Markovian Repeated Interaction Quantum Systems

Jean-François Bougron[1,2], Alain Joye[2], Claude-Alain Pillet[3]

[1] CY Cergy Paris Université, CNRS, AGM, 95000 Cergy-Pontoise, France

[2] Univ. Grenoble Alpes, CNRS, Institut Fourier, 38000 Grenoble, France

[3] Aix Marseille Univ., Université de Toulon, CNRS, CPT, Marseille, France

Abstract. We study a class of dynamical semigroups $(\mathbb{L}^n)_{n\in\mathbb{N}}$ that emerge, by a Feynman–Kac type formalism, from a random quantum dynamical system $(\mathscr{L}_{\omega_n}\circ\cdots\circ\mathscr{L}_{\omega_1}(\rho_{\omega_0}))_{n\in\mathbb{N}}$ driven by a Markov chain $(\omega_n)_{n\in\mathbb{N}}$. We show that the almost sure large time behavior of the system can be extracted from the large n asymptotics of the semigroup, which is in turn directly related to the spectral properties of the generator $\mathbb{L}$. As a physical application, we consider the case where the $\mathscr{L}_\omega$'s are the reduced dynamical maps describing the repeated interactions of a system $\mathscr{S}$ with thermal probes $\mathscr{E}_\omega$. We study the full statistics of the entropy in this system and derive a fluctuation theorem for the heat exchanges and the associated linear response formulas.

Contents

1 Introduction **2**

2 The Feynman–Kac Formalism of MRIS **5**

2.1 Notations and Conventions . . . 5
2.2 Markovian Repeated Interaction Quantum Systems . . . 7
2.3 Extended Observables and Extended States . . . 8
2.4 The Semigroup . . . 8
2.5 Primitivity and Irreducibility as Ergodic Properties . . . 10

3 Main Abstract Results **12**

3.1 A Pointwise Ergodic Theorem . . . 12
3.2 Inhomogeneous MRIS in the Adiabatic Limit . . . 14

4 Entropic Fluctuations and Linear Response Theory **15**

4.1 Entropy Balance . . . 16
4.2 Full Statistics of Entropy . . . 19
4.3 The Fluctuation Theorem . . . 22
4.4 Linear Response . . . 23

5 Proofs **26**

5.1 Proof of Proposition 4.1 . . . 26
5.2 Proof of Theorem 4.2 . . . 28
5.3 Proof of Theorem 4.4 . . . 31
5.4 Proof of Theorem 4.6 . . . 31

1 Introduction

Repeated Interaction Systems

From a physical perspective, a Repeated Interaction System, RIS for short, is a quantum system $\mathscr{S}$, together with a sequence[1] $(\mathscr{E}_n)_{n\in\mathbb{N}^*}$ of quantum probes. $\mathscr{S}$, the system of interest, is described by a Hilbert space $\mathscr{H}_{\mathscr{S}}$. Each probe $\mathscr{E}_n$ is a quantum system characterized by a Hilbert space $\mathscr{H}_{\mathscr{E}_n}$ and a (normal) state or density matrix $\rho_{\mathscr{E}_n}$. Unless otherwise stated, all Hilbert spaces in this work are supposed to be finite dimensional.

The dynamics of the compound system is defined as follows: the system $\mathscr{S}$, initially in a state ρ_0, interacts for a certain time with the first probe $\mathscr{E}_1$ initially in its state $\rho_{\mathscr{E}_1}$. Tracing out the degrees of freedom of the first probe $\mathscr{E}_1$, one gets a state ρ_1 for $\mathscr{S}$ which is then put in contact with the next probe $\mathscr{E}_2$ in its initial state $\rho_{\mathscr{E}_2}$. Repeating the procedure defines a sequence $(\rho_n)_{n\in\mathbb{N}}$ where ρ_n is the state of $\mathscr{S}$ after n interactions.

A well known example of Repeated Interaction System in physics is the one-atom maser experiment: $\mathscr{S}$ is then the quantized electromagnetic field of a cavity, and the probes are atoms coming from an oven and passing through this cavity, see, e.g., [FJM86, MWM85, RBH01, WBKM00].

Typically, the interaction process is described as follows: the free Hamiltonian of the small system, resp. of the probe number n, is a self-adjoint operator denoted by $H_{\mathscr{S}}$, resp. $H_{\mathscr{E}_n}$, and the interaction between $\mathscr{S}$ and $\mathscr{E}_n$ is induced by a coupling V_n which is a self-adjoint operator on $\mathscr{H}_{\mathscr{S}} \otimes \mathscr{H}_{\mathscr{E}_n}$. The total Hamiltonian governing the interaction of $\mathscr{S}$ and $\mathscr{E}_n$ is thus

$$H_n := H_{\mathscr{S}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathscr{E}_n} + V_n \tag{1.1}$$

acting on $\mathscr{H}_{\mathscr{S}} \otimes \mathscr{H}_{\mathscr{E}_n}$. In the sequel, we will drop the symbols $\otimes\mathbb{1}$ and $\mathbb{1}\otimes$ when the context is clear. The reduced propagator $\mathscr{L}_n$, obtained by tracing out the probe's degrees of freedom,

$$\mathscr{L}_n\rho := \mathrm{tr}_{\mathscr{H}_{\mathscr{E}_n}}(\mathrm{e}^{-\mathrm{i}\tau_n H_n}(\rho \otimes \rho_{\mathscr{E}_n})\mathrm{e}^{\mathrm{i}\tau_n H_n}),$$

describes the evolution of the state of $\mathscr{S}$ due to its interaction with $\mathscr{E}_n$ for a duration $\tau_n > 0$.

By construction, $\mathscr{L}_n$ is a Completely Positive Trace Preserving (CPTP for short) map on the set $\mathscr{B}^1(\mathscr{H}_{\mathscr{S}})$ of trace class operators on $\mathscr{H}_{\mathscr{S}}$. Consequently, the state ρ_n of the system $\mathscr{S}$ after interacting with the first n probes is given by

$$\rho_n := \mathscr{L}_n \dots \mathscr{L}_1 \rho_0.$$

RIS have been introduced and developed in the non-exhaustive list of papers [AP06, BJM06, BJM08, BP09, BJM10b, BDBP11, NP12, Bru14, BJM14, HJPR17, HJPR18, BB20, MS19, MS20].

To take into account the uncontrollable odds that may affect the probes and their interaction with the system in real physical applications, it makes sense to generalize the above setup and consider random dynamical systems

$$\rho_n(\boldsymbol{\omega}) = \mathscr{L}_{\omega_n} \cdots \mathscr{L}_{\omega_1} \rho_{\omega_0}, \tag{1.2}$$

where $\omega \ni \Omega \mapsto \rho_\omega \in \mathscr{B}^1(\mathscr{H}_{\mathscr{S}})$, $\omega \ni \Omega \mapsto \mathscr{L}_\omega$ takes its values in CPTP-maps on $\mathscr{B}^1(\mathscr{H}_{\mathscr{S}})$ and $\boldsymbol{\omega} = (\omega_n)_{n\in\mathbb{N}} \in \Omega^{\mathbb{N}}$ denotes the sample path of a stochastic process.

One way to think about this setup is to consider that, according to some probabilistic rules, the probes are drawn from a pool which is partitioned into a family $(\mathscr{R}_\omega)_{\omega\in\Omega}$, each $\mathscr{R}_\omega$ containing identical probes. Assuming that all the probes are in thermal equilibrium, the physical

[1] We denote by $\mathbb{N}^*$ the positive integers and by $\mathbb{N} = \mathbb{N}^* \cup \{0\}$ the non-negative ones.

situation is then close to the one of a small system $\mathscr{S}$ interacting with several thermal reservoirs $\mathscr{R}_\omega$. We shall henceforth call the $\mathscr{R}_\omega$'s *reservoirs*. For a given ω, $\mathscr{L}_\omega$ represents the effect on $\mathscr{S}$ of an interaction with the reservoir $\mathscr{R}_\omega$.

As an example, consider the previously mentioned one-atom maser. The interaction times τ_n depend on the flight velocities of the atoms which fluctuate. In this case, a reasonable model would be

$$\mathscr{L}_\tau \rho := \mathrm{tr}_{\mathscr{H}_\mathscr{E}}(\mathrm{e}^{-\mathrm{i}\tau H}\, \rho \otimes \rho_\mathscr{E}\, \mathrm{e}^{\mathrm{i}\tau H}),$$

the probe Hilbert space $\mathscr{H}_\mathscr{E}$, the total Hamiltonian H and the probe state $\rho_\mathscr{E}$ being independent of n, and $\boldsymbol{\tau} = (\tau_n)_{n\in\mathbb{N}^*}$ being an i.i.d. sequence of random variables. Now assume that the atoms are in thermal equilibrium with the oven, *i.e.*, $\rho_\mathscr{E}^\beta = \mathrm{e}^{-\beta H_\mathscr{E}} / \mathrm{tr}(\mathrm{e}^{-\beta H_\mathscr{E}})$ where β denotes the oven's inverse temperature. To take fluctuations of the latter into account, we should consider a process where $\omega_n = (\tau_n, \beta_n)$ and

$$\mathscr{L}_{(\tau,\beta)} \rho := \mathrm{tr}_{\mathscr{H}_\mathscr{E}}(\mathrm{e}^{-\mathrm{i}\tau H}\, \rho \otimes \rho_\mathscr{E}^\beta\, \mathrm{e}^{\mathrm{i}\tau H}).$$

In this case, it may be more realistic to allow for some correlations between successive probes.

The purpose of this article is to study the large n behavior of the random dynamical system (1.2) when the driving stochastic process is a Markov chain with a transition matrix P and initial probability vector π on a finite state space Ω. We will call such a system a *Markovian Repeated Interaction System*, MRIS for short. We will first consider (1.2) in the abstract setting where $(\mathscr{L}_\omega)_{\omega\in\Omega}$ is a family of arbitrary CPTP-maps on $\mathscr{B}^1(\mathscr{H}_\mathscr{S})$ for a finite dimensional Hilbert space $\mathscr{H}_\mathscr{S}$. From a mathematical perspective, our main result is a pointwise ergodic theorem for MRIS. We will then apply this abstract result to the more concrete and physically motivated case where each $\mathscr{L}_\omega$ describes the interaction of a system $\mathscr{S}$ with a probe $\mathscr{E}_\omega$. We shall restrict our attention to MRIS with thermal reservoirs, *i.e.*, to the cases where each probe $\mathscr{E}_\omega$ is in thermal equilibrium at a given inverse temperature β_ω. We will investigate energy transfers between the system $\mathscr{S}$ and the reservoirs and the induced entropy production. Under microscopic time-reversibility, we will derive a strong form of nonequilibrium fluctuation relations and their connections with linear response. Our framework also allows us to consider situations where the driving Markov chain is not homogeneous in time, meaning that the transition matrix P depends on the time step. Assuming the variations between successive transition matrices is small, we derive the large time asymptotics of the corresponding random dynamical system, in this instance of the adiabatic regime.

Several limiting cases of MRIS have been addressed in previous works: these are the deterministic case with a single reservoir, the periodic case where P is a cyclic permutation matrix, and the i.i.d. case where $\boldsymbol{\omega}$ is a sequence of independent and identically distributed random variables. The deterministic case — the simplest example of RIS — was introduced and developed in [AP06, BJM06]. In particular, the latter reference shows the strict positivity of entropy production even in this simple setup. The periodic case, mentioned in [BJM14], provides another toy model of RIS allowing for a study of its nonequilibrium thermodynamics. The i.i.d. case has been analyzed in [BJM08, BJM10a], see also [NP12]. The nonequilibrium thermodynamics of both these models has been recently investigated in [BB20]. The papers [MS19, MS20] address more general abstract situations where the underlying stochastic process is an ergodic dynamical system, and the large time asymptotics of the states is investigated by means of methods of dynamical systems.

Remark. We consider Markovian randomness in the RIS framework to take into account (classical) correlations between fluctuating probes. The case of quantum entanglement between the probes has been studied, for instance, in [HJ17, Raq20, AJR21].

Feynman–Kac Formalism

In quantum mechanics one is generally interested in quantum expectation values of observables which can depend on time and, in particular, in the large time behavior of those expectations. In our model we will focus on observables which depend on which reservoir $\mathscr{S}$ is currently interacting with. Consequently, we will study the large n properties of

$$\mathrm{tr}(\rho_n(\boldsymbol{\omega})\ X(\omega_{n+1})), \tag{1.3}$$

where $\Omega \ni \omega \mapsto X(\omega) \in \mathscr{B}(\mathscr{H}_{\mathscr{S}})$. The idea to deal with this model is to work in the Hilbert space

$$\mathscr{K} = \ell^2(\Omega;\mathscr{H}_{\mathscr{S}})$$

of square-summable, $\mathscr{H}_{\mathscr{S}}$-valued functions on Ω. Considering the C^*-algebra $\mathfrak{A} \subset \mathscr{B}(\mathscr{K})$ of functions $X:\Omega \to \mathscr{B}(\mathscr{H}_{\mathscr{S}})$ acting as a multiplication operator on $\mathscr{K}$, a straightforward computation shows that

$$\mathbb{E}\left[\mathrm{tr}\left(\rho_n(\boldsymbol{\omega})X(\omega_{n+1})\right)\right] = \mathrm{tr}_{\mathscr{K}}\left(X\mathbb{L}^n R\right), \tag{1.4}$$

where $\mathbb{E}$ denotes the expectation w.r.t. the Markov chain, and $\mathbb{L}$ is a CPTP map on $\mathscr{B}^1(\mathscr{K})$, constructed with the maps $\mathscr{L}_\omega$ and the transition matrix P, see (2.8), and[2]

$$R(\omega) := \mathbb{E}[\rho_{\omega_0} 1_{\{\omega_1=\omega\}}].$$

Consequently, the dynamics of $\mathscr{S}$ can be modeled with the discrete time semigroup $(\mathbb{L}^n)_{n\in\mathbb{N}}$. This structure is called a *Feynman–Kac formalism*. Following [Pil85, Pil86b, Pil86a], it turns the study of the large n behavior of (1.3) into a spectral problem for $\mathbb{L}$. See also [KS09] for a discussion of a somewhat related problem concerning random time dependent Lindbladians. We note that, in the periodic and i.i.d. cases mentioned above, the MRIS dynamics can be modeled by a semigroup acting on $\mathscr{B}^1(\mathscr{H}_{\mathscr{S}})$, that is, without the need for a Feynman–Kac formalism.

In the i.i.d. case, it has been proved in [BJM08] that, if the random CPTP map $\mathscr{L}_{\omega_0}$ is primitive with nonzero probability, then (1.3) converges almost surely in Cesàro mean to the deterministic value $\mathrm{tr}(\rho_+\mathbb{E}[X(\omega_0)])$, ρ_+ being the unique invariant state of the CPTP map $\mathbb{E}[\mathscr{L}_{\omega_0}]$. This result was extended to a more general class of i.i.d. random dynamical systems in [BB20]. In our Markovian framework, we shall see in Theorem 3.3 that the mere irreducibility of $\mathbb{L}$ implies that (1.3) converges almost surely in Cesàro mean to $\mathrm{tr}(R_+X)$, where R_+ is the unique invariant state of $\mathbb{L}$. This is in keeping with the recent results in [MS19, MS20] about ergodic positive linear map valued processes. However, our spectral point of view allowed by the Feynman–Kac formalism is distinct from the approach there, which makes essential use of positivity properties. Moreover, the spectral approach turns out to be instrumental in the subsequent analysis of the thermodynamic properties of MRIS.

A natural extension of the above results concern the case of an inhomogeneous Markov chain. A first step in this direction, dealt with in Section 3.2, is the case of a Markov chain whose transition matrix changes infinitely slowly with time, the so-called adiabatic inhomogeneous case. Making use of the analogy with the adiabatic RIS studied in [HJPR17, HJPR18], we get Theorem 3.4, that describes the large n asymptotics of the expectation (1.4) in this framework.

At last, the MRIS framework is well suited to study nonequilibrium thermodynamic properties of RIS when each reservoir $\mathscr{R}_\omega$ is in thermal equilibrium. In Section 4, we derive the entropy balance relation of MRIS process and investigate its consequences. We study the large time asymptotics of the full statistics of entropy and energy transfers and give sufficient conditions ensuring the validity of entropic fluctuation relations. For processes running near appropriately defined equilibrium, we derive the main ingredients of linear response theory — Green–Kubo formula, Onsager reciprocity relations and fluctuation-dissipation relations.

[2] We shall denote by 1_A the indicator function of a set A

Organization of the Paper

The core of this article is organized as follows. After setting up some conventions and notations, Section 2 introduces the Feynman–Kac formalism associated to the abstract framework of Markovian repeated interaction quantum systems. The section ends with a few useful results on the spectral properties of the Feynman–Kac generator $\mathbb{L}$. In Section 3, we formulate and prove our main results on abstract MRIS: a pointwise ergodic theorem and an adiabatic theorem. The nonequilibrium thermodynamics of concrete MRIS is the subject of Section 4, while Section 5 is devoted to the proofs.

Acknowledgements. The research of J.-F.B. was partially funded by the Agence Nationale de la Recherche under the programme "Investissements d'avenir" (ANR-15-IDEX-02), Cross Disciplinary Program "Quantum Engineering Grenoble". A.J. and C.-A.P. acknowledge support from Agence Nationale de la Recherche grant NONSTOPS (ANR-17-CE40-0006-01).

2 The Feynman–Kac Formalism of MRIS

2.1 Notations and Conventions

We start by setting up some notations and conventions that will be used in this work. We also recall some basic definitions concerning positive maps on C^*-algebras and their state spaces.

Let $\mathscr{H}$ be a complex Hilbert space. The inner product of two vectors $\varphi, \psi \in \mathscr{H}$ will be denoted by $\langle\varphi,\psi\rangle$, and supposed to be anti-linear in the first argument and linear in the second one. We use Dirac's notation: $|\psi\rangle\langle\varphi|$ stands for the linear map $\chi \mapsto \langle\varphi,\chi\rangle\psi$ on $\mathscr{H}$.

$\mathscr{B}(\mathscr{H})$ is the C^*-algebra of all linear operators on $\mathscr{H}$, $\mathbb{1}$ denotes its unit, and $\mathrm{sp}(X)$ is the spectrum of $X \in \mathscr{B}(\mathscr{H})$. Self-adjoint elements of $\mathscr{B}(\mathscr{H})$ with spectrum in $\mathbb{R}_+$ are said to be non-negative. $\mathscr{B}(\mathscr{H})$ is ordered by the proper convex cone $\mathscr{B}_+(\mathscr{H})$ of these non-negative elements, *i.e.,* $X \geq Y$ iff $X - Y \in \mathscr{B}_+(\mathscr{H})$. If $X \geq \delta\mathbb{1}$ for some $\delta > 0$ we say that X is positive and write $X > 0$.

The $*$-ideal of all trace class operators on $\mathscr{B}(\mathscr{H})$ is denoted $\mathscr{B}^1(\mathscr{H})$. Equipped with the trace norm $\|\alpha\|_1 = \mathrm{tr}(\sqrt{\alpha^*\alpha})$, $\mathscr{B}^1(\mathscr{H})$ is a Banach space, the predual of $\mathscr{B}(\mathscr{H})$. We write the corresponding duality as

$$\langle\alpha, X\rangle = \mathrm{tr}(\alpha^* X). \tag{2.1}$$

Since $\mathscr{H}$ is finite dimensional, $\mathscr{B}^1(\mathscr{H})$ and $\mathscr{B}(\mathscr{H})$ are identical topological vector spaces, but are distinct normed vector spaces. $\mathscr{B}^1(\mathscr{H})$ is ordered by the proper convex cone $\mathscr{B}^1_+(\mathscr{H})$ of its non-negative elements. A *state* on $\mathscr{B}(\mathscr{H})$ is an element $\rho \in \mathscr{B}^1_+(\mathscr{H})$ normalized by $\|\rho\|_1 = \langle\rho,\mathbb{1}\rangle = 1$. We denote by $\mathscr{B}^1_{+1}(\mathscr{H})$ the closed convex set of these states. Its elements are also known as *density matrices* or *statistical operators*. A state ρ is said to be faithful whenever $\rho > 0$, and pure whenever it is of rank one.

More generally, if $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$ is a unital C^*-subalgebra[3], we denote by $\mathscr{A}_+ = \mathscr{A} \cap \mathscr{B}_+(\mathscr{H})$ the proper cone of non-negative elements of $\mathscr{A}$, by $\mathscr{A}_*$ its predual and by $\mathscr{A}_{*+1}$ the set of states of $\mathscr{A}$. The duality (2.1) allows us to identify $\mathscr{A}_*$ with the vector space $\mathscr{A}$ equipped with the trace norm, and $\mathscr{A}_{*+1}$ with the subset of density matrices in $\mathscr{A}$.

A linear map $\phi : \mathscr{A} \to \mathscr{A}$ is *positivity preserving,* or simply *positive,* written $\phi \geq 0$, whenever $\phi(\mathscr{A}_+) \subset \mathscr{A}_+$. ϕ is *positivity improving,* written $\phi > 0$, if $\phi(X) > 0$ for $X \geq 0$, $X \neq 0$. A positive map ϕ is said to be *primitive* if $\phi^n > 0$ for some (and hence all sufficiently large) $n \in \mathbb{N}$, and *irreducible* if $\mathrm{e}^{t\phi} > 0$ for some (and hence all) $t > 0$. Note that positivity improving implies primitivity, which implies irreducibility.

[3] Any finite dimensional unital C^*-algebra can be realized in this way.

A positive map ϕ on $\mathscr{A}$ is said to be *completely positive* when $\phi\otimes\mathrm{Id}$ is a positive map on $\mathscr{A}\otimes\mathscr{B}(\mathbb{C}^n)$ for all $n\in\mathbb{N}$. In view of the identification of $\mathscr{A}\otimes\mathscr{B}(\mathbb{C}^n)$ with the set $\mathscr{M}_n(\mathscr{A})$ of $n\times n$-matrices with entries in $\mathscr{A}$, the complete positivity of ϕ means that for any finite set J and any square matrix $A=[A_{ij}]_{i,j\in J}$ associated to a non-negative map on $\mathscr{H}\otimes\mathbb{C}^J$, the map associated to the matrix $[\phi(A_{ij})]_{i,j\in J}$ is non-negative. One easily shows that the latter condition is equivalent to

$$\sum_{i,j\in J} B_i^*\phi(A_i^*A_j)B_j\geq 0 \tag{2.2}$$

for any finite families $(A_j)_{j\in J}$ and $(B_j)_{j\in J}$ in $\mathscr{A}$ [Tak79, Corollary 3.4]. Any CP map ϕ on $\mathscr{A}$ has a (non-unique) *Kraus representation*[4]

$$\phi(A)=\sum_{j\in J}V_j^*AV_j,$$

where the finite family $\mathscr{V}=(V_j)_{j\in J}\subset\mathscr{B}(\mathscr{H})$ is called a Kraus family of ϕ. $\mathscr{V}$ is said to be irreducible whenever any non-zero element of $\mathscr{H}$ is cyclic for the algebra generated by $\mathscr{V}\cup\{\mathbb{1}\}$. $\mathscr{V}$ is primitive when there exists $n>0$ such that any non-zero element of $\mathscr{H}$ is cyclic for the linear span of $\mathscr{V}^n=\{V_{j_1}\cdots V_{j_n}\mid j_1,\ldots,j_n\in J\}$. The CP map ϕ is irreducible/primitive iff one (and hence any) of its Kraus family is irreducible/primitive.

