Modèles dispersifs en océanographie côtière : Enjeux et avancées récentes

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Travail en collaboration avec Gaël L. Richard
INRAE, Université Grenoble Alpes.

Outline

Framework

Introduction - Context and issues

Numerical treatment of the LcT model

Results and outlook

Framework

Collaboration with the SHOM (*Service Hydrographique et Océanographie de la Marine*).

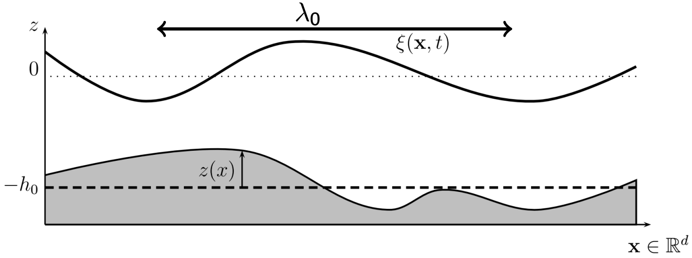
Tolosa : Open source simulation platform for free-surface models, applications in coastal and large-scale oceanography (Shallow Water, multilayer SW and dispersive models).

Contributors :

- ▷ **R. Baraille, M. Ciavaldini, F. Couderc, P. Noble, J.P. Vila** - Toulouse
- ▷ **B. Fabrèges, K. Msheik** - Lyon
- ▷ **F. Marche** - Montpellier
- ▷ **M. Kazakova, Y. C. Hung** - Chambéry
- ▷ **G.L. Richard** - Grenoble
- ▷ **V. Duchêne** - Rennes

Outline

A brief overview of free-surface flows



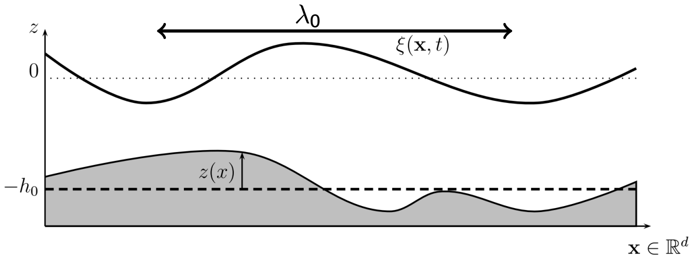
Free surface Euler equations

$$\nabla . \mathbf{v} = 0\,,$$
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}.\nabla)\,\mathbf{v} = -\nabla P\,.$$

$\mathbf{v} = \mathbf{v}(\mathbf{x}, z, t) \in \mathbb{R}^{d+1} \times \mathbb{R}^{+}$.

$\Omega_t = \left\{(\mathbf{x}, z) \in \mathbb{R}^{d+1}\,,\, -h_0 + z(\mathbf{x}) < z < \xi(\mathbf{x}, t)\right\}$.

A brief overview of free-surface flows



I - Depth-averaged models

Integration along the vertical coordinate + BC :

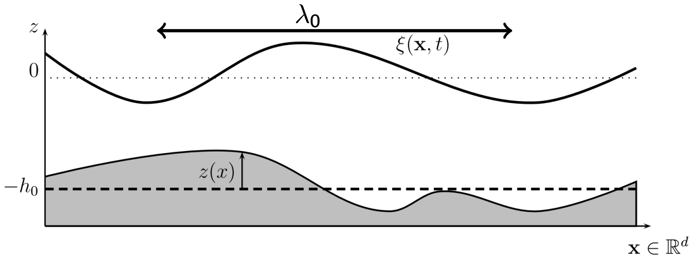
$$\mathbf{u}(\mathbf{x},t) = \frac{1}{h}\int_{z(\mathbf{x})}^{\xi(\mathbf{x},t)} \mathbf{v}_h(\mathbf{x},z,t)dz\,.$$

- $\mathcal{O}(1)$: Shallow Water
- $\mathcal{O}(\mu)$: Boussinesq, Serre-Green-Naghdi,...

$$\mu = \left(\frac{h_0}{\lambda_0}\right)^2 \ll 1$$

▷ **D. Lannes**, *The Water Waves problem : mathematical analysis and asymptotics*, 2013.

A brief overview of free-surface flows



II - Potential flows

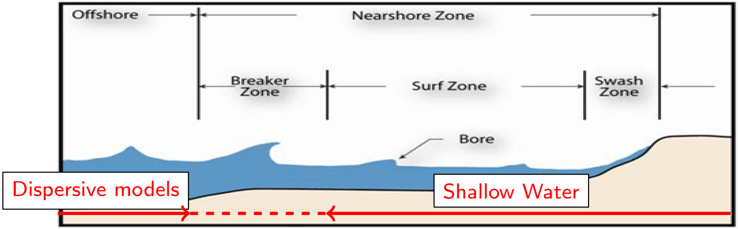
$$\mathbf{v}(\mathbf{x}, z, t) = \nabla\phi(\mathbf{x}, z, t)\,.$$

$$\psi(\mathbf{x}, t) = \phi(\mathbf{x}, z = \xi(\mathbf{x}, t), t)\,.$$

- DtN operator : evolution equations on ξ and ψ.

▷ **D. Lannes**, *The Water Waves problem : mathematical analysis and asymptotics*, 2013.

Application context - Issues and difficulties



- Density stratification, low Mach/Froude regimes.

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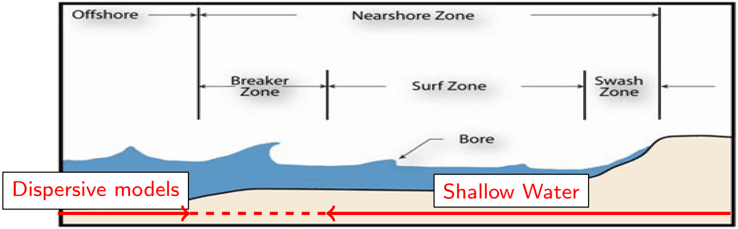

- Density stratification, low Mach/Froude regimes.
- Boundary conditions.

