

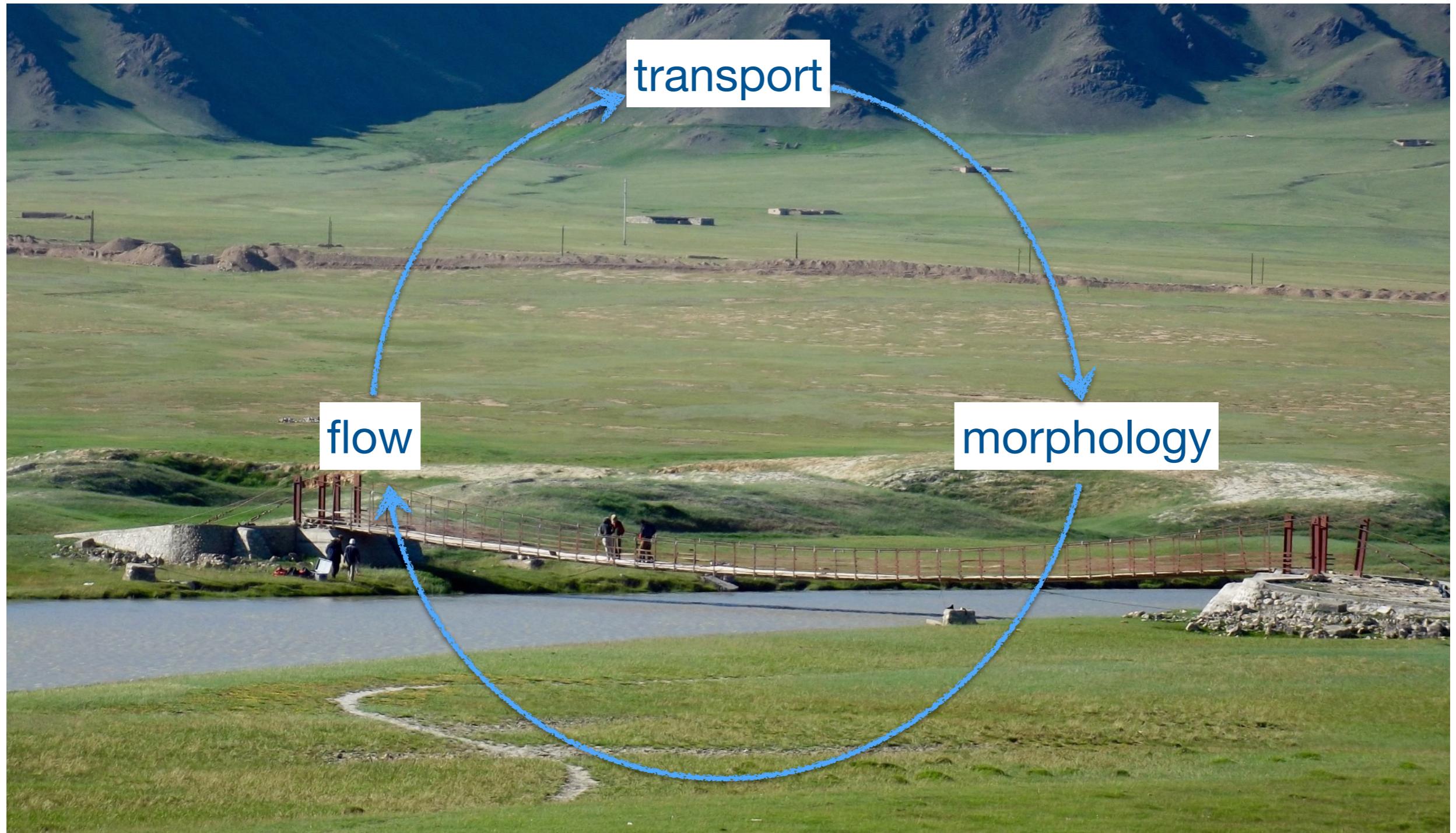
# Equilibrium shape and size of alluvial rivers

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F. Métivier, P. Popović, G. Seizilles

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# Alluvial river



Kaidu river, Tian-Shan, China

# Bedforms



Ripples, Urumqi river  
(chinese Tian-Shan)



Alternate bars, Ornain, Bar le Duc

# Channel size



Kaidu river, Tian-Shan, China

# Channel

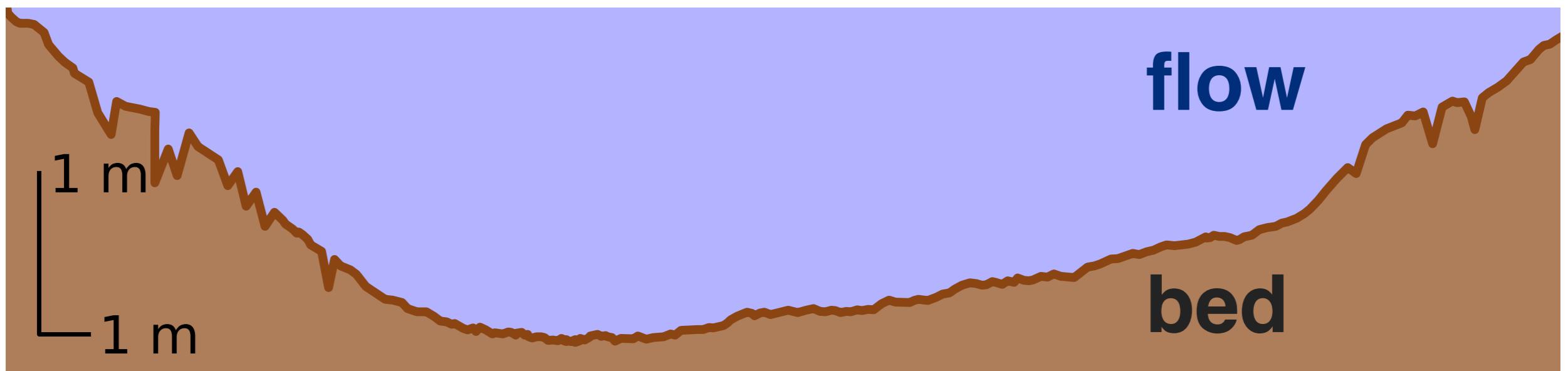


Acoustic Doppler Current Profiler

# Channel shape



Kaidu river, Tian-Shan, China



# Channel shape



Kaidu river, Tian-Shan, China

What selects the shape and the size of an alluvial channel?

Leopold & Maddock [1953], Parker [1978], Vigilar & Diplas [1997], Cao & Knight [1998], ...

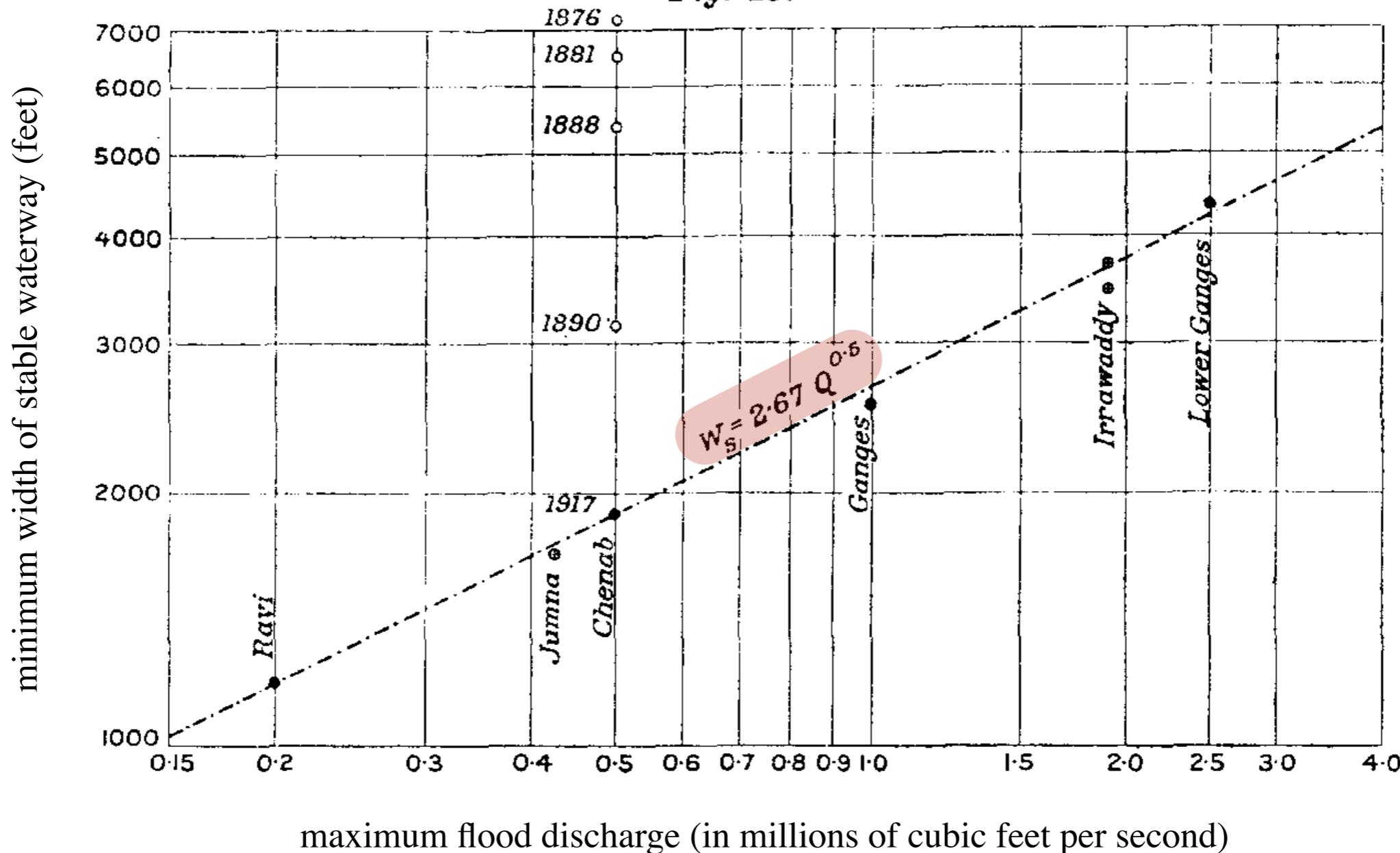
# Lacey's law

“Stable Channels in Alluvium.”

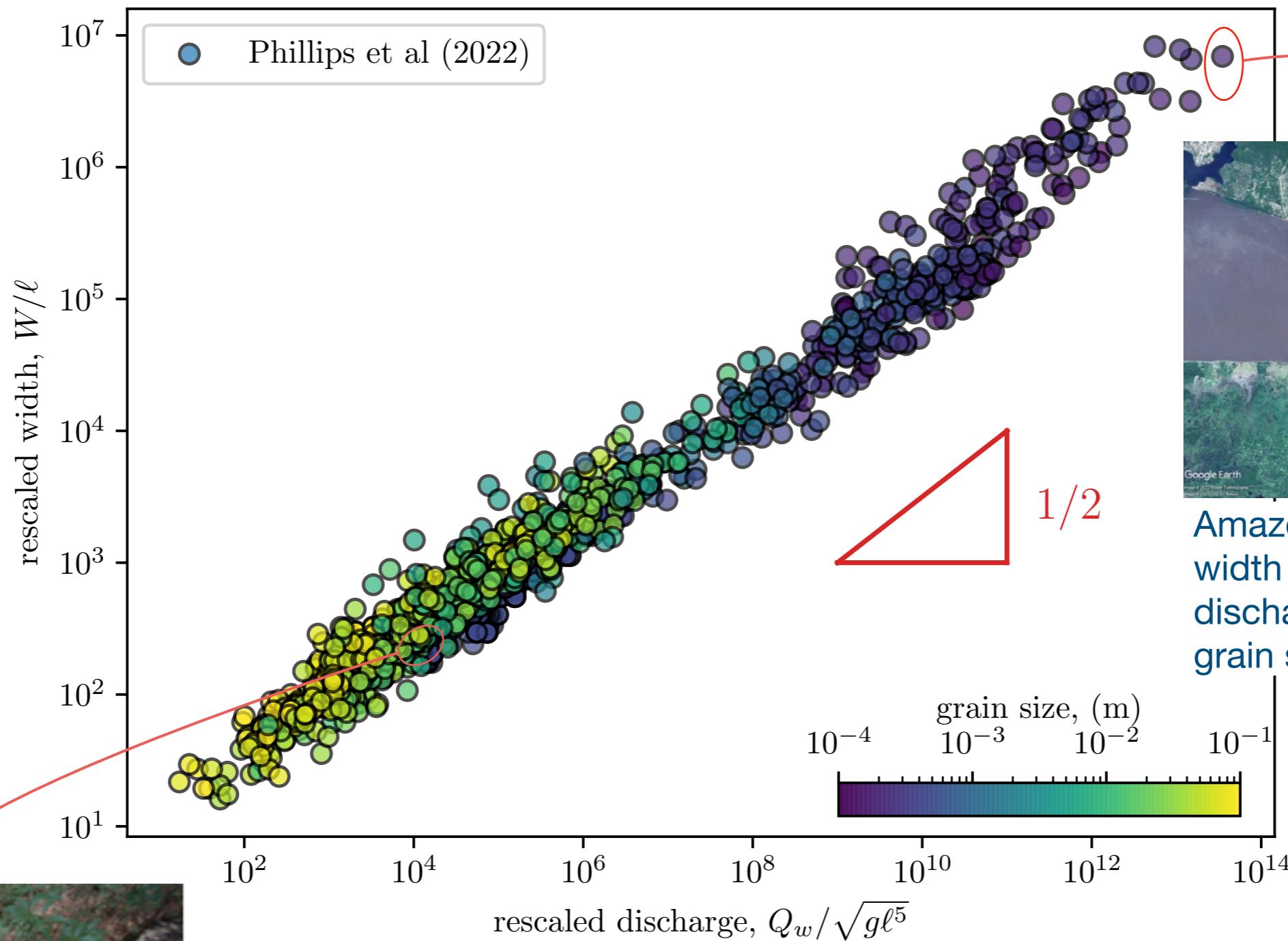
By GERALD LACEY, B.Sc., Assoc. M. Inst. C.E.

*Minutes of the Proceedings of the Institution of Civil Engineers, Vol. 229 (1930)*

Fig. 13.



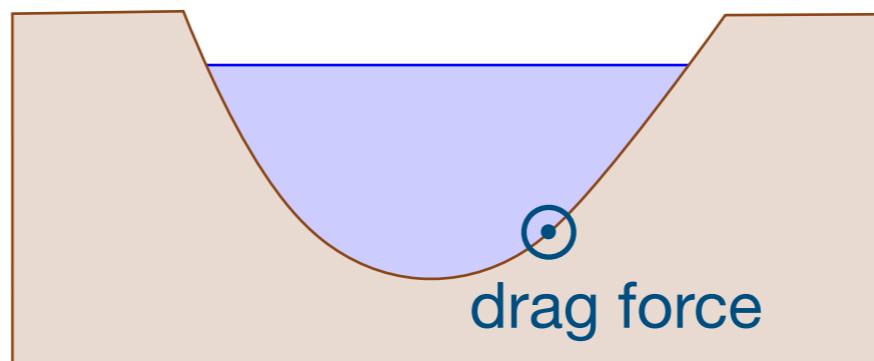
# Lacey's law



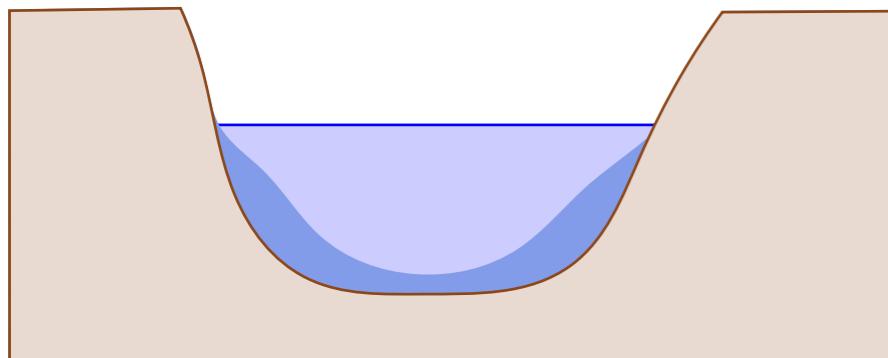
sand stream in Florida  
width = 10 cm  
discharge = 2 L/s  
grain size = 450 microns

# Threshold theory

Glover and Florey [1951] & Henderson [1961]

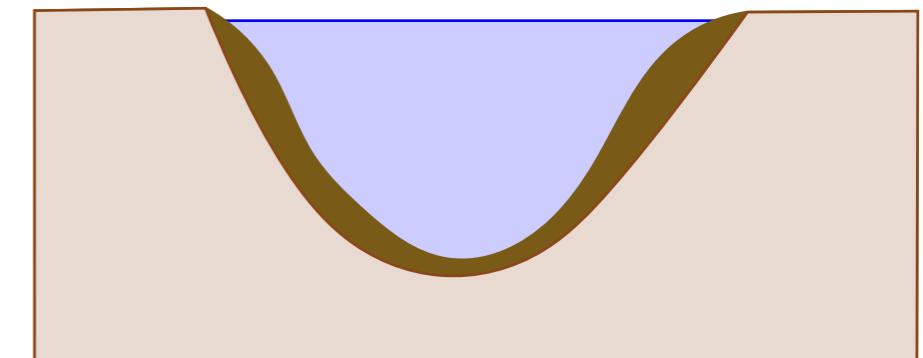


force > threshold



erosion

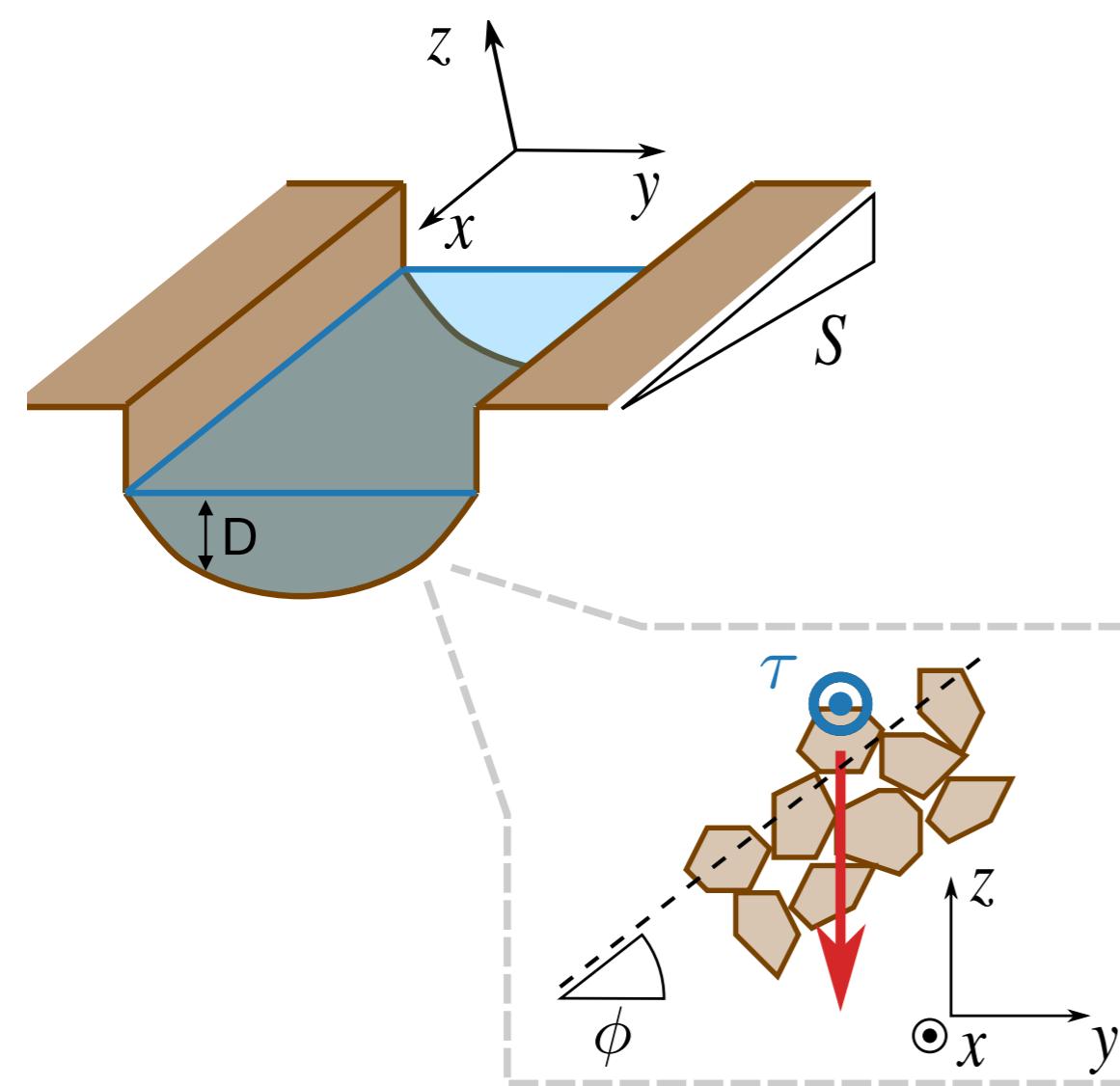
force < threshold



aggradation

→ river builds its bed near the threshold of entrainment.

# Threshold theory

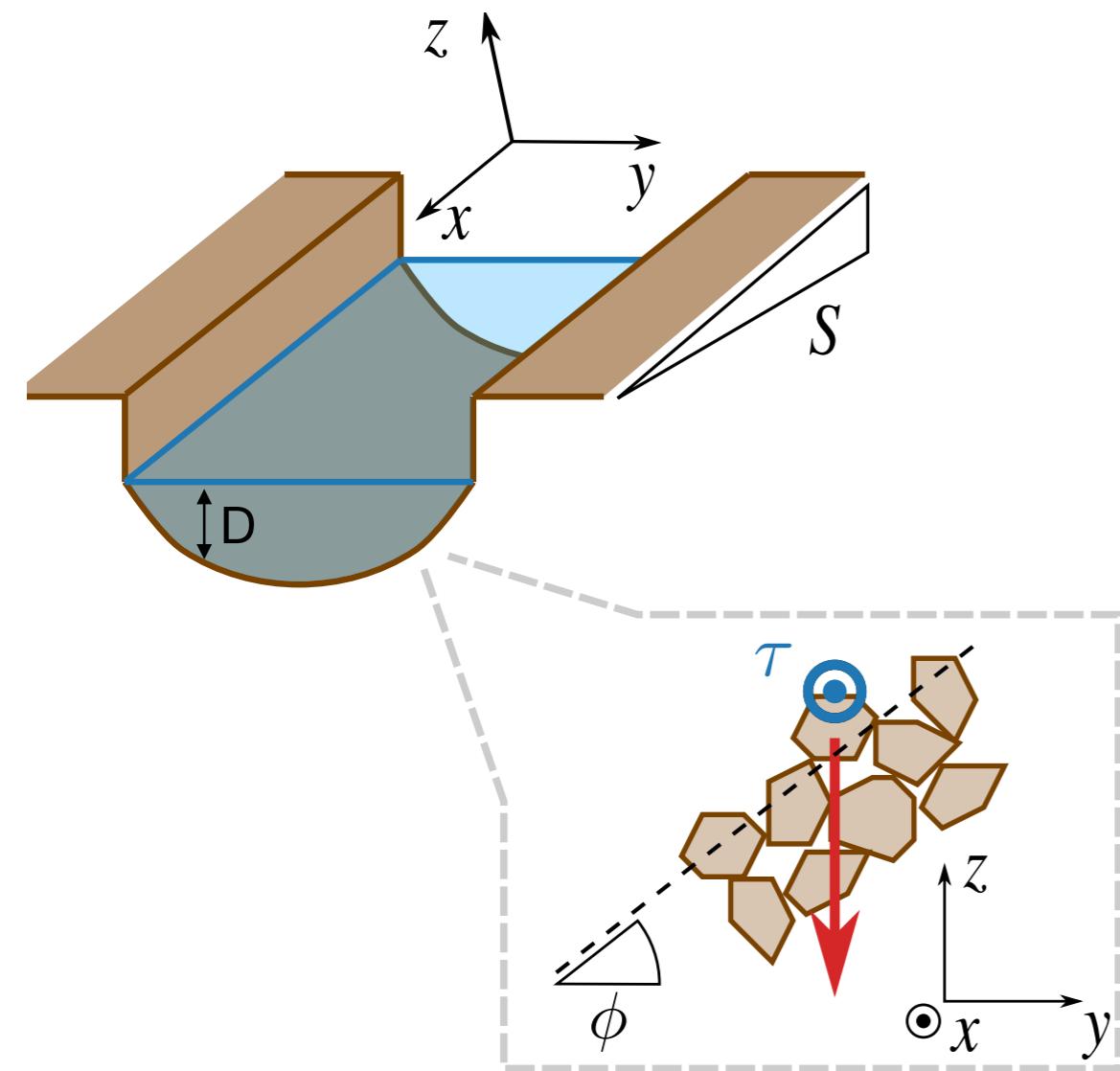


- gravity force:  $F_g \propto (\rho_s - \rho_f) g d_s^3$
  - drag force:  $F_d \propto \tau d_s^2$
  - shallow water approximation:  $\tau = \rho g D S$
- density of sediment  
fluid density  
grain size  
shear stress  
flow depth  
slope

Coulomb's law of friction  $\rightarrow \frac{\text{tangential force}}{\text{normal force}} = \mu_t$

friction coefficient

# Threshold theory



threshold Shields stress

$$L = \frac{\theta_t}{\mu_t} \frac{\rho_s - \rho_f}{\rho_f} d_s$$

friction coef.

grain size

Coulomb's law of friction  $\rightarrow$

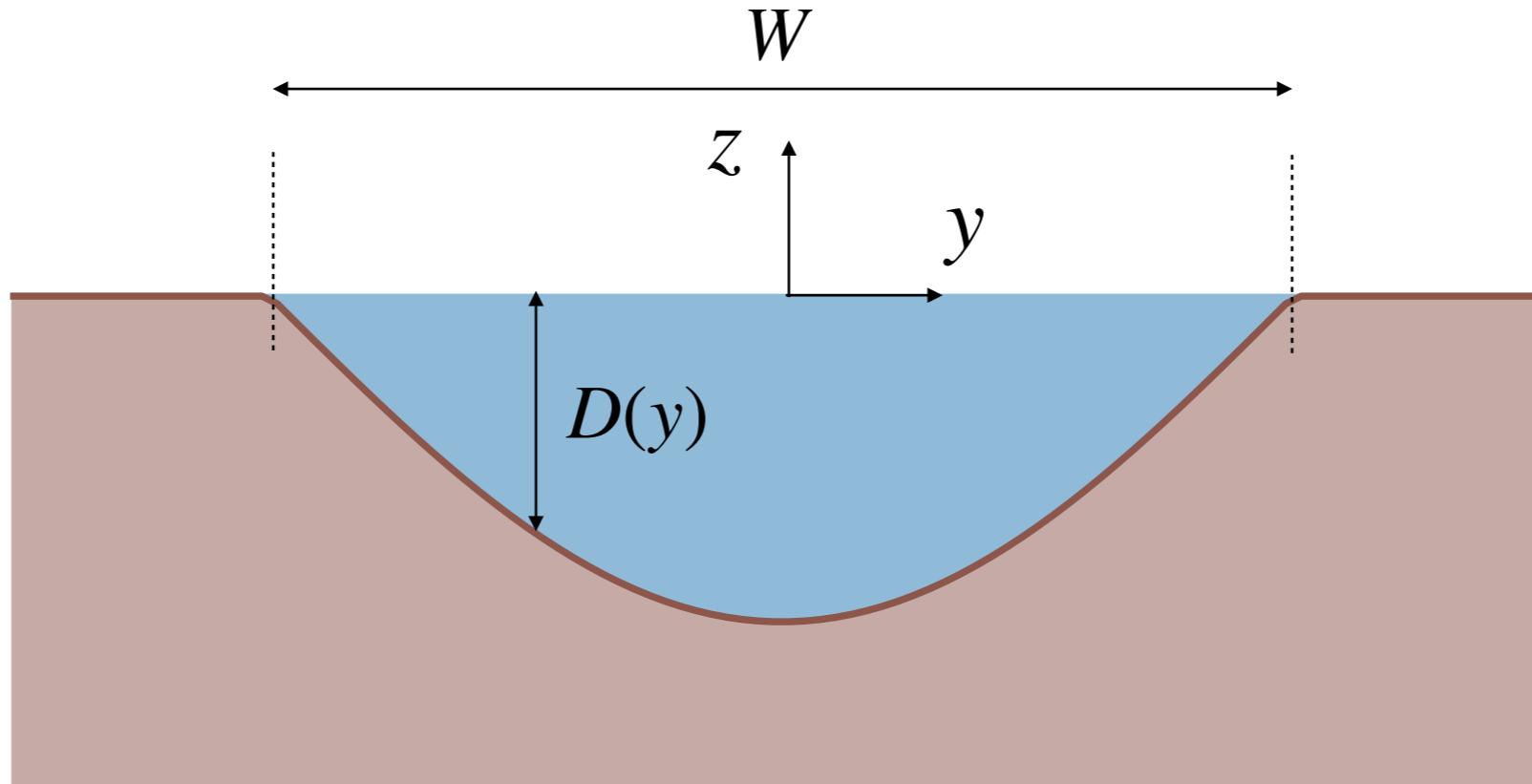
tangential/normal force

$$\sqrt{\left(\frac{SD}{L}\right)^2 + \left(\frac{dD}{dy}\right)^2} = \mu_t$$

drag force      gravity

friction coef.