In view of the above mentioned identification of the sets $\mathscr{A}$ and $\mathscr{A}_*$, the various notions of positivity just introduced on endomorphisms of $\mathscr{A}$ also apply to linear maps $\mathscr{L}:\mathscr{A}_*\to\mathscr{A}_*$. A linear map on $\mathscr{A}$ is *unital* when $\phi(\mathbb{1})=\mathbb{1}$ and CPU when completely positive and unital. A linear map on $\mathscr{A}_*$ is *trace preserving* when $\mathrm{tr}\circ\phi=\mathrm{tr}$, and CPTP when completely positive and trace preserving.

Decomposing $\alpha\in\mathscr{A}_*$ into the positive and negative parts of its real and imaginary parts, one easily shows that any affine map $\mathscr{L}:\mathscr{A}_{*+1}\to\mathscr{A}_{*+1}$ has a unique linear extension to $\mathscr{A}_*$. By construction, this extension is positive and trace preserving. Reciprocally, any linear, positive and trace preserving map on $\mathscr{A}_*$ restricts to an affine map on $\mathscr{A}_{*+1}$. Hence, we shall identify these two kinds of maps in the following. Denoting by $\mathscr{L}^*:\mathscr{A}\to\mathscr{A}$ the adjoint of $\mathscr{L}:\mathscr{A}_*\to\mathscr{A}_*$ w.r.t. the duality (2.1), $\mathscr{L}^*$ is positive (resp. CPU) iff $\mathscr{L}$ is positive (resp. CPTP).

When $\mathscr{A}$ is associated to a quantum mechanical system, self-adjoint elements X of $\mathscr{A}$ are observables of this system and density matrices $\rho\in\mathscr{A}_{*+1}$ describe its physical states. The physical quantity described by the observable X can only take numerical values in $\mathrm{sp}(X)$. Denoting $E_X(I)$ the spectral projection of X to the part of its spectrum contained in I, when the system's state is ρ, the probability to observe the value of X in I is given by $\langle\rho,E_X(I)\rangle$. In particular, the quantum mechanical expectation value of X in the state ρ is given by $\langle\rho,X\rangle$.

We shall consider finite matrices $P\in\mathbb{C}^{J\times J}$ as linear maps on the vector space $\mathbb{C}^J$ interpreted as the commutative unital C^*-algebra of diagonal $J\times J$-matrices. The unit of this algebra is $\mathbf{1}=(1,\ldots,1)\in\mathbb{C}^J$. In this context, P is positive (resp. positivity improving) whenever $P_{ij}\geq 0$ (resp. $P_{ij}>0$) for all $i,j\in J$. We note that these are also the conditions of the classical Perron-Frobenius theory. We observe also that, due to the commutativity of the underlying C^*-algebra (see [Tak79, Corollary 3.5]), P is completely positive iff it is positive and CPU iff it is a right stochastic matrix.

Given a Banach space $\mathscr{X}$, a finite set Ω and $p\in[1,\infty]$, we denote by $\ell^p(\Omega;\mathscr{X})$ the vector

[4] The Kraus representation of CP maps on $\mathscr{B}(\mathbb{C}^n)$ is discussed in any textbook on quantum information, see, e.g., [Pet08, Theorem 2.2]. It is a simple corollary of Stinespring's dilation theorem [Tak79, Theorem 3.6]. The slightly more general form used here is a direct consequence of Arveson's extension theorem [Arv69, Theorem 1.2.3] which asserts that ϕ extends to a CP map on $\mathscr{B}(\mathscr{H})$.

space $\mathscr{X}^\Omega$ equipped with the p-norm

$$\|x\|_p = \begin{cases} \left(\sum_{\omega\in\Omega} \|x(\omega)\|^p\right)^{1/p} & \text{for } p<\infty; \\ \max_{\omega\in\Omega} \|x(\omega)\| & \text{for } p=\infty. \end{cases}$$

We note that if $\mathscr{X}^*$ is dual to $\mathscr{X}$, with duality $\mathscr{X}^*\times\mathscr{X}\ni(\varphi,x)\mapsto\langle\varphi,x\rangle$ and $\frac{1}{p}+\frac{1}{q}=1$, then $\ell^q(\Omega;\mathscr{X}^*)$ is dual to $\ell^p(\Omega;\mathscr{X})$ with duality $\sum_{\omega\in\Omega}\langle\varphi(\omega),x(\omega)\rangle$.

The product topology induced on $\mathbf{\Omega}=\Omega^{\mathbb{N}}$ by the discrete topology of the finite set Ω is metrizable, the compatible metric

$$d((\omega_k)_{k\in\mathbb{N}},(\nu_k)_{k\in\mathbb{N}}) = 2^{-\inf\{k\in\mathbb{N}\,|\,\omega_k\neq\nu_k\}}$$

makes $\mathbf{\Omega}$ a compact metric space. We denote by $\mathscr{O}_n$ the finite algebra generated by the cylinder sets

$$[\nu_0\cdots\nu_n]=\{\boldsymbol{\omega}\in\mathbf{\Omega}\mid\omega_k=\nu_k \text{ for } 0\le k\le n\},\qquad(\nu_0,\ldots,\nu_n\in\Omega),$$

and by

$$\mathscr{O}=\bigvee_{n\in\mathbb{N}}\mathscr{O}_n$$

the generated σ-algebra.[5] We recall that $\mathscr{O}$ coincides with the Borel algebra of $\mathbf{\Omega}$.

2.2 Markovian Repeated Interaction Quantum Systems

Let $\mathscr{S}$ be a quantum mechanical system described by a finite dimensional Hilbert space $\mathscr{H}_{\mathscr{S}}$, denote by $\mathscr{A}=\mathscr{B}(\mathscr{H}_{\mathscr{S}})$ the associated C^*-algebra, set $\mathscr{A}_*=\mathscr{B}^1(\mathscr{H}_{\mathscr{S}})$, let $(\rho_\omega)_{\omega\in\Omega}\subset\mathscr{A}_*$ be a finite family of states and $(\mathscr{L}_\omega)_{\omega\in\Omega}$ a finite family of CPTP maps on $\mathscr{A}_*$.

For a probability vector $\pi\in\mathbb{R}^\Omega$, and a right stochastic matrix $P\in\mathbb{R}^{\Omega\times\Omega}$, we consider the probability space $(\mathbf{\Omega},\mathscr{O},\mathbb{P})$ where $\mathscr{O}$ is the Borel σ-algebra of $\mathbf{\Omega}:=\Omega^{\mathbb{N}}$, and $\mathbb{P}$ is the homogeneous Markovian probability measure on $(\mathbf{\Omega},\mathscr{O})$ uniquely determined by

$$\mathbb{P}([\omega_0\cdots\omega_n])=\pi_{\omega_0}P_{\omega_0\omega_1}\ldots P_{\omega_{n-1}\omega_n}, \tag{2.3}$$

for any cylinder set $[\omega_0\cdots\omega_n]$. In particular, the probability vector $\pi^{(n)}$ describing the distribution of the random variable ω_n is given by

$$\pi^{(n)}_\nu=\mathbb{P}(\omega_n=\nu)=\sum_{\omega_0,\ldots,\omega_{n-1}\in\Omega}\mathbb{P}([\omega_0\cdots\omega_{n-1}\nu])=(\pi P^n)_\nu.$$

We denote by $\mathbb{E}[\cdot]$ the expectation functional associated with $\mathbb{P}$. Given $A\in\mathscr{O}$, 1_A is the characteristic function of A and $\mathbb{E}[\cdot|A]=\mathbb{E}[\cdot\,1_A]/\mathbb{E}[1_A]$ is the conditional expectation given A. Finally, letting σ be the left-shift on $\mathbf{\Omega}$, we recall that whenever π is a left eigenvector of P to the eigenvalue 1, then $\mathbb{P}$ is σ-invariant, *i.e.,* the Markov chain is stationary.

We associate to $(\pi,P,(\rho_\omega)_{\omega\in\Omega},(\mathscr{L}_\omega)_{\omega\in\Omega})$ a random dynamics on the system $\mathscr{S}$ as follows: Each sample path $\boldsymbol{\omega}\in\mathbf{\Omega}$ of our Markov chain induces a sequence of states $(\rho_n(\boldsymbol{\omega}))_{n\in\mathbb{N}}$, where

$$\rho_n(\boldsymbol{\omega}):=\mathscr{L}_{\omega_n}\cdots\mathscr{L}_{\omega_1}\rho_{\omega_0},$$

is the state of $\mathscr{S}$ started in ρ_{ω_0} after interacting with the sequence of reservoirs $\mathscr{R}_{\omega_1},\ldots,\mathscr{R}_{\omega_n}$.

By Stinespring's dilation Theorem [Tak79, Theorem 3.6], any such random dynamical system describes a MRIS, *i.e.,* there exists a family of probes $(\mathscr{E}_\omega)_{\omega\in\Omega}$ such that

$$\mathscr{L}_\omega(\rho)=\operatorname{tr}_{\mathscr{H}_{\mathscr{E}_\omega}}(U_\omega(\rho\otimes\rho_{\mathscr{E}_\omega})U_\omega^*)$$

for some probe states $\rho_{\mathscr{E}_\omega}\in\mathscr{B}^1_{+1}(\mathscr{H}_{\mathscr{E}_\omega})$, unitary propagators U_ω on $\mathscr{H}_{\mathscr{S}}\otimes\mathscr{H}_{\mathscr{E}_\omega}$ and all $\omega\in\Omega$.

[5] By convention the cylinder with empty base [ ] is $\mathbf{\Omega}$.

2.3 Extended Observables and Extended States

The natural semigroup structure of autonomous dynamical systems gets lost in this random setting. However, using the Markov property, it is possible to restore this structure at the level of "classical" expectations. To this end, let us introduce the extended Hilbert space of $\mathcal{H}_{\mathcal{S}}$-valued functions on the noise space Ω

$$\mathcal{K} = \ell^2(\Omega;\mathcal{H}_{\mathcal{S}}).$$

Identifying $\mathcal{K}$ with $\mathcal{H}_{\mathcal{S}} \otimes \mathbb{C}^\Omega$ leads to the identification of $\mathcal{B}(\mathcal{K})$ with $\mathcal{A}^{\Omega\times\Omega}$, the C^*-algebra of $\Omega\times\Omega$ matrices with entries in $\mathcal{A}$. Further, identifying $X \in \ell^\infty(\Omega;\mathcal{A})$ with a diagonal matrix in $\mathcal{A}^{\Omega\times\Omega}$ gives a natural isometric injection

$$\mathfrak{A} = \ell^\infty(\Omega;\mathcal{A}) \hookrightarrow \mathcal{B}(\mathcal{K}), \tag{2.4}$$

which makes $\mathfrak{A}$ a C^*-subalgebra of $\mathcal{B}(\mathcal{K})$. We shall say that elements of $\mathfrak{A}$ are *extended observables* of the MRIS associated with $(\pi, P, (\rho_\omega)_{\omega\in\Omega}, (\mathcal{L}_\omega)_{\omega\in\Omega})$. The map

$$X \mapsto \operatorname{Tr} X = \sum_{\omega\in\Omega} \operatorname{tr} X(\omega),$$

is the restriction to $\mathfrak{A}$ of the trace of $\mathcal{B}(\mathcal{K})$ and defines a trace on $\mathfrak{A}$, *i.e.,* a positive linear functional such that $\operatorname{Tr}(XY) = \operatorname{Tr}(YX)$ for all $X, Y \in \mathfrak{A}$.

Dual to the injection (2.4) is the surjection

$$\mathcal{B}^1(\mathcal{K}) \to \mathfrak{A}_* = \ell^1(\Omega;\mathcal{A}_*), \tag{2.5}$$

which sends $R = [R_{\omega\nu}]_{\omega,\nu\in\Omega} \in \mathcal{A}_*^{\Omega\times\Omega}$ to the diagonal map $\omega \mapsto R_{\omega\omega}$. Note that elements of $\mathcal{B}^1_{+1}(\mathcal{K})$ are sent to positive valued elements of $\ell^1(\Omega;\mathcal{A}_*)$ of 1-norm 1. These are precisely the states of $\mathfrak{A}$ w.r.t. the duality

$$\langle R, X\rangle = \operatorname{Tr}(R^* X) = \sum_{\omega\in\Omega} \langle R(\omega), X(\omega)\rangle. \tag{2.6}$$

Positive and normalized elements of $\mathfrak{A}_*$, images of states $R \in \mathcal{B}^1_{+1}(\mathcal{K})$ by (2.5), will be called *extended states* of the MRIS, the set of these extended states is denoted $\mathfrak{A}_{*+1}$. We shall associate to $(\pi, P, (\rho_\omega)_{\omega\in\Omega}, (\mathcal{L}_\omega)_{\omega\in\Omega})$ the sequence of extended state $(R_n)_{n\in\mathbb{N}}$ defined by

$$R_n(\omega) = \pi_\omega^{(n+1)} \mathbb{E}[\rho_n(\boldsymbol{\omega}) \,|\, \omega_{n+1} = \omega] = \mathbb{E}[\rho_n(\boldsymbol{\omega}) 1_{\omega_{n+1}=\omega}],$$

and in particular

$$R_0(\omega) = \sum_{\nu\in\Omega} \pi_\nu P_{\nu\omega} \rho_\nu.$$

Observe that this state has "marginals"

$$\pi_\omega^{(n+1)} = \operatorname{tr} R_n(\omega), \qquad \bar\rho_n = \mathbb{E}[\rho_n(\boldsymbol{\omega})] = \sum_{\omega\in\Omega} R_n(\omega).$$

2.4 The Semigroup

Lemma 2.1. *Let $(R_n)_{n\in\mathbb{N}} \subset \mathfrak{A}_*$ be the sequence of extended states of the MRIS associated to $(\pi, P, (\rho_\omega)_{\omega\in\Omega}, (\mathcal{L}_\omega)_{\omega\in\Omega})$. Then, for $n \in \mathbb{N}$, one has*

$$R_n = \mathbb{L}^n R_0, \tag{2.7}$$

where $\mathbb{L} : \mathfrak{A}_* \to \mathfrak{A}_*$ *is the CPTP map defined by*

$$(\mathbb{L}R)(\omega) = \sum_{\nu\in\Omega} P_{\nu\omega}\mathscr{L}_\nu R(\nu). \tag{2.8}$$

In particular, given an extended observable $X \in \mathfrak{A}$*, one has*

$$\mathbb{E}[\langle\rho_n(\boldsymbol{\omega}), X(\omega_{n+1})\rangle] = \langle\mathbb{L}^n R_0, X\rangle. \tag{2.9}$$

Proof. The relation (2.7) clearly holds for $n = 0$. For any $n \in \mathbb{N}^*$ and $\omega \in \Omega$ one has

$$\begin{aligned} R_n(\omega) &= \sum_{\omega_0,\ldots,\omega_n\in\Omega} \pi_{\omega_0} P_{\omega_0\omega_1}\cdots P_{\omega_n\omega}\mathscr{L}_{\omega_n}\cdots\mathscr{L}_{\omega_1}\rho_{\omega_0} \\ &= \sum_{\omega_n\in\Omega} P_{\omega_n\omega}\mathscr{L}_{\omega_n} \sum_{\omega_0,\ldots,\omega_{n-1}\in\Omega} \pi_{\omega_0} P_{\omega_0\omega_1}\cdots P_{\omega_{n-1}\omega_n}\mathscr{L}_{\omega_{n-1}}\cdots\mathscr{L}_{\omega_1}\rho_{\omega_0} \\ &= \sum_{\omega_n\in\Omega} P_{\omega_n\omega}\mathscr{L}_{\omega_n} R_{n-1}(\omega_n) = (\mathbb{L}R_{n-1})(\omega), \end{aligned}$$

which implies (2.7) for all $n \in \mathbb{N}$. Consequently, one has[6]

$$\begin{aligned} \mathbb{E}[\langle\rho_n(\boldsymbol{\omega}), X(\omega_{n+1})\rangle] &= \mathbb{E}[\langle\mathbb{E}[\rho_n(\boldsymbol{\omega})\,|\,\omega_{n+1}], X(\omega_{n+1})\rangle] \\ &= \sum_{\omega\in\Omega} \pi^{(n+1)}_\omega \langle\mathbb{E}[\rho_n(\boldsymbol{\omega})\,|\,\omega_{n+1}=\omega], X(\omega)\rangle \\ &= \sum_{\omega\in\Omega} \langle R_n(\omega), X(\omega)\rangle = \langle\mathbb{L}^n R_0, X\rangle, \end{aligned}$$

which proves (2.9). An elementary calculation shows that the adjoint of $\mathbb{L}$ w.r.t. the duality (2.6) is given by

$$(\mathbb{L}^* X)(\omega) = \sum_{\nu\in\Omega} P_{\omega\nu}\mathscr{L}^*_\omega X(\nu).$$

One also easily checks (e.g., using (2.2)) that $\mathbb{L}^*$ is a CPU-map on $\mathfrak{A}$. By duality, $\mathbb{L}$ is a CPTP-map on $\mathfrak{A}_*$. □

By Brouwer's fixed point theorem, $\mathbb{L}$ has a fixed point on the convex set $\mathfrak{A}_{*+1}$ of extended states. Such a fixed point will be called *Extended Steady State* (ESS) of the MRIS.

Let $R_+ \in \mathfrak{A}_{*+1}$ be an ESS, and set

$$\pi_{+\omega} = \operatorname{tr} R_+(\omega), \qquad \rho_{+\omega} = \begin{cases} \dfrac{\mathscr{L}_\omega R_+(\omega)}{\pi_{+\omega}} & \text{if } \pi_{+\omega} \neq 0, \\ \dfrac{\mathbb{1}}{\operatorname{tr}\mathbb{1}} & \text{otherwise.} \end{cases} \tag{2.10}$$

Since R_+ is positive with $\operatorname{Tr} R_+ = 1$, π_+ is a probability vector, and it follows from

$$\pi_{+\omega} = \operatorname{tr} R_+(\omega) = \operatorname{tr}(\mathbb{L}R_+)(\omega) = \sum_{\nu\in\Omega} P_{\nu\omega}\operatorname{tr}\mathscr{L}_\nu R_+(\nu) = \sum_{\nu\in\Omega} P_{\nu\omega}\operatorname{tr} R_+(\nu) = \sum_{\nu\in\Omega} \pi_{+\nu}P_{\nu\omega}$$

that it is an invariant probability for the driving Markov chain. By construction, $\rho_+ = (\rho_{+\omega})_{\omega\in\Omega}$ is a family of states on $\mathscr{A}$. Since $R_+(\omega) \geq 0$, $R_+(\omega) = 0$ whenever $\pi_{+\omega} = 0$, so that the identity $\mathscr{L}_\omega R_+(\omega) = \pi_{+\omega}\rho_{+\omega}$ holds for all $\omega \in \Omega$. The formula

$$\sum_{\nu\in\Omega} P_{\nu\omega}\pi_{+\nu}\rho_{+\nu} = \sum_{\nu\in\Omega} P_{\nu\omega}\mathscr{L}_\nu R_+(\nu) = (\mathbb{L}R_+)(\omega) = R_+(\omega), \tag{2.11}$$

thus allows to reconstruct the ESS R_+ from the invariant probability π_+ and the family ρ_+.

[6] We denote by $\mathbb{E}[\cdot\,|\omega_n]$ the conditional expectation w.r.t. the random variable ω_n.

Reciprocally, given a P-invariant probability π_+ and a family of states $(\rho_{+\omega})_{\omega\in\Omega}$ on $\mathscr{A}$ such that

$$\mathscr{L}_\omega \sum_{\mu\in\Omega} P_{\mu\omega}\pi_{+\mu}\rho_{+\mu} = \pi_{+\omega}\rho_{+\omega}$$

for all $\omega\in\Omega$, the extended state R_+ given by (2.11) satisfies

$$(\mathbb{L}R_+)(\omega) = \sum_{\nu\in\Omega} P_{\nu\omega}\mathscr{L}_\nu \sum_{\mu\in\Omega} P_{\mu\nu}\pi_{+\mu}\rho_{+\mu} = \sum_{\nu\in\Omega} P_{\nu\omega}\pi_{+\nu}\rho_{+\nu} = R_+(\omega).$$

Denoting by $\mathbb{P}_+$ the stationary Markov measure on $(\boldsymbol{\Omega},\mathscr{O})$ associated to (π_+,P), one has, with $\rho_{+n}(\boldsymbol{\omega}) = \mathscr{L}_{\omega_n}\cdots\mathscr{L}_{\omega_1}\rho_{+\omega_0}$,

$$\begin{aligned}\mathbb{E}_+[\rho_{+n}(\boldsymbol{\omega})1_{\omega_{n+1}=\omega}] &= \sum_{\omega_0,\dots,\omega_n\in\Omega} \pi_{+\omega_0}P_{\omega_0\omega_1}\cdots P_{\omega_n\omega}\mathscr{L}_{\omega_n}\cdots\mathscr{L}_{\omega_1}\rho_{+\omega_0}\\ &= \sum_{\omega_0,\dots,\omega_n\in\Omega} P_{\omega_n\omega}\mathscr{L}_{\omega_n}\cdots P_{\omega_0\omega_1}\mathscr{L}_{\omega_0}R_+(\omega_0)\\ &= (\mathbb{L}^{n+1}R_+)(\omega) = R_+(\omega)\end{aligned}$$

so that

$$\mathbb{E}_+[\langle\rho_{+n}(\boldsymbol{\omega}), X(\omega_{n+1})\rangle] = \langle R_+, X\rangle, \tag{2.12}$$

for all $X\in\mathfrak{A}$ and $n\in\mathbb{N}_0$.

2.5 Primitivity and Irreducibility as Ergodic Properties

The irreducibility and primitivity properties of CP maps ϕ will play a central role in our approach. As introduced in Section 2.1, they are intimately linked to the natural order structure of a C^*-algebra. In this section, we recall their important connections with the spectral properties of ϕ and the ergodic properties of the semigroup $(\phi^n)_{n\in\mathbb{N}}$. We also discuss some issues more specific to the CPTP map (2.8)

Let ϕ be a CPTP map on the predual $\mathscr{C}_*$ of the finite dimensional C^*-algebra $\mathscr{C}\subset\mathscr{B}(\mathscr{H})$. The following statements are well known (see, e.g., [EHK78] for details):

(i) $\mathrm{sp}(\phi)$ is a subset of the closed unit disk containing 1, and the eigenspace of ϕ associated to the eigenvalue 1 contains a state ρ.