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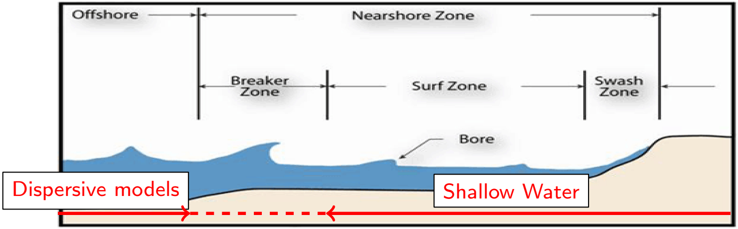

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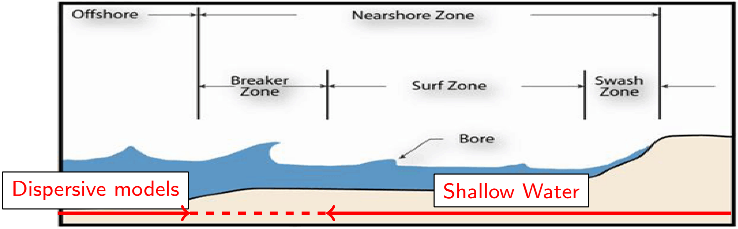

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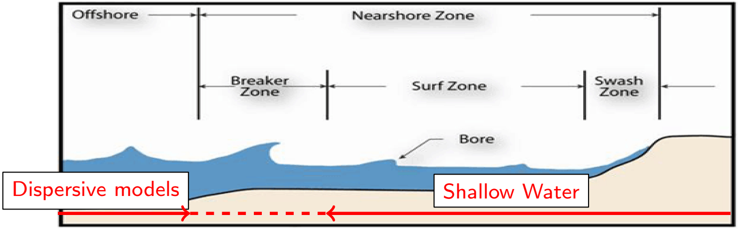

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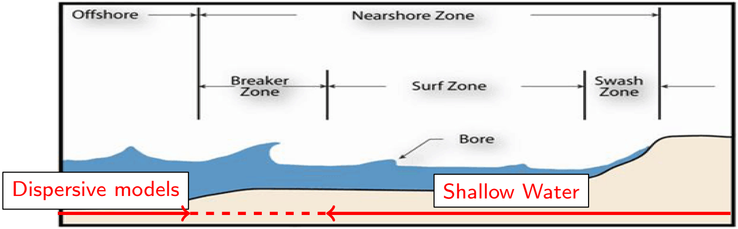

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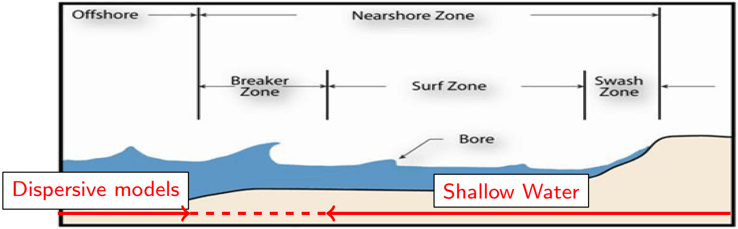

- Density stratification, low Mach/Froude regimes.
- Boundary conditions.
- Wave breaking, coupling, turbulence.
- Dynamics in the surf zone, morphodynamics.
- Fluid/structure interactions.
- Numerical modelling, operational simulation.
- Measures, deep learning, A.I.

Focus on integrated models

Shallow Water and SGN equations

$$\begin{cases} \partial_t h + \partial_x(hu) = 0\,, \\ \partial_t hu + \partial_x\Big(hu^2 + \frac{1}{2}gh^2 + \underbrace{\frac{1}{3}h^2\ddot{h} + \Pi}_{\mathcal{O}(\mu)}\Big) = -gh\partial_x z - \underbrace{f}_{\mathcal{O}(\mu)}\,. \end{cases}$$

- Notations

$$\dot{h} = \frac{Dh}{Dt} = \partial_t h + u\partial_x h \quad , \quad \ddot{h} = \frac{D\dot{h}}{Dt}\,.$$

$$\Pi = \frac{h^2}{2}\frac{D[u\partial_x z]}{Dt} \quad , \quad f = h\partial_x z\left(\frac{\ddot{h}}{2} + \frac{D[u\partial_x z]}{Dt}\right)\,.$$

Reformulation of the SGN équations (1)

Elliptic operator

$$\begin{cases} \partial_t h + \partial_x(hu) = 0\,, \\ (I + \mathcal{T}[h,z])\left(\partial_t hu + \partial_x(hu^2)\right) + gh\partial_x\xi + h\mathcal{Q}[h,u,z] = 0\,. \end{cases}$$

▷ **P. Bonneton *et al.***, *A splitting approach for the fully nonlinear and weakly dispersive Green-Naghdi model*, 2011.

▷ **D. Lannes, F. Marche**, *A new class of fully nonlinear and weakly dispersive Green–Naghdi models for efficient 2D simulations*, 2015.

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- We set :

$$\mathcal{D} = gh\partial_x\xi - (I + \mathcal{T}[h,z])^{-1}\left(gh\partial_x\xi + h\mathcal{Q}\right)\,.$$

- Shallow Water with source term :

$$\begin{cases} \partial_t h + \partial(hu) = 0\,, \\ \partial_t hu + \partial_x(hu^2) + gh\partial_x\xi = \mathcal{D}\,. \end{cases}$$

▷ **P. Bonneton *et al.***, *A splitting approach for the fully nonlinear and weakly dispersive Green-Naghdi model*, 2011.

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Reformulation of the SGN equations (2)

Hyperbolic problem with constraint

$$\begin{cases} \partial_t h + \partial_x (hu) = 0\,, \\ \partial_t hu + \partial_x \left(hu^2 + \dfrac{1}{2} g h^2 + \dfrac{1}{3} h^2 \ddot{h} \right) = 0\,. \end{cases}$$

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- We set $p = h\ddot{h}$.

Reformulation of the SGN equations (2)

Hyperbolic problem with constraint

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- We set $p = h\ddot{h}$.

$$\partial_t hw + \partial_x(huw) = p\,,$$

where $w = -h\partial_x u \quad \rightsquigarrow \quad$ **constraint**.

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$$\partial_t hw + \partial_x(huw) = p\,,$$

where $w = -h\partial_x u \quad \rightsquigarrow \quad$ **constraint**.

