

Path large deviations for classical kinetic theories: beyond the Boltzmann, the Landau and the Balescu–Guernsey–Lenard kinetic equations

F. BOUCHET (CNRS) and O. FELIACHI - ENS de Lyon

09/2021 – MathsInFluid – Lyon.

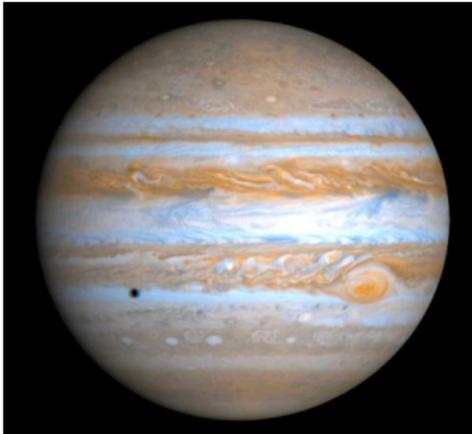


Outline

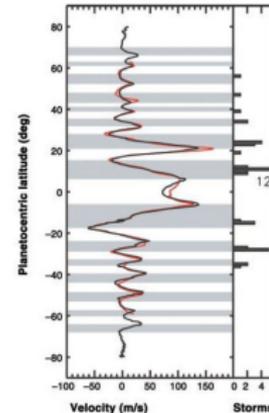
- 1 Introduction
 - Abrupt transitions in turbulent flows
 - Introduction to path large deviations for kinetic theories
- 2 Particles with long range interactions, the Landau and the Balescu–Lenard–Guernsey equations
 - Particles with long range interactions and the Vlasov equation
 - The Balescu–Guernsey–Lenard and the Landau equations
 - Sketch of a derivation of the Balescu–Lenard–Guernsey eq.
- 3 Path large deviations for the Balescu–Lenard–Guernsey equation
 - Quasilinear dynamics and large deviations for slow/fast dynamics
 - Path large deviations for quadratic forms of Gaussian processes using the Szegö–Widom theorem
 - Hamiltonian for the Balescu–Lenard–Guernsey equation

Jupiter's Zonal Jets

We look for a theoretical description of zonal jets



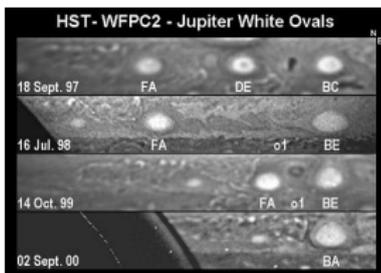
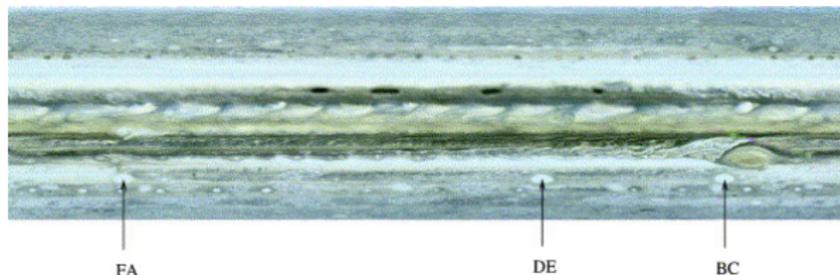
Jupiter's troposphere



Jupiter's zonal winds (Voyager and Cassini, from Porco et al 2003)

Jupiter's Abrupt Climate Change

Have we lost one of Jupiter's jets ?



Jupiter's white ovals (see Youssef and Marcus 2005)

The white ovals appeared in 1939-1940 (Rogers 1995). Following an instability of one of the zonal jets?

The Barotropic Quasi-Geostrophic Equations

- The simplest model for geostrophic turbulence.
- Quasi-Geostrophic equations with random forces

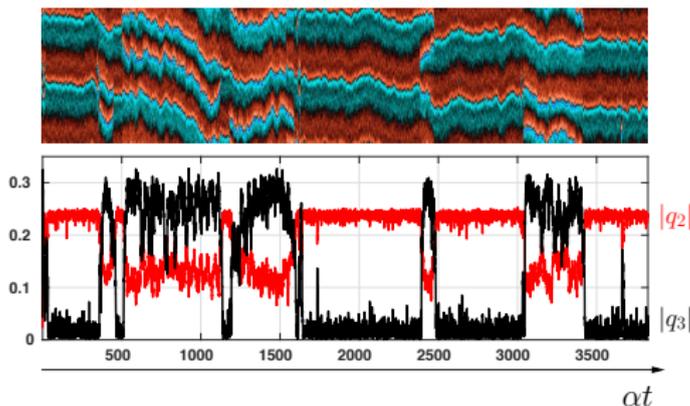
$$\frac{\partial q}{\partial t} + \mathbf{v} \cdot \nabla q = \mathbf{v} \Delta \omega - \alpha \omega + \sqrt{2\alpha} f_s,$$

where $\omega = (\nabla \wedge \mathbf{v}) \cdot \mathbf{e}_z$ is the vorticity, $q = \omega + \beta y$ is the Potential Vorticity (PV), f_s is a random Gaussian field with correlation $\langle f_s(\mathbf{x}, t) f_s(\mathbf{x}', t') \rangle = C(\mathbf{x} - \mathbf{x}') \delta(t - t')$, ε is the average energy input rate, λ is the Rayleigh friction coefficient.

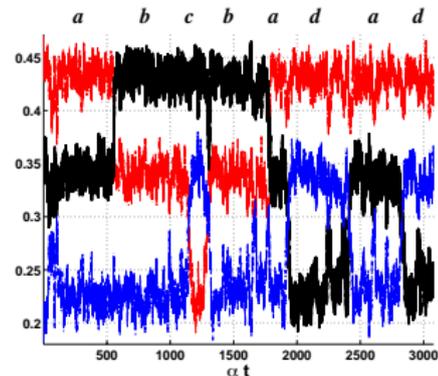
- Spin up or spin down time = $1/\alpha \ll 1$ = jet inertial time scale.
- A reasonable model for Jupiter's zonal jets.

