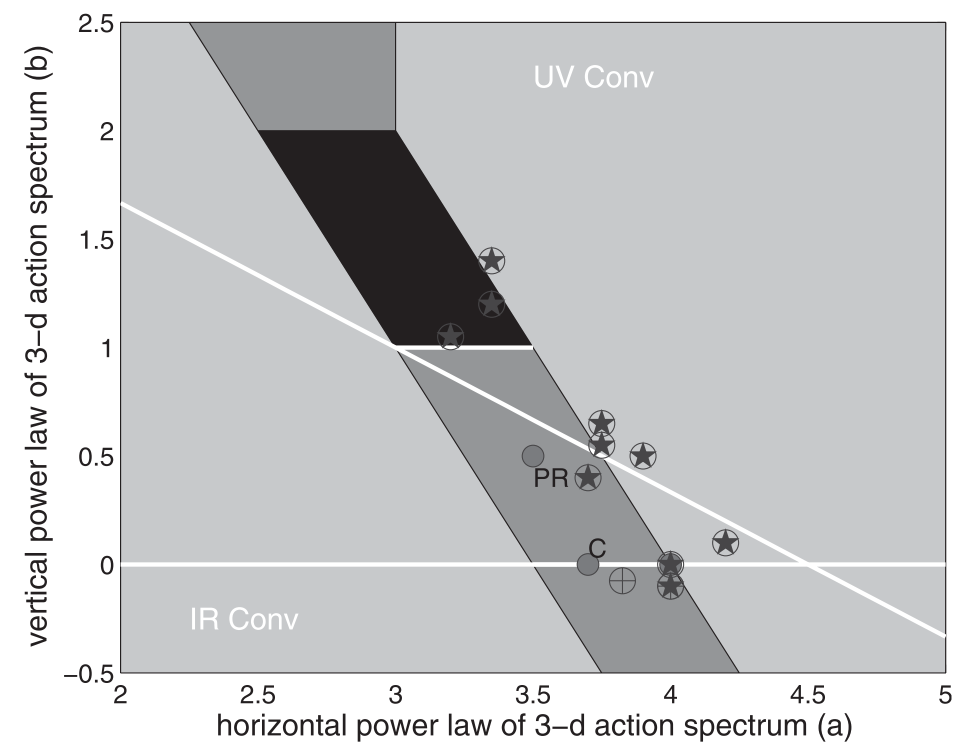
(Manuscript received 28 August 2008, in final form 28 July 2010)

ABSTRACT

Steady scale-invariant solutions of a kinetic equation describing the statistics of oceanic internal gravity waves based on wave turbulence theory are investigated. It is shown in the nonrotating scale-invariant limit that $\boxed{\text{the collision integral in the kinetic equation diverges for almost all spectral power-law exponents.}}$ These divergences come from resonant interactions with the smallest horizontal wavenumbers and/or the largest horizontal wavenumbers with extreme scale separations.

A small domain is identified in which the scale-invariant collision integral converges and numerically find a convergent power-law solution. This numerical solution is close to the Garrett–Munk spectrum. Power-law exponents that potentially permit a balance between the infrared and ultraviolet divergences are investigated. The balanced exponents are generalizations of an exact solution of the scale-invariant kinetic equation, the Pelinovsky–Raevsky spectrum. A small but finite Coriolis parameter representing the effects of rotation is introduced into the kinetic equation to determine solutions over the divergent part of the domain using rigorous asymptotic arguments. This gives rise to the induced diffusion regime.

The derivation of the kinetic equation is based on an assumption of weak nonlinearity. Dominance of the nonlocal interactions puts the self-consistency of the kinetic equation at risk. However, these weakly nonlinear stationary states are consistent with much of the observational evidence.



small region of exponents for which the divergences may cancel

BUT: issues with the KZ spectra

J. Phys. Oceanogr. 2012

Resonant and Near-Resonant Internal Wave Interactions

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(Manuscript received 19 August 2008, in final form 25 March 2011)

ABSTRACT

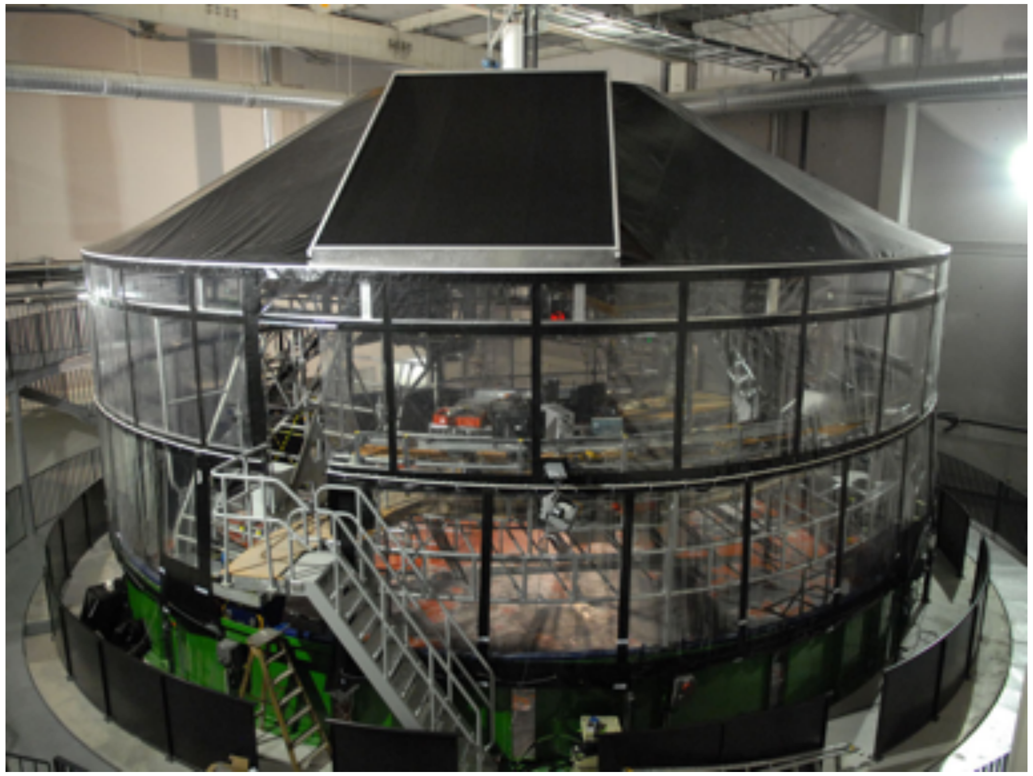
The spectral energy density of the internal waves in the open ocean is considered. The Garrett and Munk spectrum and the resonant kinetic equation are used as the main tools of the study. Evaluations of a resonant kinetic equation that suggest the slow time evolution of the Garrett and Munk spectrum is not in fact slow are reported. Instead, nonlinear transfers lead to evolution time scales that are smaller than one wave period at high vertical wavenumber. Such values of the transfer rates are inconsistent with the viewpoint expressed in papers by C. H. McComas and P. Müller, and by P. Müller et al., which regards the Garrett and Munk spectrum as an approximate stationary state of the resonant kinetic equation. It also puts the self-consistency of a resonant kinetic equation at a serious risk. The possible reasons for and resolutions of this paradox are explored. Inclusion of near-resonant interactions decreases the rate at which the spectrum evolves. Consequently, this inclusion shows a tendency of improving of self-consistency of the kinetic equation approach.

Our fundamental result is that the GM spectrum is not stationary with respect to the resonant interaction approximation. This result is contrary to the point of view expressed in McComas and Müller (1981a) and Müller et al. (1986) and gave us cause to review published results concerning resonant internal wave interactions. We also arrived at the point where we can say that the resonant kinetic equation does not constitute a self-consistent approach. We then included near-resonant interactions and found significant reductions in the temporal evolution of the GM spectrum.