# Threshold theory



friction coef.

$$D = \mu_t \frac{L}{S} \cos \left( \frac{Sy}{L} \right)$$

$\sim d_s$

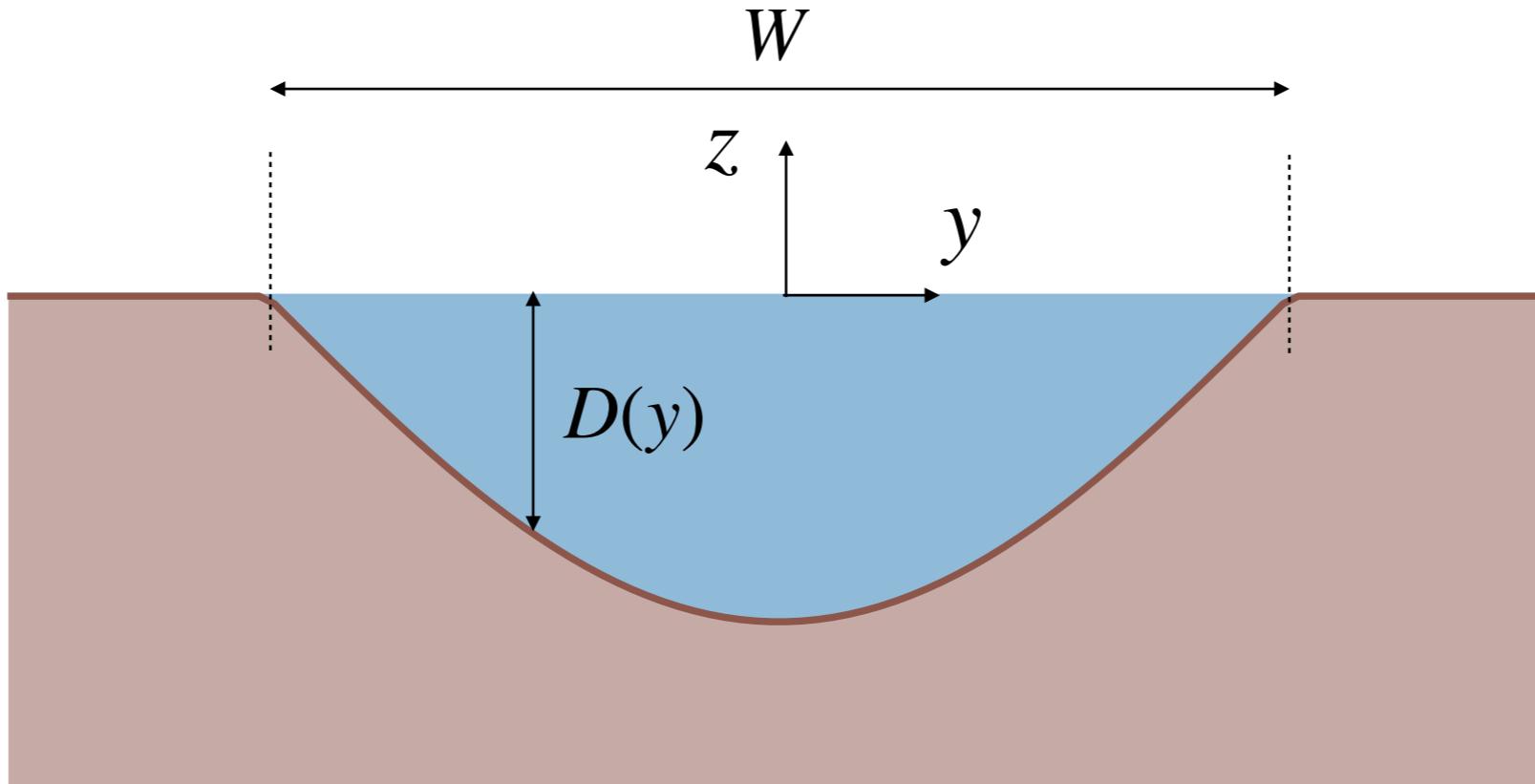
Glover and Florey [1951], Henderson [1961], Seizilles *et al.* [2013]

width and depth  $\propto \frac{d_s}{S}$

— grain size  
— river slope  $\sim 10^{-5}-10^{-2}$

→ channel size  $\gg$  grain size

# Threshold theory



friction coef.

$$D = \mu_t \frac{L}{S} \cos \left( \frac{Sy}{L} \right)$$

slope

$\sim d_s$

Glover and Florey [1951], Henderson [1961], Seizilles et al. [2013]

Shallow-water theory -> channel shape independent of the nature of the flow.

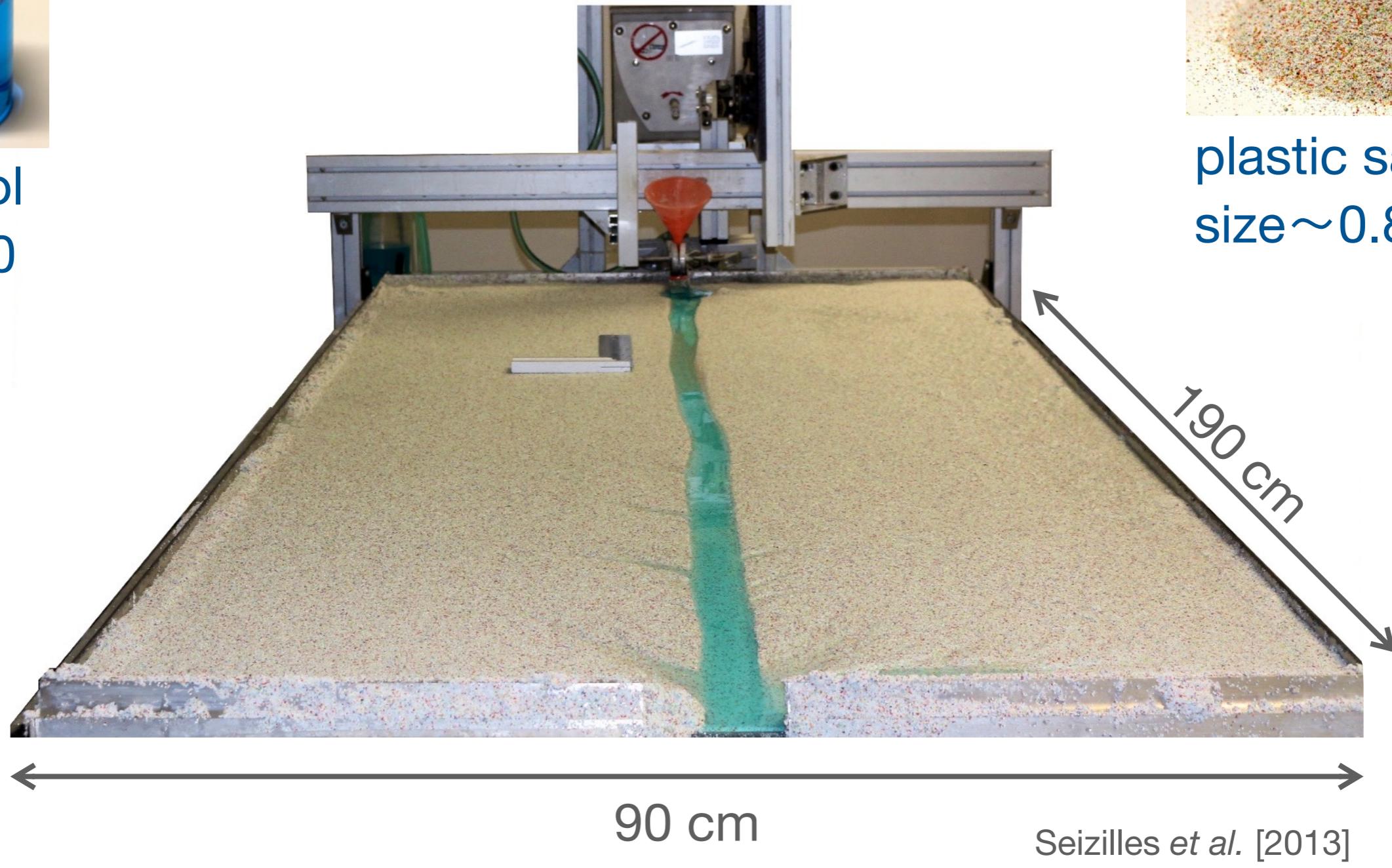
# Laboratory laminar river



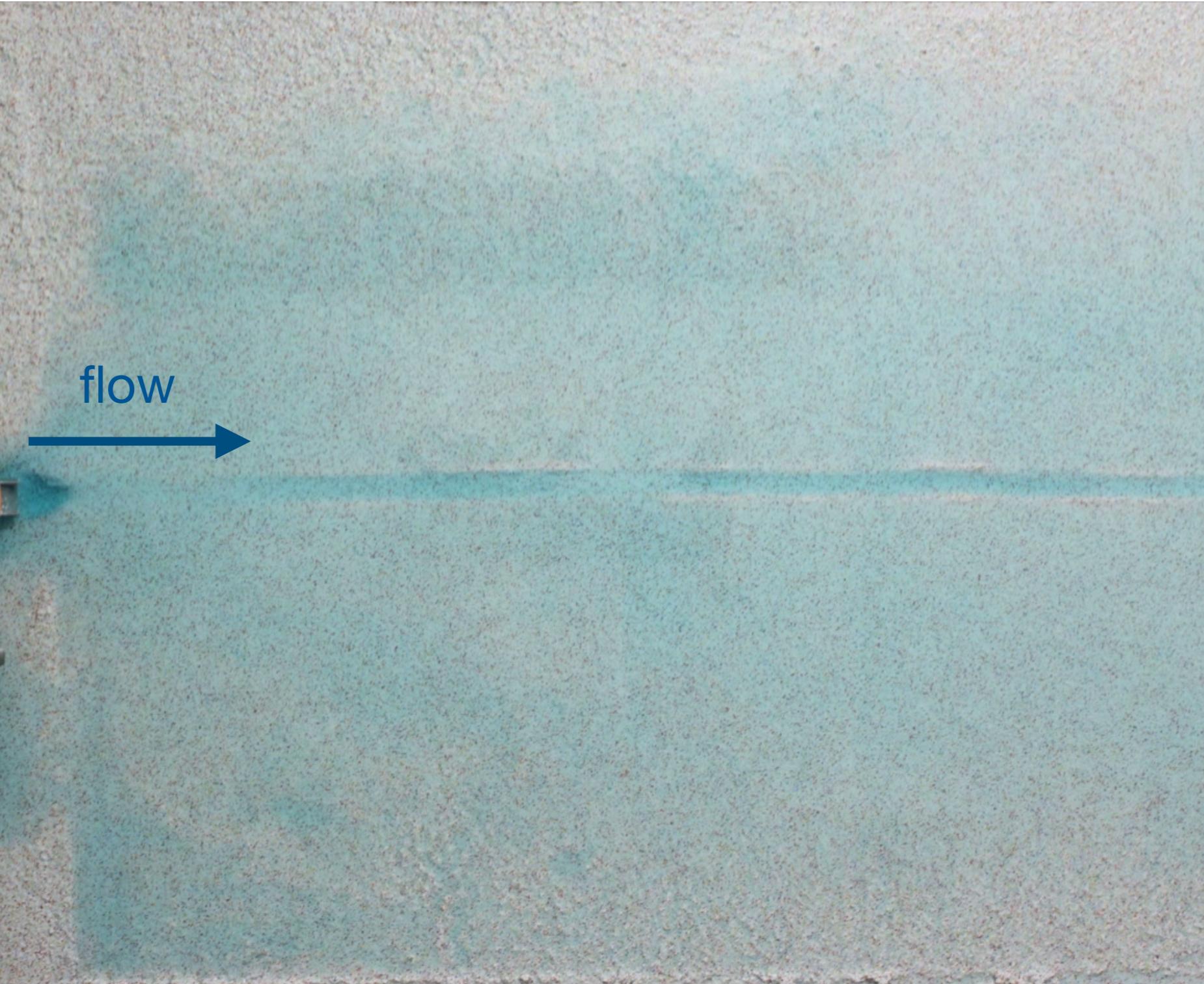
glycerol  
 $Re \sim 10$



plastic sand  
size  $\sim 0.83$  mm

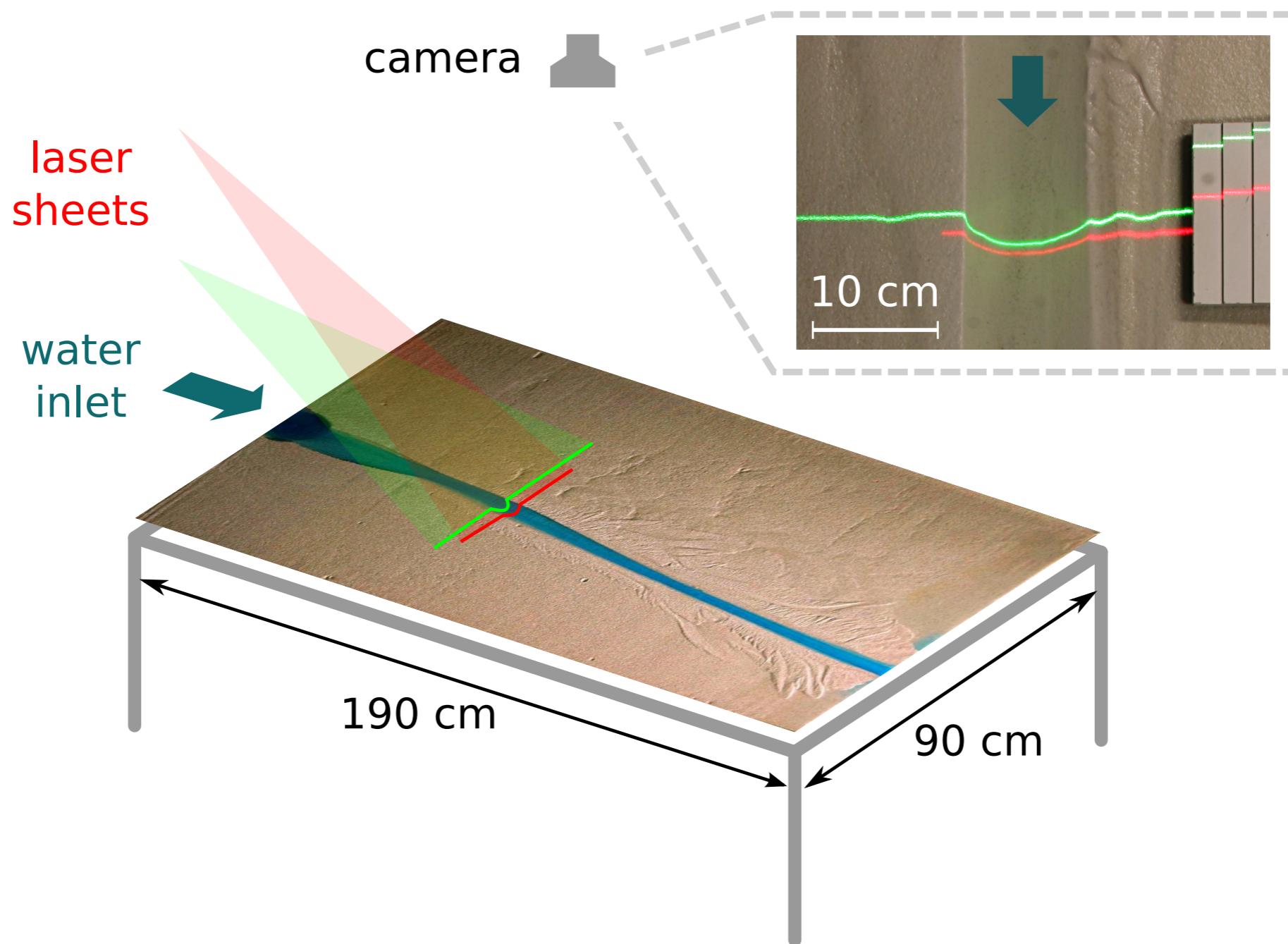


# Laboratory laminar river

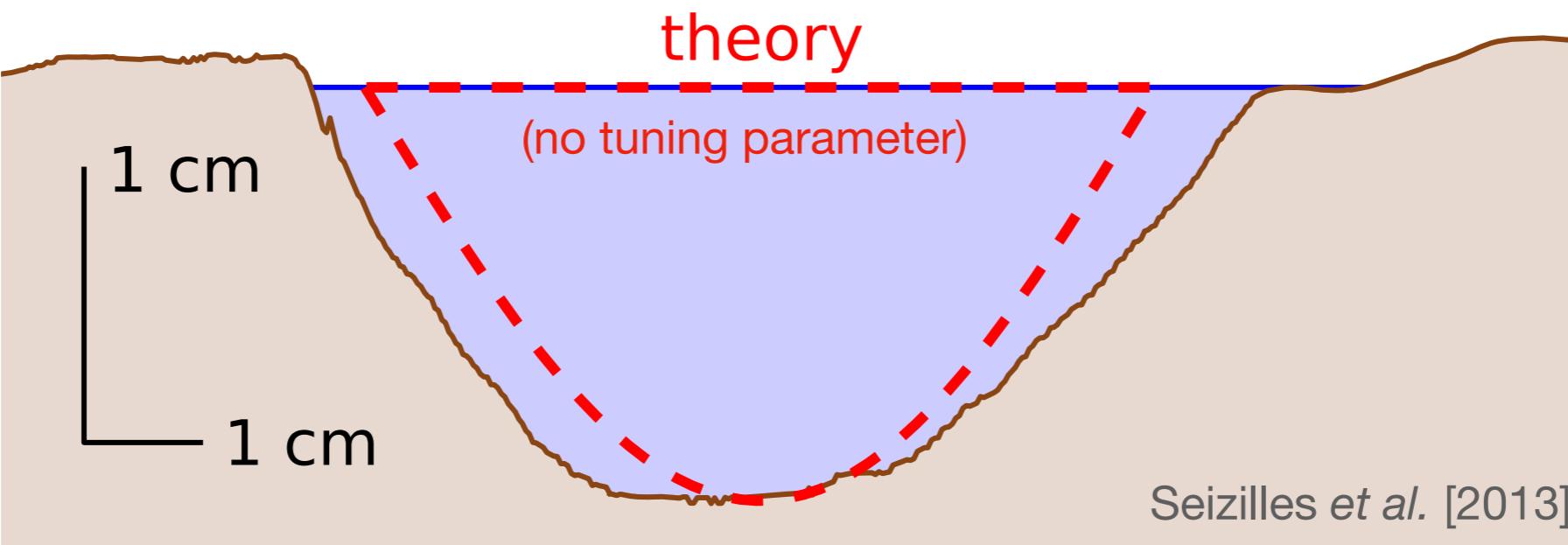


1 image every 5 minutes  
duration ~ 20 hours

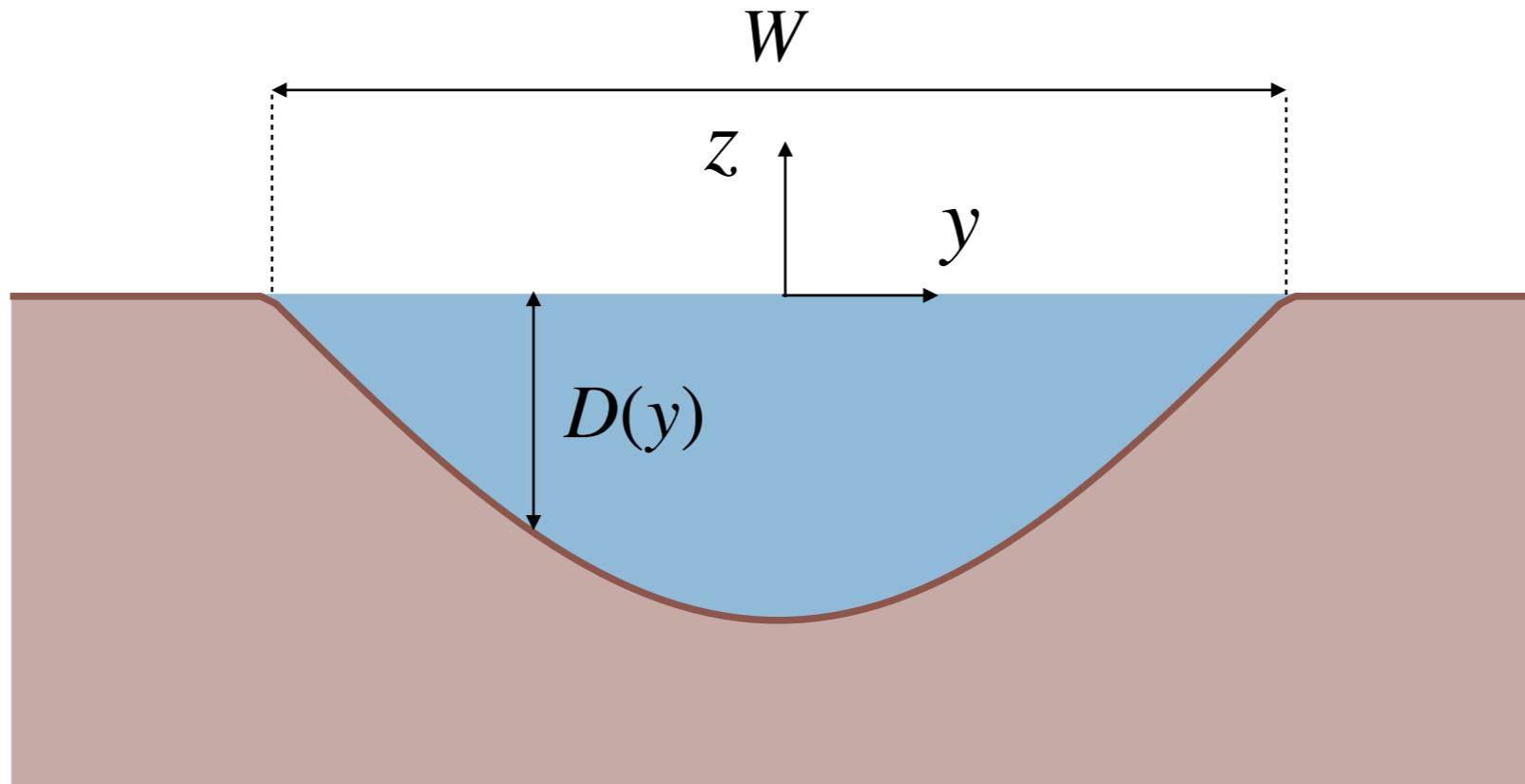
# Laminar channel : shape



# Laminar channel : shape



# Width vs discharge?



discharge : 
$$Q_w = \int_{-W/2}^{+W/2} U D \, dy$$

depth-averaged velocity | depth

lubrication theory : 
$$U = \frac{g S D^2}{3\nu}$$

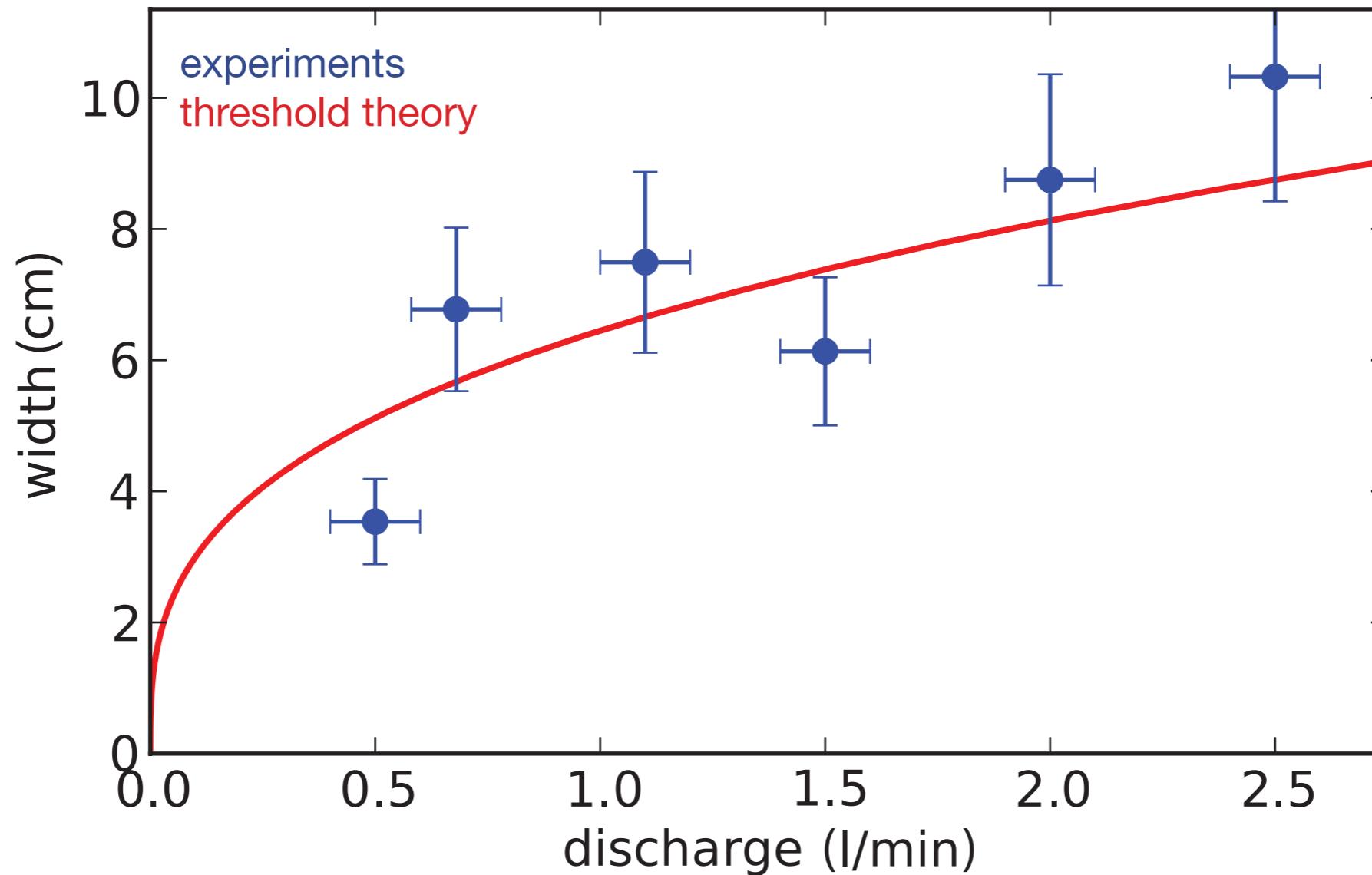
slope | viscosity  
depth-averaged velocity

# Laminar channel : width vs discharge

$$\frac{W}{\ell} = \pi \left( \frac{9}{4 \theta_t \mu_t^2} \right)^{1/3} \left( \frac{Q_w}{\ell^4 g / \nu} \right)^{1/3}$$

width  
characteristic length :  $\ell = \frac{\Delta \rho}{\rho} d_s$   
threshold Shields stress  
friction coef.  
discharge  
kinematic viscosity

laminar flow



# What of the field ?



river near Bayin-Buluk, chinese Tian-Shan

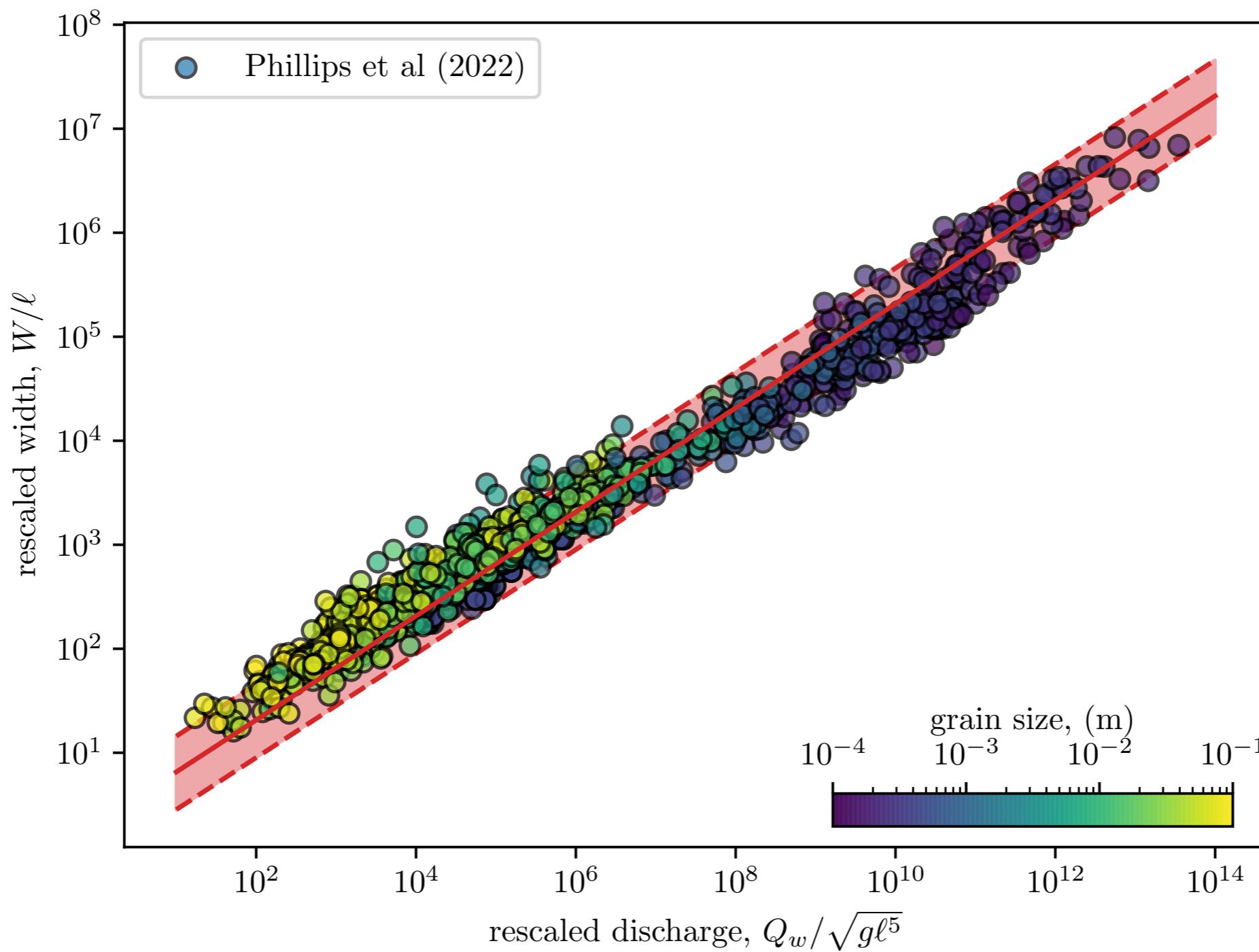
turbulent flow  $\rightarrow U = \left( \frac{gSD}{C_f} \right)^{1/2}$

slope      depth  
               \     /  
               |  
depth-averaged velocity    turbulent friction coefficient

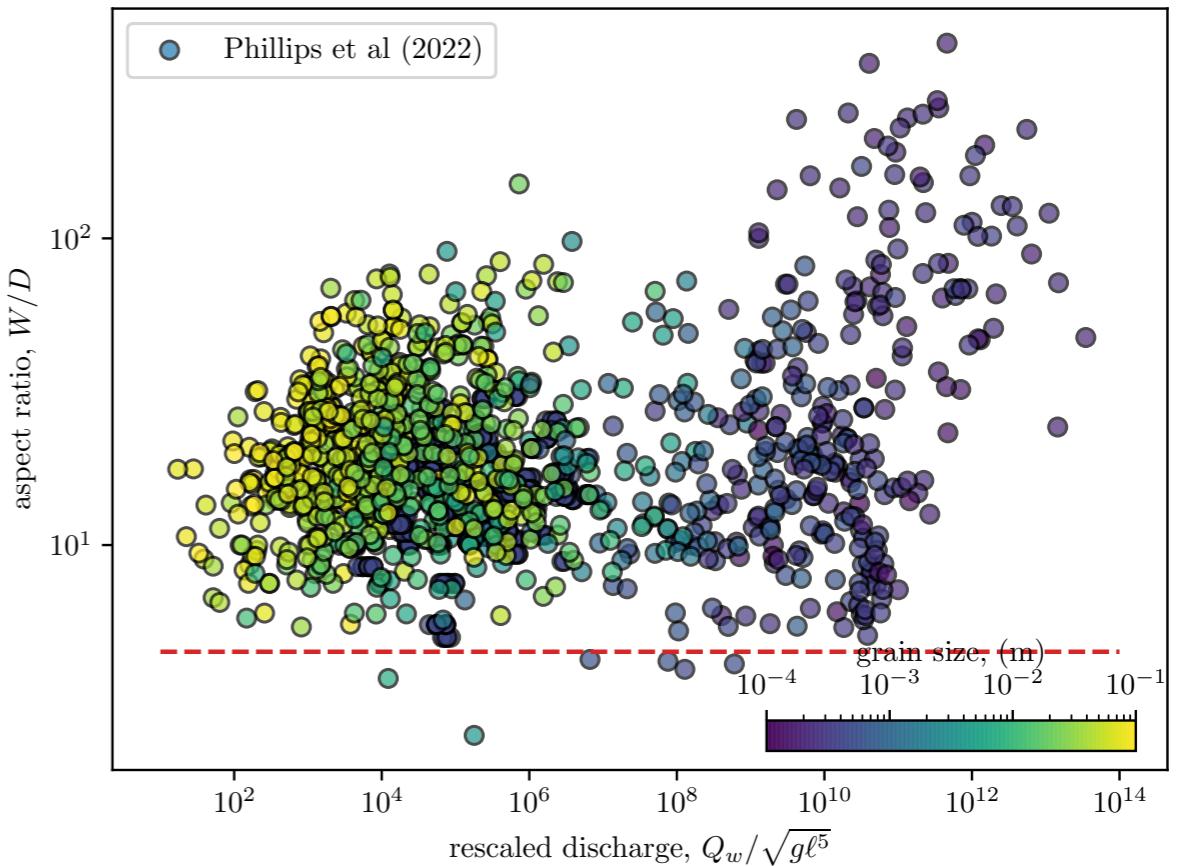
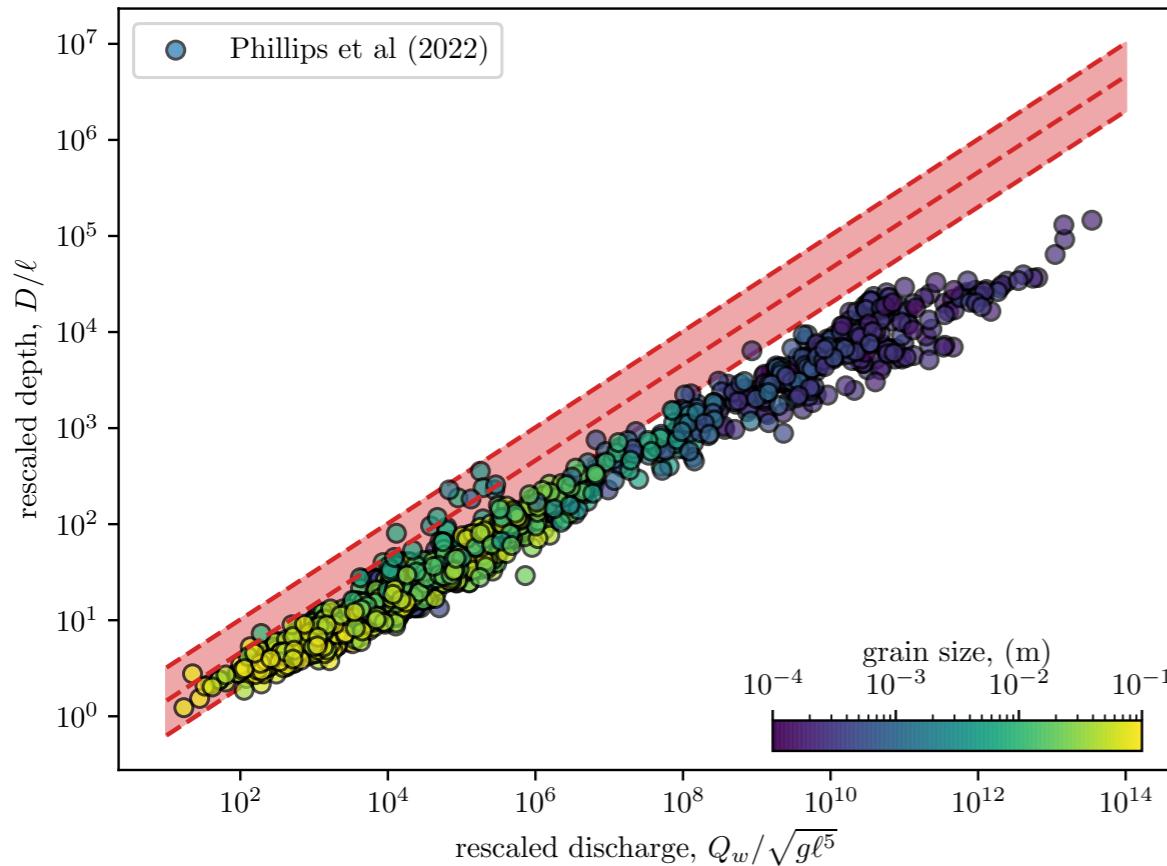
# What of the field ?

$$\frac{\Delta\rho}{\rho} d_s = \frac{W}{\ell} = \frac{\pi}{\sqrt{I}} \left( \frac{C_f}{\theta_t \mu_t^2} \right)^{1/4} \left( \frac{Q_w}{\sqrt{g \ell^5}} \right)^{1/2}$$

width      turbulent friction coef.      discharge      turbulent flow



# Limits of the threshold theory



The threshold theory accounts for the width of rivers  
but not for their shape.