Rare Transitions in Geostrophic Turbulent Flows

Rare transitions for quasigeostrophic jets



Bistability between 2 and 3 jets

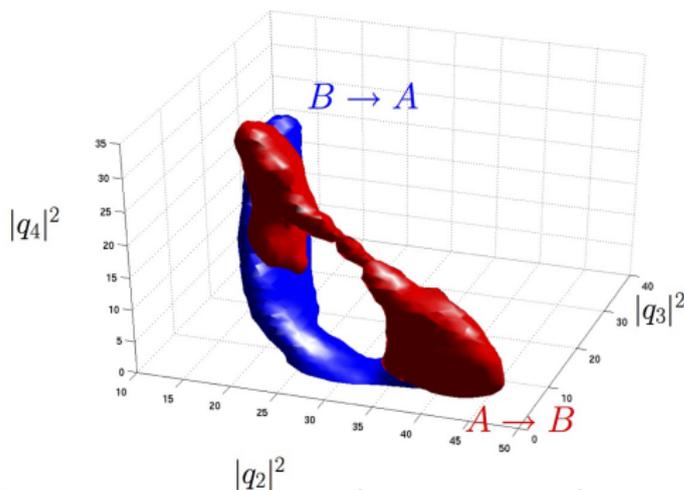


Between 3 jet states

(F. Bouchet, J. Rolland and E. Simonnet, PRL 2018 and JAS 2021)

Atmosphere Jet “Instantons” Computed using the AMS

AMS: an algorithm to compute rare events, for instance rare reactive trajectories



Transition trajectories between 2 and 3 jet states

- The dynamics of turbulent transitions is predictable.
- Asymmetry between forward and backward transitions.

Path Large Deviation for Zonal Jets

- Time scale separation. We decompose into slow (zonal flows) and fast (eddy turbulence) variables

$$v_\alpha = U_\alpha(y)e_x + \sqrt{\alpha}v_m \text{ with } U_\alpha(y)e_x \equiv \frac{1}{2\pi} \int_{\mathcal{D}} dx v_\alpha$$

- We expect:

$$\mathbb{P}[\{U_\alpha(t)\}_{0 \leq t < T} = \{U(t)\}_{0 \leq t < T}] \underset{\alpha \downarrow 0}{\asymp} \exp\left(-\frac{\sup_p \int_0^T dt \left\{ \int \dot{U} p dr dv - H[U, p] \right\}}{\alpha}\right).$$

- Can we compute H ? Through kinetic theory and large deviation theory for slow fast systems.
- We have a series of partial results on the subject, and numerical computations of some properties of H , but no complete explicit formula yet for H .

Outline

- 1 Introduction
 - Abrupt transitions in turbulent flows
 - Introduction to path large deviations for kinetic theories
- 2 Particles with long range interactions, the Landau and the Balescu–Lenard–Guernsey equations
 - Particles with long range interactions and the Vlasov equation
 - The Balescu–Guernsey–Lenard and the Landau equations
 - Sketch of a derivation of the Balescu–Lenard–Guernsey eq.
- 3 Path large deviations for the Balescu–Lenard–Guernsey equation
 - Quasilinear dynamics and large deviations for slow/fast dynamics
 - Path large deviations for quadratic forms of Gaussian processes using the Szegö–Widom theorem
 - Hamiltonian for the Balescu–Lenard–Guernsey equation

Large Deviation Theory

- **Large deviation theory** is a general framework to describe probability distributions in asymptotic limits

$$P(x, \varepsilon) = \mathbb{P}[X_\varepsilon = x] \underset{\varepsilon \ll 1}{\asymp} e^{-\frac{\mathcal{F}[x]}{\varepsilon}}.$$

For equilibrium statistical mechanics, \mathcal{F} is the free energy, and $\varepsilon = k_B T/N$.

Maths: Cramer 30', Sanov 50', Lanford 70', Freidlin–Wentzell 70' and 80', Varadhan, ... In parallel with theoretical physicists.

Feng and Kurtz book for stochastic processes.

Path Large Deviation Theory

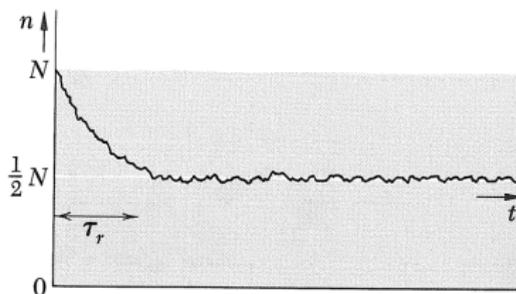
$$g_N(r, v, t) \equiv \frac{1}{N} \sum_{n=1}^N \delta(v - v_n(t)) \delta(r - r_n(t)).$$

- For many kinetic theories one expects:

$$\mathbb{P}[\{g_N(t)\}_{0 \leq t < T} = \{g(t)\}_{0 \leq t < T}] \underset{\varepsilon \downarrow 0}{\asymp} \exp\left(-\frac{\sup_p \int_0^T dt \{ \int \dot{g} p dr dv - H[g, p] \}}{\varepsilon}\right).$$

- What is ε ? Can we compute H ?
- This is a statistical field theory for the effective large scale dynamics.
- H summarizes all the relevant statistical information. This is the Holy Grail of any modern statistical mechanician.
- This gives the most probable evolution, the Gaussian fluctuations (stochastic differential or partial differential equations) and the rare events beyond Gaussian fluctuations.

Motivation 1: Joule Expansion and Large Deviations



(Figures: Reif)

- What is the probability of a dynamical rare fluctuation? The answer is not known within the classical statistical equilibrium framework.