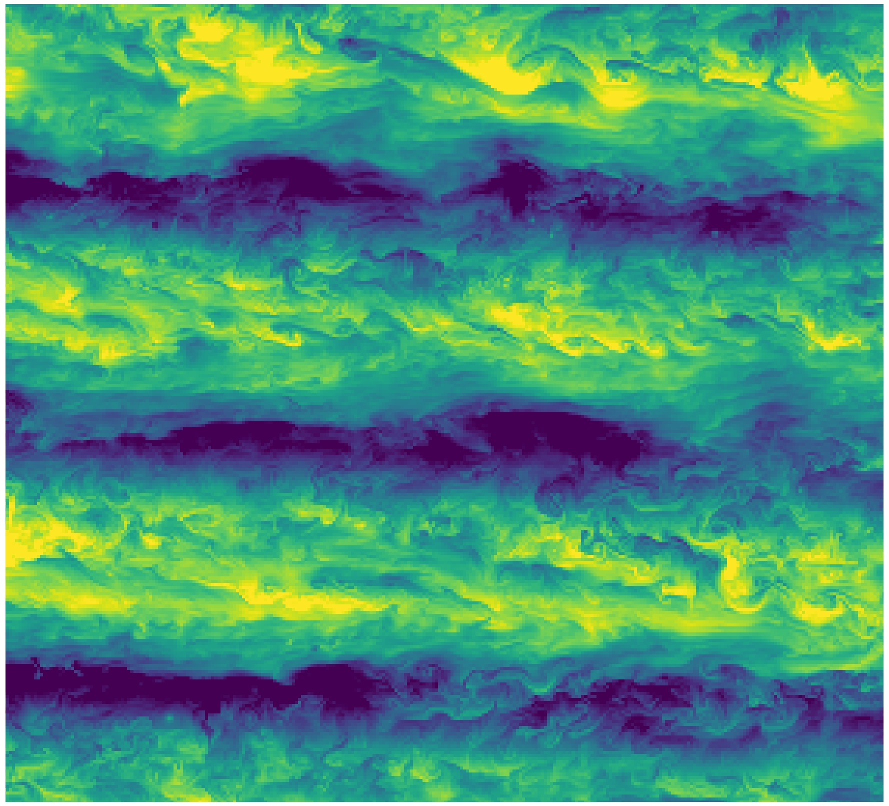
no weak turbulence of internal waves ?

