Enhanced dissipation and axisymmetrization of two-dimensional viscous vortices

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Abstract

This paper is devoted to the stability analysis of the Lamb-Oseen vortex in the regime of high circulation Reynolds numbers. When strongly localized perturbations are applied, it is shown that the vortex relaxes to axisymmetry in a time proportional to $Re^{2/3}$, which is substantially shorter than the diffusion time scale given by the viscosity. This enhanced dissipation effect is due to the differential rotation inside the vortex core. Our result relies on a recent work by Li, Wei, and Zhang [29], where optimal resolvent estimates for the linearized operator at Oseen's vortex are established. A comparison is made with the predictions that can be found in the physical literature, and with the rigorous results that were obtained for shear flows using different techniques.

1 Introduction

It is a well known experimental fact that isolated vortices in two-dimensional viscous flows relax to axisymmetry in a relatively short time, because the differential rotation in the vortex core creates small spatial scales in the radial direction which substantially enhance the viscous dissipation [31, 40, 9]. This effect can be quantified in terms of the circulation Reynolds number $Re = |\Gamma|/\nu$, where Γ denotes the total circulation of the vortex and ν is the kinematic viscosity of the fluid. Indeed, radially symmetric vortex patches of size $\mathcal{O}(R)$ evolve diffusively toward the Gaussian Lamb-Oseen vortex on a time scale of the order of TRe, where $T = R^2/|\Gamma|$ is the inviscid turnover time. In contrast, it is observed that non-axisymmetric perturbations that preserve the first moment of vorticity decay to zero in a much shorter time, proportional to $TRe^{1/3}$ [9, 3]. Actually, the enhanced dissipation effect is necessarily weaker near the vortex center where the differential rotation degenerates, and it can be shown that axisymmetrization occurs in that region on some intermediate time scale, typically of the order of $TRe^{1/2}$ [1]. Similar conclusions are reached when considering the distribution of a passive scalar advected by a vortex or by a shear flow [3, 1].

From a mathematical point of view, however, it is quite difficult to obtain rigorous results that describe under which assumptions axisymmetrization occurs for two-dimensional vortices, and at which rate. In the physical literature, the perturbations of a radially symmetric vortex are mostly studied at the level of the linearized equations, and both the time evolution of the underlying vortex and the deformation of its streamlines are often neglected, so that vorticity is considered as a passive scalar advected by a stationary flow. In this paper, we focus on the archetypal example of the Lamb-Oseen vortex, and we establish the first stability result that describes and exploits the enhanced dissipation effect for the full nonlinear problem. Our analysis shows that non-axisymmetric perturbations that preserve the first moment of vorticity disappear in a time proportional to $TRe^{2/3}$, which is longer than the physical scales $TRe^{1/3}$ and $TRe^{1/2}$ obtained in [31, 9, 3], but still substantially smaller than the diffusive scale. The origin of the new exponent 2/3 will be explained in Section 2.5 below, once the resolvent estimates for the linearized operator will be presented.

It is important to realize that axisymmetrization at high Reynolds numbers plays a crucial role not only in the stability analysis of isolated vortices in freely decaying turbulence, but also in a number of related problems. For instance, Burgers vortices are stationary solutions of the three-dimensional Navier-Stokes equations in the presence of a linear strain field, and it is observed that the streamlines of these vortices become more and more circular in the limit of large Reynolds numbers, even if the strain is not axisymmetric [41, 35, 27]. Rigorous results in this direction have been obtained by C.E. Wayne and the author [24], and by Y. Maekawa [33, 34]. Another example is the evolution of a two-dimensional vortex in a time-dependent strain field, such as the velocity field produced by a collection of other vortices. Accurate asymptotic expansions [43, 42] and rigorous error estimates [18] show that, for large Reynolds numbers, the vortex is nearly axisymmetric and is deformed in such a way that the self-interaction exactly compensates for the action of the exterior strain, except for a rigid translation. It should be mentioned, however, that the results presented below require a much deeper understanding of the enhanced dissipation effect than what is necessary to prove axisymmetrization in [24, 18]. The main new ingredient in our proof is the beautiful resolvent estimate recently obtained by Li, Wei, and Zhang [29] for the linearized operator at Oseen's vortex.