Boltzmann Equation for dilute gases

$$\frac{\partial g}{\partial t} + v \cdot \frac{\partial g}{\partial r} = \int dv_2 dv'_1 dv'_2 w(v'_1, v'_2; v, v_2) [g(v'_1, r) g(v'_2, r) - g(v, r) g(v_2, r)].$$

- A cornerstone of physics.
- The irreversibility paradox and the 19th century controversy (Loschmidt, Zermelo, Poincaré).
- Classical explanation of the paradox by Boltzmann, theoretical physicists of the 20th century, **Lanford work (1973)**.
- **It is a very active contemporary subject both in physics and mathematics.**

The Boltzmann Equation is a Law of Large Numbers

- We consider the empirical distribution

$$g_N(r, v, t) = \frac{1}{N} \sum_{n=1}^N \delta(r - r_n(t), v - v_n(t))$$

- We consider an ensemble of initial conditions $\{r_n, v_n\}_{1 \leq n \leq N}$ where each $g_N(t=0)$ is close to g_0 .
- The Boltzmann equation is a law of large numbers:

$$\lim_{N \rightarrow \infty} g_N(t) = g(t),$$

where g solves the Boltzmann equation with $g(t=0) = g_0$.

- For large enough N , “for almost all initial conditions” and for a finite time, $g_N(t)$ remains close to $g(t)$ where g solves the Boltzmann equation with $g(t=0) = g_0$.
- We should study the probabilities of g_N , beyond the law of large numbers. May be Gaussian fluctuations, but even more interesting large deviations.

Path Large Deviations for the Boltzmann Equation

- Dynamical large deviations for the empirical distribution:

$$P[\{g_\varepsilon(t)\}_{0 \leq t < T} = \{g(t)\}_{0 \leq t < T}] \underset{\varepsilon \downarrow 0}{\asymp} \exp\left(-\frac{\sup_p \int_0^T dt \left\{ \int \dot{g} p d\mathbf{r} d\mathbf{v} - H_B[g, p] \right\}}{\varepsilon}\right).$$

ε is the inverse of the number of particles in a volume of size the mean free path.

- The large deviation Hamiltonian is $H_B = H_C + H_T$ with the collision part H_C given by

$$H_C[g, p] = \frac{1}{2} \int d\mathbf{r} d\mathbf{v}_{1,2,1',2'} w(\mathbf{v}'_1, \mathbf{v}'_2; \mathbf{v}_1, \mathbf{v}_2) g(\mathbf{r}, \mathbf{v}_1) g(\mathbf{r}, \mathbf{v}_2) \left\{ e^{[p(\mathbf{r}, \mathbf{v}_1) + p(\mathbf{r}, \mathbf{v}_2) - p(\mathbf{r}, \mathbf{v}'_1) - p(\mathbf{r}, \mathbf{v}'_2)]} - 1 \right\}.$$

F. Rezakhanlou for a simpler stochastic dynamics with Boltzmann like behavior (1998).

F. Bouchet (2020) for dilute gases.

T. Bodineau, I. Gallagher, L. Saint-Raymond and S. Simonella for a mathematical proof for short times (2020).

Motivation 3: Large Deviations and the Geometric Structure of Kinetic Equations

- In many cases (diffusion, Vlasov–Mac-Kean, and so on), kinetic equations can be seen as Gradient dynamics

$$\frac{\partial \rho}{\partial t} = -\text{Grad}_\rho \mathcal{F},$$

where the gradient is with respect to the Wasserstein distance (Otto-Villani).

- Where does this structure come from ?
- From large deviation theory. The metric is given by the dynamical large deviation Hamiltonian, \mathcal{F} is the quasipotential. There is then a natural generalization

$$\frac{\partial \rho}{\partial t} = -\text{Grad}_\rho \mathcal{F} + \mathcal{G} \text{ with } (\text{Grad}_\rho \mathcal{F}, \mathcal{G}) = 0.$$

Outline

- 1 Introduction
 - Abrupt transitions in turbulent flows
 - Introduction to path large deviations for kinetic theories
- 2 Particles with long range interactions, the Landau and the Balescu–Lenard–Guernsey equations
 - Particles with long range interactions and the Vlasov equation
 - The Balescu–Guernsey–Lenard and the Landau equations
 - Sketch of a derivation of the Balescu–Lenard–Guernsey eq.
- 3 Path large deviations for the Balescu–Lenard–Guernsey equation
 - Quasilinear dynamics and large deviations for slow/fast dynamics
 - Path large deviations for quadratic forms of Gaussian processes using the Szegö–Widom theorem
 - Hamiltonian for the Balescu–Lenard–Guernsey equation

Outline

- 1 Introduction
 - Abrupt transitions in turbulent flows
 - Introduction to path large deviations for kinetic theories
- 2 Particles with long range interactions, the Landau and the Balescu–Lenard–Guernsey equations
 - Particles with long range interactions and the Vlasov equation
 - The Balescu–Guernsey–Lenard and the Landau equations
 - Sketch of a derivation of the Balescu–Lenard–Guernsey eq.
- 3 Path large deviations for the Balescu–Lenard–Guernsey equation
 - Quasilinear dynamics and large deviations for slow/fast dynamics
 - Path large deviations for quadratic forms of Gaussian processes using the Szegö–Widom theorem
 - Hamiltonian for the Balescu–Lenard–Guernsey equation

Particles with Mean–Field Interactions

$$\frac{dr_n}{dt} = v_n \text{ and } \frac{dv_n}{dt} = -\frac{1}{N} \sum_{m=1}^N \frac{dW}{dx}(r_n - r_m). \quad (r_n \in \mathbb{R}^d \text{ or } r_n \in \mathbb{T}^d).$$

- Energy

$$H_N = \frac{1}{2} \sum_{n=1}^N \frac{v_n^2}{2} + \frac{1}{2N} \sum_{n,m=1}^N W(r_n - r_m).$$

- The empirical distribution $g_N(r, v, t) = \frac{1}{N} \sum_{n=1}^N \delta(r - r_n(t); v - v_n(t))$ formally solves the Klimontovich equation

$$\frac{\partial g_N}{\partial t} + v \cdot \frac{\partial g_N}{\partial r} - \frac{dV[g_N]}{dr} \cdot \frac{\partial g_N}{\partial v} = 0 \text{ with } V[g_N](r) = \int dr' dv W(r - r') g_N(r, v).$$

- Coulomb interaction: $W(r) = -1/r^2$ and N is the number of particles in a volume of the size of the Debye length. N is related to the plasma parameter Γ .

