Long-time behaviour in the 2D inhomogeneous incompressible fluids near a stably stratified Couette flow

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EPFL

Outline

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2D inhomogeneous incompressible Euler

The inhomogeneous and incompressible Euler equations are

$$\begin{aligned} &\partial_t \rho + \boldsymbol{u} \cdot \nabla \rho = 0, \qquad (x, y) \in \mathbb{T} \times \mathbb{R}, \ t \geq 0 \\ &\rho(\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}) + \nabla \mathrm{P} = -\rho(0, g), \\ &\mathrm{div}(\boldsymbol{u}) = 0. \end{aligned}$$

- Well posedness? Problem: for $w := \nabla^\perp \cdot \boldsymbol{u} = \partial_x u^y - \partial_y u^x$ there is the *baroclinic vorticity production* by $\rho^{-2} \nabla^\perp \rho \cdot \nabla \mathrm{P}$

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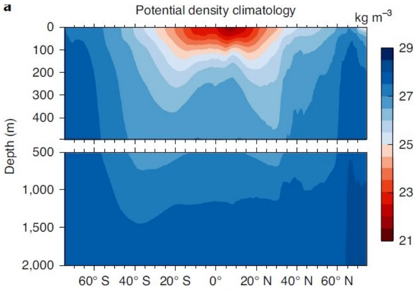
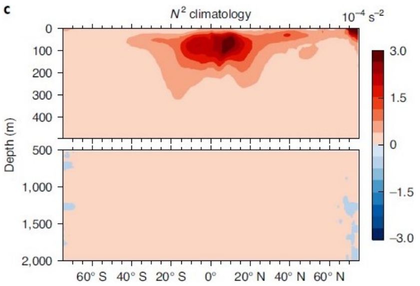
- *Boussinesq approximation* (A. Oberbeck 1879 and J. Boussinesq 1903):

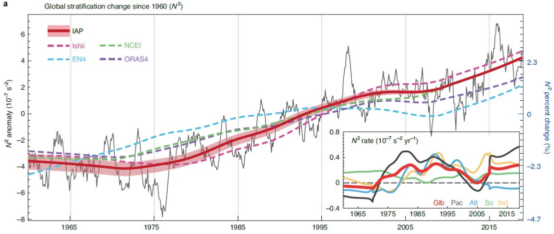
$$\bar{\rho}(\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}) + \nabla \mathrm{P} = -\rho(0, g)$$

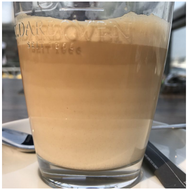
  with $\bar{\rho}$ constant
- Well posedness Boussinesq? ...Blow-up Elgindi '19, Chen/Hou '20,'22, Elgindi/Pasqualotto '23...
- Long-time behavior Boussinesq? ...Elgindi/Widmayer '14, Castro/Cordoba/Lear '18, Kukavica/Wang '19, Kiselev/Park/Yao '22 Zillinger '20, Masmoudi/Said-Houari/Zhao '20...

Stable steady state: $\boldsymbol{u}_E = 0,\ \rho_E'(y) < 0$









Images from G. Li et al. *Nature climate change* '20 (not the coffee).

Perturbing a stably stratified shear flow

Consider the equilibrium

$$\boldsymbol{u}_E = (U(y), 0), \qquad \rho_E = \mathrm{e}^{-by} \qquad (\textit{Boussinesq}: \rho_E = 1 - \gamma y)$$

with $b > 0$ $(\gamma > 0)$. Let $w = -U'(y) + \omega$, $\rho = \rho_E + \tilde{\rho}$. Define

$$\theta = \frac{\tilde{\rho}}{-\rho_E'}, \quad \beta^2 = -\frac{\rho_E'}{\rho_E} g = bg \qquad \left(\theta = \frac{\tilde{\rho}}{-\rho_E'}, \quad \beta^2 = \gamma g\right)$$

$$(\partial_t + U(y)\partial_x)\theta = \partial_x \psi + \mathrm{NL}_\theta$$
$$(\partial_t + U(y)\partial_x)(\omega - b\partial_x\psi) = -\beta^2 \partial_x \theta + (U'' - bU')\partial_x \psi + \mathrm{NL}_\omega$$
$$\Delta\psi = \omega, \qquad \boldsymbol{v} = \nabla^\perp \psi, \qquad (b = 0)$$