excite stratified turbulence with waves

experiments in the Coriolis facility



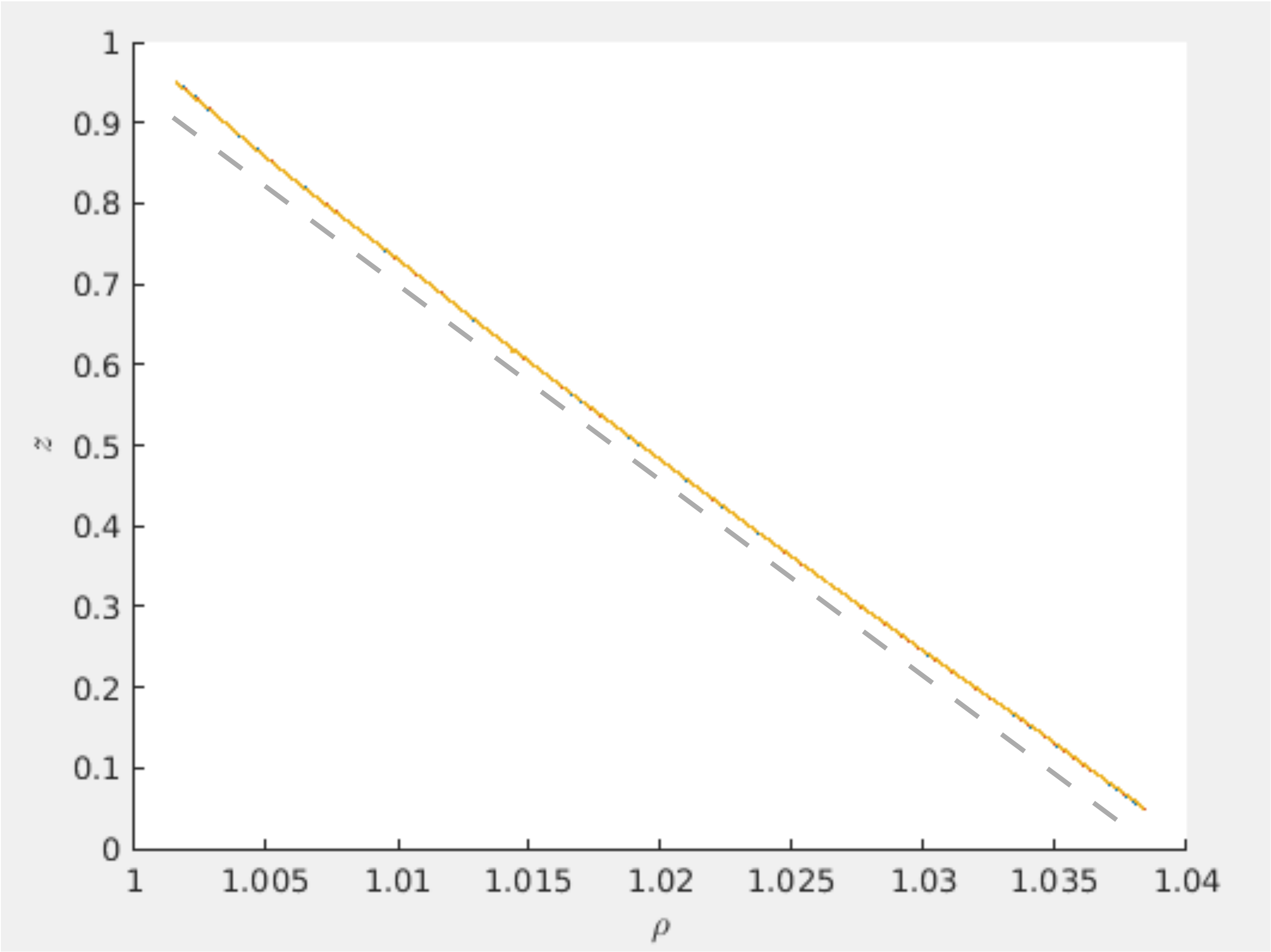
numerical simulations



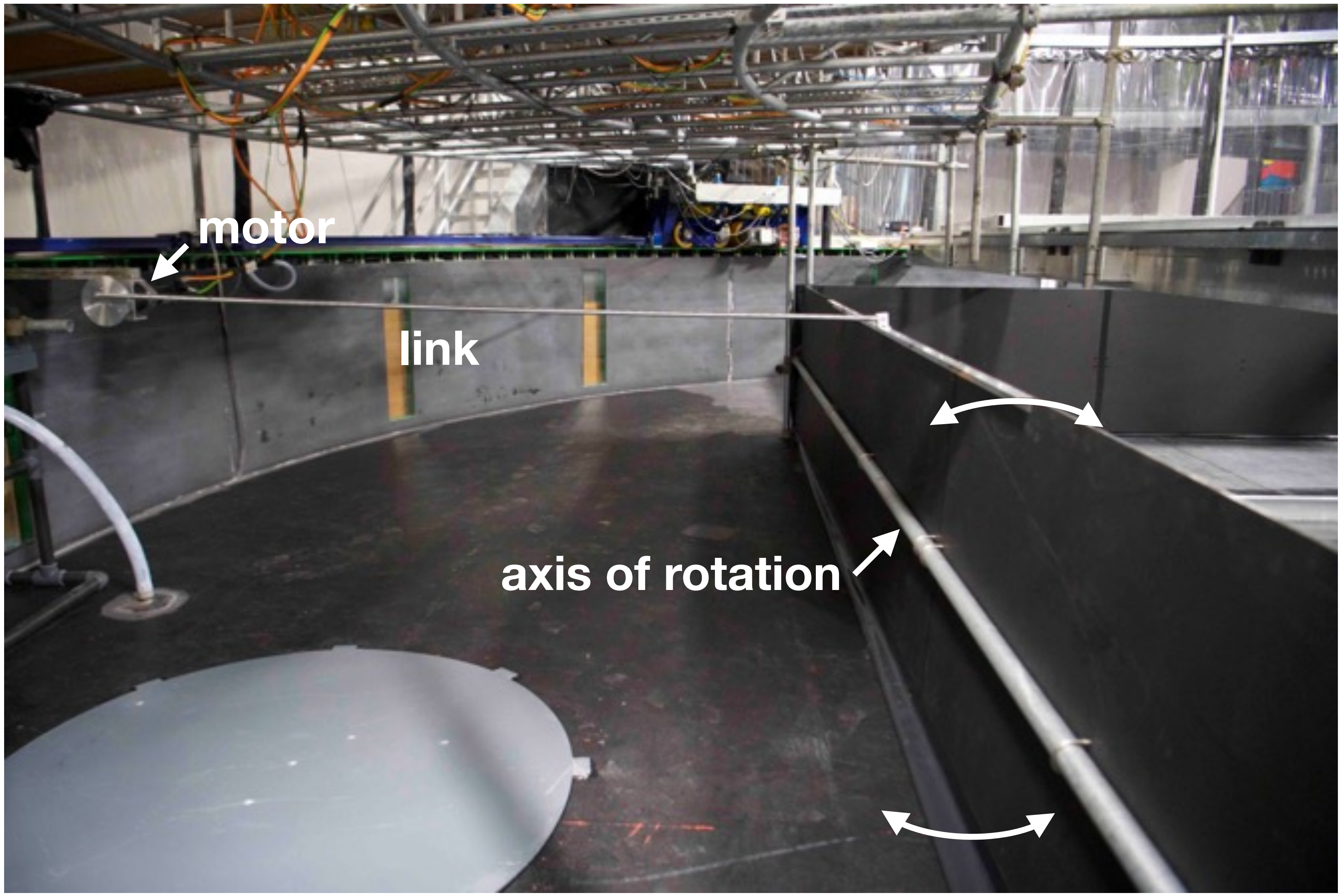
2D and 3D pseudospectral DNS

first results, work in progress…

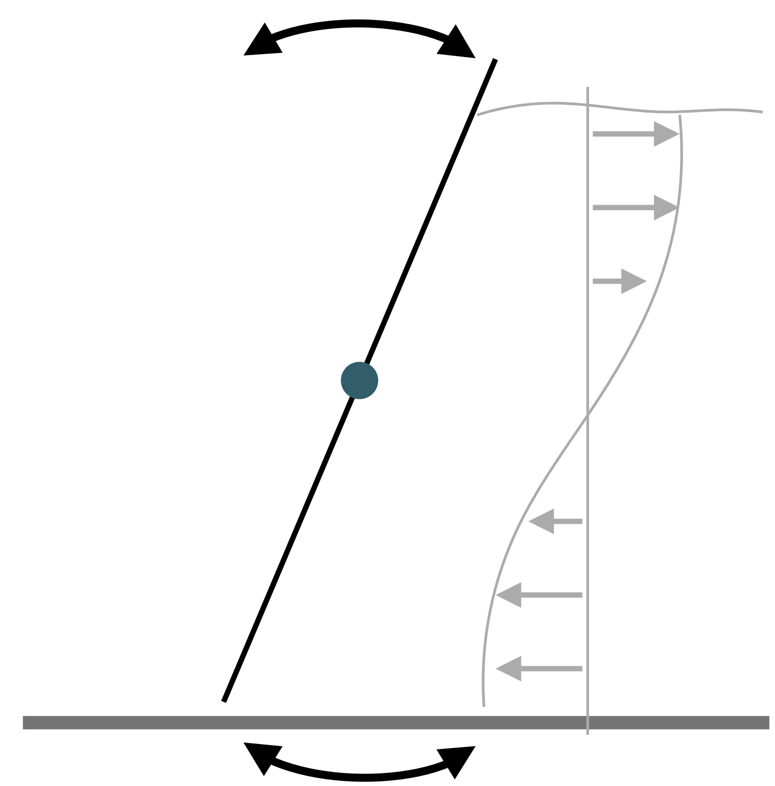
linear stratification