Vlasov Equation

- We suppose an ensemble of initial conditions $\{g_N\}$ where each g_N is close to g_0 .
- Law of large numbers: “for almost all initial conditions”
 $\lim_{N \rightarrow \infty} g_N = g$ where g solves the Vlasov equation

$$\frac{\partial g}{\partial t} + \mathbf{v} \cdot \frac{\partial g}{\partial \mathbf{r}} - \frac{dV[g]}{d\mathbf{r}} \cdot \frac{\partial g}{\partial \mathbf{v}} = 0 \text{ with } V[g](\mathbf{r}, t) = \int d\mathbf{r}' d\mathbf{v}' W(\mathbf{r} - \mathbf{r}') g(\mathbf{r}', \mathbf{v}', t).$$

- This is actually a stability result for the Vlasov (Klimontovich) equation ([Braun and Hepp, 1977](#) for smooth interactions).
- This equation is Hamiltonian, conserves the energy and an infinite number of Casimir conserved quantities.
- It could still converge to the Boltzmann distribution in a weak sense, but it does not.

Stationary Solutions of the Vlasov Equation

- The Vlasov equation has an infinite number of stable stationary solutions, for instance **homogenous solutions** $g(\mathbf{r}, \mathbf{v}) = f_0(\mathbf{v})$ such that for any \mathbf{k} and $\omega \in \mathbb{R}$, $\varepsilon(\mathbf{k}, \omega) > 0$ with

$$\varepsilon[f_0](\mathbf{k}, \omega) = 1 - \hat{W}(\mathbf{k}) \int d\mathbf{v} \frac{\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}(\mathbf{v})}{\mathbf{k} \cdot \mathbf{v} - \omega - i0^+}$$

- ε is called the dielectric susceptibility and plays the role of a dispersion relation for the linearized dynamics.
- \hat{W} is the Fourier transform of W .
- Those stable homogeneous distributions $f_0(\mathbf{v})$ play the role of attractors for the Vlasov equation.

Stationary Solutions of the Vlasov Equation

- The Vlasov equation has an infinite number of stable stationary solutions: homogeneous distributions $f_0(v)$.
- Those stable homogeneous distributions $f_0(v)$ play the role of attractors for the Vlasov equation.
- What will happen for the N particle dynamics if we start from an ensemble of initial conditions $\{f_N\}$ which is close to a homogeneous stable $f_0(v)$?
- The distribution are stable on the Vlasov time scale (of order 1), however an evolution will occur on a time scale of order $\tau = N$. This evolution is governed by the Balescu–Guernsey–Lenard equation.

Outline

- 1 Introduction
 - Abrupt transitions in turbulent flows
 - Introduction to path large deviations for kinetic theories
- 2 Particles with long range interactions, the Landau and the Balescu–Lenard–Guernsey equations
 - Particles with long range interactions and the Vlasov equation
 - The Balescu–Guernsey–Lenard and the Landau equations
 - Sketch of a derivation of the Balescu–Lenard–Guernsey eq.
- 3 Path large deviations for the Balescu–Lenard–Guernsey equation
 - Quasilinear dynamics and large deviations for slow/fast dynamics
 - Path large deviations for quadratic forms of Gaussian processes using the Szegö–Widom theorem
 - Hamiltonian for the Balescu–Lenard–Guernsey equation

The Balescu–Guernsey–Lenard Equation

- We suppose an ensemble of initial conditions $\{g_N\}$ where each g_N is close to a stable homogeneous $f_0(v)$.
- **Law of large numbers:** after time rescaling $t = N\tau$, “for almost all initial conditions”, $\lim_{N \rightarrow \infty} g_N = f$, where f solves the Balescu–Lenard–Guernsey equation

$$\frac{\partial f}{\partial \tau} = LB[f] \text{ with } LB[f] = \frac{\partial}{\partial v} \cdot \int dv_2 \overleftrightarrow{B}[f](v, v_2) \left(-\frac{\partial f}{\partial v_2} f(v) + f(v_2) \frac{\partial f}{\partial v} \right).$$

with

$$\overleftrightarrow{B}[f](v_1, v_2) = \frac{\pi}{L^3} \int_{-\infty}^{+\infty} d\omega \sum_{\mathbf{k}} \delta(\omega - \mathbf{k} \cdot \mathbf{v}_1) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_2) \frac{\mathbf{k} \mathbf{k} \hat{W}(\mathbf{k})^2}{|\mathcal{E}[f](\omega, \mathbf{k})|}.$$

- First derived by **R.L. Guernsey (1960)**. (or Bogolyubov?)
- For smooth enough W , there is an exact derivation of the Balescu–Guernsey–Lenard equation in the sense of theoretical physicists (see **Lifshitz–Pitaevskii’s book on kinetic theory**, or **Nicholson’s book on plasma**).

The Landau Equation

- The Landau equation is an approximation the Balescu–Guernsey–Lenard equation neglecting collective effects, or equivalently assuming $\varepsilon(\mathbf{k}, \omega) = 1$.
- Landau equation:

$$\frac{\partial f}{\partial \tau} = \frac{\partial}{\partial \mathbf{v}} \cdot \int d\mathbf{v}_2 \overleftrightarrow{\mathbf{B}}(\mathbf{v}, \mathbf{v}_2) \left(-\frac{\partial f}{\partial \mathbf{v}_2} f(\mathbf{v}) + f(\mathbf{v}_2) \frac{\partial f}{\partial \mathbf{v}} \right),$$

where $\overleftrightarrow{\mathbf{B}}$ does not depend on f :

$$\overleftrightarrow{\mathbf{B}}(\mathbf{v}_1, \mathbf{v}_2) = \frac{\pi}{L^3} \int_{-\infty}^{+\infty} d\omega \sum_{\mathbf{k}} \delta(\omega - \mathbf{k} \cdot \mathbf{v}_1) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_2) \frac{\mathbf{k} \mathbf{k} \hat{W}(\mathbf{k})^2}{|\varepsilon[f](\omega, \mathbf{k})|} = \frac{1}{2} \int d\mathbf{q} w(\mathbf{v}_1, \mathbf{v}_2; \mathbf{q}) \mathbf{q} \otimes \mathbf{q},$$

see Lifshitz–Pitaevskii's book on kinetic theory, or Nicholson's book on plasma.

