Radiative Transfer in Fluids: From Mathematical Analysis to Numerical Simulations

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What is Radiative Transfer?

Radiative Transfer studies the interaction of electromagnetic radiation viewed as a gas of photons with a background medium (fluid or plasma, e.g. stellar or planetary atmospheres)

How the various frequencies in the radiation coming from the Sun interact with the atmosphere of the Earth is important in the understanding of

- the blue color of a cloudless sky (J.W. Strutt Rayleigh 1871)
- the greenhouse effect (J. Fourier 1824)

The Radiative Intensity

(Specific) radiative intensity

$$I_\nu(t,\vec{x},\vec{\omega}) := ch\nu \ \underbrace{f(t,\vec{x},\vec{\omega},\nu)}_{\text{photon \# density}} \ , \quad \vec{x} \in \mathbf{R}^3 \, , \ |\vec{\omega}| = 1$$

$$J_\nu(t,\vec{x}) := \frac{1}{4\pi} \int_{\mathbf{S}^2} I_\nu(t,\vec{x},\vec{\omega}) d\vec{\omega} \qquad \text{mean intensity}$$

Example: Planck's function for a black body at temperature T

$$B_\nu(T) := \frac{2h\nu^3}{c^2(e^{h\nu/kT} - 1)}$$

Stefan-Boltzmann law

$$\pi \int_0^\infty B_\nu(T) d\nu = \Sigma_R T^4 \, , \qquad \Sigma_R := \frac{2\pi^5 k^4}{15 c^2 h^3}$$

Planck's Function

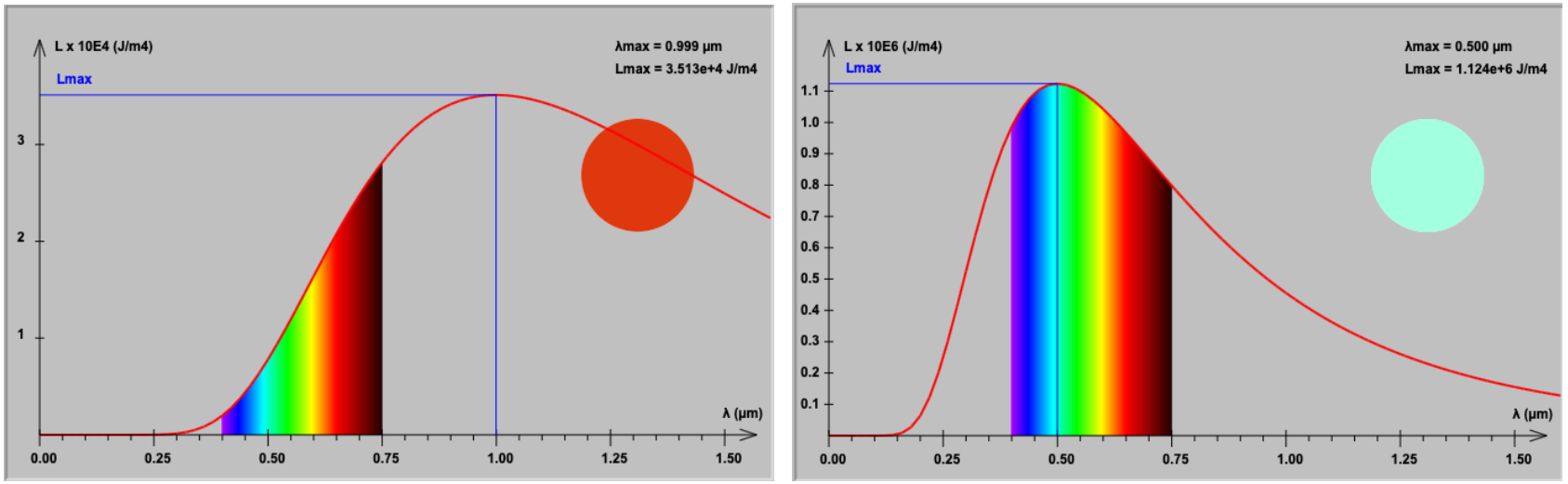


Figure: Planck's function in terms of the wavelength at temperatures $2900K$ (left) and $5800K$ (right). For a black body at $288K$ (Earth's mean temperature), most of the emitted radiation is infrared.

Source: ressources.univ-lemans.fr

Radiation/Matter Interaction

There are (at least) 3 different processes

(a) Scattering Photons suddenly change direction, and sometimes even frequency (for instance, Compton scattering, not considereed here), due to collisions with electrons in the material;

(b) Absorption A photon is absorbed by one electron in the material, which is excited to a higher energy level;

(c) Emission Conversely, an electron in the material can fall back to a lower energy level and emit one photon

Scattering

We shall mainly consider the two following scattering terms

$$\rho\bar{\kappa}_\nu^s \int_{\mathbf{S}^2} (I_\nu(t,\vec{x},\vec{\omega}') - I_\nu(t,\vec{x},\vec{\omega}))\frac{d\vec{\omega}'}{4\pi} = \rho\bar{\kappa}_\nu^s (J_\nu(t,\vec{x}) - I_\nu(t,\vec{x},\vec{\omega}))$$

(isotropic)

$$\rho\bar{\kappa}_\nu^s \int_{\mathbf{S}^2} \frac{3}{16\pi}(1+(\vec{\omega}\cdot\vec{\omega}')^2)(I_\nu(t,\vec{x},\vec{\omega}') - I_\nu(t,\vec{x},\vec{\omega}))d\vec{\omega}'$$

(Rayleigh phase function)

Rayleigh scattering (by a dielectric sphere of polarizability α)

$$\bar{\kappa}_\nu^{s,Rayleigh} = \frac{128\pi^5\alpha^2\nu^4}{3c^2}$$

Absorption/Emission

Assume local thermodynamic equilibrium (LTE): at each point x, and for all time t, there exists a(n electronic) temperature $T(t,x)$

Absorption term

$$-\rho\sigma_\nu(T(t,\vec{x}))I_\nu(t,\vec{x},\vec{\omega})$$

Emission term

$$\rho\sigma_\nu(T(t,\vec{x}))B_\nu(T(t,\vec{x}))$$

where B_ν is the Planck function and $\sigma_\nu(T)$ is the opacity of matter at temperature T to radiation of frequency ν

Henceforth set

$$\bar{\kappa}_\nu := \underbrace{\sigma_\nu + \bar{\kappa}^s_\nu}_{\text{absorption rate}} \quad , \qquad a_\nu := \underbrace{\frac{\bar{\kappa}^s_\nu}{\sigma_\nu + \bar{\kappa}^s_\nu}}_{\text{scattering albedo}} \in [0,1]$$

Opacity of a Boron Plasma at $T = 40eV$

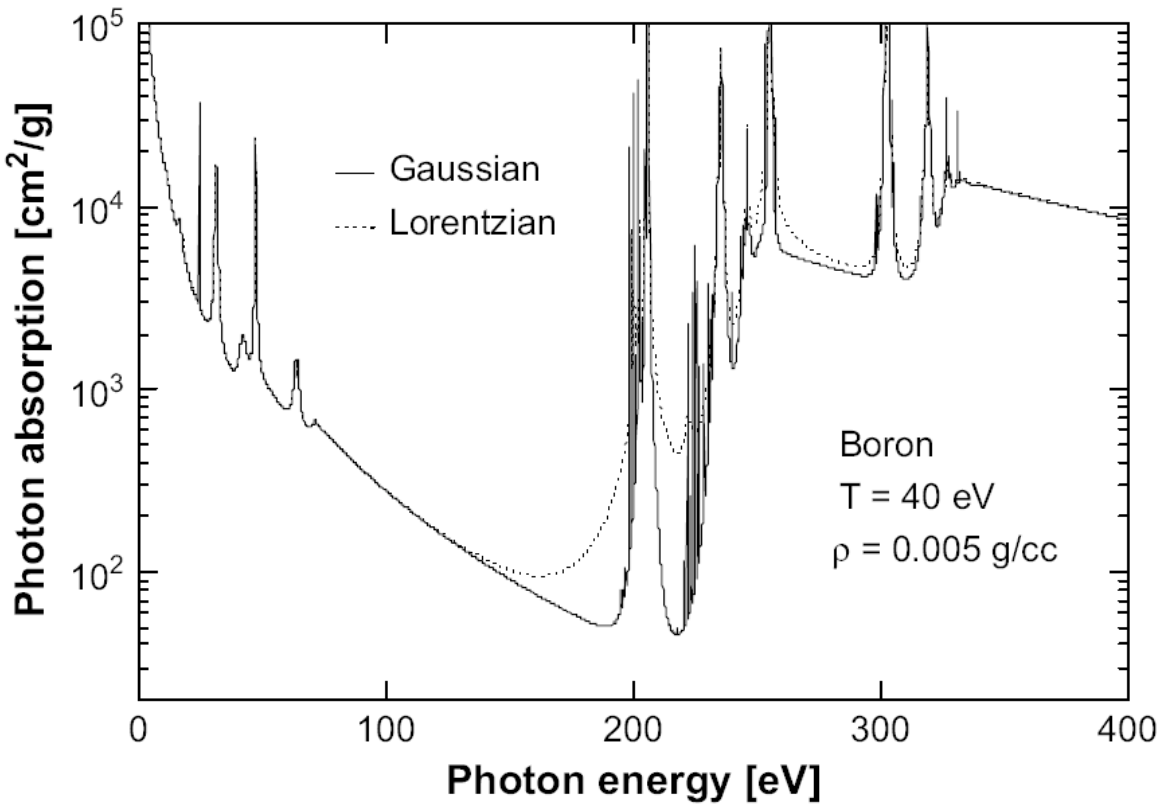


Figure: Opacity of boron in terms of frequency (UV-X rays)
Source: Iglesias CA, Sonnad V, Wilson BG, Castor JI. High Energy Density Physics 2009; 5:97-104.

