Shear flows in stratified fluids

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Stably stratified fluids near shear flows

$$\text{(E)} \quad \begin{cases} \partial_t \rho + \mathbf{u} \cdot \nabla \rho = 0 \\ \rho \left(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} \right) + \nabla P = -\rho \begin{pmatrix} 0 \\ \mathfrak{g} \end{pmatrix} \end{cases} \qquad \nabla \cdot \mathbf{u} = 0 \qquad \text{for} \quad (x,y) \in \mathbb{T} \times [0,1]$$

where $\mathbf{u} = (u^x, u^y)$ $\qquad u^y|_{y=0} = u^y|_{y=1} = 0$

$\Rightarrow$ **Stationary solutions** of the form

$(\bar\rho_{eq}(y), U(y)\mathbf{e}_1, \bar P_{eq}(y))$ where $\bar P'_{eq}(y) = -\mathfrak{g}\bar\rho_{eq}(y)$ [**hydrostatic balance**]

Hydrodynamic stability from the end of the 19th century: Rayleigh, Kelvin, Taylor, Goldstein…

Questions

- Are these solutions "**stable**"?
- What is the asymptotic behavior of the perturbations in time, with or without dissipation?

Stably stratified fluids near shear flows

$\Rightarrow$ perturbed solutions: $\begin{cases} \rho(t,x,y) = \bar{\rho}_{eq}(y) + \widetilde{\rho}(t,x,y) \\ \mathbf{u}(t,x,y) = U(y)\mathbf{e}_1 + \widetilde{\mathbf{u}}(t,x,y) \\ p(t,x,y) = \bar{P}_{eq}(y) + \widetilde{p}(t,x,y) \end{cases}$

$\Rightarrow$ linearized system $\partial_t \begin{pmatrix} \widetilde{\rho} \\ \widetilde{\mathbf{u}} \end{pmatrix} = \mathbf{L}(t,x,y) \begin{pmatrix} \widetilde{\rho} \\ \widetilde{\mathbf{u}} \end{pmatrix}$

$\Rightarrow$ eigenvalues of $\mathbf{L}(t,x,y)$?

normal mode analysis, take $\begin{cases} \widetilde{\rho}(t,x,y) = \rho(y)e^{st+ikx} \\ \widetilde{\mathbf{u}}(t,x,y) = \mathbf{u}(y)e^{st+ikx} \\ \widetilde{p}(y) = p(y)e^{st+ikx} \end{cases}$

Taylor-Goldstein Equation and Miles-Howard criterion

The triple $(\rho(y), \mathbf{u}(y), p(y))$ satisfies, for $\gamma(y) = s + ikU(y)$

Taylor-Goldstein Equation

$$\begin{cases} \gamma(y)\rho + \bar{\rho}'_{eq}(y)u^y = 0 \\ \bar{\rho}_{eq}(y)(\gamma(y)u^x + U'(y)u^y) = -ik\rho \\ \bar{\rho}_{eq}(y)\gamma(y)u^y = -p' - \mathfrak{g}\rho \\ iku^x + u^{y'} = 0 \end{cases} \Rightarrow \; -(\bar{\rho}_{eq}(y)u^{y'})' + k^2\bar{\rho}_{eq}(y)u^y + \frac{ik}{\gamma(y)}(\bar{\rho}_{eq}(y)U'(y))'u^y - \frac{k^2\mathfrak{g}}{\gamma^2(y)}\bar{\rho}'_{eq}(y)u^y = 0$$

$\Rightarrow$ introducing the variable $v(y)$ such that $u^y = v(y)\sqrt{\gamma(y)}$ and multiplying by $\bar{v}(y)$ (complex conj) gives

$$\mathrm{Re}(s)\int_0^1 \bar{\rho}_{eq}(y)(|v'|^2 + k^2|v|^2) + \frac{k^2\bar{\rho}_{eq}(y)(U'(y))^2}{|\gamma(y)|^2}\left(\mathrm{Ri}(y) - \frac{1}{4}\right)|v|^2\,dy = 0$$

$\mathrm{Ri}(y) = \left(\frac{\beta(y)}{U'(y)}\right)^2$ '**Richardson number**' and $\beta^2(y) = \frac{-\bar{\rho}'_{eq}(y)\mathfrak{g}}{\bar{\rho}_{eq}(y)}$ '**Brunt-Väisälä frequency**' if $\underbrace{\bar{\rho}'_{eq}(y) < 0}_{\text{stably stratified}}$

Miles-Howard criterion: if $\mathrm{Ri}(y) \geq 1/4 \Rightarrow \mathrm{Re}(s) = 0$ [NO any unstable mode]

'Rigidity' of the Miles-Howard condition

The Miles-Howard condition

- Is sharp in the sense that the value 1/4 is sharp
- But it is only a sufficient condition (ex. Homogeneous case)

However, it persists under

- The Boussinesq approximation
- The hydrostatic approximation

- Taylor-Goldstein Equation under the Boussinesq approximation $\bar{\rho}_{eq}(y) = \bar{\rho}_c - by, \quad b > 0$

$$-(u^y)'' + k^2 u^y + \frac{ik}{\gamma(y)} U''(y) u^y + \frac{k^2}{\gamma^2(y)} \underbrace{\frac{\mathfrak{g} b}{\bar{\rho}_c}}_{\beta^2} u^y = 0$$

- Taylor-Goldstein Equation under the hydrostatic (and Boussinesq) approximation $x = \frac{\tilde{x}}{\varepsilon}$

$$-(u^y)'' + \varepsilon^2 k^2 u^y + \frac{ik}{\gamma(y)} U''(y) u^y + \frac{k^2}{\gamma^2(y)} \underbrace{\frac{\mathfrak{g} b}{\bar{\rho}_c}}_{\beta^2} u^y = 0$$

homogeneous Vs non-homogeneous (Boussinesq)

Let $\gamma(y) = s + ikU(y) = ik\left(U(y) - \frac{is}{k}\right) = ik(U-c)$ where $c = is/k$

Homogeneous density: **Rayleigh Equation**

$$-(u^y)'' + k^2 u^y + \frac{U''}{(U-c)} u^y = 0$$

NONhomogeneous density: **Taylor-Goldstein**

$$-(u^y)'' + k^2 u^y + \frac{U''(y)}{(U-c)} u^y - \frac{\beta^2}{(U-c)^2} u^y = 0$$

*** Rayleigh Equation has a singularity of order 1 in $(U-c)$ while TG has a singularity of order 2 ***

- This does not change under the Boussinesq approximation
- The different orders of singularity determine a different time decay of the perturbation

Let us consider the simplest shear flow, namely the **Couette flow** $U(y) = y$

The 2D Boussinesq equations around the Couette flow

* The inviscid Euler-Boussinesq equations in $\mathbb{T}\times\mathbb{R}$ read

$$\begin{cases} \partial_t \rho + \mathbf{u}\cdot\nabla\rho = 0 \quad (x,y)\in\mathbb{T}\times\mathbb{R} \\ \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = -\rho\mathbf{g}, \qquad \nabla\cdot\mathbf{u} = 0 \end{cases} \qquad (\rho = \frac{\tilde\rho}{\bar\rho_c}, P = \frac{\tilde P}{\bar\rho_c})$$

* Stationary solutions $(\bar\rho_{eq}(y), \bar{\mathbf{u}}_{eq}(y), \bar p_{eq}(y))$ stratified Couette flow

$\bar\rho_{eq}(y) = \bar\rho_c - by, \quad b > 0$ [stable]; $\quad \bar{\mathbf{u}}_{eq} = (y, 0)$ [Couette flow]; $\quad \partial_y \bar p_{eq} = -\mathfrak{g}\bar\rho_{eq}$

For $\theta = \mathfrak{g}\rho/\bar\rho_c$ [buoyancy forcing] the linearized system in vorticity

$$\begin{cases} \partial_t\omega + y\partial_x\omega = -\partial_x\theta - (\mathbf{u}\cdot\nabla)\omega \\ \partial_t\theta + y\partial_x\theta = \beta^2\partial_x\psi - \mathbf{u}\cdot\nabla\theta \end{cases} \qquad (x,y)\in\mathbb{T}\times\mathbb{R}$$

$\beta = \sqrt{b\mathbf{g}/\bar\rho_c}$ Brunt-Väisälä frequency

Some mathematical results

'Spectral stability is not enough' and a steady state is stable if, given two spaces X, Y, perturbations decay

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \|(\rho^{in}, \mathbf{u}^{in})\|_X < \delta \Rightarrow \|(\rho(t), \mathbf{u}(t))\|_Y \leq \varepsilon$$

- Asymptotic stability of the 2D Boussinesq system near the Couette flow, inviscid (our results in the 2D infinite strip with Coti Zelati, Dolce and Bedrossian; linear results in the finite channel by Nualart '23)
- Asymptotic stability of the 2D Boussinesq system near the Couette flow with viscosity, no diffusivity [Masmoudi et al '20]
- Construction of echo chains for the 2D Boussinesq system near the Couette flow with viscosity [Zillinger '21]
- Enhanced dissipation with viscosity and diffusivity [Del Zotto '23] and transition threshold in Sobolev [Masmoudi et al '22]
- Stability threshold for the 3D equations with viscosity and diffusivity in Sobolev [Del Zotto, in preparation]
- Spectral instability and ill-posedness of the hydrostatic-Boussinesq equations near a shear flow violating the Miles-Howard criterion [with Lucas Ertzbischoff and Coti Zelati, in preparation]

2D Euler equations linearized

- [Bedrossian-Masmoudi 2015] nonlinear, Couette flow
- [Ionescu-Jia 2020] nonlinear, monotone shear flows
- [Wei, Zheng, Zhao 2020, after Bouchet-Morita 2010] linear, near the Kolmogorov flow $(\sin y, 0)$

A step back: the Euler equations in 2D

Let's focus on **vorticity mixing** : consider the Euler equations near the 2D Couette flow $\bar{\mathbf{u}}_{couette} = (y,0)$

Like any shear flow, Couette is a **steady state** of 2D Euler. **Q:** "Is it stable to perturbation?"