→ effect of sediment transport ?

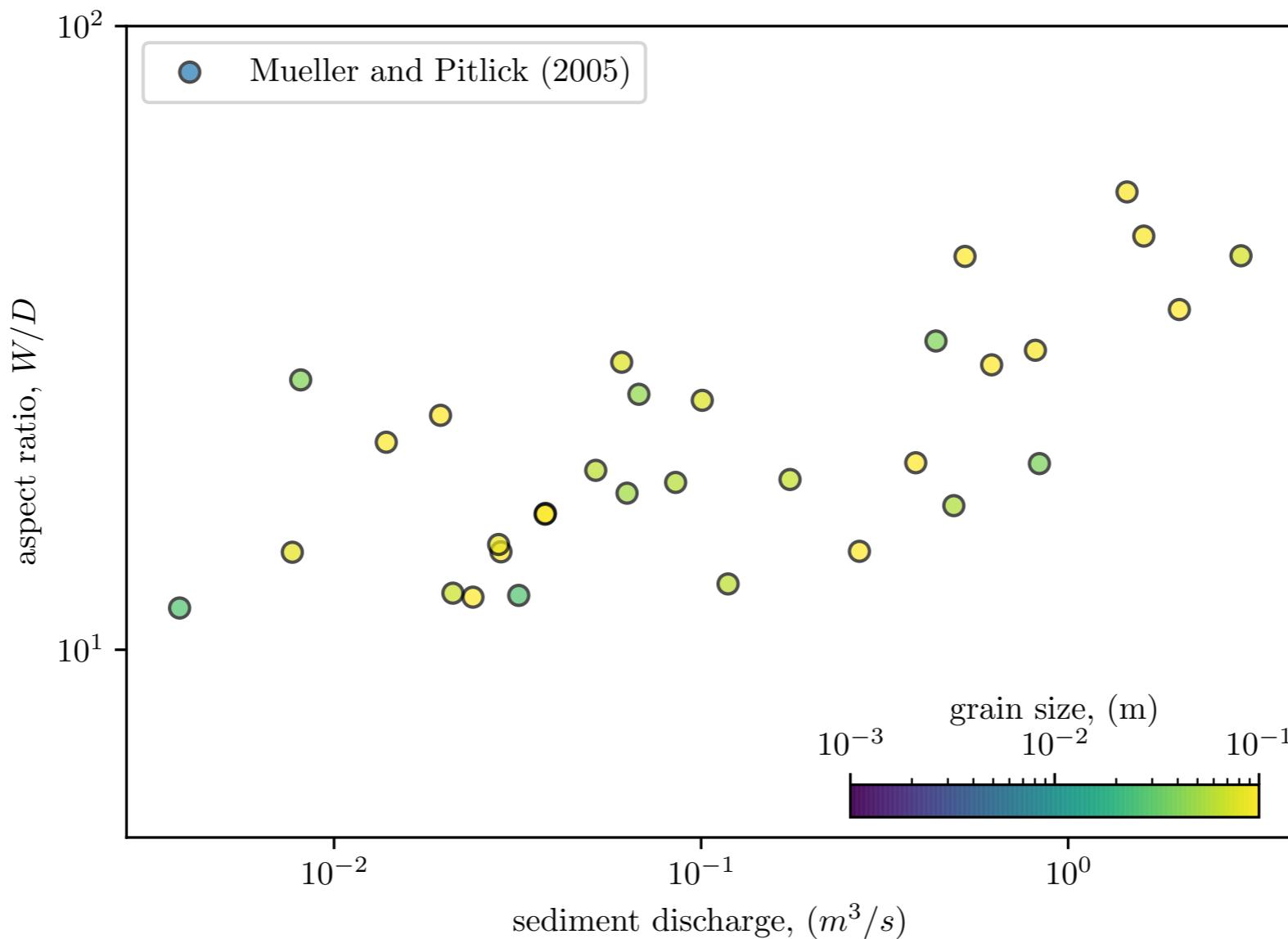
# Rivers transport sediments



Sacramento river  
© Ethan Mora

Does sediment transport influence the shape of rivers?

# How does sediment transport influence the shape of rivers?



The sediment size and the river discharge vary from one data point to the other.

# Rivers transport sediments



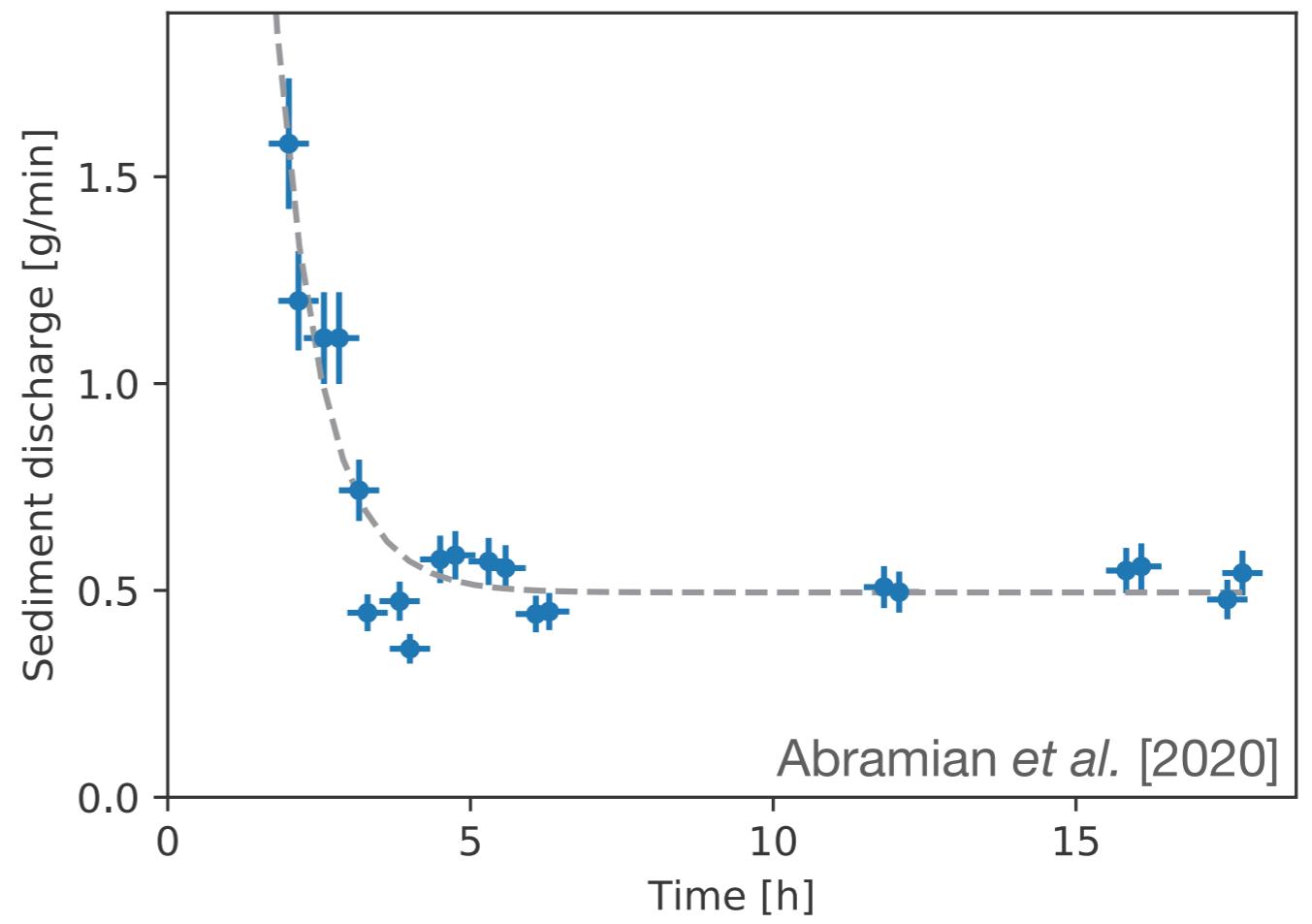
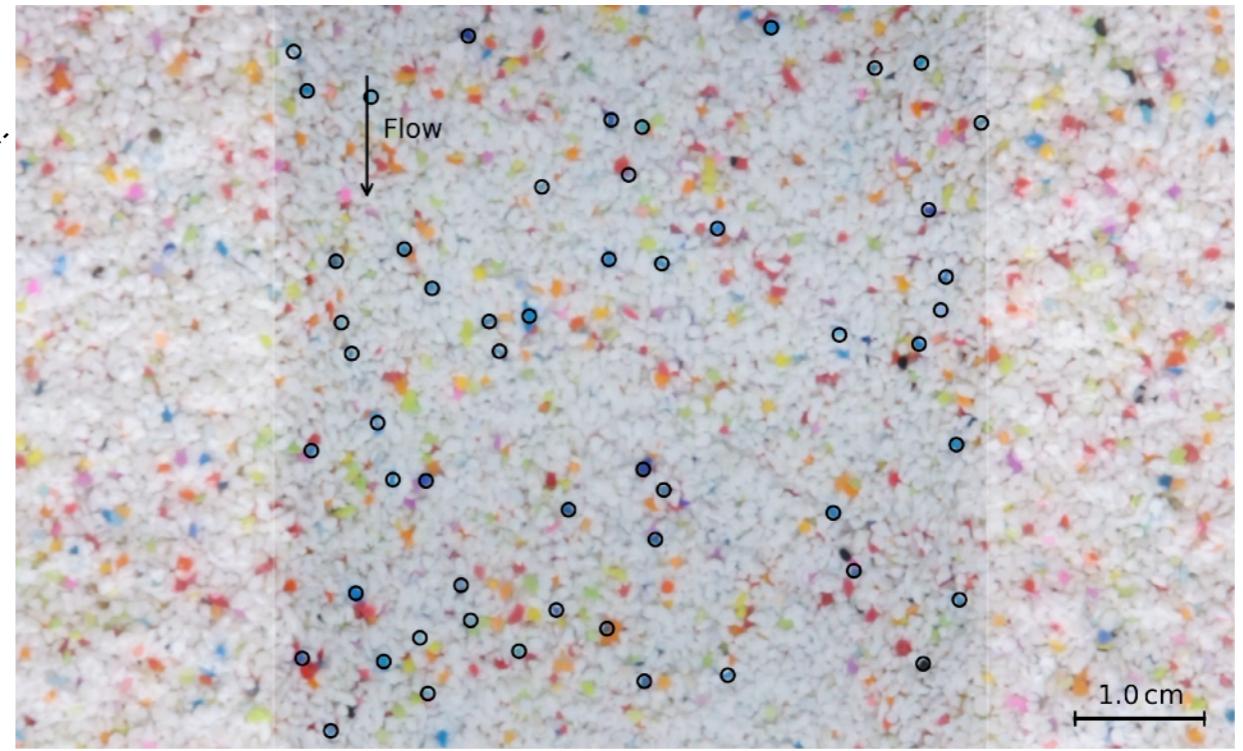
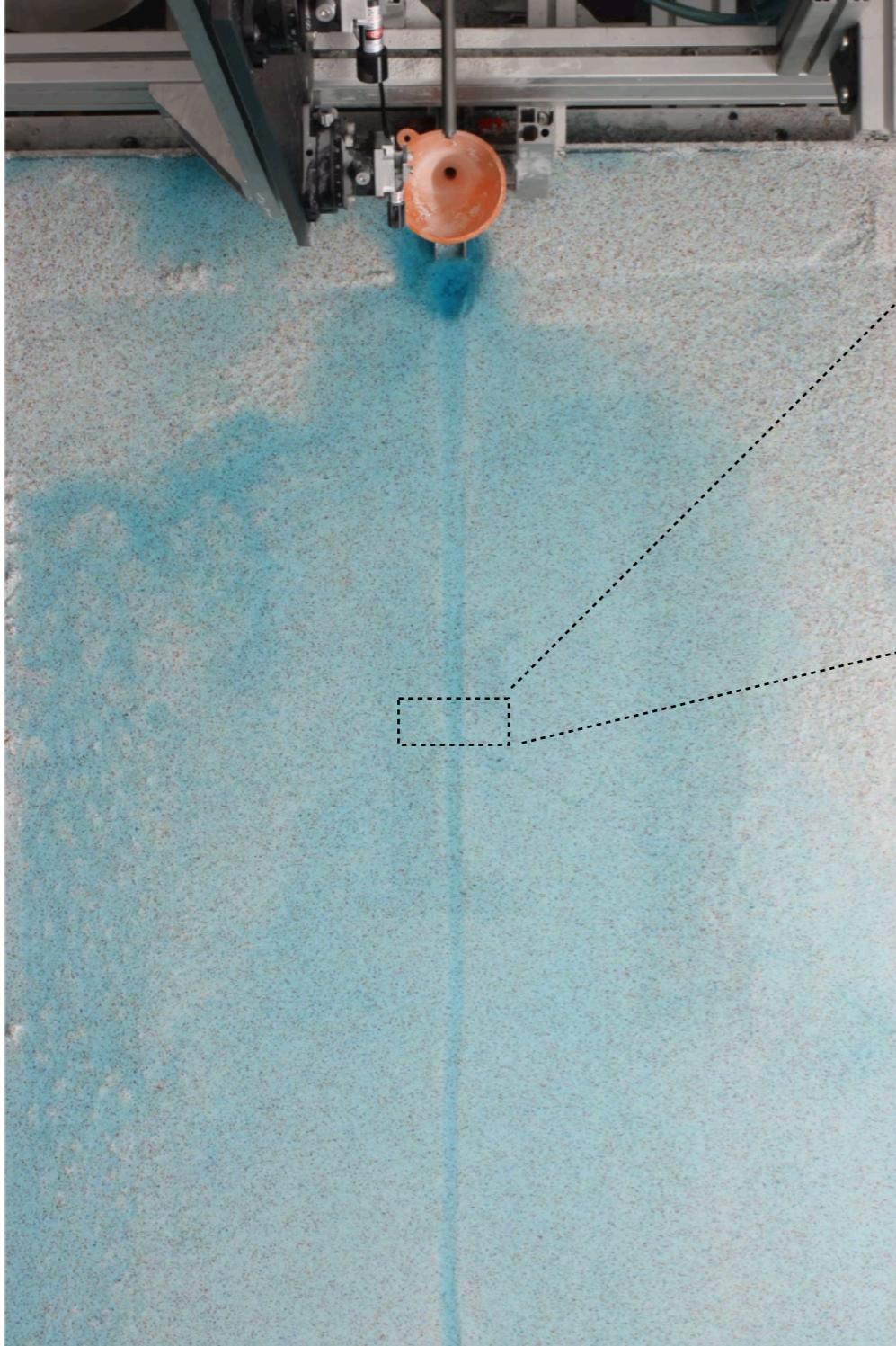
Urümqi He, Chinese Tian-Shan

# Active laboratory river



- constant flow discharge
- constant sediment discharge

# Active laboratory river

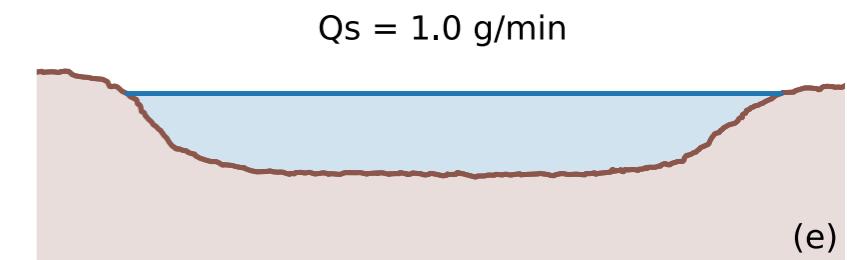
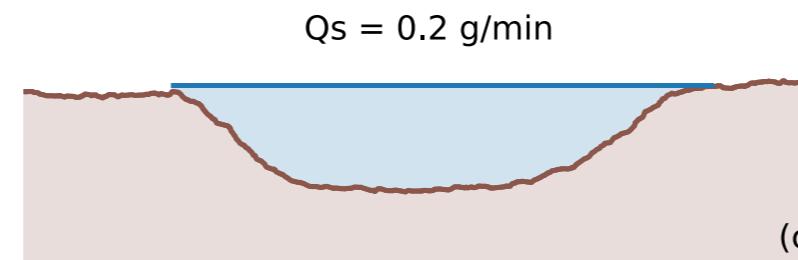
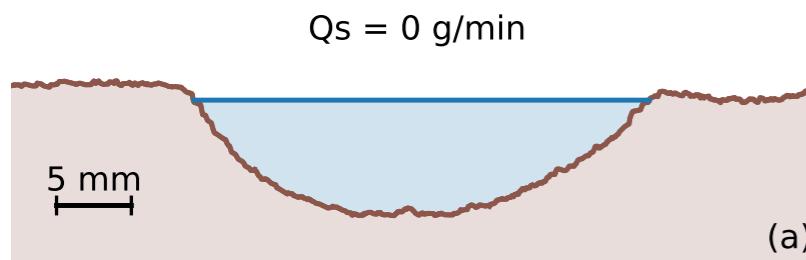


- constant flow discharge
- constant sediment discharge

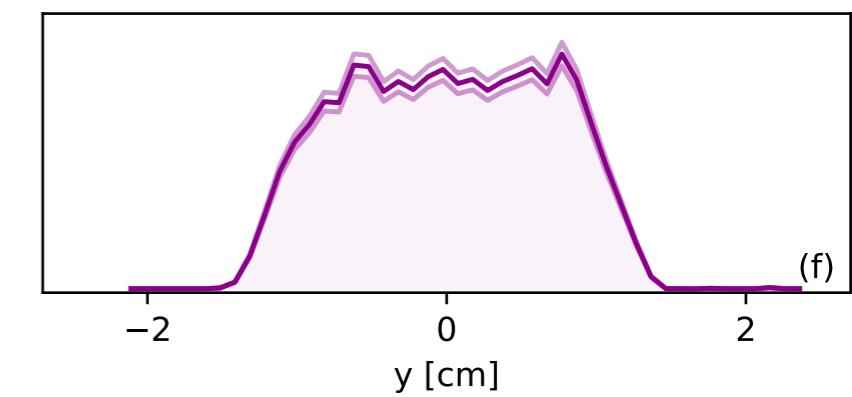
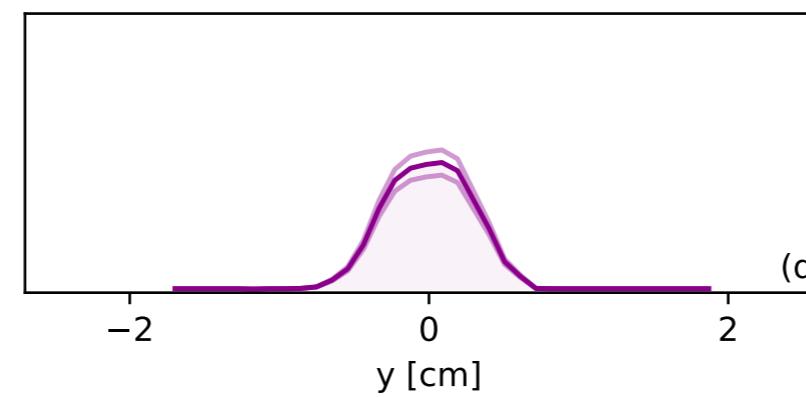
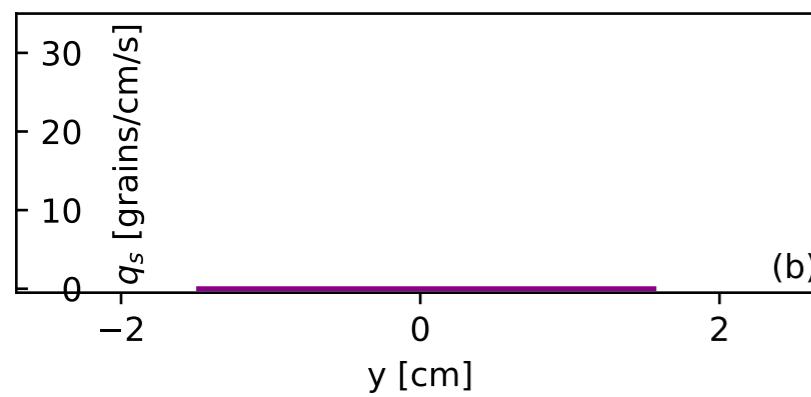
# Influence of sediment transport

channel cross-section  
flow discharge = 0.97 L/min

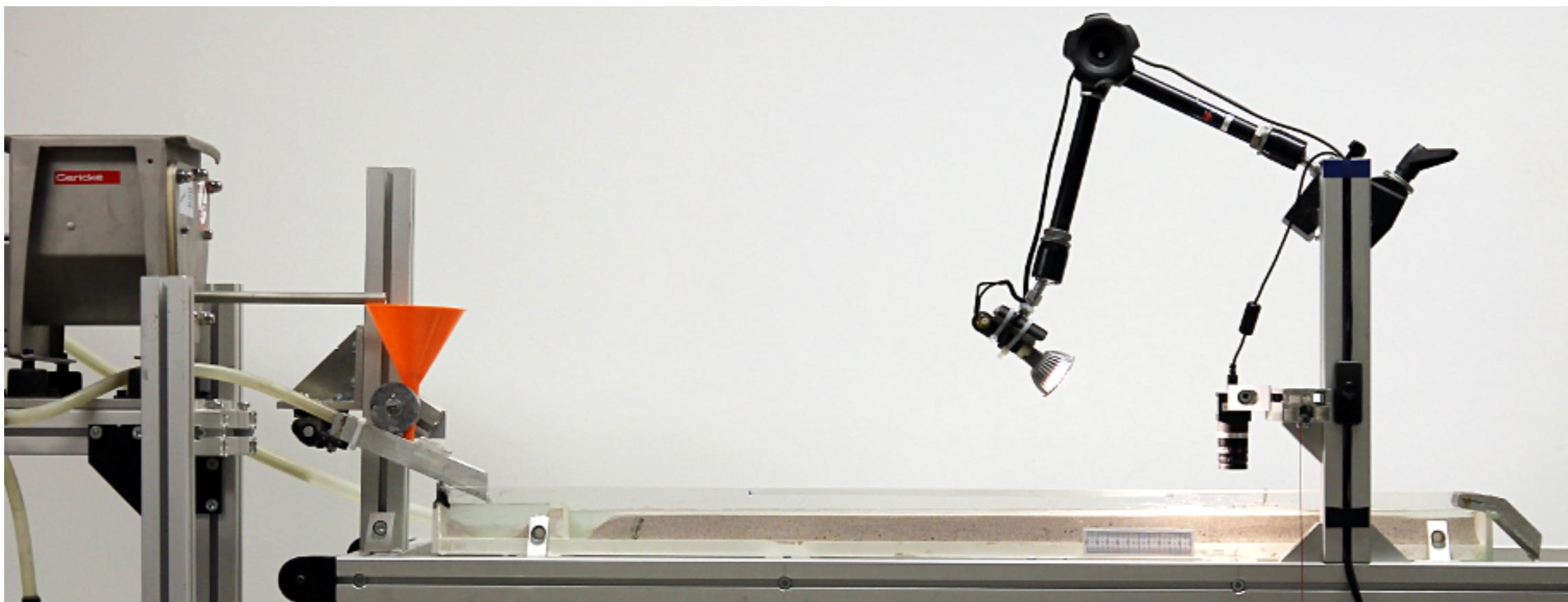
Abramian et al. [2020]



profile of sediment flux



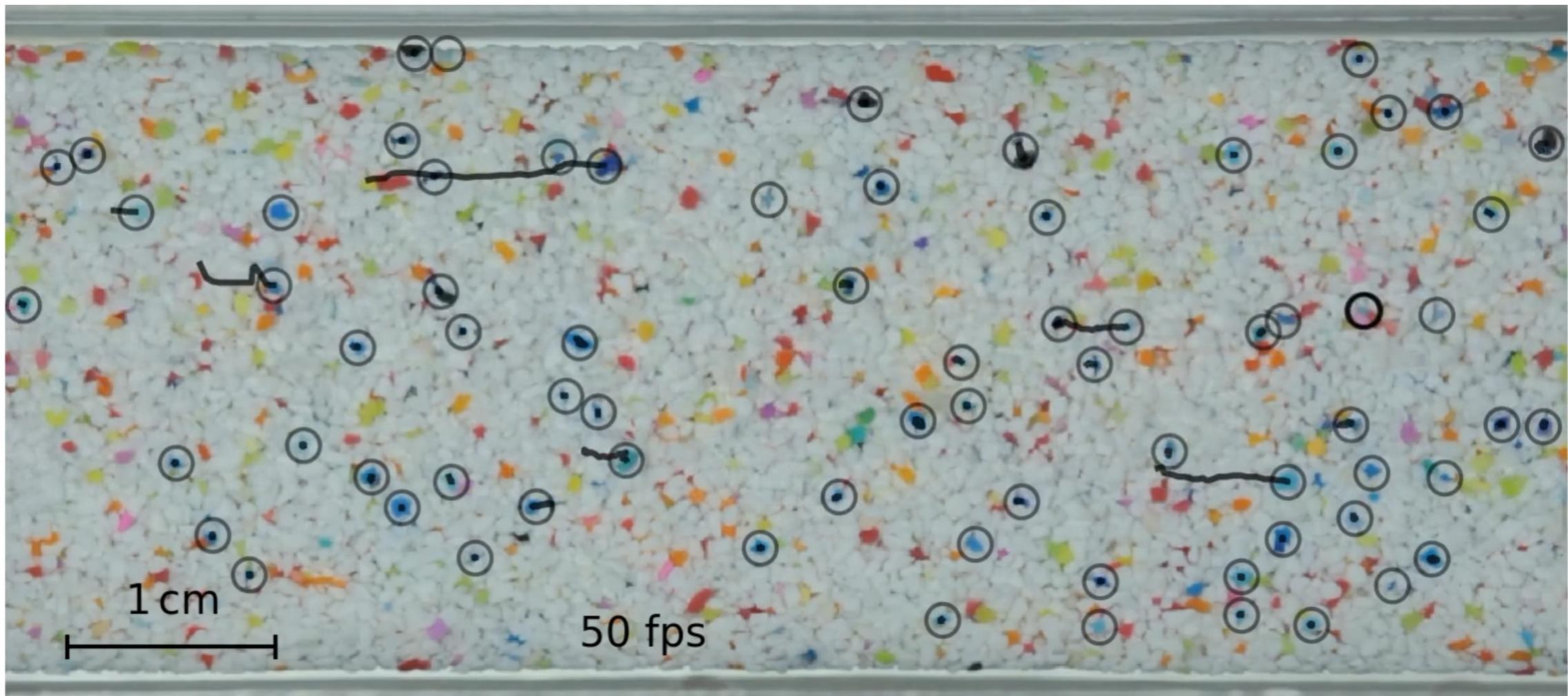
# Bedload transport in a laboratory Flume