(ii) ϕ is irreducible iff its eigenvalue 1 is simple. In this case, the eigenspace of ϕ associated to the eigenvalue 1 contains a unique state ρ. Moreover, ρ is faithful and for any $R\in\mathscr{C}_*$ one has

$$\left|\frac{1}{n}\sum_{k=0}^{n-1}\phi^k(R) - \langle R,\mathbb{1}\rangle\rho\right| = \|R\|\mathscr{O}(n^{-1}),$$

as $n\to\infty$.

(iii) ϕ is primitive, iff its simple eigenvalue 1 is gapped, *i.e.,*

$$\Delta = -\log\max\{|z|\mid z\in\mathrm{sp}(\phi)\setminus\{1\}\} > 0.$$

Moreover, for any $R\in\mathscr{C}_*$ and $\epsilon>0$ one has

$$|\phi^n(R) - \langle R,\mathbb{1}\rangle\rho| = \|R\|\mathscr{O}(\mathrm{e}^{-n(\Delta-\epsilon)}),$$

as $n\to\infty$.

We conclude this section with some useful connections between the properties of P and $\mathscr{L}_\omega$ and that of $\mathbb{L}$.

Lemma 2.2. *Set*

$$\bar{\mathscr{L}} = \sum_{\omega\in\Omega} \pi_\omega \mathscr{L}_\omega,$$

where π is a faithful probability vector, and note that $\bar{\mathscr{L}}$ is a CPTP map on $\mathfrak{A}_*$.

(i) $\mathbb{L}$ is positivity improving iff P and every $\mathscr{L}_\omega$ are.

(ii) If $\mathbb{L}$ is irreducible (resp. primitive), so are P and $\bar{\mathscr{L}}$.

(iii) If P is positivity improving, then $\mathbb{L}$ is irreducible (resp. primitive) iff $\bar{\mathscr{L}}$ is.

(iv) If every $\mathscr{L}_\omega$ is positivity improving, then $\mathbb{L}$ is irreducible (resp. primitive) iff P is.

Proof. (i) Assume that P and each $\mathscr{L}_\omega$ are positivity improving. Given non-zero $R \in \mathfrak{A}_{*+}$ and $X \in \mathfrak{A}_+$, there exists $\omega, \nu \in \Omega$ such that both $R(\omega)$ and $X(\nu)$ are non-negative and non-zero. It follows that $\mathscr{L}_\omega R(\omega) > 0$ and hence

$$\langle \mathbb{L}R, X\rangle \ge P_{\omega\nu}\langle \mathscr{L}_\omega R(\omega), X(\nu)\rangle > 0,$$

which shows that $\mathbb{L}$ is positivity improving. Reciprocally, let $\mathbb{L}$ be positivity improving, $\xi, \eta \in \Omega$, and $\rho \in \mathscr{A}_{*+}$, $A \in \mathscr{A}_+$ be non-zero. With $R(\omega) = \delta_{\omega\xi}\rho$ and $X(\omega) = \delta_{\omega\eta}A$, one has

$$0 < \langle \mathbb{L}R, X\rangle = P_{\xi\eta}\langle \mathscr{L}_\xi \rho, A\rangle,$$

from which we can conclude that P and all $\mathscr{L}_\omega$ are positivity improving.

(ii) For $R \in \mathfrak{A}_*$ and $n \in \mathbb{N}$ one easily checks that

$$(\mathbb{L}^n R)(\omega) = \sum_{\omega_1,\ldots,\omega_n\in\Omega} P_{\omega_1\omega_2}\cdots P_{\omega_n\omega}\mathscr{L}_{\omega_n}\cdots\mathscr{L}_{\omega_1}R(\omega_1). \tag{2.13}$$

For $u \in \mathbb{R}^\Omega$, let us set $R_u(\omega) = u_\omega\rho$ and $X_u(\omega) = u_\omega \mathbb{1}$ where ρ is an arbitrary density matrix. Formula (2.13) gives that

$$\langle \mathbb{L}^n R_u, X_v\rangle = \langle u, P^n v\rangle$$

for any $u, v \in \mathbb{R}^\Omega$ and $n \in \mathbb{N}$, and it follows that P is irreducible (resp. primitive) if $\mathbb{L}$ is. Setting $R(\omega) = \pi_\omega\rho$ and $\epsilon = \min_\omega \pi_\omega$, and using the fact that the matrix elements of P are all bounded above by 1, we further deduce from (2.13)

$$\begin{aligned}(\mathbb{L}^n R)(\omega) &\le \sum_{\omega_1,\ldots,\omega_n\in\Omega} \mathscr{L}_{\omega_n}\cdots\mathscr{L}_{\omega_1}\pi_{\omega_1}\rho\\ &\le \epsilon^{-n+1}\sum_{\omega_1,\ldots,\omega_n\in\Omega} \pi_{\omega_n}\mathscr{L}_{\omega_n}\cdots\pi_{\omega_1}\mathscr{L}_{\omega_1}\rho = \epsilon^{-n+1}\bar{\mathscr{L}}^n\rho,\end{aligned}$$

from which one easily concludes that $\bar{\mathscr{L}}$ is irreducible (resp. primitive) whenever $\mathbb{L}$ is.

(iii) If P is positivity improving, then there exists δ such that $0 < \delta \le P_{\nu\omega}$ for all $\omega, \nu \in \Omega$. Setting again $R(\omega) = \pi_\omega\rho$, it follows from (2.13) that

$$\delta^n \bar{\mathscr{L}}^n \rho \le (\mathbb{L}^n R)(\omega),$$

from which it follows that $\mathbb{L}$ is irreducible (resp. primitive) whenever $\bar{\mathscr{L}}$ is. The reciprocal property follows from Part (ii).

(iv) Let $R \in \mathfrak{A}_{*+}$ be non-zero, so that $R(\nu)$ is non-zero for some $\nu \in \Omega$. Formula (2.13) yields

$$(\mathbb{L}^n R)(\omega) \ge \sum_{\omega_2,\ldots,\omega_n\in\Omega} P_{\nu\omega_2}\cdots P_{\omega_n\omega}\mathscr{L}_{\omega_n}\cdots\mathscr{L}_{\omega_2}\mathscr{L}_\nu R(\nu),$$

and if every $\mathscr{L}_\omega$ is positivity improving, then

$$\delta = \min_{\omega_2,\ldots,\omega_n\in\Omega} \min \mathrm{sp}(\mathscr{L}_{\omega_n}\cdots\mathscr{L}_{\omega_2}\mathscr{L}_\nu R(\nu)) > 0,$$

so that

$$(\mathbb{L}^n R)(\omega) \geq (P^n)_{\nu\omega}\delta\mathbb{1},$$

from which we can conclude that $\mathbb{L}$ is irreducible (resp. primitive) whenever P is. The reciprocal property again follows from Part (ii). □

3 Main Abstract Results

3.1 A Pointwise Ergodic Theorem

Central to our results are consequences of spectral properties of the map $\mathbb{L}$ on the large time behavior of MRIS. Below, we formulate a pointwise ergodic theorem for MRIS which applies when the driving Markov chain is stationary. The case of an inhomogeneous driving process is considered in Section 3.2 in the adiabatic limit. Applications of these results to the nonequilibrium thermodynamic properties of MRIS will be given in the subsequent Section 4.

Our ergodic theorem relies on the minimal

Assumption (STAT). Either the Markov chain is stationary, i.e., $\pi P = \pi$, or it admits a faithful stationary state π_+.

We note that whenever $\mathbb{L}$ is irreducible (or primitive, or positivity improving), then, by Lemma 2.2 (ii), the unique left eigenvector of P is faithful, so that Assumption (**STAT**) is satisfied.

We shall invoke the following vector-valued random ergodic theorem of Beck and Schwartz.

Theorem 3.1 ([BS57, Theorem 2]). *Let $\mathscr{X}$ be a reflexive Banach-space and let (S,Σ,m) be a σ-finite measure space. Let there be defined a strongly measurable function $S \ni s \mapsto T_s$ with values in the Banach-space $\mathscr{B}(\mathscr{X})$ of bounded linear operators on $\mathscr{X}$. Suppose that $\|T_s\| \leq 1$ for all $s \in S$. Let h be a measure-preserving transformation on (S,Σ,m). Then for each $X \in L^1(S,\mathscr{X})$ there is an $\bar{X} \in L^1(S,\mathscr{X})$ such that*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} T_s T_{h(s)}\cdots T_{h^{n-1}(s)} X(h^n(s)) = \bar{X}(s),$$

strongly in $\mathscr{X}$, a.e. in S, and

$$\bar{X}(s) = T_s(\bar{X}(h(s)))$$

a.e. in S. Moreover, if $m(S) < \infty$, then $\mathscr{X}$ is also the limit in the mean of order 1.

Let us start by assuming that the Markov chain is stationary. A direct application of Theorem 3.1 shows that the limit

$$\bar{X}(\boldsymbol{\omega}) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\mathscr{L}_{\omega_1}^*\cdots\mathscr{L}_{\omega_n}^* X(\omega_{n+1})$$

exists $\mathbb{P}$-a.e. and in $L^1(\boldsymbol{\Omega},\mathbb{P};\mathfrak{A})$. Moreover, the covariance relation $\bar{X}(\boldsymbol{\omega}) = \mathscr{L}^*_{\omega_1}\bar{X}\circ\sigma(\boldsymbol{\omega})$ implies

$$\begin{aligned}
\bar{\bar{X}}(\omega) :=& \mathbb{E}[\bar{X}(\boldsymbol{\omega})\,|\,\omega_1=\omega]\\
=& \mathbb{E}[\mathscr{L}^*_{\omega_1}\bar{X}(\sigma(\boldsymbol{\omega}))\,|\,\omega_1=\omega]\\
=& \mathscr{L}^*_{\omega}\mathbb{E}[\bar{X}(\sigma(\boldsymbol{\omega}))\,|\,\omega_1=\omega]\\
=& \mathscr{L}^*_{\omega}\mathbb{E}[\bar{X}(\boldsymbol{\omega})\,|\,\omega_0=\omega]\\
=& \mathscr{L}^*_{\omega}\mathbb{E}[\mathbb{E}[\bar{X}(\boldsymbol{\omega})\,|\,\omega_1]\,|\,\omega_0=\omega]\\
=& \mathscr{L}^*_{\omega}\mathbb{E}[\bar{\bar{X}}(\omega_1)\,|\,\omega_0=\omega]\\
=& \sum_{\nu\in\Omega} P_{\omega\nu}\mathscr{L}^*_{\omega}\bar{\bar{X}}(\nu) = (\mathbb{L}^*\bar{\bar{X}})(\omega).
\end{aligned}$$

Assuming now that π_+ is a faithful stationary state, *i.e.,* $\pi_+P=\pi_+>0$, then the stationary Markov measure $\mathbb{P}_+$ on $(\boldsymbol{\Omega},\mathscr{O})$ with transition matrix P and faithful invariant probability π_+ satisfies

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}_+}(\boldsymbol{\omega}) = \frac{\pi(\omega_0)}{\pi_+(\omega_0)},$$

which gives that $\mathbb{P}$ is absolutely continuous w.r.t. $\mathbb{P}_+$. Consequently, any $\mathbb{P}_+$-a.s. property also holds $\mathbb{P}$-almost surely. Summarizing, we have proved

Theorem 3.2. *Under Assumption* (**STAT**)*, the limit*

$$\bar{X}(\boldsymbol{\omega}) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\mathscr{L}^*_{\omega_1}\cdots\mathscr{L}^*_{\omega_n}X(\omega_{n+1}) \tag{3.1}$$

exists $\mathbb{P}$-almost surely and in $L^1(\boldsymbol{\Omega},\mathbb{P};\mathfrak{A})$ for any $X\in\mathfrak{A}$. The limiting function is such that

$$\mathscr{L}^*_{\omega_1}\bar{X}(\sigma(\boldsymbol{\omega})) = \bar{X}(\boldsymbol{\omega}). \tag{3.2}$$

Moreover, the extended observable $\bar{\bar{X}}\in\mathfrak{A}$ defined by

$$\bar{\bar{X}}(\omega) = \mathbb{E}[\bar{X}(\boldsymbol{\omega})\,|\,\omega_1=\omega],$$

satisfies

$$\mathbb{L}^*\bar{\bar{X}} = \bar{\bar{X}}. \tag{3.3}$$

As an immediate corollary, we get that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\langle\rho_n(\boldsymbol{\omega}),X(\omega_{n+1})\rangle = \langle\rho_{\omega_0},\bar{X}(\boldsymbol{\omega})\rangle = \langle\rho_k(\boldsymbol{\omega}),\bar{X}\circ\sigma^k(\boldsymbol{\omega})\rangle \tag{3.4}$$

$\mathbb{P}$-almost surely and in $L^1(\boldsymbol{\Omega},\mathbb{P})$, for any $k\in\mathbb{N}$.

A less immediate corollary of Theorem 3.2 is the following, which only requires spectral properties of the map $\mathbb{L}$.

Theorem 3.3. *If $\mathbb{L}$ is irreducible, its unique ESS R_+ is faithful and for any $X\in\mathfrak{A}$ one has*[7]

$$\bar{X}(\boldsymbol{\omega}) = \lim_{N\to+\infty}\frac{1}{N}\sum_{n=0}^{N-1}\mathscr{L}^*_{\omega_1}\cdots\mathscr{L}^*_{\omega_n}X(\omega_{n+1}) = \langle R_+,X\rangle\mathbb{1}$$

[7] When $\mathbb{L}$ is irreducible, we sometimes omit to mention the dependence of $\bar{X}(\boldsymbol{\omega})$ on $\boldsymbol{\omega}$ and write its $\mathbb{P}$-a.s. value as $\bar{X}$.

for $\mathbb{P}$-almost every $\boldsymbol{\omega} \in \boldsymbol{\Omega}$ and in $L^1(\boldsymbol{\Omega}, \mathbb{P}; \mathfrak{A})$. In particular, for any initial state $\rho \in \mathfrak{A}_{*+1}$ and any $X \in \mathfrak{A}$,

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{n=0}^{N-1} \langle \rho_n(\boldsymbol{\omega}), X(\omega_{n+1})\rangle = \langle R_+, X\rangle$$

holds $\mathbb{P}$-almost surely and in $L^1(\boldsymbol{\Omega}, \mathbb{P})$. Moreover, if $\mathbb{L}$ is primitive, then

$$\lim_{n\to\infty} \mathbb{E}_+[\langle \rho_n(\boldsymbol{\omega}), X(\omega_{n+1})\rangle] = \langle R_+, X\rangle.$$

Proof. Since $\mathbb{L}$ is irreducible, so is P by Lemma 2.2 (ii), and hence Assumption (**STAT**) is satisfied and Theorem 3.2 applies. Iterating the covariance relation (3.2) and invoking the Markov property lead us to

$$\begin{aligned}\mathbb{E}[\bar{X}(\boldsymbol{\omega}) \,|\, \mathscr{O}_n] &= \mathbb{E}[\mathscr{L}^*_{\omega_1} \cdots \mathscr{L}^*_{\omega_{n-1}} \bar{X} \circ \sigma^{n-1}(\boldsymbol{\omega}) \,|\, \mathscr{O}_n] \\ &= \mathscr{L}^*_{\omega_1} \cdots \mathscr{L}^*_{\omega_{n-1}} \mathbb{E}[\bar{X} \circ \sigma^{n-1}(\boldsymbol{\omega}) \,|\, \omega_n] \\ &= \mathscr{L}^*_{\omega_1} \cdots \mathscr{L}^*_{\omega_{n-1}} \bar{\bar{X}}(\omega_n).\end{aligned} \tag{3.5}$$

$\mathbb{L}$ being irreducible, it has a simple eigenvalue 1 with left/right eigenvector $\mathbb{1}/R_+$.[8] From $\mathbb{L}^*\mathbb{1} = \mathbb{1}$ and (3.3) we deduce that $\bar{\bar{X}} = \langle \lambda, X\rangle \mathbb{1}$ for some $\lambda \in \mathfrak{A}_*$. Relation (3.5) further yields

$$\mathbb{E}[\bar{X}(\boldsymbol{\omega}) \,|\, \mathscr{O}_n] = \langle \lambda, X\rangle \mathbb{1},$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ we get

$$\bar{X}(\boldsymbol{\omega}) = \lim_{n\to\infty} \mathbb{E}[\bar{X}(\boldsymbol{\omega}) \,|\, \mathscr{O}_n] = \langle \lambda, X\rangle \mathbb{1},$$

and in particular,

$$\langle \rho_{\omega_0}, \bar{X}(\boldsymbol{\omega})\rangle = \langle \lambda, X\rangle \langle \rho_{\omega_0}, \mathbb{1}\rangle = \langle \lambda, X\rangle.$$

Finally, using (2.9) and invoking the mean ergodic theorem gives

$$\mathbb{E}[\langle \rho_{\omega_0}, \bar{X}(\boldsymbol{\omega})\rangle] = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}[\langle \rho_n(\boldsymbol{\omega}), X(\omega_{n+1})\rangle] = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \langle \mathbb{L}^n R_0, X\rangle = \langle R_+, X\rangle,$$

from which we conclude that $\lambda = R_+$ and $\bar{X}(\boldsymbol{\omega}) = \langle R_+, X\rangle \mathbb{1}$. The remaining statements are obvious. □

3.2 Inhomogeneous MRIS in the Adiabatic Limit

In this section, we consider the possibility of driving our MRIS with an *inhomogeneous* Markov chain, *i.e.,* we allow the transition matrix P to depend on the time step. More specifically, we address the regime of slow variation from one time step to the next, known as the *adiabatic regime*, by applying the results of [HJPR17, HJPR18] to control the expectations

$$\mathbb{E}[\langle \rho_n(\boldsymbol{\omega}), X(\omega_{n+1})\rangle]$$

when n goes to infinity.

Notice that the whole construction that has been made in Section 2 does not rely on the homogeneity of the Markov chain, so that, given a sequence $(P^{(n)})_{n\in\mathbb{N}^*}$ of right stochastic matrices, we can replace (2.3) with

$$\mathbb{P}([\omega_0 \cdots \omega_n]) = \pi_{\omega_0} P^{(1)}_{\omega_0\omega_1} P^{(2)}_{\omega_1\omega_2} \cdots P^{(n)}_{\omega_{n-1}\omega_n}.$$

[8] Here, $\mathbb{1}$ denotes the unit of $\mathfrak{A}$.

Further, defining the sequence $(\mathbb{L}_n)_{n\in\mathbb{N}^*}$ by

$$(\mathbb{L}_n R)(\omega) = \sum_{\nu\in\Omega} P^{(n)}_{\nu\omega}\mathscr{L}_\nu R(\nu),$$

one can write, in analogy with Lemma 2.1,

$$\mathbb{E}[\langle\rho_n(\boldsymbol{\omega}), X(\omega_{n+1})\rangle] = \langle\mathbb{L}_n\cdots\mathbb{L}_1 R_0, X\rangle. \tag{3.6}$$

At such a level of generality, not much can be said of (3.6). Hence, we consider an adiabatic regime in which successive transition matrices are obtained by sampling a smooth family of transition matrices defined on $[0,1]$, with smaller and smaller variations between successive transition matrices, following [HJPR17, HJPR18].

As before, let $(\mathscr{L}_\omega)_{\omega\in\Omega}$ be a finite family of CPTP maps on $\mathscr{A}_*$. Let $[0,1]\ni s\mapsto P(s)$ be a C^2 function taking its values in the set of right stochastic $\Omega\times\Omega$-matrices. Define

$$(\mathbb{L}(s)R)(\omega) = \sum_{\nu\in\Omega} P_{\nu\omega}(s)\mathscr{L}_\nu R(\nu)$$

so that $[0,1]\ni s\mapsto\mathbb{L}(s)$ is also of class C^2. For $\epsilon = 1/N$ denote by $\mathbb{P}_\epsilon$ the Markov probability measure on Ω^{N+1} defined as

$$\mathbb{P}_\epsilon(\boldsymbol{\omega} = (\omega_0,\dots,\omega_N)) = \pi_{\omega_0}P_{\omega_0\omega_1}(0)P_{\omega_1\omega_2}(\epsilon)\cdots P_{\omega_{N-1}\omega_N}((N-1)\epsilon).$$

It was shown in [HJPR18] that, under spectral hypotheses on the family $(\mathbb{L}(s))_{s\in[0,1]}$, the product

$$\mathbb{L}^{(n)}_\epsilon = \mathbb{L}(n\epsilon)\mathbb{L}((n-1)\epsilon)\cdots\mathbb{L}(\epsilon)$$

admits an asymptotics for $n\in\{0,\dots,N\}$, in the adiabatic limit $\epsilon\downarrow 0$. More precisely, we have

Theorem 3.4 ([HJPR18, Corollary 3.14])**.** *Assume that, for all $s\in[0,1]$, $\mathbb{L}(s)$ is primitive. Denote by $R_+(s)$ its unique (and faithful) invariant state, and fix $R\in\mathfrak{A}_{*+1}$. Then, there exist $0<l<1$, $0<\epsilon_0<1$ and $C<\infty$, such that, for all $\epsilon\in]0,\epsilon_0]$ with $1/\epsilon\in\mathbb{N}$, and all $n\in\{0,\dots,1/\epsilon\}\subset\mathbb{N}_0$,*

$$\|\mathbb{L}^{(n)}_\epsilon R - R_+(n\epsilon)\| \le C\left(\frac{\epsilon}{(1-l)} + l^{n+1}\right).$$

In particular, for any $X\in\mathfrak{A}$,

$$\mathbb{E}_\epsilon[\langle\rho_n(\boldsymbol{\omega}), X(\omega_{n+1})\rangle] = \langle R_+(n\epsilon), X\rangle + \|X\|\,O(\epsilon/(1-l) + l^{n+1}).$$

If, moreover, we write $n = t/\epsilon$, with $0<t<1$, then

$$\mathbb{E}_\epsilon[\langle\rho_{\frac{t}{\epsilon}}(\boldsymbol{\omega}), X(\omega_{\frac{t+\epsilon}{\epsilon}})\rangle] = \langle R_+(t), X\rangle + \|X\|\,O(\epsilon).$$

Remark. A similar statement holds when the $\mathbb{L}(s)$'s are irreducible, with a more complicated expression for the asymptotic state, in terms of all eigenvectors corresponding to the peripheral eigenvalues.