⇒ **It depends pretty much on the regularity of the perturbation**:

look at the **linearized** 2D Euler equations in vorticity form near Couette

$$\begin{cases} \partial_t \omega + y \partial_x \omega = 0 \\ \omega(0,x,y) = \omega_{in}(x,y) \end{cases}$$

in the domain $\mathbb{T} \times \mathbb{R}$

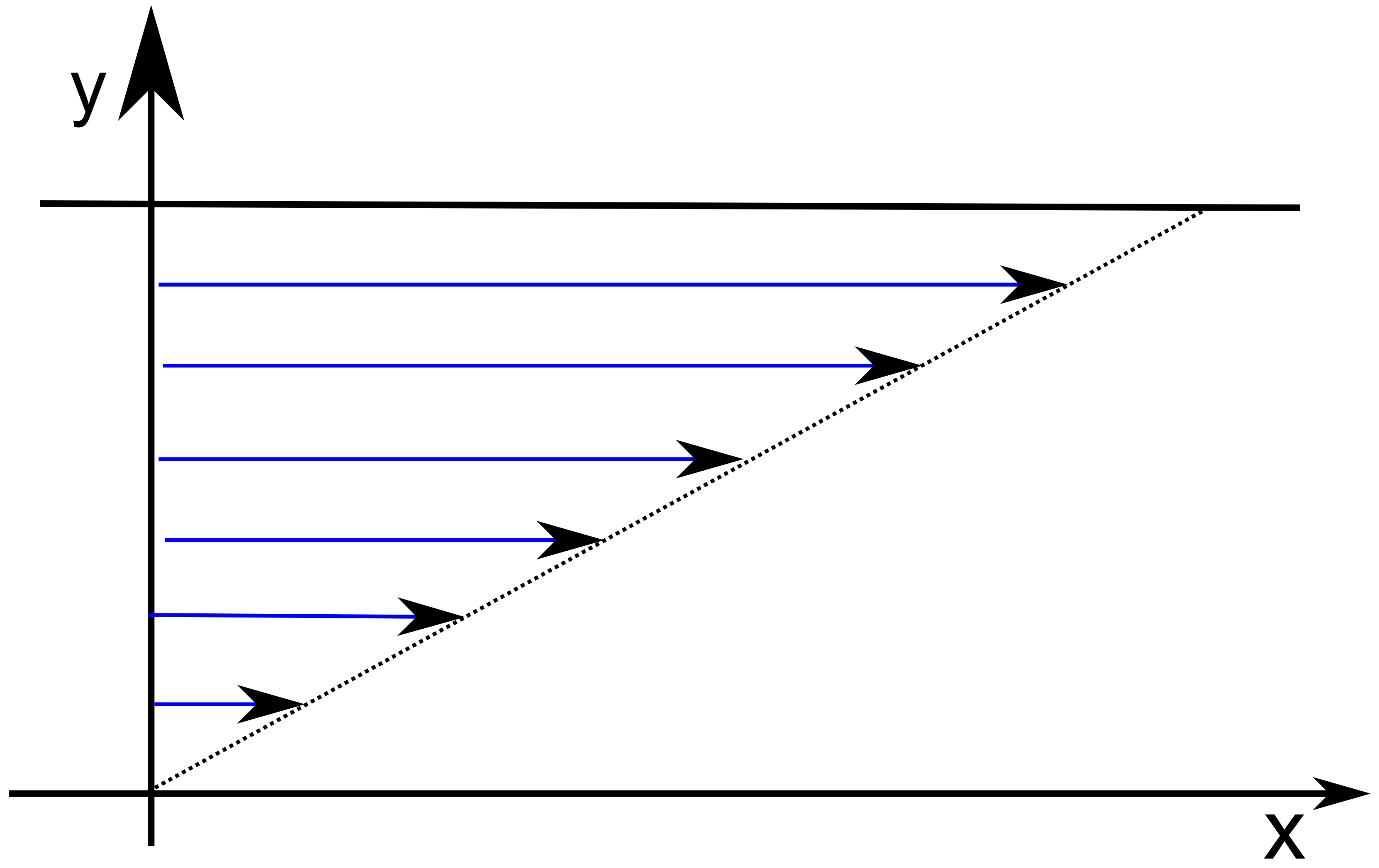
- Spectrally stable in L^2 [continuous spectrum, the imaginary axis]
- Lyapunov stable in L^2
- Lyapunov unstable from $H^s \to H^s$, $s > 0$
- Lyapunov stable from $H^1_\star \to H^{-1}$ ($\star$=zero average in x)

"Lypunov stability - time decay - requires loss of regularity"

Mixing by shear flows in the Euler equations

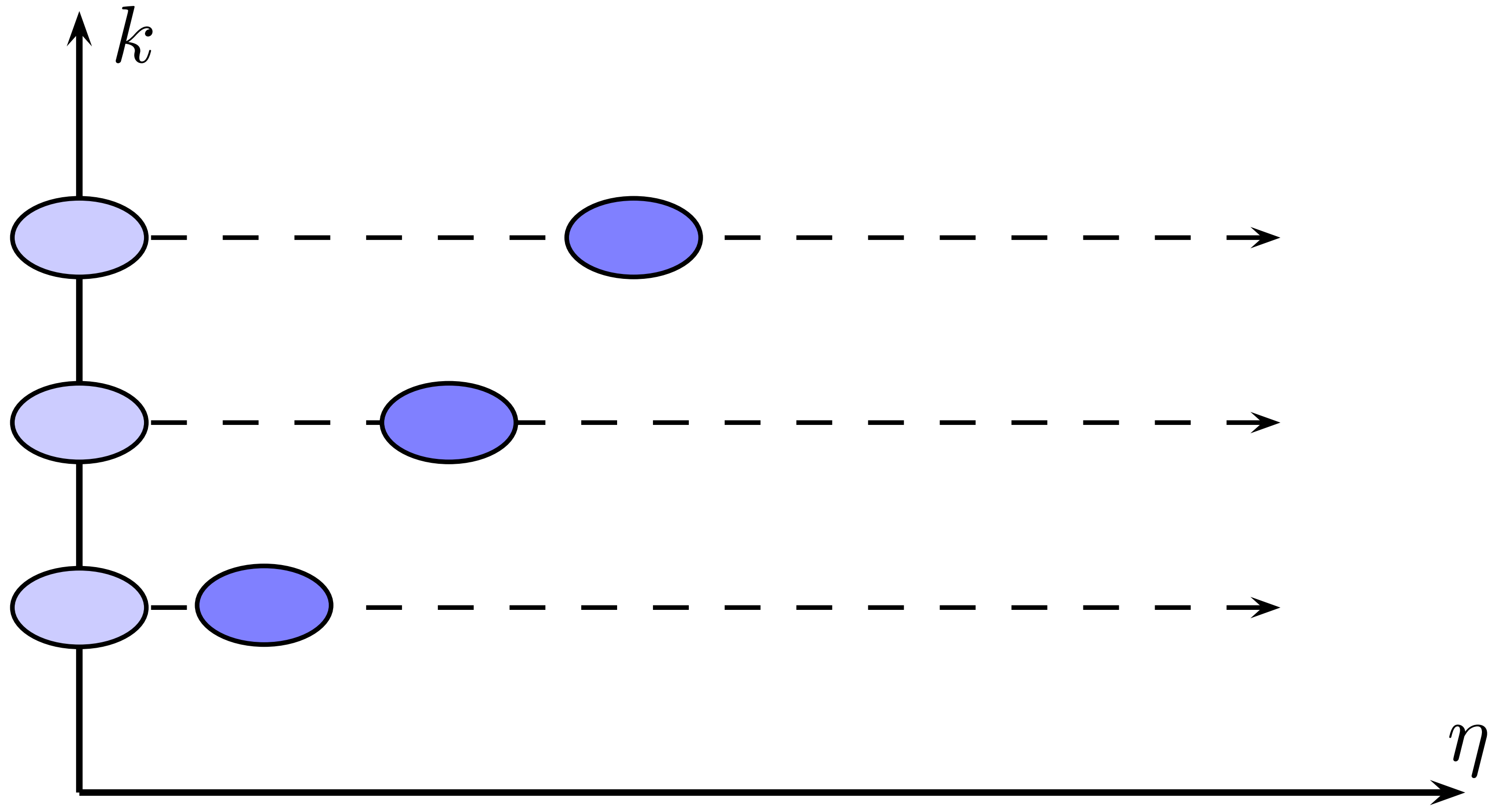
Physical space

- Consider $\theta = 1$ and the Euler equations $\partial_t \omega + (\mathbf{u} \cdot \nabla)\omega = 0$
- Linearized around Couette $\mathbf{u} = (y, 0)$ i.e. $\partial_t \omega + y \partial_x \omega = 0$
- Explicit solution $\omega(t, x, y) = \omega_{\text{in}}(x - yt)$