The (LTE) Radiative Transfer Equation

Kinetic equation for the radiative intensity

$$\left(\frac{1}{c}\partial_t + \vec{\omega}\cdot\nabla_{\vec{x}}\right)I_\nu + \underbrace{\rho\bar{\kappa}_\nu I_\nu}_{\text{absorption}} = \underbrace{\rho\bar{\kappa}_\nu(1-a_\nu)B_\nu(T)}_{\text{LTE reemission}} + \underbrace{\rho\bar{\kappa}_\nu a_\nu J_\nu}_{\text{scattering}}$$

mean radiative intensity $\quad J_\nu(t,\vec{x}) := \frac{1}{4\pi}\int_{\mathbf{S}^2} I_\nu(t,\vec{x},\vec{\omega})d\vec{\omega}$

where

- $\rho =$ background fluid density
- $\bar{\kappa}_\nu =$ total absorption rate
- $[0,1] \ni a_\nu =$ scattering albedo

Local Thermodynamic Equilibrium (LTE) relaxation to the Planck function, analogous to a BGK model in the kinetic theory of gases

Transmittance of Earth's Atmosphere

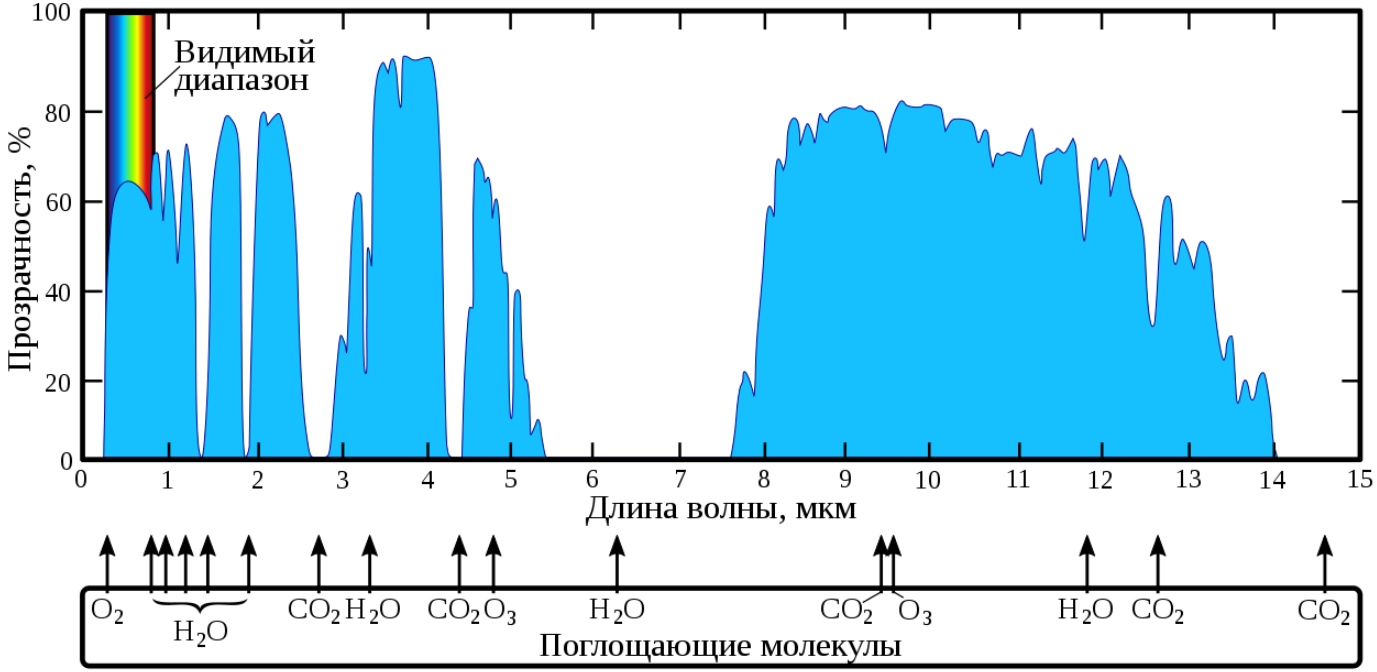


Figure: Transmittance of Earth's atmosphere in terms of the wavelength.
Source: commons.wikimedia.org/wiki/File:Atmosfaerisk spredning-ru.svg

Coupling Radiation to the Fluid Energy Balance

Energy balance in an incompressible fluid with radiation

$$\partial_t \underbrace{\left(\rho(\tfrac{1}{2}|\vec{u}|^2 + c_V T) + \tfrac{4\pi}{c}\int_0^\infty J_\nu d\nu\right)}_{\text{kinetic+internal+radiative energy}} + \nabla_{\vec{x}} \cdot ((p - \rho\vec{g}\cdot\vec{x})\vec{u})$$

$$+\nabla_{\vec{x}} \cdot \underbrace{\left(\rho\vec{u}(\tfrac{1}{2}|\vec{u}|^2 + c_V T) + \int_0^\infty \int_{\mathbf{S}^2} \vec{\omega} I_\nu d\vec{\omega} d\nu\right)}_{\text{kinetic+internal+radiative energy flux}}$$

$$= \nabla_{\vec{x}}(\rho c_P \kappa_T \nabla_{\vec{x}} T) + \nabla_{\vec{x}}(\mu_F(\nabla_{\vec{x}} u + (\nabla_{\vec{x}} u)^T)\cdot\vec{u})$$

where

- c_V, c_P = specific heats
- κ_T = heat diffusivity
- μ_F = viscosity
- $\vec{g}$ = gravitational field

Momentum balance in an incompressible fluid with radiation

$$\partial_t \underbrace{\left(\rho\vec{u} + \frac{1}{c^2}\int_0^\infty\int_{\mathbf{S}^2} \vec{\omega} I_\nu d\vec{\omega} d\nu\right)}_{\text{fluid momentum+radiation flux density}} + \nabla_{\vec{x}} \cdot (\rho\vec{u} \otimes \vec{u})$$

$$+\nabla_{\vec{x}} \cdot \underbrace{\left(p + \frac{1}{c}\int_0^\infty\int_{\mathbf{S}^2} \vec{\omega} \otimes \vec{\omega} I_\nu d\vec{\omega} d\nu\right)}_{\text{fluid+radiation pressure}} = \nabla_{\vec{x}} \cdot (\mu_F(\nabla_{\vec{x}}\vec{u} + (\nabla_{\vec{x}}\vec{u})^T))$$

$$+\rho\vec{g} - \underbrace{\frac{1}{c}\int_0^\infty \rho\bar{\kappa}_\nu \int_{\mathbf{S}^2} \vec{\omega} I_\nu d\vec{\omega} d\nu}_{\text{momentum deposited in the fluid}}$$

Neglecting $O(1/c)$ Terms

In the momentum balance equality above, we neglect all the terms (formally) of order $1/c$:

$$\partial_t \left(\rho \vec{u} + \cancel{\frac{1}{c^2} \int_0^\infty \int_{\mathbf{S}^2} \vec{\omega} I_\nu d\vec{\omega} d\nu} \right) + \nabla_{\vec{x}} \cdot (\rho \vec{u} \otimes \vec{u})$$

$$+ \nabla_{\vec{x}} \cdot \left(p + \cancel{\frac{1}{c} \int_0^\infty \int_{\mathbf{S}^2} \vec{\omega} \otimes \vec{\omega} I_\nu d\vec{\omega} d\nu} \right) = \nabla_{\vec{x}} \cdot \left(\mu_F \left(\nabla_{\vec{x}} \vec{u} + (\nabla_{\vec{x}} \vec{u})^T \right) \right)$$

$$+ \rho \vec{g} - \cancel{\frac{1}{c} \int_0^\infty \rho \bar{\kappa}_\nu \int_{\mathbf{S}^2} \vec{\omega} I_\nu d\vec{\omega} d\nu}$$

In other words, the contribution of radiation to the fluid **momentum** balance is considered negligible, but **not** the contribution of radiation to the **energy** balance.

Heat Equation

Substracting the kinetic energy balance leads to the heat equation

$$\rho c_V(\partial_t + \vec{u}\cdot\nabla_{\vec{x}})T = 4\pi\int_0^\infty \rho\bar{\kappa}_\nu(1-a_\nu)(J_\nu - B_\nu(T))d\nu + \nabla_{\vec{x}}\cdot(\rho c_P \kappa_T \nabla_{\vec{x}} T) + \underbrace{\tfrac{1}{2}\mu_F|\nabla_{\vec{x}}\vec{u} + (\nabla_{\vec{x}}\vec{u})^T|^2}_{\text{viscous heating}}$$

Simplifying assumptions Henceforth assume

- that $|\vec{u}| \ll 1$ and neglect viscous heating
- the radiative intensity is quasi-static

$$\frac{1}{c}|\partial_t I_\nu(t,\vec{x},\vec{\omega})| \ll 1$$

- the radiative intensity is slowly varying in the horizontal variables

$$\vec{x} = (x,y,z) \text{ and } |\partial_x I_\nu(t,\vec{x},\vec{\omega})| + |\partial_y I_\nu(t,\vec{x},\vec{\omega})| \ll 1$$

$\Longrightarrow$ **stratified** radiative transfer

The Coupled Radiative Transfer+Heat Equations

With $\vec{\omega} = (\omega_x, \omega_y, \omega_z)$, set

$$\mu = \omega_z = \cos\theta \quad \text{and} \quad (\omega_x, \omega_y) = \sin\theta(\cos\alpha, \sin\alpha)$$

and

$$\mathcal{I}_\nu(t, \vec{x}, \mu) := \frac{1}{2\pi}\int_0^{2\pi} I_\nu(t, \vec{x}, \vec{\omega})d\alpha \implies J_\nu(t, \vec{x}) = \frac{1}{2}\int_{-1}^{1} \mathcal{I}_\nu(t, \vec{x}, \mu)d\mu$$

The coupled radiative transfer+temperature equation becomes

$$\begin{cases} (\partial_t + \vec{u}\cdot\nabla_{\vec{x}})T - \frac{c_P}{c_V}\kappa_T \Delta_{\vec{x}} T = 4\pi \int_0^\infty \frac{\bar{\kappa}_\nu}{c_V}(1-a_\nu)(J_\nu - B_\nu(T))d\nu \\ \mu\partial_z \mathcal{I}_\nu + \rho\bar{\kappa}_\nu \mathcal{I}_\nu = \rho\bar{\kappa}_\nu a_\nu J_\nu + \rho\bar{\kappa}_\nu(1-a_\nu)B_\nu(T) \end{cases}$$

Radiation Heating of a Pool/Lake

Assume that $(x, y) \in \mathcal{O}$ (bounded domain in $\mathbf{R}^2$) and $z \in (0, Z)$, and set $\Omega := \mathcal{O} \times (0, Z)$.