- The system is rewritten :

$$\partial_t V + \partial_x A(V) = \Psi(p)\,,$$

with $V = \begin{pmatrix} h \\ u \\ w \end{pmatrix}$ and $V \in \mathbb{A}_h := \left\{ {}^t(h, u, w) \in L^2(\Omega)\,,\ w = -h\partial_x u \right\}$

Reformulation of the SGN equations (2)

- General frame

$$U = \begin{pmatrix} h \\ u \end{pmatrix} \rightsquigarrow V = \begin{pmatrix} h \\ u \\ w \\ \dots \end{pmatrix},$$

$$\partial_t V + \partial_x A(V) = \Psi \quad , \quad V \in \mathbb{A}_h \, .$$

Reformulation of the SGN equations (2)

- General frame

$$U = \begin{pmatrix} h \\ u \end{pmatrix} \rightsquigarrow V = \begin{pmatrix} h \\ u \\ w \\ \dots \end{pmatrix},$$

$$\partial_t V + \partial_x A(V) = \Psi \qquad , \qquad V \in \mathbb{A}_h \,.$$

- Numerical resolution : splitting.

$$V^n \underset{S_1}{\longrightarrow} V^* \underset{S_2}{\longrightarrow} V^{n+1}$$

$$\begin{array}{lll} S_1 & : \partial_t V + \partial_x A(V) = 0 & \longrightarrow V^* \\ S_2 & : \partial_t V = \Psi & \longrightarrow V^{n+1} = \Pi_h[V^*] \end{array}$$

▷ **E.D. Fernandez-Nieto, M. Parisot, Y. Penel, J. Sainte-Marie**, *A hierarchy of dispersive layer-averaged approximations of Euler equations for free surface flows*, 2018.
▷ **C. Escalante, T. Morales de Luna, M.J. Castro**, *Non-hydrostatic pressure shallow flows : GPU implementation using finite volume and finite difference scheme*, 2018.
▷ **S. Noelle, M. Parisot, T. Tscherpel**, *A class of boundary conditions for time-discrete Green–Naghdi equations with bathymetry*, 2022.

Reformulation of the SGN equations (3)

Hyperbolic problem - Relaxation

$$\partial_t h + \partial_x hu = 0\,,$$

$$\partial_t hu + \partial_x \left(hu^2 + \frac{1}{2} gh^2 + \frac{1}{3} h p \right) = 0\,,$$

$$\partial_t hw + \partial_x (huw) = p\,,$$

$$w = -h\partial_x u \qquad \rightsquigarrow \text{constraint}$$

▷ **N. Favrie, S. Gavrilyuk**, *A rapid numerical method for solving Serre–Green–Naghdi equations describing long free surface gravity waves*, 2017.

▷ **V. Duchêne**, *Rigorous justification of the Favrie–Gavrilyuk approximation to the Serre-Green–Naghdi model*, 2019.

▷ **C. Escalante *et al.***, *On high order ADER Discontinuous Galerkin schemes for first order hyperbolic reformulations of nonlinear dispersive systems*, 2019.

▷ **G. Richard**, *An extension of the Boussinesq-type models to weakly compressible flows*, 2021.

Reformulation of the SGN equations (3)

Hyperbolic problem - Relaxation

$$
\begin{aligned}
&\partial_t h + \partial_x hu = 0\,,\\
&\partial_t hu + \partial_x \left(hu^2 + \frac{1}{2} gh^2 + \frac{1}{3} hp \right) = 0\,,\\
&\partial_t hw + \partial_x (huw) = p\,,\\
&\partial_t hp + \partial_x (hup) = -\lambda \left(w + h\partial_x u \right)\,, \qquad \lambda \gg 1\,.
\end{aligned}
$$

▷ **N. Favrie, S. Gavrilyuk**, *A rapid numerical method for solving Serre–Green–Naghdi equations describing long free surface gravity waves*, 2017.

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On dispersive properties - Motivation

Equations SGN

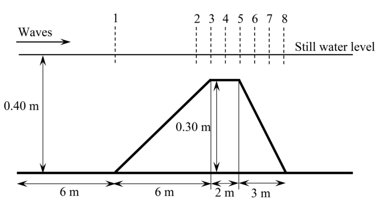
$$\begin{cases} \partial_t h + \partial_x(hu) = 0\,, \\ (I + \mathcal{T}[h,z])\left(\partial_t hu + \partial_x(hu^2)\right) + gh\partial_x\xi + h\mathcal{Q} = 0\,. \end{cases}$$

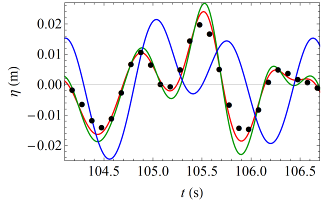
- Dispersion relation :

$$\omega_{GN}^2(k) = gk^2 h_0 \left(\frac{1}{1+(kh_0)^2/3}\right)\,.$$

- ω_{GN} *vs.* linear theory (Stokes) : $\omega_S^2(k) = gk\tanh(kh_0)$.

Beji & Battjes test case





▷ **S. Beji, J. Battjes**, *Numerical simulation of nonlinear wave propagation over a bar*, 1994.

Improvement of dispersive properties

Equations SGN

$$\partial_t h + \partial_x hu = 0\,,$$

$$\left(I + \underbrace{\mathcal{T}[h,z]}_{\mathcal{O}(\mu)}\right)\left(\partial_t hu + \partial_x(hu^2)\right) + gh\partial_x\xi + \underbrace{h\mathcal{Q}}_{\mathcal{O}(\mu)} = 0\,.$$

▷ **P. Bonneton *et al.***, *A splitting approach for the fully nonlinear and weakly dispersive Green-Naghdi model*, 2011.

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- $\partial_t hu = -\partial_x(hu^2) - gh\partial_x\xi + \mathcal{O}(\mu)$.

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- $\partial_t hu = -\partial_x(hu^2) - gh\partial_x\xi + \mathcal{O}(\mu)$.
- Introduction of the parameter α

$$\partial_t hu = \alpha\partial_t hu + (1-\alpha)\big(-\partial_x(hu^2) - gh\partial_x\xi\big) + \mathcal{O}(\mu)\,.$$

▷ **P. Bonneton *et al.***, *A splitting approach for the fully nonlinear and weakly dispersive Green-Naghdi model*, 2011.