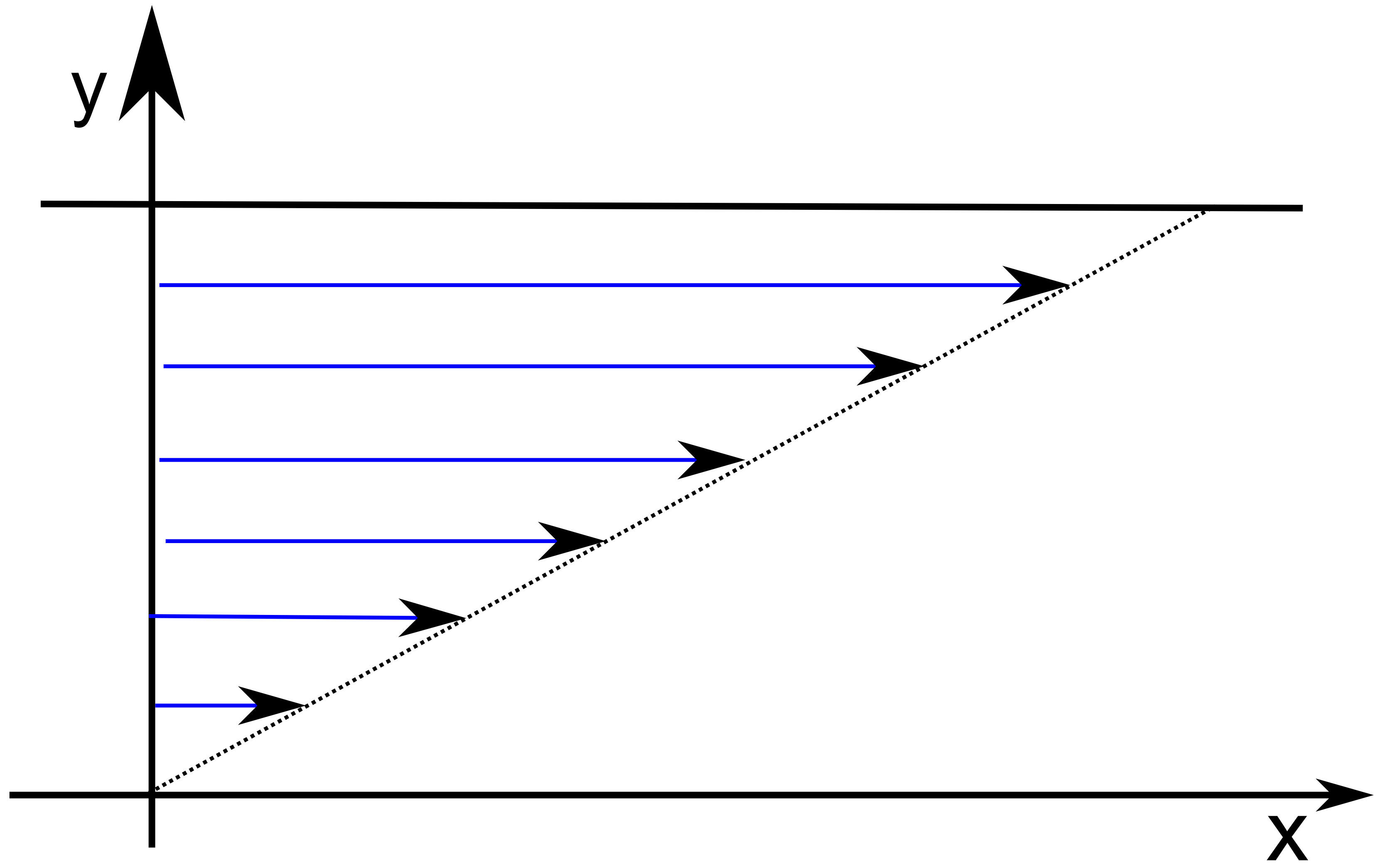
Fourier Dynamics

- $\partial_t \widehat{\omega} + k \partial_\eta \widehat{\omega} = 0$
- Explicit solution $\widehat{\omega}(t, k, \eta) = \widehat{\omega_{\text{in}}}(\eta + kt)$



Linear inviscid damping

$$\begin{cases} \partial_t \omega + y \partial_x \omega = 0 \\ \omega|_{t=0} = \omega_{\text{in}} \end{cases}$$



- $\widehat{\omega}(t,k,\eta) = \widehat{\omega}_{\text{in}}(k,\eta+kt)$
- $\widehat{\omega}(t,k,\eta-kt) = \widehat{\omega}_{\text{in}}(k,\eta)$

  $\widehat{\Delta\psi}(t,k,\eta-kt) = \widehat{\omega}_{\text{in}}(k,\eta-kt)$
- $$\Rightarrow \quad \begin{cases} \widehat{u^y} \sim k\,\widehat{\psi} = \dfrac{k\,\widehat{\omega}_{\text{in}}}{k^2+(\eta-kt)^2} = \dfrac{k(k^2+\eta^2)\,\widehat{\omega}_{\text{in}}}{(k^2+(\eta-kt)^2)(k^2+\eta^2)} \lesssim O(t^{-2}) \\ \widehat{u^x} \lesssim O(t^{-1}) \end{cases}$$

Linear enhanced dissipation with $\nu\Delta$

Mixing by shear flows transports energy at **high frequencies** where the Laplacian is **stronger**

- **Navier-Stokes** at Couette $\partial_t \omega + y\partial_x \omega = \nu \Delta \omega$
- Explicitly solvable $\partial_t \widehat{\omega} + k\partial_\eta \widehat{\omega} = -\nu(k^2+\eta^2)\widehat{\omega}$
- $$\widehat{\omega}(t,k,\eta-kt) \lesssim \mathbf{e}^{\int_0^t -\nu(k^2+(\eta-k\tau)^2)\,d\tau} \lesssim \mathbf{e}^{-c\nu t^3}$$

For more general shear flows it is more complicated, enhanced dissipation rates obtained through suitable **modified energy functionals** (hypocoercivity method)

Back to Boussinesq and linear dynamics in the infinite strip $\mathbb{T}\times\mathbb{R}$

Theorem [RB, Coti Zelati, Dolce '20] Let $\beta > 1/2$. Define

$$C_\beta := \left[\frac{2\beta+1}{2\beta-1}\exp\left(\frac{1}{2\beta-1}\right)\right]^{1/2}.$$

Then there hold the linear inviscid damping estimates

$$\|\theta_{\neq}(t)\|_{L^2} + \left\|u^x_{\neq}(t)\right\|_{L^2} \lesssim C_\beta \langle t\rangle^{-1/2}\left[\left\|\omega^{in}_{\neq}\right\|_{L^2} + \left\|\theta^{in}_{\neq}\right\|_{H^1}\right],$$

$$\|u^y(t)\|_{L^2} \lesssim C_\beta \langle t\rangle^{-\frac{3}{2}}\left[\left\|\omega^{in}_{\neq}\right\|_{H^1} + \left\|\theta^{in}_{\neq}\right\|_{H^2}\right],$$

and the shear-buoyancy instability estimate

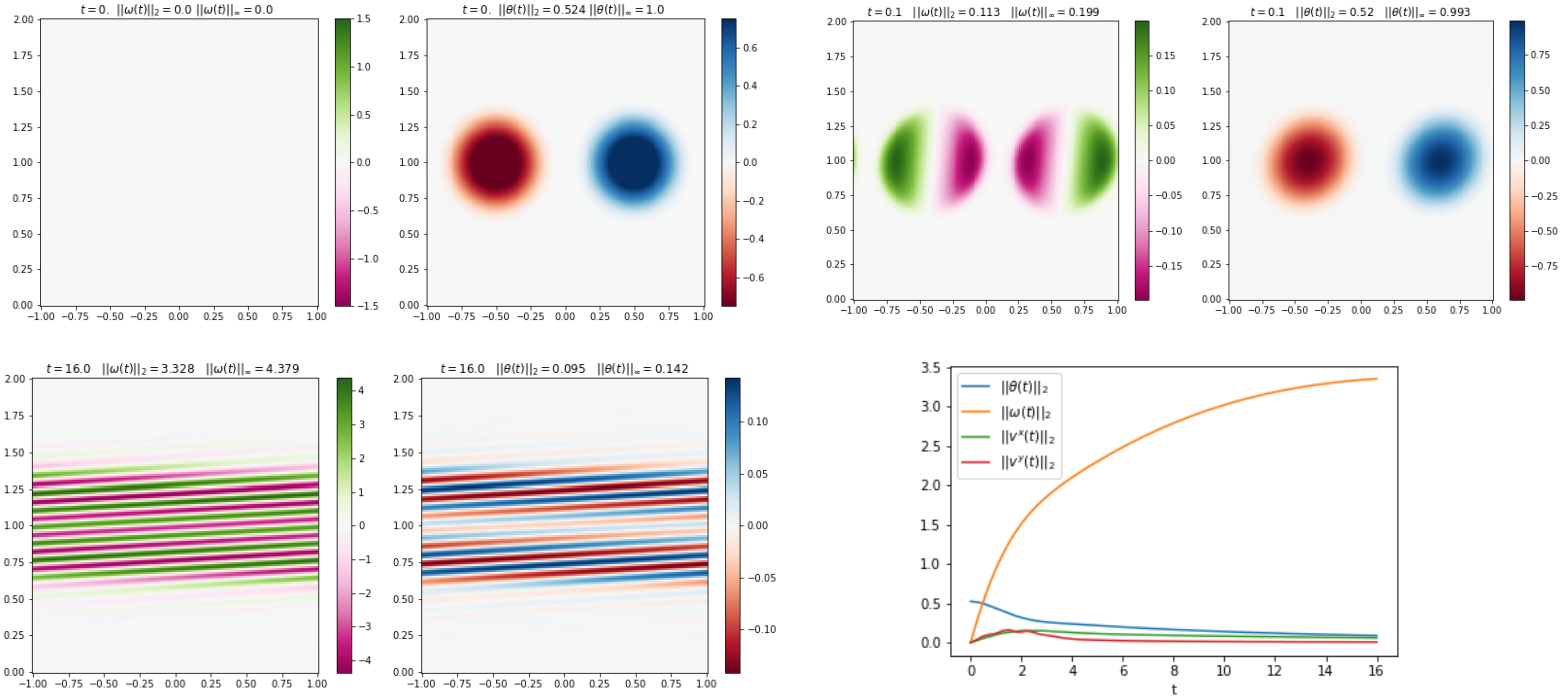
$$\|\omega_{\neq}(t)\|_{L^2} + \|\nabla\theta_{\neq}(t)\|_{L^2} \gtrsim \boxed{\frac{1}{C_\beta}\langle t\rangle^{1/2}}\left[\left\|\omega^{in}_{\neq}\right\|_{H^{-1}} + \left\|\theta^{in}_{\neq}\right\|_{L^2}\right],$$

for every $t \geq 0$.