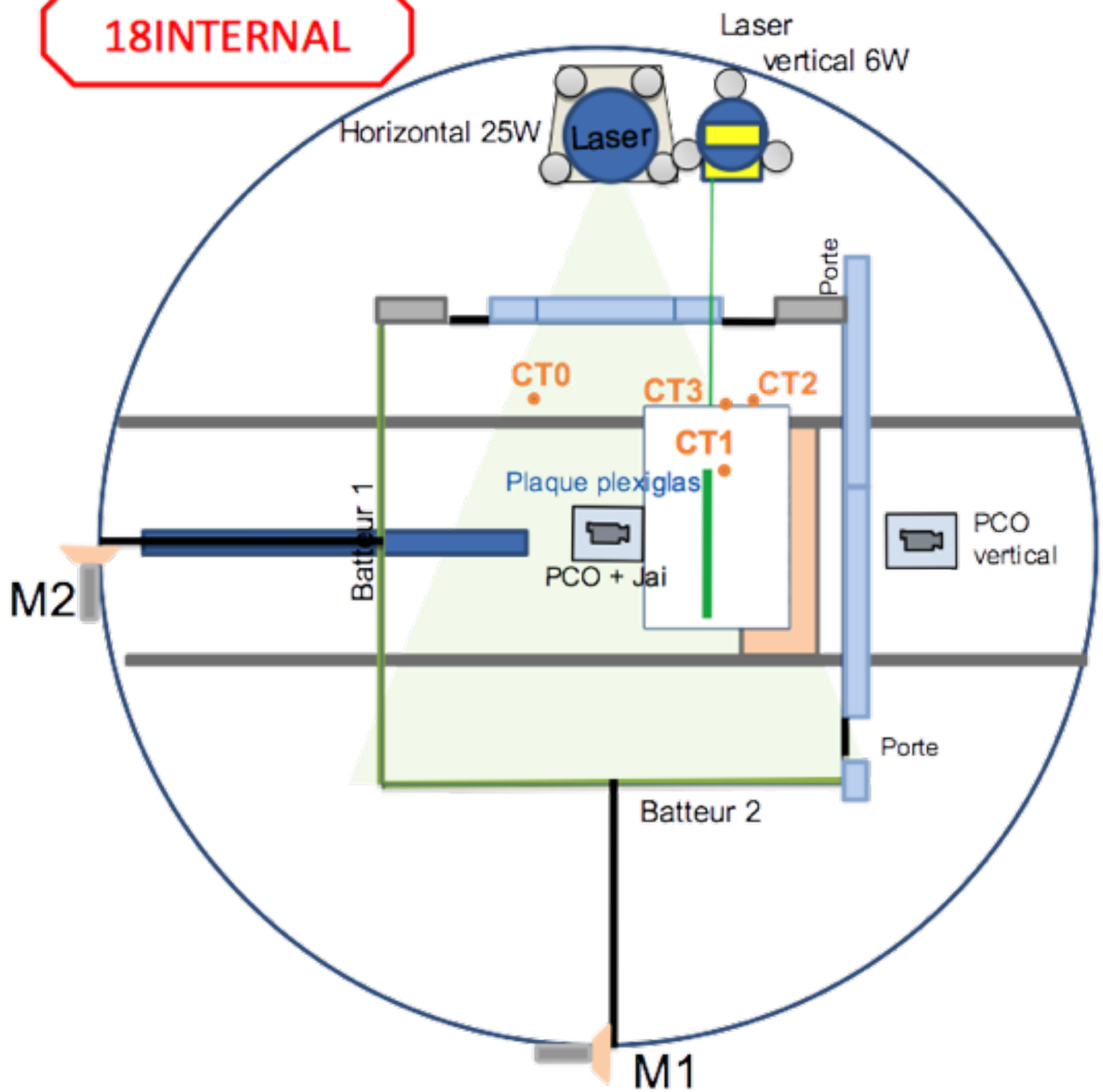
forcing of the waves



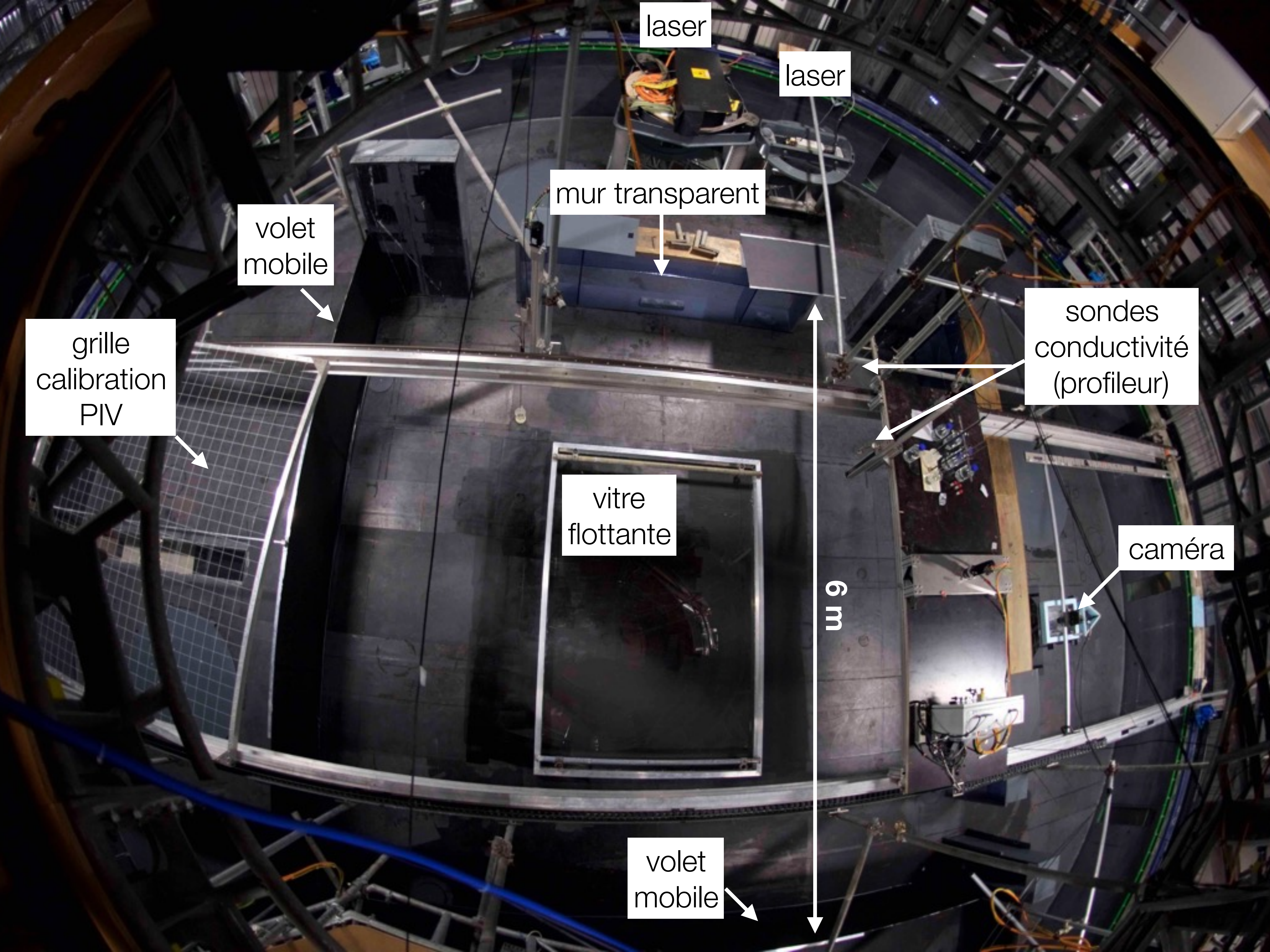
generation by an oscillating panel



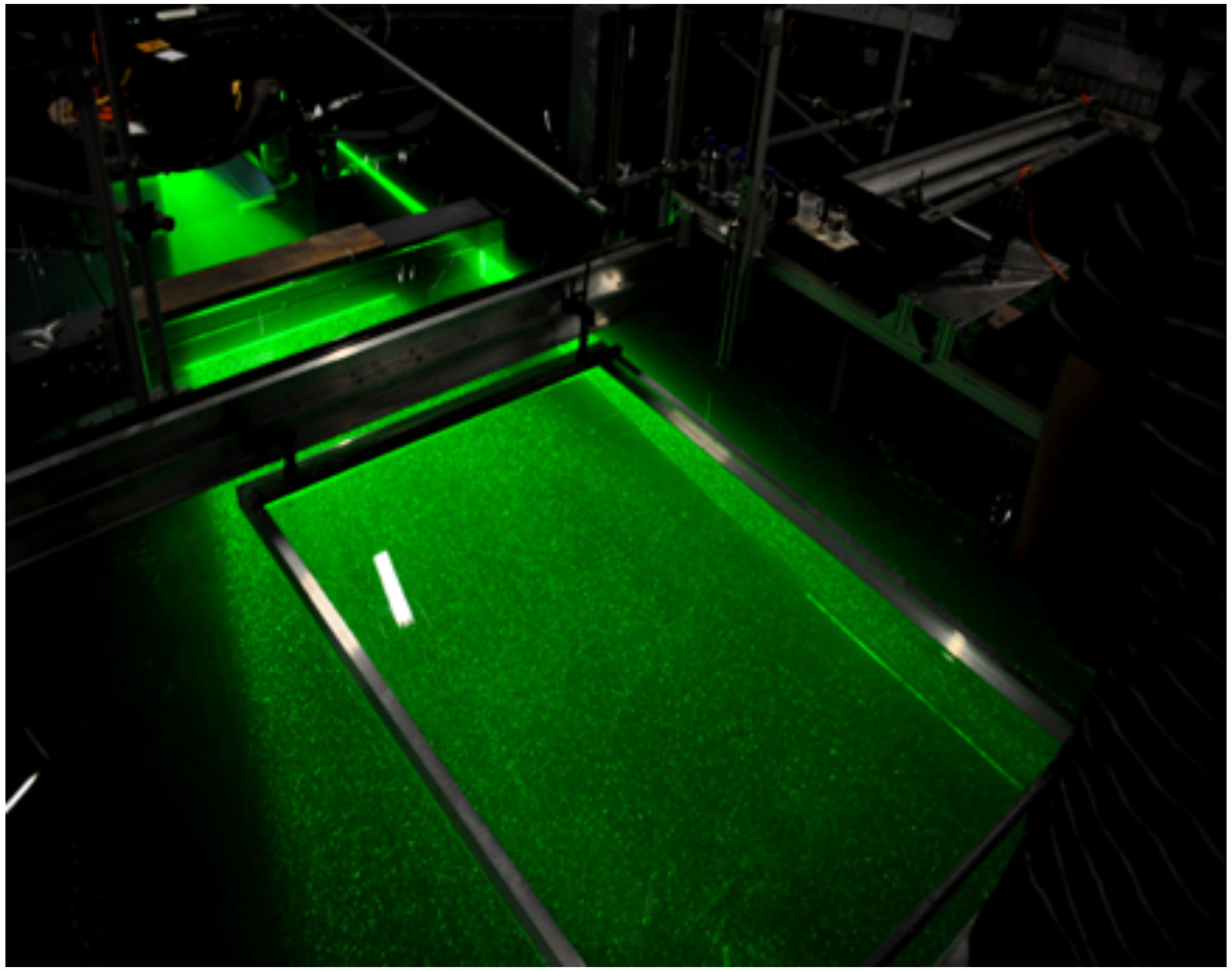
18INTERNAL



Laser
vertical 6W
Horizontal 25W
Laser
Porte
CT0
CT3
CT2
CT1
Plaque plexiglas
Batteur 1
PCO + Jai
PCO
vertical
M2
Porte
Batteur 2
M1