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$$\partial_t hu = \alpha\partial_t hu + (1-\alpha)\big(-\partial_x(hu^2) - gh\partial_x\xi\big) + \mathcal{O}(\mu)\,.$$

- Momentum equation

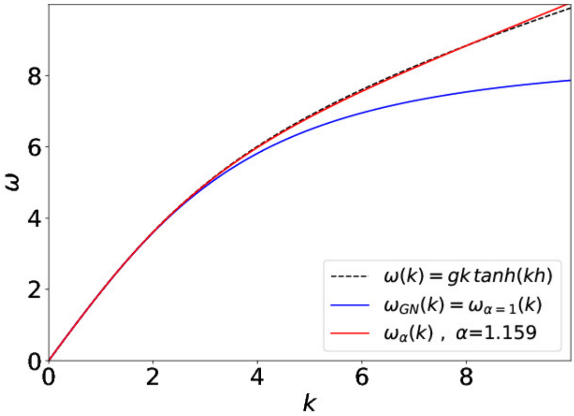
$$\big(I + \alpha\mathcal{T}[h,z]\big)\left(\partial_t hu + \partial_x(hu^2) + \frac{\alpha-1}{\alpha}gh\partial_x\xi\right) + \frac{1}{\alpha}gh\partial_x\xi + h\mathcal{Q}_1 = 0\,.$$

▷ **P. Bonneton *et al.***, *A splitting approach for the fully nonlinear and weakly dispersive Green-Naghdi model*, 2011.

Improvement of dispersive properties

- Dispersion relation :

$$\omega_{\alpha}^2(k) = gk^2 h_0 \left(\frac{1 + (1-\alpha)(kh_0)^2/3}{1 + \alpha (kh_0)^2/3} \right) .$$



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Outline

The *Leucothéa* model (LcT) - G.L. Richard, 2021

1d version, flat bottom

$$(LcT) \quad \begin{cases} \partial_t h + \partial_x(hu) = 0\,, \\ \partial_t(hu) + \partial_x(hu^2 + \frac{1}{2}gh^2 + hP) = 0\,, \\ \partial_t(hW) + \partial_x(huW) = \frac{3}{2}P\,, \\ \partial_t(hP) + \partial_x(huP) = -a^2(2W + h\partial_x u)\,. \end{cases}$$

a : acoustic speed
W : depth-averaged vertical speed
P : depth-averaged non-hydrostatic pressure

▷ **G. Richard**, *An extension of the Boussinesq-type models to weakly compressible flows*, 2021.

Model features

▶ Hyperbolicity

$$\partial_t V + A(V)\partial_x V = S(V)\,,$$

$$\lambda_{1,2} = u \qquad , \qquad \lambda_{3,4} = u \pm \sqrt{gh + a^2}\,.$$

Model features

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$$\partial_t V + A(V)\partial_x V = S(V)\,,$$

$$\lambda_{1,2} = u \quad , \quad \lambda_{3,4} = u \pm \sqrt{gh + a^2}\,.$$

▶ Energy

$$\partial_t E + \partial_x \left((E + \frac{1}{2}gh^2 + hP)u \right) = 0\,,$$

$$E = \frac{1}{2}hu^2 + \frac{1}{2}gh^2 + \frac{2}{3}hW^2 + \frac{1}{2a^2}hP^2\,.$$

Model features

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▶ Dispersion relation

$$\frac{h_0^2}{3a^2}\omega^4 - \omega^2 \left(1 + \frac{k^2 h_0^2}{3}\left(1 + \frac{gh_0}{a^2}\right)\right) + k^2 gh_0 = 0\,.$$

Improvement of dispersive properties

Introduction of the parameter α

$$(LcT_\alpha) \quad \begin{cases} \partial_t h + \partial_x(hu) = 0\,, \\ \partial_t(hu) + \partial_x(hu^2 + \frac{1}{2}gh^2 + hP) = 0\,, \\ \partial_t(h\mathcal{W}) + \partial_x(hu\mathcal{W}) = \frac{3}{2}P + \frac{\alpha - 1}{2\alpha}gh^2\partial_x S\,, \\ \partial_t(hP) + \partial_x(huP) = -a^2(2\mathcal{W} + \alpha h\partial_x u)\,, \\ \partial_t(hS) + \partial_x(huS) = 2h\partial_x\mathcal{W} + \frac{2}{\alpha}\mathcal{W}S\,. \end{cases}$$

- Formally : $LcT = LcT_\alpha + \mathcal{O}(\mu^2)$.

Improvement of dispersive properties

Introduction of the parameter α

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- Formally : $LcT = LcT_\alpha + \mathcal{O}(\mu^2)$.

$\rightsquigarrow$ Look for an energy of the form :

$$E = \frac{1}{2}gh^2 + \frac{1}{2}hu^2 + \frac{2}{3\alpha}h\mathcal{W}^2 + \frac{1}{2\alpha a^2}hP^2 + E_S\,.$$

Improvement of dispersive properties

Introduction of the parameter α

$$(LcT_\alpha) \quad \begin{cases} \partial_t h + \partial_x(hu) = 0\,, \\ \partial_t(hu) + \partial_x(hu^2 + \frac{1}{2}gh^2 + hP) = 0\,, \\ \partial_t(h\mathcal{W}) + \partial_x(hu\mathcal{W}) = \frac{3}{2}P + \frac{\alpha - 1}{2\alpha} gh^2 \partial_x S\,, \\ \partial_t(hP) + \partial_x(huP) = -a^2(2\mathcal{W} + \alpha h \partial_x u)\,, \\ \partial_t(hS) + \partial_x(huS) = 2h\partial_x \mathcal{W} + \frac{2}{\alpha}\mathcal{W}S\,. \end{cases}$$

- Formally : $LcT = LcT_\alpha + \mathcal{O}(\mu^2)$.

$\rightsquigarrow$ Look for an energy of the form :

$$E = \frac{1}{2}gh^2 + \frac{1}{2}hu^2 + \frac{2}{3\alpha}h\mathcal{W}^2 + \frac{1}{2\alpha a^2}hP^2 + E_S\,.$$

- No energy equation for LcT_α !

$\rightsquigarrow$ Set $E_S = \kappa h B^2$ and assume $\alpha - 1 = \mathcal{O}(\mu^{1/2})$.