4 Entropic Fluctuations and Linear Response Theory

This section is devoted to applications to linear response theory and entropic fluctuations within our Markovian framework, generalizing [BB20] which addresses these aspects for *periodic* and *uniform i.i.d.* (u.i.i.d. for short) repeated interaction systems.

Within our setup, a periodic RIS (called cyclic in [BB20]) is a MRIS such that P is the permutation matrix associated to a cycle $(12\cdots m) \in S_m$, so that $\boldsymbol{\omega}$ is the repetition of the word $12\cdots m$ with probability 1. A i.i.d. RIS corresponds to the case $P_{\nu\omega} = \pi_\omega$ for all $\omega \in \Omega$ where π is a probability vector. In this case $\boldsymbol{\omega}$ is a sequence of independent random variables [BJM14]. The special case of uniform distribution is simply called a Random RIS in [BB20].

In both cases, the reduced dynamics of the small system yields a discrete time quantum dynamical semigroup on $\mathscr{A}_* = \mathscr{B}^1(\mathscr{H}_{\mathscr{S}})$, with $\dim \mathscr{H}_{\mathscr{S}} < \infty$. Indeed, in the periodic case,

$$\rho_{nm}(\boldsymbol{\omega}) = (\mathscr{L}_m \cdots \mathscr{L}_1)^n \rho$$

while, in the random case,

$$\mathbb{E}[\rho_n(\boldsymbol{\omega})] = \left(\frac{1}{|\Omega|}\sum_{\omega\in\Omega}\mathscr{L}_\omega\right)^n \rho.$$

The results of [BB20] are largely based on this semigroup structure. We shall invoke Lemma 2.1, to extend these results to the MRIS framework.

Remark. Continuous time semigroups of CPTP maps, often also called *quantum Markov semigroups,* are generated by Lindbladians. As dynamics of a small quantum system $\mathscr{S}$ interacting with several extended thermal reservoirs, they emerge in the van Hove weak coupling limit [Dav74, Dav76a, Dav75, Dav76b]. The interested reader should consult [LS78, JPW14] for discussions of the nonequilibrium thermodynamics of such systems.

4.1 Entropy Balance

Returning to the concrete MRIS with CPTP and dual CPU maps

$$\mathscr{L}_\omega \rho = \mathrm{tr}_{\mathscr{H}_{\mathscr{E}_\omega}}\left(U_\omega(\rho\otimes\rho_{\mathscr{E}_\omega})U_\omega^*\right), \qquad \mathscr{L}_\omega^* X = \mathrm{tr}_{\mathscr{H}_{\mathscr{E}_\omega}}\left(U_\omega^*(X\otimes\mathbb{1})U_\omega(\mathbb{1}\otimes\rho_{\mathscr{E}_\omega})\right), \tag{4.1}$$

where $U_\omega = \mathrm{e}^{-\mathrm{i}\tau_\omega(H_{\mathscr{S}}+H_{\mathscr{E}_\omega}+V_\omega)}$, we shall now assume that the reservoirs are in thermal equilibrium, more precisely

Assumption (KMS). There is $\boldsymbol{\beta} = (\beta_\omega)_{\omega\in\Omega} \in \mathbb{R}_+^\Omega$ such that, for all $\omega \in \Omega$,

$$\rho_{\mathscr{E}_\omega} = \mathrm{e}^{-\beta_\omega(H_{\mathscr{E}_\omega}-F_\omega)},$$

where

$$F_\omega = -\frac{1}{\beta_\omega}\log\mathrm{tr}\left(\mathrm{e}^{-\beta_\omega H_{\mathscr{E}_\omega}}\right)$$

is the free energy of a probe from reservoir $\mathscr{R}_\omega$.

Remark. Given $\boldsymbol{\beta} \in \mathbb{R}_+^\Omega$ and assuming each $\rho_{\mathscr{E}_\omega}$ to be faithful, it is of course possible to redefine the probe Hamiltonians in such a way that (**KMS**) holds, at the cost of absorbing the change of $H_{\mathscr{E}_\omega}$ in the interaction V_ω. However, thermodynamic considerations require the propagators U_ω to be independent of $\boldsymbol{\beta}$, so that such circumstances are excluded in the following.

The energy lost by the system $\mathscr{S}$ during the $n+1^{\mathrm{th}}$ interaction, which we interpret as the amount of heat dumped in the reservoir $\mathscr{R}_{\omega_{n+1}}$, is

$$\Delta Q_{n+1}(\boldsymbol{\omega}) = \mathrm{tr}\left(\rho_n(\boldsymbol{\omega})\otimes\rho_{\mathscr{E}_{\omega_{n+1}}}(U_{\omega_{n+1}}^* H_{\mathscr{E}_{\omega_{n+1}}} U_{\omega_{n+1}} - H_{\mathscr{E}_{\omega_{n+1}}})\right) = -\langle\rho_n(\boldsymbol{\omega}), J(\omega_{n+1})\rangle$$

with

$$J(\omega) = \mathrm{tr}_{\mathscr{H}_{\mathscr{E}_\omega}}(U_\omega^*[U_\omega, H_{\mathscr{E}_\omega}](\mathbb{1}\otimes\rho_{\mathscr{E}_\omega})).$$

Further setting

$$J_\nu : \Omega \ni \omega \mapsto \delta_{\nu\omega} J(\nu), \tag{4.2}$$

yields an extended observable $J_\nu \in \mathfrak{A}$ describing the energy transferred from reservoir $\mathscr{R}_\nu$ to the system $\mathscr{S}$ during a single interaction.

Accordingly, the time-averaged quantum mechanical expectation values of the heat extracted from the reservoir $\mathscr{R}_\nu$ during a single interaction is given by

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \langle \rho_n(\boldsymbol{\omega}), J_\nu(\omega_{n+1})\rangle. \tag{4.3}$$

If $\mathbb{L}$ is irreducible, then Theorem 3.3 implies that this limit exists for $\mathbb{P}$-almost every $\boldsymbol{\omega} \in \boldsymbol{\Omega}$, and coincides with the ESS ensemble average

$$\langle R_+, J_\nu\rangle = \mathbb{E}_+[\langle \rho_+(\omega_0), J_\nu(\omega_1)\rangle]. \tag{4.4}$$

The von Neumann entropy [Pet08, Section 3.3] of the system $\mathscr{S}$ after completion of the n^{th} interaction is

$$S(\rho_n(\boldsymbol{\omega})) := -\langle \rho_n(\boldsymbol{\omega}), \log \rho_n(\boldsymbol{\omega})\rangle,$$

so that, during the $n+1^{\text{th}}$ interaction, the system entropy decreases by

$$\Delta S_{n+1}(\boldsymbol{\omega}) = S(\rho_n(\boldsymbol{\omega})) - S(\rho_{n+1}(\boldsymbol{\omega})).$$

Recall that the relative entropy of a state $\mu \in \mathscr{A}_*$ relatively to another state $\rho \in \mathscr{A}_*$ is

$$\mathrm{Ent}(\mu|\rho) := \begin{cases} \mathrm{tr}(\mu(\log\mu - \log\rho)) & \text{if } \mathrm{Ran}\,\mu \subset \mathrm{Ran}\,\rho; \\ +\infty & \text{otherwise,} \end{cases}$$

and that $\mathrm{Ent}(\mu|\rho) \geq 0$ with equality iff $\mu = \rho$ (see [Pet08, Section 3.4]). Setting

$$\mathrm{ep}_n(\boldsymbol{\omega}) = \mathrm{Ent}\left(U_{\omega_{n+1}} \left(\rho_n(\boldsymbol{\omega}) \otimes \rho_{\mathscr{E}_{\omega_{n+1}}}\right) U^*_{\omega_{n+1}} \,\middle|\, \rho_{n+1}(\boldsymbol{\omega}) \otimes \rho_{\mathscr{E}_{\omega_{n+1}}}\right),$$

an elementary calculation yields

$$\begin{aligned} \mathrm{ep}_n(\boldsymbol{\omega}) = &- S_n(\boldsymbol{\omega}) - \beta_{\omega_{n+1}} \mathrm{tr}\left(\left(\rho_n(\boldsymbol{\omega}) \otimes \rho_{\mathscr{E}_{\omega_{n+1}}}\right)(H_{\mathscr{E}_{\omega_{n+1}}} - F_{\omega_{n+1}})\right) \\ &+ S_{n+1}(\boldsymbol{\omega}) + \beta_{\omega_{n+1}} \mathrm{tr}\left(\left(\rho_n(\boldsymbol{\omega}) \otimes \rho_{\mathscr{E}_{\omega_{n+1}}}\right) U^*_{\omega_{n+1}}(H_{\mathscr{E}_{\omega_{n+1}}} - F_{\omega_{n+1}}) U_{\omega_{n+1}}\right), \end{aligned}$$

which can be rewritten as the one-step entropy balance relation

$$\Delta S_{n+1}(\boldsymbol{\omega}) + \mathrm{ep}_n(\boldsymbol{\omega}) = \beta_{\omega_{n+1}} \Delta Q_{n+1}(\boldsymbol{\omega}). \tag{4.5}$$

Identifying the right-hand side of this identity with the amount of entropy dissipated in the reservoir $\mathscr{R}_{\omega_{n+1}}$, $\mathrm{ep}_n(\boldsymbol{\omega})$ can be interpreted as the entropy produced by the interaction process. The inequality $\mathrm{ep}_n(\boldsymbol{\omega}) \geq 0$ thus becomes the expression of the 2^{nd}–law of thermodynamics, and yields Landauer's lower bound

$$\Delta Q_{n+1}(\boldsymbol{\omega}) \geq \frac{\Delta S_{n+1}(\boldsymbol{\omega})}{\beta_{\omega_{n+1}}},$$

on the energetic cost of a reduction of the system entropy (see [RW14, JP14, HJPR17, HJPR18] for more details and discussions). Summing over n we get

$$\frac{S_0(\boldsymbol{\omega}) - S_N(\boldsymbol{\omega})}{N} + \frac{1}{N}\sum_{n=0}^{N-1} \mathrm{ep}_n(\boldsymbol{\omega}) = -\sum_{\nu\in\Omega} \beta_\nu \frac{1}{N} \sum_{n=0}^{N-1} \langle \rho_n(\boldsymbol{\omega}), J_\nu(\omega_{n+1})\rangle.$$

Observing that $0 \le S_n(\boldsymbol{\omega}) \le \log\dim\mathcal{H}_{\mathcal{S}}$ and recalling (3.4), we deduce that whenever the limit (4.3) exists, the following expression of the time-averaged entropy production

$$\overline{\mathrm{ep}}(\boldsymbol{\omega}) := \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\mathrm{ep}_n(\boldsymbol{\omega}) = -\sum_{\nu\in\Omega}\beta_\nu\langle\rho_{\omega_0},\bar{J}_\nu(\boldsymbol{\omega})\rangle \ge 0 \tag{4.6}$$

holds, where $\bar{J}_\nu$ is the ergodic average (3.1). This applies, in particular, under Assumption (**STAT**), and expresses then the 2nd–law in the context of steady-state thermodynamics. If, moreover, $\mathbb{L}$ is irreducible, then (4.4) yields

$$\overline{\mathrm{ep}}(\boldsymbol{\omega}) = -\sum_{\nu\in\Omega}\beta_\nu\langle R_+,J_\nu\rangle$$

$\mathbb{P}$-a.s.

Remark. Assuming (**STAT**) but replacing Assumption (**KMS**) by the faithfulness of the probe states $\rho_{\mathcal{E}_\omega}$, setting[9]

$$J_S(\omega) := \mathrm{tr}_{\mathcal{H}_{\mathcal{E}_\omega}}\left(U_\omega^*[S_{\mathcal{E}_\omega},U_\omega](\mathbb{1}\otimes\rho_{\mathcal{E}_\omega})\right),\qquad S_{\mathcal{E}_\omega} := -\log\rho_{\mathcal{E}_\omega},$$

and repeating the previous calculation leads to the entropy balance equation

$$\Delta S_{n+1}(\boldsymbol{\omega}) + \mathrm{ep}_n(\boldsymbol{\omega}) = \langle\rho_n(\boldsymbol{\omega}),J_S(\omega_{n+1})\rangle \tag{4.7}$$

with the time-average

$$\overline{\mathrm{ep}}(\boldsymbol{\omega}) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\mathrm{ep}_n(\boldsymbol{\omega}) = \bar{J}_S(\boldsymbol{\omega}).$$

The irreducibility of $\mathbb{L}$ further leads to the $\mathbb{P}$-a.s. identity

$$\overline{\mathrm{ep}}(\boldsymbol{\omega}) = \langle R_+,J_S\rangle. \tag{4.8}$$

The extended observable J_S describes the entropy dumped into the reservoirs during a single interaction.

The vanishing of entropy production is a signature of thermodynamic equilibrium, as such, it will play a central role in our discussion of linear response in Section 4.4. Our next result is a quite general necessary and sufficient condition for the vanishing of entropy production in MRIS.

Proposition 4.1. *Assume that the probe states $\rho_{\mathcal{E}_\omega}$ are faithful and that $\mathbb{L}$ is irreducible. Then, the time averaged entropy production (4.8) vanishes $\mathbb{P}$-a.s. iff the family of states $(\rho_{+\omega})_{\omega\in\Omega}$ associated with the unique ESS R_+ in (2.10) satisfies*

$$U_\omega(\rho_{+\nu}\otimes\rho_{\mathcal{E}_\omega})U_\omega^* = \rho_{+\omega}\otimes\rho_{\mathcal{E}_\omega} \tag{4.9}$$

*for all pairs $(\nu,\omega)\in\Omega\times\Omega$ such that $P_{\nu\omega}>0$. In this case, further assuming (**KMS**), the entropy balance equation*

$$S(\rho_{+\omega_{n+1}}) - S(\rho_{+\omega_n}) = \beta_{\omega_{n+1}}\langle\rho_{+\omega_n},J(\omega_{n+1})\rangle = 0. \tag{4.10}$$

holds $\mathbb{P}_+$-a.s., and in particular,

$$\bar{J}_\nu = \langle R_+,J_\nu\rangle = 0 \tag{4.11}$$

for all $\nu\in\Omega$.

[9] Recall the convention made after (1.1)

Remarks. 1. Whenever $P_{\nu\omega} > 0$ and $P_{\nu'\omega} > 0$, it follows from (4.9) that $\rho_{+\nu} = \rho_{+\nu'}$.

2. The very special case where all reservoirs are identical in the sense that all the CPTP maps $\mathscr{L}_\omega$ coincide with some $\mathscr{L}$ will be of interest in Section 4.4. One easily checks that in this circumstance the mean state

$$\bar{\rho}_+ = \sum_{\omega\in\Omega} R_+(\omega) = \sum_{\omega\in\Omega} \pi_{+\omega}\rho_{+\omega}$$

satisfies $\mathscr{L}\bar{\rho}_+ = \bar{\rho}_+$. Setting $R(\omega) = \pi_{+\omega}\bar{\rho}_+$, we derive

$$(\mathbb{L}R)(\omega) = \sum_{\nu\in\Omega} P_{\nu\omega}\mathscr{L}R(\nu) = \sum_{\nu\in\Omega} \pi_{+\nu}P_{\nu\omega}\mathscr{L}\bar{\rho}_+ = \sum_{\nu\in\Omega} \pi_{+\nu}P_{\nu\omega}\bar{\rho}_+ = \pi_{+\omega}\bar{\rho}_+ = R(\omega).$$

If $\mathbb{L}$ is irreducible, then we can conclude that $R = R_+$, and (2.10) yields

$$\pi_{+\omega}\rho_{+\omega} = \mathscr{L}R(\omega) = \pi_{+\omega}\mathscr{L}\bar{\rho}_+ = \pi_{+\omega}\bar{\rho}_+,$$

so that the states $\rho_{+\omega}$ all coincide with $\bar{\rho}_+$.

4.2 Full Statistics of Entropy

While the entropy balance equation (4.5) and the related Landauer bound are clearly supporting the proposed interpretation of the quantum expectation value of the observable J_ν, as defined in (4.2), the fact that a real measurement of the energy transfer between the system $\mathscr{S}$ and the reservoir $\mathscr{R}_\nu$ is unlikely to proceed through an "instantaneous" measurement of J_ν casts some doubts on the physical meaning of its higher moments, and more generally of its spectral measure. This issue is not specific to RIS but is relevant to more general open quantum systems (see [TLH07] and [JOPP12, Section 5.10]). In this section, we adopt an operational point of view and consider a two-time measurement protocol of the *entropy observables*

$$S_{\mathscr{E}_\omega} = -\log\rho_{\mathscr{E}_\omega} = \beta_\omega(H_{\mathscr{E}_\omega} - F_\omega). \tag{4.12}$$

Set $\Sigma = \cup_{\omega\in\Omega}\mathrm{sp}(S_{\mathscr{E}_\omega})$ and for $\varsigma \in \Sigma$ let $\Pi_\varsigma(\omega)$ denote the spectral projection of $S_{\mathscr{E}_\omega}$ for the eigenvalue ς.[10] Further, let $\mathscr{X}$ be the σ-algebra generated by the cylinder sets of $\Xi = (\Sigma\times\Sigma)^{\mathbb{N}^*}$.

The state of the joint system before their coupling is the product state $\rho = \rho_n(\boldsymbol{\omega})\otimes\rho_{\mathscr{E}_{\omega_{n+1}}}$. A first measurement of $S_{\mathscr{E}_{\omega_{n+1}}}$ before interaction of the probe $\mathscr{E}_{\omega_{n+1}}$ with the system $\mathscr{S}$ thus yields the value $\varsigma\in\Sigma$ with probability

$$p(\varsigma) = \mathrm{tr}\left(\rho\Pi_\varsigma(\omega_{n+1})\right)$$

and leaves the joint system in the state

$$\rho_{|\varsigma} = \frac{\Pi_\varsigma(\omega_{n+1})\rho\Pi_\varsigma(\omega_{n+1})}{\mathrm{tr}\left(\rho\Pi_\varsigma(\omega_{n+1})\right)}.$$

Once the interaction between $\mathscr{E}_{\omega_{n+1}}$ and $\mathscr{S}$ is complete, the state of the joint system has evolved to

$$\rho'_{|\varsigma} = U_{\omega_{n+1}}\rho_{|\varsigma}U^*_{\omega_{n+1}} = \frac{U_{\omega_{n+1}}\Pi_\varsigma(\omega_{n+1})\rho\Pi_\varsigma(\omega_{n+1})U^*_{\omega_{n+1}}}{\mathrm{tr}\left(\rho\Pi_\varsigma(\omega_{n+1})\right)}$$

and the probability for the outcome of a second measurement of $S_{\mathscr{E}_{\omega_{n+1}}}$ to be $\varsigma'\in\Sigma$ is

$$p(\varsigma'\,|\,\varsigma) = \mathrm{tr}(\rho'_{|\varsigma}\Pi_{\varsigma'}(\omega_{n+1})) = \frac{\mathrm{tr}\left(U_{\omega_{n+1}}\Pi_\varsigma(\omega_{n+1})\rho\Pi_\varsigma(\omega_{n+1})U^*_{\omega_{n+1}}\Pi_{\varsigma'}(\omega_{n+1})\right)}{\mathrm{tr}\left(\rho\Pi_\varsigma(\omega_{n+1})\right)}.$$

[10] The convention here is that $\Pi_\varsigma(\omega) = 0$ whenever $\varsigma\notin\mathrm{sp}(S_{\mathscr{E}_\omega})$.

Applying Bayes' rule, we conclude that the joint probability for the two successive measurements of $S_{\mathcal{E}_{\omega_{n+1}}}$ to have the outcome $\xi = (\varsigma, \varsigma') \in \Sigma \times \Sigma$ is

$$p(\xi) = p(\varsigma' | \varsigma) p(\varsigma) = \operatorname{tr}\left(U_{\omega_{n+1}} \Pi_{\varsigma}(\omega_{n+1}) \rho \Pi_{\varsigma}(\omega_{n+1}) U^*_{\omega_{n+1}} \Pi_{\varsigma'}(\omega_{n+1}) \right).$$

This leads us to define the CP maps

$$\mathcal{L}_{\omega,\xi} \rho := \mathrm{e}^{-\varsigma} \operatorname{tr}_{\mathcal{H}_{\mathcal{E}_\omega}} ((\mathbb{1} \otimes \Pi_{\varsigma'}(\omega)) U_\omega (\rho \otimes \Pi_{\varsigma}(\omega)) U^*_\omega), \tag{4.13}$$

in terms of which we can introduce the conditional probability measure on $\mathscr{X}$ characterized by

$$\mathbb{Q}([\xi_1, \ldots, \xi_n] | \boldsymbol{\omega}) = \operatorname{tr}(\mathcal{L}_{\omega_n, \xi_n} \cdots \mathcal{L}_{\omega_1, \xi_1} \rho_{\omega_0}).$$

We can now associate to the MRIS $(\pi, P, (\rho_\omega)_{\omega \in \Omega}, (\mathcal{L}_\omega)_{\omega \in \Omega})$ the probability measure on $(\boldsymbol{\Omega} \times \Xi, \mathscr{O} \times \mathscr{X})$ defined by

$$\widehat{\mathbb{P}}(\mathrm{d}\boldsymbol{\xi} \mathrm{d}\boldsymbol{\omega}) := \mathbb{Q}(\mathrm{d}\boldsymbol{\xi} | \boldsymbol{\omega}) \mathbb{P}(\mathrm{d}\boldsymbol{\omega}),$$

with the corresponding expectation functional $\widehat{\mathbb{E}}$. Since we are mostly interested in entropy increments, for $\xi = (\varsigma, \varsigma') \in \Sigma \times \Sigma$ we shall set

$$\delta \xi := \varsigma' - \varsigma.$$

The *full statistics* of the entropy increments in the reservoirs during the first N interactions is the random vector $\mathscr{I}_N := (\mathscr{I}_{N,\nu})_{\nu \in \Omega}$ under the law $\widehat{\mathbb{P}}$, where

$$\mathscr{I}_{N,\nu} := \sum_{n=1}^{N} 1_{\{\omega_n = \nu\}} \delta \xi_n.$$

Our next results concern the large N asymptotics of $\mathscr{I}_N$. We shall see, in particular, that at the level of expectations this asymptotics is governed by the observables (4.2). To formulate these results we need to introduce a few new objects. For $\alpha \in \mathbb{R}$ let

$$\mathcal{L}^{[\alpha]}_\omega := \sum_{\xi \in \Sigma \times \Sigma} \mathrm{e}^{-\alpha \delta \xi} \mathcal{L}_{\omega, \xi}, \tag{4.14}$$

and for $\boldsymbol{\alpha} = (\alpha_\nu)_{\nu \in \Omega} \in \mathbb{R}^\Omega$, define a CP map on $\mathfrak{A}_*$ by

$$(\mathbb{L}^{[\boldsymbol{\alpha}]} R)(\omega) := \sum_{\nu \in \Omega} P_{\nu\omega} \mathcal{L}^{[\alpha_\nu]}_\nu R(\nu). \tag{4.15}$$

Finally, denote by $\ell(\boldsymbol{\alpha})$ the spectral radius of $\mathbb{L}^{[\boldsymbol{\alpha}]}$, which coincides with its dominant eigenvalue.