Assume that $\rho = \text{Const.}$, that $\vec{u}$ satisfies $\nabla_{\vec{x}} \cdot \vec{u} = 0$ is a solution of the Navier-Stokes equations. Denoting $\kappa_\nu = \rho \bar{\kappa}_\nu$

$$\begin{cases} \vec{u} \cdot \nabla_{\vec{x}} T - \frac{c_P}{c_V} \kappa_T \Delta_{\vec{x}} T = \frac{4\pi}{\rho c_V} \displaystyle\int_0^\infty \kappa_\nu (1 - a_\nu)(J_\nu - B_\nu(T)) d\nu \\ \mu \partial_z \mathcal{I}_\nu + \kappa_\nu \mathcal{I}_\nu = \kappa_\nu a_\nu J_\nu + \kappa_\nu (1 - a_\nu) B_\nu(T) \\ \\ \mathcal{I}_\nu(0, \mu) = \mu Q_\nu^+, \quad \mathcal{I}_\nu(Z, -\mu) = \mu Q_\nu^-, \quad 0 < \mu < 1 \\ \dfrac{\partial T}{\partial n}\Big|_{\partial\Omega} = 0, \quad \vec{u}\Big|_{\partial\Omega} = 0 \end{cases}$$

Algorithm: Iteration on the Sources

(1) Data: $Q_\nu^\pm$ and

$$S_\nu(\vec{x}) := \frac{1}{2}\int_0^1 \mu(e^{-\kappa_\nu z/\mu} Q_\nu^+(x,y) + e^{-\kappa_\nu (Z-z)/|\mu|} Q_\nu^-(x,y))d\mu$$

(2) Choose $T^0(\vec{x})$ and $J_\nu^0(\vec{x}) = S_\nu(\vec{x})$ for all $\vec{x} \in \Omega$

(3) For all $(x,y) \in \mathcal{O}$ do

(a) for all $0 < z < Z$ and $\nu > 0$, compute $J^n(\vec{x})$ by

$$J_\nu^n(\vec{x}) = S_\nu(\vec{x}) + \int_0^Z \frac{\kappa_\nu}{2} E_1(\kappa_\nu |z-\zeta|)(a_\nu J_\nu^{n-1} + (1-a_\nu) B_\nu(T^{n-1})))(x,y,\zeta)d\zeta$$

(b) compute $T^n(\vec{x})$ by solving

$$\begin{cases} \vec{u}\cdot\nabla_{\vec{x}} T^n - \frac{c_P}{c_V}\kappa_T \Delta_{\vec{x}} T^n = \frac{4\pi}{\rho c_V}\int_0^\infty \kappa_\nu (1-a_\nu)(J_\nu^n - B_\nu(T^n))d\nu \\ \frac{\partial T}{\partial n}\Big|_{\partial\Omega} = 0 \end{cases}$$

(4) end for; (5) return T

Notation for the exponential integral

$$E_1(\theta) := \int_\theta^\infty \frac{e^{-y}}{y} dy\,, \qquad C_1(k) := \int_0^{kZ/2} E_1(\theta) d\theta$$

Thm A Assume $0 \le a_\nu \le a_M < 1$ and $0 < \kappa_m \le \kappa_\nu \le \kappa_M$, and $0 \le Q_\nu^\pm \le B_\nu(T_M)$, and set $T^0(\vec{x}) = 0$.
(a) One has

$$0 \le S_\nu = J^0 \le J^1 \le \ldots \le J_\nu^n \le J_\nu^{n+1} \le B_\nu(T_M)$$
$$0 = T^0 \le T^1 \le \ldots \le T^n \le T^{n+1} \le T_M$$

(b) One has $(J_\nu^n, T^n) \to (J_\nu, T)$ solution of the radiative transfer+heat equation system in the limit as $n \to \infty$

Pbms Convergence rate? Uniqueness of the solution?

Uniqueness+Convergence Rate

Thm B Under the same assumptions as Thm A, and if

$$\sup_{\nu>0}((1-a_\nu)C_1(\kappa_\nu)) + \sup_{\nu>0}(a_\nu C_1(\kappa_\nu)) = \gamma < 1$$

(a) The algorithm above converges exponentially fast

$$\int_\Omega \int_0^\infty (|J_\nu - J_\nu^n| + \kappa_\nu(1-a_\nu)|B_\nu(T) - B_\nu(T^n)|)d\nu d\vec{x}$$
$$\leq \frac{\gamma^n|\Omega|}{1-\gamma}(1 + \frac{1}{\kappa_m(1-a_M)}) \int_0^\infty \kappa_\nu(1-a_\nu)B_\nu(T_M)d\nu$$

(b) There exists at most one solution such that

$$0 \leq T \in L^\infty(\Omega) \quad \text{and} \quad I_\nu \geq 0 \text{ a.e. on } \Omega \times \mathbf{S}^2 \times (0,\infty)$$

Solving the Heat Equation for T^n

Consider

$$\mathcal{B}(T) := \int_0^\infty \kappa_\nu (1 - a_\nu) B_\nu(T) d\nu \,, \qquad \text{increasing on } (0, +\infty)$$

Lemma 1 For all $R \in L^{6/5}(\Omega)$ there exists a weak solution of

$$(H) \qquad -\lambda \Delta_{\vec{x}} T + \vec{u} \cdot \nabla_{\vec{x}} T + \mathcal{B}(T_+) = R \,, \qquad \frac{\partial T}{\partial n}\Big|_{\partial\Omega} = 0$$

(1) If $R \geq 0$ a.e. on Ω and $|\{\vec{x} \in \Omega \text{ s.t. } R(\vec{x}) > 0\}| > 0$, the solution T of (H) is unique and $T \geq 0$ a.e. on Ω;

(2) If $R' \in L^{6/5}(\Omega)$ and $R' \geq R$ a.e. on Ω the weak solution T' of (H) with r.h.s. R' satisfies $T \leq T'$ a.e. on Ω.

Proof of Lemma 1

For all $\epsilon > 0$, apply the Leray-Lions theorem [BSMF 1965] to the (nonlinear) functional $\mathcal{A}_\epsilon : H^1(\Omega) \to H^1(\Omega)'$ defined by the formula

$$\langle \mathcal{A}_\epsilon T, \phi \rangle := \int_\Omega (\epsilon T \phi + \nabla_{\vec{x}} T \cdot (\lambda \nabla_{\vec{x}} \phi + \phi u) + \mathcal{B}(T_+)\phi) d\vec{x}$$

Since B_ν is increasing for each $\nu > 0$, the function $\mathcal{B}$ is increasing

$$\langle \mathcal{A}_\epsilon T - \mathcal{A}_\epsilon T', T - T' \rangle = \epsilon \| T - T' \|_{L^2}^2 + \| \nabla_{\vec{x}} (T - T') \|_{L^2}^2 + \int_\Omega \underbrace{(T - T')(\mathcal{B}(T) - \mathcal{B}(T'))}_{\geq 0} d\vec{x}$$

Let $\epsilon \to 0$, by Rellich and Banach-Alaoglu, find T_{ϵ_n} s.t. for $p \in [1, 6)$

$$\| T_\epsilon \|_{L^2} = O(\tfrac{1}{\sqrt{\epsilon}}), \quad T_{\epsilon_n,+} \to T_+ \text{ in } L^p(\Omega), \quad \nabla_{\vec{x}} T_{\epsilon_n} \rightharpoonup \nabla_{\vec{x}} T \text{ in } L^2(\Omega)$$

and pass to the limit in $\mathcal{A}_{\epsilon_n}$ as $\epsilon_n \to 0$ $\square$

Lemma 2 The function

$$(0,+\infty)\ni k\mapsto C_1(k)=\int_0^{kZ/2}E_1(\theta)d\theta\in(0,1) \text{ is increasing}$$

and

$$\int_0^Z\int_0^Z \tfrac{1}{2}kE_1(k|z-\zeta|)|f(\zeta)|d\zeta dz \le C_1(k)\int_0^Z |f(\zeta)|d\zeta$$

Proof Since E_1 is decreasing, by symmetric rearrangement

$$\sup_{0<\zeta<Z}\int_0^Z kE_1(k|\zeta-z|)dz = \sup_{0<\zeta<Z}\int_{\mathbf{R}} kE_1(k|\zeta-z|)\mathbf{1}_{[0,Z]}(z)dz$$

$$\le \int_{\mathbf{R}} kE_1(k|z|)\underbrace{(\mathbf{1}_{[0,Z]})^*}_{=\mathbf{1}_{[-\frac{Z}{2},\frac{Z}{2}]}}(z)dz = \int_{-kZ/2}^{kZ/2}E_1(\theta)d\theta = 2C_1(k)$$

□

Monotonicity of Radiative Transfer (Mercier SIMA1987)