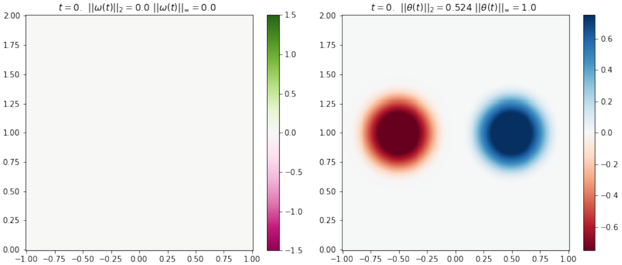
Goal: study the long-time behavior of (ω, θ).

Outline

Linearized Boussinesq Couette: $t = 0$

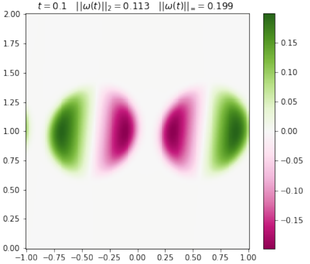
Let us consider the Boussinesq case $b = 0$ with $U(y) = y$ (the Couette flow).

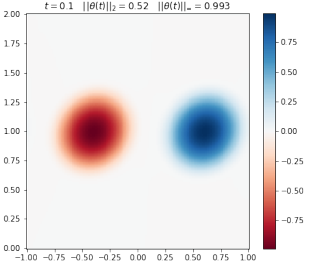
$$(\partial_t + y\partial_x)\theta = \partial_x \Delta^{-1}\omega,$$
$$(\partial_t + y\partial_x)\omega = -\beta^2 \partial_x \theta,$$
$$\omega^{in} = 0,$$
$$\left\|\theta^{in}\right\|_{L^2} = 0.524,$$

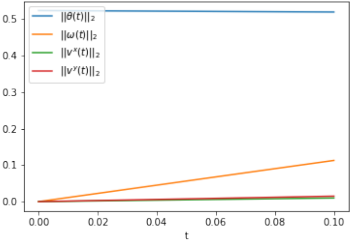


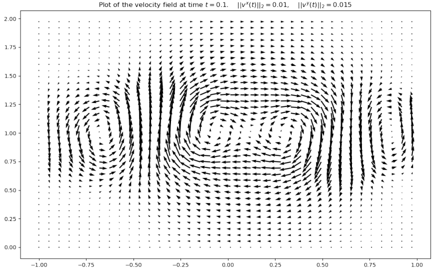
$t = 0.1$

$$(\partial_t + y\partial_x)\theta = \partial_x \Delta^{-1}\omega,$$
$$(\partial_t + y\partial_x)\omega = -\beta^2 \partial_x \theta.$$



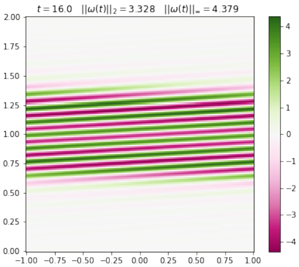


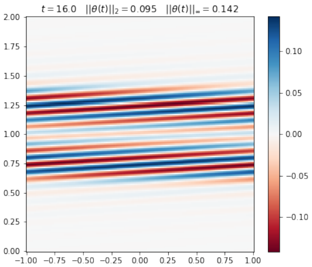




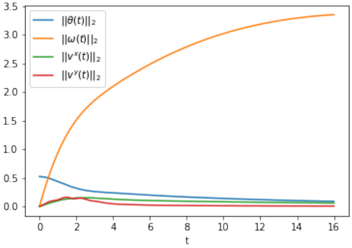
Plots of $\|\theta\|_2, \|\omega\|_2, \|v^x\|_2, \|v^y\|_2$ on the left, velocity field on the right.