laser
laser
mur transparent
volet
mobile
grille
calibration
PIV
sondes
conductivité
(profileur)
vitre
flottante
6 m
caméra
volet
mobile

time-resolved Particle Image Velocimetry (PIV) measurement

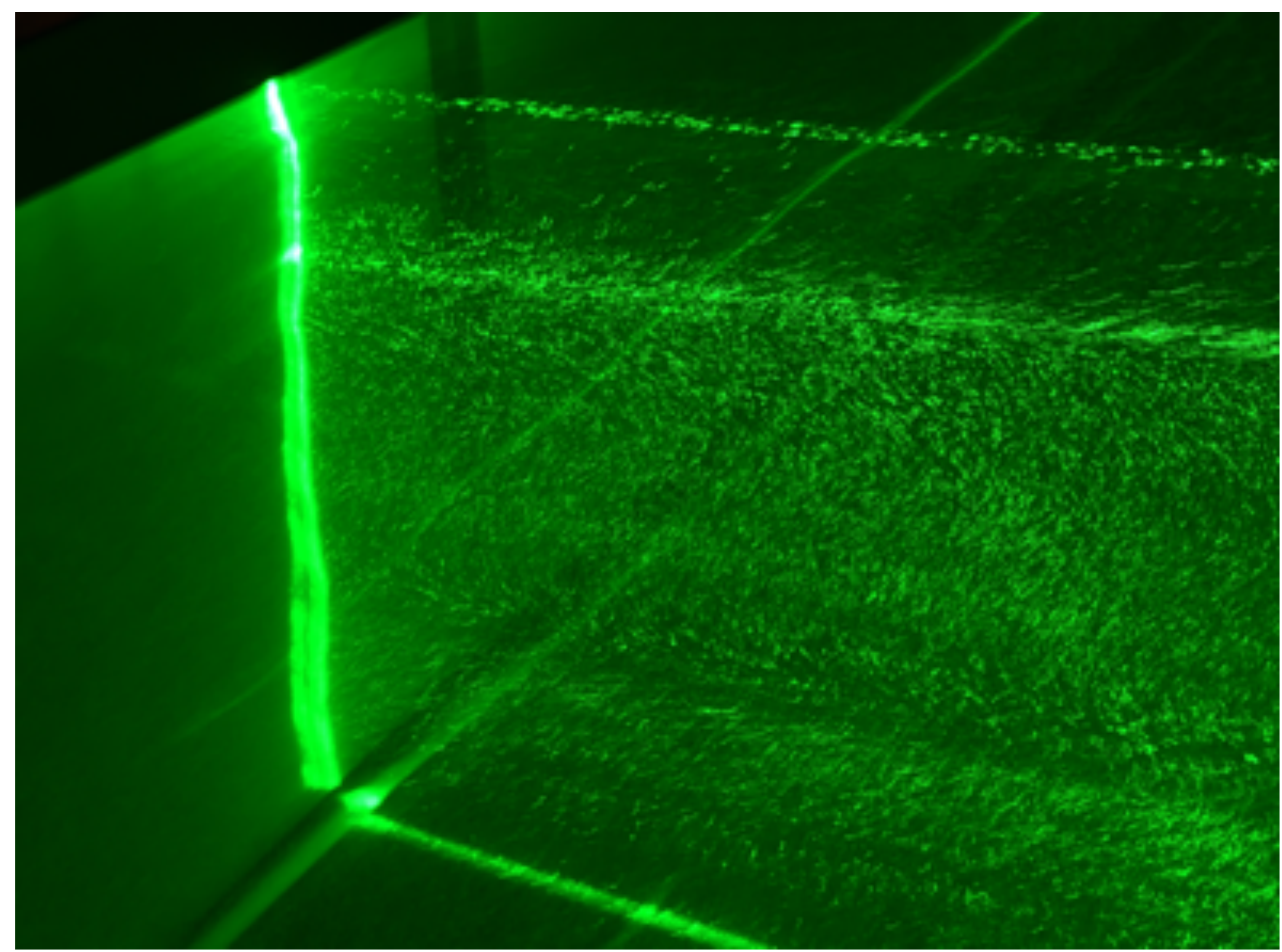


horizontal laser sheet:
horizontal velocity

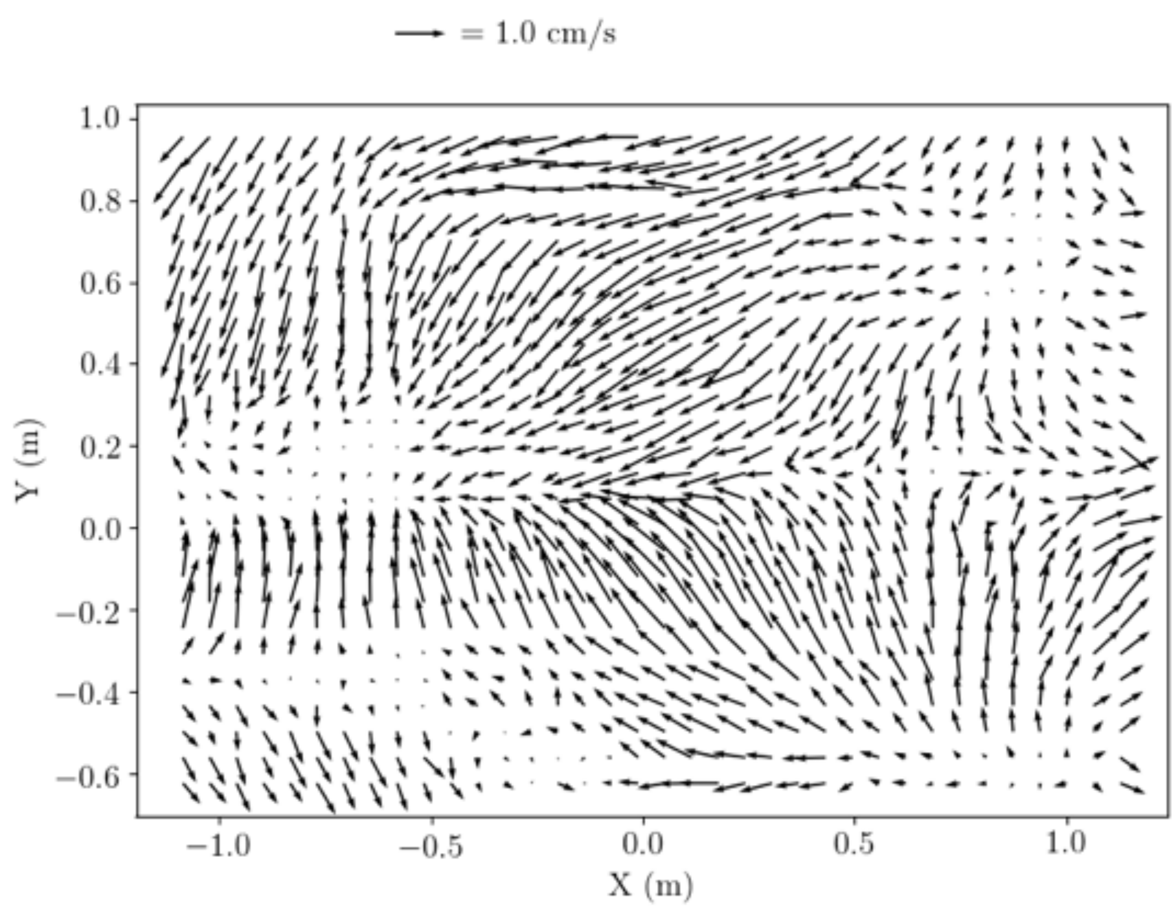
now: laser sheet static
future : scanned vertically for 3D measurement