Energy conservation

Final system ($B = \sqrt{h}S$)

$$\begin{cases} \partial_t h + \partial_x(hu) = 0\,, \\ \partial_t(hu) + \partial_x(hu^2 + \frac{1}{2}gh^2 + hP) = 0\,, \\ \partial_t(h\mathcal{W}) + \partial_x(hu\mathcal{W}) = \frac{3}{2}P + \frac{\alpha - 1}{2\alpha} gh^{3/2}\partial_x B\,, \\ \partial_t(hP) + \partial_x(huP) = -a^2(2\mathcal{W} + \alpha h\partial_x u)\,, \\ \partial_t(hB) + \partial_x(huB) = \partial_x(2h^{3/2}\mathcal{W})\,. \end{cases}$$

Energy equation

$$\partial_t E + \partial_x \left((E + \frac{1}{2}gh^2 + hP + \Pi_B)u \right) = 0\,,$$

$$E = \frac{1}{2}hu^2 + \frac{1}{2}gh^2 + \frac{2}{3\alpha}h\mathcal{W}^2 + \frac{1}{2\alpha a^2}hP^2 + \frac{\alpha - 1}{6\alpha^2}ghB^2\,,$$

$$\Pi_B = -\frac{2}{3}\frac{\alpha - 1}{\alpha^2}gh^{3/2}\mathcal{W}B\,.$$

Numerical scheme

Hyperbolic / acoustic splitting

$$\begin{cases} \partial_t h + \partial_x(hu) = 0\,, \\ \partial_t(hu) + \partial_x(hu^2 + \frac{1}{2}gh^2 + hP) = 0\,, \\ \partial_t(h\mathcal{W}) + \partial_x(hu\mathcal{W}) = \frac{3}{2}P + \frac{\alpha - 1}{2\alpha}gh^{3/2}\partial_x B\,, \\ \partial_t(hP) + \partial_x(huP) = -a^2(2\mathcal{W} + \alpha h\partial_x u)\,, \\ \partial_t(hB) + \partial_x(huB) = \partial_x(2h^{3/2}\mathcal{W})\,. \end{cases}$$

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$$\begin{cases} \partial_t h = 0\,, \\ \partial_t(hu) = -\partial_x(hP)\,, \\ \partial_t(h\mathcal{W}) = \frac{3}{2}P + \frac{\alpha - 1}{2\alpha}gh^{3/2}\partial_x B\,, \\ \partial_t(hP) = -a^2(2\mathcal{W} + \alpha h\partial_x u)\,, \\ \partial_t(hB) = \partial_x(2h^{3/2}\mathcal{W})\,. \end{cases}$$

$$\partial_t E + \partial_x\left((E + \frac{1}{2}gh^2)u\right) = 0\,.$$

$$\partial_t E + \partial_x\left((hP + \Pi_B)u\right) = 0\,.$$

Framework
Introduction - Context and issues
Free-surface flows - Overview
Application context - Issues and difficulties
Shallow Water equations and dispersive models
Improvement of dispersive properties
Numerical treatment of the LcT model
The LcT model
Numerical scheme
Hyperbolic step
Acoustic step
Justification of the LcT model
Results and outlook

Hyperbolic step

Shallow Water with passive transport

$$\begin{cases} \partial_t h + \partial_x(hu) = 0\,, \\ \partial_t(hu) + \partial_x(hu^2 + \frac{1}{2}gh^2) = 0\,, \\ \partial_t(h\mathcal{W}) + \partial_x(hu\mathcal{W}) = 0\,, \\ \partial_t(hP) + \partial_x(huP) = 0\,, \\ \partial_t(hB) + \partial_x(huB) = 0\,. \end{cases}$$

$$\partial_t E + \partial_x\left((E + \frac{1}{2}gh^2)u\right) = 0\,.$$

Discrete counterpart of the energy equation :

$$E_K^{n+1} \leq E_K^n - \frac{\Delta t}{\Delta x}\left(\mathcal{G}_{K+1/2}^{SW} - \mathcal{G}_{K-1/2}^{SW}\right)\,.$$

Hyperbolic step

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$$E_K^{n+1} \leq E_K^n - \frac{\Delta t}{\Delta x} \left(\mathcal{G}^{SW}_{K+1/2} - \mathcal{G}^{SW}_{K-1/2} \right)\,.$$

Requirements :

- Explicit methods.
- Inclusion of topography terms.
- Minimise diffusion.
- Extension in 2d on unstructured meshes.

Continuous frame

- Hyperbolic systems

$$\partial_t w + \partial_x f(w) = 0 \quad , \quad w \in \mathbb{R}^d \subset \Omega \, .$$

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- Shallow water equations : $w = {}^t(h, hu)$.

$$\begin{cases} \partial_t h + \partial_x (hu) = 0\,, \\ \partial_t (hu) + \partial_x (hu^2 + \frac{1}{2} g h^2) = 0\,. \end{cases}$$

$$\eta(w) = \mathcal{E}_P + \mathcal{E}_K = \frac{1}{2} g h^2 + \frac{1}{2} h u^2$$

Discrete frame

- Numerical scheme

$$\tilde{\partial}_t w_{\mathfrak{h}} + \tilde{\partial}_x f(w_{\mathfrak{h}}) = 0 \quad , \quad w_{\mathfrak{h}} \in \mathbb{R}^d \subset \Omega \, .$$

▷ **C. Berthon *et al.***, *An easy control of the artificial numerical viscosity to get discrete entropy inequalities when approximating hyperbolic systems of conservation laws*, 2020.

▷ **C. Berthon *et al.***, *Improvement of the hydrostatic reconstruction scheme to get fully discrete entropy inequalities*, 2019.

Discrete frame

- Numerical scheme

$$\tilde{\partial}_t w_{\mathfrak{h}} + \tilde{\partial}_x f(w_{\mathfrak{h}}) = 0 \quad , \quad w_{\mathfrak{h}} \in \mathbb{R}^d \subset \Omega \, .$$

- Local discrete entropy estimate :

$$\tilde{\partial}_t \eta(w_{\mathfrak{h}}) + \tilde{\partial}_x G(w_{\mathfrak{h}}) \leq R_{\mathfrak{h}} \, .$$

▷ **C. Berthon *et al.***, *An easy control of the artificial numerical viscosity to get discrete entropy inequalities when approximating hyperbolic systems of conservation laws*, 2020.