Notation for $f \equiv f(\nu) \in L^1((0,\infty))$ and $g \equiv g(\mu) \in L^1([-1,1])$

$$\langle f \rangle := \int_0^\infty f(\nu)d\nu \qquad \tilde{g} := \tfrac{1}{2}\int_{-1}^{1} g(\mu)d\mu\,, \qquad \langle\!\langle h \rangle\!\rangle := \langle \tilde{h} \rangle$$

Lemma 3 Let (I_ν, T) and (I'_ν, T') satisfy RT+heat equations; then

$$\partial_z \langle\!\langle \mu(\mathcal{I}_\nu - \mathcal{I}'_\nu)_+ \rangle\!\rangle + \vec{u}\cdot\nabla_{\vec{x}}(T-T')_+ - \lambda\Delta_{\vec{x}}(T-T')\mathbf{1}_{T>T'} = -D_1 - D_2$$

where $\mathcal{I}_\nu$ is the average of I_ν in (ω_x, ω_y) and

$$\begin{cases} D_1 = \langle\!\langle \kappa_\nu(1-a_\nu)((\mathcal{I}_\nu - \mathcal{I}'_\nu) - (B_\nu(T) - B_\nu(T')))(\mathbf{1}_{\mathcal{I}_\nu > \mathcal{I}'_\nu} - \mathbf{1}_{T>T'}) \rangle\!\rangle \geq 0 \\ D_2 = \langle\!\langle \kappa_\nu a_\nu((\mathcal{I}_\nu - \mathcal{I}'_\nu) - (J_\nu - J'_\nu))\mathbf{1}_{\mathcal{I}_\nu > \mathcal{I}'_\nu} \rangle\!\rangle \geq 0 \end{cases}$$

NB in Lemma 3 we have set $\lambda := \frac{\rho C_P \kappa_T}{4\pi}$ and replaced $\vec{u}$ with $\frac{\rho C_V}{4\pi}\vec{u}$

Proof of the Monotonicity Lemma

•Multiplying both sides of the RT equation by $\mathbf{1}_{\mathcal{I}_\nu > \mathcal{I}'_\nu}$ and integrating the resulting expressions in μ and ν, and both sides of the heat equation by $\mathbf{1}_{T>T'}$ leads to

$$\partial_z \langle\!\langle \mu(\mathcal{I}_\nu - \mathcal{I}'_\nu)_+ \rangle\!\rangle + \vec{u} \cdot \nabla_{\vec{x}} (T - T')_+ - \lambda \Delta_{\vec{x}} (T - T') \mathbf{1}_{T>T'} = -D_1 - D_2$$

•Since $\widetilde{(\mathcal{I}_\nu - \mathcal{I}'_\nu)} = (J_\nu - J'_\nu)$ and $\mathbf{1}_{J_\nu > J'_\nu}$ is independent of μ

$$D_2 = \langle\!\langle \kappa_\nu a_\nu ((\mathcal{I}_\nu - \mathcal{I}'_\nu) - (J_\nu - J'_\nu))(\mathbf{1}_{\mathcal{I}_\nu > \mathcal{I}'_\nu} - \mathbf{1}_{J_\nu > J'_\nu}) \rangle\!\rangle$$

Since $z \mapsto \mathbf{1}_{z>0}$ is nondecreasing, one has

$$(a - b)(\mathbf{1}_{a>0} - \mathbf{1}_{b>0}) \geq 0$$

so that

$$((\mathcal{I}_\nu - \mathcal{I}'_\nu) - (J_\nu - J'_\nu))(\mathbf{1}_{\mathcal{I}_\nu > \mathcal{I}'_\nu} - \mathbf{1}_{J_\nu > J'_\nu}) \geq 0 \implies D_2 \geq 0$$

•Since $T \mapsto B_\nu(T)$ is increasing on $(0, +\infty)$ for all $\nu > 0$, one has

$$\mathbf{1}_{T>T'} = \mathbf{1}_{B_\nu(T)>B_\nu(T')} \text{ is independent of } \mu, \nu$$

Since $z \mapsto \mathbf{1}_{z>0}$ is nondecreasing,

$$\begin{aligned} &((\mathcal{I}_\nu - \mathcal{I}'_\nu) - (B_\nu(T) - B_\nu(T')))(\mathbf{1}_{\mathcal{I}_\nu > \mathcal{I}'_\nu} - \mathbf{1}_{T>T'}) \\ &= ((\mathcal{I}_\nu - \mathcal{I}'_\nu) - (B_\nu(T) - B_\nu(T')))(\mathbf{1}_{\mathcal{I}_\nu > \mathcal{I}'_\nu} - \mathbf{1}_{B_\nu(T)>B_\nu(T')}) \geq 0 \end{aligned}$$

This implies that

$$D_1 \geq 0$$

NUMERICAL SIMULATIONS

The Absorption Coefficient κ_ν

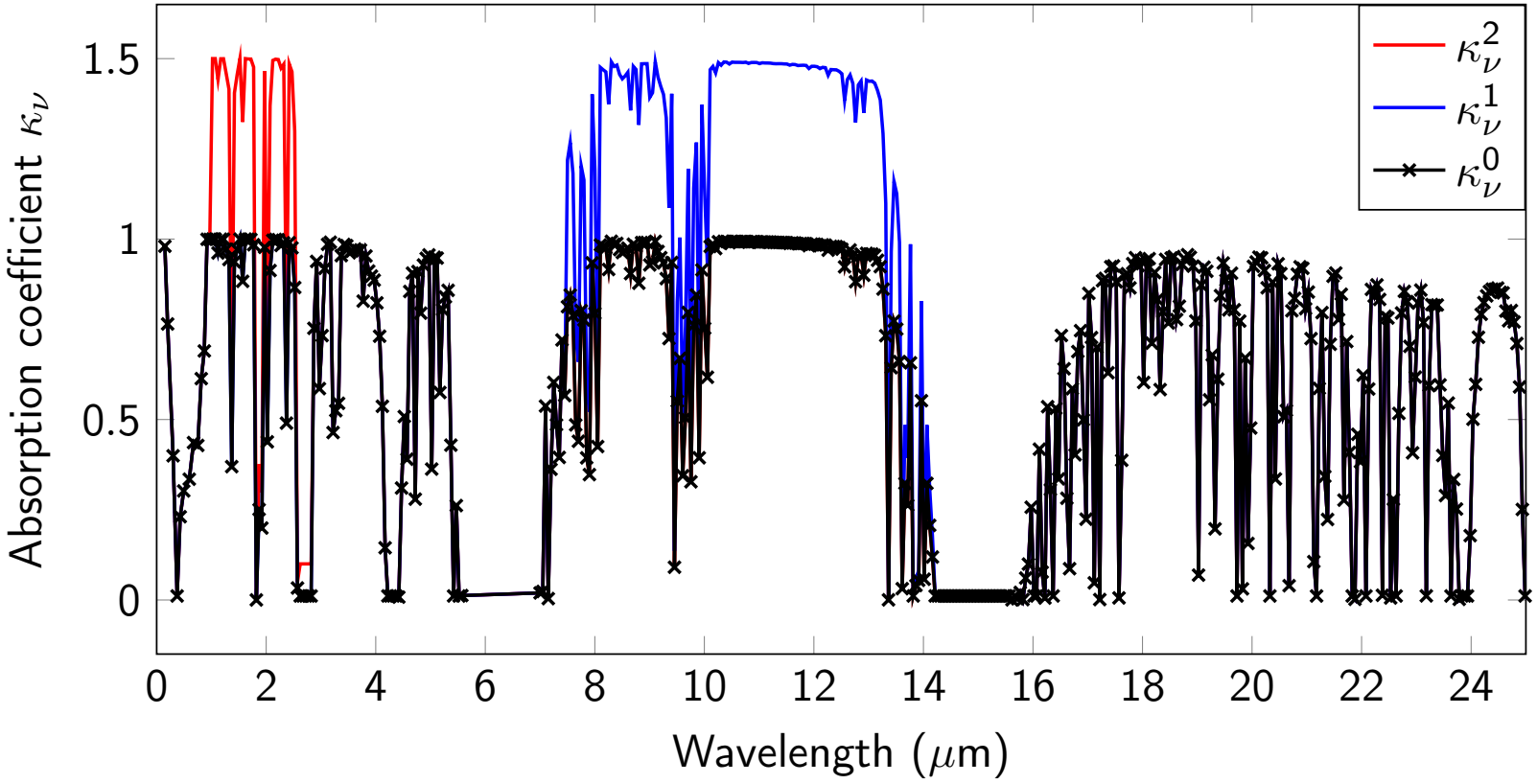


Figure: Absorption κ_ν^0 read from Gemini measurements. Set enhanced absorption: $\kappa_\nu^1 > \kappa_\nu^0$ in the infrared range $2-3\mu m$ and $\kappa_\nu^2 > \kappa_\nu^0$ in the range $8-14\mu m$. The $\times$ marks are the 487 grid points for the ν-integrals. Enhanced high values are truncated at $\kappa = 1.5$.