polystyrene particles matched in density
(for stratified flows)

vertical laser sheet:
vertical velocity

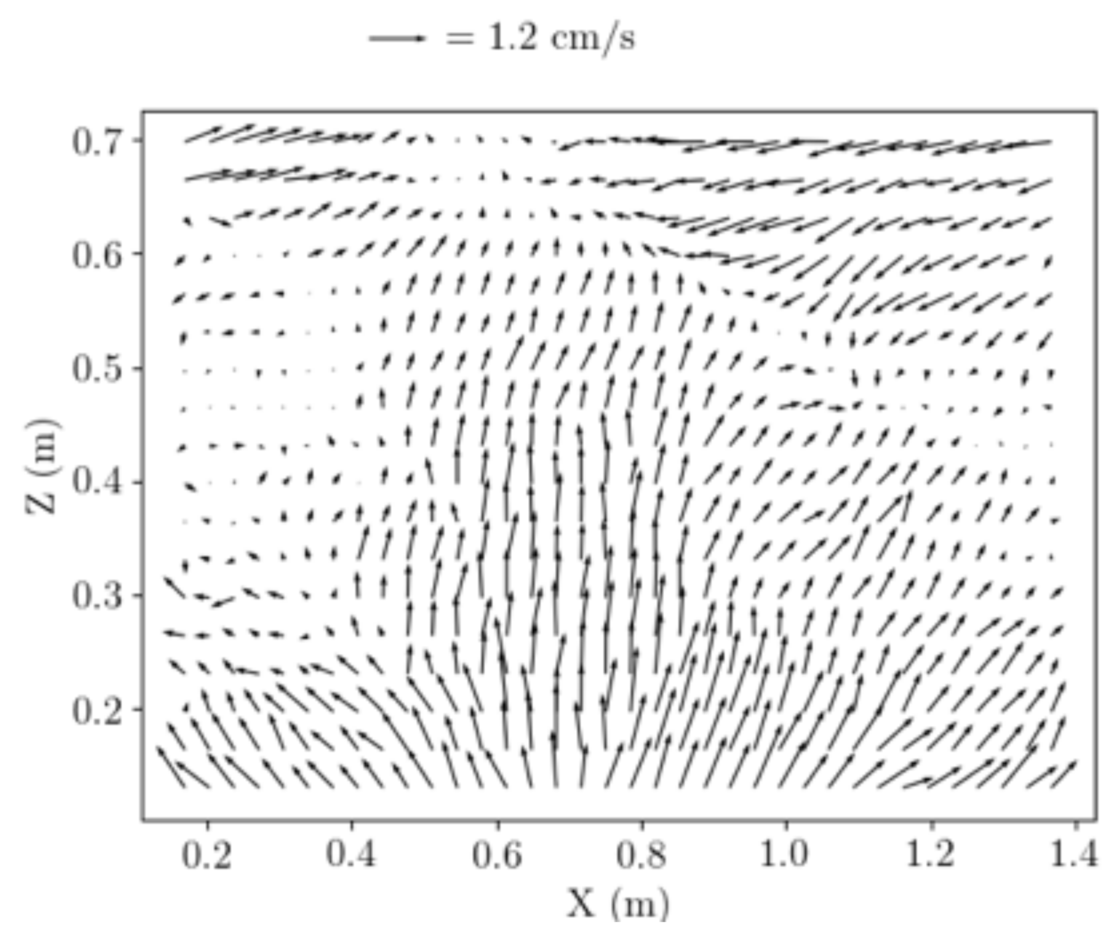


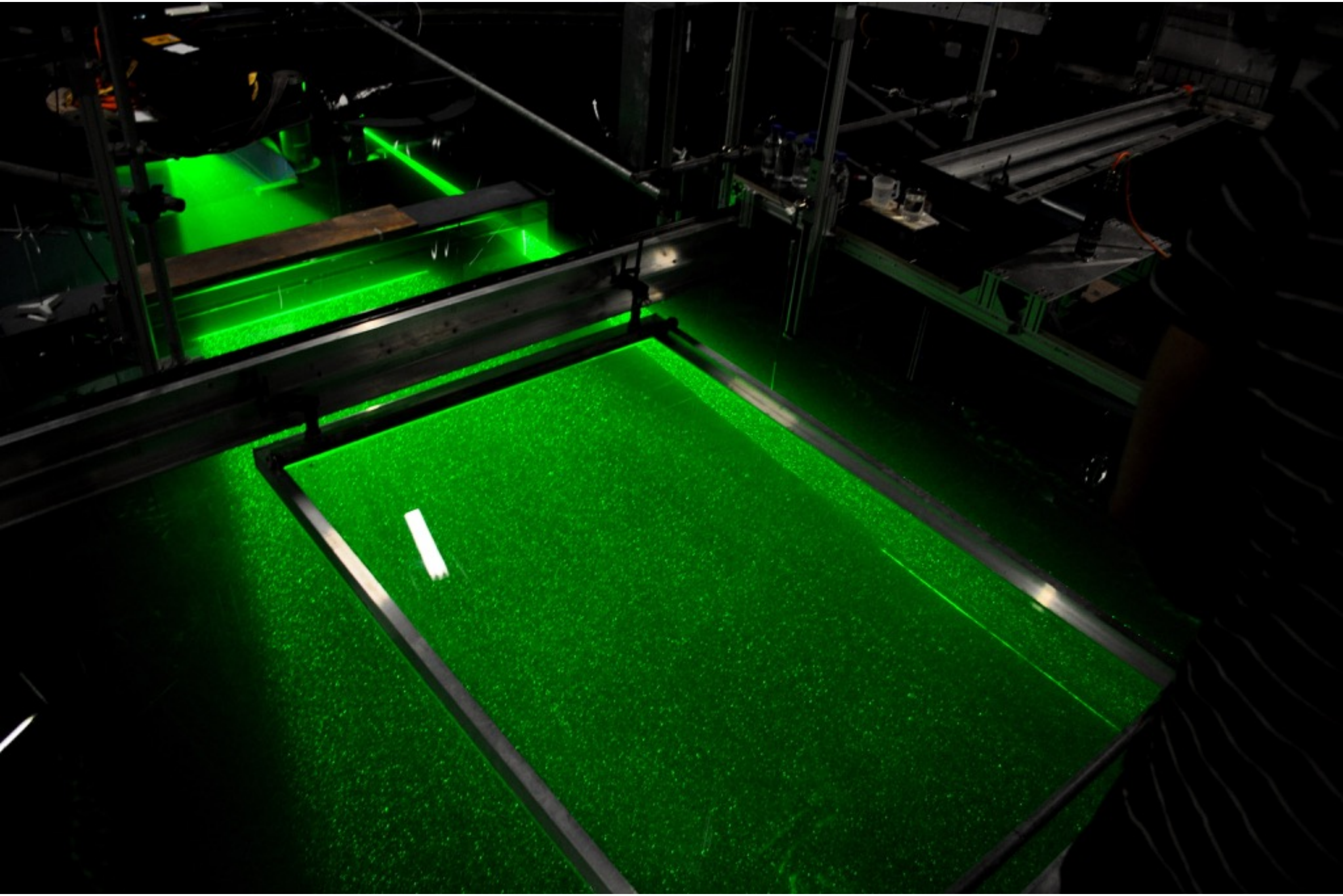
Particle Image Velocimetry (PIV) measurement: velocity field