The Properties of the Balescu–Guernsey–Lenard equation

$$\frac{\partial f}{\partial \tau} = LB[f] \text{ with } LB[f] = \frac{\partial}{\partial \mathbf{v}} \cdot \int d\mathbf{v}_2 \overleftrightarrow{B}[f](\mathbf{v}, \mathbf{v}_2) \left(-\frac{\partial f}{\partial \mathbf{v}_2} f(\mathbf{v}) + f(\mathbf{v}_2) \frac{\partial f}{\partial \mathbf{v}} \right),$$

$$\text{with } \overleftrightarrow{B}[f](\mathbf{v}_1, \mathbf{v}_2) = \frac{\pi}{L^3} \int_{-\infty}^{+\infty} d\omega \sum_{\mathbf{k}} \delta(\omega - \mathbf{k} \cdot \mathbf{v}_1) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_2) \frac{\mathbf{k} \mathbf{k} \hat{W}(\mathbf{k})^2}{|\varepsilon[f](\omega, \mathbf{k})|}.$$

- It conserves mass, momentum and kinetic energy

$$E = \int d\mathbf{v} \frac{v^2}{2} f.$$

- It increases monotonically the entropy

$$S = - \int d\mathbf{v} f \log f.$$

- It converges towards the Boltzmann distribution for the corresponding energy

$$f_B(\mathbf{v}) = \frac{\beta^{d/2}}{(2\pi)^{d/2}} \exp\left(-\beta \frac{v^2}{2}\right).$$

Outline

- 1 Introduction
 - Abrupt transitions in turbulent flows
 - Introduction to path large deviations for kinetic theories
- 2 Particles with long range interactions, the Landau and the Balescu–Lenard–Guernsey equations
 - Particles with long range interactions and the Vlasov equation
 - The Balescu–Guernsey–Lenard and the Landau equations
 - Sketch of a derivation of the Balescu–Lenard–Guernsey eq.
- 3 Path large deviations for the Balescu–Lenard–Guernsey equation
 - Quasilinear dynamics and large deviations for slow/fast dynamics
 - Path large deviations for quadratic forms of Gaussian processes using the Szegö–Widom theorem
 - Hamiltonian for the Balescu–Lenard–Guernsey equation

Derivation of the Balescu–Lenard–Guernsey eq. 1: Projection on homogeneous distributions

- We decompose

$$g_N(r, v, t) \equiv \frac{1}{N} \sum_{n=1}^N \delta(r - r_n(t); v - v_n(t)) = f_N(v, t) + \frac{1}{\sqrt{N}} \delta g_N(r, v, t),$$

with the projection over homogeneous distributions:

$$f_N(v, t) = \frac{1}{L^3} \int dr g_N(r, v, t) = \frac{1}{NL^3} \sum_{n=1}^N \delta(v - v_n(t)).$$

- The dynamics (Klimontovich equation) then reads

$$\frac{\partial f_N}{\partial t} = \frac{1}{NL^3} \int dr \left(\frac{\partial V[\delta g_N]}{\partial r} \cdot \frac{\partial \delta g_N}{\partial v} \right) \text{ and}$$

$$\frac{\partial \delta g_N}{\partial t} = -v \cdot \frac{\partial \delta g_N}{\partial r} + \frac{\partial V[\delta g_N]}{\partial r} \cdot \frac{\partial f_N}{\partial v} + \frac{1}{\sqrt{N}} \left[\frac{\partial V[\delta g_N]}{\partial r} \cdot \frac{\partial \delta g_N}{\partial v} - \frac{1}{L^3} \int dr \left(\frac{\partial V[\delta g_N]}{\partial r} \cdot \frac{\partial \delta g_N}{\partial v} \right) \right].$$

Derivation of the BGL eq. 2: Quasilinear approximation

$$\frac{\partial f_N}{\partial t} = \frac{1}{NL^3} \int dr \left(\frac{\partial V[\delta g_N]}{\partial r} \cdot \frac{\partial \delta g_N}{\partial v} \right) \text{ and}$$

$$\frac{\partial \delta g_N}{\partial t} = -v \cdot \frac{\partial \delta g_N}{\partial r} + \frac{\partial V[\delta g_N]}{\partial r} \cdot \frac{\partial f_N}{\partial v} + \frac{1}{\sqrt{N}} \left[\frac{\partial V[\delta g_N]}{\partial r} \cdot \frac{\partial \delta g_N}{\partial v} - \frac{1}{L^3} \int dr \left(\frac{\partial V[\delta g_N]}{\partial r} \cdot \frac{\partial \delta g_N}{\partial v} \right) \right].$$

- Neglecting the non-linear terms in the second equation and rescaling time $\tau = t/N$, gives the quasilinear approximation

$$\begin{aligned} \frac{\partial f_N}{\partial \tau} &= \frac{1}{L^3} \int dr \left(\frac{\partial V[\delta g_N]}{\partial r} \cdot \frac{\partial \delta g_N}{\partial v} \right), \\ \frac{\partial \delta g_N}{\partial \tau} &= N \left(-v \cdot \frac{\partial \delta g_N}{\partial r} + \frac{\partial V[\delta g_N]}{\partial r} \cdot \frac{\partial f_N}{\partial v} \right). \end{aligned}$$

- Solving this set of equations with the Bogoliubov hypothesis (averaging a slow/fast set of equation) gives the Balescu–Guernsey–Lenard equation.

Beyond the Law of Large Numbers: The Large Deviation Action.