Case 1: No Fluid Coupling ($\vec{u} = 0$ and $\kappa_T = 0$)

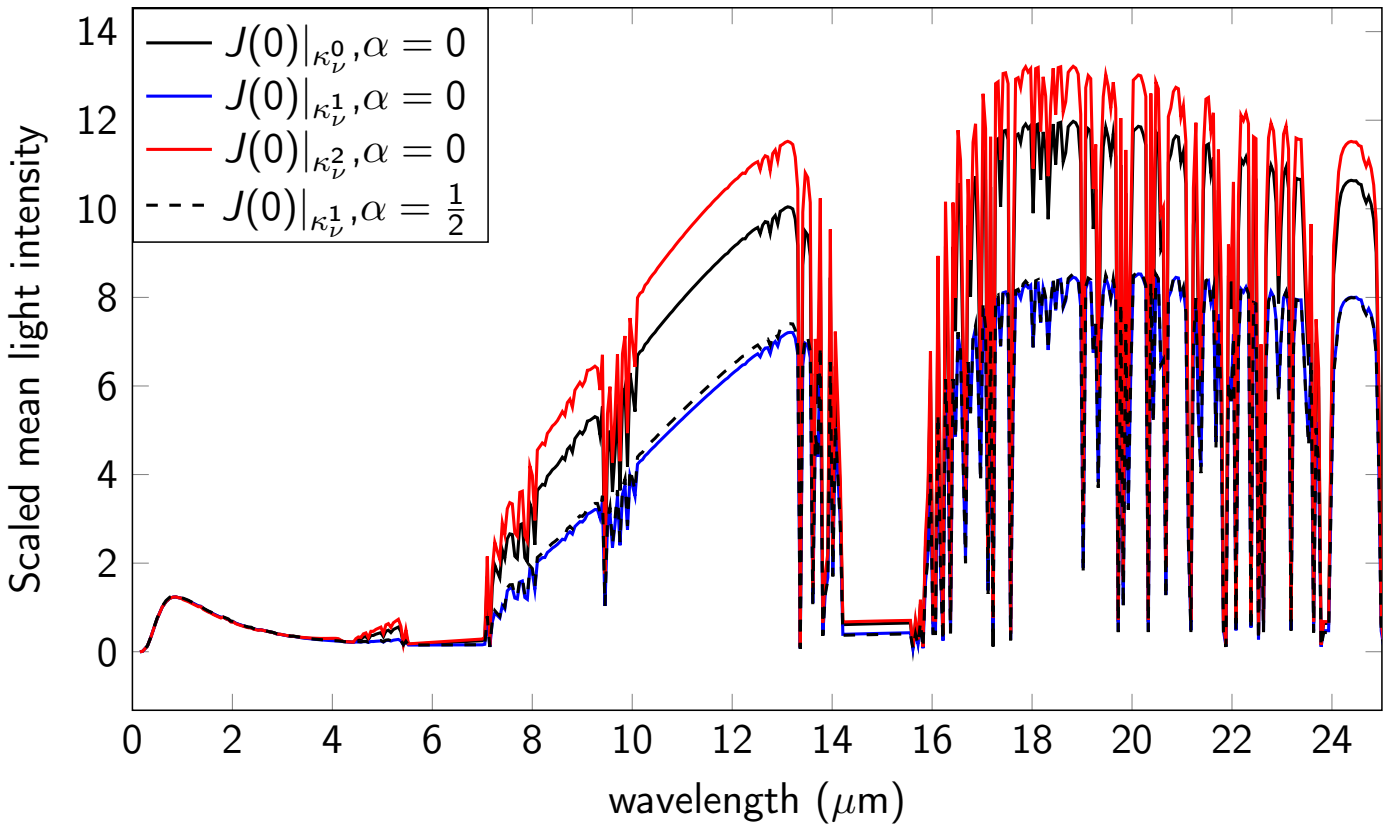


Figure: Computed mean radiation intensities $J_\nu(0)$ at the ground level for κ_ν^0, κ_ν^1, κ_ν^2 without scattering and for κ_ν^0 with isotropic scattering at altitude 6-9km and Rayleigh phase function above 9km, with scattering albedo $a_\nu = \alpha = \frac{1}{2}$.

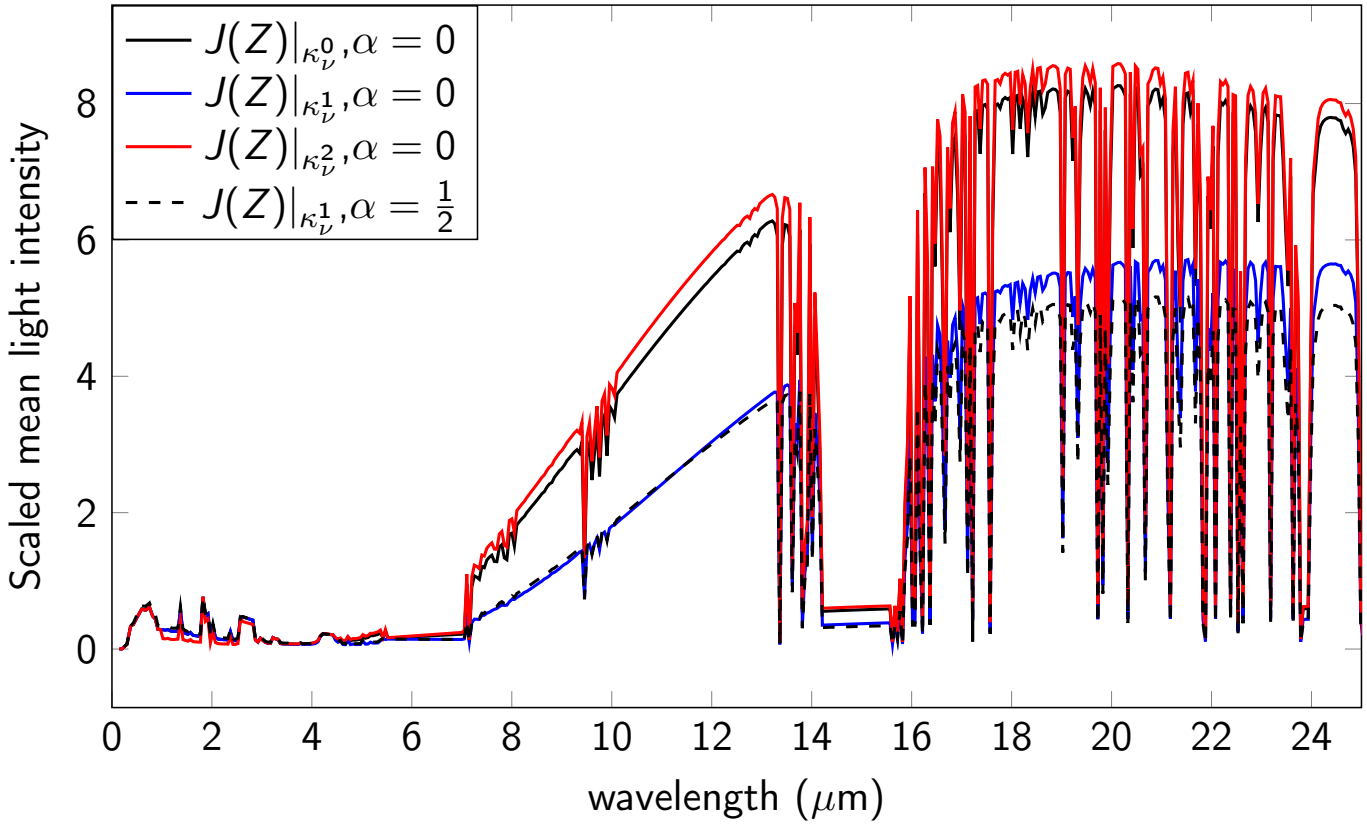


Figure: Computed mean radiation intensities $J_\nu(Z)$ at the top of the troposphere for κ_ν^0, κ_ν^1, κ_ν^2 without scattering and for κ_ν^0 with isotropic scattering at altitude 6-9km and Rayleigh phase function above 9km, with scattering albedo $a_\nu = \alpha = \frac{1}{2}$.

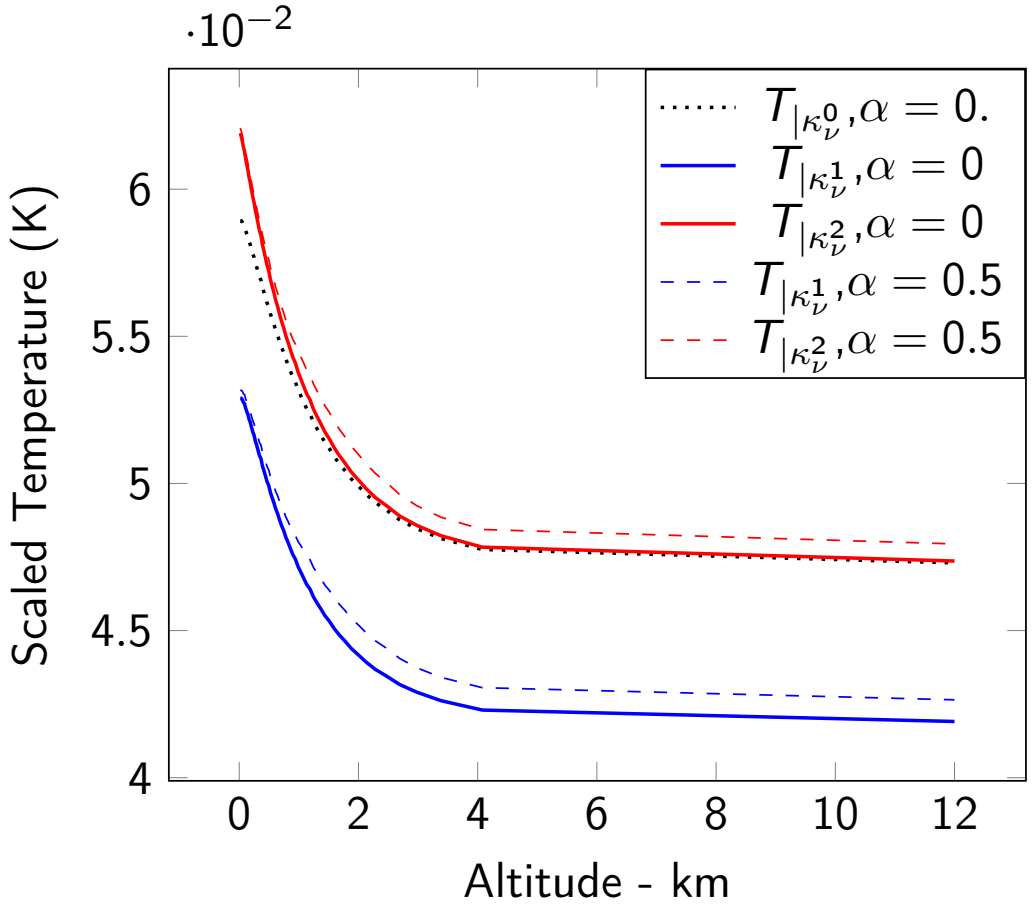


Figure: Temperatures $z \to T(z)$ in Kelvin divided by 4798 computed with κ_ν^0, κ_ν^1 and κ_ν^2 without scattering ($\alpha = 0$) and with a scattering albedo $a_\nu = \alpha = \frac{1}{2}$.