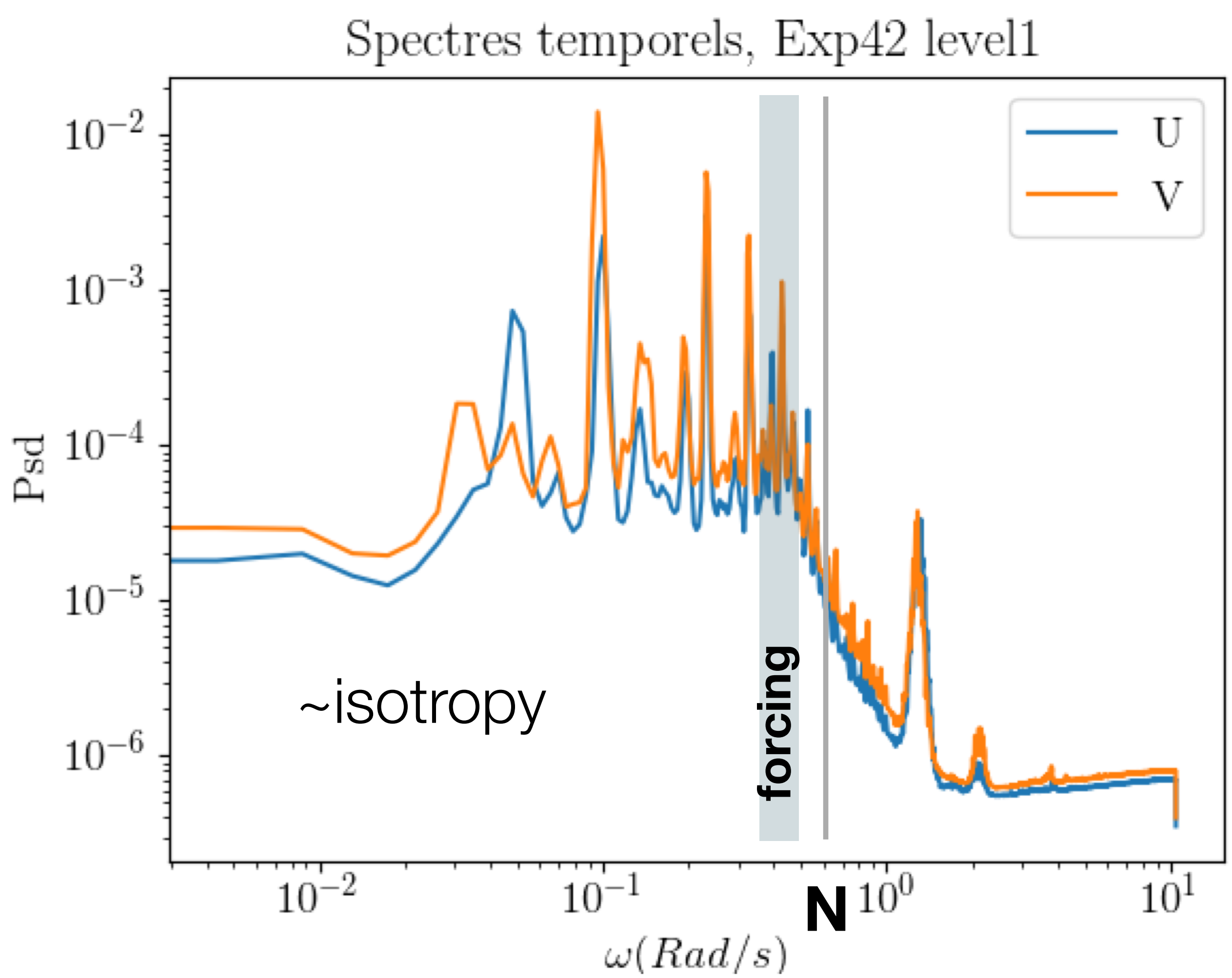
horizontal laser sheet:
horizontal velocity