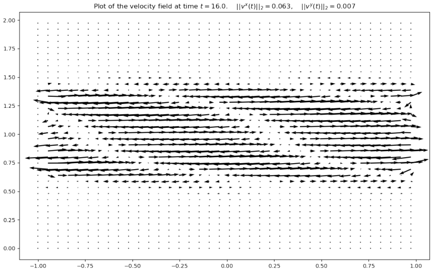
$t = 16$





$$(\partial_t + y\partial_x)\theta = \partial_x \Delta^{-1}\omega,$$
$$(\partial_t + y\partial_x)\omega = -\beta^2 \partial_x \theta.$$





Plots of $\|\theta\|_2, \|\omega\|_2, \|v^x\|_2, \|v^y\|_2$ on the left, velocity field on the right.

Theorem (Bianchini/Coti Zelati/D '20)

Let $\beta^2 > 1/4$, $b \geq 0$, $\rho^{in}, \mathbf{v}^{in} \in H^{10}(\mathbb{T}\times\mathbb{R})$. *Assuming that* $U(y) \approx y$, *i.e.*

$$\|U' - 1\|_{H^6} + \|U''\|_{H^5} \leq \varepsilon, \qquad \textit{we have}$$

$$\|(\theta - \langle\theta\rangle_x)(t)\|_{L^2} + \|(v^x - \langle v^x\rangle_x)(t)\|_{L^2} + (1+t)\,\|v^y(t)\|_{L^2} \lesssim \frac{C^{in}}{(1+t)^{1/2-\sqrt{\varepsilon}}}.$$

When $U(y) = y$, *i.e.* $\varepsilon = 0$, *we have*

$$\|(\omega - \langle\omega\rangle_x)(t)\|_{L^2} + \|\nabla(\theta - \langle\theta\rangle_x)(t)\|_{L^2} \approx c^{in}\sqrt{1+t}.$$

- Upper bounds Boussinesq: Hartman '75 with asymptotics of hypergeometric functions for Couette (rigorous by Yang/Lin '18).
  Nualart/Coti Zelati '23 also in the channel $\mathbb{T}\times[0,1]$ with spectral method.
- Proof based on energy method in the spirit of Antonelli/D/Marcati '20 for compressible fluids around Couette with constant density.
  Works in other models as well (also with dissipation): 2D NS-Boussinesq Zhai-Zhao, 2D NS-MHD D 23, 3D NS-Boussinesq Coti Zelati/Del Zotto/Widmayer '24

Linearized Boussinesq Couette: Fourier analysis

$$
\begin{aligned}
&(\partial_t + y\partial_x)\partial_x\theta = \partial_{xx}\Delta^{-1}\omega, \qquad (x,y)\in\mathbb{T}\times\mathbb{R}\\
&(\partial_t + y\partial_x)\omega = -\beta^2\partial_x\theta.
\end{aligned}
$$

Change of coordinates: $z = x - yt, v = y$. Ω, Θ in the new frame.