Theorem 4.2 (Limit Theorems and Large Deviations Principle)**.**

(i) Under Assumption (**STAT**) *one has*

$$\lim_{N \to \infty} \widehat{\mathbb{E}} \left[\frac{\mathscr{I}_{N,\nu}}{N} \right] = -\beta_\nu \mathbb{E}[\langle \rho_{\omega_0}, \bar{J}_\nu(\boldsymbol{\omega}) \rangle],$$

and whenever $\mathbb{L}$ *is irreducible, the right-hand side of this identity equals* $-\beta_\nu \langle R_+, J_\nu \rangle$.

(ii) If $\mathbb{L}$ *is irreducible, then the weak law of large numbers holds, i.e., the limit*

$$\lim_{N \to \infty} \frac{\mathscr{I}_{N,\nu}}{N} = -\beta_\nu \langle R_+, J_\nu \rangle,$$

exists in probability.

(iii) *If $\mathbb{L}$ is irreducible, then the central limit theorem holds, i.e., as $N \to \infty$*

$$\frac{1}{\sqrt{N}}\left(\mathscr{I}_N - \widehat{\mathbb{E}}[\mathscr{I}_N]\right)$$

converges in law towards a centered Gaussian vector with covariance matrix

$$C_{\omega\nu} = \ell_{\omega\nu} - \ell_\omega \ell_\nu, \tag{4.16}$$

where

$$\ell_\omega = (\partial_{\alpha_\omega}\ell)(0), \qquad \ell_{\omega\nu} = (\partial_{\alpha_\nu}\partial_{\alpha_\omega}\ell)(0).$$

(iv) *If $\mathbb{L}$ is primitive, then the limit*

$$e(\boldsymbol{\alpha}) = \lim_{N\to\infty}\frac{1}{N}\log\widehat{\mathbb{E}}[\mathrm{e}^{-\boldsymbol{\alpha}\cdot\mathscr{I}_N}] \tag{4.17}$$

exists and defines a real analytic function on $\mathbb{R}^\Omega$. Moreover, one has $e(\boldsymbol{\alpha}) = \log\ell(\boldsymbol{\alpha})$ for all $\boldsymbol{\alpha} \in \mathbb{R}^\Omega$.

(v) *If $\mathbb{L}$ is primitive, then the sequence of random vectors $(\mathscr{I}_N)_{N\in\mathbb{N}}$ satisfies a large deviation principle. More precisely, for any Borel set $G \subset \mathbb{R}^\Omega$,*

$$\begin{aligned} -\inf_{\boldsymbol{\varsigma}\in\mathring{G}} I(\boldsymbol{\varsigma}) &\le \liminf_{N\to\infty}\frac{1}{N}\log\widehat{\mathbb{P}}\left(\frac{\mathscr{I}_N}{N}\in G\right) \\ &\le \limsup_{N\to\infty}\frac{1}{N}\log\widehat{\mathbb{P}}\left(\frac{\mathscr{I}_N}{N}\in G\right) \le -\inf_{\boldsymbol{\varsigma}\in\overline{G}} I(\boldsymbol{\varsigma}), \end{aligned} \tag{4.18}$$

where $\mathring{G}/\overline{G}$ denote the interior/closure of G and the good rate function $\boldsymbol{\varsigma} \mapsto I(\boldsymbol{\varsigma})$ is given by the Legendre-Fenchel transform of the function $\boldsymbol{\alpha} \mapsto e(-\boldsymbol{\alpha})$,

$$I(\boldsymbol{\varsigma}) := \sup_{\boldsymbol{\alpha}\in\mathbb{R}^\Omega}(\boldsymbol{\alpha}\cdot\boldsymbol{\varsigma} - e(-\boldsymbol{\alpha})). \tag{4.19}$$

4.3 The Fluctuation Theorem

We are now in position to formulate the Fluctuation Relations (FRs for short) satisfied by the full statistics $(\mathscr{I}_N)_{N\in\mathbb{N}^*}$. The first FR dates back to 1905 and the celebrated work of Einstein on Brownian motion. The reader is referred to [RMM07] for an overview of the subsequent developments pertaining to classical physical systems. We shall follow the mathematical formulation of FRs through large deviation estimates initiated by Gallavotti and Cohen in their foundational works on steady-state FRs in chaotic dynamics [GC95a, GC95b]. See also [JPRB11] for a general approach to transient and steady-state FRs in the framework of classical dynamical systems. The direct quantization of a classical observable obeying a FR fails to satisfy a "quantum FR", see [JOPP12, Section 5.10] for concrete examples. In fact, to obtain an operationally meaningful extension of FRs to the quantum regime, one has to take into account the peculiar status of measurements in quantum mechanics. This precludes, in particular, the existence of steady-state FRs at the quantum scale. We refer the interested reader to [EHM09, CHT11] for exhaustive reviews and extensive lists of references to the physics literature. The mathematically oriented reader can also consult [DRM06, DDRM08, CM12, JPW14] for a study of entropic FRs in the Markovian approximation of open quantum systems, [DR09, JLP13, JPPP15, BJP+15, BPR19, BPP20] for studies of two-time measurement protocols and to [BJPP18, BCJP21] for extensions to repeated quantum measurements.

Fluctuation relations are deeply linked with microscopic time-reversal invariance, as is already apparent in Onsager's theory of irreversible processes [Ons31a, Ons31b]. Hence, we need to assume that our MRIS has some form of time-reversal invariance.

The driving Markov chain (π, P) is said to be *reversible* whenever the probability $\mathbb{P}$ is invariant upon reversing the chronological order of events, that is, for any cylinder $[\omega_0,\dots,\omega_n]$ one has

$$\mathbb{P}([\omega_0,\dots,\omega_n]) = \mathbb{P}([\omega_n,\dots,\omega_0]).$$

It is well known, and straightforward to check, that reversibility is equivalent to the so-called *Detailed Balance* condition, namely:

$$\pi_\omega P_{\omega\nu} = \pi_\nu P_{\nu\omega}$$

for all $\omega,\nu\in\Omega$. Note that under this condition one has $\pi P = \pi$, so that the following assumption implies (**STAT**).

Assumption (DB). The driving Markov chain (π, P) satisfies detailed balance.

Our second, complementary assumption ensures the reversibility of the interaction processes.

Assumption (TRI). There are anti-unitary involutions θ and θ_ω acting on $\mathscr{H}_{\mathscr{S}}$ and $\mathscr{H}_{\mathscr{E}_\omega}$, such that

$$\theta_\omega H_{\mathscr{E}_\omega} = H_{\mathscr{E}_\omega}\theta_\omega, \qquad (\theta\otimes\theta_\omega)U_\omega = U_\omega^*(\theta\otimes\theta_\omega),$$

for all $\omega\in\Omega$.

Whenever Assumption (**TRI**) holds, we denote $\Theta : X \mapsto \theta X\theta$ the map induced on $\mathscr{A}$ and on $\mathfrak{A}$.

Theorem 4.4 (Fluctuation Theorem)**.** *If $\mathbb{L}$ is primitive, then Assumptions* (**TRI**) *and* (**DB**) *imply that the rate function of the large deviation principle* (4.18) *satisfies the Fluctuation Relation*

$$I(-\boldsymbol{\varsigma}) - I(\boldsymbol{\varsigma}) = \sum_{\omega\in\Omega}\varsigma_\omega,$$

for all $\boldsymbol{\varsigma} \in \mathbb{R}^{\Omega}$. *This relation is associated with the following symmetry of the cumulant generating function*

$$e(\mathbf{1}-\boldsymbol{\alpha}) = e(\boldsymbol{\alpha}), \tag{4.20}$$

for all $\boldsymbol{\alpha} \in \mathbb{R}^{\Omega}$, *with* $\mathbf{1} = (1,1,\ldots,1)$.

Since $\mathscr{I}_{N,\nu}$ accounts for the entropy produced by the interaction with reservoir $\mathscr{R}_\nu$ during the N first interactions, the total entropy produced during this period is given by

$$\sigma_N = \sum_{\nu\in\Omega} \mathscr{I}_{N,\nu} = \mathbf{1}\cdot\mathscr{I}_N.$$

Applying the contraction principle [DZ98, Theorem 4.2.1] to the map $\mathscr{I} \mapsto \mathbf{1}\cdot\mathscr{I}$ immediately yields the following

Corollary 4.5. *Under the assumptions of Theorem 4.4 the sequence of entropy production random variable* $(\sigma_N)_{N\in\mathbb{N}}$ *satisfies a large deviation principle*

$$-\inf_{s\in\mathring{S}} \bar{I}(s) \le \liminf_{N\to\infty} \frac{1}{N}\log\widehat{\mathbb{P}}\left(\frac{\sigma_N}{N}\in S\right) \le \limsup_{N\to\infty}\frac{1}{N}\log\widehat{\mathbb{P}}\left(\frac{\sigma_N}{N}\in S\right) \le -\inf_{s\in\bar{S}}\bar{I}(s),$$

for any Borel set $S\subset\mathbb{R}$, *with the rate function*

$$\bar{I}(s) = \inf\{I(\boldsymbol{\varsigma}) \mid \boldsymbol{\varsigma}\in\mathbb{R}^{\Omega}, \sum_{\omega\in\Omega}\varsigma_\omega = s\}$$

satisfying the Fluctuation Relation

$$\bar{I}(-s) - \bar{I}(s) = s.$$

The last relation implies

$$\lim_{\delta\downarrow 0}\lim_{N\to\infty}\left(\frac{\widehat{\mathbb{P}}\left(\left|\frac{\sigma_N}{N}+s\right|<\delta\right)}{\widehat{\mathbb{P}}\left(\left|\frac{\sigma_N}{N}-s\right|<\delta\right)}\right)^{1/N} = \mathrm{e}^{-s},$$

which is often written as

$$\frac{\widehat{\mathbb{P}}\left(\frac{\sigma_N}{N}=-s\right)}{\widehat{\mathbb{P}}\left(\frac{\sigma_N}{N}=s\right)} \simeq \mathrm{e}^{-sN}$$

in the physics literature. It shows that negative values of the entropy production rate are exponentially suppressed relative to positive values, thus providing a deep refinement of the Second Law (4.6),

Remark. Theorems 4.2 and 4.4 as well as Corollary 4.5 concern the full statistics of entropy fluxes into the reservoirs. Relation (4.12) links entropy to energy and can be used to convert these results into statements on the full statistics of energy fluxes $\mathscr{J}_N = (\mathscr{J}_{N,\omega})_{\omega\in\Omega}$, where $\mathscr{J}_{N,\omega} = -\mathscr{I}_{N,\omega}/\beta_\omega$. We shall leave the details of this alternative formulation to the reader.

4.4 Linear Response

Linear Response Theory deals with the reaction of physical systems to a small deviation from a thermal equilibrium situation, the latter being characterized by a well-defined temperature and vanishing entropy production. Since RIS in general, and MRIS in particular, evolve according to a time-dependent Hamiltonian, they are not expected to reach thermal equilibrium, even when all reservoirs are in thermal equilibrium *at the same temperature.* In this section we will provide a simple example of a MRIS admitting a steady state qualifying as a thermal equilibrium in the above sense. Moreover, we shall see that the usual linear response properties near thermal equilibrium, *Green–Kubo Formula, Onsager Reciprocity Relations* and *Fluctuation–Dissipation Relations,* hold in these cases.

Our starting point is a MRIS satisfying the following *equilibrium conditions*

Assumption (EQU).

- (**KMS**) is satisfied with all reservoirs at the same temperature: $\boldsymbol{\beta} = \bar{\beta}\mathbf{1}$.
- $\mathbb{L}$ is irreducible, with the unique ESS R_+.
- Entropy production vanishes: $\langle R_+, J_S\rangle = \bar{\beta} \sum_{\omega\in\Omega} \langle R_+, J_\omega\rangle = 0$.

From the family of state ρ_+ associated to R_+ by (2.10), we define the observables

$$S_{+\omega} := -\log \rho_{+\omega}.$$

We also introduce the family $(\widehat{J}_\nu)_{\nu\in\Omega}$ of extended observables

$$\widehat{J}_\nu(\omega) := \delta_{\omega\nu}\,\mathrm{tr}_{\mathcal{H}_{\mathcal{E}_\omega}}((U_\omega H_{\mathcal{E}_\omega} U_\omega^* - H_{\mathcal{E}_\omega})(\mathbb{1}\otimes\rho_{\mathcal{E}_\omega})),$$

obtained from (4.2) by flipping sign and exchanging the roles of U_ω and U_ω^*, and the CPU maps

$$\widehat{\mathcal{L}}_\omega^* : X \mapsto \mathrm{tr}_{\mathcal{H}_{\mathcal{E}_\omega}}(U_\omega(X\otimes\mathbb{1})U_\omega^*(\mathbb{1}\otimes\rho_{\mathcal{E}_\omega})),$$

given by (4.1) after exchanging the roles of U_ω and U_ω^*, *i.e.*, reversing time in the interaction processes. In fact, Assuming (**TRI**), one has $\widehat{J}_\nu = -\Theta J_\nu$ and $\widehat{\mathcal{L}}_\omega^* = \Theta\mathcal{L}_\omega^*\Theta$.

We consider the family of MRIS obtained from the previous one by modifying the reservoir temperatures, *i.e.*, by replacing the "equilibrium" probe states by the family

$$\rho_{\mathcal{E}_\omega,\boldsymbol{\zeta}} = \mathrm{e}^{-(\bar{\beta}-\zeta_\omega)(H_{\mathcal{E}_\omega}-F_{\omega,\boldsymbol{\zeta}})}, \qquad F_{\omega,\boldsymbol{\zeta}} = -\frac{1}{\bar{\beta}-\zeta_\omega}\log \mathrm{tr}\,\mathrm{e}^{-(\bar{\beta}-\zeta_\omega)H_{\mathcal{E}_\omega}},$$

parametrized by $\boldsymbol{\zeta}\in\mathbb{R}^\Omega$. To express the dependence of a quantity A w.r.t. the parameter $\boldsymbol{\zeta}$, we will write $A_{\boldsymbol{\zeta}}$. In particular, $A_{\mathbf{0}}$ denotes an equilibrium quantity.

As the name suggest, linear response theory aims at expressing the changes of various properties of the perturbed ESS $R_{+\boldsymbol{\zeta}}$ to first order in the perturbation parameter $\boldsymbol{\zeta}$. Of particular interest are the so-called *kinetic coefficients*

$$L_{\omega\nu} := \partial_{\zeta_\nu}\bar{J}_{\omega\boldsymbol{\zeta}}|_{\boldsymbol{\zeta}=0},$$

where (recall (4.3) (4.4))

$$\bar{J}_{\omega\boldsymbol{\zeta}} = \langle R_{+\boldsymbol{\zeta}}, J_{\omega\boldsymbol{\zeta}}\rangle$$

is the steady-state ensemble average of the energy flux into reservoir $\mathcal{R}_\omega$.

Theorem 4.6. *Under Assumption* (**EQU**), *the following statements hold for all* $\boldsymbol{\zeta}\in\mathbb{R}^\Omega$*:*

(i) $\mathbb{L}_{\boldsymbol{\zeta}}$ *is irreducible. It is primitive whenever* $\mathbb{L}$ *is.*

(ii) For $\mathbb{P}_{\boldsymbol{\zeta}}$*-almost every* $\boldsymbol{\omega}\in\boldsymbol{\Omega}$ *one has*

$$\sum_{\nu\in\Omega}\bar{J}_{\nu\boldsymbol{\zeta}}(\boldsymbol{\omega}) = 0.$$

(iii) With $\boldsymbol{\beta}^{-1} = ((\bar{\beta}-\zeta_\omega)^{-1})_{\omega\in\Omega}$*, the limit*

$$\lim_{N\to\infty}\frac{\boldsymbol{\beta}^{-1}\cdot\mathcal{I}_{N\zeta}}{N} = 0$$

holds in probability under the law $\widehat{\mathbb{P}}_{\boldsymbol{\zeta}}$*.*

(iv) The Gaussian measure obtained in Theorem 4.2 (iii) as the limiting law of

$$\frac{1}{\sqrt{N}}\left(\mathscr{I}_{N\boldsymbol{\zeta}} - \widehat{\mathbb{E}}_{\boldsymbol{\zeta}}\left[\mathscr{I}_{N\boldsymbol{\zeta}}\right]\right)$$

as $N \to \infty$ is supported by the hyperplane

$$\mathfrak{Z}_{\boldsymbol{\zeta}} = \{\boldsymbol{\varsigma} \in \mathbb{R}^{\Omega} \mid \boldsymbol{\beta}^{-1} \cdot \boldsymbol{\varsigma} = 0\}.$$

In the following, suppose in addition that $\mathbb{L}$ is primitive.

(v) The cumulant generating function (4.17) satisfies the translation symmetry

$$e_{\boldsymbol{\zeta}}(\boldsymbol{\alpha} + \gamma\boldsymbol{\beta}^{-1}) = e_{\boldsymbol{\zeta}}(\boldsymbol{\alpha}) \tag{4.21}$$

for all $\boldsymbol{\alpha} \in \mathbb{R}^{\Omega}$ and $\gamma \in \mathbb{R}$.

(vi) The rate function of the LDP (4.18) satisfies

$$I_{\boldsymbol{\zeta}}(\boldsymbol{\varsigma}) = +\infty$$

whenever $\boldsymbol{\varsigma} \notin \mathfrak{Z}_{\boldsymbol{\zeta}}$.

*Finally, assuming also that the time-reversal symmetries (**TRI**) and (**DB**) hold, one has:*

(vii) A Green–Kubo type formula holds in the following form

$$\begin{aligned} L_{\omega\nu} =& \frac{1}{2}\sum_{n\geq 0}\mathbb{E}_{+}\left[\langle \rho_{+\omega_0}, \widehat{J}_{\nu}(\omega_0)\mathscr{L}^{*}_{\omega_1}\cdots\mathscr{L}^{*}_{\omega_n}J_{\omega}(\omega_{n+1})\rangle + \langle \omega \leftrightarrows \nu\rangle\right] \\ & \frac{1}{2}\delta_{\omega\nu}\mathbb{E}_{+}\left[\langle \rho_{+\omega_0}, \widehat{\mathscr{L}}^{*}_{\omega_1}(S^2_{+\omega_1}) - \widehat{\mathscr{L}}^{*}_{\omega_1}(S_{+\omega_1})S_{+\omega_0} - S_{+\omega_0}\widehat{\mathscr{L}}^{*}_{\omega_1}(S_{+\omega_1}) + S^2_{+\omega_0}\rangle 1_{\omega_1=\omega}\right], \end{aligned} \tag{4.22}$$

all the quantities on the right-hand side of the equality (4.22) being evaluated at equilibrium $\boldsymbol{\zeta} = 0$.

(viii) The Onsager Reciprocity Relations

$$L_{\omega\nu} = L_{\nu\omega},$$

hold. In addition, the kinetic coefficients are related to the covariance matrix of the limiting Gaussian measure in Theorem 4.2 (iii) by the Fluctuation-Dissipation Relations

$$L_{\omega\nu} = \frac{1}{2\bar{\beta}^2}C_{\omega\nu}.$$

Example. We consider a MRIS with 3 reservoirs and transition matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}.$$

One easily checks that $P^5 > 0$, so P is primitive (its non-dominant eigenvalues are $e^{\pm 3i\pi/4}/\sqrt{2}$). The invariant probability is $\pi_+ = [2/5, 2/5, 1/5]$. However, P fails to satisfy the Detailed Balance condition (**DB**). Denote by σ_x, σ_y, σ_z the usual Pauli matrices and let $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$. The system $\mathscr{S}$ and the reservoirs $\mathscr{R}_\omega$ are 2-level systems with Hamiltonians

$$H_{\mathscr{S}} = \epsilon\sigma_z, \qquad H_{\mathscr{E}_\omega} = \gamma\sigma_z,$$

with the corresponding propagators

$$U_\omega = e^{-i\tau(H_{\mathscr{S}} + H_{\mathscr{E}_\omega} + V_\omega)}, \qquad V_\omega = \lambda(\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+).$$

We further assume (**KMS**), all inverse temperatures being given by $\bar\beta$. It is an exercise to show that all the induced CPTP maps $\mathscr{L}_\omega$ coincide with

$$\mathscr{L} : \begin{bmatrix} \rho_{++} & \rho_{+-} \\ \rho_{-+} & \rho_{--} \end{bmatrix} \mapsto \begin{bmatrix} \rho_{++} - A^2 \frac{\rho_{++}e^{\bar\beta\gamma} - \rho_{--}e^{-\bar\beta\gamma}}{e^{\bar\beta\gamma} + e^{-\bar\beta\gamma}} & B\rho_{+-} \\ \bar B\rho_{-+} & \rho_{--} + A^2 \frac{\rho_{++}e^{\bar\beta\gamma} - \rho_{--}e^{-\bar\beta\gamma}}{e^{\bar\beta\gamma} + e^{-\bar\beta\gamma}} \end{bmatrix},$$

with

$$A = \frac{\lambda \sin(\Lambda\tau)}{\Lambda}, \quad B = e^{-i(\epsilon+\gamma)\tau}\left(\cos(\Lambda\tau) - i(\epsilon-\gamma)\frac{\sin(\Lambda\tau)}{\Lambda}\right), \qquad \Lambda = \sqrt{\lambda^2 + (\epsilon-\gamma)^2}.$$

Discarding the trivial case $A = 0$ and $B = 1$, the unique fixed point of $\mathscr{L}$ is the Gibbs state

$$\rho_+ = \frac{e^{-\tilde\beta\sigma_z}}{\operatorname{tr} e^{-\tilde\beta\sigma_z}}, \qquad \tilde\beta = \gamma\bar\beta.$$

Moreover, one checks that the observable $N = \sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z$ is invariant under the interaction dynamics. It follows that

$$U_\omega e^{-\tilde\beta N} U_\omega^* = e^{-\tilde\beta N},$$

and since

$$\rho_+ \otimes \rho_{\mathscr{E}_\omega} = \frac{e^{-\tilde\beta N}}{\operatorname{tr} e^{-\tilde\beta N}},$$

according to the remark after Proposition 4.1, Assumption (**EQU**) is satisfied with $\rho_{+\omega} = \rho_+$ for all ω. A simple calculation gives

$$\bar\beta J(\omega) = 2A^2\tilde\beta\sigma_z \frac{e^{\tilde\beta\sigma_z}}{\operatorname{tr} e^{\tilde\beta\sigma_z}} = -\bar\beta\widehat{J}(\omega),$$

so that

$$\bar\beta\bar J = \bar\beta\langle R_+, J\rangle = \sum_\omega \pi_{+\omega}\langle\rho_+, J(\omega)\rangle = \frac{2A^2\tilde\beta \operatorname{tr}(\sigma_z)}{\operatorname{tr} e^{\tilde\beta\sigma_z}\operatorname{tr} e^{-\tilde\beta\sigma_z}} = 0.$$

5 Proofs

5.1 Proof of Proposition 4.1

Since $\mathbb{L}$ is irreducible, it has a unique ESS R_+ and the transition matrix P has the unique and faithful invariant probability π_+. We set $Q_{\nu\omega} = \pi_{+\omega}P_{\omega\nu}\pi_{+\nu}^{-1}$, observing that Q is again a right stochastic matrix with unique and faithful invariant probability π_+. Define the following maps on $\mathfrak{A}_*$,

$$\begin{array}{rclrcl} (\mathscr{L}R)(\omega) & = & \mathscr{L}_\omega R(\omega), & (\Pi R)(\omega) & = & \pi_{+\omega}R(\omega), \\ (\mathscr{P}R)(\omega) & = & \sum\limits_{\nu\in\Omega} P_{\nu\omega}R(\nu), & (\mathscr{Q}R)(\omega) & = & \sum_{\nu\in\Omega} Q_{\nu\omega}R(\nu), \end{array} \tag{5.1}$$

so that the factorizations $\mathbb{L} = \mathscr{P}\mathscr{L}$ and $\mathscr{Q} = \Pi\mathscr{P}^*\Pi^{-1}$ hold.