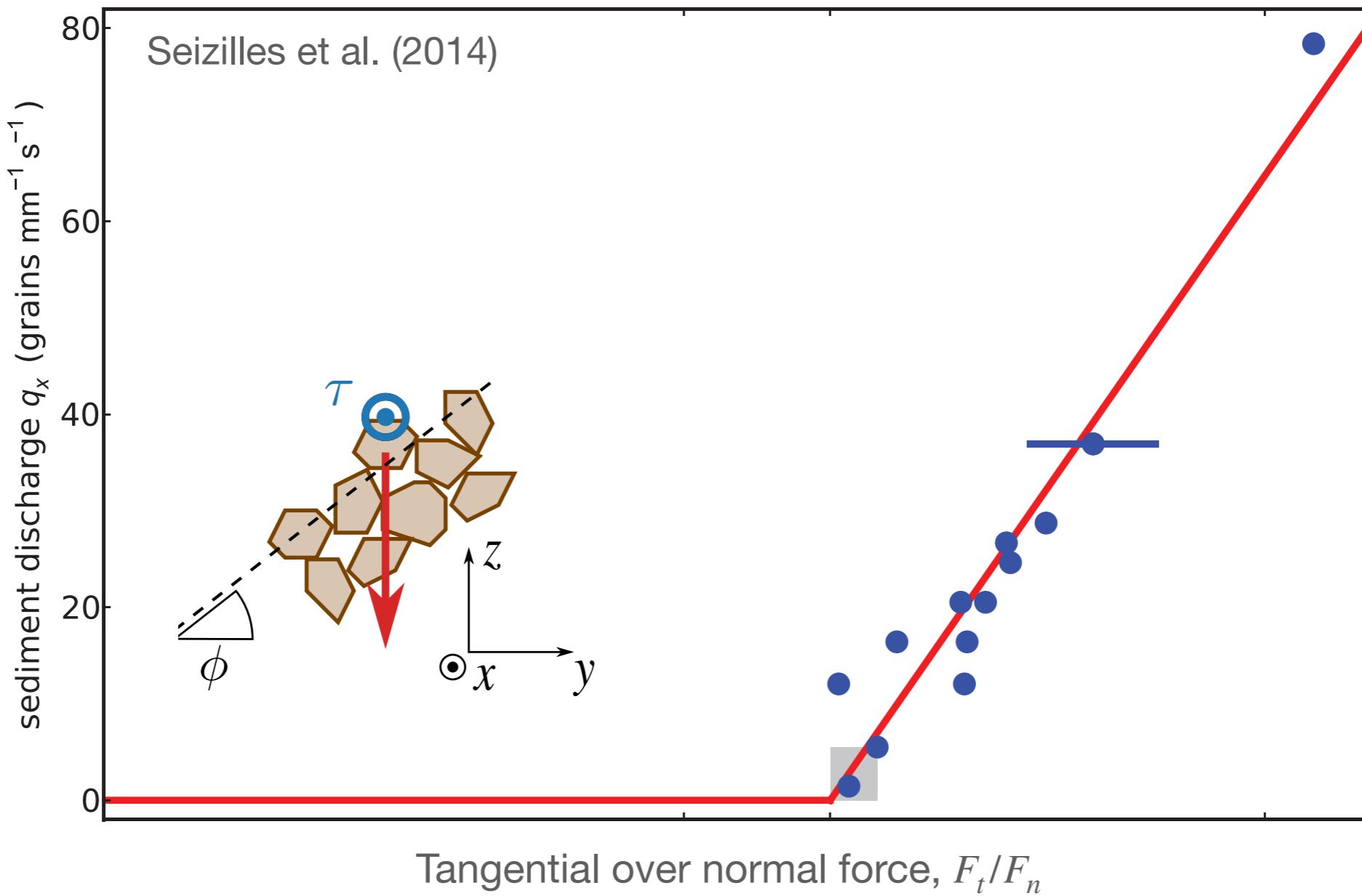
- plastic grains ( $d_s \sim 0.830\text{mm}$ )
- water-glycerol mixture ( $\text{Re} \sim 10$ )
- constant flow and sediment discharges

Seizilles *et al.* (2014), Abramian *et al.* (2019)

# Bedload transport in a laboratory Flume



# Streamwise bedload flux



Near threshold :

$$q_s \propto \left( \frac{F_t}{F_n} - \mu_t \right)$$

tangential force

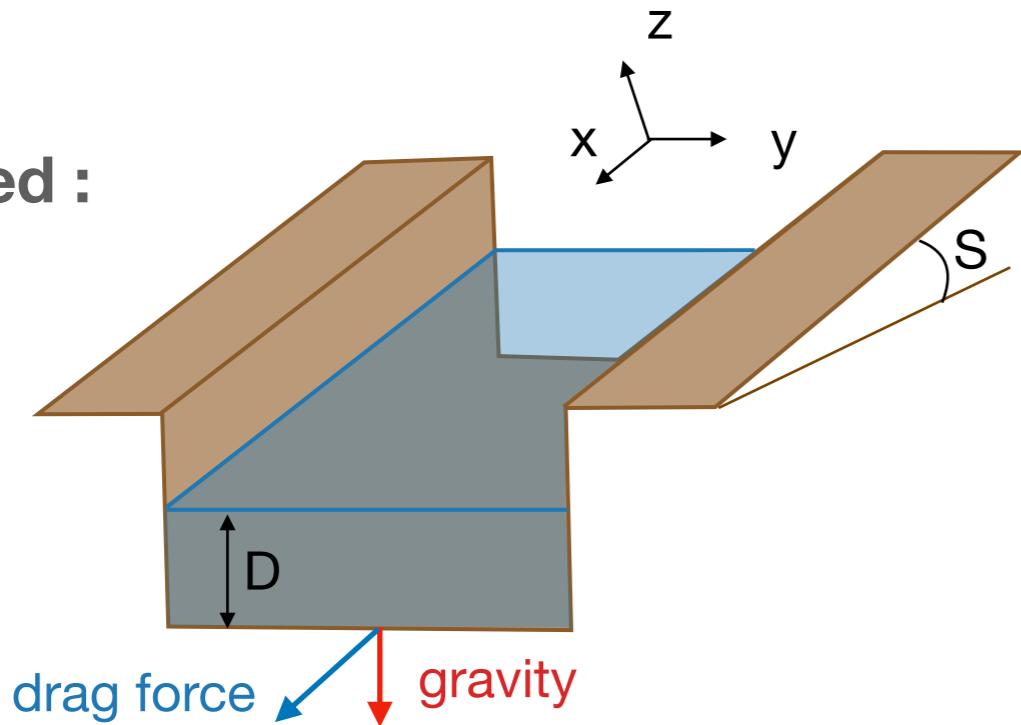
normal force

friction coefficient

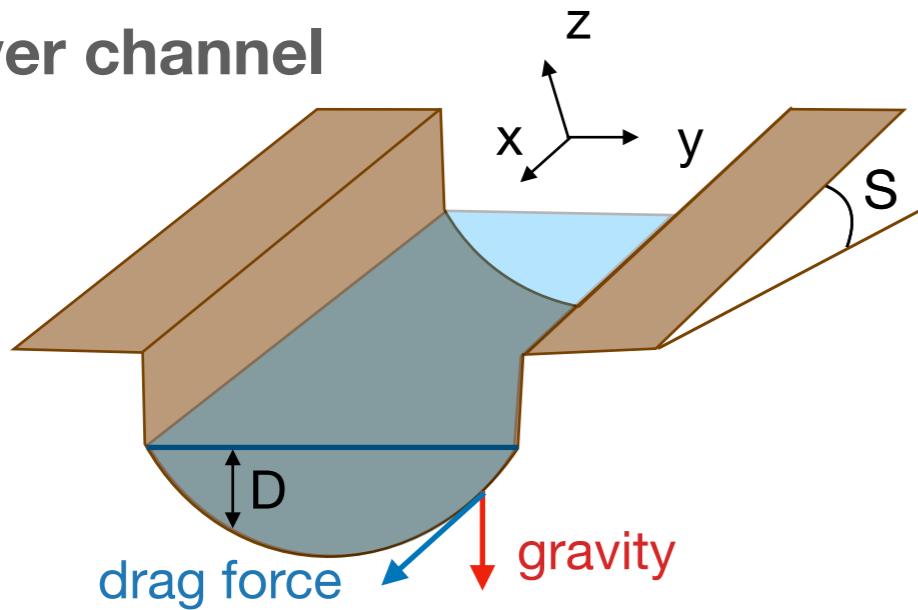
Charru et al. (2004), seizilles et al. (2014)

# Streamwise bedload flux

- Flat bed :



- River channel



Shields number |

$$\frac{F_t}{F_n} \propto \frac{\tau}{\Delta \rho g d_s} = \theta$$

drag force

weight

The equation shows the relationship between the ratio of transport force to normal force ( $F_t/F_n$ ) and the Shields number ( $\theta$ ). The term  $\tau/\Delta \rho g d_s$  is highlighted with a red oval. A bracket labeled 'drag force' is placed over the term  $\tau$ , and another bracket labeled 'weight' is placed over the term  $\Delta \rho g d_s$ .

shear stress

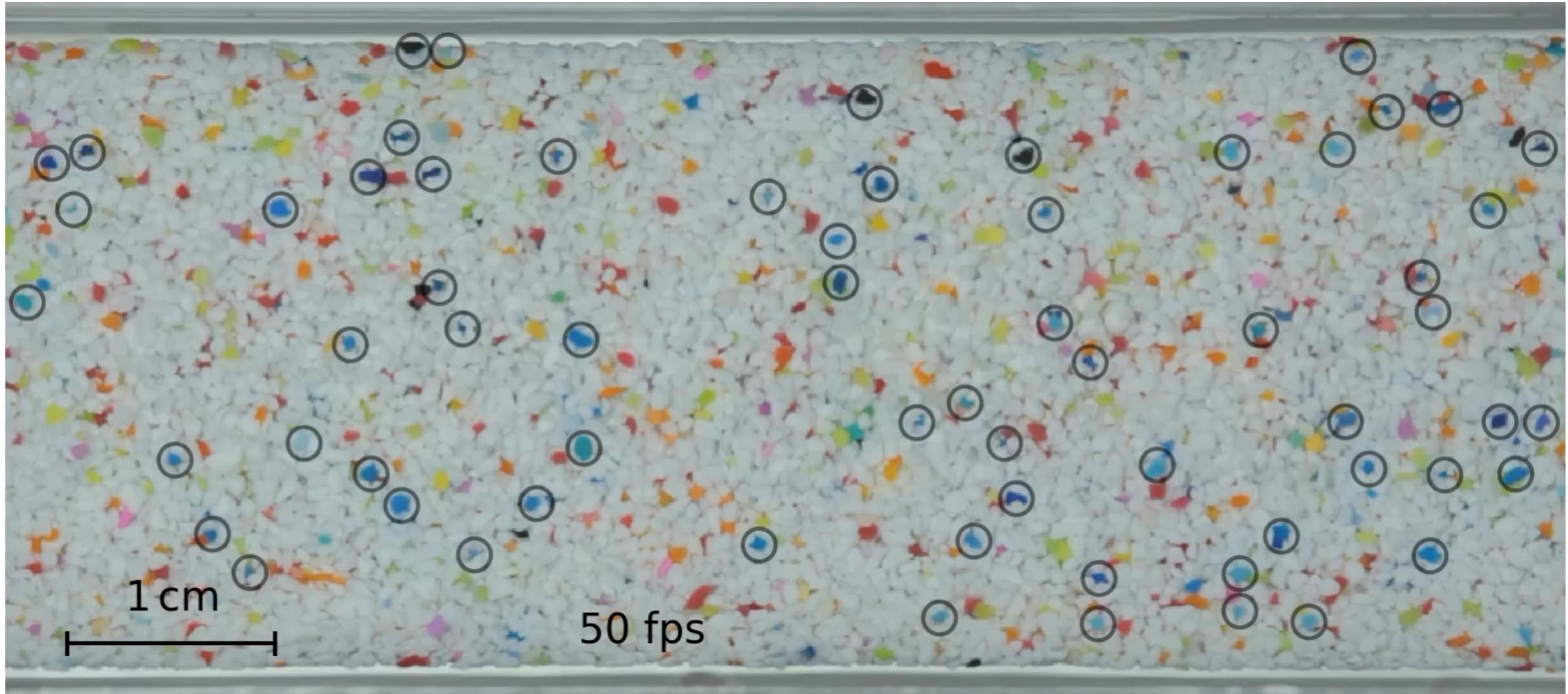
$$\frac{F_t}{F_n} = \sqrt{\left( \frac{\mu_t}{\theta_t \cos \phi} \frac{\tau}{\Delta \rho g d_s} \right)^2 + D'^2}$$

drag force

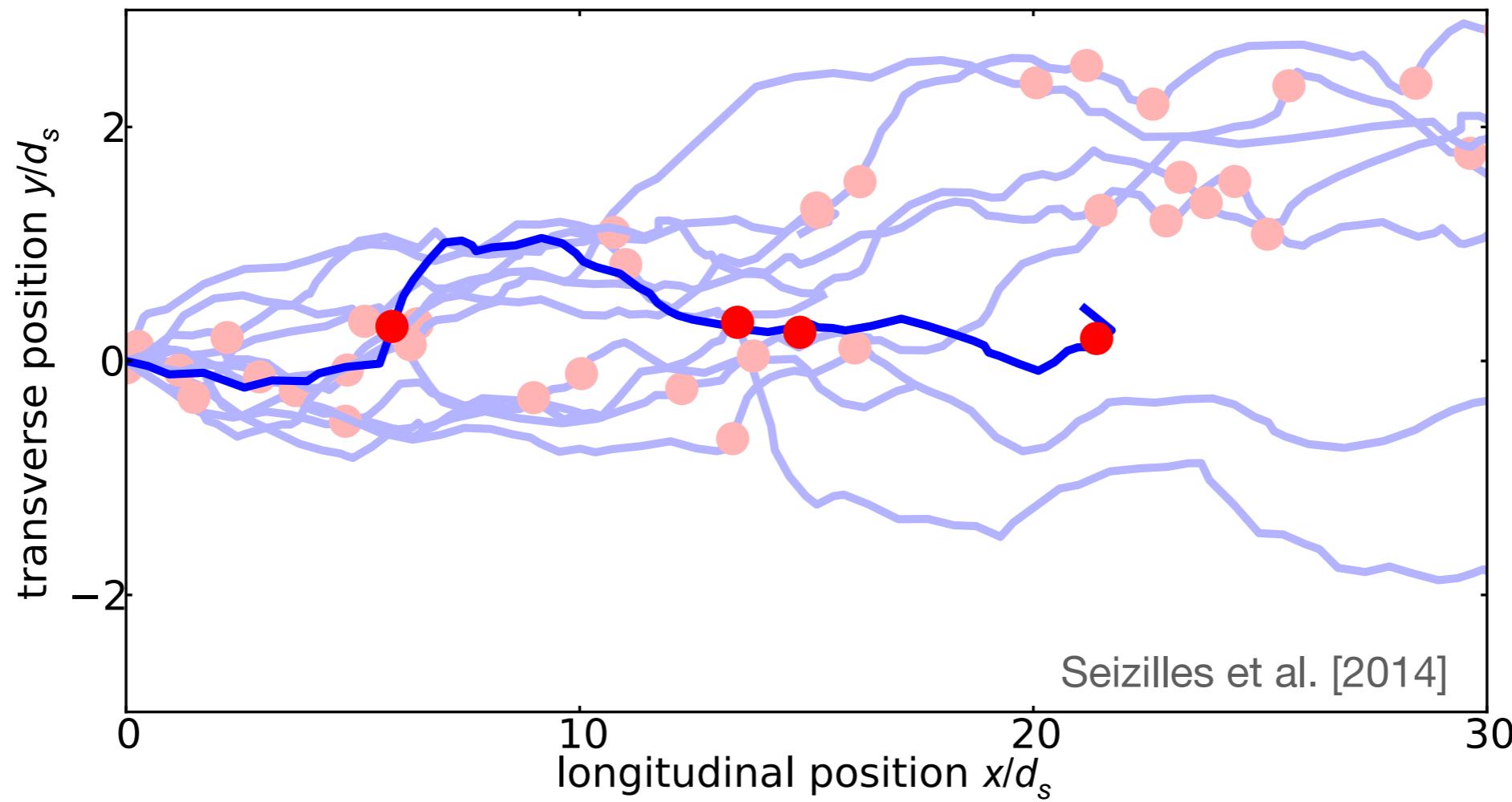
gravity

The equation provides a detailed formula for the ratio of transport force to normal force in a river channel. It includes terms for friction coefficient ( $\mu_t$ ), angle of friction ( $\phi$ ), shear stress ( $\tau$ ), density difference ( $\Delta \rho$ ), gravitational acceleration ( $g$ ), and particle diameter ( $d_s$ ). The term  $D'$  is highlighted with a red oval. Brackets indicate the components of the square root: the first part is labeled 'drag force' and the second part is labeled 'gravity'.

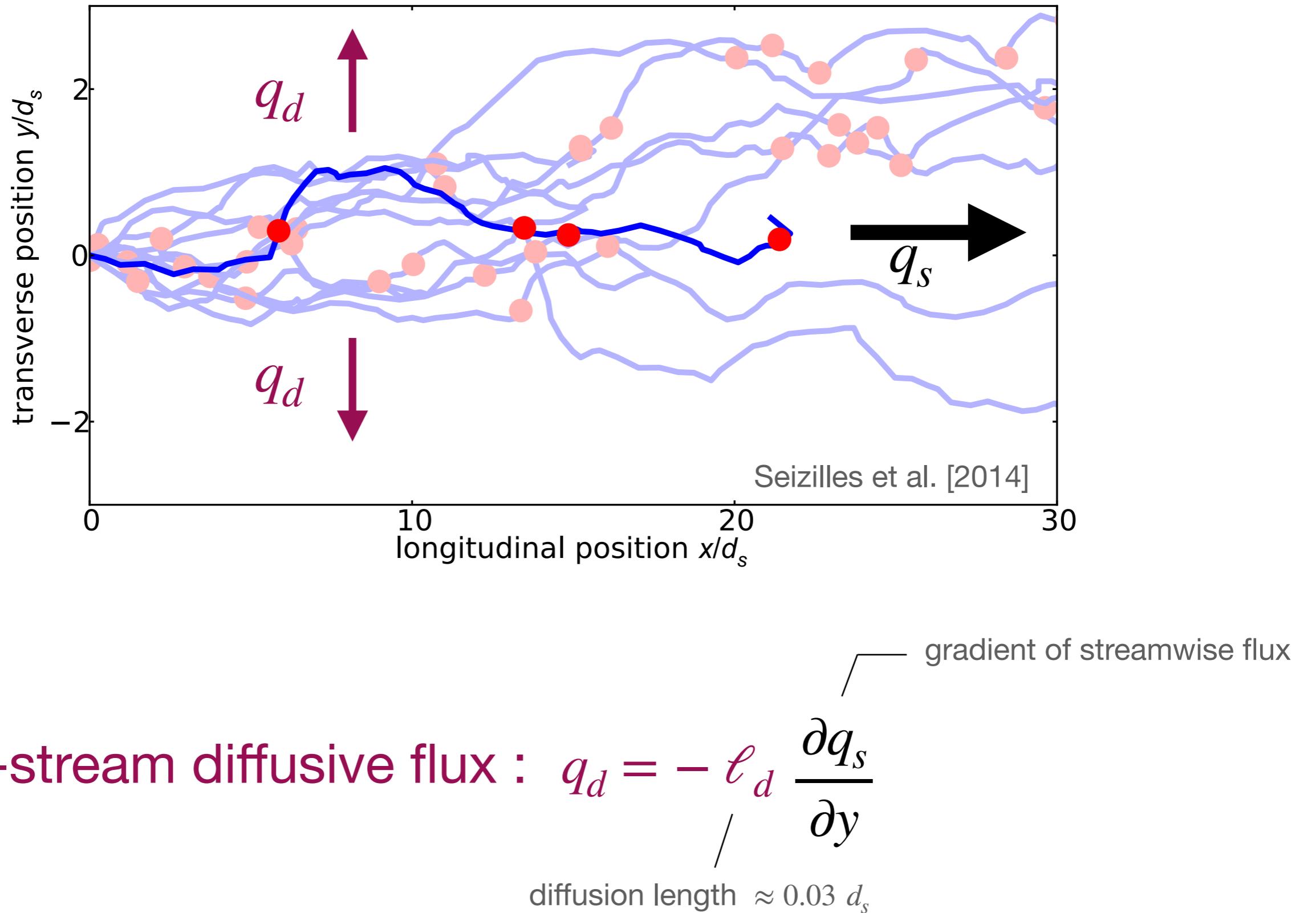
# Cross-stream bedload diffusion



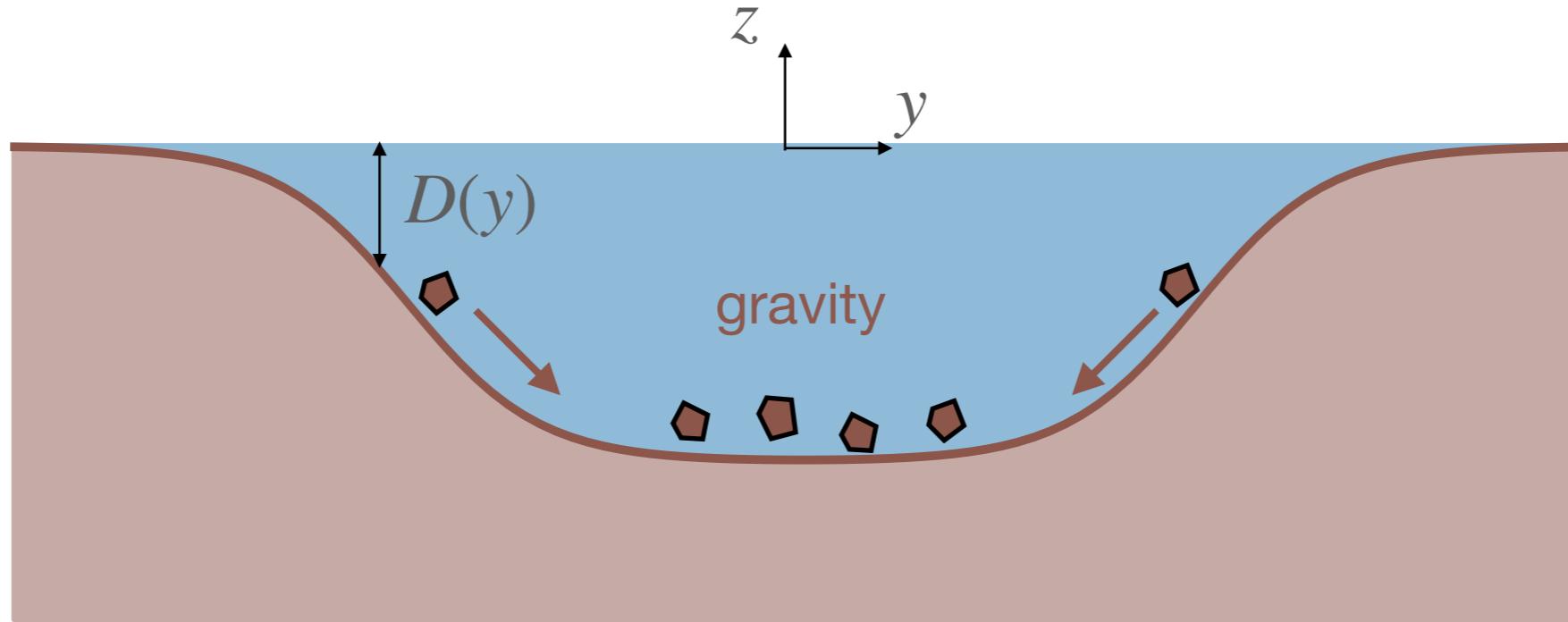
# Cross-stream bedload diffusion



# Cross-stream bedload diffusion



# Cross-stream gravity flux

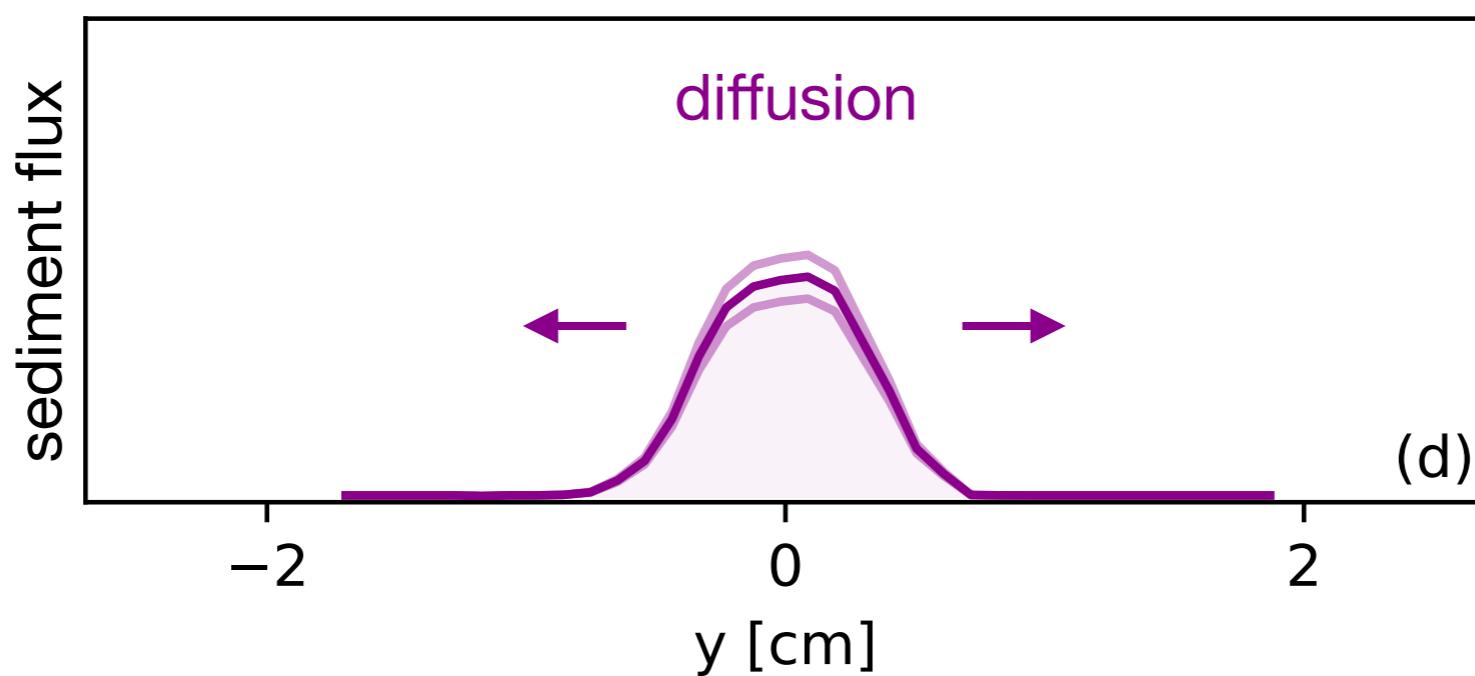
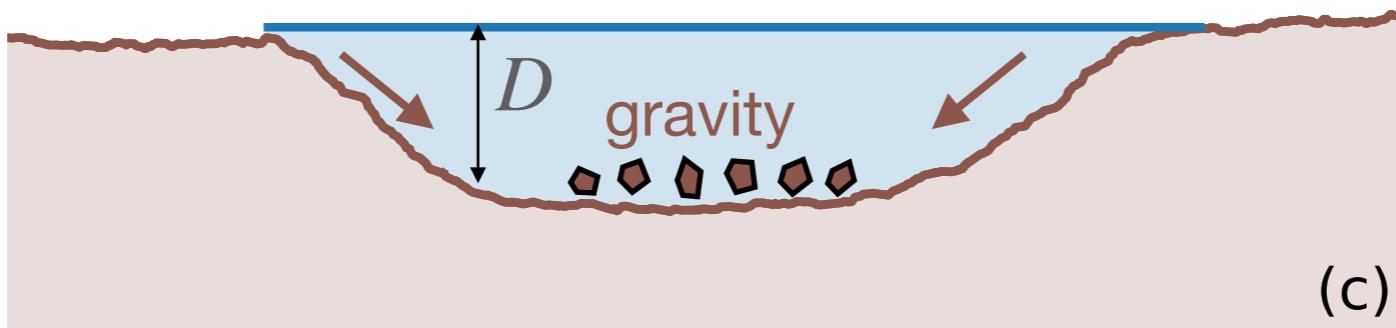


cross-stream gravity flux :

$$q_g = \alpha q_s \frac{\partial D}{\partial y}$$

streamwise flux      ↘  
                                constant      ↗ bed slope

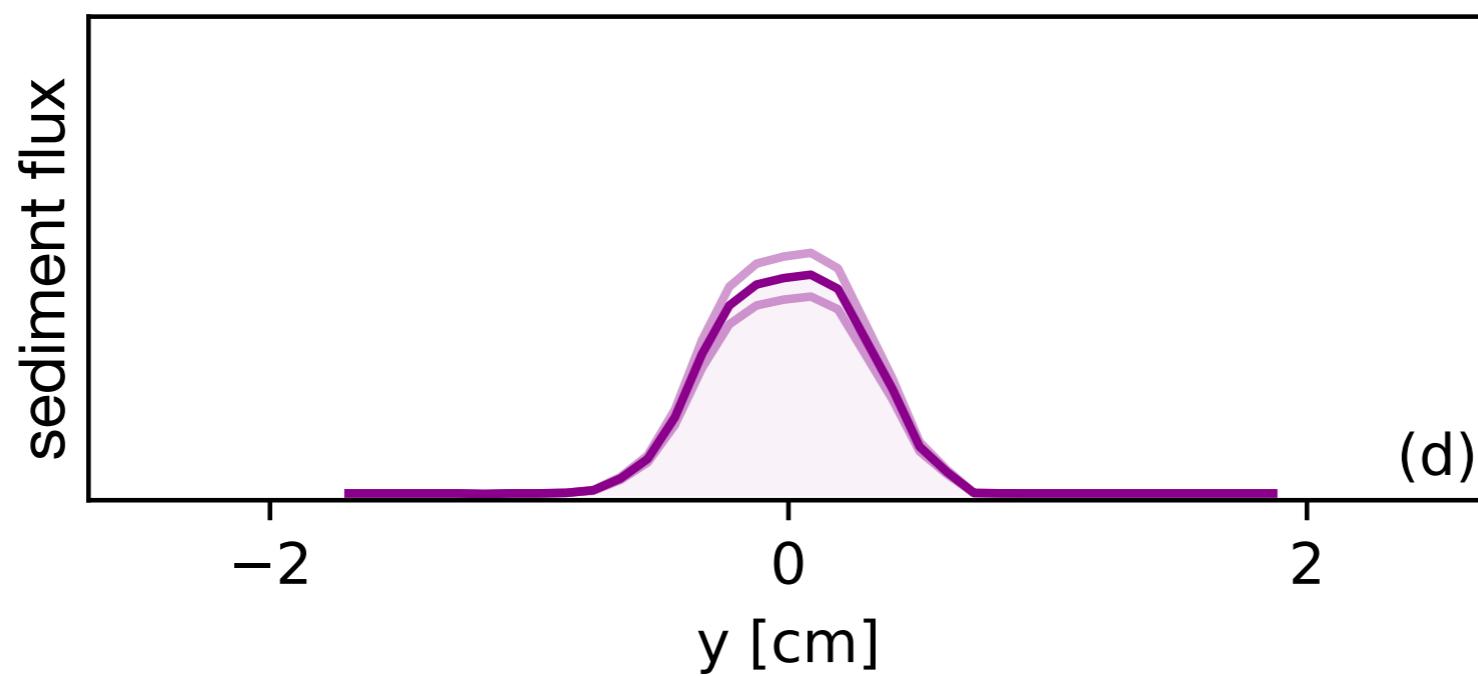
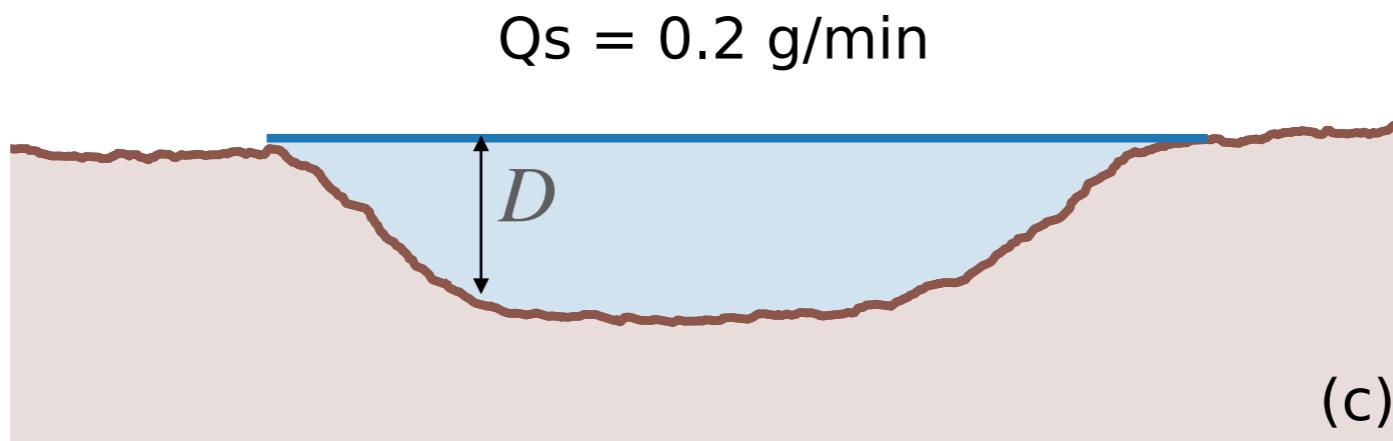
# Equilibrium condition in an active channel