Fourier transform: $\Omega(z,v) = (2\pi)^{-1}\sum_{k\in\mathbb{Z}} e^{ikz}\int_{\mathbb{R}} e^{i\eta v}\widehat{\Omega}_k(\eta)\mathrm{d}\eta$.

$$
\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\widehat{\partial_z\Theta}_k\\ \widehat{\Omega}_k\end{pmatrix} = \begin{pmatrix}0 & \dfrac{1}{p_k(t,\eta)}\\ -\beta^2 & 0\end{pmatrix}\begin{pmatrix}\widehat{\partial_z\Theta}_k\\ \widehat{\Omega}_k\end{pmatrix}
$$

where p is the symbol of $\partial_{xx}^{-1}\Delta$ in the new coordinates

$$
p_k(t,\eta) = 1 + (\eta/k - t)^2
$$

The symmetrization scheme

$p_k(t,\eta) = 1 + (\eta/k - t)^2$. Define

$$Z(t) = (p^{-1/4}\widehat{\Omega})_k(t,\eta), \qquad Q(t) = \beta(p^{1/4}\widehat{\partial_z\Theta})_k(t,\eta).$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} Q(t) \\ Z(t) \end{pmatrix} = \begin{pmatrix} d(t) & a(t) \\ -a(t) & -d(t) \end{pmatrix}\begin{pmatrix} Q(t) \\ Z(t) \end{pmatrix}, \quad d(t) = \frac{1}{4}\frac{\partial_t p}{p}, \quad a(t) = \frac{\beta}{\sqrt{p}}.$$

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For $k \neq 0$, let

$$E(t) = \frac{1}{2}\left(|Z|^2 + |Q|^2 + 2\frac{d}{a}\mathrm{Re}(Z\overline{Q})\right)(t)$$

Since $|d/a| \leq 1/(2\beta)$, E is coercive if $\beta^2 > 1/4$ (Miles-Howard criterion). With a Grönwall type estimate we deduce

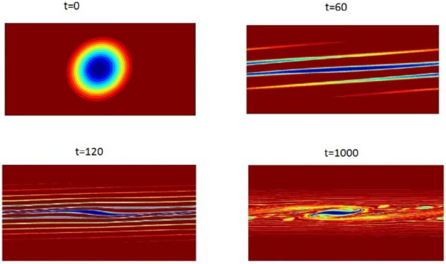
$$c_\beta E^{in} \leq E(t) \leq C_\beta E^{in}, \quad \Longrightarrow \quad |Z(t)| + |Q(t)| \approx_\beta E^{in}$$

Outline

Orr mechanism and the echoes in a nutshell

Since $\mathrm{div}(\boldsymbol{v}) = 0 \implies \boldsymbol{v} \cdot \nabla = v_0^x \partial_x + \boldsymbol{v}_{\neq} \cdot \nabla$. 2D Euler around Couette is

$$\partial_t \omega + (y + v_0^x(t,y))\partial_x \omega = -\boldsymbol{v}_{\neq} \cdot \nabla \omega, \qquad \boldsymbol{v} = \nabla^{\perp} \Delta^{-1} \omega.$$



An echo. Numerics by Shnirelman '13.
See also Vanneste et al. '98.
High-to-low (inverse) frequencies cascade

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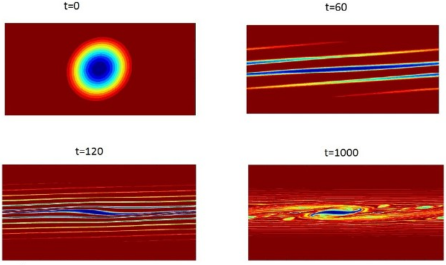


An echo. Numerics by Shnirelman '13. See also Vanneste et al. '98.

Toy model used by Bedrossian/Masmoudi '13
See also Ionescu/Jia '19, '20, Masmoudi/Zhao '20

Toy Model: $z = x - yt$. Then $\nabla^\perp \Delta^{-1}\omega \cdot \nabla\omega \to \nabla^\perp \Delta_L^{-1}\Omega \cdot \nabla\Omega$.
Approximation:

$$\partial_t \widehat{\Omega}_k \approx \mathcal{F}(\partial_y \Delta_L^{-1}\Omega\ \partial_z \Omega)_k.$$

Bad term for $t \approx \eta/k$ and $\eta/k^2 \gg 1$

$$\mathcal{F}(\partial_v \Delta_L^{-1}\Omega)_k = \frac{\eta}{k^2}\frac{1}{1+|\eta/k - t|^2}\widehat{\Omega}_k.$$