▷ **C. Berthon *et al.***, *Improvement of the hydrostatic reconstruction scheme to get fully discrete entropy inequalities*, 2019.

Discrete frame

- Numerical scheme

$$\tilde{\partial}_t w_{\mathfrak{h}} + \tilde{\partial}_x f(w_{\mathfrak{h}}) = \gamma \tilde{\partial}_{xx} w_{\mathfrak{h}} \quad , \quad w_{\mathfrak{h}} \in \mathbb{R}^d \subset \Omega \,.$$

- Local discrete entropy estimate :

$$\tilde{\partial}_t \eta(w_{\mathfrak{h}}) + \tilde{\partial}_x G(w_{\mathfrak{h}}) \leq R_{\mathfrak{h}} \; -\gamma D_{\mathfrak{h}} \,.$$

- We take $\gamma = \max(0, R_{\mathfrak{h}}/D_{\mathfrak{h}})$.

▷ **C. Berthon *et al.***, *An easy control of the artificial numerical viscosity to get discrete entropy inequalities when approximating hyperbolic systems of conservation laws*, 2020.
▷ **C. Berthon *et al.***, *Improvement of the hydrostatic reconstruction scheme to get fully discrete entropy inequalities*, 2019.

The CPR approach

Back to the Shallow Water equations

$$\begin{cases} \partial_t h + \partial_x(hu) = 0\,, \\ \partial_t(hu) + \partial_x(hu^2) + h\partial_x\phi = 0\,, \end{cases} \qquad \phi = g(h+z)\,.$$

- Energy equation : $E = \frac{1}{2}hu^2 + \frac{1}{2}gh^2 + gz.$

$$\partial_t E + \partial_x\left(\left(E + \frac{1}{2}gh^2\right)u\right) = 0\,.$$

The CPR approach

Back to the Shallow Water equations

$$\begin{cases} \partial_t h + \partial_x (h(u - \delta u)) = 0\,, \\ \partial_t (hu) + \partial_x (hu(u - \delta u)) + h\partial_x \phi = 0\,, \end{cases} \qquad \phi = g(h+z)\,.$$

- Energy equation : $E = \frac{1}{2}hu^2 + \frac{1}{2}gh^2 + gz.$

$$\partial_t E + \partial_x \left(\left(E + \frac{1}{2}gh^2 \right) (u - \delta u) \right) = -\, h\partial_x \phi \delta u\,.$$

The CPR approach

Back to the Shallow Water equations

$$\begin{cases} \partial_t h + \partial_x(h(u - \delta u)) = 0\,, \\ \partial_t(hu) + \partial_x(hu(u - \delta u)) + h\partial_x\phi = 0\,, \end{cases} \qquad \phi = g(h+z)\,.$$

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$$\partial_t E + \partial_x\left(\left(E + \frac{1}{2}gh^2\right)(u - \delta u)\right) = -h\partial_x\phi\delta u\,.$$

Numerical scheme

$$h_K^{n+1} = h_K^n - \Delta t\partial_K(hu^*)$$
$$(hu)_K^{n+1} = (hu)_K^n - \Delta t\partial_K^{up}(u, hu^*) - \Delta t h_K^{n+1}\partial_K^c\phi\,.$$

- $\partial_K(hu^*) = \frac{1}{\Delta x}\left(hu^*_{K+1/2} - hu^*_{K-1/2}\right)$, $hu^*_{K-1/2} = \overline{hu}_{K+1/2} - \Pi_{K+1/2}$

$\overline{hu}_{K+1/2} = \frac{1}{2}\left(hu^n_{K+1} + hu^n_K\right)$, $\Pi_{K+1/2} = \gamma\Delta t\bar{h}^{n+1}_{K+1/2}\left(\frac{\phi^{n+1}_{K+1/2} - \phi^{n+1}_{K-1/2}}{\Delta x}\right)$

▷ **M. Parisot, J.P. Vila**, *Centered-potential regularization for the advection upstream splitting method*, 2016.

The CPR approach

Back to the Shallow Water equations

$$\begin{cases} \partial_t h + \partial_x (h(u - \delta u)) = 0\,, \\ \partial_t (hu) + \partial_x (hu(u - \delta u)) + h\partial_x \phi = 0\,, \end{cases} \qquad \phi = g(h+z)\,.$$

Explicit version

$$h_K^{n+1} = h_K^n - \Delta t \partial_K (hu^*)$$
$$(hu)_K^{n+1} = (hu)_K^n - \Delta t \partial_K^{up}(u, hu^*) - \Delta t h_K^n \partial_K \phi^*\,.$$

- $\partial_K (hu^*) = \frac{1}{\Delta x}\left(hu^*_{K+1/2} - hu^*_{K-1/2}\right)$, $hu^*_{K-1/2} = \overline{hu}_{K+1/2} - \Pi_{K+1/2}$,

$$\Pi_{K+1/2} = \gamma \Delta t \bar{h}^n_{K+1/2} \left(\frac{\phi^n_{K+1/2} - \phi^n_{K-1/2}}{\Delta x} \right).$$

- $\partial_K \phi^* = \frac{1}{\Delta x}\left(\phi^*_{K+1/2} - \phi^*_{K-1/2}\right)$, $\phi^*_{K-1/2} = \overline{\phi}_{K+1/2} - \Lambda_{K+1/2}$,

$$\Lambda_{K+1/2} = \alpha g \Delta t \left(\frac{hu_{K+1/2} - hu_{K-1/2}}{\Delta x} \right).$$

▷ **F. Couderc, A.D., J.P. Vila**, *An explicit asymptotic preserving low Froude scheme for the multilayer shallow water model with density stratification*, 2017.
▷ **A.D.**, *Revisiting energy estimates of the CPR scheme for the Shallow Water equations*, 2023.