- We expect a large deviation principle for the empirical distribution dynamics

$$P[f_N = f] \underset{N \rightarrow \infty}{\asymp} \exp \left(-NL^3 \sup_p \int_0^T dt \left\{ \int \dot{f} p dv - H[f, p] \right\} \right).$$

- Why is N the large deviation parameter?
- Expected properties of H ? First the Lenard–Balescu equation should be the most probable evolution

$$\frac{\partial f}{\partial \tau} = LB[f] = \frac{\delta H}{\delta p} [f, p = 0].$$

Conserved Quantities

- If C is equal to either the mass

$$M = \int dr dv f,$$

or the momentum

$$P = \int dr dv v f,$$

or the kinetic energy

$$E = \frac{1}{2} \int dr dv v^2 f,$$

H should have the symmetry related to the conservation law:

$$\text{for any } f \text{ and } p, \int dr dv \frac{\delta H}{\delta p(v)} [f, p] \frac{\delta C}{\delta f(v)} = 0.$$

The Quasipotential is the Entropy Constrained by the Conserved Quantities

- We expect from equilibrium statistical, that

$$U[f] = \begin{cases} -S[f] & \text{if } M[f] = 1, P[f] = 0, \text{ and } E[f] = E_0; \\ -\infty & \text{otherwise.} \end{cases}$$

Time Reversal Symmetry and Detailed Balance

- We expect

$$H[f, -p] = H\left[f, p - \frac{\delta S}{\delta f}\right].$$

- As a consequence, $-S$ solves the Hamilton–Jacobi equation $H\left[f, -\frac{\delta S}{\delta f}\right] = 0$, and the quasipotential is the entropy constrained by the conserved quantities.
- How to find the large deviation Hamiltonian H ?

Outline

- 1 Introduction
 - Abrupt transitions in turbulent flows
 - Introduction to path large deviations for kinetic theories
- 2 Particles with long range interactions, the Landau and the Balescu–Lenard–Guernsey equations
 - Particles with long range interactions and the Vlasov equation
 - The Balescu–Guernsey–Lenard and the Landau equations
 - Sketch of a derivation of the Balescu–Lenard–Guernsey eq.
- 3 Path large deviations for the Balescu–Lenard–Guernsey equation
 - Quasilinear dynamics and large deviations for slow/fast dynamics
 - Path large deviations for quadratic forms of Gaussian processes using the Szegö–Widom theorem
 - Hamiltonian for the Balescu–Lenard–Guernsey equation

Quasilinear Dynamics and Large Deviation Principle

- With time rescaling $\tau = t/N$, we have the slow/fast dynamics

$$\frac{\partial f_N}{\partial \tau} = \frac{1}{L^3} \int dr \left(\frac{\partial V[\delta g_N]}{\partial r} \cdot \frac{\partial \delta g_N}{\partial v} \right),$$

$$\frac{\partial \delta g_N}{\partial \tau} = N \left\{ -v \cdot \frac{\partial \delta g_N}{\partial r} + \frac{\partial V[\delta g_N]}{\partial r} \cdot \frac{\partial f_N}{\partial v} \right\}.$$

- This is a slow fast dynamical system. Large deviations for slow-fast dynamical systems?

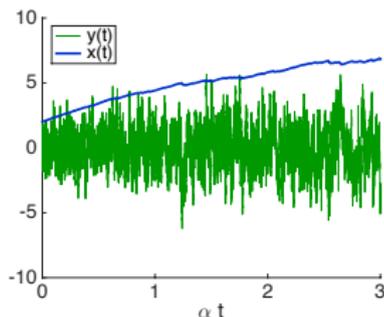
Our plan to compute explicitly H

- 1 Describe the path large deviations for slow/fast dynamics.
- 2 Compute path large deviations for quadratic observables of Gaussian processes using Szegő–Widom theorems.
- 3 Compute explicitly functional determinants and determinants over infinite dimensional space.
- 4 Write the formula for H and verify all its symmetry properties (time reversal symmetries, conservation laws, entropy and quasipotential).
- 5 Justify the quasi-linear approximation (or check the self-consistency of this hypothesis).

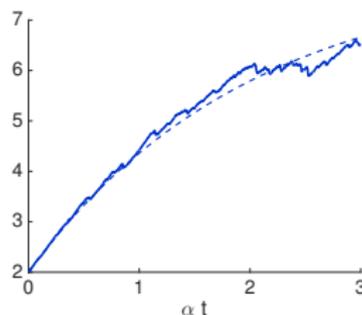
Stochastic Averaging for Slow/Fast Dynamical Systems

$$\begin{cases} \frac{dX_\varepsilon}{dt} = f(X_\varepsilon, Y_\varepsilon) \\ \frac{dY_\varepsilon}{dt} = \frac{1}{\varepsilon} g(X_\varepsilon, Y_\varepsilon) + \frac{1}{\sqrt{\varepsilon}} h(X_\varepsilon, Y_\varepsilon) \frac{dW}{dt} \end{cases}$$

- Time scale separation: $\varepsilon \ll 1$. Y_ε is the fast variable and X_ε is the slow one.



A slow/fast dynamics.



The averaged evolution.

Path Large Deviations for Slow/Fast Dynamics

- We have the large deviation principle

$$\mathbb{P}(X_\varepsilon = x) \underset{\varepsilon \rightarrow 0}{\asymp} \exp\left(-\frac{\int_0^T L(x, \dot{x}) dt}{\varepsilon}\right) \text{ and } L(x, \dot{x}) = \sup_p \{\dot{x}p - H(x, p)\},$$

$$\text{with } H(x, p) = \lim_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E} \left\{ \exp \left[p \int_0^T f(x, Y_x(t)) dt \right] \right\}.$$

(See for instance Freidlin–Wentzell for stochastic processes, and Kifer and Veretnikov for chaotic dynamical systems.)

- For quadratic in y f , and linear g and h , H solves a nonlinear Lyapunov eq.