** density induces creation of vorticity and hence an $\mathbf{L}^2$ growth in time **

Initial vorticity

Initial buoyancy



Shearing effect at later times

Growth of the vorticity

Linear enhanced dissipation

THEOREM 2 (Linear enhanced dissipation). *Let $\beta > 1/2$, assume that $\nu, \kappa > 0$ satisfy*

$$\frac{\max\{\nu,\kappa\}}{\min\{\nu,\kappa\}} < 4\beta - 1,$$

and define the strictly positive number

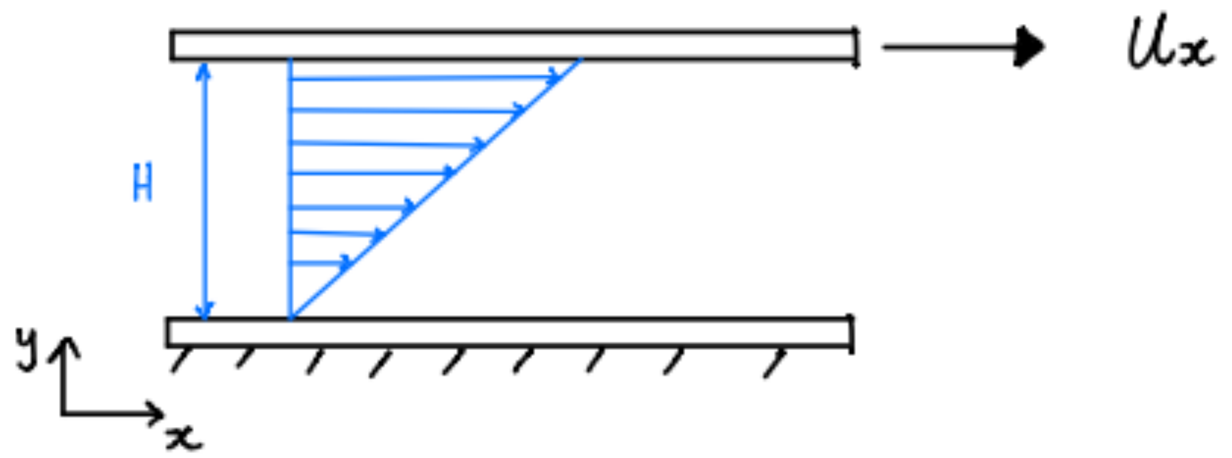
$$\lambda_{\nu,\kappa} := \min\{\nu,\kappa\}\left(1 - \frac{1}{4\beta} - \frac{1}{4\beta}\frac{\max\{\nu,\kappa\}}{\min\{\nu,\kappa\}}\right).$$

Then

$$\|\omega_{\neq}(t)\|_{L^2} + \langle t\rangle \|\theta_{\neq}(t)\|_{L^2} \lesssim C_\beta \langle t\rangle^{1/2} \boxed{e^{-\frac{1}{24}\lambda_{\nu,\kappa}k^2t^3}} \left[\|\omega^{in}_{\neq}\|_{L^2} + \|\theta^{in}_{\neq}\|_{H^1}\right]$$

$\Rightarrow$ Transition threshold $\nu^{1/2}$ in Sobolev spaces H^s [Zhai & Zhao '22] while for the homogeneous case $\nu^{1/3}$

This is related to the asymptotic $t \sim \nu^{-1/3} \Rightarrow \|f_{\neq}\| \lesssim \sqrt{\langle t\rangle}e^{-c\nu t^3} \lesssim \nu^{-1/6}$ and $\nu^{1/3+1/6} = \nu^{1/2}$



Symmetrization and energy method

The transport $y\partial_x$ suggests changing coordinate $z = x - yt$ and variables $\begin{cases} \Omega(t,z,y) = \omega(t,x,y) \\ \Theta(t,z,y) = \theta(t,x,y) \\ \Psi(t,z,y) = \psi(t,x,y) \end{cases}$

In this moving frame $\Delta_L \Psi = \Omega$ where $\Delta_L = \partial_{zz} + (\partial_y - t\partial_z)^2$ [in Fourier $\mathbf{p} = \mathbf{k}^2 + (\boldsymbol{\eta} - \mathbf{k}t)^2$] and

$$\underbrace{\partial_t \begin{pmatrix} \Omega \\ \Theta \end{pmatrix} = \begin{pmatrix} 0 & -ik\beta^2 \\ -ik\mathbf{p}^{-1} & 0 \end{pmatrix} \begin{pmatrix} \Omega \\ \Theta \end{pmatrix}}_{\nu=\kappa=0} \qquad \underbrace{\partial_t \begin{pmatrix} \Omega \\ \Theta \end{pmatrix} = \begin{pmatrix} -\mathbf{p}\nu & -ik\beta^2 \\ -ik\mathbf{p}^{-1} & -\mathbf{p}\kappa \end{pmatrix} \begin{pmatrix} \Omega \\ \Theta \end{pmatrix}}_{\nu,\kappa>0}$$

Symmetric variables: $\begin{cases} Z = (\mathbf{p}/k^2)^{-1/4}\Omega \\ Q = ik\beta(\mathbf{p}/k^2)^{1/4}\Theta \end{cases}$

Energy in the moving frame - inviscid

In terms of the symmetric variables

$$\partial_t \begin{pmatrix} Z \\ Q \end{pmatrix} = \begin{pmatrix} -\frac{1}{4}\frac{\partial_t p}{p} & -\beta |k| p^{-1/2} \\ \beta |k| p^{-1/2} & \frac{1}{4}\frac{\partial_t p}{p} \end{pmatrix} \begin{pmatrix} Z \\ Q \end{pmatrix}$$

we can define the energy functional

$$\mathsf{E}(t,k,\eta) = \frac{1}{2}\left[|Z|^2 + |Q|^2 + \frac{1}{2\beta}\frac{\partial_t p}{|k| p^{1/2}} \mathrm{Re}(Z\bar{Q}) \right]$$

is coercive provided that $\beta > 1/2$ [Miles-Howard]

$$\frac{d}{dt}E(t) = \frac{1}{4\beta}\partial_t\left(\frac{\partial_t p}{|k| p^{1/2}}\right)\mathrm{Re}(Z\bar{Q})$$

$$\Downarrow$$

$$-\frac{\mathsf{E}}{2(1-2\beta)}\left|\partial_t\left(\frac{\partial_t p}{|k|p^{1/2}}\right)\right| \leq \frac{d}{dt}\mathsf{E} \leq \frac{\mathsf{E}}{2(1-2\beta)}\left|\partial_t\left(\frac{\partial_t p}{|k|p^{1/2}}\right)\right|$$

$$\Downarrow$$

Upper and lower bounds, point wise in (k,η)

$$\mathsf{E}(t) \approx_\beta \mathsf{E}(0)$$

Linear inviscid damping by the energy method

$\mathsf{E}(t) \approx_\beta \mathsf{E}(0)$ reads, more explicitely

$$|p^{-1/4}\Omega(t,k,\eta)|^2 + |p^{1/4}\Theta(t,k,\eta)|^2 \approx_\beta |(k^2+\eta^2)^{-1/4}\Omega(0,k,\eta)|^2 + |(k^2+\eta^2)^{1/4}\Theta(0,k,\eta)|^2$$

$$\Downarrow$$

[damping]
[instability]

$$\begin{cases} \|\theta(t) - \langle\theta\rangle_x\|_{L^2} + \|u^x(t) - \langle u^x\rangle_x\|_{L^2} + \langle t\rangle\|u^y(t)\|_{L^2} \lesssim \langle \mathbf{t}\rangle^{-\frac{1}{2}}(\|\omega_{in} - \langle\omega_{in}\rangle_x\|_{H^1} + \|\theta_{in} - \langle\theta_{in}\rangle_x\|_{H^2} \\ \|\omega - \langle\omega\rangle_x\|_{L^2} + \|\nabla\theta - \langle\nabla\theta\rangle_x\|_{L^2} \approx C_{in}\langle \mathbf{t}\rangle^{\mathbf{1/2}} \end{cases}$$