High-to-low cascade $k \to k-1 \to \dots 1$

$$\left(\frac{\eta}{k^2}\right)\left(\frac{\eta}{(k-1)^2}\right)\dots\left(\frac{\eta}{1^2}\right) \approx \mathrm{e}^{\sqrt{\eta}}.$$

Rewrite 2D Euler-Boussinesq as

$$\partial_t\theta + (y + v_0^x(t,y))\partial_x\theta = \partial_x\psi - \boldsymbol{v}_{\neq}\cdot\nabla\theta,$$
$$\partial_t\omega + (y + v_0^x(t,y))\partial_x\omega = -\beta^2\partial_x\theta - \boldsymbol{v}_{\neq}\cdot\nabla\omega,$$
$$\Delta\psi = \omega, \qquad \boldsymbol{v} = \nabla^\perp\psi.$$

Consider an initial perturbation $\theta^{in}, \omega^{in} \approx \varepsilon$ (in some space).

- v_0^x: perturbative at most on a time-scale $O(\varepsilon^{-1})$. Change of coordinates

$$z = x - vt, \qquad v = y + \frac{1}{t}\int_0^t v_0^x(\tau, y)\mathrm{d}\tau, \qquad \text{(mix Eulerian \& Lagrangian)}$$

- $\boldsymbol{v}_{\neq}$: echoes instability (at least). Some decay from inviscid damping.
- From the linearized dynamics, $\omega, \nabla\theta \approx t^{1/2}$
  On a time-scale $O(\varepsilon^{-2})$ they might become of size $O(1)$.
  Out of a perturbative regime.

Claim: the linearized behavior persists up to $t = O(\varepsilon^{-2})$.

Fix $s > 1/2$, the Gevrey-$1/s$ is $\|f\|^2_{\mathcal{G}^\lambda} = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} e^{2\lambda(|k|+|\eta|)^s} |\widehat{f}_k(\eta)|^2 \mathrm{d}\eta$.

Theorem (Bedrossian/Bianchini/Coti Zelati/D '21)

Let $\lambda_0 > \lambda' > 0$ *and* $0 < \varepsilon \ll \delta < 1$. *Assume* $\|\boldsymbol{v}^{in}\|_{L^2} + \|\omega^{in}\|_{\mathcal{G}^{\lambda_0}} + \|\theta^{in}\|_{\mathcal{G}^{\lambda_0}} < \varepsilon$. *For all* $0 \le t \le \delta^2\varepsilon^{-2}$ *we have* $\|v_0^x(t)\|_{\mathcal{G}^{\lambda'}} \lesssim \varepsilon$ *and*

$$\|\omega(t, x+vt, y)\|_{\mathcal{G}^{\lambda'}} + (1+t)\|(\partial_x\theta)(t, x+vt, y)\|_{\mathcal{G}^{\lambda'}} \lesssim \varepsilon(1+t)^{\frac{1}{2}},$$

$$\|\partial_x\theta(t)\|_{L^2} + \|(v^x - \langle v^x\rangle_x)(t)\|_{L^2} + (1+t)\|v^y(t)\|_{L^2} \lesssim \frac{\varepsilon}{(1+t)^{\frac{1}{2}}}.$$

There exists $K = K(\beta, \lambda_0, s) > 0$ *such that if* $\|\omega^{in}\|_{H^{-1}} + \|\theta^{in}\|_{L^2} \ge K\delta\varepsilon$ *then*

$$\|(\omega - \langle\omega\rangle_x)(t)\|_{L^2} + \|\nabla(\theta - \langle\theta\rangle_x)(t)\|_{L^2} \approx \varepsilon(1+t)^{\frac{1}{2}}.$$

- Stratified fluids without gravity: same setting of us, modified scattering ($t \to \infty$) and nonlinear inviscid damping Chen/Wei/Zhang/Zhang '23. Zhao '23 in $\mathbb{T} \times [0,1]$, general strictly monotone shear and density ρ_E.