F. Bouchet, T. Tangarife, and E. Vanden-Eijnden, J. Stat. Phys., 2016.

Quasilinear Dynamics and Large Deviation Principle

- With time rescaling $\tau = t/N$, we have the slow/fast dynamics

$$\frac{\partial f_N}{\partial \tau} = \frac{1}{L^3} \int dr \left(\frac{\partial V[\delta g_N]}{\partial r} \cdot \frac{\partial \delta g_N}{\partial v} \right),$$

$$\frac{\partial \delta g_N}{\partial \tau} = N \left\{ -v \cdot \frac{\partial \delta g_N}{\partial r} + \frac{\partial V[\delta g_N]}{\partial r} \cdot \frac{\partial f_N}{\partial v} \right\}.$$

- Then we have the large deviation principle

$$P(f_N = f) \underset{N \rightarrow \infty}{\asymp} e^{-NL^3 \text{Sup}_p \int_0^T \{ \int dr dv \dot{f} p - H[f, p] \}}, \text{ with}$$

$$H[f, p] = \lim_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_f \left[\exp \left(\frac{1}{L^3} \int_0^T d\tau \int dr dv p(v) \frac{\partial V[\delta g_N]}{\partial r} \cdot \frac{\partial \delta g_N}{\partial v} \right) \right].$$

- Those are path large deviations for time averages of quadratic functionals of a Gaussian process.

Outline

- 1 Introduction
 - Abrupt transitions in turbulent flows
 - Introduction to path large deviations for kinetic theories
- 2 Particles with long range interactions, the Landau and the Balescu–Lenard–Guernsey equations
 - Particles with long range interactions and the Vlasov equation
 - The Balescu–Guernsey–Lenard and the Landau equations
 - Sketch of a derivation of the Balescu–Lenard–Guernsey eq.
- 3 Path large deviations for the Balescu–Lenard–Guernsey equation
 - Quasilinear dynamics and large deviations for slow/fast dynamics
 - Path large deviations for quadratic forms of Gaussian processes using the Szegö–Widom theorem
 - Hamiltonian for the Balescu–Lenard–Guernsey equation

Quasilinear Dynamics and Large Deviation Principle

- With time rescaling $\tau = t/N$, we have the slow/fast dynamics

$$\frac{\partial f_N}{\partial \tau} = \frac{1}{L^3} \int dr \left(\frac{\partial V[\delta g_N]}{\partial r} \cdot \frac{\partial \delta g_N}{\partial v} \right),$$

$$\frac{\partial \delta g_N}{\partial \tau} = N \left\{ -v \cdot \frac{\partial \delta g_N}{\partial r} + \frac{\partial V[\delta g_N]}{\partial r} \cdot \frac{\partial f_N}{\partial v} \right\}.$$

- Then we have the large deviation principle

$$P(f_N = f) \underset{N \rightarrow \infty}{\asymp} \exp \left[-NL^3 \text{Sup}_p \int_0^T \left(\int dr dv \dot{f} p - H[f, p] \right) \right], \text{ with}$$

$$H[f, p] = \lim_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_f \left[\exp \left(\frac{1}{L^3} \int_0^T d\tau \int dr dv p(v) \int dr' \frac{\partial V[\delta g_N]}{\partial r'} \cdot \frac{\partial \delta g_N}{\partial v} \right) \right].$$

- This is the large deviation for time averages of quadratic functionals of a Gaussian process.

Large deviations for quadratic functionals of stationary Gaussian processes

- Let Y_t be a stationary Gaussian process with values on \mathbb{C}^n . We denote $C(t) = \mathbb{E}(Y_t^* \otimes Y_0)$ and assume $\mathbb{E}(Y_t \otimes Y_0) = 0$.
- Let M is an Hermitian matrix of size $n \times n$.
- Then

$$\log \mathbb{E} \exp \left(\int_0^T dt Y_t^{*\top} M Y_t \right) \underset{T \rightarrow \infty}{\sim} -\frac{T}{2\pi} \int d\omega \log \det_{\mathcal{M}_{n,n}} \left(I_n - M \tilde{C}(\omega) \right), \quad (1)$$

where $\tilde{C}(\omega) = \int_{\mathbb{R}} e^{i\omega t} C(t) dt$ is the Fourier transform of C .

Gaussian integration of quadratic functionals

- Let Y_t be a stationary Gaussian process with values on \mathbb{C}^n . We denote $C(t) = \mathbb{E}(Y_t^* \otimes Y_0)$ and assume $\mathbb{E}(Y_t \otimes Y_0) = 0$.
- Then

$$\log \mathbb{E} \exp \left(\int_0^T dt Y_t^{*T} M Y_t \right) = -\log_{\mathcal{F}([0, T], \mathbb{C}^n)} \det (\text{Id} - \overline{MC}_T),$$

where \overline{MC}_T is the integral operator over $\mathcal{F}([0, T], \mathbb{C}^n)$ defined by

$$\text{If } X \in \mathcal{F}([0, T], \mathbb{C}^n) \text{ then } \overline{MC}_T(t)[X] = \int_0^T MC(t-s)X(s)ds.$$

The Szegő–Widom theorem

- Let \bar{K}_T be an integral operator on $\mathcal{F}([0, T], \mathbb{C}^n)$ defined by

$$\bar{K}_T X(t) = \int_0^T K(t-s) X(s) ds,$$

where $K \in \mathcal{F}([0, T], \mathbb{C}^n)$ is called the kernel of the operator \bar{K}_T .

- Then

$$\log_{\mathcal{F}([0, T], \mathbb{C}^n)} \det (\text{Id} + \bar{K}_T) \underset{T \rightarrow \infty}{\sim} \frac{T}{2\pi} \int d\omega \log \det_{\mathcal{M}_{n,n}} \left(I_n + \int_{\mathbb{R}} e^{i\omega t} K(t) dt \right).$$

Large deviations for quadratic functionals of stationary Gaussian processes

- Let Y_t be a stationary Gaussian process with values on \mathbb{C}^n . We denote $C(t) = \mathbb{E}(Y_t^* \otimes Y_0)$ and assume $\mathbb{E}(Y_t \otimes Y_0) = 0$.
- Let M is an Hermitian matrix of size $n \times n$.
- Then

$$\log \mathbb{E} \exp \left(\int_0^T dt Y_t^{*\top} M Y_t \right) \underset{T \rightarrow \infty}{\sim} -\frac{T}{2\pi} \int d\omega \log \det_{\mathcal{M}_{n,n}} \left(I_n - M \tilde{C}(\omega) \right), \quad (2)$$

where $\tilde{C}(\omega) = \int_{\mathbb{R}} e^{i\omega t} C(t) dt$ is the Fourier transform of C .