The energy method applies to the case of exponentially stratified fluids $\bar{\rho}_{eq}(y) = e^{-by}$ without the Boussinesq approximation and to shear flows close to Couette $U'_{eq}(y) \sim 1,\ U''_{eq}(y) \sim 0$ such that

$$\|U'_{eq}(y) - 1\|_{H^s} = O(\varepsilon), \quad \|U''_{eq}(y)\|_{H^{s-1}} = O(\varepsilon)$$

Enhanced dissipation via the energy method for $\nu, \kappa > 0$

The same energy functional $\mathsf{E}(t,k,\eta) = \frac{1}{2}\left[|Z|^2 + |Q|^2 + \frac{1}{2\beta}\frac{\partial_t p}{|k|p^{1/2}}\mathrm{Re}(Z\bar{Q})\right]$ applied to the dissipative case

$$\Downarrow$$

$$\frac{d}{dt}\mathsf{E} \leq \frac{1}{2(1-2\beta)}\partial_t\left(\frac{\partial_t p}{|k|p^{1/2}}\right) - \frac{4\beta}{2\beta+1}\lambda_{\nu,\kappa} p \mathsf{E}$$

where $\lambda_{\nu,\kappa} = \min\left\{\kappa - \frac{\nu+\kappa}{4\beta}, \nu - \frac{\nu+\kappa}{4\beta}\right\}$

$$\Downarrow$$

$$\mathsf{E}(t) \leq \exp\left(\frac{1}{2\beta-1}\right)\exp\left(-\frac{\beta}{3(2\beta+1)}\lambda_{\nu,\kappa}k^2t^3\right)\mathsf{E}(0)$$

Partial dissipation $\kappa = 0, \quad \nu > 0$

$$\begin{cases} \partial_t \Omega = -i\beta k \Theta - \nu p \Omega \\ \partial_t \Theta = -\frac{ik}{p}\Omega \end{cases}$$

[Masmoudi et al '20]

THEOREM 3 (Stability in the non-diffusive case, [**19**]). *Let $\beta, \nu > 0$ and $\kappa = 0$ in* (1.11)-(1.12). *Then there hold the asymptotic stability estimates*

$$\|\omega_{\neq}(t)\|_{L^2} + \langle t\rangle \left\|u^x_{\neq}(t)\right\|_{L^2} + \langle t\rangle^2 \|u^y(t)\|_{L^2} \lesssim \langle t\rangle^{-2} \left[\left\|\omega^{in}_{\neq}\right\|_{H^4} + \left\|\theta^{in}_{\neq}\right\|_{H^5}\right],$$

and

$$\|\theta_{\neq}(t)\|_{L^2} \lesssim \left\|\omega^{in}_{\neq}\right\|_{H^2} + \left\|\theta^{in}_{\neq}\right\|_{H^1},$$

- θ does not decay at all
- The velocity decays faster
- No any enhanced dissipation since $\lesssim$ depends very badly ν

Good unknown $\Sigma = -ik\beta\Theta - \nu p \Omega$

Performing an energy estimate $|\Sigma(t)|^2 + |\Theta(t)|^2 \lesssim_{\nu,\beta} |\Sigma(0)|^2 + |\Theta(0)|^2 \Rightarrow |\Omega(t)| \lesssim_{\beta,\nu} (k^2 + (\eta - kt)^2)^{-1}$

In “more general” domains, another approach
“Limiting Absorption Principle”

- Doing resolvent estimates by hands [Jia 2019, Zhao…]
- Using the conjugate operator method [Grenier at al 2020]

Sketch of the approach via the LAP

The LAP [Limiting Absorption Principle] tells that the resolvent $\mathbf{R}(c)$ of an operator $\mathbf{L}$ is uniformly bounded in terms of the spectral parameter c [and it actually decays in time] if it acts on a suitable weighted space

Linear inviscid damping of the 2D Boussinesq equations near the Couette flow in the periodic channel $\mathbb{T} \times [0,1]$ [Marc Nualart '23]

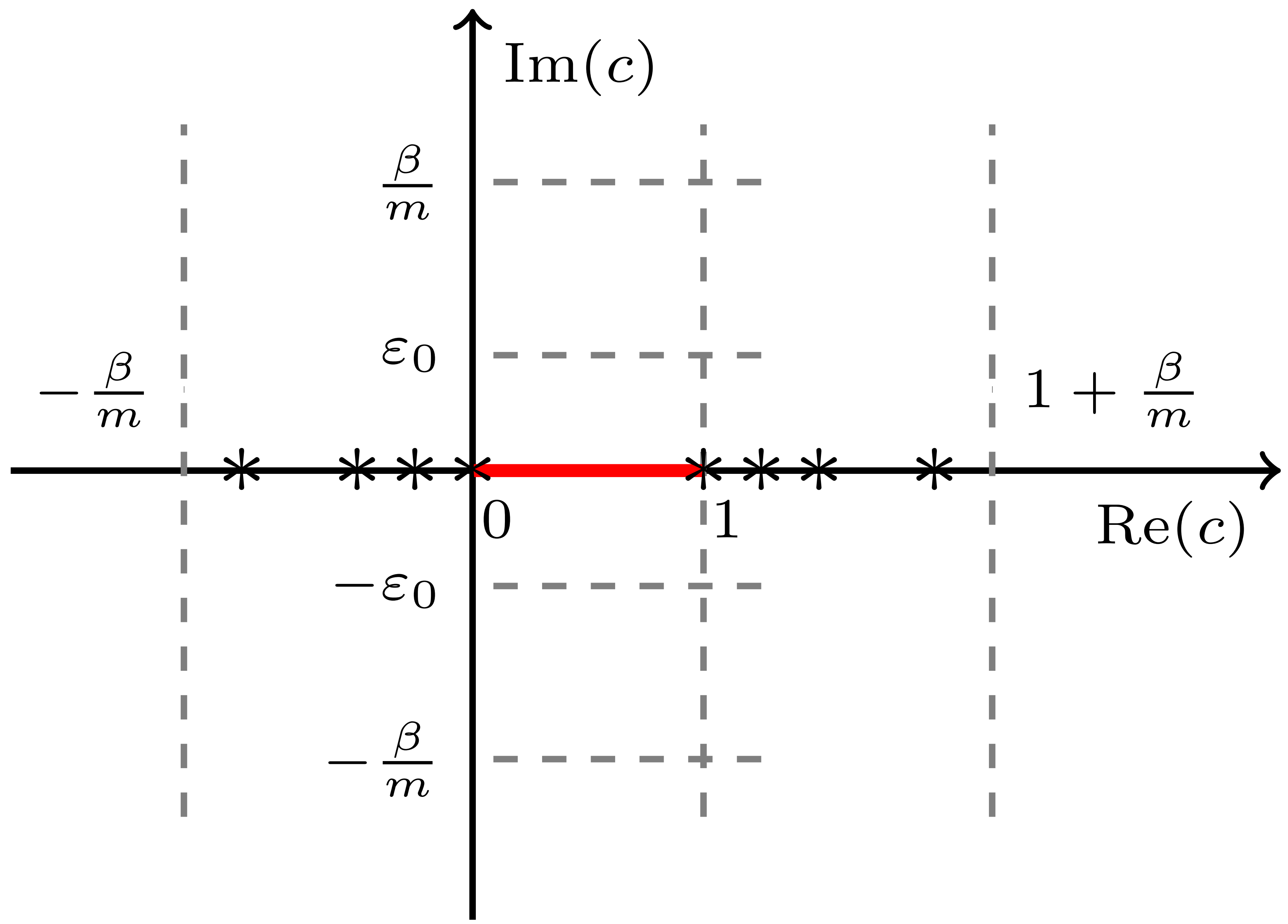
- Mode decomposition in x $\Rightarrow \partial_t \begin{pmatrix} \psi_k \\ \rho_k \end{pmatrix} + i\mathbf{L}_k \begin{pmatrix} \psi_k \\ \rho_k \end{pmatrix} = 0 \quad \Rightarrow \begin{pmatrix} \psi_k(t) \\ \rho_k(t) \end{pmatrix} = e^{i\mathbf{L}_k t} \begin{pmatrix} \psi_k^0 \\ \rho_k^0 \end{pmatrix}$
- Write down $e^{i\mathbf{L}_k t}$ using Dunford [Cauchy integral] formula for operators

$$\begin{pmatrix} \psi_k \\ \rho_k \end{pmatrix} = \mathbf{e}^{i\mathbf{L}_k t} \begin{pmatrix} \psi_k^0 \\ \rho_k^0 \end{pmatrix} = \int_{\partial D} e^{-ikct} \underbrace{(c - \mathbf{L}_k)^{-1} \begin{pmatrix} \psi_k^0 \\ \rho_k^0 \end{pmatrix}}_{\text{solutions to TG}} dc \sim \int \frac{\cdots}{(y-c)^{\mu}} dc$$

[singular integral, OK only in the sense of the Principal Value]

Sketch of the approach via the LAP

- The decay estimates are obtained by a non stationary phase method (integration by parts in the spectral parameter c) that applies to this problem since the spectrum is continuous for $\beta > 1/2$



RMK

- **Rates of decay** in time depend on the **order of singularity** of the ODE equation TG (Taylor-Goldstein). The singularity of TG is worse than the Rayleigh equation (homogeneous case), then the decay is slower
- The **TG equation has the same order of singularity with and without the Boussinesq approximation**, then **the Boussinesq approximation should not affect the time behavior of the perturbations**