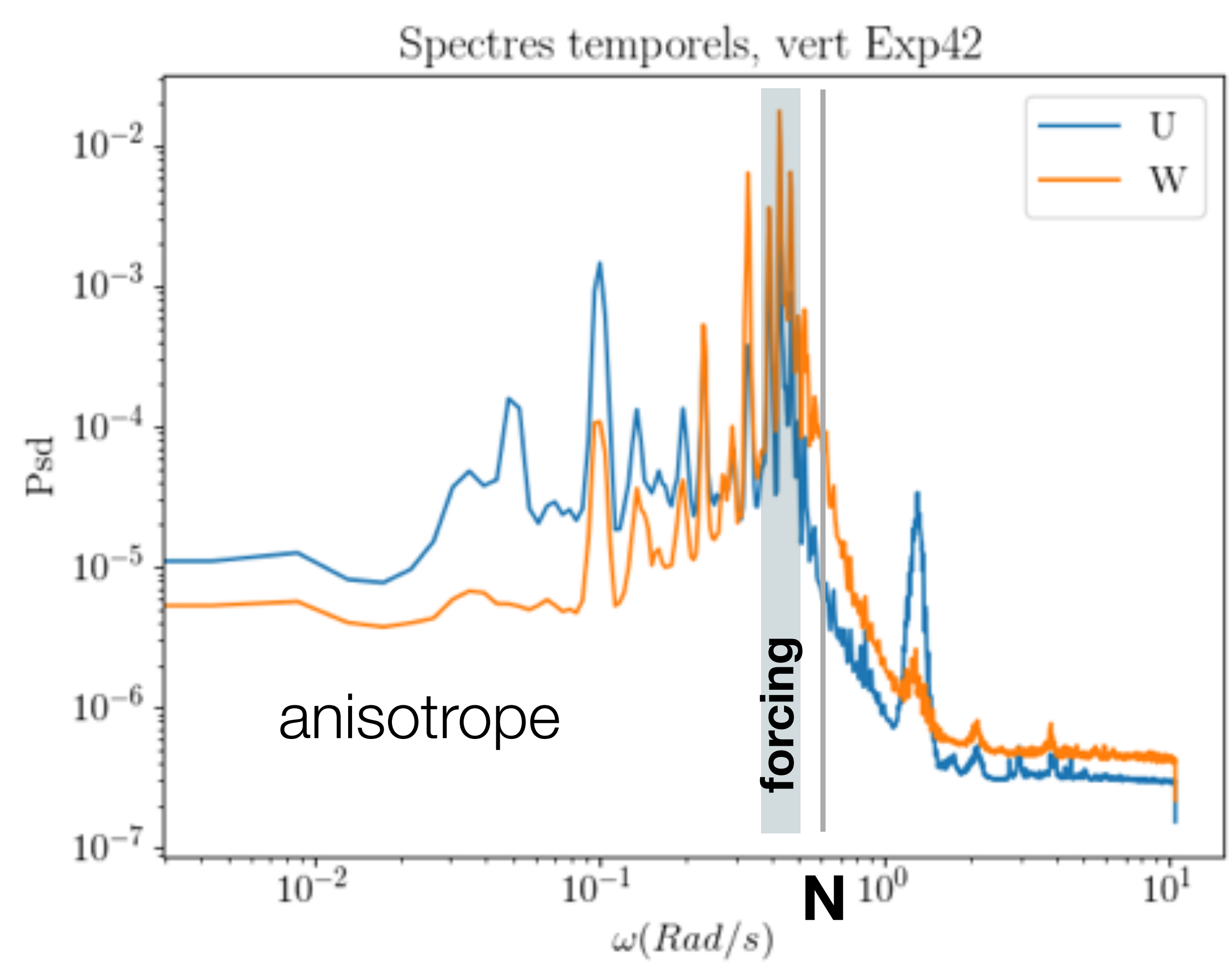
vertical laser sheet:
vertical velocity





spectral content





peaks in spectrum: modes of the tank

➤ **waves**

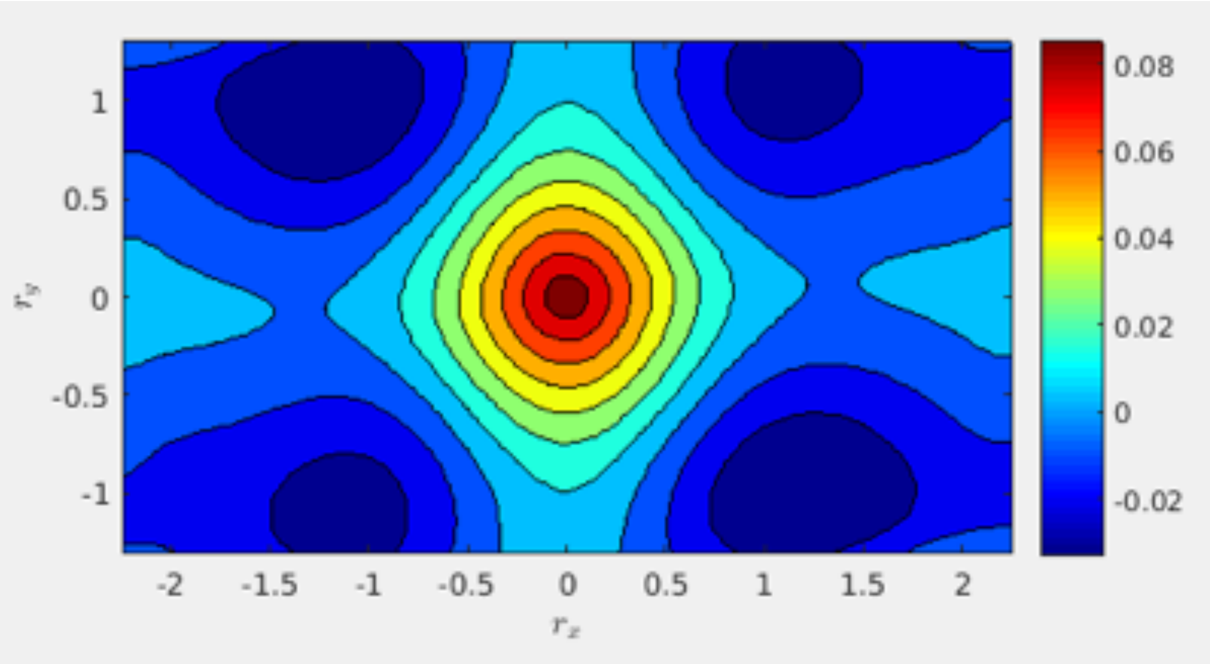
& nonlinear !

what about the continuum ?

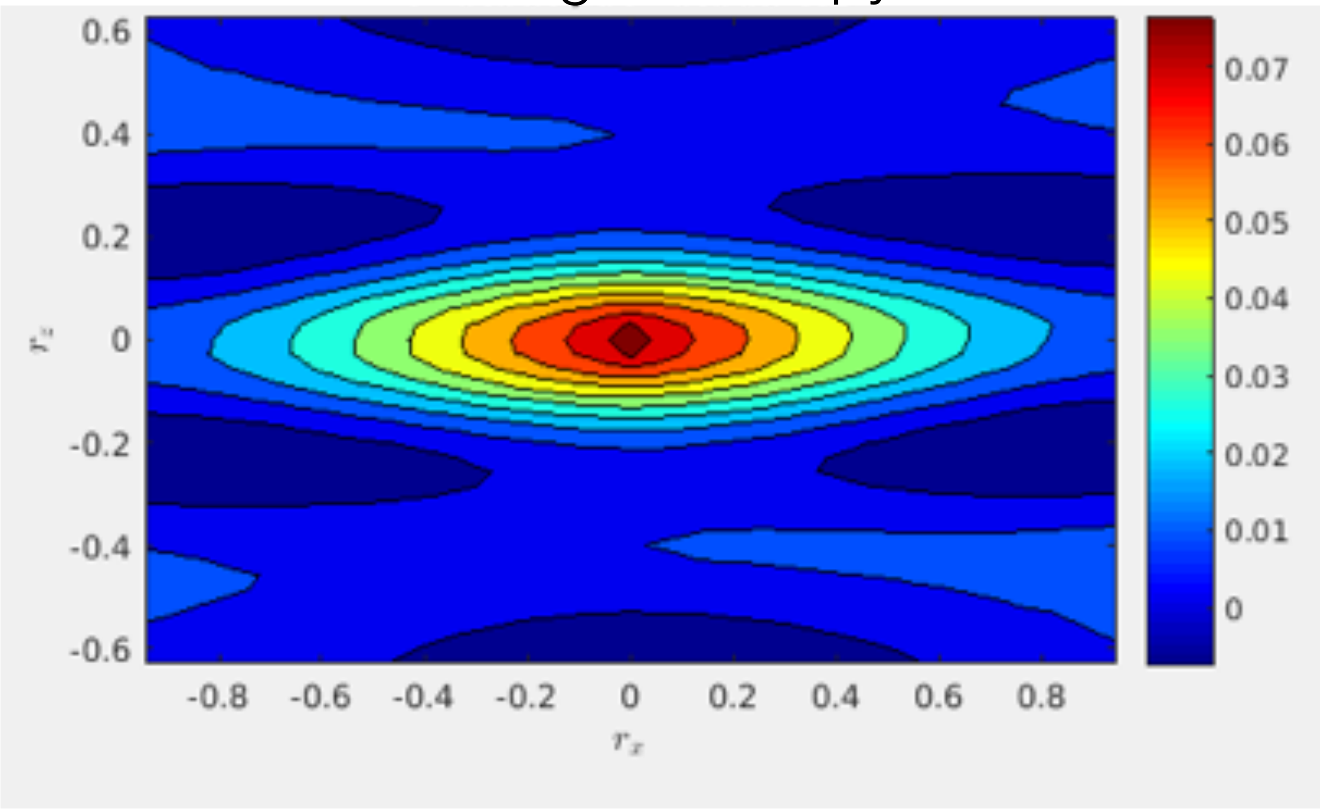
space correlations, full field

$$\langle \mathbf{v}(\mathbf{x}+\mathbf{r},t) v(\mathbf{x},t) \rangle$$

horizontal plane



vertical plane
strong anisotropy



waves or vortices ?

issue: how to disentangle waves from vortices ?

look if energy lies on the dispersion relation

issue: 2D cut of the flow ➤ requires additional assumption
ex: axisymmetry

see Campagne *et al.* 2015 for inertial waves in a rotating fluid

better: full 3D measurement !
maybe in Spring 2019 !!

Conclusion

not very clear…

experiments: it is non linear and there are waves, more analysis needed

simulations: complex…
not simply a wave cascade, but there are waves !
time-space analysis required

work in progress…