Towards a Mathematical Explanation of Case no. 1?

Fact

Increasing κ_ν in frequency ranges corresponding to greenhouse gases in the RT equation does not lead to an enhanced greenhouse effect

Milne Pbm

$$\begin{cases} \mu\partial_z I_\nu^\epsilon + \sigma_\nu^\epsilon(I_\nu^\epsilon - b_\nu(\Phi^\epsilon)) + \lambda(I_\nu^\epsilon - \tilde{I}_\nu^\epsilon) = 0\,, & z,\nu > 0\,,\ |\mu| < 1 \\ \langle \sigma_\nu^\epsilon(\tilde{I}_\nu^\epsilon - b_\nu(\Phi^\epsilon))\rangle = 0 & \\ I_\nu^\epsilon(0,\mu) = I_\nu^b(\mu)\,, & 0 < \mu < 1 \end{cases}$$

with the notation

$$\Phi := \int_0^\infty B_\nu(T)d\nu = \tfrac{\Sigma_R}{\pi}T^4\,, \qquad b_\nu(\Phi) := B_\nu(T)$$

and

$$\tilde{\phi} := \tfrac{1}{2}\int_{-1}^1 \phi(\mu)d\mu\,, \qquad \langle\psi\rangle := \int_0^\infty \psi(\nu)d\nu$$

Enhancing Absorption in some Frequency Range

For $\sigma > 0$ constant, set

$$\sigma_\nu^\epsilon := \sigma + \epsilon\delta\sigma_\nu > 0\,, \qquad \text{with} \qquad \delta\sigma_\nu := \sigma \mathbf{1}_{\nu_m<\nu<\nu_M}$$

Consider the total radiative intensity (integrated over frequencies)

$$u^\epsilon(z,\mu) := \langle I_\nu^\epsilon(z,\mu)\rangle$$

Case $\epsilon = 0$. This is the grey case" (σ independent of ν). One easily checks that u^0 is the solution of the **linear** Milne problem

$$\begin{cases} \mu\partial_z u^0(z,\mu) + (\sigma+\lambda)(u^0(z,\mu) - \tilde{u}^0(z)) = 0\,, & z > 0 \\ u(0,\mu) = \langle I_\nu^b\rangle\,, & \mu\in(0,1) \end{cases}$$

Once $\tilde{u}^0$ is known, recover I_ν^0 by solving explicitly

$$\begin{cases} \mu\partial_z I_\nu^0 + (\sigma+\lambda)I_\nu^0 - \lambda\tilde{I}_\nu^0 = \sigma b_\nu(\Phi^0) = \sigma b_\nu(\tilde{u}^0)\,, & z,\nu > 0 \\ I_\nu^\epsilon(0,\mu) = I_\nu^b(\mu)\,, & \mu\in(0,1) \end{cases}$$

Effect of the Perturbation of the Absorption Coefficient

Seek to compute

$$I'_\nu(z,\mu) := \frac{\partial I^\epsilon_\nu(z,\mu)}{\partial \epsilon}\bigg|_{\epsilon=0}, \qquad \Phi'(z) := \frac{\partial \Phi^\epsilon(z)}{\partial \epsilon}\bigg|_{\epsilon=0}$$

and

$$u'(z,\mu) := \frac{\partial u^\epsilon(z,\mu)}{\partial \epsilon}\bigg|_{\epsilon=0} = \langle I'_\nu(z,\mu)\rangle$$

Differentiate in ϵ both sides of the RT equations at $\epsilon = 0$:

$$\mu\partial_z I'_\nu + \sigma(I'_\nu - \dot{b}_\nu(\Phi^0)\Phi') + \lambda(I'_\nu - \tilde{I}'_\nu) + \delta\sigma_\nu(I^0_\nu - b_\nu(\Phi^0)) = 0$$

$$\sigma\langle \tilde{I}'_\nu\rangle - \sigma\underbrace{\langle \dot{b}_\nu(\Phi^0)\rangle}_{=1}\Phi' + \langle \delta\sigma_\nu(\tilde{I}^0_\nu - b_\nu(\Phi^0))\rangle = 0$$

Integrating in ν both sides of the first equality leads to

$$\begin{cases} \mu\partial_z u'(z,\mu)+(\sigma+\lambda)(u'(z,\mu)-\tilde{u}'(z)) \\ \qquad\qquad\qquad =-\langle\delta\sigma_\nu(I_\nu^0(z,\mu)-\tilde{I}_\nu^0(z))\rangle\,, \quad z>0\,,\ |\mu|<1\,, \\ \\ u'(0,\mu)=0\,, \qquad 0<\mu<1\,, \end{cases}$$

and recast the second equality as

$$\begin{aligned} \Phi'(z) =& \tilde{u}'(z)+\tfrac{1}{\sigma}\langle\delta\sigma_\nu(\tilde{I}_\nu^0(z)-b_\nu(\Phi^0(z)))\rangle \\ =& \tilde{u}'(z)+\tfrac{1}{\sigma}\langle\delta\sigma_\nu(\tilde{I}_\nu^0(z)-b_\nu(\tilde{u}^0(z)))\rangle \end{aligned}$$

A (Plausible?) Scenario

Start from the formula

$$\Phi'(z) = \underbrace{\tilde{u}'(z)}_{=A} + \underbrace{\frac{1}{\sigma}\langle \delta\sigma_\nu(\tilde{I}^0_\nu(z) - b_\nu(\tilde{u}^0(z)))\rangle}_{=B}$$

(1) Unless $\tilde{u}^0$ is a constant, one has $\tilde{I}^0_\nu(z) \neq b_\nu(\tilde{u}^0(z))$ and

$$\langle \tilde{I}^0_\nu \rangle = \tilde{u}^0 = \langle b_\nu(\tilde{u}^0)\rangle \implies \langle (\tilde{I}^0_\nu(z) - b_\nu(\tilde{u}^0(z)))\rangle = 0$$

Therefore the sign of the function $\nu \mapsto \tilde{I}^0_\nu(z) - b_\nu(\tilde{u}^0(z))$ must change as ν runs through $(0, +\infty)$, and one can find two different frequency intervals (ν_m, ν_M) and (ν'_m, ν'_M) such that

$$\langle \underbrace{\mathbf{1}_{\nu_m<\nu<\nu_M}}_{=\frac{1}{\sigma}\delta\sigma_\nu}(\tilde{I}^0_\nu(z) - b_\nu(\tilde{u}^0(z)))\rangle \langle \underbrace{\mathbf{1}_{\nu'_m<\nu<\nu'_M}}_{=\frac{1}{\sigma}\delta\sigma'_\nu}(\tilde{I}^0_\nu(z) - b_\nu(\tilde{u}^0(z)))\rangle < 0$$

(2) By uniqueness+stability for the half-space problem for $u'(z,\mu)$:

$$I_\nu^0 \simeq \tilde{I}_\nu^0 \implies |u'(z,\mu)| \simeq 0$$

If $\lambda \gg \sigma = O(1)$, one expects that

$$|I_\nu^0(z,\mu) - \tilde{I}_\nu^0(z)| \ll |\tilde{I}_\nu^0(z) - b_\nu(\tilde{u}^0(z))| > 0$$

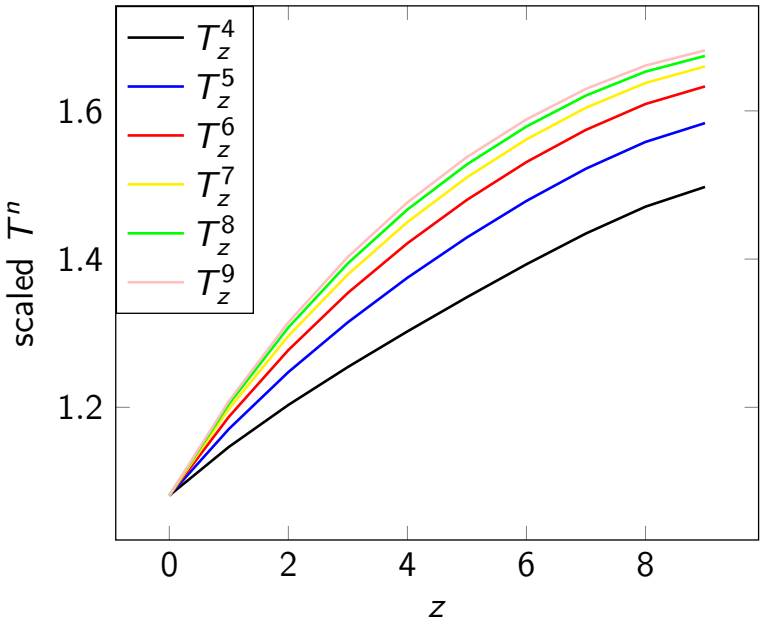
(This remains to be rigorously checked). If so

$$|A| \ll |B| \implies \mathrm{sign}(\Phi'(z)) = \mathrm{sign}(B)$$

Conclusion depending on the **sign** of the **unperturbed** solution (grey case, explicitly computable by Wiener-Hopf techniques) on the **support of the variation of opacity**, the term B in the temperature variation satisfies

$$B > 0 \quad \text{or} \quad B < 0 \implies \Phi'(z) > 0 \quad \text{or} \quad \Phi'(z) < 0$$