Stability in 3D

[Del Zotto' 23] $u=(u^1,u^2,u^3)$ and β is the Brunt-Väisälä frequency. The 3D Couette flow is $(y,0,0)$

$$\begin{cases} \partial_t u + y\partial_x u + u^2\hat{x} + 2\nabla(-\Delta)^{-1}\partial_x u^2 + \beta\nabla(-\Delta)^{-1}\partial_y\theta = \nu\Delta u - \beta\theta\hat{y}, \\ \partial_t\theta + y\partial_x\theta - \beta u^2 = \kappa\Delta\theta, \end{cases}$$

Linear enhanced dissipation

Assume $\beta > 1/2$ and $\nu = \kappa > 0$. Define $C_\beta^2 = \frac{2\beta+1}{2\beta-1}\exp\left(\frac{1}{2\beta-1}\right)$ and $\lambda_\nu := \nu\left(1-\frac{1}{2\beta}\right)$

Then $\begin{cases} \|(u^1,u^3)_{\neq}\|_{L^2} + \langle t\rangle^{3/2}\|u^2_{\neq}\|_{L^2} + \langle t\rangle^{1/2}\|\theta_{\neq}\|_{L^2} \lesssim e^{-\lambda_\nu t^3}(\|u_{\neq}(0)\|_{H^3} + \|\theta_{\neq}(0)\|_{L^2}) \\ \|(u_0,\theta_0)(t)\|_{L^2} \lesssim_\beta e^{-\nu t}\|(u_0^{\text{in}},\theta_0^{\text{in}})\|_{H^4} \end{cases}$

RMK The last estimate implies suppression of the lift-up effect

Stability in 3D in the homogeneous setting

In the homogeneous setting, the x-average of the Navier-Stokes equations satisfies

$$\begin{cases} \partial_t u_0^1 + u_0^2 = \nu \Delta_{y,z} u_0^1 \\ \partial_t u_0^i = \nu \Delta_{y,z} u_0^i, \qquad i = 1,2 \end{cases} \qquad u_0(t) = e^{\nu \Delta_{y,z} t}(u_0^1(0) - t u_0^2(0), u_0^2(0), u_0^3(0))$$

Thus the x-average displays a linear growth in time as $\nu \to 0$

*****

In contrast, in the nonhomogeneous setting all the components decay at the rate of the heat equation as soon as $\beta > 0$

$$\|(u,\theta)(t)\|_{\mathbf{L}^2} \lesssim \nu^{-8/9} \|(u,\theta)(0)\|_{H^4} \leq C \quad \text{if} \quad \|(u,\theta)(0)\|_{H^4} \lesssim \nu^{8/9}$$

While in the homogeneous case ν replaces $\nu^{8/9}$

The nonlinear 2D inviscid problem in $\mathbb{T} \times \mathbb{R}$

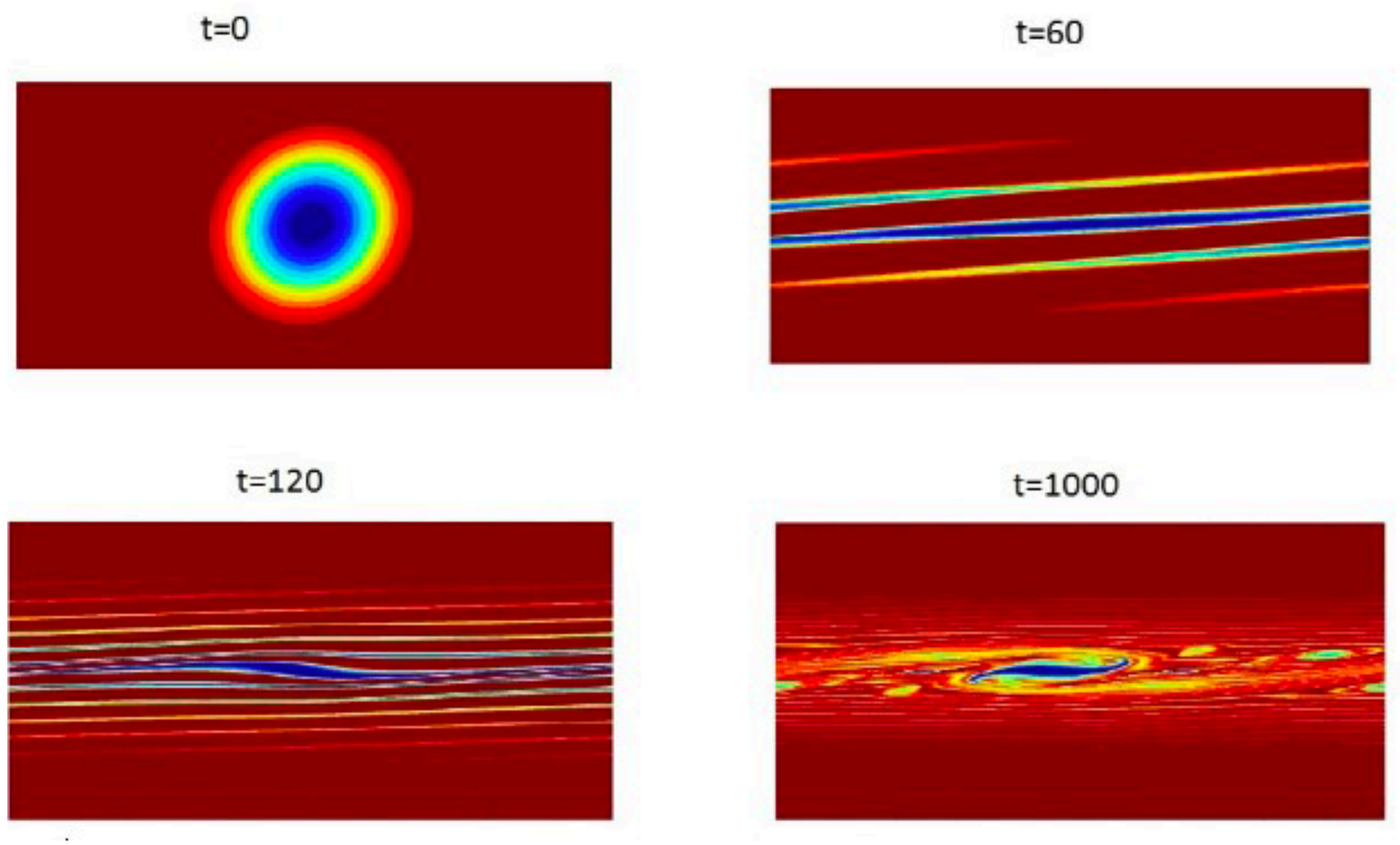
Echoes in the homogeneous-density case

Since $\nabla \cdot \mathbf{u} = 0 \Rightarrow \mathbf{u} \cdot \nabla = u_0^x \partial_x + \mathbf{u}_{\neq} \cdot \nabla$

Homogeneous 2D Euler around Couette

$$\begin{cases} \partial_t \omega + (y + u_0^x(t,y))\partial_x \omega = -\mathbf{u}_{\neq} \cdot \nabla \omega \\ \mathbf{u} = \nabla^{\perp} \Delta^{-1} \omega \end{cases}$$

- The 0-th mode does not decay
- It's time average + y is the final shear flow
- The nonlinear term is important: **toy model**



An echo: Shinrelmann 2013

Toy model: X=x-yt; Y=y

$$\nabla^{\perp}\Delta^{-1}\omega \cdot \nabla \omega \rightarrow \nabla^{\perp}\Delta_L^{-1}\Omega \cdot \nabla \Omega$$

$$\partial_t \widehat{\Omega}_k \approx \mathcal{F}(\partial_v \Delta_L^{-1}\Omega \partial_z \Omega)_k$$

$$\mathcal{F}(\partial_v \Delta_L^{-1}\Omega)_k = \frac{\eta}{k^2} \frac{\widehat{\Omega}_k}{1 + |t - \eta/k|^2}$$

- **Paraproduct decomposition**

$$(\mathbf{u} \cdot \nabla)\omega = (\mathbf{u} \cdot \nabla)\omega_{\text{High-Low}} + (\mathbf{u} \cdot \nabla)\omega_{\text{LH}} + (\mathbf{u} \cdot \nabla)\omega_{\text{HH}}$$

Echo cascade heuristic

- Initial perturbation by a single mode $\delta \exp(kx + i\eta y)$

$$\partial_t f_{k-1} \sim \frac{\eta}{k^2} \frac{f_k}{1 + (t - \eta/k)^2}$$

excites $(k-1, \eta)$ at the resonant time $t_k = \eta/k$

$$\partial_t f_{k-2} \sim \frac{\eta}{(k-1)^2} \frac{f_{k-1}}{1 + (t - \eta/(k-1))^2}$$

excites $(k-2, \eta)$ at the resonant time $t_{k-1} = \eta/(k-1)$

For $t \sim \eta/k$ and $\eta/k^2 \gg 1$

High-to-low frequency cascade may happen

$$k \to k-1 \to \cdots 1$$

$$(\eta/k^2)(\eta/(k-1)^2)\cdots(\eta/1^2) \sim e^{\sqrt{\eta}}$$

Gevrey 2 regularity

Notion of damping

- The **velocity** converges strongly in $\mathbf{L}^2$ as $t \to +\infty$

$$u^x(t,x,y) \to_{\mathbf{L}^2} u_0^x = \int_{\mathbb{T}} u^x(\,\cdot\,,x)\,dx \quad \text{at rate} \quad t^{-1}$$

$$u^y(t,x,y) \to_{\mathbf{L}^2} 0 \quad \text{at rate} \quad t^{-2}$$

- The **vorticity** converges only weakly

$$\omega(t,x,y) \rightharpoonup \omega_\infty(t,x-tu_\infty(y),y)$$

Where $\omega_\infty(t,x-tu_\infty(y),y)$ [scattering profile] solves a **linear** problem