Stability conditions

Conditions

$$\gamma, \alpha \in \mathcal{S} := \left\{ x \in \mathbb{R}\,,\, p(x) = (\xi_{K+1/2})^2 x^2 - x + 1 \leq 0 \right\}\ ,$$
$$\xi_{K+1/2} = 2\frac{\Delta t}{\Delta x}\sqrt{g h_{K+1/2}}\,.$$

$$\Delta \geq 0 \Leftrightarrow \frac{\Delta t}{\Delta x}\sqrt{gh} \leq \frac{1}{4}\ \text{(Condition CFL)}\ .$$

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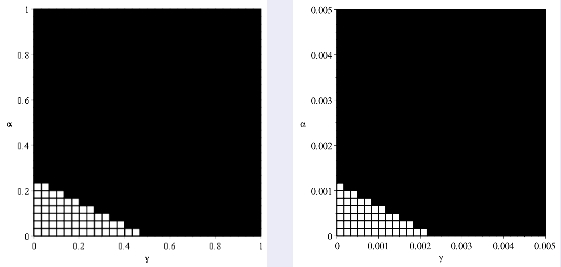
Choice of viscosity parameters :

▶ $2 \in \mathcal{S} : \alpha = \gamma = 2$ ensures stability.

▶ Search for optimal conditions : linear stability analysis and numerical exploration.

Stability cartography

First and second order time schemes (1/2-CFL)

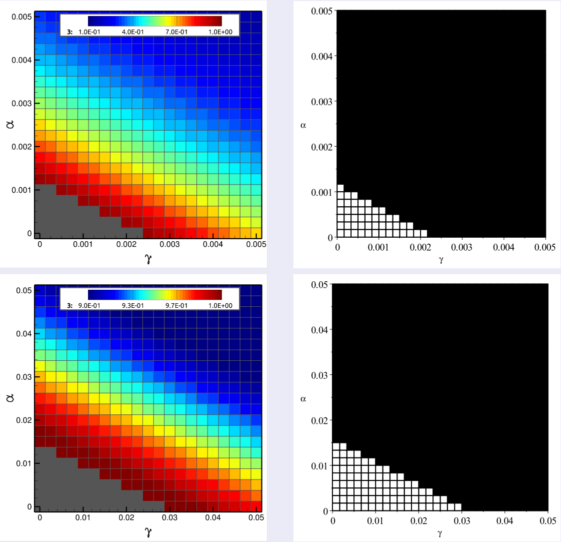


- Linear stability condition : $\tilde{\alpha} + \gamma \geq 0.5$.
- The (1/4-CFL) condition seems not optimal, as well as

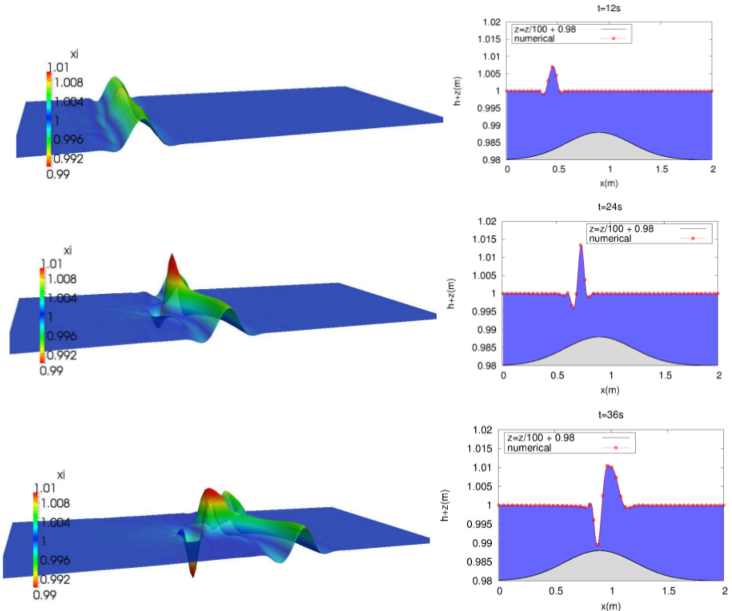
$\alpha, \gamma \geq 2$.

Stability cartography

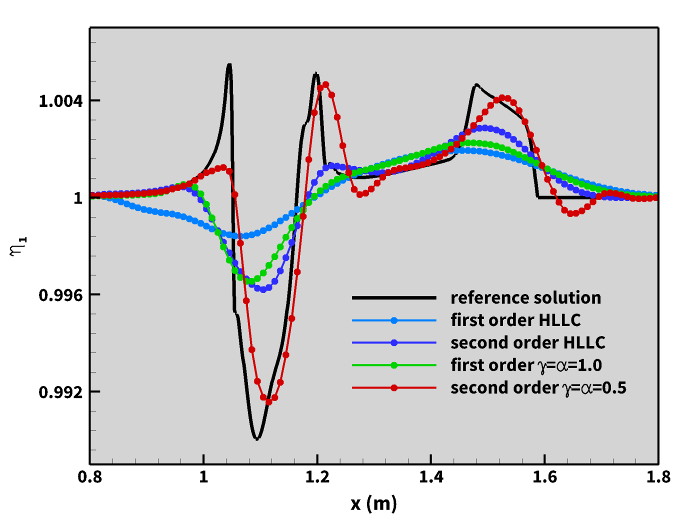
Linear stability anlaysis vs. numercis. First and second order space schemes, RK2 time scheme



Small perturbation of a steady state (1)



Small perturbation of a steady state (2)



Acoustic step (1)

Acoustic system

$$\begin{cases} \partial_t h = 0\,, \\ \partial_t(hu) = -\partial_x(hP)\,, \\ \partial_t(h\mathcal{W}) = \dfrac{3}{2}P + \dfrac{\alpha - 1}{2\alpha} g h^{3/2}\partial_x B\,, \\ \partial_t(hP) = -a^2(2\mathcal{W} + \alpha h \partial_x u)\,, \\ \partial_t(hB) = \partial_x(2h^{3/2}\mathcal{W})\,. \end{cases}$$

- Energy equation :

$$\partial_t E + \partial_x\left((hP + \Pi_B)u\right) = 0\,.$$

- Discrete counterpart :

$$E_K^{n+1} \leq E_K^n - \frac{\Delta t}{\Delta x}\left(\mathcal{G}_{K+1/2}^{ac} - \mathcal{G}_{K-1/2}^{ac}\right)\,.$$