Case 2: Heating of a Pool (2D, $\vec{u} = 0$)



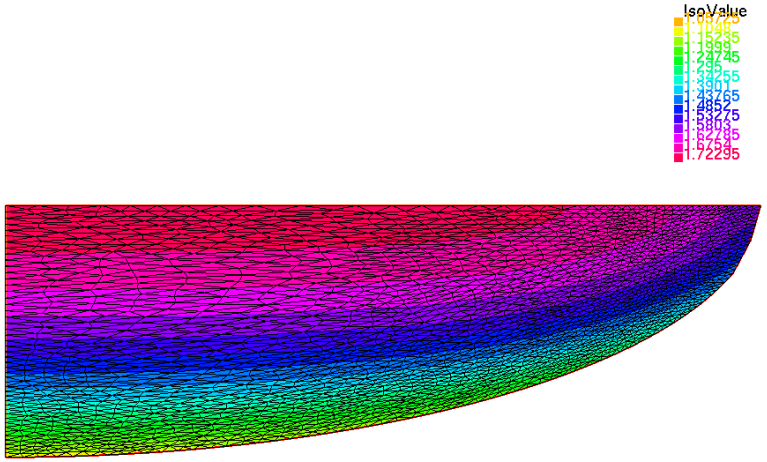


Figure: Left: Convergence of the sequence T^n. Right: Color map of $T(x, z)$ at iteration 10.

Double iteration loop; inner loop of 3 iterations to resolve the T^4 nonlinearity in the heat equation (coming from the Stefan-Boltzmann law if κ and a are both independent of ν)

Case 3: Heating of a Pool (3D, constant wind)

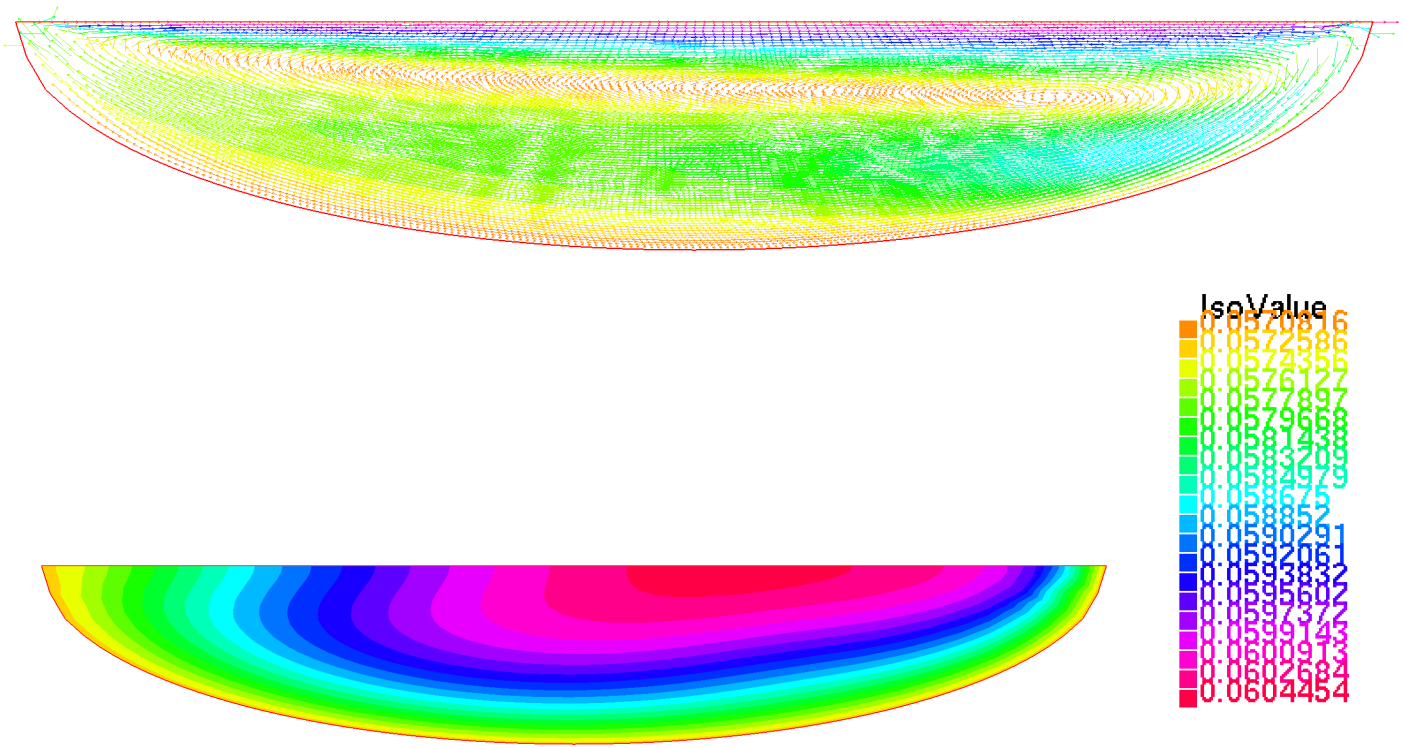


Figure: Velocity field and temperature. Wind velocity $(10, 0)^T$, velocity field solution of Navier-Stokes. Temperature given at the bottom. Solution of the heat equation computed by a time-marching algorithm based on the formula of characteristics for the drift + a quasi-Newton method for a variational formulation of temperature diffusion.

Case 4: Lake Geneva (3D, potential flow)

Set $\vec{u} = \nabla_{\vec{x}} p$, where $\Delta_{\vec{x}} p = 0$ with Dirichlet conditions for p on the red part of the boundary (inlet and outlet of the Rhône) and $\partial_n p = 0$ elsewhere. FEM P1 method with 33810 tetrahedra, and 1287 triangles on the surface.

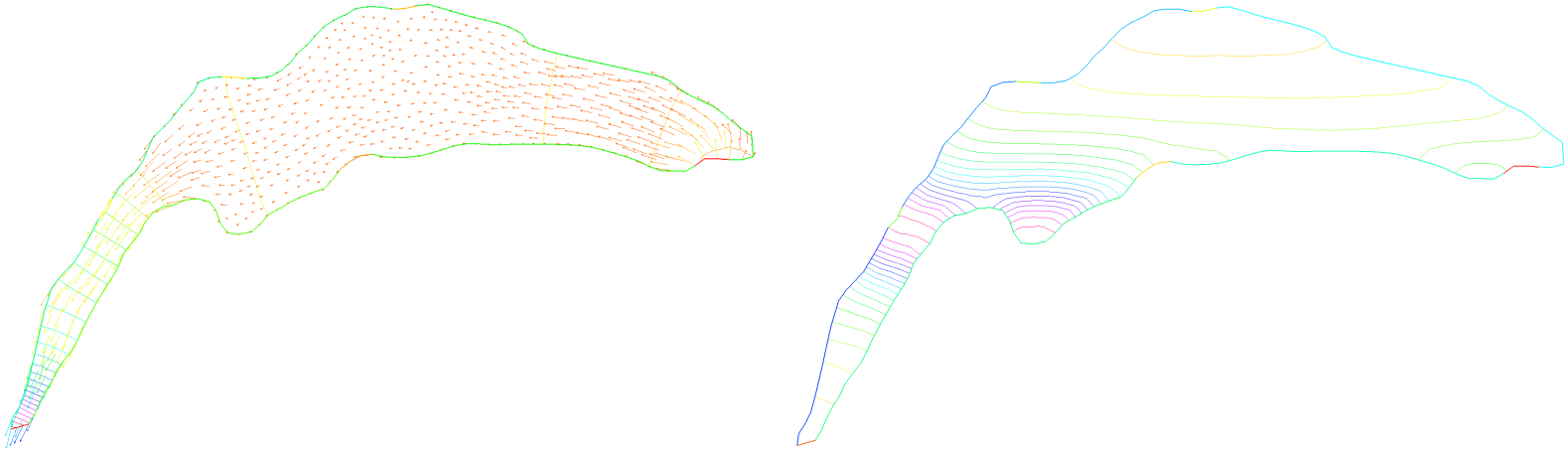


Figure: Left: velocity vectors and pressure isolines at the surface of the lake. Right: isolines of the surface temperature.