($\Leftarrow$) If $P_{\nu\omega} > 0$ implies (4.9), then it follows from (2.11) and (4.8) that the time-averaged entropy production vanishes $\mathbb{P}$-almost surely

$$\begin{aligned}\overline{\mathrm{ep}}(\boldsymbol{\omega}) = \langle R_+, J_S\rangle &= \sum_{\omega\in\Omega} \mathrm{tr}\big((R_+(\omega)\otimes\rho_{\mathscr{E}_\omega})U_\omega^*[S_{\mathscr{E}_\omega}, U_\omega]\big)\\ &= \sum_{\omega,\nu\in\Omega} \pi_{+\nu}P_{\nu\omega}\,\mathrm{tr}\big((U_\omega(\rho_{+\nu}\otimes\rho_{\mathscr{E}_\omega})U_\omega^* - \rho_{+\nu}\otimes\rho_{\mathscr{E}_\omega})S_{\mathscr{E}_\omega}\big)\\ &= \sum_{\omega,\nu\in\Omega} \pi_{+\nu}P_{\nu\omega}\,\mathrm{tr}\big((\rho_{+\omega}-\rho_{+\nu})\otimes\rho_{\mathscr{E}_\omega}S_{\mathscr{E}_\omega}\big) = 0.\end{aligned}$$

($\Rightarrow$) Assuming now that the time-averaged entropy production vanishes $\mathbb{P}$-almost surely, we get, using (4.8), (2.12) and the entropy balance (4.7),

$$\begin{aligned}0 = \overline{\mathrm{ep}}(\boldsymbol{\omega}) &= \langle R_+, J_S\rangle = \mathbb{E}_+[\langle\rho_{+\omega_0}, J_S(\omega_1)\rangle]\\ &= \mathbb{E}_+[\mathrm{Ent}(U_{\omega_1}(\rho_{+\omega_0}\otimes\rho_{\mathscr{E}_{\omega_1}})U_{\omega_1}^*|\mathscr{L}_{\omega_1}\rho_{+\omega_0}\otimes\rho_{\mathscr{E}_{\omega_1}}) + S(\rho_{+\omega_0}) - S(\mathscr{L}_{\omega_1}\rho_{+\omega_0})].\end{aligned}\tag{5.2}$$

Recalling the definition of the right stochastic matrix Q, we can write

$$\mathbb{E}_+[S(\mathscr{L}_{\omega_1}\rho_{+\omega_0})] = \sum_{\nu,\omega\in\Omega}\pi_{+\nu}P_{\nu\omega}S(\mathscr{L}_\omega\rho_{+\nu}) = \sum_{\omega\in\Omega}\pi_{+\omega}\sum_{\nu\in\Omega}Q_{\omega\nu}S(\mathscr{L}_\omega\rho_{+\nu}).$$

Invoking the concavity of the entropy map $\rho\mapsto S(\rho)$ [Pet08, Theorem 3.7], we derive

$$\mathbb{E}_+[S(\mathscr{L}_{\omega_1}\rho_{+\omega_0})] \le \sum_{\omega\in\Omega}\pi_{+\omega}S\left(\sum_{\nu\in\Omega}Q_{\omega\nu}\mathscr{L}_\omega\rho_{+\nu}\right) = \sum_{\omega\in\Omega}\pi_{+\omega}S((\mathscr{L}\mathscr{Q}^*\rho_+)(\omega)).$$

By (2.10), we have

$$\mathscr{L}\mathscr{Q}^*\rho_+ = \mathscr{L}\mathscr{Q}^*\Pi^{-1}\mathscr{L}R_+ = \mathscr{L}\Pi^{-1}\mathscr{P}\mathscr{L}R_+ = \Pi^{-1}\mathscr{L}\mathbb{L}R_+ = \Pi^{-1}\mathscr{L}R_+ = \rho_+,\tag{5.3}$$

so that

$$\mathbb{E}_+[S(\mathscr{L}_{\omega_1}\rho_{+\omega_0})] \le \mathbb{E}_+[S(\rho_{+\omega_0})].$$

It follows that $\mathbb{E}_+[S(\rho_{+\omega_0}) - S(\mathscr{L}_{\omega_1}\rho_{+\omega_0})] \ge 0$. This inequality, together with the non-negativity of relative entropy, allow us to deduce from (5.2) that

$$\mathbb{E}_+[\mathrm{Ent}(U_{\omega_1}(\rho_{+\omega_0}\otimes\rho_{\mathscr{E}_{\omega_1}})U_{\omega_1}^*|\mathscr{L}_{\omega_1}\rho_{+\omega_0}\otimes\rho_{\mathscr{E}_{\omega_1}})] = 0 = \mathbb{E}_+[S(\rho_{+\omega_0}) - S(\mathscr{L}_{\omega_1}\rho_{+\omega_0})].\tag{5.4}$$

Writing the second identity as

$$\sum_{\omega\in\Omega}\pi_{+\omega}\left(S\left(\sum_{\nu\in\Omega}Q_{\omega\nu}\mathscr{L}_\omega\rho_{+\nu}\right) - \sum_{\nu\in\Omega}Q_{\omega\nu}S(\mathscr{L}_\omega\rho_{+\nu})\right) = 0,$$

and combining the facts that von Neumann's entropy is strictly convex [Car10, Theorem 2.10] and π_+ faithful, we derive that for all $\nu,\omega\in\Omega$ such that $Q_{\omega\nu} > 0$, one has

$$\mathscr{L}_\omega\rho_{+\nu} = \sum_{\mu\in\Omega}Q_{\omega\mu}\mathscr{L}_\omega\rho_{+\mu} = \rho_{+\omega},$$

where we used (5.3) to justify the last identity.

Since $Q_{\omega\nu} > 0$ iff $P_{\nu\omega} > 0$, the first equality in (5.4) further yields that

$$U_\omega(\rho_{+\nu}\otimes\rho_{\mathscr{E}_\omega})U_\omega^* = \mathscr{L}_\omega\rho_{+\nu}\otimes\rho_{\mathscr{E}_\omega} = \rho_{+\omega}\otimes\rho_{\mathscr{E}_\omega},$$

provided $P_{\nu\omega} > 0$. Thus, the family of states $(\rho_{+\omega})_{\omega\in\Omega}$ has the required property.

The first equality in (4.10) follows from the fact that, whenever $\rho_k(\boldsymbol{\omega}) = \rho_{+\omega_k}$ for $k \in \{n, n+1\}$, then the previous identity implies that $\mathrm{ep}_n(\boldsymbol{\omega}) = 0$. Assuming (**KMS**), the simple calculation

$$\begin{aligned}\langle \rho_{+\nu}, J(\omega)\rangle &= \mathrm{tr}\left((\rho_{+\nu}\otimes\rho_{\mathcal{E}_\omega})(H_{\mathcal{E}_\omega} - U_\omega^* H_{\mathcal{E}_\omega} U_\omega)\right)\\ &= \mathrm{tr}\left((\rho_{+\nu} - \rho_{+\omega})\otimes \rho_{\mathcal{E}_\omega} H_{\mathcal{E}_\omega}\right) = 0,\end{aligned}$$

shows that the same identity leads to the second equality in (4.10). Finally, (4.4) immediately leads to (4.11).

Remark. If $\mathbb{L}$ is irreducible and the family of states $(\varkappa_\omega)_{\omega\in\Omega}$ satisfies

$$U_\omega(\varkappa_\nu\otimes\rho_{\mathcal{E}_\omega})U_\omega^* = \varkappa_\omega\otimes\rho_{\mathcal{E}_\omega}$$

for all pairs $(\nu,\omega)\in\Omega\times\Omega$ such that $P_{\nu\omega} > 0$, then $\varkappa_\omega = \rho_{+\omega}$ for all $\omega\in\Omega$.

To see this, we first note that

$$P_{\nu\omega}\mathscr{L}_\omega\pi_{+\nu}\varkappa_\nu = P_{\nu\omega}\pi_{+\nu}\varkappa_\omega,$$

for any pair $(\nu,\omega)\in\Omega\times\Omega$. Summing both sides of this identity over ν yields

$$\mathscr{L}\mathscr{P}\Pi\varkappa = \Pi\varkappa, \tag{5.5}$$

from which we deduce

$$\mathbb{L}(\mathscr{P}\Pi\varkappa) = \mathscr{P}(\mathscr{L}\mathscr{P}\Pi\varkappa) = \mathscr{P}\Pi\varkappa,$$

and since $\mathbb{L}$ is irreducible, we can conclude that its unique fixed point in $\mathfrak{A}_{*+1}$ is $R_+ = \mathscr{P}\Pi\varkappa$. The second relation in (2.10) further gives

$$\rho_+ = \mathscr{L}\Pi^{-1}R_+ = \mathscr{L}\Pi^{-1}\mathscr{P}\Pi\varkappa$$

which, combined with (5.5) yields

$$\varkappa = \Pi^{-1}\mathscr{L}\mathscr{P}\Pi\varkappa = \mathscr{L}\Pi^{-1}\mathscr{P}\Pi\varkappa = \rho_+.$$

5.2 Proof of Theorem 4.2

We start with some preparations. For $\omega\in\Omega$ denote by $(\phi_{\omega,s})_{s\in\Sigma_\omega}$ an orthonormal eigenbasis of $S_{\mathcal{E}_\omega}$, with $S_{\mathcal{E}_\omega}\phi_{\omega,s} = \varsigma_{\omega,s}\phi_{\omega,s}$. One easily derives the following Kraus representation of the TPCP-map $\mathscr{L}_\omega$

$$\mathscr{L}_\omega\rho = \sum_{s,s'\in\Sigma_\omega} V_{\omega,s,s'}\rho V_{\omega,s,s'}^*,$$

where $V_{\omega,s,s'}\in\mathscr{B}(\mathscr{H}_\mathscr{S})$ is the operator associated to the sesquilinear form

$$\langle\chi, V_{\omega,s,s'}\psi\rangle = \mathrm{e}^{-\varsigma_{\omega,s}/2}\langle\chi\otimes\phi_{\omega,s'}, U_\omega\psi\otimes\phi_{\omega,s}\rangle.$$

Further setting

$$\mathbb{V}_{\omega,s,s',\nu} = p_\omega^*\sqrt{P_{\nu\omega}}V_{\nu,s,s'}p_\nu, \tag{5.6}$$

where $p_\omega : \ell^2(\Omega;\mathscr{H}_\mathscr{S})\ni\psi\mapsto\psi(\omega)\in\mathscr{H}_\mathscr{S}$, we can write a Kraus decomposition of $\mathbb{L}$ as

$$\mathbb{L}R = \sum_{\substack{\omega,\nu\in\Omega\\ s,s'\in\Sigma_\nu}}\mathbb{V}_{\omega,s,s',\nu}R\,\mathbb{V}_{\omega,s,s',\nu}^*.$$

Recalling (4.14) and (4.15), for $\boldsymbol{\alpha}\cdot\mathscr{I}_n=\sum_{\omega\in\Omega}\alpha_\omega\mathscr{I}_{n,\omega}$, and $X\in\mathfrak{A}$ one has

$$\widehat{\mathbb{E}}[\mathrm{e}^{-\boldsymbol{\alpha}\cdot\mathscr{I}_n}X(\omega_{n+1})]=\mathbb{E}\left[\mathrm{tr}\left(X(\omega_{n+1})\mathscr{L}_{\omega_n}^{[\alpha_{\omega_n}]}\cdots\mathscr{L}_{\omega_1}^{[\alpha_{\omega_1}]}\rho_{\omega_0}\right)\right]=\langle\mathbb{L}^{[\boldsymbol{\alpha}]n}R,X\rangle$$

where $\mathbb{L}^{[\boldsymbol{\alpha}]}$ has the representation

$$\mathbb{L}^{[\boldsymbol{\alpha}]}R=\sum_{\substack{\omega,\nu\in\Omega\\ s,s'\in\Sigma_\nu}}\mathrm{e}^{-\alpha_\nu(\varsigma_{\nu,s'}-\varsigma_{\nu,s})}\mathbb{V}_{\omega,s,s',\nu}R\mathbb{V}_{\omega,s,s',\nu}^*.$$

Note that this representation extends $\mathbb{L}^{[\boldsymbol{\alpha}]}$ to a CP map on $\mathscr{B}^1(\mathscr{H}_{\mathscr{S}})$ whose range is in $\mathfrak{A}_*$. In particular, the left/right eigenvectors of $\mathbb{L}^{[\boldsymbol{\alpha}]}$ to non-zero eigenvalues are in $\mathfrak{A}/\mathfrak{A}_*$. Comparing the Kraus families of $\mathbb{L}$ and $\mathbb{L}^{[\boldsymbol{\alpha}]}$, we conclude that $\mathbb{L}^{[\boldsymbol{\alpha}]}$ is irreducible/primitive for all $\boldsymbol{\alpha}\in\mathbb{R}^\Omega$ iff $\mathbb{L}$ is.

(i) Observing that

$$\mathscr{L}_\omega^{[\alpha]}\rho=\mathrm{tr}_{\mathscr{H}_{\mathscr{E}_\omega}}\left((\mathbb{1}\otimes\mathrm{e}^{-\alpha S_{\mathscr{E}_\omega}})U_\omega(\rho\otimes\mathrm{e}^{-(1-\alpha)S_{\mathscr{E}_\omega}})U_\omega^*\right),\tag{5.7}$$

it is clear that the map $\boldsymbol{\alpha}\mapsto\mathbb{L}^{[\boldsymbol{\alpha}]}$ is real analytic, and differentiation yields

$$\partial_{\alpha_\nu}\langle\mathbb{L}^{[\boldsymbol{\alpha}]}R,X\rangle|_{\boldsymbol{\alpha}=0}=\langle\mathbb{L}_\nu R,X\rangle,\tag{5.8}$$

with

$$(\mathbb{L}_\nu R)(\omega)=P_{\nu\omega}\,\mathrm{tr}_{\mathscr{H}_{\mathscr{E}_\nu}}\left([U_\nu,S_{\mathscr{E}_\nu}](R(\nu)\otimes\rho_{\mathscr{E}_\nu})U_\nu^*\right).$$

In particular, invoking (4.2),

$$\langle\mathbb{L}_\nu R,\mathbb{1}\rangle=\beta_\nu\langle R,J_\nu\rangle,\tag{5.9}$$

and hence

$$\begin{aligned}\widehat{\mathbb{E}}[\mathscr{I}_{N,\omega}]=-\partial_{\alpha_\omega}\widehat{\mathbb{E}}[\mathrm{e}^{-\boldsymbol{\alpha}\cdot\mathscr{I}_N}]|_{\boldsymbol{\alpha}=0}&=-\partial_{\alpha_\omega}\langle\mathbb{L}^{[\boldsymbol{\alpha}]N}R,\mathbb{1}\rangle|_{\boldsymbol{\alpha}=0}\\&=\sum_{k=0}^{N-1}\langle\mathbb{L}^{[\boldsymbol{\alpha}](N-k-1)}(-\partial_{\alpha_\omega}\mathbb{L}^{[\boldsymbol{\alpha}]})\mathbb{L}^{[\boldsymbol{\alpha}]k}R,\mathbb{1}\rangle|_{\boldsymbol{\alpha}=0}\\&=-\sum_{k=0}^{N-1}\beta_\omega\langle\mathbb{L}^kR,J_\omega\rangle=-\sum_{k=0}^{N-1}\beta_\omega\mathbb{E}[\langle\rho_k(\boldsymbol{\omega}),J_\omega(\omega_{k+1})\rangle].\end{aligned}$$

The assertions thus follow from Theorems 3.2 and 3.3.

(ii) If $\mathbb{L}$ is irreducible, then its peripheral spectrum is the circle-subgroup generated by a primitive m-th root of unity ϑ. Each eigenvalue ϑ^k is simple and there is a unitary $U\in\mathfrak{A}$ such that $U^m=\mathbb{1}$, $U^k\neq\mathbb{1}$ for $k\not\equiv0\pmod m$, $[U,R_+]=0$, and in terms of which the left/right eigenvalue equations for $\mathbb{L}$ write

$$\mathbb{L}(R_+U^k)=\vartheta^k(R_+U^k),\qquad\mathbb{L}^*U^{-k}=\vartheta^kU^{-k},$$

for $k\in\{0,\dots,m-1\}$, see [EHK78, Theorem 4.2].[11]

Invoking [HJPR18, Proposition A.3], or more precisely an easy extension of this result to the multi-parameter case, we get that the peripheral spectrum of $\mathbb{L}^{[\boldsymbol{\alpha}]}$ consists of m simple eigenvalues $\ell(\boldsymbol{\alpha})\vartheta^k$, $k\in\{0,\dots,m-1\}$, where $\ell(\boldsymbol{\alpha})$ is the spectral radius of $\mathbb{L}^{[\boldsymbol{\alpha}]}$. The corresponding left/right eigenvectors can be expressed as $X(\boldsymbol{\alpha})U^{-k}$ and $R(\boldsymbol{\alpha})U^k$, with $\langle R(\boldsymbol{\alpha}),X(\boldsymbol{\alpha})\rangle=1$, and $[R(\boldsymbol{\alpha}),U]=[X(\boldsymbol{\alpha}),U]=0$. Since the eigenvalues belonging to the peripheral spectrum of $\mathbb{L}^{[\boldsymbol{\alpha}]}$ are simple, analytic perturbation theory, [Kat13, Chapter 2, Theorem 1.8] implies that

[11] As a CPU map, $\mathbb{L}^*$ is Schwarz, *i.e.*, $\mathbb{L}^*(A^*A)\geq(\mathbb{L}^*A)^*(\mathbb{L}^*A)$ for all $A\in\mathfrak{A}$.

$\ell(\boldsymbol{\alpha}) \in \mathbb{R}_+$, $R(\boldsymbol{\alpha}) \in \mathfrak{A}_{*+}$ and $X(\boldsymbol{\alpha}) \in \mathfrak{A}_+$ are real analytic functions of $\boldsymbol{\alpha}$. The spectral decomposition of $\mathbb{L}^{[\boldsymbol{\alpha}]}$ yields

$$\langle \mathbb{L}^{[\boldsymbol{\alpha}]N} R, \mathbb{1}\rangle = \ell(\boldsymbol{\alpha})^N \sum_{k=0}^{N-1} \vartheta^{kN} \langle R, X(\boldsymbol{\alpha}) U^{-k}\rangle \langle R(\boldsymbol{\alpha}) U^k, \mathbb{1}\rangle + \langle \mathbb{L}_<^{[\boldsymbol{\alpha}]N} R, \mathbb{1}\rangle \tag{5.10}$$

where $\mathbb{L}_<^{[\boldsymbol{\alpha}]}$ is the part of $\mathbb{L}^{[\boldsymbol{\alpha}]}$ associated to its non-peripheral spectrum. Let $\mathcal{U} \subset \mathbb{C}^\Omega$ be a neighborhood of 0 such that $\ell(\boldsymbol{\alpha})$, $R(\boldsymbol{\alpha})$, $X(\boldsymbol{\alpha})$ and $\mathbb{L}_<^{[\boldsymbol{\alpha}]}$ are analytic on $\mathcal{U}$. Since $\ell(0) = 1$, w.l.o.g. we can assume that, for $\boldsymbol{\alpha} \in \mathcal{U}$, $\|\mathbb{L}_<^{[\boldsymbol{\alpha}]N}\| = O(\varrho^N)$ for some $\varrho < 1$. Since both sides of (5.10) are analytic in $\mathcal{U}$, this identity holds on $\mathcal{U}$. Using the fact that $\langle R(0) U^k, \mathbb{1}\rangle = 0$ for $k \not\equiv 0 \pmod m$, a first order Taylor expansion yields

$$\langle \mathbb{L}^{[\boldsymbol{\alpha}]N} R, \mathbb{1}\rangle = (1 + \boldsymbol{\alpha}\cdot(\nabla\ell)(0) + O(|\boldsymbol{\alpha}|^2))^N (1 + O(|\boldsymbol{\alpha}|)) + O(\varrho^N)$$

as $\boldsymbol{\alpha} \to 0$. Moreover, first order perturbation theory gives

$$(\partial_{\alpha_\nu}\ell)(0) = \langle \mathbb{L}_\nu R_+, \mathbb{1}\rangle = \beta_\nu \langle R_+, J_\nu\rangle. \tag{5.11}$$

Thus, we have

$$\begin{aligned}\lim_{N\to\infty} \widehat{\mathbb{E}}[\mathrm{e}^{-\mathrm{i}\boldsymbol{\alpha}\cdot\mathscr{I}_N/N}] &= \lim_{N\to\infty} (1 + \mathrm{i}\boldsymbol{\alpha}\cdot(\nabla\ell)(0) N^{-1} + O(N^{-2}))^N \left(1 + O(N^{-1})\right) + O(\varrho^N)\\ &= \mathrm{e}^{\mathrm{i}\boldsymbol{\alpha}\cdot(\nabla\ell)(0)} = \mathrm{e}^{\mathrm{i}\sum_{\omega\in\Omega}\alpha_\omega\beta_\omega\langle R_+, J_\omega\rangle},\end{aligned} \tag{5.12}$$

and the claim follows from the Lévy–Cramér continuity theorem [Bil68, Theorem 7.6].