Nonlinear Boussinesq in 2d

$$\begin{cases} \partial_t \omega + (y + u_0^x)\partial_x \omega = -\beta^2 \partial_x \theta - \mathbf{u}_{\neq} \cdot \nabla \omega \\ \partial_t \theta + (y + u_0^x)\partial_x \theta = \partial_x \psi - \mathbf{u}_{\neq} \cdot \nabla \theta \\ \mathbf{u} = \nabla^\perp \psi \qquad \Delta \psi = \omega \end{cases} \qquad \mathbb{T} \times \mathbb{R}$$

1) Pro/contro shared with the Euler

- Decay of $\mathbf{u}_{\neq}$ due to inviscid damping
- u_0^x does not decay → nonlinear change of coordinates to go beyond the linear time-scale $O(\varepsilon)$
- $\mathbf{u}_{\neq}$ may create echoes at resonant times

Guess: linear behavior persists for data of Gevrey norm $O(\varepsilon)$ and $t \sim \varepsilon^{-2}$

2) Peculiarities of linear Boussinesq

- Slower damping rates
- $\|\omega\| + \|\nabla \theta\|_{L^2(\mathbb{T}\times\mathbb{R})} \sim \varepsilon t^{1/2}$ in $L^2(\mathbb{T}\times\mathbb{R})$ and $\sim O(1)$ when $t \sim \varepsilon^{-2}$
- $\partial_t(t(v' - 1)) = \omega_0 \sim \varepsilon t^{1/2} < \delta$ if $t = O(\delta \varepsilon^{-2})$

NONLINEAR INVISCID DAMPING

Denote $\|f\|^2_{\mathcal{G}^\lambda} = \sum_{k\in\mathbb{Z}} \int e^{2\lambda(|k|+|\eta|)^s} |\hat{f}_k(\eta)|^2 d\eta$ and $f_0(y) = \frac{1}{2\pi}\int_{\mathbb{T}} f(x,y)dx, \quad f_{\neq} = f - f_0$

Theorem [J. Bedrossian, R. Bianchini, M. Coti Zelati, M. Dolce]

Let $\beta > 1/2$. For all $1/2 < s \leq 1$, $\lambda_0 > \lambda' > 0$, there exist $\varepsilon_0 \ll \delta < 1 \quad [\delta^2 \sim (\beta^2 - 1/4)]$ such that for $\varepsilon \leq \varepsilon_0$ and ω^{in}, θ^{in} mean-free initial data with

$$\|\mathbf{u}^{in}\|_{L^2} + \|\omega^{in}\|_{\mathcal{G}^{\lambda_0}} + \|\theta^{in}\|_{\mathcal{G}^{\lambda_0}} \leq \varepsilon\,.$$

Define the phase shift $\Phi(t,y) = \int_0^t u_0^x(\tau,y)d\tau$. Then, for all $\mathbf{0 \leq t \leq \delta^2\varepsilon^{-2}}$

$$\|u_0^x(t)\|_{\mathcal{G}^{\lambda'}} + \|\theta_0(t)\|_{\mathcal{G}^{\lambda'}} \lesssim \varepsilon$$

$$\|\omega(t,x+ty+\Phi(t,y),y)\|_{\mathcal{G}^{\lambda'}} + \langle t\rangle \|\theta_{\neq}(t,x+ty+\Phi(t,y),y)\|_{\mathcal{G}^{\lambda'}} \lesssim \varepsilon\langle t\rangle^{1/2}$$

Therefore

$$\|u^x_{\neq}(t)\|_{L^2} + \|\theta_{\neq}(t)\|_{L^2} + \langle \mathbf{t}\rangle \|u^y_{\neq}(t)\|_{L^2} \lesssim \varepsilon\langle \mathbf{t}\rangle^{-\frac{1}{2}}$$

Shear-Bouyancy Instability

Theorem [J. Bedrossian, R. Bianchini, M. Coti Zelati, M. Dolce 2021]

Same hypotheses. There exists $K > 0$ such that, if

$$\|\omega_{\neq}^{in}\|_{H^{-1}} + \|\theta_{\neq}^{in}\|_{L^2} \geq K\varepsilon\delta$$

Then

$$\|\omega_{\neq}(t)\|_{L^2} + \|\nabla\theta_{\neq}(t)\|_{L^2} \approx \varepsilon\langle \mathbf{t}\rangle^{\frac{1}{2}} \quad \text{for all} \quad \mathbf{0 \leq t \leq} \delta^{2}\varepsilon^{-2}$$

Nonlinear dynamics: the change of coordinates

$$\begin{cases} \partial_t\omega + y\partial_x\omega = -\beta^2\partial_x\theta - \mathbf{u}\cdot\nabla\omega \\ \partial_t\theta + y\partial_x\theta = \partial_x\psi - \mathbf{u}\cdot\nabla\theta \\ \mathbf{u} = \nabla^\perp\psi \qquad \Delta\psi = \omega \end{cases}$$

in $\mathbb{T}\times\mathbb{R}$

3 main ingredients:

1) nonlinear **change of coordinates**

2) **change of variables** ["symmetrization"] to handle the linear dynamics

3) **dynamical weight** inside the norm to control echo chains

$$\nabla\cdot\mathbf{u} = 0 \quad \Rightarrow \quad \mathbf{u}\cdot\nabla = u_0^x\partial_x + \mathbf{u}_{\neq}\cdot\nabla$$

where $\mathbf{u}_{\neq}$ should decay

$$v = y + \frac{1}{t}\int_0^t u_0^x(s,y)\,ds \qquad z = x - vt$$

1) change of coordinates

$$\begin{cases} \Omega(t,z,v) = \omega(t,x,y) \\ \Theta(t,z,v) = \theta(t,x,y) \\ \Psi(t,z,v) = \psi(t,x,y) \end{cases}$$

RMK: to be invertible, the zero mode needs to stay small, thus we need a **coordinate system control**

The system in the new coordinates

$$\begin{cases} \partial_t\Omega = -\beta^2\partial_z\Theta - \mathsf{U}\cdot\nabla\Omega \\ \partial_t\Theta = \partial_z\Psi - \mathsf{U}\cdot\nabla\Theta \\ \mathsf{U} = (0,\dot v) + v'\nabla^\perp\Psi_{\neq} \qquad \Delta_t\Psi = \Omega \end{cases}$$

$$\dot v = \partial_t v\ ; \quad \nabla = \nabla_{z,v}; \quad \Delta_t := \partial_{zz} + (v')^2(\partial_v - t\partial_z)^2 + v''(\partial_v - t\partial_z)$$

Zoom in on the construction of the toy model

- high-low term $(\partial_v \Delta_L^{-1} \Omega)_{Hi} \cdot (\partial_z \Omega)_{lo}$ [as $\partial_z \Omega$ is low $\Rightarrow \eta \sim 0$]
- nearest interaction $k \Rightarrow k-1$
- near critical times "resonant interval" $|t - \eta/k| \leq \eta/k^2$

For any $t > 0$ there is at most a critical k such that $t \approx \frac{\eta}{k} \Rightarrow \{k, k-1\}$. The toy model reduces to

$$\partial_t Z_k \approx \epsilon t^{1/2} \left(\frac{k^2}{\eta}\right)^{\frac{1}{2}} \frac{Z_{k-1}(\eta)}{(1 + |t - \eta/k|^2)^{\frac{1}{4}}}$$

$$\partial_t Z_{k-1} \approx \epsilon t^{1/2} \left(\frac{\eta}{k^2}\right)^{\frac{1}{2}} \frac{Z_k(\eta)}{(1 + |t - \eta/k|^2)^{\frac{3}{4}}}$$

Construct a weight $\mathbf{w_k(t, \eta)}$ encoding the maximal growth predicted by the toy model