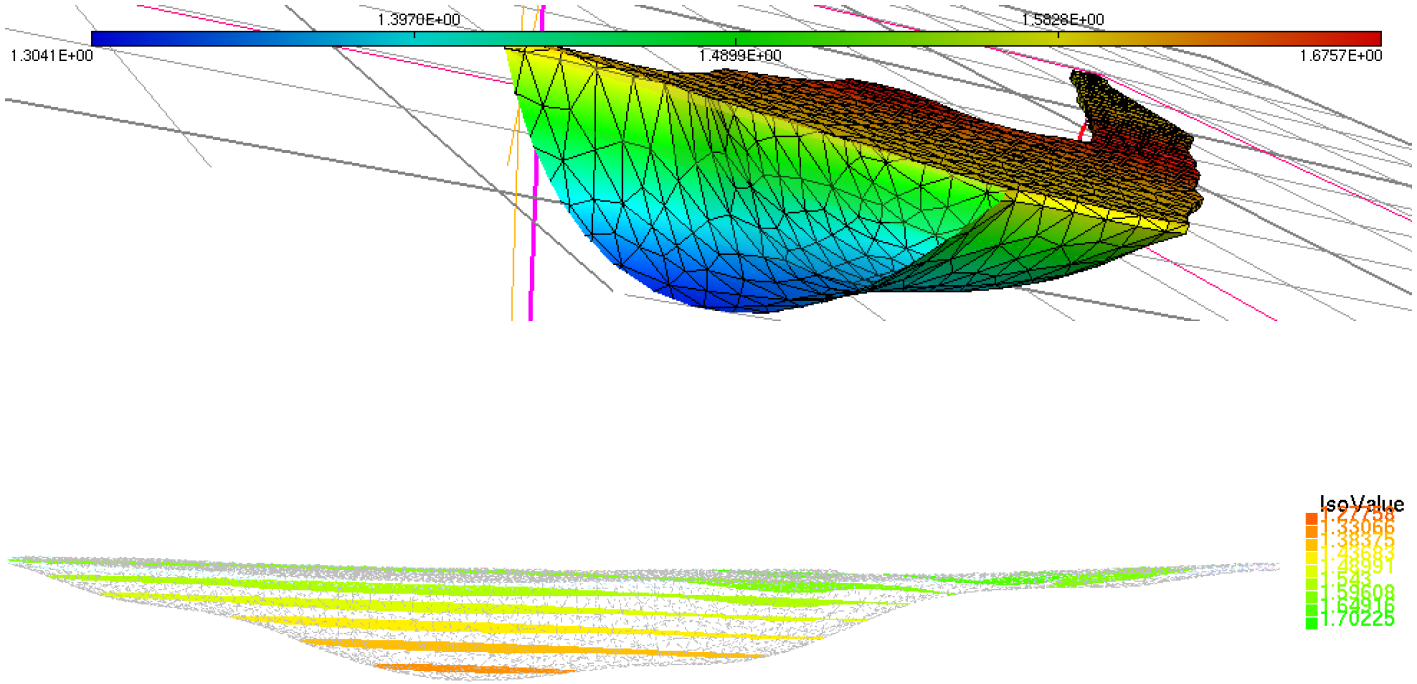


Figure: Top: perspective view of a 3D color map of the temperature on the side of the lake past a middle vertical plane. Bottom: perspective view showing some temperature level surfaces inside the lake.

Discretization of Lake Geneva: F. Hecht (New Developments in FreeFem++ J. Numer. Math. 2012)

Case 5: Planetary Atmosphere Heated by the Sun

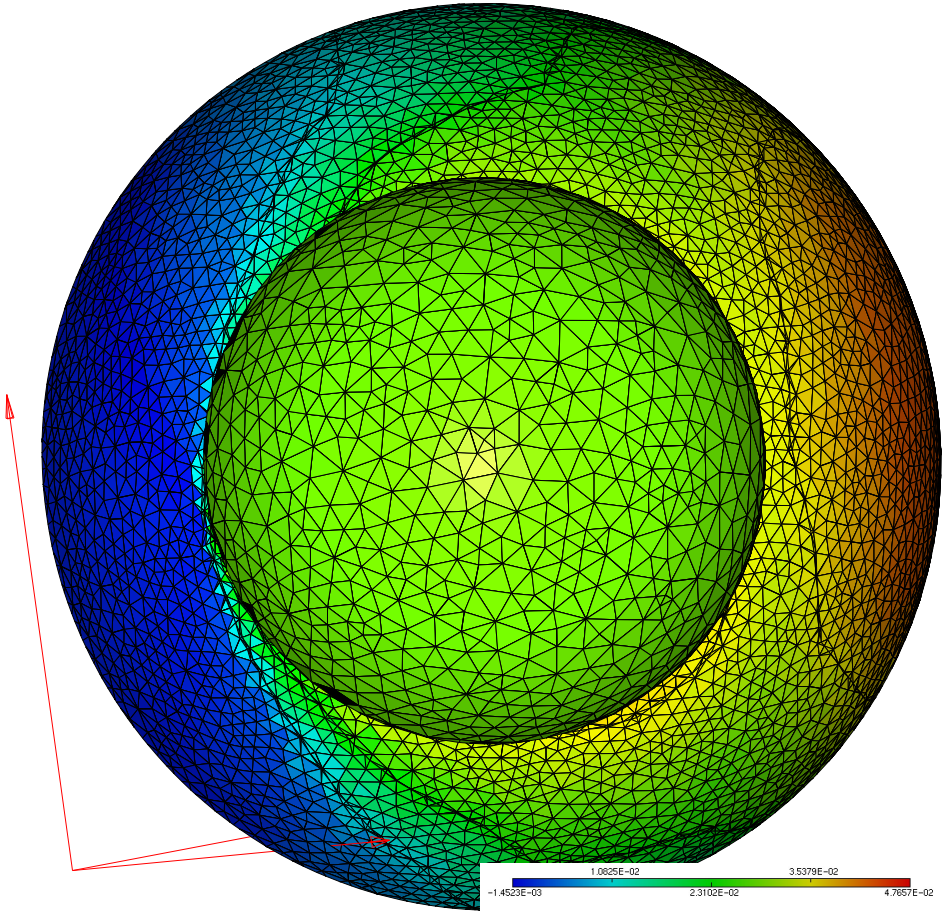


Figure: Temperature in the atmosphere of a planet heated by the Sun. Thermal diffusion propagates heat in unlit regions, with or without the presence of a counterclockwise rotating wind. (Aspect ratio enlarged.)

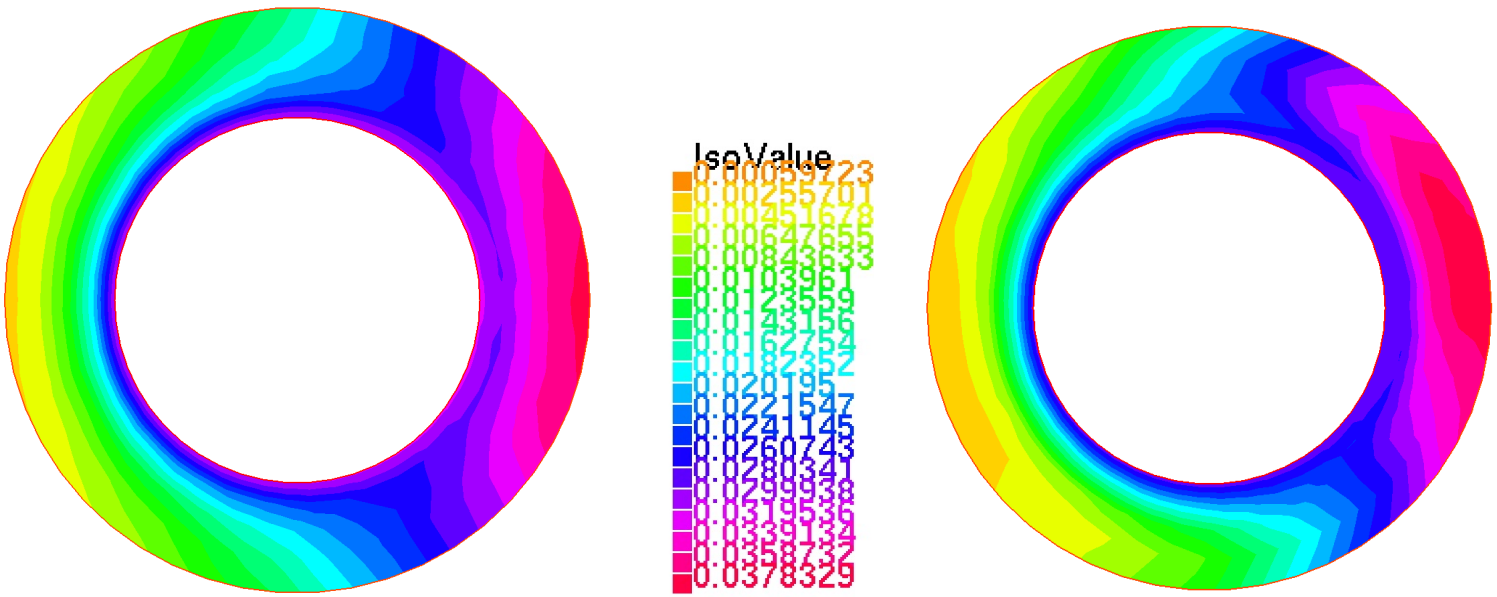


Figure: Temperature in the atmosphere of a planet heated by the Sun on the right with (right) and without (left) almost counterclockwise rotating wind (the axis of rotation is not perpendicular to the figure).

Wind velocity=rotating Poiseuille flow, axis≠direction of the Sun
RT boundary condition: with Q deduced from $Q_{Sun} = 1300 W/m^2$

$$\mathcal{I}_\nu(Z, -\mu) = 0 \text{ and } \mathcal{I}_\nu(0, \mu) = Q\mu B_\nu(T_{Sun}), \quad 0 < \mu < 1$$

Conclusions

•Existence for the radiative transfer equation with frequency dependence coupled with incompressible Navier-Stokes-Fourier system
•Monotonicity structure of RT discovered by Mercier (SIMA1987) adapted to this setting leading to uniqueness and exponential convergence of an algorithm based on iteration on the sources
•Fast numerical simulations in the case of stratified RT (integral equation for the angle averaged radiative intensity) coupled with the fluid equations solved with FreeFEM https://freefem.org
•Extension to Rayleigh's phase function (scattering kernel)

$$p(\vec{\omega}, \vec{\omega}') = \frac{3}{16\pi}(1 + (\vec{\omega} \cdot \vec{\omega}')^2)$$

•Increasing κ_ν in frequency ranges corresponding to greenhouse gases in the RT equation **alone** seems insufficient to explain the enhancement of greenhouse effect

References

•**Textbooks on RT**

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G.C. Pomraning "Radiation Hydrodynamics" Pergamon Press 1973

A. Fowler "Mathematical Geoscience" chap. 2 Springer 2011

•**Papers**

F.G., O. Pironneau:

https://arxiv.org/abs/2107.13857

https://hal.sorbonne-universite.fr/hal-03419670v1

Comments on the temperature computation in case 1 in

C.Bardos, F.G., O. Pironneau:

https://arxiv.org/abs/2111.14167