At equilibrium :  $-\ell_d \frac{\partial q_s}{\partial y} + \alpha q_s \frac{\partial D}{\partial y} = 0$

diffusion	gravity
-----------	---------

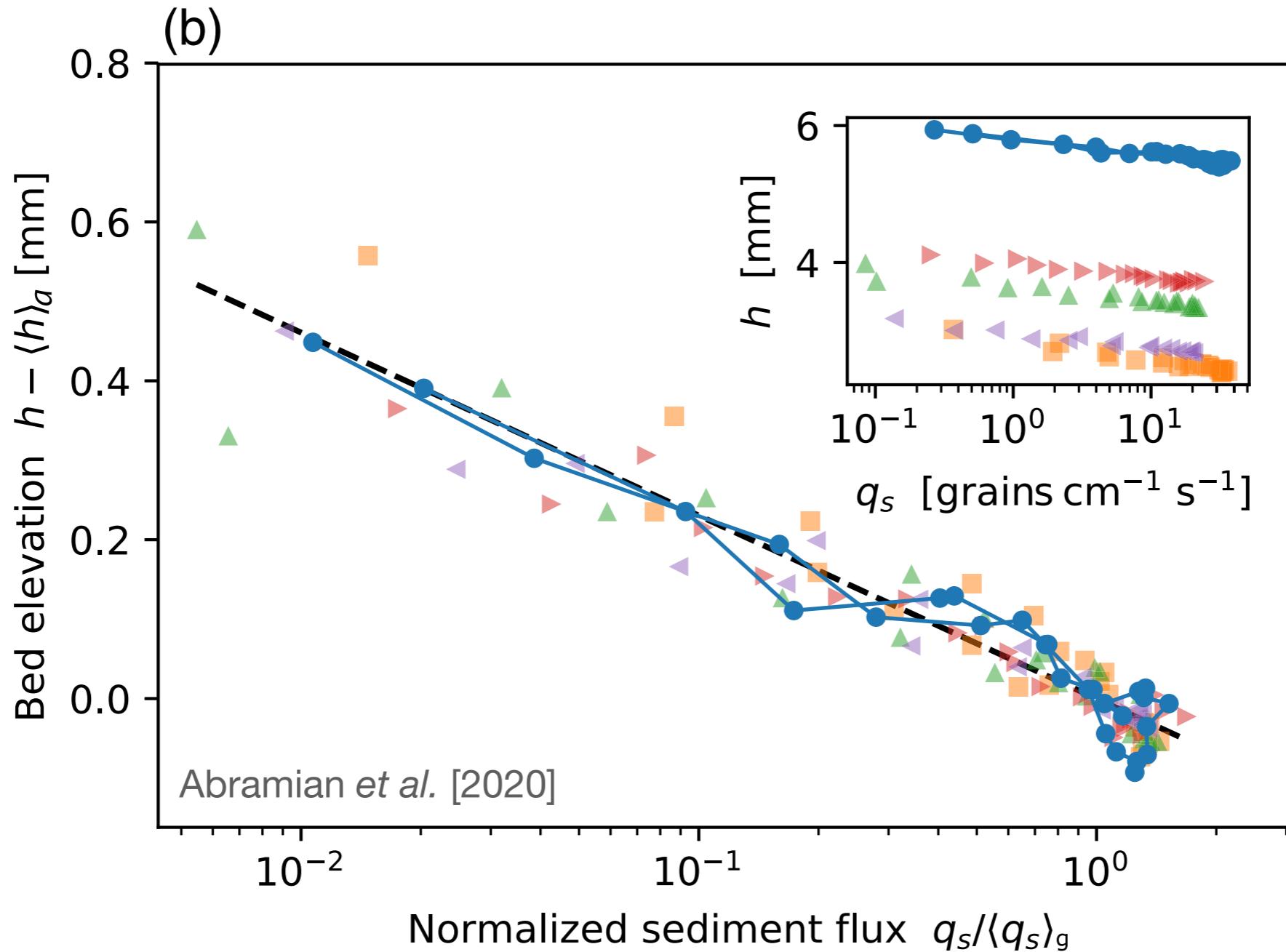
# Equilibrium condition in an active channel



At equilibrium :  $q_s \propto e^{D/\lambda}$

depth  $\lambda \sim$  diffusion length = damping height

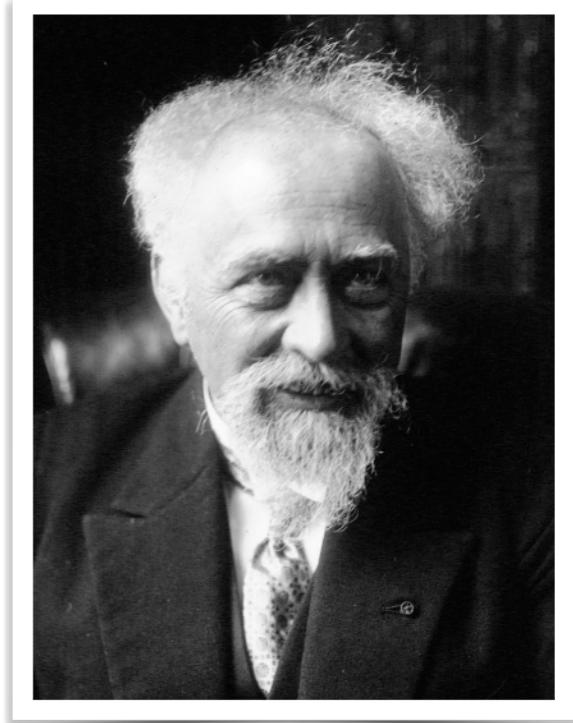
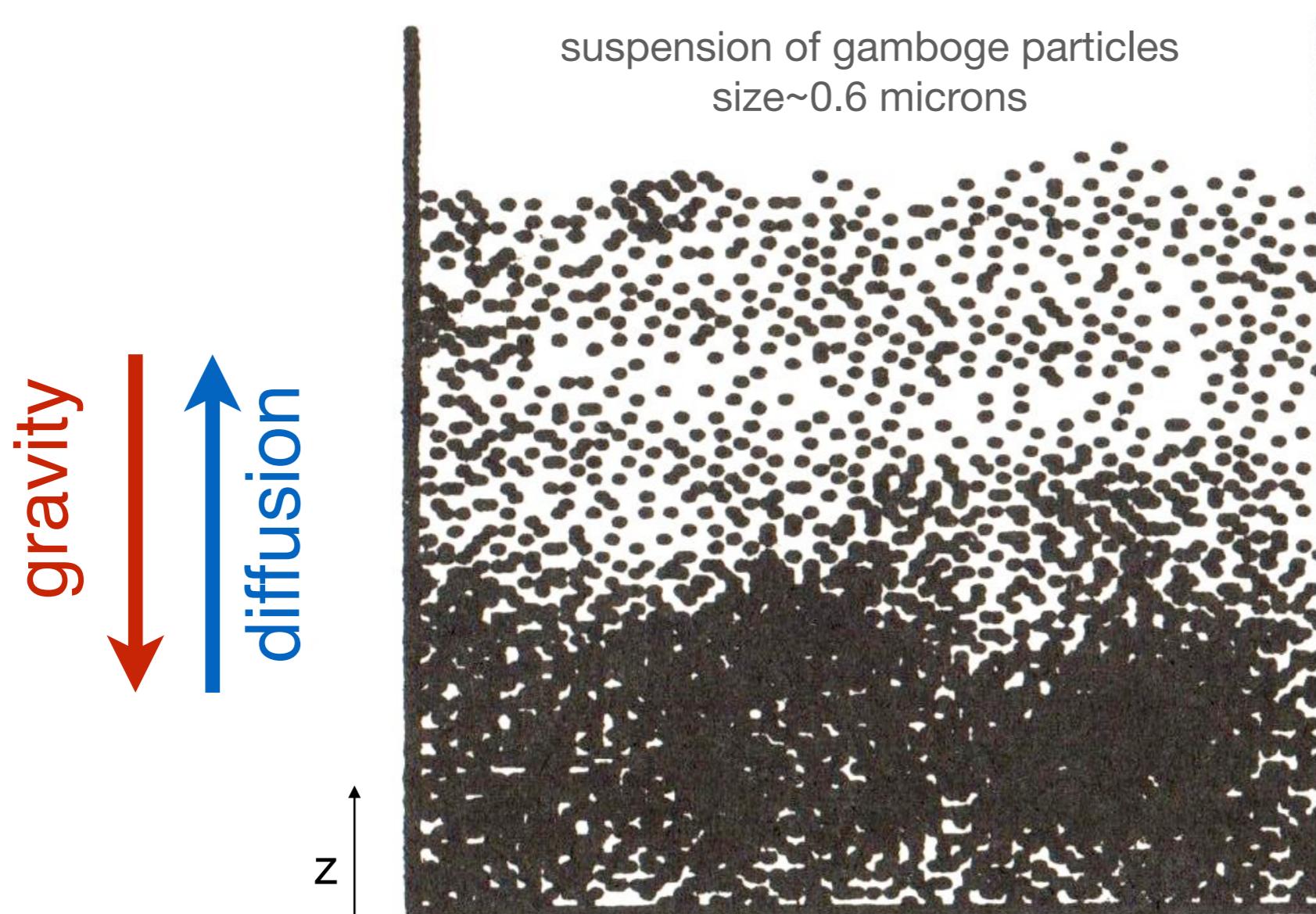
# Boltzmann equilibrium



At equilibrium :  $q_s \propto e^{D/\lambda}$

damping length  $\lambda \sim 0.12 d_S$

# Gravity vs diffusion

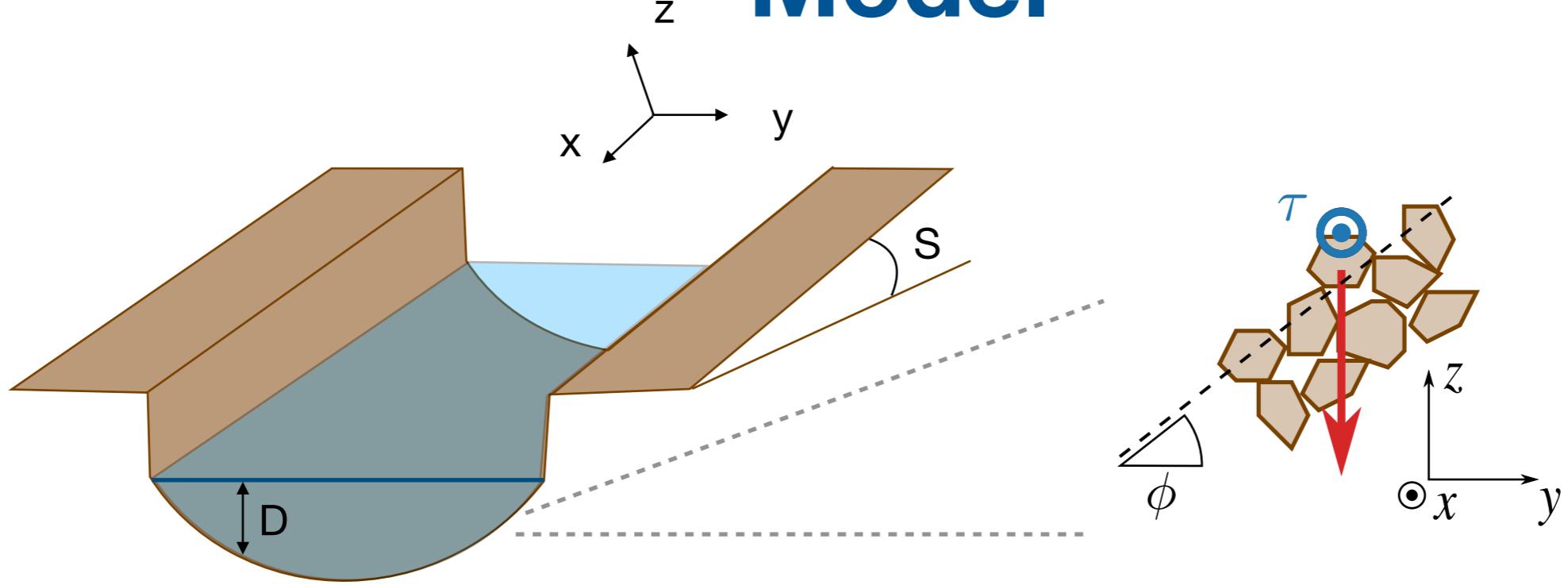


J. Perrin [1908]

At equilibrium, diffusion = gravity flux [Einstein, 1905]

$$c \sim e^{-z/kT}$$

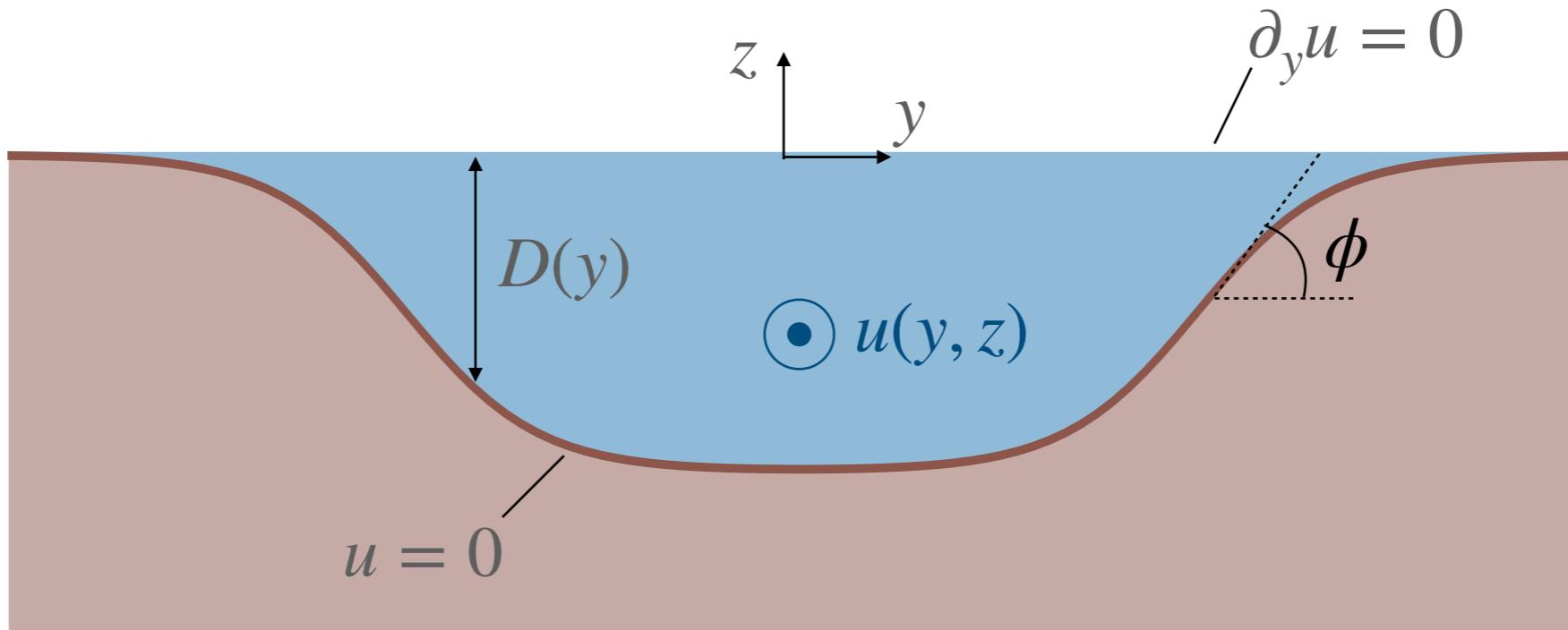
# Model



- Diffusion vs gravity :  $q_s \propto e^{D/\lambda}$
- Transport law :  $q_s \propto \left( \left( \frac{\mu_t}{\theta_t \cos \phi} \frac{\tau}{\Delta \rho g d_s} \right)^2 + \left( \frac{dD}{dy} \right)^2 \right)^{1/2} - \mu_t$
- Shear stress :  $\tau = f(\text{channel shape})$

→ Free boundary problem

# Cross-stream diffusion of momentum



Stokes equation :

$$\partial_{yy}u + \partial_{zz}u = -\frac{gS}{\nu}$$

velocity | slope  
| viscosity

for large aspect ratios  $\rightarrow \tau = \rho g S \left( D + \frac{1}{3} (D^3)'' \right) \cos \phi$

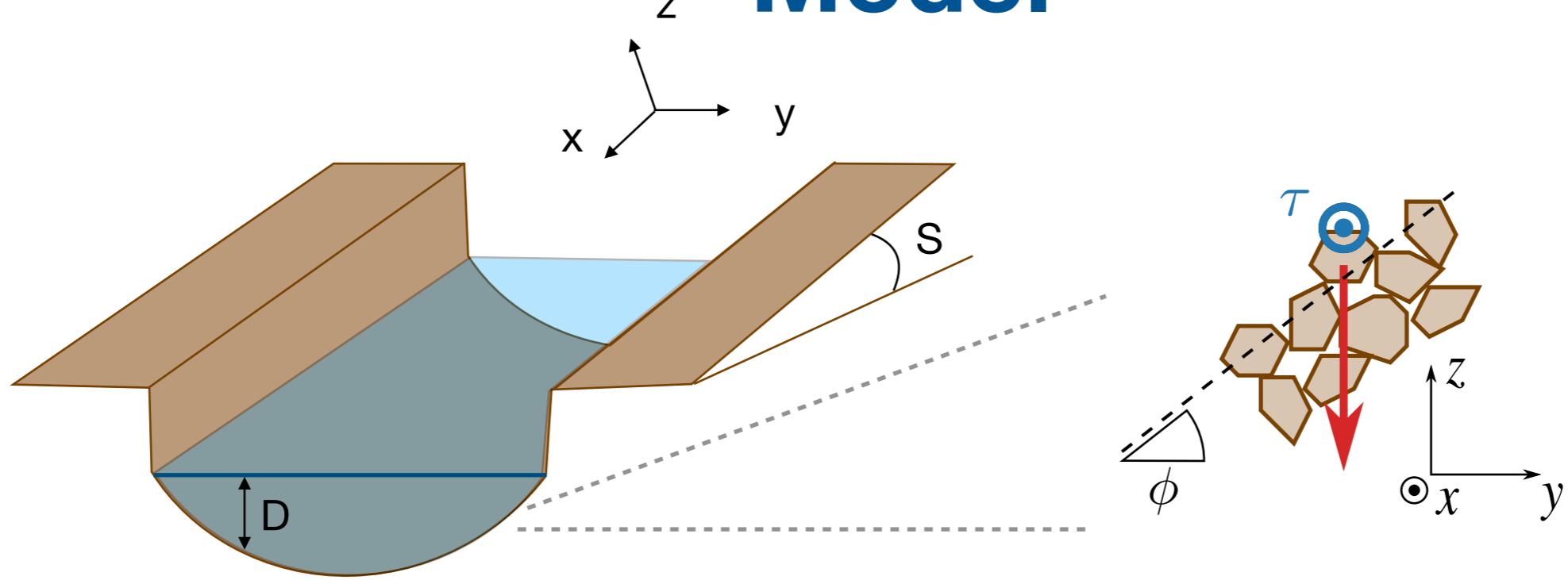
Devauchelle et al. [2021], Popovic et al. [2021]

weight of water column (shallow water)

cross-stream diffusion of momentum ( $\sim 1/a^2$ )

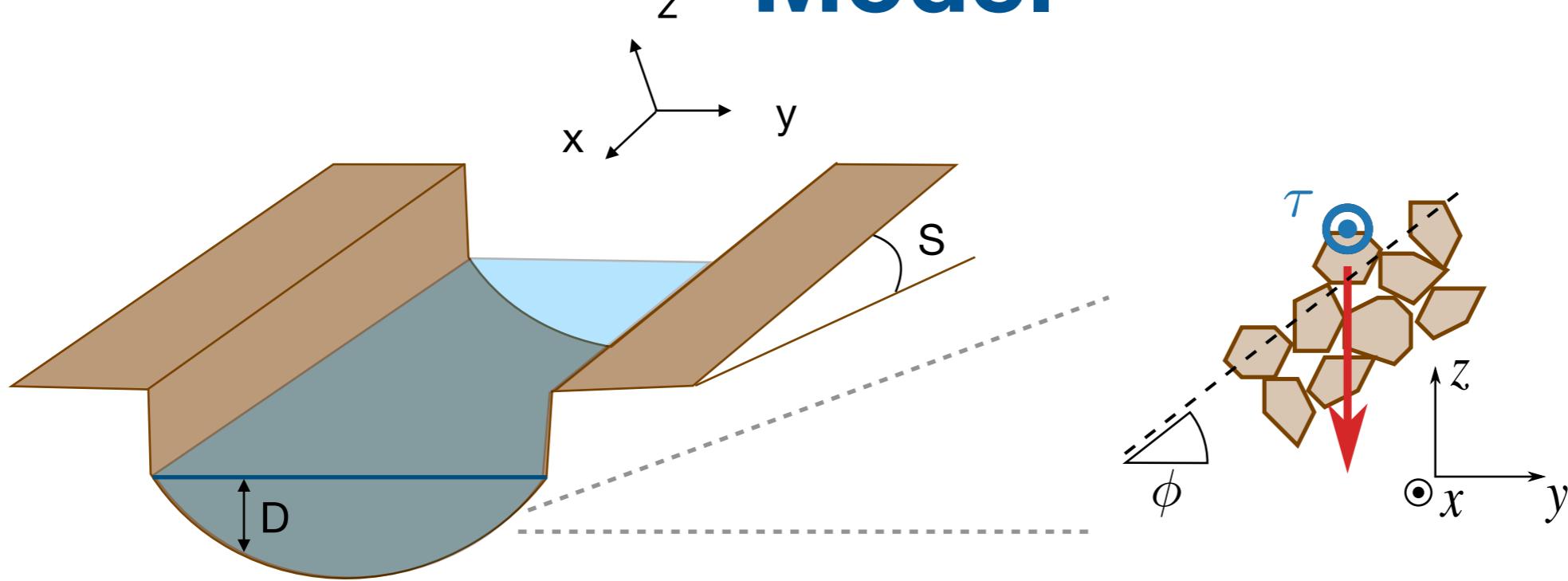
orientation of the bed surface

# Model



- Diffusion vs gravity :  $q_s \propto e^{D/\lambda}$
- Transport law :  $q_s \propto \left( \left( \frac{\mu_t}{\theta_t \cos \phi} \frac{\tau}{\Delta \rho g d_s} \right)^2 + D^2 \right)^{1/2} - \mu_t$
- Shear stress :  $\tau = \rho g S \left( D + \frac{1}{3} (D^3)'' \right) \cos \phi$

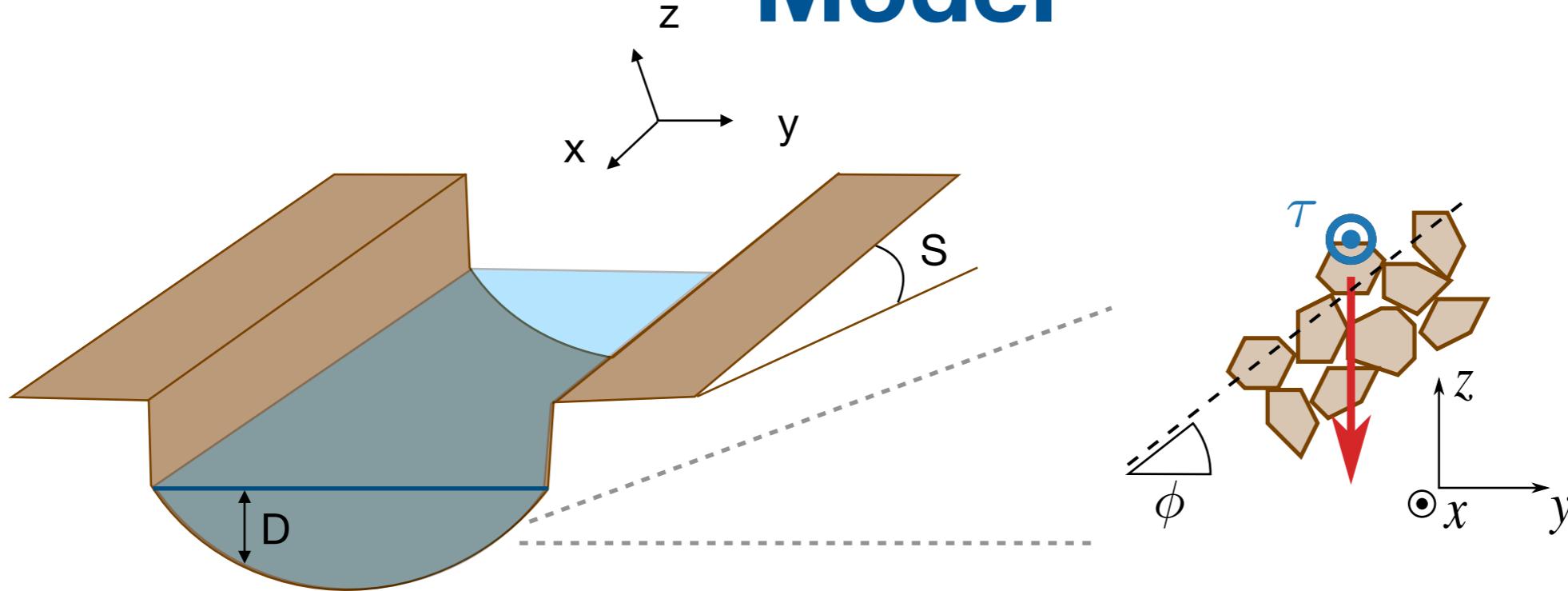
# Model



$$\sqrt{\frac{S^2}{L^2} \left( D + \frac{1}{3}(D^3)'' \right)^2 + D'^2} - \mu_t = e^{(D-\xi)/\lambda}$$

slope  
 depth  
 friction coefficient  
 characteristic length  
 ~ grain size  
**transport law**  
**equilibrium condition**  
 ~ diffusion length  
 integration constant related to the total sediment discharge

# Model

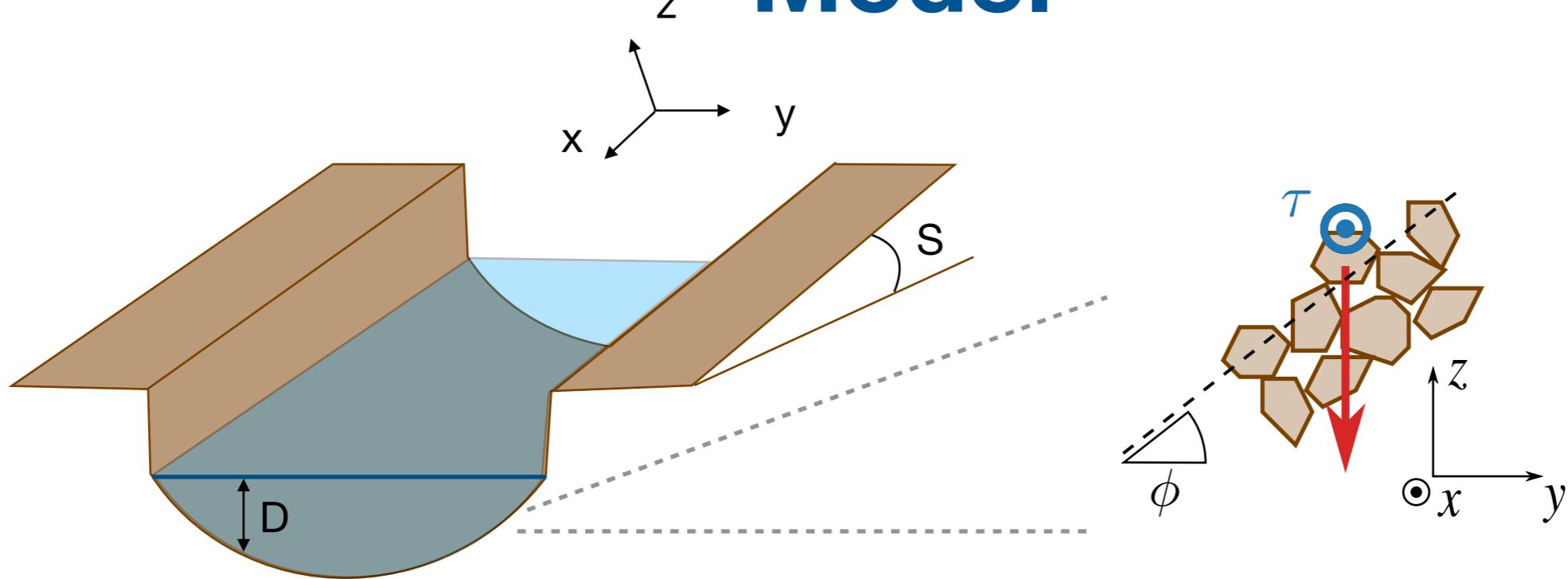


$$\sqrt{\frac{S^2}{L^2} \left( D + \frac{1}{3}(D^3)'' \right)^2 + D'^2} - \mu_t = e^{(D-\xi)/\lambda}$$

slope  
 depth  
 friction coefficient  
 characteristic length  
 ~ grain size  
**transport law**  
**equilibrium condition**  
 ~ diffusion length  
 integration constant related to the total sediment discharge

characteristic length :  $\frac{L}{S}$

# Model

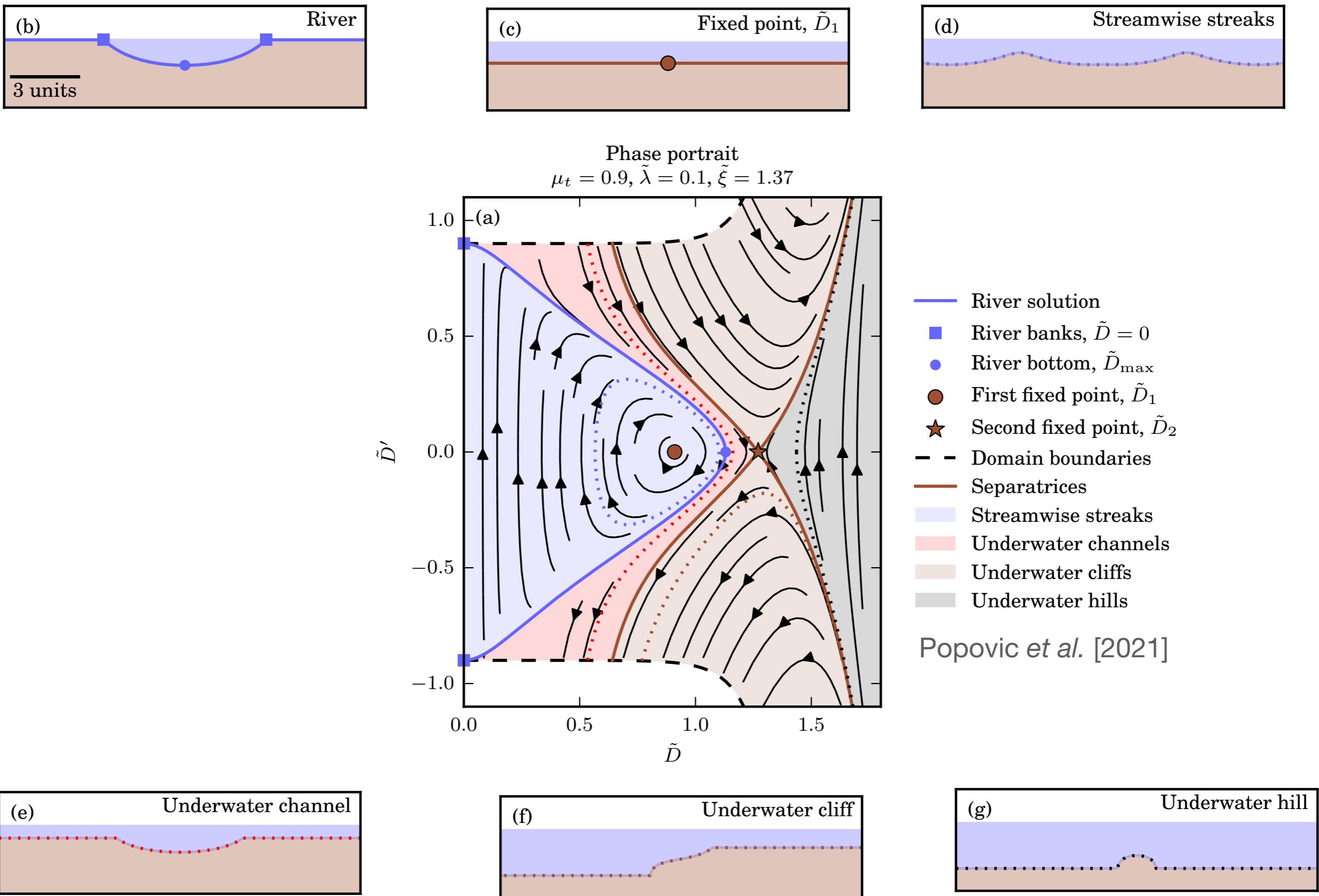


$$\sqrt{\left(\tilde{D} + \frac{1}{3}(\tilde{D}^3)''\right)^2 + \tilde{D}'^2} - \mu_t = e^{(\tilde{D} - \tilde{\xi})/\tilde{\lambda}}$$