Outline

- 1 Introduction
 - Abrupt transitions in turbulent flows
 - Introduction to path large deviations for kinetic theories
- 2 Particles with long range interactions, the Landau and the Balescu–Lenard–Guernsey equations
 - Particles with long range interactions and the Vlasov equation
 - The Balescu–Guernsey–Lenard and the Landau equations
 - Sketch of a derivation of the Balescu–Lenard–Guernsey eq.
- 3 Path large deviations for the Balescu–Lenard–Guernsey equation
 - Quasilinear dynamics and large deviations for slow/fast dynamics
 - Path large deviations for quadratic forms of Gaussian processes using the Szegö–Widom theorem
 - Hamiltonian for the Balescu–Lenard–Guernsey equation

Large Deviation Principle

- We have the large deviation principle

$$P(f_N = f) \underset{N \rightarrow \infty}{\asymp} e^{-NL^3 \text{Sup}_p \int_0^T \{ \int dr dv \dot{f} p - H[f, p] \}}, \text{ with}$$

$$H[f, p] = \lim_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_f \left[\exp \left(\frac{1}{L^3} \int_0^T d\tau \int dr dv p(v) \frac{\partial V[\delta g_N]}{\partial r} \cdot \frac{\partial \delta g_N}{\partial v} \right) \right].$$

- This is the large deviation for time averages of quadratic functionals of a Gaussian process.

$$H[f, p] = -\frac{T}{2\pi} \int d\omega \log \det_{\mathcal{L}_v} \left(I_n - M \tilde{C}(\omega) \right).$$

Computing determinants on the space of complex functions of the velocity space

- We need to compute the determinant of an operators U that acts on complex-function φ over the velocity space:

$$U[\varphi](v_1) = \varphi(v_1) + i\hat{W}(k)k \cdot \int dv_2 dv_3 \tilde{C}_{GG}(k, \omega, v_2, v_3) \left\{ \frac{\partial p}{\partial v}(v_2) - \frac{\partial p}{\partial v}(v_1) \right\} \varphi(v_3).$$

- A critical remark: U is the identity plus a rank two linear operator

$$U : \varphi \longmapsto \varphi + (w, Q\varphi)v + (v, Q\varphi)w,$$

then

$$\det U = 1 + 2\Re[(v, Qw)] + (v, Qw)(v, Qw)^* - (w, Qw)(v, Qv).$$

- The determinant of U only depends on the two-point correlation function of the quasi-linear problem.

The Large Deviation Hamiltonian for the Lenard–Balescu equation

- The large deviation Hamiltonian reads

$$H[f, \rho] = -\frac{1}{4\pi} \sum_{\mathbf{k}} \int d\omega \log \{1 - \mathcal{J}[f, \rho](\mathbf{k}, \omega)\},$$

with

$$\begin{aligned} \mathcal{J}[f, \rho](\mathbf{k}, \omega) &= 4\pi \int dv_1 dv_2 \frac{\partial \rho}{\partial v_1} \cdot \overleftrightarrow{A}(\mathbf{k}, \omega, v_1, v_2) \cdot \left\{ \frac{\partial f}{\partial v_2} f(v_1) - f(v_2) \frac{\partial f}{\partial v_1} \right\} \\ &+ 4\pi \int dv_1 dv_2 \left\{ \frac{\partial \rho}{\partial v_1} \frac{\partial \rho}{\partial v_1} - \frac{\partial \rho}{\partial v_1} \frac{\partial \rho}{\partial v_2} \right\} : \overleftrightarrow{A}(\mathbf{k}, \omega, v_1, v_2) f(v_1) f(v_2), \quad (3) \end{aligned}$$

where

$$\overleftrightarrow{A}(\mathbf{k}, \omega, v_1, v_2) = \pi \frac{\mathbf{k} \mathbf{k} \hat{W}(\mathbf{k})^2}{|\varepsilon(\omega, \mathbf{k})|^2} \delta(\omega - \mathbf{k} \cdot v_1) \delta(\omega - \mathbf{k} \cdot v_2).$$

Conclusion: The Landau and Balescu–Lenard–Guernsey Large Deviation Hamiltonians

- With O. Feliachi, we have derived the Hamiltonian for the path large deviations for the empirical density of systems with long range interactions (related to the BLG equation).
- We have justified the Hamiltonian for the path large deviations for the Landau equation, both from the Boltzmann and from the BLG Hamiltonians.

$$H_{LB}[f, p] = \underbrace{\int d\mathbf{v}_1 f \left\{ \mathbf{b}[f] \cdot \frac{\partial p}{\partial \mathbf{v}_1} + \frac{\partial}{\partial \mathbf{v}_1} \left(\overleftrightarrow{\mathcal{D}}[f] \frac{\partial p}{\partial \mathbf{v}_1} \right) + \overleftrightarrow{\mathcal{D}}[f] : \frac{\partial p}{\partial \mathbf{v}_1} \frac{\partial p}{\partial \mathbf{v}_1} \right\}}_{H_{MF}[f, p]} - \underbrace{\int d\mathbf{v}_1 d\mathbf{v}_2 f(\mathbf{v}_1) f(\mathbf{v}_2) \overleftrightarrow{\mathcal{B}}[f](\mathbf{v}_1, \mathbf{v}_2) \frac{\partial p}{\partial \mathbf{v}_1} \frac{\partial p}{\partial \mathbf{v}_2}}_{H_I[f, p]}.$$

- The large deviations are non-Gaussian for BLG and Gaussian for Landau. We can identify a gradient structure for both.
- The Hamiltonians are time reversal symmetric, conserve mass, momentum and energy. Entropy is the quasipotential.