The nonlinear change of coordinates

The shear flow profile is now $y + u_0^x(t,y)$. The natural change of coordinates is

$$z = x - vt, \qquad v = y + \frac{1}{t}\int_0^t u_0^x(\tau,y)\mathrm{d}\tau$$
$$\dot{v} := \partial_t v, \quad v' := \partial_y v, \quad \nabla = (\partial_z, \partial_v),$$
$$\Delta_{NL} := \partial_{zz} + (v')^2(\partial_v - t\partial_z)^2 + v''(\partial_v - t\partial_z).$$

Let $(\Omega, \Theta, \Psi)(t,z,v) = (\omega, \theta, \psi)(t,x,y)$. Then

$$\partial_t\Theta = \partial_z\Psi - \boldsymbol{U}\cdot\nabla\Theta,$$
$$\partial_t\Omega = -\beta^2\partial_z\Theta - \boldsymbol{U}\cdot\nabla\Omega,$$
$$\boldsymbol{U} := (0,\dot{v}) + v'\nabla^{\perp}\Psi_{\neq}, \qquad \Delta_{NL}\Psi = \Omega.$$

Remark: Equations for the coordinate change are the same as 2D Euler, but

$$\partial_t(t(1-v')) = -\omega_0 \lesssim \varepsilon t^{\frac{1}{2}} \quad \Longrightarrow \quad |1-v'| \lesssim \varepsilon t^{\frac{1}{2}}$$

Nonlinear Symmetrization scheme

Recall that $p_k(t,\eta) = 1 + (\eta/k - t)^2$ and $Z = p^{-1/4}\widehat{\Omega}$, $Q = \beta p^{1/4}\widehat{\partial_z \Theta}$.

$$\partial_t Z = -\frac{1}{4}\frac{\partial_t p}{p} Z - \frac{\beta}{\sqrt{p}} Q - p^{-1/4}\mathcal{F}(\boldsymbol{U}\cdot\nabla\Omega)$$

$$\partial_t Q = \frac{\beta}{\sqrt{p}} Z + \frac{1}{4}\frac{\partial_t p}{p} Q - \beta p^{1/4}\mathcal{F}(\partial_z(\boldsymbol{U}\cdot\nabla\Theta)) - \beta p^{3/4}\mathcal{F}((\Delta_{NL} - \Delta_L)\Psi)$$

- **Goal**: $\|Z(t)\|_{\mathcal{G}^\lambda} + \|Q(t)\|_{\mathcal{G}^\lambda} \lesssim \varepsilon$. A direct estimate does not work (there are regularity losses).
- Blue term is subtle: upper bound $O(\varepsilon^2 t^{-1/2})$. Integrated in time $O(\varepsilon^2 t^{1/2}) = O(\varepsilon\delta)$.
- Red terms are clearly dangerous: toy model to estimate high-to-low cascade.
- Based on toy model, design a weight $A_k(t,\eta)$ to weaken the norms in energy estimates (e.g. Nirenberg, Foias/Temam, Alinhac...).
  Weight $\sim$ artificial damping: choose $A > 0$ s.t. $\partial_t A < 0$:

$$\partial_t(AZ) = -|\partial_t A/A|(AZ) + A\partial_t Z$$

- Use the **linear energy functional** (at least).