(iii) Starting again with (5.10), for $\boldsymbol{\alpha} \in \mathcal{U}$ we can write

$$\frac{1}{N}\log\widehat{\mathbb{E}}[\mathrm{e}^{-\boldsymbol{\alpha}\cdot\mathscr{I}_N}] = \log\ell(\boldsymbol{\alpha}) + \frac{1}{N}\log\left(1 + O(|\boldsymbol{\alpha}|) + O\left(\frac{\varrho}{|\ell(\boldsymbol{\alpha})|}\right)^N\right).$$

Thus, on a neighborhood of 0 in $\mathbb{C}^\Omega$ one has

$$\lim_{N\to\infty}\frac{1}{N}\log\widehat{\mathbb{E}}[\mathrm{e}^{-\boldsymbol{\alpha}\cdot(\mathscr{I}_N - \widehat{\mathbb{E}}[\mathscr{I}_N])}] = \log\ell(\boldsymbol{\alpha}) - \boldsymbol{\alpha}\cdot(\nabla\ell)(0)$$

and the result follows from [Bry93, Proposition 1].

(iv) Assuming $\mathbb{L}$ to be primitive, this property is shared with $\mathbb{L}^{[\boldsymbol{\alpha}]}$ for all $\boldsymbol{\alpha} \in \mathbb{R}^\Omega$. Thus, $\ell(\boldsymbol{\alpha})$ is the only peripheral eigenvalue of $\mathbb{L}^{[\boldsymbol{\alpha}]}$, and it follows that

$$\widehat{\mathbb{E}}[\mathrm{e}^{-\boldsymbol{\alpha}\cdot\mathscr{I}_N}] = \langle \mathbb{L}^{[\boldsymbol{\alpha}]N} R, 1\rangle = \ell(\boldsymbol{\alpha})^N \langle R, X(\boldsymbol{\alpha})\rangle \langle R(\boldsymbol{\alpha}), \mathbb{1}\rangle + o(\ell(\boldsymbol{\alpha})^N),$$

and hence the limit

$$e(\boldsymbol{\alpha}) = \lim_{N\to\infty}\frac{1}{N}\log\widehat{\mathbb{E}}[\mathrm{e}^{-\boldsymbol{\alpha}\cdot\mathscr{I}_N}] = \log\ell(\boldsymbol{\alpha})$$

exists for all $\boldsymbol{\alpha} \in \mathbb{R}^\Omega$ and defines a real analytic function.

(v) Follows from (iv) and the Gärtner–Ellis theorem [DZ98, Theorem 2.3.6].

5.3 Proof of Theorem 4.4

On $\mathscr{A}_*$ define the involution $\Theta : \rho \mapsto \theta\rho\theta$. Note that, for all $\omega \in \Omega$, Assumption (**TRI**) implies $[\theta_\omega, S_{\mathscr{E}_\omega}] = 0$, and hence $[\theta_\omega, \Pi_\varsigma(\omega)] = 0$ for any $\varsigma \in \Sigma$. By definition (4.13), for $\rho \in \mathscr{A}_*$, $X \in \mathscr{A}$, $\xi = (\varsigma, \varsigma') \in \Sigma \times \Sigma$ and $\widehat{\xi} = (\varsigma', \varsigma)$,

$$\begin{aligned}
\langle \mathscr{L}_{\omega,\xi}\Theta\rho, X\rangle &= \mathrm{e}^{-\varsigma}\,\mathrm{tr}\big((X\otimes\Pi_{\varsigma'}(\omega))U_\omega(\theta\otimes\theta_\omega)(\rho^*\otimes\Pi_\varsigma(\omega))(\theta\otimes\theta_\omega)U_\omega^*\big)\\
&= \mathrm{e}^{-\varsigma}\,\mathrm{tr}\big((X\otimes\Pi_{\varsigma'}(\omega))(\theta\otimes\theta_\omega)U_\omega^*(\rho^*\otimes\Pi_\varsigma(\omega))U_\omega(\theta\otimes\theta_\omega)\big)\\
&= \mathrm{e}^{-\varsigma}\,\overline{\mathrm{tr}\big((\rho^*\otimes\Pi_\varsigma(\omega))U_\omega(\theta\otimes\theta_\omega)(X\otimes\Pi_{\varsigma'}(\omega))(\theta\otimes\theta_\omega)U_\omega^*\big)}\\
&= \mathrm{e}^{\delta\xi}\,\overline{\langle\rho, \mathscr{L}_{\omega,\widehat{\xi}}\theta X\theta\rangle} = \mathrm{e}^{\delta\xi}\langle\mathscr{L}^*_{\omega,\widehat{\xi}}\rho, \Theta X\rangle = \langle\mathrm{e}^{\delta\xi}\Theta\mathscr{L}^*_{\omega,\widehat{\xi}}\rho, X\rangle,
\end{aligned}$$

so that

$$\mathscr{L}_{\omega,\xi}\Theta = \mathrm{e}^{\delta\xi}\Theta\mathscr{L}^*_{\omega,\widehat{\xi}}.$$

It further follows from Definition (4.14) that, for any $\alpha \in \mathbb{R}$,

$$\begin{aligned}
\mathscr{L}_\omega^{[\alpha]}\Theta &= \sum_{\xi\in\Sigma\times\Sigma}\mathrm{e}^{-\alpha\delta\xi}\mathscr{L}_{\omega,\xi}\Theta = \sum_{\xi\in\Sigma\times\Sigma}\mathrm{e}^{(1-\alpha)\delta\xi}\Theta\mathscr{L}^*_{\omega,\widehat{\xi}}\\
&= \sum_{\xi\in\Sigma\times\Sigma}\mathrm{e}^{-(1-\alpha)\delta\xi}\Theta\mathscr{L}^*_{\omega,\xi} = \Theta\mathscr{L}_\omega^{[1-\alpha]*}.
\end{aligned}\tag{5.13}$$

We will use the notations introduced in (5.1), as well as the following ones

$$(\Theta R)(\omega) \;=\; \Theta R(\omega), \qquad (\mathscr{L}^{[\boldsymbol{\alpha}]}R)(\omega) \;=\; \mathscr{L}_\omega^{[\alpha_\omega]}R(\omega). \tag{5.14}$$

Observe that (5.13) implies $\mathscr{L}^{[\boldsymbol{\alpha}]}\Theta = \Theta\mathscr{L}^{[\mathbf{1}-\boldsymbol{\alpha}]*}$, while, obviously, $[\Theta, \mathscr{P}] = 0$. Factorizing $\mathbb{L}^{[\boldsymbol{\alpha}]} = \mathscr{P}\mathscr{L}^{[\boldsymbol{\alpha}]}$, we derive from Definition (4.15)

$$\Theta\mathbb{L}^{[\boldsymbol{\alpha}]}\Theta = \Theta\mathscr{P}\mathscr{L}^{[\boldsymbol{\alpha}]}\Theta = \Theta\mathscr{P}\Theta\mathscr{L}^{[\mathbf{1}-\boldsymbol{\alpha}]*} = \mathscr{P}\mathscr{L}^{[\mathbf{1}-\boldsymbol{\alpha}]*},$$

for all $\boldsymbol{\alpha} \in \mathbb{R}^\Omega$. Writing Assumption (**DB**) as $\mathscr{P} = \Pi\mathscr{P}^*\Pi^{-1}$ and using the fact that $[\Pi, \mathscr{L}^{[\boldsymbol{\alpha}]}] = 0$ we further get

$$\Theta\mathbb{L}^{[\boldsymbol{\alpha}]}\Theta = \Pi\mathscr{P}^*\Pi^{-1}\mathscr{L}^{[\mathbf{1}-\boldsymbol{\alpha}]*} = \Pi(\mathscr{L}^{[\mathbf{1}-\boldsymbol{\alpha}]}\mathscr{P})^*\Pi^{-1}.$$

Since $\mathrm{sp}(\mathscr{L}^{[\mathbf{1}-\boldsymbol{\alpha}]}\mathscr{P})\setminus\{0\} = \mathrm{sp}(\mathscr{P}\mathscr{L}^{[\mathbf{1}-\boldsymbol{\alpha}]})\setminus\{0\}$, we conclude that $\mathbb{L}^{[\boldsymbol{\alpha}]}$ and $\mathbb{L}^{[\mathbf{1}-\boldsymbol{\alpha}]}$ have identical spectral radii, *i.e.*, the symmetry (4.20) holds. Theorem 4.4 now follows from Theorem 4.2 (iv)-(v), in particular (4.19) yields

$$I(-\boldsymbol{\varsigma}) = \sup_{\boldsymbol{\alpha}\in\mathbb{R}^\Omega}\left(-\boldsymbol{\varsigma}\cdot\boldsymbol{\alpha} - e(-\boldsymbol{\alpha})\right),$$

and the symmetry (4.20) gives,

$$I(-\boldsymbol{\varsigma}) = \sup_{\boldsymbol{\alpha}\in\mathbb{R}^\Omega}\left(-\boldsymbol{\varsigma}\cdot\boldsymbol{\alpha} - e(\mathbf{1}+\boldsymbol{\alpha})\right) = \sup_{\boldsymbol{\alpha}'\in\mathbb{R}^\Omega}\big(\boldsymbol{\varsigma}\cdot(\mathbf{1}+\boldsymbol{\alpha}') - e(-\boldsymbol{\alpha}')\big) = I(\boldsymbol{\varsigma}) + \boldsymbol{\varsigma}\cdot\mathbf{1},$$

with $\mathbf{1}+\boldsymbol{\alpha} = -\boldsymbol{\alpha}'$.

5.4 Proof of Theorem 4.6

(i) One has

$$\mathbb{L}_{\boldsymbol{\zeta}}R = \sum_{\substack{\omega,\nu\in\Omega\\ s,s'\in\Sigma_\nu}}\mathbb{V}_{\omega,s,s',\nu,\boldsymbol{\zeta}}R\,\mathbb{V}^*_{\omega,s,s',\nu,\boldsymbol{\zeta}}$$

where, in terms of the Kraus family (5.6),

$$\mathbb{V}_{\omega,s,s',\nu,\boldsymbol{\zeta}} = \mathrm{e}^{\zeta_\nu \varsigma_{\nu,s}/2\bar\beta-(\bar\beta-\zeta_\nu)(F_{\nu,\mathbf{0}}-F_{\nu,\zeta})}\mathbb{V}_{\omega,s,s',\nu}.$$

Thus, arguing as in the proof of Theorem 4.2, we conclude that $\mathbb{L}_{\boldsymbol{\zeta}}$ is irreducible/primitive iff $\mathbb{L}$ is.

(ii) Besides (5.1) and (5.14), for $\gamma\in\mathbb{R}$ define

$$(\mathscr{R}^\gamma R)(\omega) = \rho_{+\omega}^{\gamma/\bar\beta}R(\omega)$$

where $\rho_+ = (\rho_{+\omega})_{\omega\in\Omega}$ is the family of states associated to the equilibrium ($\boldsymbol{\zeta}=\mathbf{0}$) ESS R_+ via (2.10). Setting $\mathbb{K}_{\boldsymbol{\zeta}}^{[\boldsymbol{\alpha}]} = \mathscr{L}_{\boldsymbol{\zeta}}^{[\boldsymbol{\alpha}]}\mathscr{P}$, we compute

$$\begin{aligned}(\mathscr{R}^\gamma\mathbb{K}_{\boldsymbol{\zeta}}^{[\boldsymbol{\alpha}]}\mathscr{R}^{-\gamma}R)(\omega) &= \sum_{\nu\in\Omega}\rho_{+\omega}^{\gamma/\bar\beta}\mathscr{L}_{\omega\boldsymbol{\zeta}}^{[\alpha_\omega]}P_{\nu\omega}\rho_{+\nu}^{-\gamma/\bar\beta}R(\nu)\\ &= \sum_{\nu\in\Omega}P_{\nu\omega}\,\mathrm{tr}_{\mathscr{H}_{\mathscr{E}_\omega}}\left((\rho_{+\omega}^{\gamma/\bar\beta}\otimes\rho_{\mathscr{E}_\omega\boldsymbol{\zeta}}^{\alpha_\omega})U_\omega(\rho_{+\nu}^{-\gamma/\bar\beta}R(\nu)\otimes\rho_{\mathscr{E}_\omega\boldsymbol{\zeta}}^{1-\alpha_\omega})U_\omega^*\right).\end{aligned}$$

Invoking Proposition 4.1, we get

$$\begin{aligned}U_\omega(\rho_{+\nu}^{-\gamma/\bar\beta}R(\nu)\otimes\rho_{\mathscr{E}_\omega\boldsymbol{\zeta}}^{1-\alpha_\omega})U_\omega^* &= U_\omega(\rho_{+\nu}\otimes\rho_{\mathscr{E}_\omega\mathbf{0}})^{-\gamma/\bar\beta}(R(\nu)\otimes\rho_{\mathscr{E}_\omega\boldsymbol{\zeta}}^{1-\alpha_\omega+\gamma/\beta_\omega})U_\omega^*\\ &= (\rho_{+\omega}\otimes\rho_{\mathscr{E}_\omega\mathbf{0}})^{-\gamma/\bar\beta}U_\omega(R(\nu)\otimes\rho_{\mathscr{E}_\omega\boldsymbol{\zeta}}^{1-\alpha_\omega+\gamma/\beta_\omega})U_\omega^*,\end{aligned}$$

so that, with $\boldsymbol{\beta}^{-1} = ((\bar\beta-\zeta_\omega)^{-1})_{\omega\in\Omega}$, we derive

$$\begin{aligned}(\mathscr{R}^\gamma\mathbb{K}_{\boldsymbol{\zeta}}^{[\boldsymbol{\alpha}]}\mathscr{R}^{-\gamma}R)(\omega) &= \sum_{\nu\in\Omega}P_{\nu\omega}\,\mathrm{tr}_{\mathscr{H}_{\mathscr{E}_\omega}}\left((\mathbb{1}\otimes\rho_{\mathscr{E}_\omega\boldsymbol{\zeta}}^{\alpha_\omega-\gamma/\beta_\omega})U_\omega(R(\nu)\otimes\rho_{\mathscr{E}_\omega\boldsymbol{\zeta}}^{1-\alpha_\omega+\gamma/\beta_\omega})U_\omega^*\right)\\ &= (\mathbb{K}_{\boldsymbol{\zeta}}^{[\boldsymbol{\alpha}-\gamma\boldsymbol{\beta}^{-1}]}R)(\omega).\end{aligned}$$

Since $\mathrm{sp}(\mathbb{L}_{\boldsymbol{\zeta}}^{[\boldsymbol{\alpha}]})\setminus\{0\} = \mathrm{sp}(\mathbb{K}_{\boldsymbol{\zeta}}^{[\boldsymbol{\alpha}]})\setminus\{0\}$, we conclude that the spectral radius of $\mathbb{L}_{\boldsymbol{\zeta}}^{[\boldsymbol{\alpha}]}$ satisfies

$$\ell_{\boldsymbol{\zeta}}(\boldsymbol{\alpha}-\gamma\boldsymbol{\beta}^{-1}) = \ell_{\boldsymbol{\zeta}}(\boldsymbol{\alpha}), \tag{5.15}$$

for all $\boldsymbol{\alpha}\in\mathbb{R}^\Omega$, $\gamma\in\mathbb{R}$ and $\boldsymbol{\zeta}\in\mathbb{R}^\Omega$.

As a first consequence of the translation symmetry (5.15), we get

$$\boldsymbol{\beta}^{-1}\cdot(\nabla\ell_{\boldsymbol{\zeta}})(0) = 0,$$

and Relation (5.11) allows us to conclude

$$\sum_{\nu\in\Omega}\bar J_{\nu\boldsymbol{\zeta}}(\boldsymbol{\omega}) = \sum_{\nu\in\Omega}\langle R_{+\boldsymbol{\zeta}}, J_{\nu\boldsymbol{\zeta}}\rangle = 0.$$

(iii) A second consequence of (5.15) is

$$\lim_{N\to\infty}\widehat{\mathbb{E}}_{\boldsymbol{\zeta}}[\mathrm{e}^{-\mathrm{i}t\boldsymbol{\beta}^{-1}\cdot\mathscr{I}_{N\zeta}/N}] = \mathrm{e}^{\mathrm{i}t\boldsymbol{\beta}^{-1}\cdot(\nabla\ell_{\zeta})(0)} = 1,$$

which follows from (5.12). As in the proof of Theorem 4.2 (ii), the claim follows from the Lévy–Cramér continuity theorem.

(iv) It follows from (5.15) that the covariance $C_{\boldsymbol{\zeta}}$ of the limiting Gaussian measure, as given by Theorem 4.2 (iii), satisfies $\boldsymbol{\beta}^{-1}\in\mathrm{Ker}\,C_{\boldsymbol{\zeta}}$, so that $\mathrm{Ran}\,C_{\boldsymbol{\zeta}}\subset\mathfrak{Z}_{\boldsymbol{\zeta}}$.

(v) As argued above, the primitivity of $\mathbb{L}$ ensures that of $\mathbb{L}_{\boldsymbol{\zeta}}$, and the claim follows from Theorem 4.2 (iv) and (5.15).

(vi) By Theorem 4.2 (v) and the symmetry (5.15) one has

$$\begin{aligned}
I_{\boldsymbol{\zeta}}(\boldsymbol{\varsigma}) &= \sup_{\boldsymbol{\alpha}\in\mathbb{R}^\Omega}\left(\boldsymbol{\alpha}\cdot\boldsymbol{\varsigma} - e_{\boldsymbol{\zeta}}(-\boldsymbol{\alpha})\right)\\
&= \sup_{\boldsymbol{\alpha}\in\mathbb{R}^\Omega}\left(\boldsymbol{\alpha}\cdot\boldsymbol{\varsigma} - e_{\boldsymbol{\zeta}}(-\boldsymbol{\alpha}+\gamma\boldsymbol{\beta}^{-1})\right)\\
&= \sup_{\boldsymbol{\alpha}\in\mathbb{R}^\Omega}\left((\boldsymbol{\alpha}+\gamma\boldsymbol{\beta}^{-1})\cdot\boldsymbol{\varsigma} - e_{\boldsymbol{\zeta}}(-\boldsymbol{\alpha})\right) = I_{\boldsymbol{\zeta}}(\boldsymbol{\varsigma})+\gamma\boldsymbol{\beta}^{-1}\cdot\boldsymbol{\varsigma}
\end{aligned}$$

for any $\gamma\in\mathbb{R}$ and hence $\boldsymbol{\beta}^{-1}\cdot\boldsymbol{\varsigma}$ has to vanish whenever $I_{\boldsymbol{\zeta}}(\boldsymbol{\varsigma})$ is finite.

(vii) By (5.11), one has

$$\bar{J}_{\omega\boldsymbol{\zeta}} = (\bar\beta-\zeta_\omega)^{-1}(\partial_{\alpha_\omega}\ell_{\boldsymbol{\zeta}})(\mathbf{0}),$$

and hence

$$\begin{aligned}
L_{\omega\nu} &= \bar\beta^{-1}\partial_{\zeta_\nu}\partial_{\alpha_\omega}\ell_{\boldsymbol{\zeta}}(\boldsymbol{\alpha})|_{\boldsymbol{\alpha}=\boldsymbol{\zeta}=\mathbf{0}} + \delta_{\omega\nu}\bar\beta^{-2}\partial_{\alpha_\omega}\ell_{\boldsymbol{\zeta}}(\boldsymbol{\alpha})|_{\boldsymbol{\alpha}=\boldsymbol{\zeta}=\mathbf{0}}\\
&= \bar\beta^{-1}\partial_{\zeta_\nu}\partial_{\alpha_\omega}\ell_{\boldsymbol{\zeta}}(\boldsymbol{\alpha})|_{\boldsymbol{\alpha}=\boldsymbol{\zeta}=\mathbf{0}} + \delta_{\omega\nu}\bar\beta^{-1}\bar{J}_{\omega\mathbf{0}}\\
&= \bar\beta^{-1}\partial_{\zeta_\nu}\partial_{\alpha_\omega}\ell_{\boldsymbol{\zeta}}(\boldsymbol{\alpha})|_{\boldsymbol{\alpha}=\boldsymbol{\zeta}=\mathbf{0}},
\end{aligned}$$

where we have invoked (4.11). Since $\ell_{\boldsymbol{\zeta}}(\boldsymbol{\alpha}) = \mathrm{e}^{e_{\boldsymbol{\zeta}}(\boldsymbol{\alpha})}$, the two symmetries (4.20) and (4.21) gives

$$(\partial_{\alpha_\omega}\ell_{\boldsymbol{\zeta}})(\boldsymbol{\alpha}) = -(\partial_{\alpha_\omega}\ell_{\boldsymbol{\zeta}})(\mathbf{1}-\boldsymbol{\alpha}+\gamma\boldsymbol{\beta}^{-1}),$$

and hence

$$(\partial_{\zeta_\nu}\partial_{\alpha_\omega}\ell_{\boldsymbol{\zeta}})(\boldsymbol{\alpha}) = -(\partial_{\zeta_\nu}\partial_{\alpha_\omega}\ell_{\boldsymbol{\zeta}})(\mathbf{1}-\boldsymbol{\alpha}+\gamma\boldsymbol{\beta}^{-1}) - (\partial_{\alpha_\nu}\partial_{\alpha_\omega}\ell_{\boldsymbol{\zeta}})(\mathbf{1}-\boldsymbol{\alpha}+\gamma\boldsymbol{\beta}^{-1})\frac{\gamma}{(\bar\beta-\zeta_\nu)^2}.$$

Setting $\boldsymbol{\alpha}=\mathbf{0}$, $\boldsymbol{\zeta}=\mathbf{0}$ and $\gamma=-\bar\beta$ yields

$$\partial_{\zeta_\nu}\partial_{\alpha_\omega}\ell_{\boldsymbol{\zeta}}(\boldsymbol{\alpha})|_{\boldsymbol{\alpha}=\boldsymbol{\zeta}=\mathbf{0}} = \frac{1}{2\bar\beta}(\partial_{\alpha_\nu}\partial_{\alpha_\omega}\ell_{\mathbf{0}})(\mathbf{0})$$

from which we conclude

$$L_{\omega\nu} = \frac{1}{2\bar\beta^2}(\partial_{\alpha_\nu}\partial_{\alpha_\omega}\ell_{\mathbf{0}})(\mathbf{0}). \tag{5.16}$$