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Reformulation

$$\begin{cases} \partial_t h = 0\,, \\ \partial_t u = -\dfrac{1}{h}\partial_x(hP)\,, \\ \partial_t \mathcal{W} = \dfrac{3}{2}\dfrac{P}{h} + \dfrac{\alpha - 1}{2\alpha} g\sqrt{h}\partial_x B\,, \\ \partial_t P = -a^2\left(2\dfrac{\mathcal{W}}{h} + \alpha\partial_x u\right)\,, \\ \partial_t B = \dfrac{1}{h}\partial_x(2h^{3/2}\mathcal{W})\,. \end{cases}$$

Acoustic step (2)

Reformulation

$$\begin{cases} \partial_t h = 0\,, \\ \partial_t u = -\dfrac{1}{h}\partial_x(hP)\,, \\ \partial_t \mathcal{W} = \dfrac{3}{2}\dfrac{P}{h} + \dfrac{\alpha - 1}{2\alpha} g\sqrt{h}\partial_x B\,, \\ \partial_t P = -a^2\left(2\dfrac{\mathcal{W}}{h} + \alpha\partial_x u\right)\,, \\ \partial_t B = \dfrac{1}{h}\partial_x(2h^{3/2}\mathcal{W})\,. \end{cases}$$

Numerical scheme

$$\begin{cases} \dfrac{u_K^{n+1} - u_K^n}{\Delta t} = -\dfrac{1}{h_K}\partial_K^c(hP^{n+1})\,, \\ \dfrac{\mathcal{W}_K^{n+1} - \mathcal{W}_K^n}{\Delta t} = \dfrac{3}{2}\dfrac{P_K^{n+1}}{h_K} + \dfrac{\alpha - 1}{2\alpha} g\sqrt{h_K}\partial_K^* B\,, \\ \dfrac{P_K^{n+1} - P_K^n}{\Delta t} = -a^2\left(2\dfrac{\mathcal{W}_K^{n+1}}{h_K} + \alpha\partial_K^* u\right)\,, \\ \dfrac{B_K^{n+1} - B_K^n}{\Delta t} = \dfrac{1}{h_K}\partial_K^c(2h^{3/2}\mathcal{W}^{n+1})\,. \end{cases}$$

Acoustic step (3)

Numerical scheme

$$\begin{cases} \dfrac{u_K^{n+1}-u_K^n}{\Delta t} = -\dfrac{1}{h_K}\partial_K^c(hP^{n+1}), \\ \dfrac{\mathcal{W}_K^{n+1}-\mathcal{W}_K^n}{\Delta t} = \dfrac{3}{2}\dfrac{P_K^{n+1}}{h_K} + \dfrac{\alpha-1}{2\alpha} g\sqrt{h_K}\partial_K^* B, \\ \dfrac{P_K^{n+1}-P_K^n}{\Delta t} = -a^2\left(2\dfrac{\mathcal{W}_K^{n+1}}{h_K} + \alpha\partial_K^* u\right), \\ \dfrac{B_K^{n+1}-B_K^n}{\Delta t} = \dfrac{1}{h_K}\partial_K^c(2h^{3/2}\mathcal{W}^{n+1}). \end{cases}$$

$$\partial_K^* B = \frac{1}{\Delta x}\left(B^*_{K+1/2} - B^*_{K-1/2}\right), \quad B^*_{K+1/2} = \bar{B}_{K+1/2} - c_B\frac{\Delta t}{\Delta x}\left[h^{3/2}W\right]_{K+1/2}.$$

$$\partial_K^* u = \frac{1}{\Delta x}\left(u^*_{K+1/2} - u^*_{K-1/2}\right), \quad u^*_{K+1/2} = \bar{u}_{K+1/2} - c_u\frac{\Delta t}{\Delta x}\left[hP\right]_{K+1/2}.$$

- Step 1 : **Explicit** resolution of $\mathcal{W}$ and P.
- Step 2 : Evolution of B and u.

Stability under the CFL condition : $\dfrac{\Delta t}{\Delta x} a\sqrt{\alpha} \leq 1/2.$

Justification of the LcT model

Symmetrizable systems

$$S_0(U)\partial_t U + \sum_{i=1}^{d} S_i(U)\partial_{x_i} U + aL^{\delta}.U = G(U)\,.$$

- Classical frame : control of the solution in H_s.

Justification of the LcT model

Symmetrizable systems

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- Classical frame : control of the solution in H_s.
- Singular limit problem - three scales : $1, a, \delta$, $(\delta = \sqrt{\mu})$.

Uniform control of :

$$\mathcal{E}_s(U) = \sum_{j=0}^{m} \|\partial_t^j U\|^2_{H^{s-j}} + \sum_{j=m+1}^{s} (a\delta)^{m-j} \|\partial_t^j U\|^2_{H^{s-j}}\,.$$

with respect to $0 < 1/a \leq \delta$.

▷ **V. Duchêne**, *Rigorous justification of the Favrie– Gavrilyuk approximation to the Serre– Green–Naghdi model*, 2019.

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▷ **V. Duchêne**, *Rigorous justification of the Favrie– Gavrilyuk approximation to the Serre– Green–Naghdi model*, 2019.

Objectives : (with K. Msheik and V. Duchêne)

- Application to the 2d LcT model with topography.
- Relax conditions on the initial data.

$$\tilde{\mathcal{E}}_s(U) = \sum_{i+j=0}^{s} \alpha_{i,j}^{-2} \|\partial_t^j U\|_{H^i}^2\,.$$

Outline

Perspectives on the Tolosa code

Numerical analysis and schemes

		$\partial_x z = 0$	mild slope	full system
1D	(LcT)	✓	✓	✓
	$(LcT)_\alpha^{cons}$	✓	✓	✗
2D NS	(LcT)	✓	(✓)	✗
	$(LcT)_\alpha^{cons}$	(✓)	(✓)	✗

Work in progress

- Numerical validations (with F. Couderc).
- Comparaisons SGN vs LcT (with F. Marche).
- Justification of the LcT model (with V. Duchêne, K. Msheik).
- Two-layer extension (with G. Richard, K. Msheik, ...).
- Wave-breaking (PHD of Y. C. Hung, Chambéry).
- High order extension (with D. Le Roux (dG), MOOD).

Perspectives on the Tolosa code

Numerical analysis and schemes

		$\partial_x z = 0$	mild slope	full system
1D	(LcT)	✓	✓	✓
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2D NS	(LcT)	✓	(✓)	✗
	$(LcT)_\alpha^{cons}$	(✓)	(✓)	✗

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MERCI !