Nonlinear growth: towards a Toy model

Consider the approximations

$$\partial_t Z_k \approx p^{1/4}\mathcal{F}(\nabla^\perp \Delta_L^{-1}\Omega \cdot \nabla\Omega)_k \approx p^{1/4}\mathcal{F}(\partial_v \Delta_L^{-1}\Omega\ \partial_z\Omega)_k.$$

Note that

$$\mathcal{F}(\partial_v \Delta_L^{-1}\Omega)_k = \frac{\eta}{k^2}\frac{1}{1+|t-\eta/k|^2}\widehat{\Omega}_k = \frac{\eta}{k^2}\frac{1}{(1+|t-\eta/k|^2)^{3/4}}Z_k$$

The Orr mechanism: if $\eta/k^2 \gg 1$ at time $t \approx \eta/k$ we have a growth.

Paraproduct: $fg = f_{Hi}g_{lo} + f_{lo}g_{Hi} + \mathcal{R}$. We are concerned with

$$\partial_t Z_k \approx p^{1/4}\mathcal{F}((\partial_v \Delta_L^{-1}\Omega)_{Hi}\ (\partial_z\Omega)_{lo})_k,$$

since we expect $(\partial_v \Delta_L^{-1}\Omega)_{lo} \approx \varepsilon t^{-3/2}$.

The toy model: resonant vs non-resonant

- $(\partial_z \Omega)_{lo} \approx \varepsilon t^{\frac{1}{2}}$: concentrated at frequencies $k \pm 1$ and $\eta = 0$.
- *High-to-low* cascade. Interactions $k \to k-1$. For times $|t - \eta/k| \leq \eta/k^2$

$$\partial_t Z_k = \left(\frac{k^2}{\eta}\right)^{1/2} \frac{\varepsilon t^{\frac{1}{2}}}{(1 + |t - \eta/k|^2)^{1/4}} Z_{k-1}$$

$$\partial_t Z_{k-1} = \left(\frac{\eta}{k^2}\right)^{1/2} \frac{\varepsilon t^{\frac{1}{2}}}{(1 + |t - \eta/k|^2)^{3/4}} Z_k$$

- When $\varepsilon t^{\frac{1}{2}} \leq 1$, maximal growth of order $(\eta/k^2)^c$.
  Then, $k-1 \to k-2$ at time $t \approx \eta/(k-1)$ will grow $(\eta/(k-1)^2)^c$. Overall

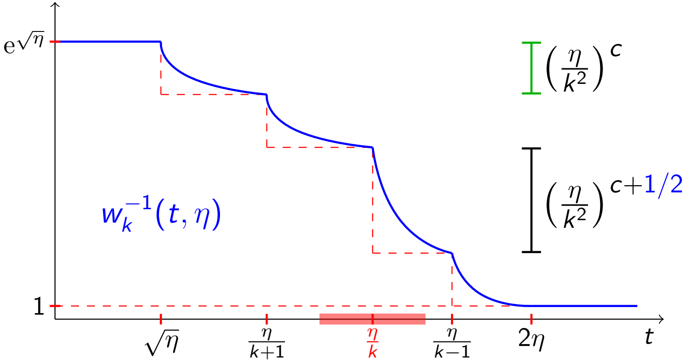
$$\left(\frac{\eta}{k^2}\frac{\eta}{(k-1)^2}\cdots\frac{\eta}{1}\right)^c = \left(\frac{\eta^k}{(k!)^2}\right)^c \approx \frac{1}{\sqrt{\eta}} e^{c\sqrt{\eta}}, \quad \text{when } k = \sqrt{\eta}$$

- Our toy model predicts a different regularity unbalance between Z_k and Z_{k-1} w.r.t. the homogeneous case (B/M '13).
  In Zillinger '22 a more refined toy model potentially usefull for $\varepsilon^{-2} < t < \varepsilon^{-q}$ (with more regularity).

Let $\sigma > 10$, $s > 1/2$. The weight A is chosen as follows

$$A_k(t,\eta) = \langle |k| + |\eta| \rangle^{\sigma} e^{\lambda(t)(|k|+|\eta|)^s} (m^{-1} w^{-1})_k(t,\eta),$$

where $\partial_t \lambda = -1/\langle t \rangle^{1+\delta}$, $m_k(t,\eta) = \exp(C_\beta \arctan(t - \eta/k))$.
w^{-1} is built on the toy model (different regularity unbalances w.r.t. B/M '13).