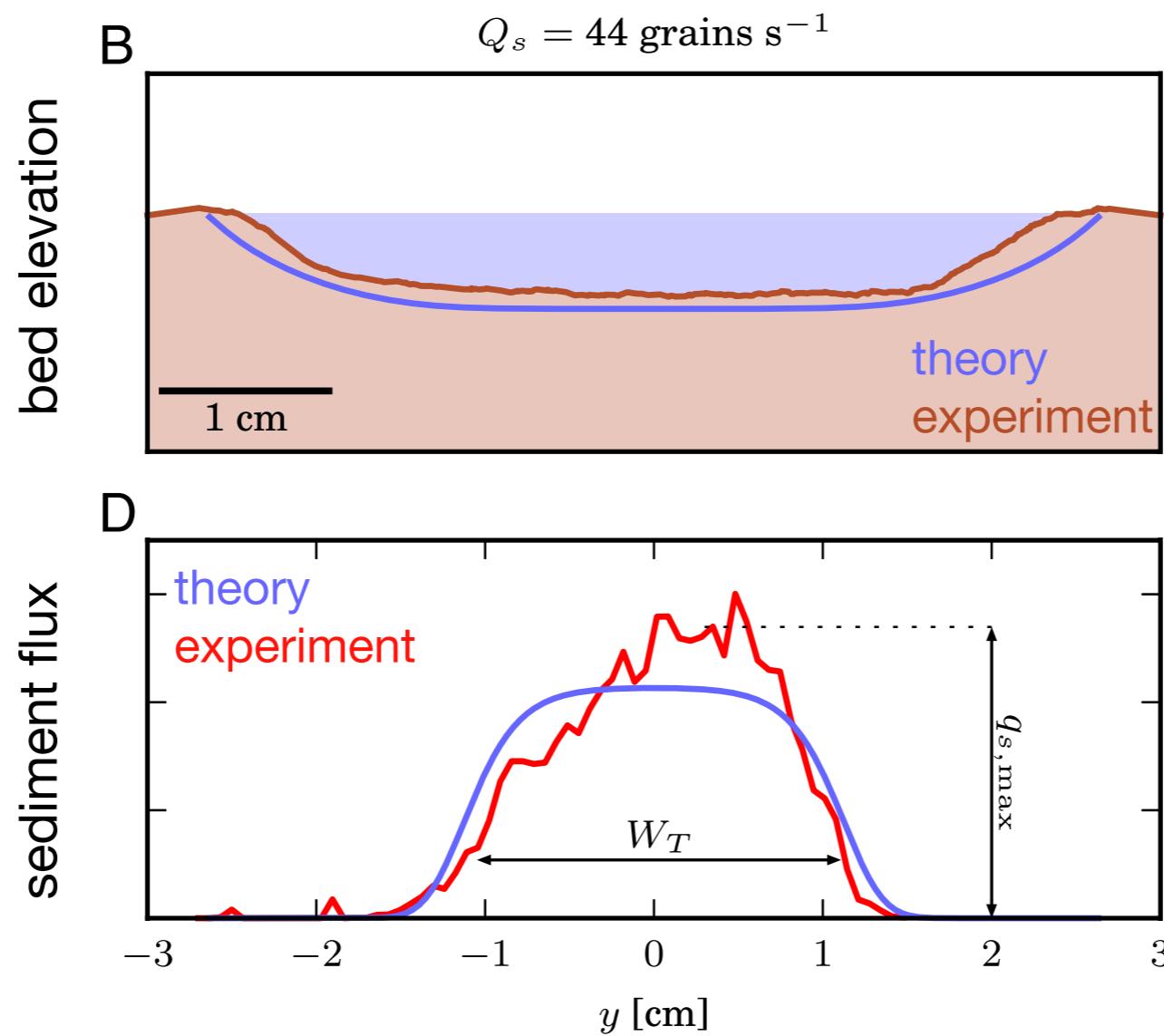
transport law

equilibrium condition

# Model

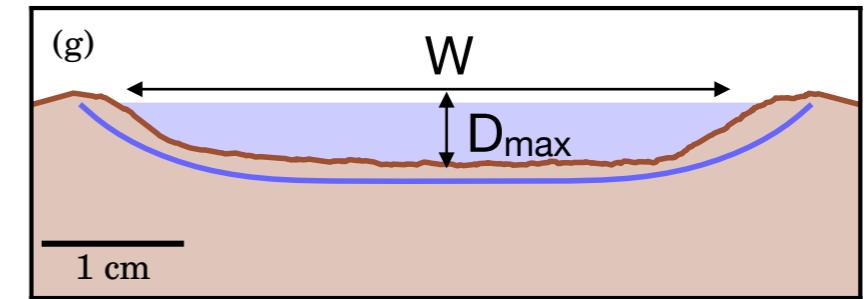
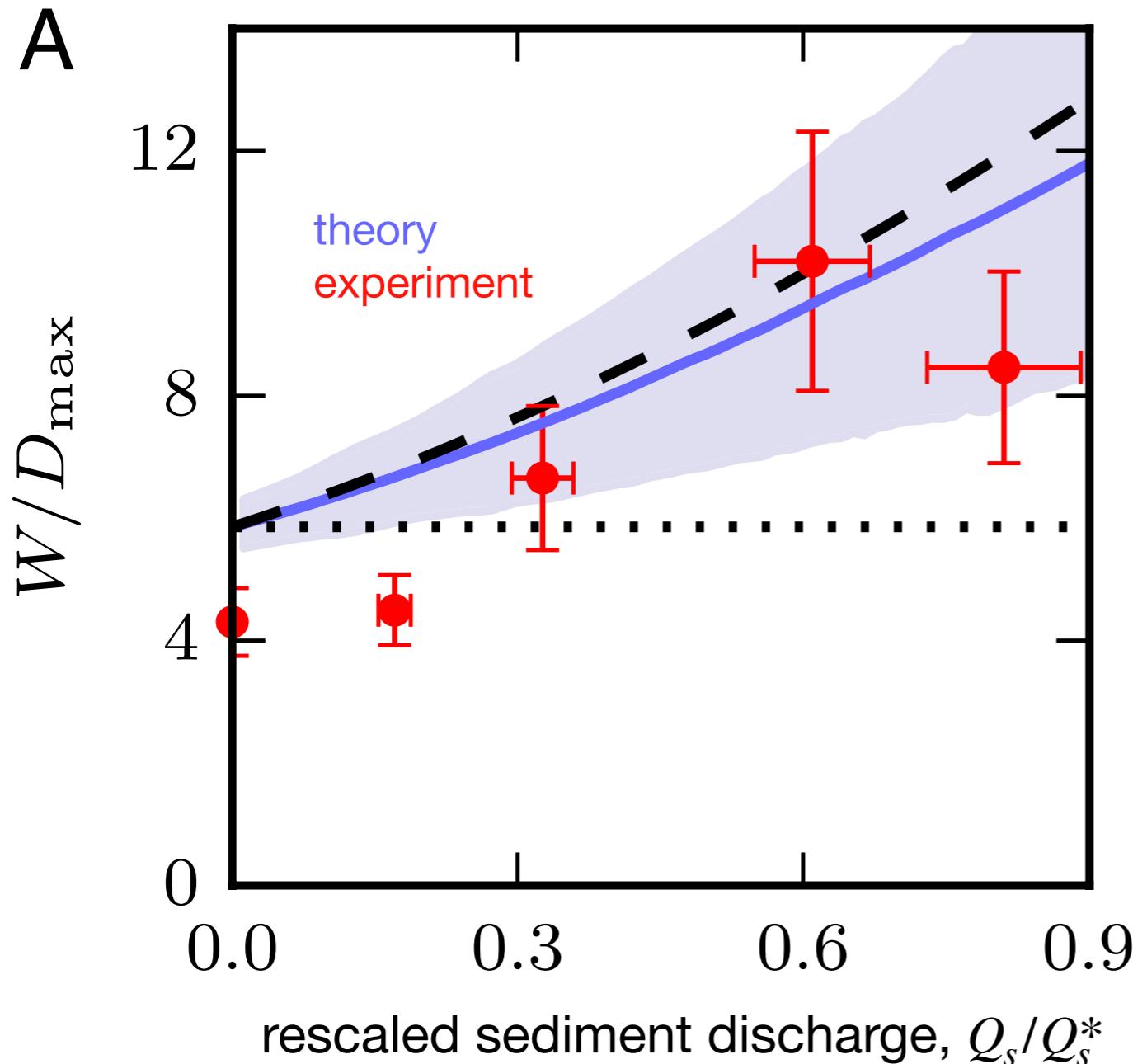


# Comparison with experiments

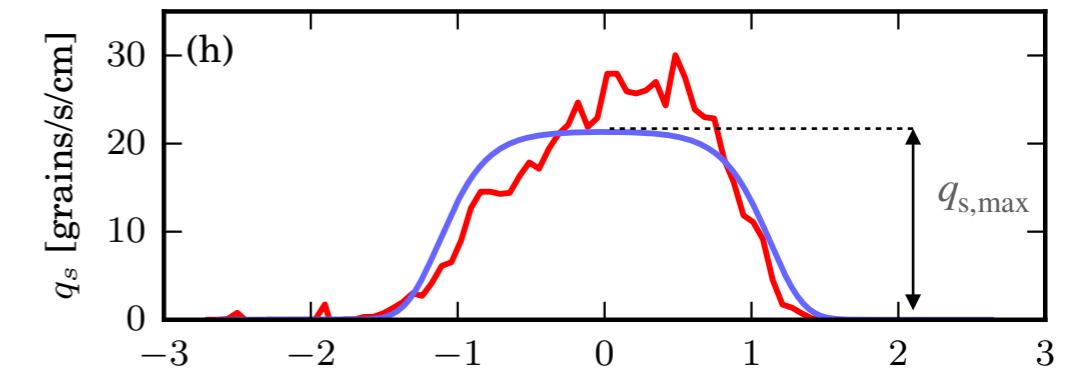
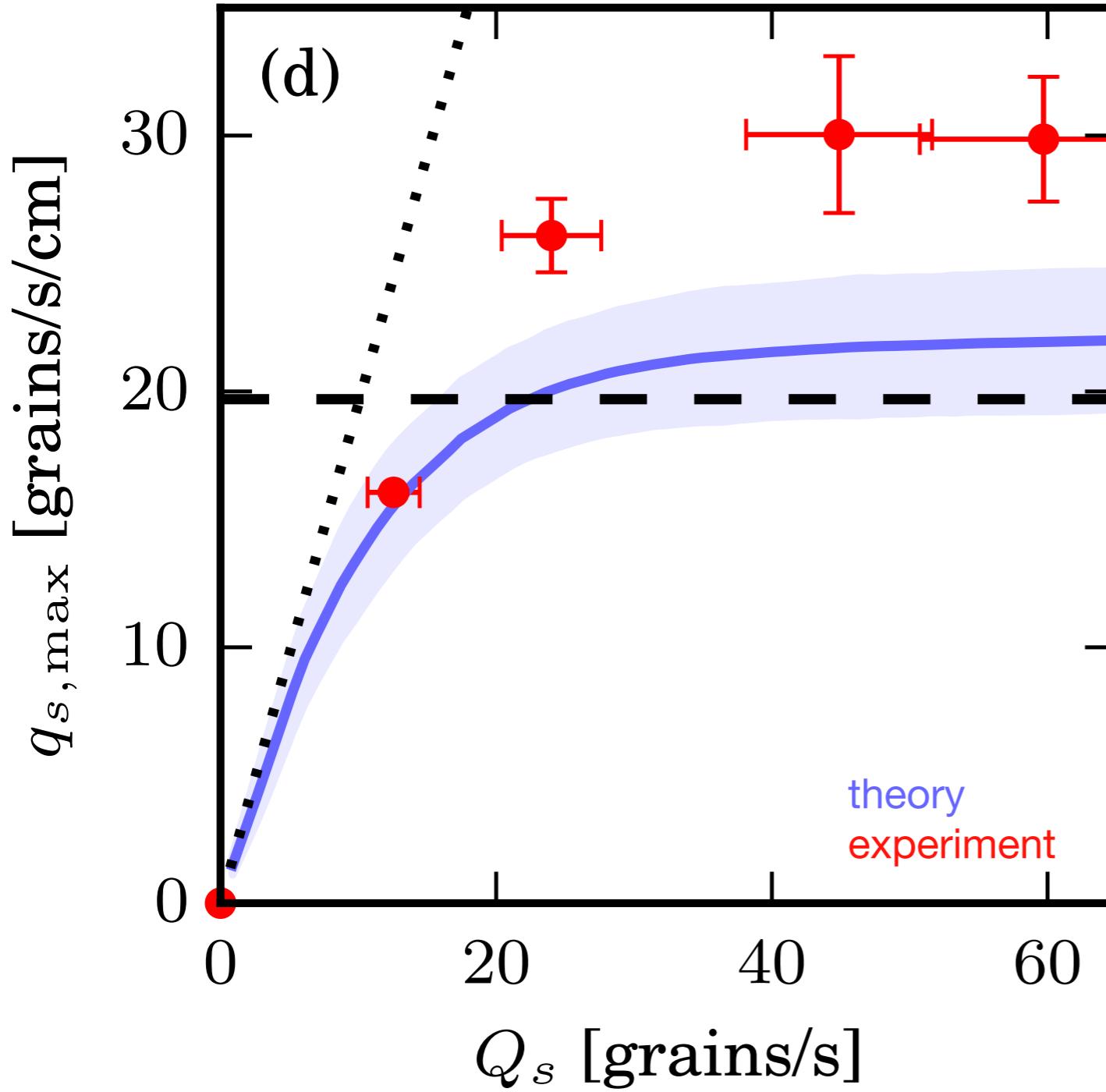


Popovic et al. [2021]

# Aspect ratio

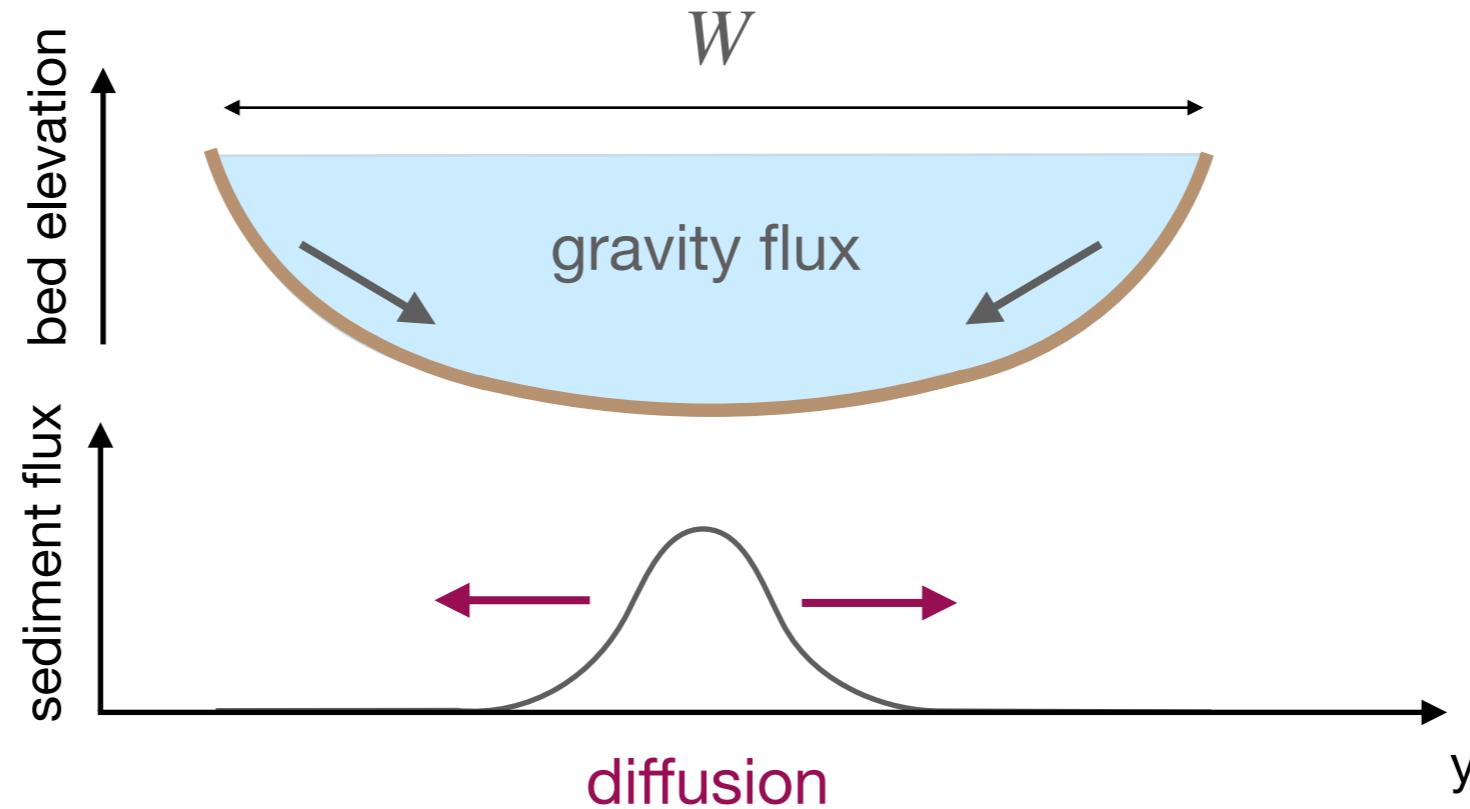


# Sediment flux

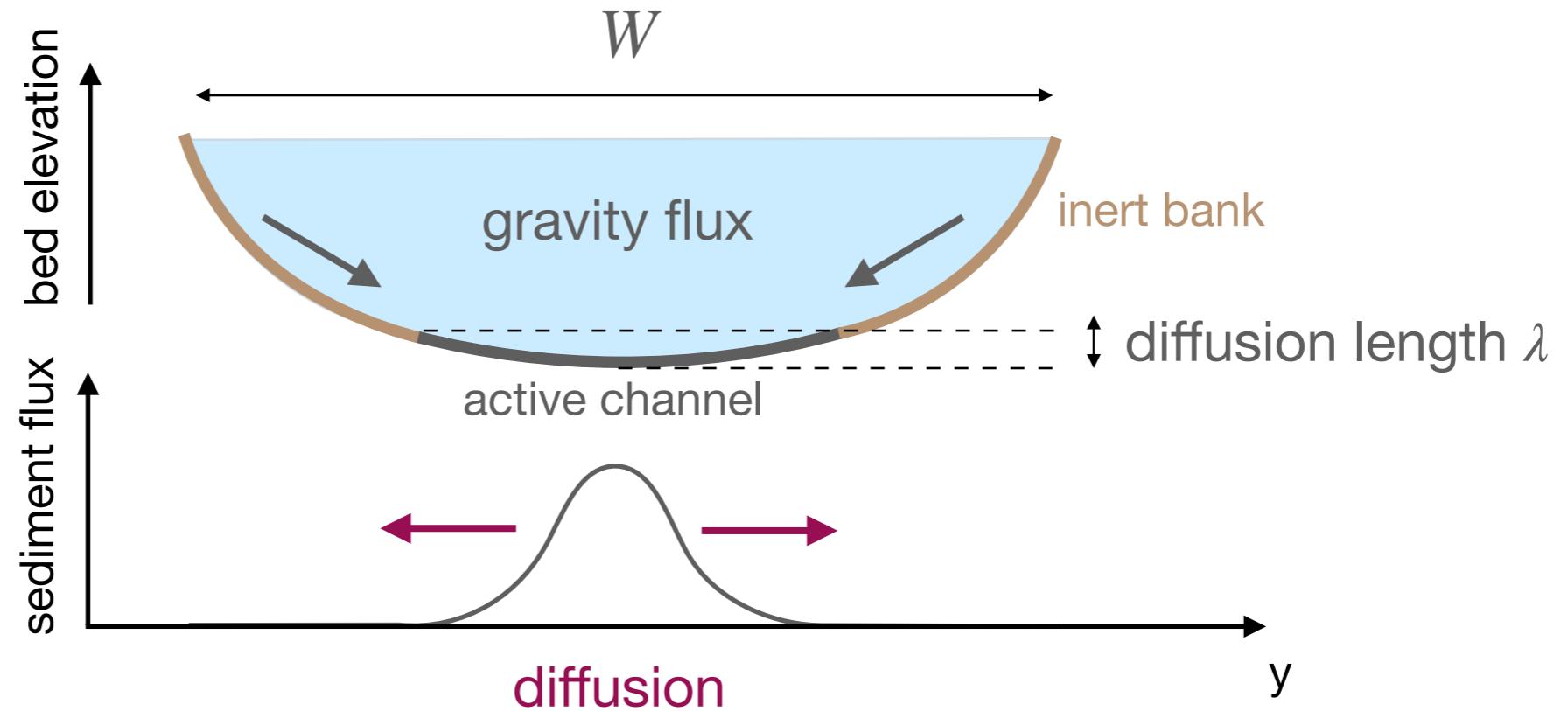


- The sediment flux saturates.
- The river must widen to accommodate an increase of sediment discharge

# Waving hand explanation

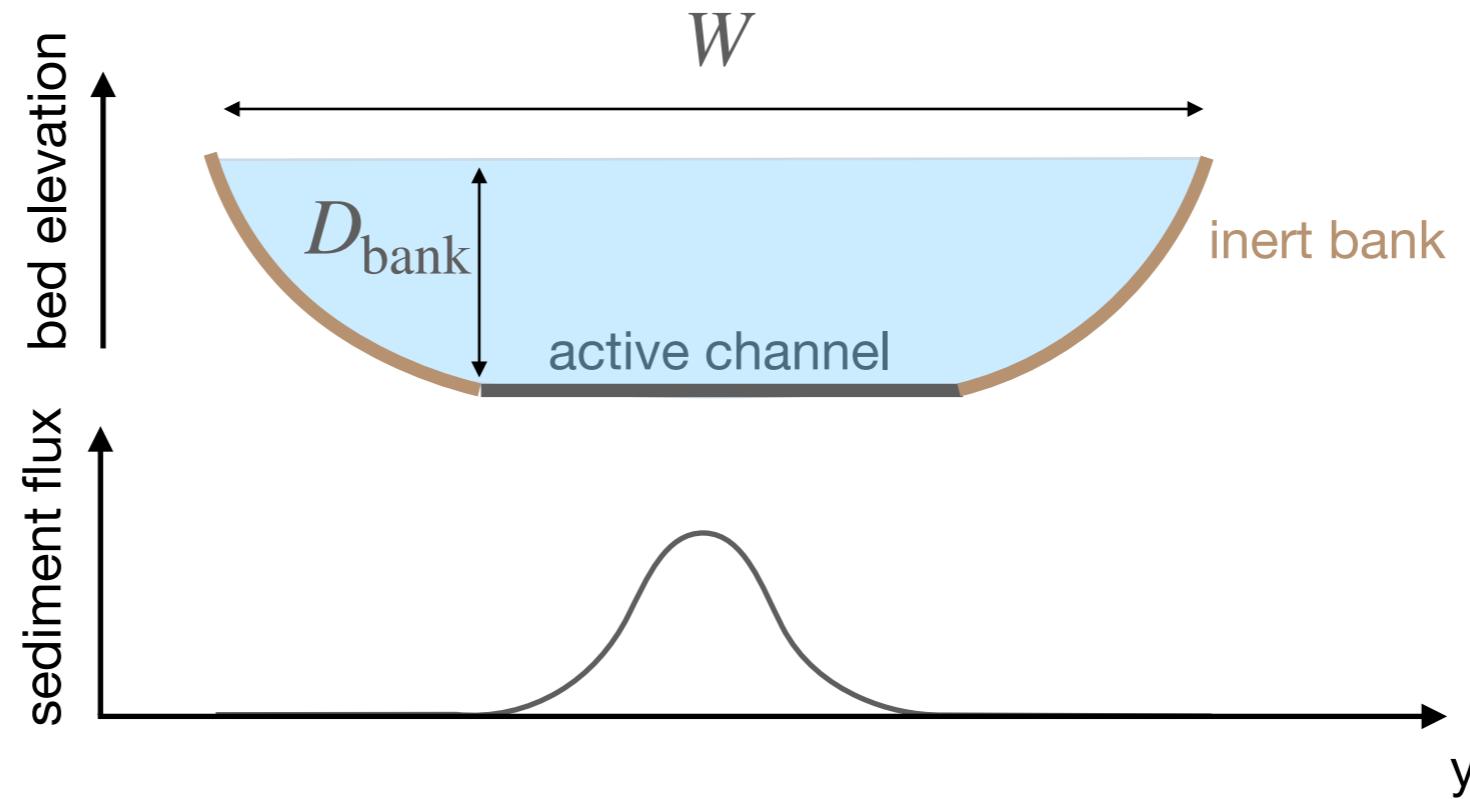


# Waving hand explanation



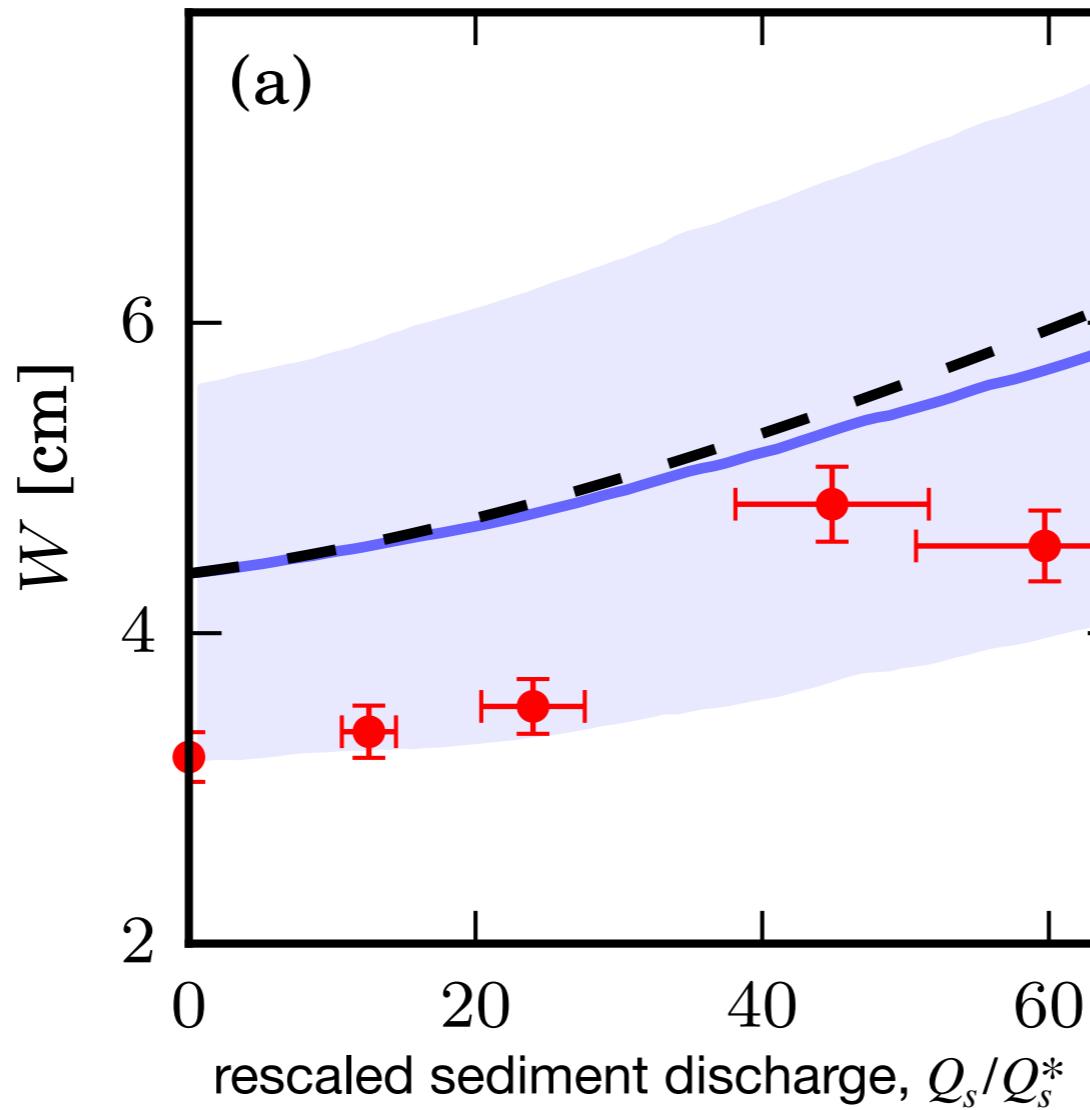
- Equilibrium  $\rightarrow$  flux  $\sim e^{D/\lambda} \rightarrow$  inert banks, at threshold
- Active channel  $\rightarrow$  lateral slope  $D' \sim \frac{\lambda}{W}$

# Waving hand explanation



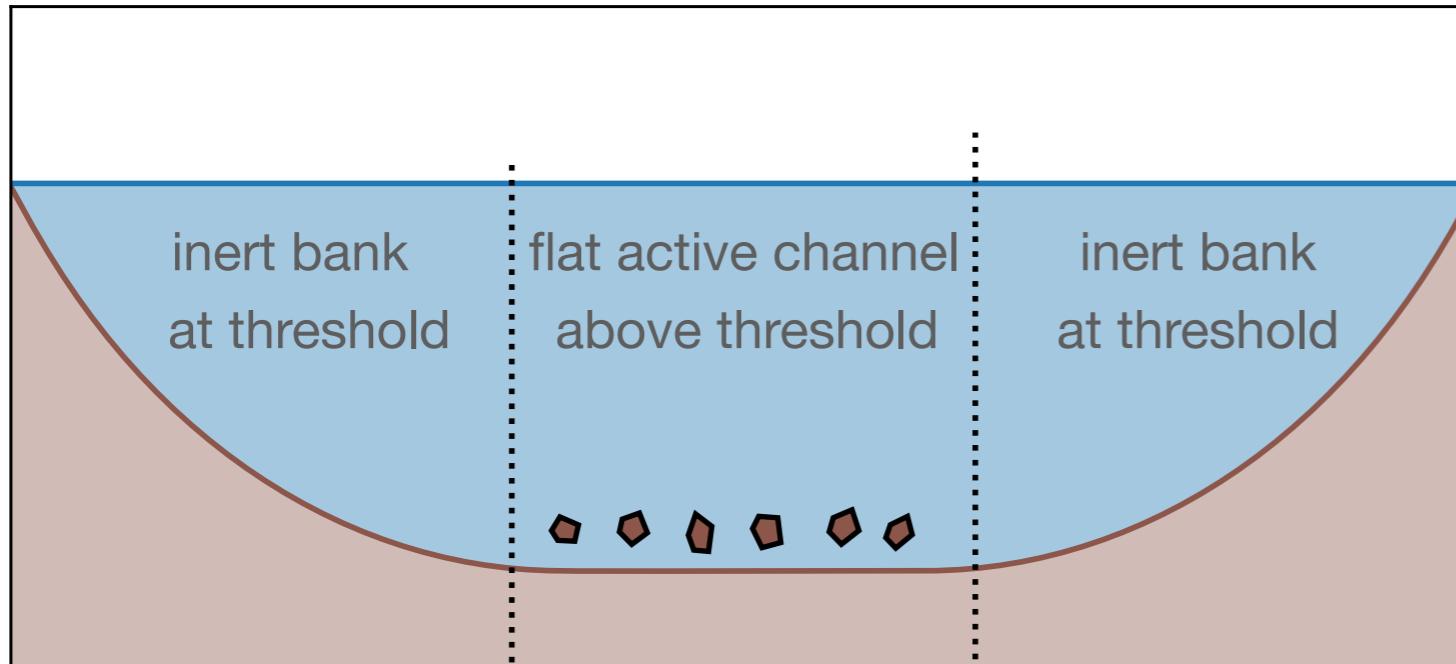
- Equilibrium  $\rightarrow$  flux  $\sim e^{D/\lambda} \rightarrow$  inert bank
  - Active channel  $\rightarrow$  lateral slope  $D' \sim \frac{\lambda}{W}$
  - $W \gg \lambda \rightarrow$  flat active channel
  - Force  $\rightarrow \frac{F_t}{F_n} \sim D_{\text{bank}} S / L$
  - inert river  $\rightarrow D_{\text{bank}} \sim \frac{d_s}{S}$
- $\left| \quad \rightarrow \quad \frac{F_t}{F_n} \sim 1.2 \mu_t \right.$

# Waving hand explanation



- As the river widens, the force exerted on the grains saturates to  $\sim 1.2 \mu_t$
- Rivers self organize near the threshold of sediment transport.
- Saturation of force  $\rightarrow$  saturation of the sediment flux
- The river widens to transport more sediment.