$w_R = Z_k; \quad w_{NR} = Z_{k-1} \quad t \in [-\eta/k^2, \eta/k^2]$

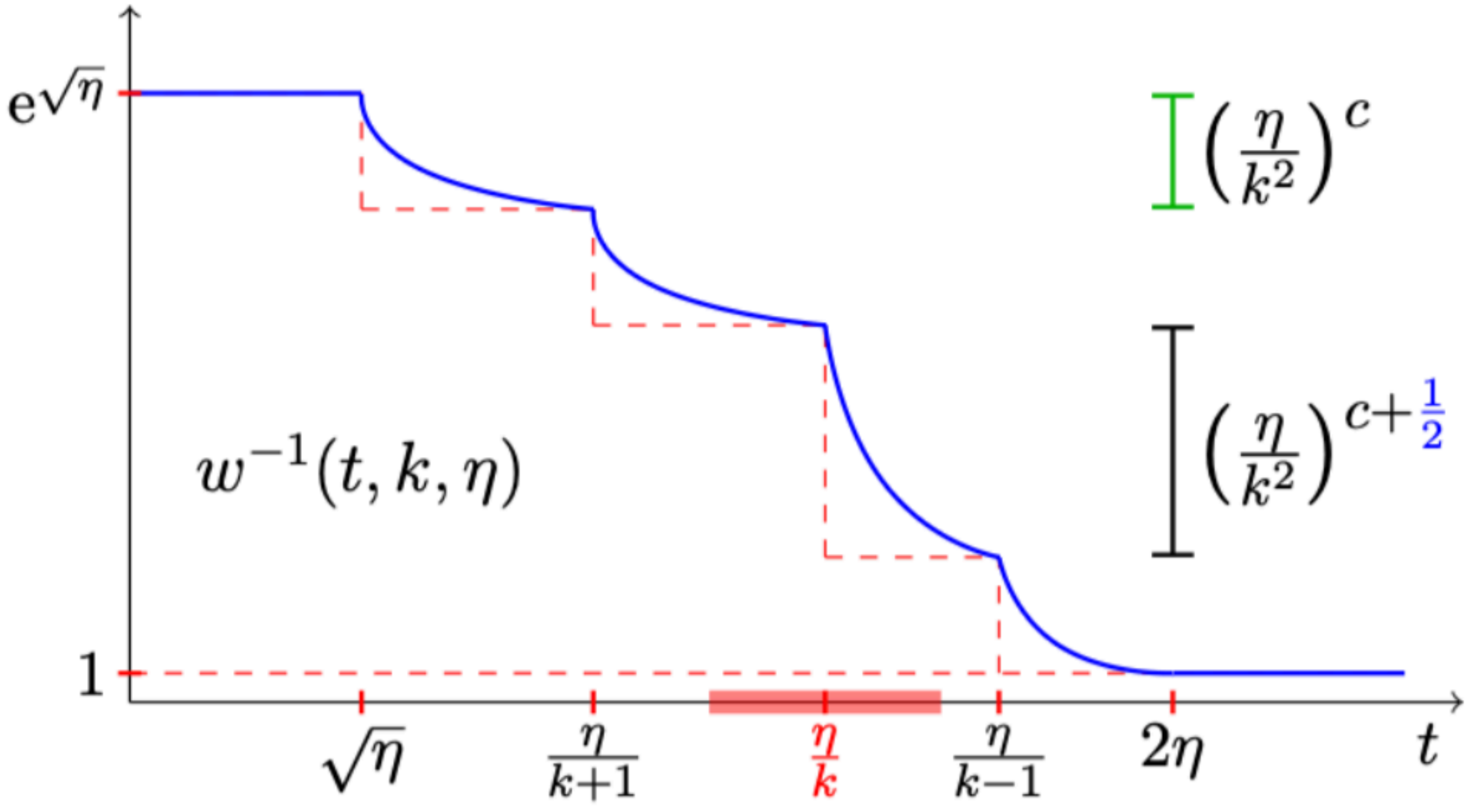
$$\partial_t w_R = \left(\frac{k^2}{\eta}\right)^{\frac{1}{2}} \frac{w_{NR}}{(1 + |t - \eta/k|^2)^{\frac{1}{4}}}$$

$$\partial_t w_{NR} = \left(\frac{\eta}{k^2}\right)^{\frac{1}{2}} \frac{w_R}{(1 + |t - \eta/k|^2)^{\frac{3}{4}}}$$

Proposition. The maximal growth of $\mathbf{w_k(t, \eta)}$

$$\frac{w_k(t_{k-1,\eta}, \eta)}{w_k(t_{k,\eta}, \eta)} \approx \frac{w_{NR}}{w_R} \lesssim \left(\frac{\eta}{k^2}\right)^{\frac{1}{2}}$$

The dynamical weight



The strategy of the proof

The proof is based on a bootstrap argument with several ingredients.

The most important are:

- "Linear" energy functional [with the dynamical weight inside]

$$\mathsf{E}_{lin}(t) = \frac{1}{2}\left[\|\mathbf{A}Z\|^2 + \|\mathbf{A}Q\|^2 + \frac{1}{2\beta}\left\langle \frac{\partial_t p}{|k|p^{\frac{1}{2}}}\mathbf{A}Z, \mathbf{A}Q \right\rangle \right]$$ where $\mathbf{A} \sim \mathbf{w_k}^{-1} e^{\sqrt{\eta}}$

- "nonlinear" energy functional [with the dynamical weight inside]

$$\mathsf{E}_{nonlin}(t) = \frac{1}{2}\left[\|\mathbf{A}\Omega\|^2 + \beta^2 \|\mathbf{A}\,\nabla_L \Theta\|^2 \right]$$

- Energy functional to control the change of coordinates

The nonlinear terms are treated by using a para-product decomposition

The main weight A

- For the variable Q we have the same bounds → we can use the same multiplier w
- In [BM15], the amplification factor is $\left(\frac{\eta}{k^2}\right)$ rather than $\left(\frac{\eta}{k^2}\right)^{\frac{1}{2}}$ → the regularity gap among resonant & non-resonant modes is different

We define the main weight:

$$A_k(t,\eta) = \langle k,\eta\rangle^{\sigma} e^{\lambda(t)|k,\eta|^s} \left(m^{-1}J\right)_k(t,\eta) \quad \text{where} \quad J_k(t,\eta) = \frac{e^{\mu|\eta|^{\frac{1}{2}}}}{w_k(t,\eta)} + e^{\mu|k|^{\frac{1}{2}}} \text{ and } m \text{ is bounded}$$

$$\partial_t A = \dot{\lambda}(t)\,|k,\eta|^s A - \frac{\partial_t w}{w}\tilde{A} - \frac{\partial_t m}{m}A$$

(in $\tilde{A}$ J_k is replaced with $\tilde{J}_k = e^{\mu|\eta|^{\frac{1}{2}}} w_k^{-1}$)

Restrict the radius of regularity by a finite a amount & continuously in time

$$\partial_t \lambda = -\langle t\rangle^{-\delta-1}$$

Control the echo chain

Artificial dissipation that absorbs the integrable remainders of the linear dynamics

$$\frac{\partial_t m}{m} = \frac{C_\beta}{1 + |t - \eta/k|^2}$$

The "linear" energy functional

Symmetrized variables to handle the linear dynamics $Z := (p/k^2)^{-\frac{1}{4}}\widehat{\Omega} \quad Q := (p/k^2)^{\frac{1}{4}} ik\beta\widehat{\Theta}$

$$E_L(t) = \frac{1}{2}\left[\|AZ\|^2 + \|AQ\|^2 + \frac{1}{2\beta}\left\langle \frac{\partial_t p}{|k|p^{\frac{1}{2}}}AZ, AQ\right\rangle\right]$$

where A is a weight encoding Gevrey regularity

$$\frac{d}{dt}E_L + \left(1 - \frac{1}{2\beta}\right)\sum_{j\in\{\lambda,w,m\}}(G_j[Z] + G_j[Q]) \leq L^{Z,Q} + NL^{Z,Q} + \mathcal{E}^{div} + \mathcal{E}^{\Delta_t}$$

$$NL^{Z,Q} = \left|\left\langle \mathcal{F}\left(\left[A\left(\frac{p}{k^2}\right)^{-\frac{1}{4}}, \mathbf{U}\right]\cdot\nabla\Omega\right), AZ + \frac{1}{4\beta}\frac{\partial_t p}{|k|p^{\frac{1}{2}}}AQ\right\rangle\right| + \frac{1}{4\beta}\left|\left\langle\left[\frac{\partial_t p}{|k|p^{\frac{1}{2}}}, \mathbf{U}\right]\cdot\nabla AZ, AQ\right\rangle\right| = NL^{Z,Q}_{\text{High-Low}} + NL^{Z,Q}_{\text{Low-High}} + NL^{Z,Q}_{\text{High-High}}$$

Several open questions

- Instability $\sim \sqrt{t}$ in $\mathbb{T}\times\mathbb{R}$ but in $\mathbb{T}\times[0,1]$?
- for $\beta \leq 1/4$ nonlinear?
- After $t \sim O(\varepsilon^{-2})$? Gevrey losses?
- More general shears? (Maybe monotone for now)
- No Boussinesq approximation?

Instability and ill-posedness near a shear with Ri$(y) < 1/4$

Choose a density profile of the form $-\bar{\rho}'(y) = \alpha(1-\alpha)(U'(y))^2$, $\alpha \in (0,1)$, $\alpha \neq \frac{1}{2}$

$\Rightarrow$ this choice forces the violation of the Miles-Howard criterion Ri$(y) < 1/4$

Choose $\psi = (U-c)^\alpha \phi$

$\Rightarrow$ the Taylor-Goldstein equation reads

$$(U-c)(\partial_y^2 - k^2)\phi + 2\alpha U' \partial_y \phi + (\alpha - 1)U''\phi = 0$$

* If $k=0$ and $\alpha = 1$, it gives the hydrostatic Rayleigh equation ($k = \varepsilon \tilde{k}$)

$$(U-c)\partial_y^2 \phi + 2\alpha U' \partial_y \phi = 0$$

$\Rightarrow$ the Taylor-Goldstein equation reads

$$(U-c)(\partial_y^2 - k^2)\phi + 2\alpha U'\partial_y \phi + (\alpha - 1)U''\phi = 0$$

* If $k = 0$ and $\alpha = 1$, it gives the hydrostatic Rayleigh equation ($k = \varepsilon \tilde{k}$)

$$(U-c)\partial_y^2 \phi + 2\alpha U'\partial_y \phi = 0$$

for which we know at least one shear flow $U(y) = \tanh(y/d), \quad 0 < d \ll 1$ having inflection point and providing an unstable eigenvalue [Renardy]

$\Rightarrow$ the perturbations are both order 0 while the main operator is order 2. Apply a perturbation approach to deduce the existence of an unstable eigenvalue for the hydrostatic Boussinesq equations and for the non-hydrostatic Boussinesq equations at small horizontal frequencies

[it should give ill-posedness in $H^s, s > 0$ of the hydrostatic equations and a proof of invalidity of the hydrostatic limit in this setting]

[ongoing project with Lucas ERTZBISCHOFF and Michele COTI ZELATI]