- w is for nonlinear errors, m for the linear ones. Useful when $\sqrt{\eta} \leq t \leq 2\eta$
- $\lambda(t)$ is "classical" and useful when $t \leq \sqrt{\eta}$ and $t \geq 2\eta$.

Energy functionals and their bounds to bootstrap

Energy estimates are made in a bootstrap scheme up to times $t \leq \delta^2\varepsilon^{-2}$.

- The **linear** energy functional (coercive if $\beta^2 > 1/4$).

$$E_L(t) = \frac{1}{2}\left(\|AZ\|_{L^2}^2 + \|AQ\|_{L^2}^2 - \frac{1}{2\beta}\left\langle \frac{\partial_t p}{\sqrt{p}} AZ, AQ \right\rangle_{L^2}\right).$$

  Goal: $E_L(t) \lesssim \varepsilon^2$.
  Bounds on Z, Q are not enough to control the nonlinearities.

- The *natural* nonlinear energy ($\nabla_L = (\partial_z, \partial_v - t\partial_z)$)

$$E_n(t) = \frac{1}{2}\left(\|A\Omega\|_{L^2}^2 + \beta^2\|A\nabla_L\Theta\|_{L^2}^2\right).$$

  Goal: $E_n(t) \lesssim \varepsilon^2 t \lesssim \delta^2$.
  At the highest level of regularity. Control on Z, Q is crucial to bound E_n.

- Energy for the coordinate change $E_v(t) = \dots$ (awful).
  Goal: $E_v(t) \lesssim \varepsilon^2 t \lesssim \delta^2$.

Instability

Call $\boldsymbol{X}(t) = (Z(t), Q(t))$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X}(t) = L(t)\boldsymbol{X}(t) + \mathcal{N}(t, \boldsymbol{X}(t)), \quad L(t) = \begin{pmatrix} -d(t) & -a(t) \\ a(t) & d(t) \end{pmatrix},$$

$d(t) = \frac{1}{4}(\partial_t p)/p$, $a(t) = \beta/\sqrt{p}$, $p = 1 + (\eta/k - t)^2$ and $\mathcal{N}$ contains all the nonlinearities.

- Let $\Phi_L(t, \tau)$ be the solution operator of the linear problem, we rewrite

$$\boldsymbol{X}(t) = \Phi_L(t, 0)\boldsymbol{X}(0) + \int_0^t \Phi_L(t, s)\mathcal{N}(s, \boldsymbol{X}(s))\mathrm{d}s.$$

- From the linearized analysis, point-wise in t, k, η we have

$$c_\beta|\boldsymbol{F}| \leq |\Phi_L(t, s)\boldsymbol{F}| \leq C_\beta|\boldsymbol{F}|.$$

- Combining these information with the stability part, we show that

$$\|(\omega - \langle\omega\rangle_x)(t)\|_{L^2} + \|\nabla(\theta - \langle\theta\rangle_x)(t)\|_{L^2} \approx \varepsilon \langle t \rangle^{\frac{1}{2}}$$

Outline

Possible future directions

- Energy approach for $\beta^2 \leq 1/4$?
- What happens for times $t > \delta\varepsilon^{-2}$? Zillinger '22 has a toy model that might be useful (but not with lower bounds).
- The analogous result **without** the Boussinesq approximation should be true. Can one use the method of Zhao '23 for inhomogeneous Euler without gravity to address general strictly monotone shear flows and densities in $\mathbb{T} \times [0, 1]$?
- Long time growth as $\|\omega^\theta/r\|_2 + \|rv^\theta\|_2 \approx \sqrt{t}$ in 3D axi-symmetric Euler?
- Many questions are still unanswered also in the 2D homogeneous case...