# Why is momentum diffusion so important ?



$$\frac{F_t}{F_n} = \mu_t$$

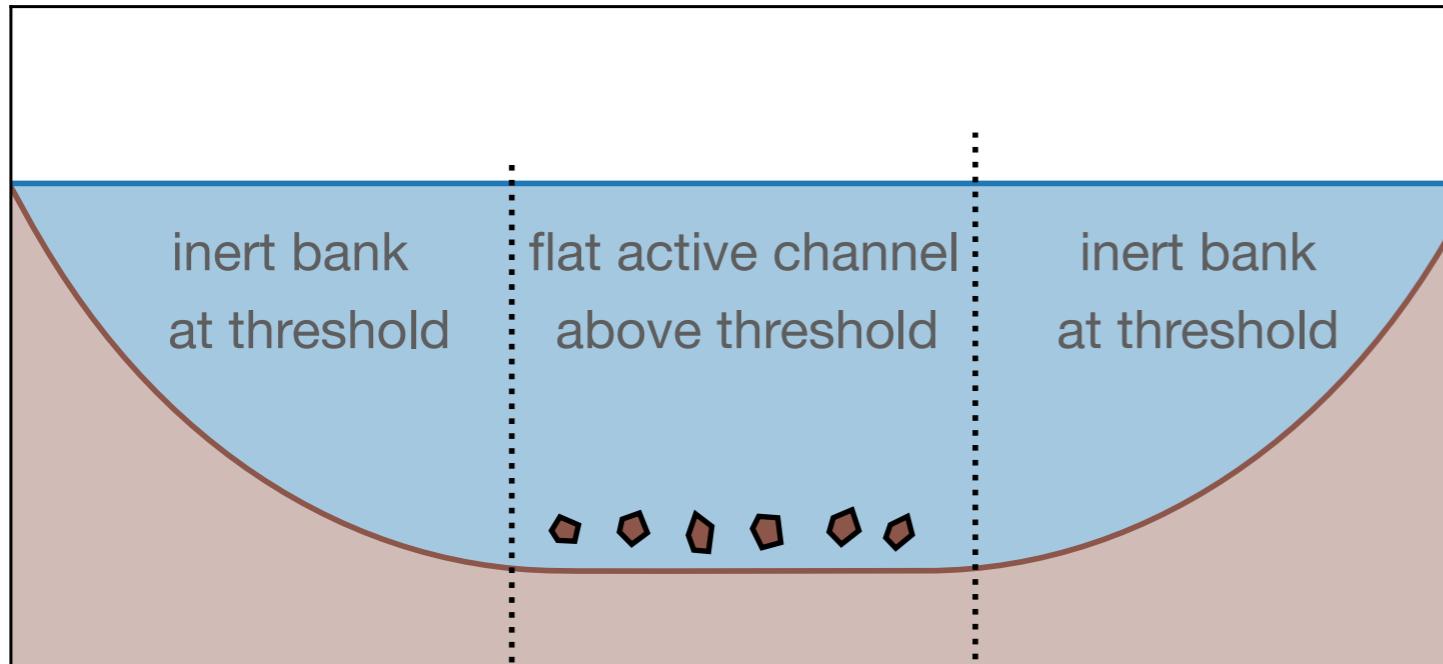
$$\frac{F_t}{F_n} > \mu_t$$

$$\frac{F_t}{F_n} = \mu_t$$

$$\frac{F_t}{F_n} = \sqrt{\frac{S^2}{L^2} \left( D + \frac{1}{3}(D^3)'' \right)^2 + D'^2}$$

shallow water      momentum diffusion      gravity

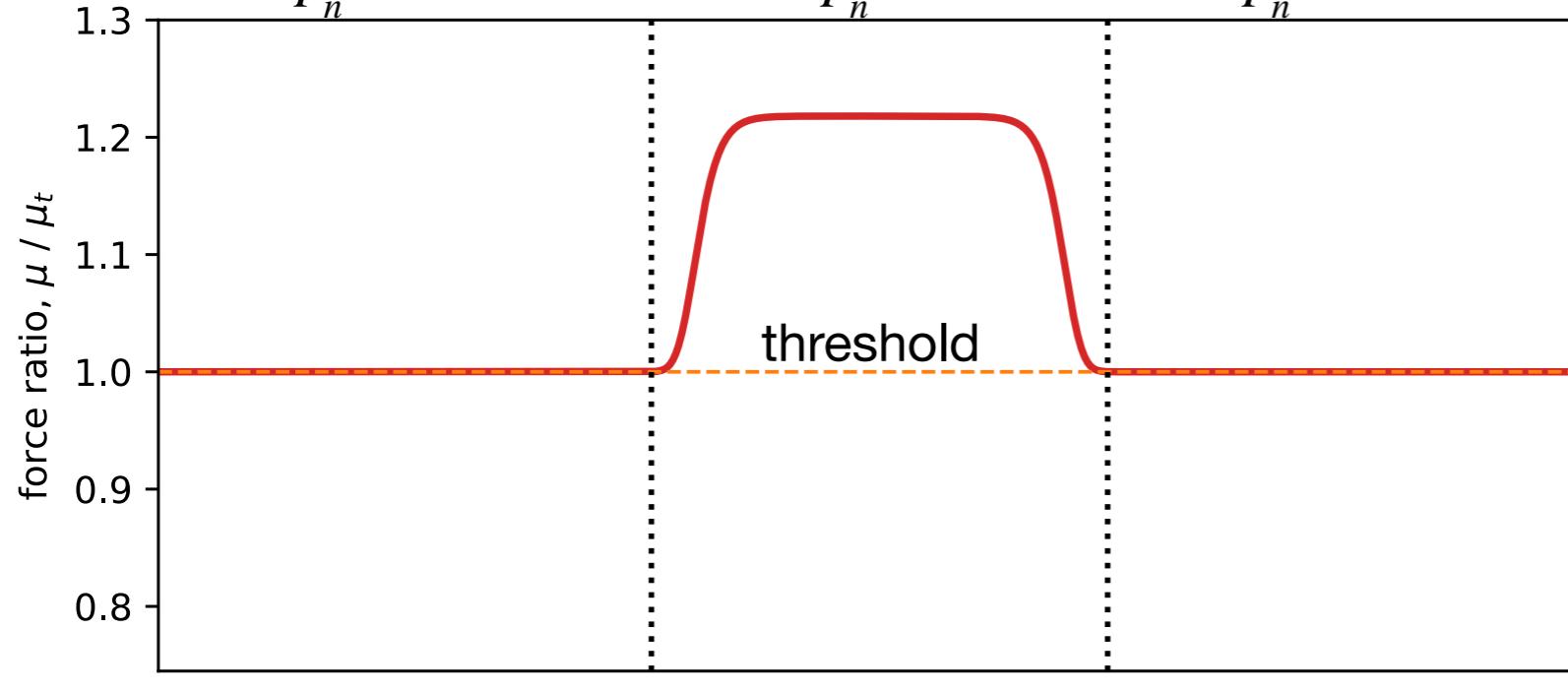
# Why is momentum diffusion so important ?



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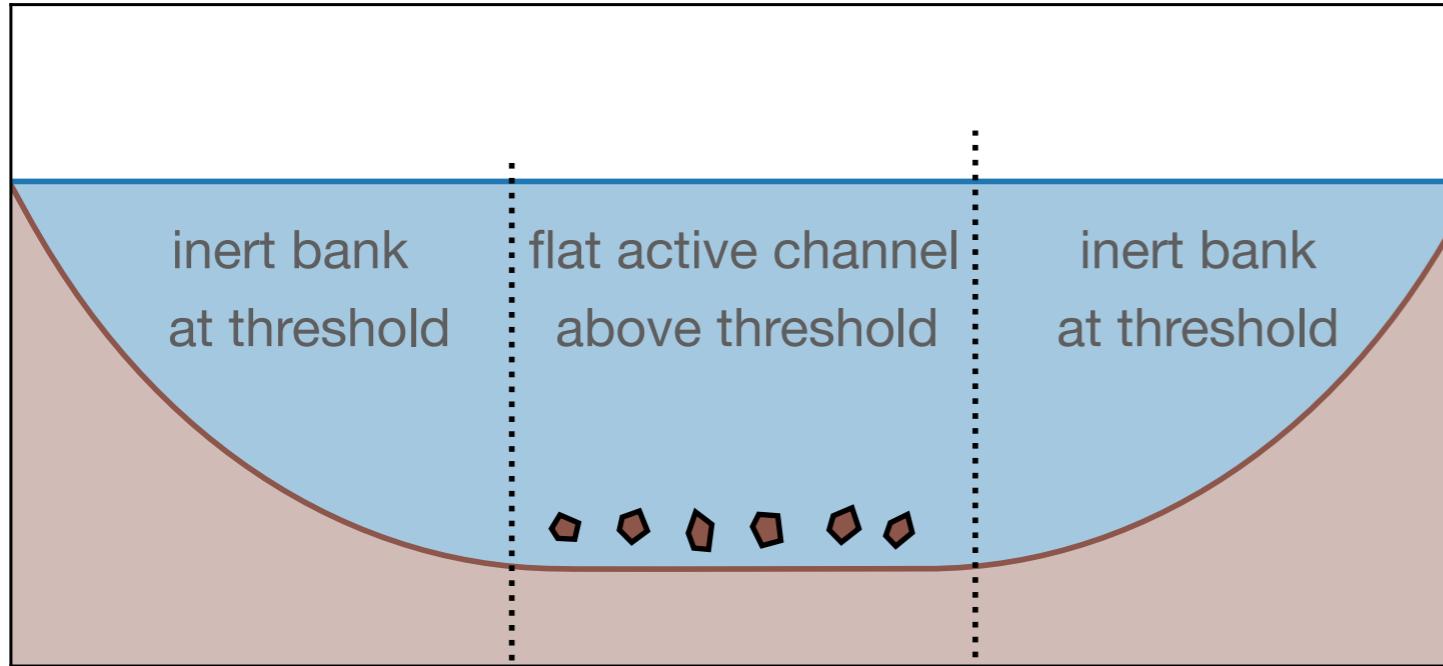


with diffusion of momentum

$$\frac{F_t}{F_n} = \sqrt{\frac{S^2}{L^2} \left( D + \frac{1}{3}(D^3)'' \right)^2 + D'^2}$$

shallow water      momentum diffusion      gravity

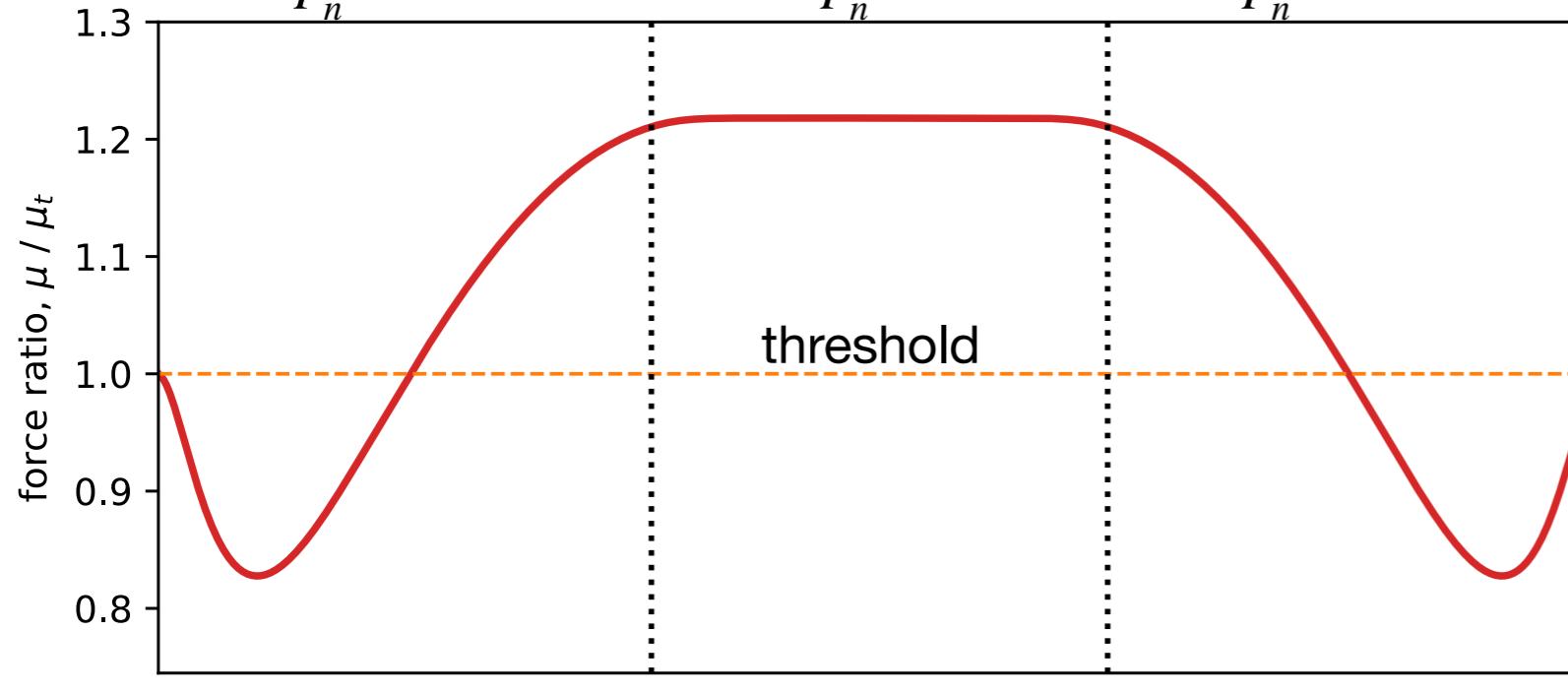
# Why is momentum diffusion so important ?



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without diffusion of momentum

$$\frac{F_t}{F_n} = \sqrt{\frac{S^2}{L^2} \left( D + \frac{1}{3}(D^3)'' \right)^2 + D'^2}$$

shallow water      momentum diffusion      gravity

# Why is momentum diffusion so important ?

## Self-formed straight rivers with equilibrium banks and mobile bed. Part 2. The gravel river

By GARY PARKER

Department of Civil Engineering, University of Alberta,  
Edmonton, Canada T6G 2G7

(Received 26 May 1977 and in revised form 3 March 1978)



G. Parker [1978]

### 11. Conclusion

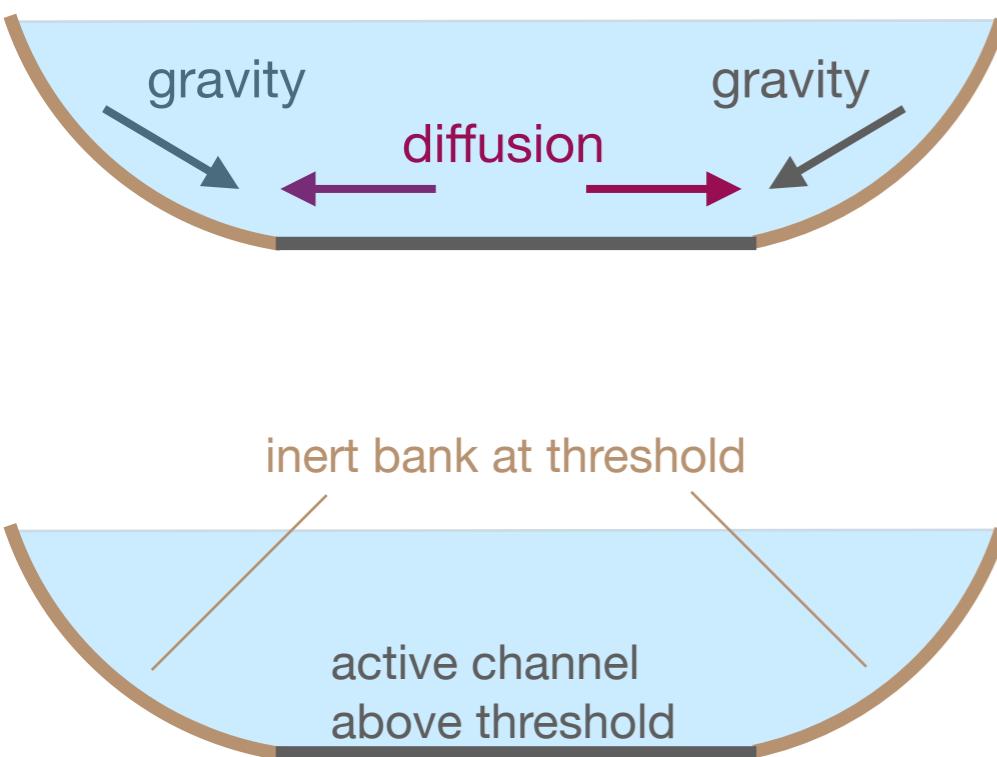
The concept of lateral transfer of downstream momentum by turbulent diffusion embodied in the work of Lundgren & Jonsson (1964) has been used together with singular perturbation techniques to explain the coexistence of stable banks and mobile beds in straight reaches of coarse gravel rivers. The analysis has been used to obtain rational regime relations for such reaches.

Points which deserve further attention are the use of more accurate closure assumptions, a treatment of secondary currents in straight channels, and the inclusion of sediment gradation effects.

# Take home messages



- Laboratory rivers construct their bed near the threshold of entrainment
- Result of the combination of 2 diffusion processes.



- Diffusion of bedload particles
- Diffusion of momentum

# Open questions

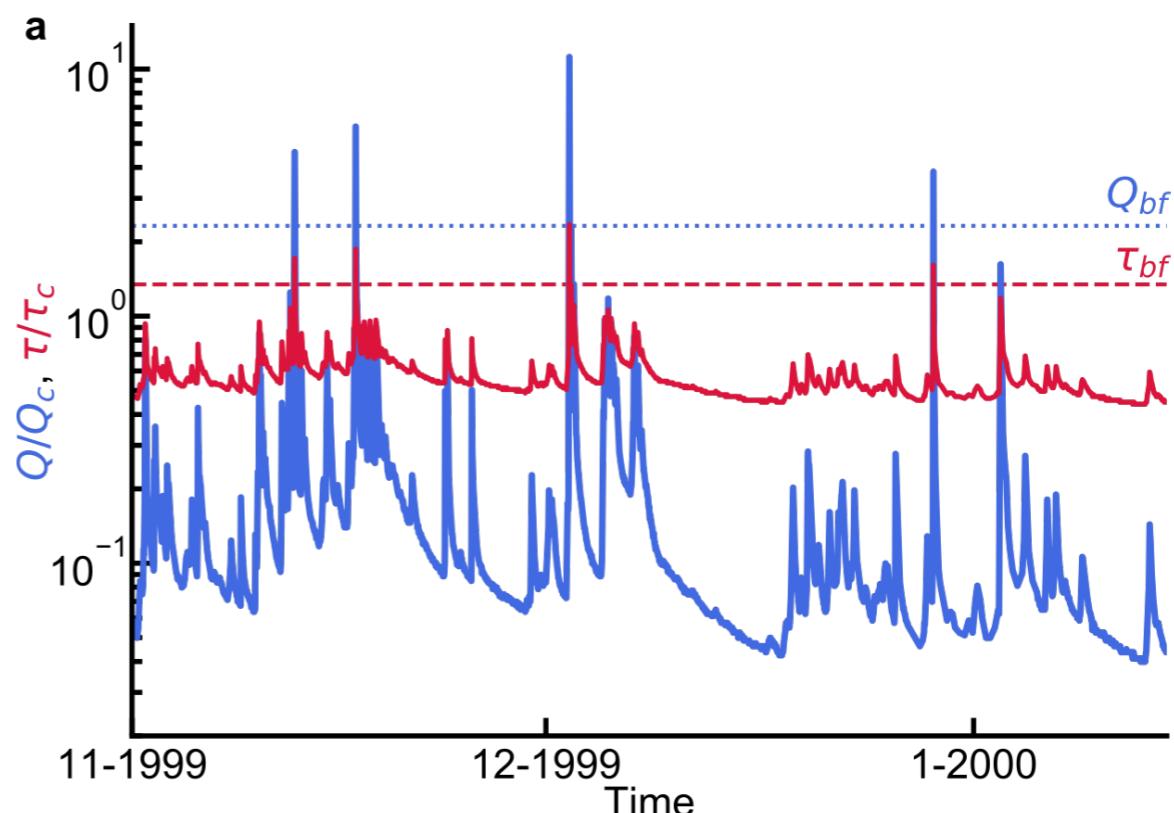


Tian-Shan, China

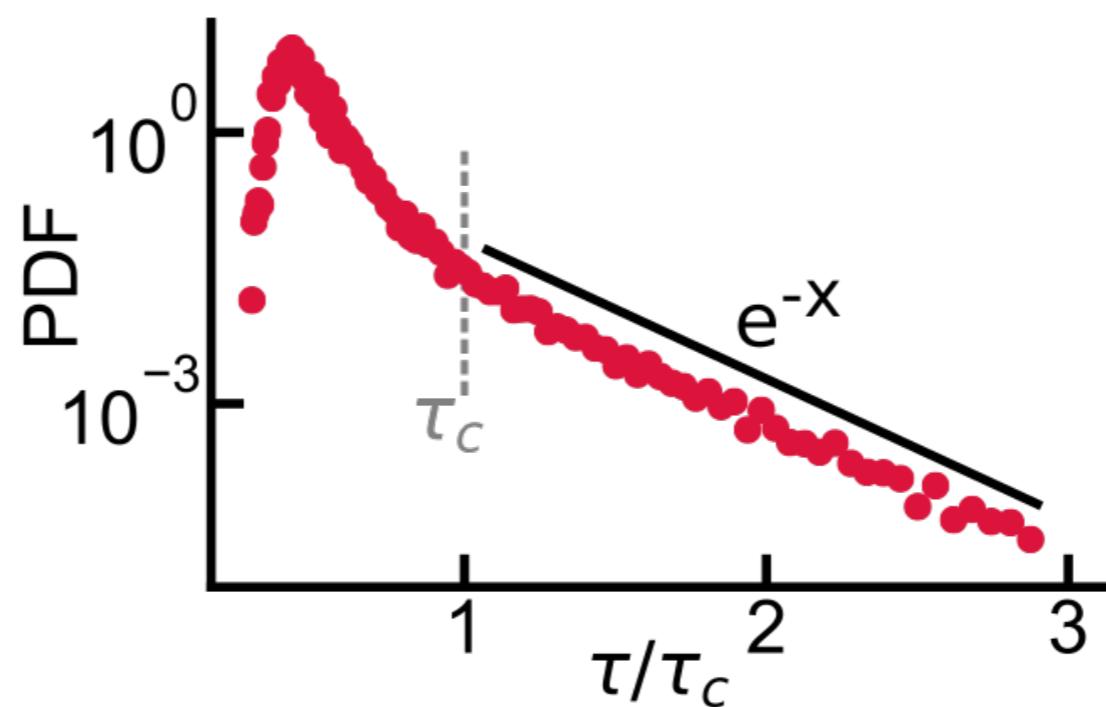
# Do alluvial rivers self-organize their bed near the threshold of motion ?



Mameyes river, Puerto-Rico



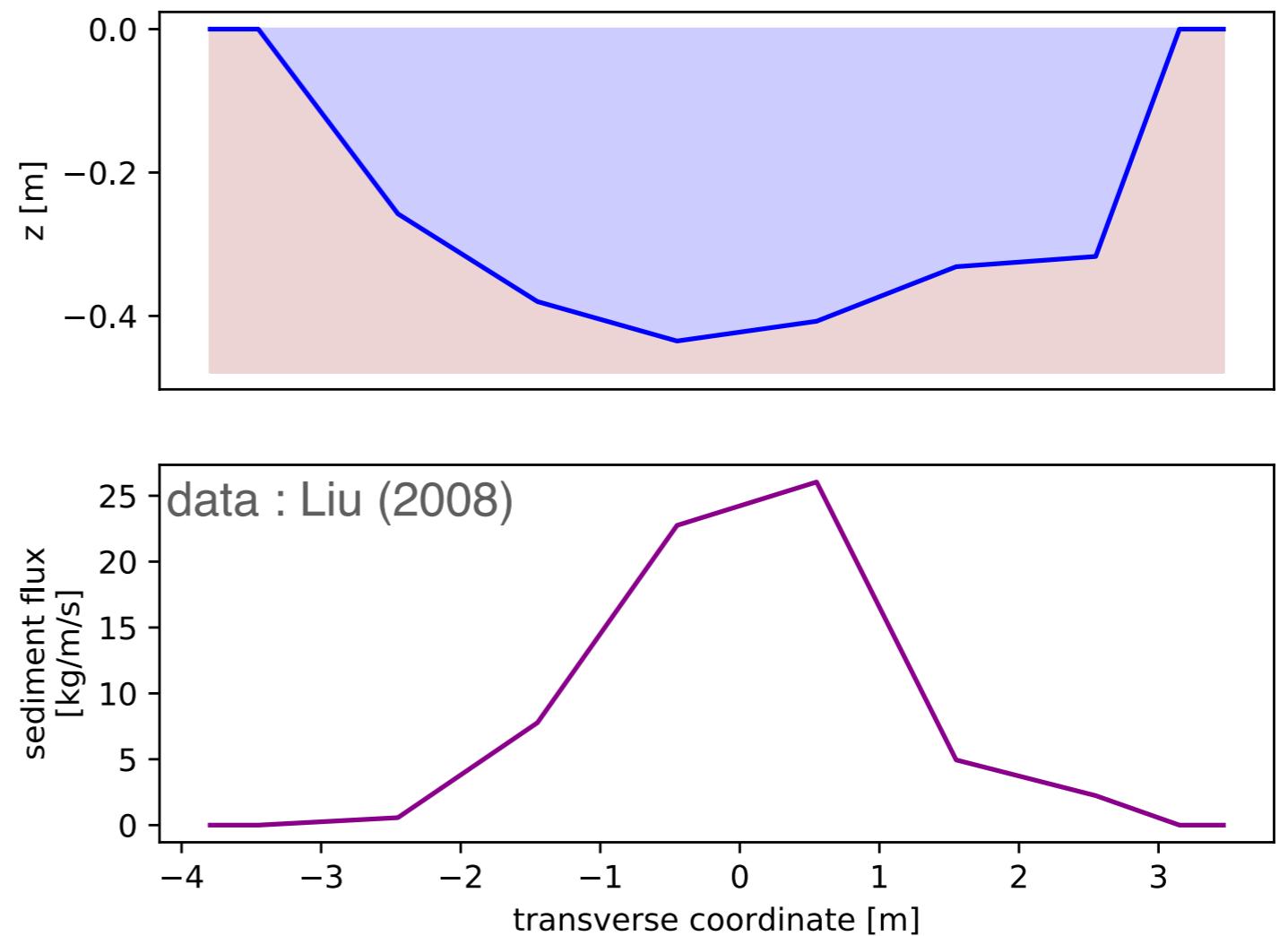
Phillips et al. [2016, 2022]



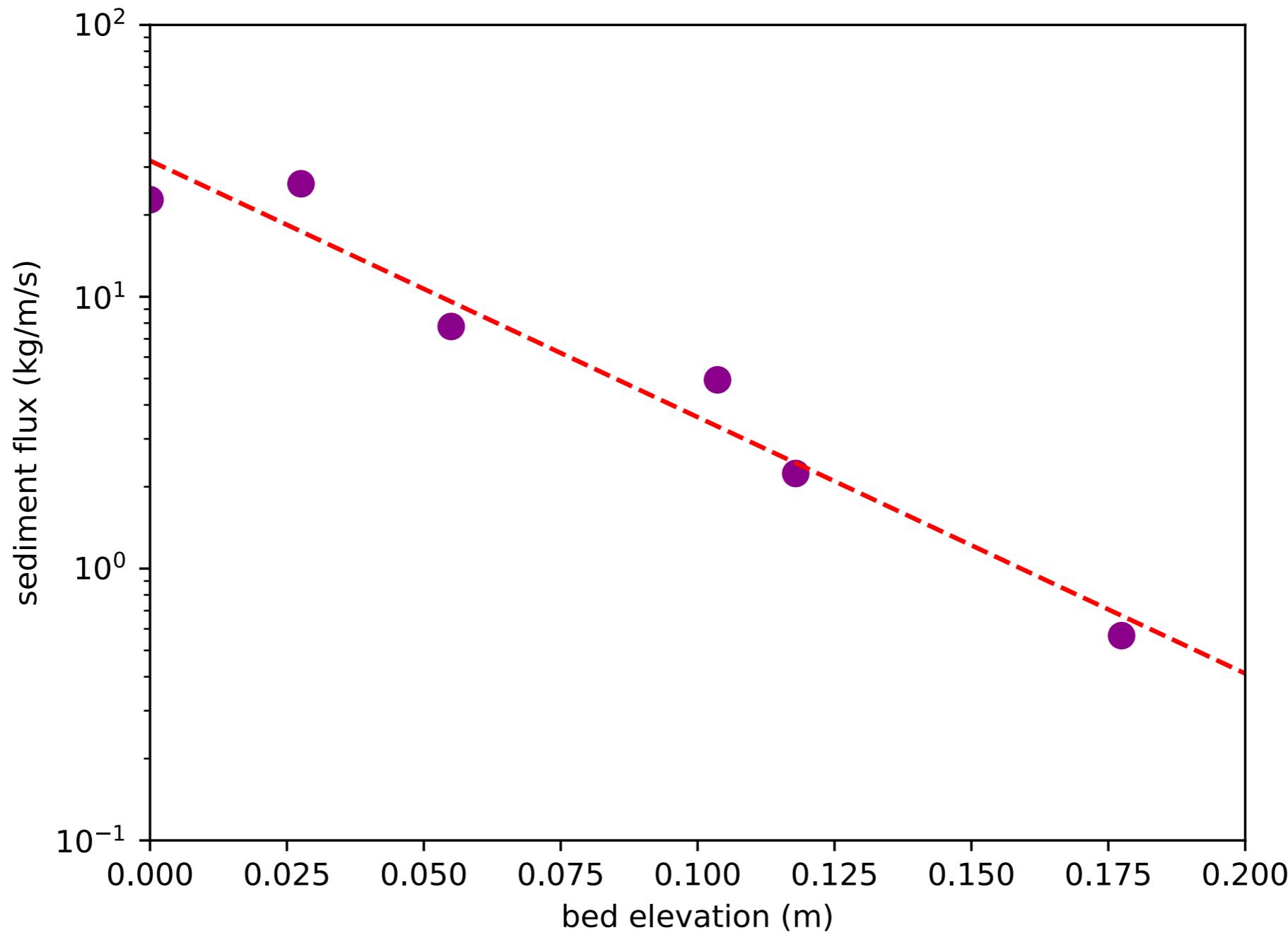
# Do alluvial rivers obey the Boltzmann-like equilibrium condition ?