The remaining part of the proof will only involve equilibrium ($\boldsymbol{\zeta}=\mathbf{0}$) quantities, and to simplify notations we will omit the subscript $\mathbf{0}$. We start with some elementary calculations. From Definition (4.15) and Relation (5.7) we have

$$\begin{aligned}
\langle\mathbb{L}^{[\boldsymbol{\alpha}]}R_+,\mathbb{1}\rangle &= \sum_{\nu\in\Omega}\langle\mathscr{L}_\nu^{\alpha_\nu}R_+(\nu),\mathbb{1}\rangle\\
&= \sum_{\nu\in\Omega}\operatorname{tr}\left((\mathbb{1}\otimes\rho_{\mathscr{E}_\nu}^{\alpha_\nu})U_\nu(R_+(\nu)\otimes\rho_{\mathscr{E}_\nu}^{1-\alpha_\nu})U_\nu^*\right)\\
&= \sum_{\nu,\mu\in\Omega}\pi_{+\mu}P_{\mu\nu}\operatorname{tr}\left((\mathbb{1}\otimes\rho_{\mathscr{E}_\nu}^{\alpha_\nu})U_\nu(\rho_{+\mu}\otimes\rho_{\mathscr{E}_\nu}^{1-\alpha_\nu})U_\nu^*\right)\\
&= \sum_{\nu,\mu\in\Omega}\pi_{+\mu}P_{\mu\nu}\operatorname{tr}\left((\mathbb{1}\otimes\rho_{\mathscr{E}_\nu}^{\alpha_\nu})U_\nu(\rho_{+\mu}^{\alpha_\nu}\otimes\mathbb{1})(\rho_{+\mu}\otimes\rho_{\mathscr{E}_\nu})^{1-\alpha_\nu}U_\nu^*\right).
\end{aligned}$$

Now, using Relation (4.9), we get

$$\begin{aligned}\langle \mathbb{L}^{[\boldsymbol{\alpha}]} R_+, \mathbb{1}\rangle &= \sum_{\nu,\mu\in\Omega} \pi_{+\mu} P_{\mu\nu} \operatorname{tr}\Big((\mathbb{1}\otimes\rho_{\mathcal{E}_\nu}^{\alpha_\nu}) U_\nu (\rho_{+\mu}^{\alpha_\nu}\otimes\mathbb{1}) U_\nu^* (\rho_{+\nu}\otimes\rho_{\mathcal{E}_\nu})^{1-\alpha_\nu}\Big)\\ &= \sum_{\nu,\mu\in\Omega} \pi_{+\mu} P_{\mu\nu} \operatorname{tr}\Big((\rho_{+\nu}^{1-\alpha_\nu}\otimes\rho_{\mathcal{E}_\nu}) U_\nu (\rho_{+\mu}^{\alpha_\nu}\otimes\mathbb{1}) U_\nu^*\Big),\end{aligned}$$

so that, with $S_{+\omega} = -\log\rho_{+\omega}$, and using again Relation (4.9), we finally get

$$\begin{aligned}\langle \mathbb{L}_{\omega\nu} R_+, \mathbb{1}\rangle &= \partial_{\alpha_\nu}\partial_{\alpha_\omega} \langle \mathbb{L}^{[\boldsymbol{\alpha}]} R_+, \mathbb{1}\rangle|_{\boldsymbol{\alpha}=\mathbf{0}}\\ &= \delta_{\omega\nu} \sum_{\mu\in\Omega} \pi_{+\mu} P_{\mu\omega} \operatorname{tr}\Big((\rho_{+\omega}\otimes\rho_{\mathcal{E}_\omega})(S_{+\omega}^2 - 2S_{+\omega} U_\omega S_{+\mu} U_\omega^* + U_\omega S_{+\mu}^2 U_\omega^*)\Big)\\ &= \delta_{\omega\nu} \sum_{\mu\in\Omega} \pi_{+\mu} P_{\mu\omega} \operatorname{tr}\Big(U_\omega^*(\rho_{+\omega}\otimes\rho_{\mathcal{E}_\omega}) U_\omega (U_\omega^* S_{+\omega}^2 U_\omega - 2U_\omega^* S_{+\omega} U_\omega S_{+\mu} + S_{+\mu}^2)\Big)\\ &= \delta_{\omega\nu} \sum_{\mu\in\Omega} \pi_{+\mu} P_{\mu\omega} \langle \rho_{+\mu}, \mathcal{L}_\omega^*(S_{+\omega}^2) - 2\mathcal{L}_\omega^*(S_{+\omega}) S_{+\mu} + S_{+\mu}^2\rangle\\ &= \delta_{\omega\nu} \mathbb{E}_+\left[\langle \rho_{+\omega_0}, S_{+\omega_0}^2 - \mathcal{L}_{\omega_1}^*(S_{+\omega_1}) S_{+\omega_0} - S_{+\omega_0}\mathcal{L}_{\omega_1}^*(S_{+\omega_1}) + \mathcal{L}_{\omega_1}^*(S_{+\omega_1}^2)\rangle 1_{\omega_1=\omega}\right].\end{aligned}$$

Let $D\subset\mathbb{C}$ be a closed disk centered at 1 and $\mathcal{U}\subset\mathbb{R}^\Omega$ a fixed neighborhood of $\mathbf{0}$ such that $|\mathrm{sp}(\mathbb{L}^{[\boldsymbol{\alpha}]})\cap D| = 1$ for $\boldsymbol{\alpha}\in\mathcal{U}$. Set

$$E_n(\boldsymbol{\alpha}) = \oint_{\partial D} (z-1)^n \langle R_+, (z-\mathbb{L}^{[\boldsymbol{\alpha}]*})^{-1}\mathbb{1}\rangle \frac{\mathrm{d}z}{2\pi\mathrm{i}},$$

so that the dominant eigenvalue $\ell(\boldsymbol{\alpha})\in\mathrm{sp}(\mathbb{L}^{[\boldsymbol{\alpha}]})\cap D$ is given by

$$\ell(\boldsymbol{\alpha}) = 1 + \frac{E_1(\boldsymbol{\alpha})}{E_0(\boldsymbol{\alpha})},$$

for $\boldsymbol{\alpha}\in\mathcal{U}$. The first derivatives of the functions E_n are readily computed,

$$(\partial_{\alpha_\omega} E_n)(\boldsymbol{\alpha}) = \oint_{\partial D} (z-1)^n \langle R_+, (z-\mathbb{L}^{[\boldsymbol{\alpha}]*})^{-1}(\partial_{\alpha_\omega}\mathbb{L}^{[\boldsymbol{\alpha}]*})(z-\mathbb{L}^{[\boldsymbol{\alpha}]*})^{-1}\mathbb{1}\rangle \frac{\mathrm{d}z}{2\pi\mathrm{i}},$$

and, using (5.8)–(5.9),

$$(\partial_{\alpha_\omega} E_n)(\mathbf{0}) = \oint_{\partial D} (z-1)^{n-2} \langle \mathbb{L}_\omega R_+, \mathbb{1}\rangle \frac{\mathrm{d}z}{2\pi\mathrm{i}} = \delta_{n,1}\bar\beta\bar J_\omega.$$

Consequently, we have

$$\begin{aligned}(\partial_{\alpha_\nu}\partial_{\alpha_\omega}\ell)(\mathbf{0}) &= (\partial_{\alpha_\nu}\partial_{\alpha_\omega}E_1)(\mathbf{0})\\ &= \oint_{\partial D} (z-1)^{-1}\langle R_+, (\mathbb{L}_\nu^*(z-\mathbb{L}^*)^{-1}\mathbb{L}_\omega^* + \mathbb{L}_\omega^*(z-\mathbb{L}^*)^{-1}\mathbb{L}_\nu^* + \mathbb{L}_{\omega\nu}^*)\mathbb{1}\rangle \frac{\mathrm{d}z}{2\pi\mathrm{i}}.\end{aligned}$$

The above Cauchy integral splits naturally in 3 terms, the first one being

$$(\partial_{\alpha_\nu}\partial_{\alpha_\omega}\ell)_I(\mathbf{0}) = \oint_{\partial D} \frac{\langle \mathbb{L}_\nu R_+, (z-\mathbb{L}^*)^{-1}\mathbb{L}_\omega^*\mathbb{1}\rangle}{z-1}\frac{\mathrm{d}z}{2\pi\mathrm{i}}.$$

An elementary calculation gives

$$\mathbb{L}_\omega^*\mathbb{1} = \bar\beta J_\omega,$$

and since the spectral projection of $\mathbb{L}^*$ to its dominant eigenvalue is $|\mathbb{1}\rangle\langle R_+|$, with

$$|\mathbb{1}\rangle\langle R_+|J_\omega = \bar J_\omega\mathbb{1} = 0,$$

one has

$$(z-\mathbb{L}^*)^{-1}\mathbb{L}^*_\omega \mathbb{1} = \bar{\beta}(z-\underline{\mathbb{L}^*})^{-1}J_\omega,$$

where $\underline{\mathbb{L}^*}$ denotes the restriction of $\mathbb{L}^*$ to its spectral subspace associated to $\mathrm{sp}(\mathbb{L}^*)\setminus\{1\}$. Applying the residue theorem, we derive

$$(\partial_{\alpha_\nu}\partial_{\alpha_\omega}\ell)_I(\mathbf{0}) = \bar{\beta}\oint_{\partial D}\frac{\langle \mathbb{L}_\nu R_+,(z-\underline{\mathbb{L}^*})^{-1}J_\omega\rangle}{z-1}\frac{\mathrm{d}z}{2\pi\mathrm{i}} = \bar{\beta}\langle \mathbb{L}_\nu R_+,(1-\underline{\mathbb{L}^*})^{-1}J_\omega\rangle.$$

Finally, using the fact that the spectral radius of $\underline{\mathbb{L}^*}$ is strictly smaller than 1, we have

$$(1-\underline{\mathbb{L}^*})^{-1} = \sum_{n\ge 0}\mathbb{L}^{*n}|_{\mathrm{Ran}(I-|\mathbb{1}\rangle\langle R_+|)},$$

and consequently

$$(\partial_{\alpha_\nu}\partial_{\alpha_\omega}\ell)_I(\mathbf{0}) = \bar{\beta}\sum_{n\ge 0}\langle \mathbb{L}_\nu R_+,\mathbb{L}^{*n}J_\omega\rangle.$$

Another elementary calculation yields

$$(\mathbb{L}_\nu R_+)(\omega) = \bar{\beta}\pi_{+\nu}P_{\nu\omega}\widehat{J}_\nu(\nu)\rho_{+\nu},$$

allowing us to deduce

$$(\partial_{\alpha_\nu}\partial_{\alpha_\omega}\ell)_I(\mathbf{0}) = \bar{\beta}^2\sum_{n\ge 0}\mathbb{E}_+\left[\langle \rho_{+\omega_0},\widehat{J}_\nu(\omega_0)\mathscr{L}^*_{\omega_1}\cdots\mathscr{L}^*_{\omega_n}J_\omega(\omega_{n+1})\rangle\right].$$

The second term $(\partial_{\alpha_\nu}\partial_{\alpha_\omega}\ell)_{II}(\mathbf{0})$ is obtained by exchanging the roles of ω and ν in the first one. The third and last term is

$$\begin{aligned}(\partial_{\alpha_\nu}\partial_{\alpha_\omega}\ell)_{III}(\mathbf{0}) &= \langle \mathbb{L}_{\omega\nu}R_+,\mathbb{1}\rangle\\ &= \delta_{\omega\nu}\mathbb{E}_+\left[\langle \rho_{+\omega_0},S^2_{+\omega_0}-\mathscr{L}^*_{\omega_1}(S_{+\omega_1})S_{+\omega_0}-S_{+\omega_0}\mathscr{L}^*_{\omega_1}(S_{+\omega_1})+\mathscr{L}^*_{\omega_1}(S^2_{+\omega_1})\rangle 1_{\omega_1=\omega}\right].\end{aligned}$$

(viii) The Onsager reciprocity relations follow directly from the Green–Kubo formula (4.22). They are also a consequence of Relation (5.16) and Clairaut's theorem on equality of mixed partial derivatives. In view of (4.11), the Fluctuation-Dissipation Relations follow from combining the formulas after (4.16) with (5.11) and (5.16).

References

[AJR21] ANDRÉYS, S., JOYE, A. and RAQUÉPAS, R.: Fermionic walkers driven out of equilibrium. J. Stat. Phys. **184**, 14 (2021).

[AP06] ATTAL, S. and PAUTRAT, Y.: From repeated to continuous quantum interactions. Ann. H. Poincaré **7**, 59–104 (2006).

[Arv69] ARVESON, W.: Subalgebras of c^*-algebras. Acta Math. **123**, 141–224 (1969).

[BB20] BOUGRON, J.-F. and BRUNEAU, L.: Linear response theory and entropic fluctuations in repeated interaction systems. J. Stat. Phys. 1636–1677 (2020), DOI:10.1007/s10955-020-02640-x.

[BCJP21] BENOIST, T., CUNEO, N., JAKŠIĆ, V. and PILLET, C.-A.: On entropy production of repeated quantum measurements II. Examples. J. Stat. Phys. **182**, 44 (2021).

[BDBP11] BRUNEAU, L., DE BIEVRE, S. and PILLET, C.-A.: Scattering induced current in a tight-binding band. J. Math. Phys. **52**, 022109 (2011).

[Bil68] BILLINGSLEY, P.: *Convergence of Probability Measures*. Wiley, New York, 1968.

[BJM06] BRUNEAU, L., JOYE, A. and MERKLI, M.: Asymptotics of repeated interaction quantum systems. J. Funct. Anal. **239**, 310–344 (2006).

[BJM08] ______: Random repeated interaction quantum systems. Commun. Math. Phys. **284**, 553–581 (2008).

[BJM10a] ______: Infinite products of random matrices and repeated interaction dynamics. Ann. IHP Proba. Stat. **46**, 442–464 (2010).

[BJM10b] ______: Repeated and continuous interactions in open quantum systems. Ann. H. Poincaré **10**, 1251–1284 (2010).

[BJM14] ______: Repeated interactions in open quantum systems. J. Math. Phys. **55**, 075204 (2014).

[BJP$^+$15] BENOIST, T., JAKŠIĆ, V., PANATI, A., PAUTRAT, Y. and PILLET, C.-A.: Full statistics of energy conservation in two-time measurement protocols. Phys. Rev. E **92**, 032115 (2015).

[BJPP18] BENOIST, T., JAKŠIĆ, V., PAUTRAT, Y. and PILLET, C.-A.: On entropy production of repeated quantum measurements I. General theory. Commun. Math. Phys. **357**, 77–123 (2018).

[BP09] BRUNEAU, L. and PILLET, C.-A.: Thermal relaxation of a QED cavity. J. Stat. Phys. **134**, 1071–1095 (2009).

[BPP20] BENOIST, T., PANATI, A. and PAUTRAT, Y.: Heat conservation and fluctuations for open quantum systems in the two-time measurement picture. J. Stat. Phys. **178**, 893–925 (2020).

[BPR19] BENOIST, T., PANATI, A. and RAQUÉPAS, R.: Control of fluctuations and heavy tails for heat variation in the two-time measurement framework. Ann. H. Poincaré **20**, 631–674 (2019).

[Bru14] BRUNEAU, L.: Mixing properties of the one-atom maser. J. Stats. Phys. **155**, 888–908 (2014).

[Bry93] BRYC, W.: A remark on the connection between the large deviation principle and the central limit theorem. Stat. Prob. Lett. **18**, 253–256 (1993).

[BS57] BECK, A. and SCHWARTZ, J. T.: A vector-valued random ergodic theorem. Proc. A.M.S. **8**, 1049–1059 (1957).

[Car10] CARLEN, E. A.: Trace inequalities and quantum entropy: An introductory course. In *Entropy and the Quantum*, Contemporary Mathematics, vol. 529, AMS, Providence, RI, 2010, p. 33–83.

[CHT11] CAMPISI, M., HÄNGGI, P., and TALKNER, P.: Quantum fluctuation relations: Foundations and applications. Rev. Mod. Phys. **83**, 771–791 (2011).

[CM12] CHETRITE, R. and MALLICK, K.: Quantum fluctuation relations for the Lindblad master equation. J. Stat. Phys. **148**, 480–501 (2012).

[Dav74] DAVIES, E. B.: Markovian master equations. Commun. Math. Phys. **39**, 91–110 (1974).

[Dav75] ______: Markovian master equations. III. Ann. Inst. H. Poincaré Proba. Stat. **11**, 265–273 (1975).

[Dav76a] ______: Markovian master equations. II. Math. Ann. **219**, 147–158 (1976).

[Dav76b] ______: *Quantum Theory of Open Systems.* Academic Press, London, 1976.

[DDRM08] DEREZIŃSKI, J., DE ROECK, W. and MAES, C.: Fluctuations of quantum currents and unravelings of master equations. J. Stat. Phys. **131**, 341–356 (2008).

[DR09] DE ROECK, W.: Large deviation generating function for currents in the Pauli–Fierz model. Rev. Math. Phys. **21**, 549–585 (2009).

[DRM06] DE ROECK, W. and MAES, C.: Steady state fluctuations of the dissipated heat for a quantum stochastic model. Rev. Math. Phys. **18**, 619–653 (2006).

[DZ98] DEMBO, A. and ZEITOUNI, O.: *Large Deviations. Techniques and Applications.* Springer, Berlin, 1998.

[EHK78] EVANS, D. and HØEGH-KROHN, R.: Spectral properties of positive maps on C*-algebras. J. London Math. Soc. **17**, 345–355 (1978).

[EHM09] ESPOSITO, M., HARBOLA, U. and MUKAMEL, S.: Nonequilibrium fluctuations, fluctuation theorems, and counting statistics in quantum systems. Rev. Mod. Phys. **81**, 1665–1702 (2009).

[FJM86] FILIPOWICZ, P., JAVANAINEN, J. and MEYSTRE, P.: Theory of a microscopic maser. Phys. Rev. A **34**, 3077–3087 (1986).

[GC95a] GALLAVOTTI, G. and COHEN, E. G. D.: Dynamical ensembles in nonequilibrium statistical mechanics. Phys. Rev. Lett. **74**, 2694–7 (1995).

[GC95b] ______: Dynamical ensembles in stationary states. J. Stat. Phys. **80**, 931–970 (1995).

[HJ17] HAMZA, E. and JOYE, A.: Thermalization of fermionic quantum walkers. J. Stat. Phys. **166**, 1365–1392 (2017).

[HJPR17] HANSON, E. P., JOYE, A., PAUTRAT, Y. and RAQUÉPAS, R.: Landauer's principle in repeated interaction systems. Commun. Math. Phys. **349**, 285–327 (2017).

[HJPR18] ______: Landauer's principle for trajectories of repeated interaction systems. Ann. H. Poincaré **19**, 1939–1991 (2018).

[JLP13] JAKŠIĆ, V., LANDON, B. and PILLET, C.-A.: Entropic fluctuations in XY chains and reflectionless Jacobi matrices. Ann. H. Poincaré **14**, 1775–1800 (2013).

[JOPP12] JAKŠIĆ, V., OGATA, Y., PILLET, C.-A. and PAUTRAT, Y.: Entropic fluctuations in quantum statistical mechanics. an introduction. In *Quantum Theory from Small to Large Scales*, Oxford University Press, Oxford, 2012, p. 213–410.

[JP14] JAKŠIĆ, V. and PILLET, C.-A.: A note on the landauer principle in quantum statistical mechanics. J. Math. Phys. **55**, 075210 (2014).

[JPPP15] JAKŠIĆ, V., PANANGADEN, J., PANATI, A. and PILLET, C.-A.: Energy conservation, counting statistics, and return to equilibrium. Lett. Math. Phys. **105**, 917–938 (2015).

[JPRB11] JAKŠIĆ, V., PILLET, C.-A. and REY-BELLET, L.: Entropic fluctuations in statistical mechanics: I. classical dynamical systems. Nonlinearity **24**, 699 (2011).

[JPW14] JAKŠIĆ, V., PILLET, C.-A. and WESTRICH, M.: Entropic fluctuations of quantum dynamical semigroups. J. Stat. Phys. **154**, 153–187 (2014).

[Kat13] KATO, T.: *Perturbation Theory for Linear Operators*. Springer, Berlin, 2013.

[KS09] KANG, Y. and SCHENKER, J.: Diffusion of wave packets in a Markov random potential. J. Stat. Phys. **134**, 1005–1022 (2009).

[LS78] LEBOWITZ, J. and SPOHN, H.: Irreversible thermodynamics for quantum systems weakly coupled to thermal reservoirs. Adv. Chem. Phys. **38**, 109–142 (1978).

[MS19] MOVASSAGH, R. and SCHENKER, J.: An ergodic theorem for homogeneously distributed quantum channels with applications to matrix product states. 2019, preprint arXiv:1909.11769.

[MS20] ———: Theory of ergodic quantum processes. 2020, preprint arXiv:2004.14397.

[MWM85] MESCHEDE, D., WALTHER, H. and MÜLLER, G.: One-atom maser. Phys. Rev. Lett. **54**, 551–554 (1985).

[NP12] NECHITA, I. and PELLEGRINI, C.: Random repeated quantum interactions and random invariant states. Prob. Theor. Rel. Fields **152**, 299–320 (2012).

[Ons31a] ONSAGER, L.: Reciprocal relations in irreversible processes I. Phys. Rev. **37**, 405 (1931).

[Ons31b] ———: Reciprocal relations in irreversible processes II. Phys. Rev. **38**, 2265 (1931).

[Pet08] PETZ, D.: *Quantum Information Theory and Quantum Statistics*. Springer, Berlin, 2008.

[Pil85] PILLET, C.-A.: Some results on the quantum dynamics of a particle in a Markovian potential. Commun. Math. Phys. **102**, 237–254 (1985).

[Pil86a] ———: Asymptotic completeness for a quantum particle in a Markovian short range potential. Commun. Math. Phys. **105**, 259–280 (1986).

[Pil86b] ———: Mécanique quantique dans un potentiel aléatoire markovien. Thesis, ETH Zürich, 1986.

[Raq20] RAQUÉPAS, R.: On fermionic walkers interacting with a correlated structured environment. Lett. Math. Phys. **110**, 121–145 (2020).

[RBH01] RAIMOND, J.-M., BRUNE, M. and HAROCHE, S.: Manipulating quantum entanglement with atoms and photons in a cavity. Rev. Mod. Phys. **73**, 565–582 (2001).

[RMM07] Rondoni, L. and Mejía-Monasterio, C.: Fluctuations in non-equilibrium statistical mechanics: models, mathematical theory, physical mechanisms. Nonlinearity **20**, 1–37 (2007).

[RW14] Reeb, D. and Wolf, M. M.: An improved Landauer principle with finite-size corrections. New J. Phys. **16**, 103011 (2014).

[Tak79] Takesaki, M.: *Theory of Operator Algebras I.* Springer, Berlin, 1979.

[TLH07] Talkner, P., Lutz, E. and Hänggi, P.: Fluctuation theorems: Work is not an observable. Phys. Rev. E **75**, 050102 (2007).

[WBKM00] Wellens, T., Buchleitner, A., Kümmerer, B. and Maassen, H.: Quantum state preparation via asymptotic completeness. Phys. Rev. Lett. **85**, 3361–3364 (2000).