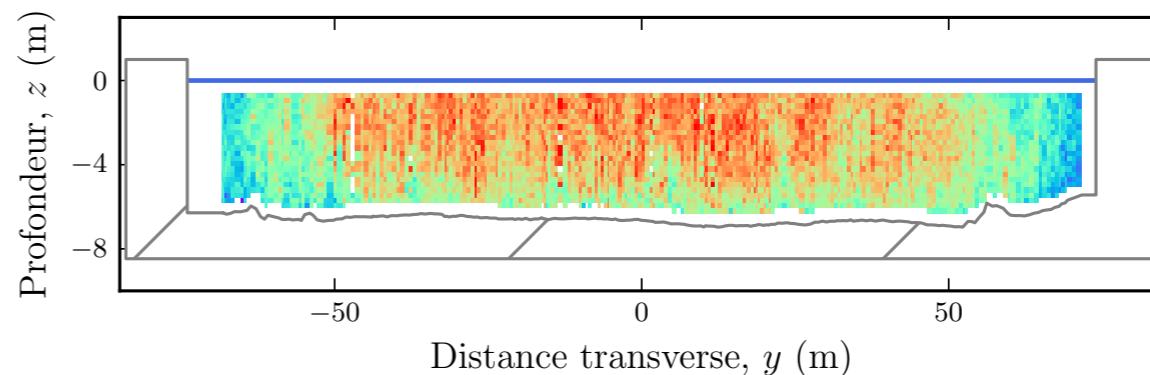
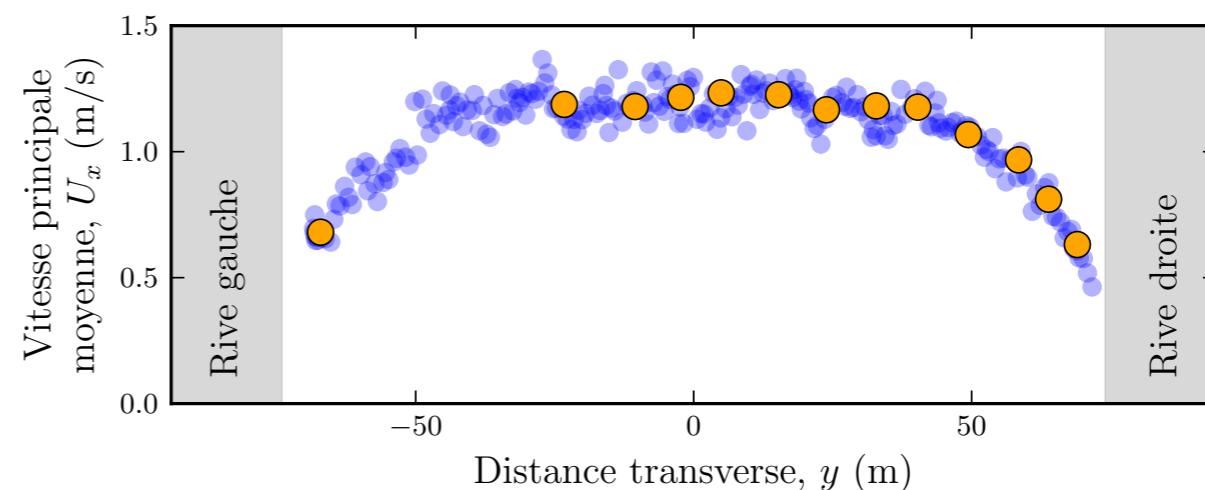
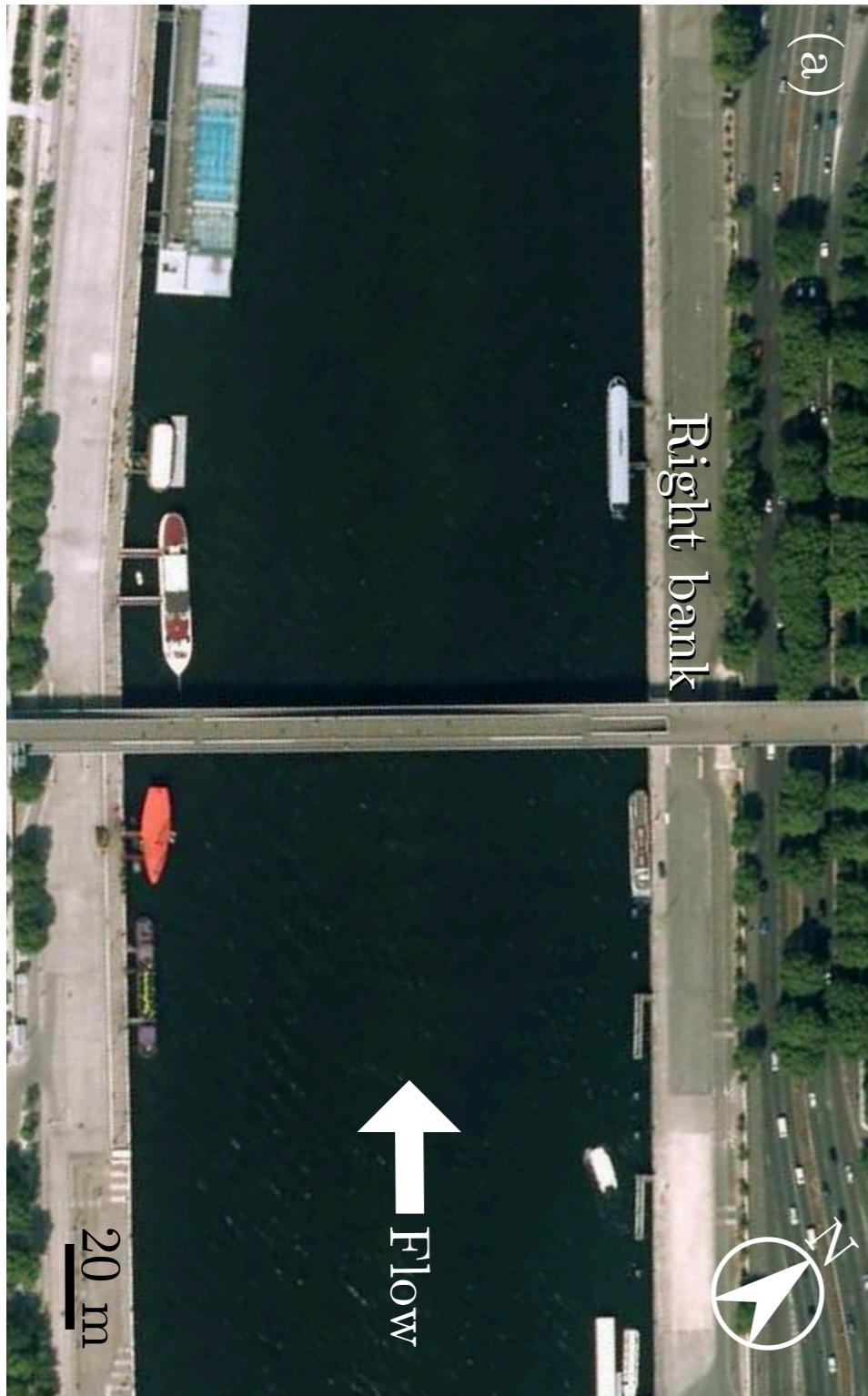
Urumqi He,  
Tian-Shan, China



# Do alluvial rivers obey the Boltzmann-like equilibrium condition ?



# Do alluvial rivers diffuse momentum in the cross-stream direction ?



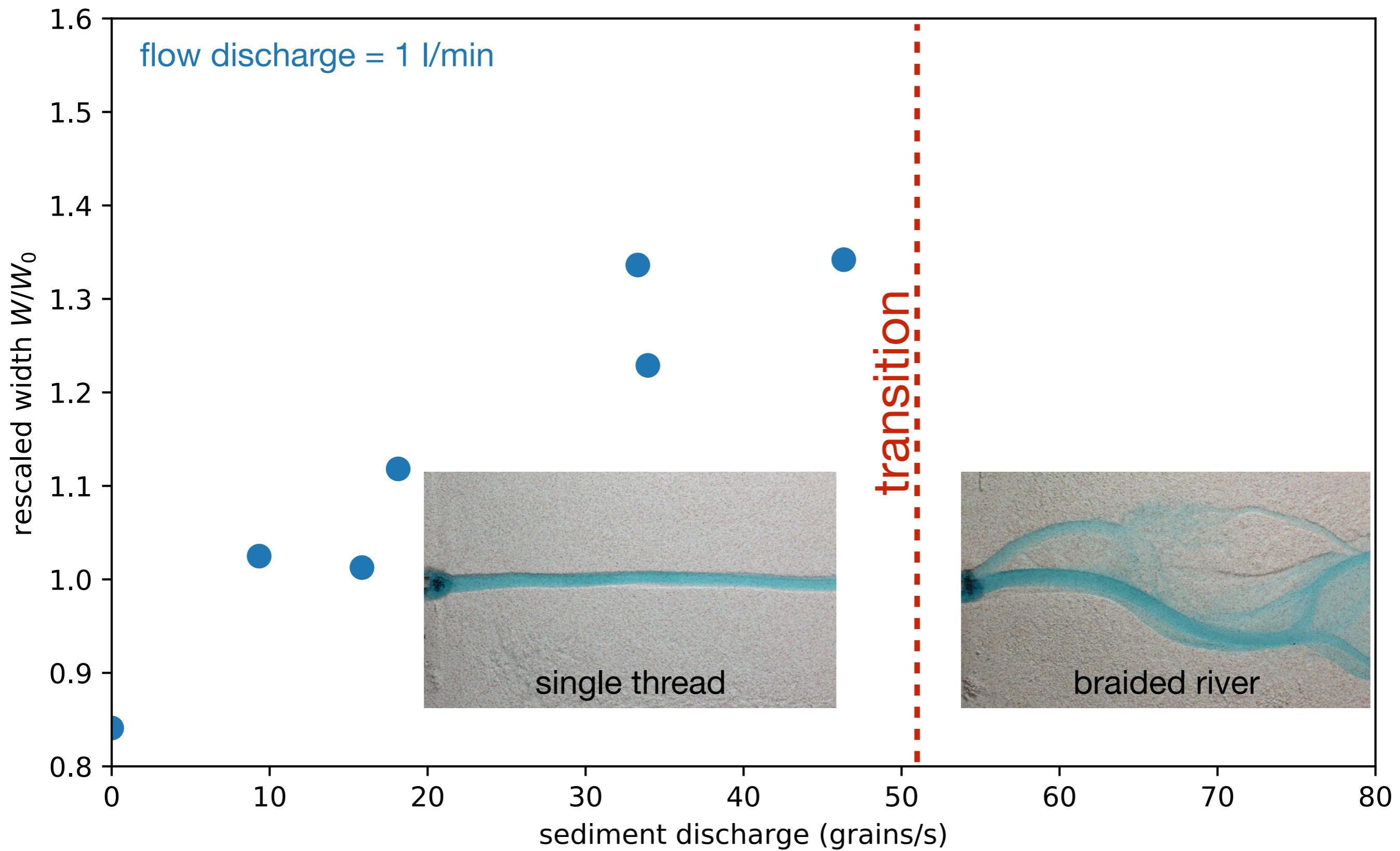
Chauvet et al. [2014]

- flat river bed
- curved depth-averaged velocity profile

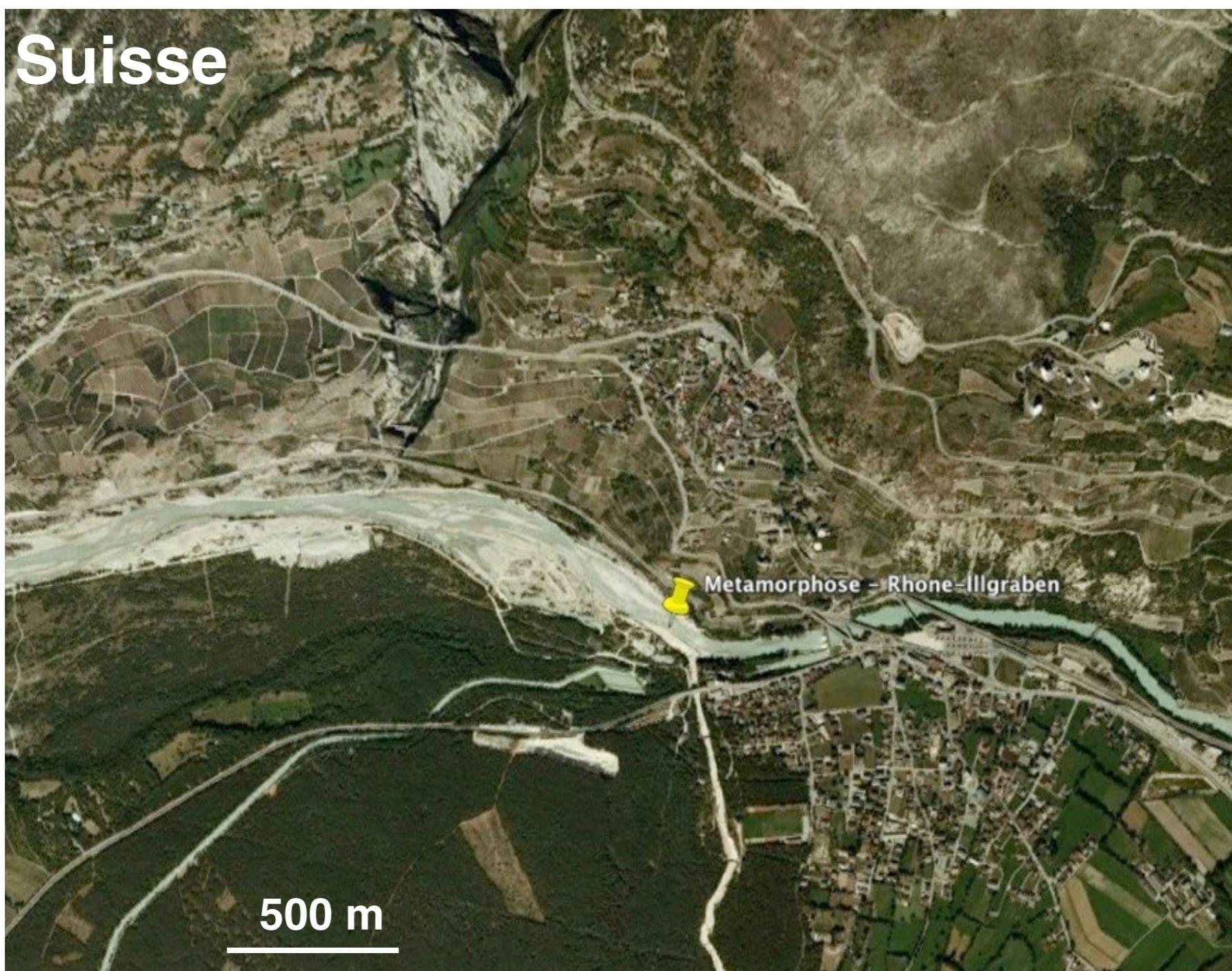
→ cross-stream diffusion of momentum

(Popović et al., sub)

# Stability of alluvial rivers ?



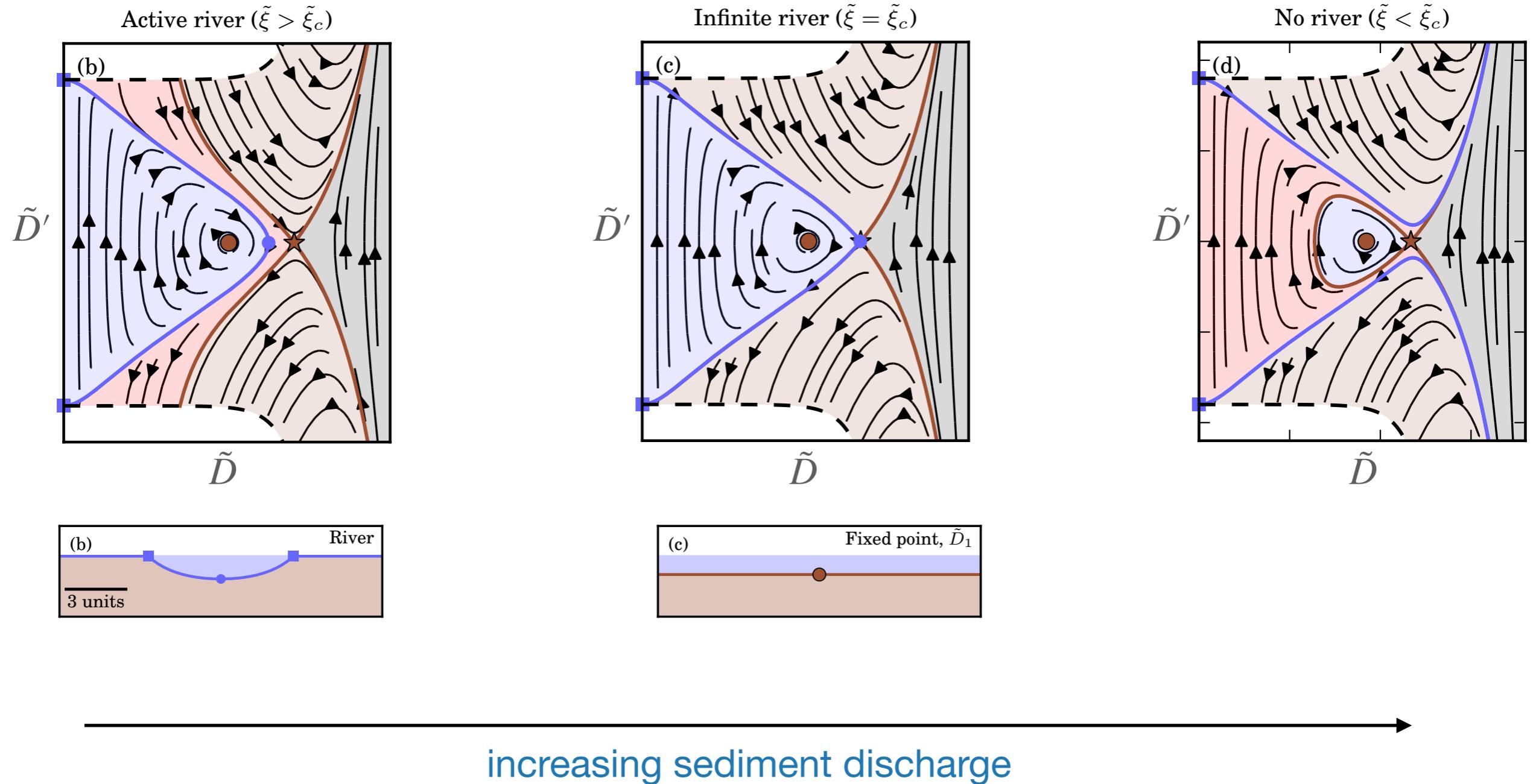
# Stability of alluvial rivers ?



# Stability of alluvial rivers ?



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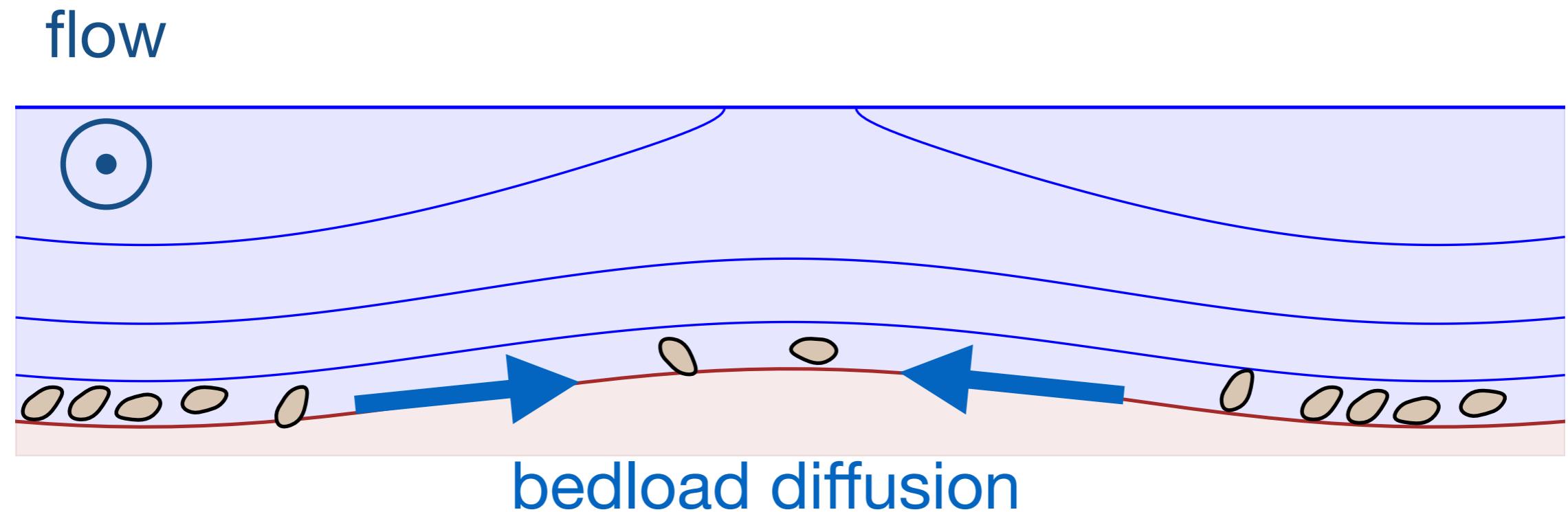
# Diffusion as an instability mechanism

flow



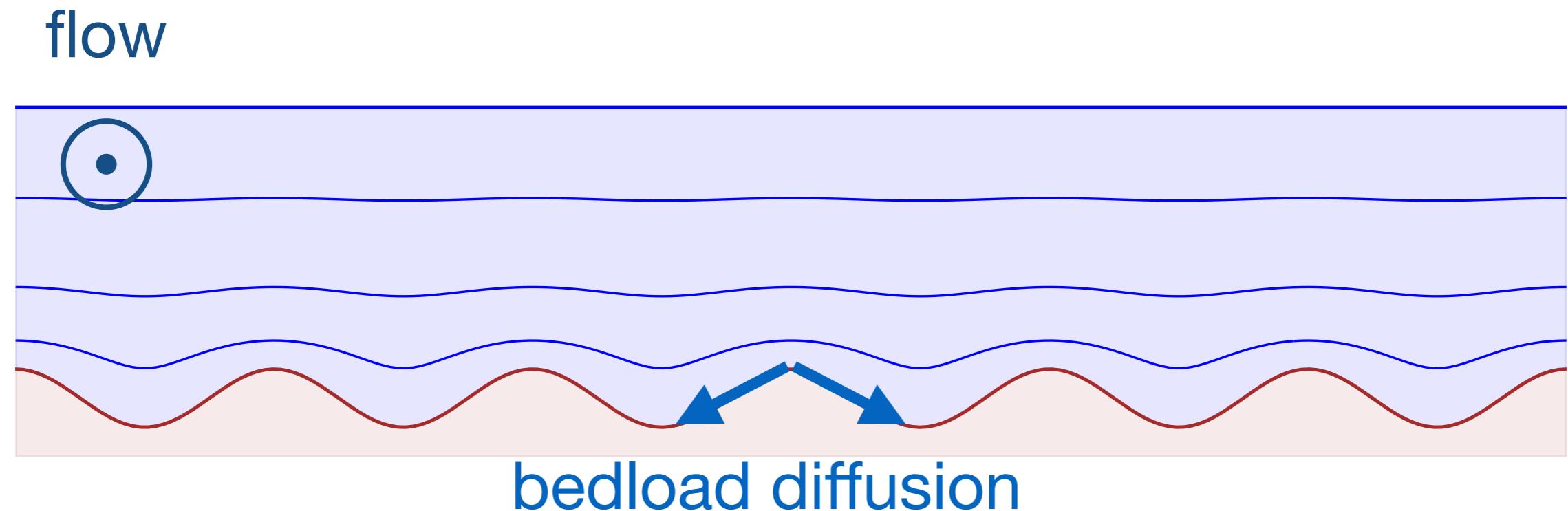
flat sediment bed  
homogeneous stress

# Diffusion as an instability mechanism



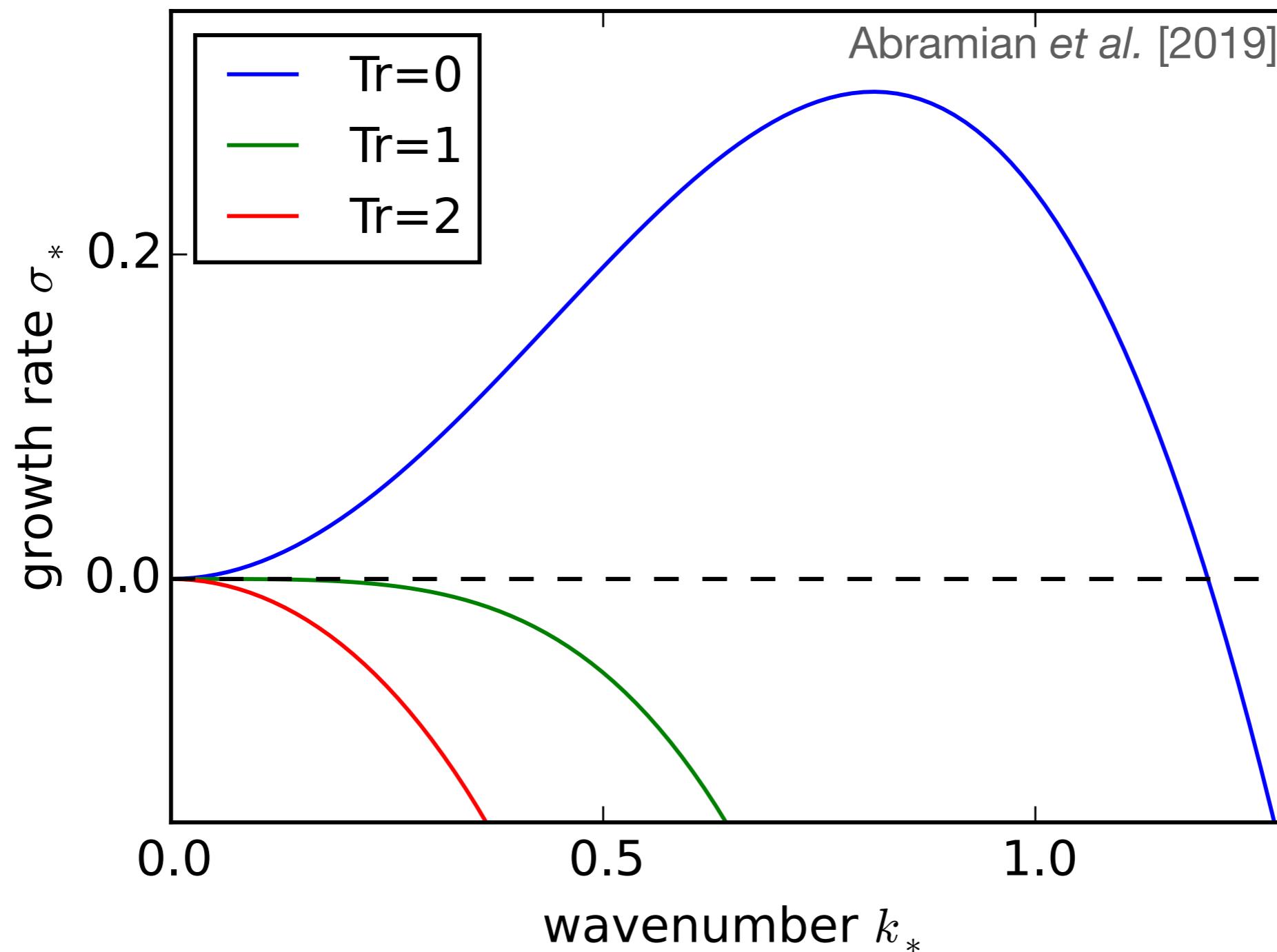
long wavelength perturbation  
weaker stress at the crest

# Diffusion as an instability mechanism



short wavelength perturbation  
stronger stress at the crest

# Dispersion relation



$$Tr = \frac{\alpha d_s^2 L n_0}{b \ell_d}$$