We now state our results in a more precise way. Our starting point is the vorticity equation in the two-dimensional plane, which reads

$$\partial_t \omega(x,t) + u(x,t)\cdot\nabla\omega(x,t) \;=\; \nu\Delta\omega(x,t)\,, \tag{1.1}$$

where $x \in \mathbb{R}^2$ is the space variable, $t \ge 0$ is the time variable, and $\nu > 0$ is the kinematic viscosity. The velocity field $u : \mathbb{R}^2\times\mathbb{R}_+ \to \mathbb{R}^2$ is obtained from the vorticity $\omega : \mathbb{R}^2\times\mathbb{R}_+ \to \mathbb{R}$ by solving the elliptic system $\operatorname{div} u = 0$, $\operatorname{curl} u \equiv \partial_1 u_2 - \partial_2 u_1 = \omega$. This leads to the two-dimensional Biot-Savart law

$$u(x,t) \;=\; \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(x-y)^\perp}{|x-y|^2}\,\omega(y,t)\,\mathrm{d}y\,, \tag{1.2}$$

which is studied in Section 5.2. In shorthand notation we write $u(\cdot,t) = K_{BS} * \omega(\cdot,t)$, where $K_{BS}(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2}$ is the Biot-Savart kernel.

The *Lamb-Oseen vortices* are self-similar solutions of equation (1.1) defined by

$$\omega(x,t) \;=\; \frac{\Gamma}{\nu t}\,G\Big(\frac{x}{\sqrt{\nu t}}\Big)\,, \qquad u(x,t) \;=\; \frac{\Gamma}{\sqrt{\nu t}}\,v^G\Big(\frac{x}{\sqrt{\nu t}}\Big)\,, \tag{1.3}$$

where the vorticity and velocity profiles have the following explicit expressions

$$G(\xi) \;=\; \frac{1}{4\pi}\,e^{-|\xi|^2/4}\,, \qquad v^G(\xi) \;=\; \frac{1}{2\pi}\,\frac{\xi^\perp}{|\xi|^2}\Big(1 - e^{-|\xi|^2/4}\Big)\,, \qquad \xi\in\mathbb{R}^2\,. \tag{1.4}$$

In particular we have $v^G = K_{BS} * G$. The parameter $\Gamma = \int_{\mathbb{R}^2}\omega(x,t)\,\mathrm{d}x \in \mathbb{R}$ is called the *total circulation* of the vortex. We are especially interested in rapidly rotating vortices, where the circulation $|\Gamma|$ is much larger than the kinematic viscosity ν. This is the regime that is most relevant for applications to two-dimensional turbulent flows.

To study the stability of the vortices (1.3), we look for solutions of (1.1) in the form

$$\begin{aligned}\omega(x,t) &= \frac{1}{t+t_0}\, w\left(\frac{x-x_0}{\sqrt{\nu(t+t_0)}}\,,\, \log\Bigl(1+\frac{t}{t_0}\Bigr)\right)\,,\\ u(x,t) &= \sqrt{\frac{\nu}{t+t_0}}\, v\left(\frac{x-x_0}{\sqrt{\nu(t+t_0)}}\,,\, \log\Bigl(1+\frac{t}{t_0}\Bigr)\right)\,,\end{aligned} \tag{1.5}$$

see [25, 19]. The parameters $x_0 \in \mathbb{R}^2$ and $t_0 > 0$ are free at this stage, but convenient choices will be made later to optimize the results. The new space and time variables are denoted by

$$\xi = \frac{x-x_0}{\sqrt{\nu(t+t_0)}} \in \mathbb{R}^2\,, \qquad \tau = \log\Bigl(1+\frac{t}{t_0}\Bigr) \geq 0\,. \tag{1.6}$$

The rescaled vorticity $w(\xi,\tau)$ satisfies the evolution equation

$$\partial_\tau w + v\cdot\nabla_\xi w = \Delta_\xi w + \frac{1}{2}\xi\cdot\nabla_\xi w + w\,, \tag{1.7}$$

where all dependent and independent variables are now dimensionless. The rescaled velocity v is again given by the Biot-Savart law (1.2), namely $v(\cdot,\tau) = K_{BS} * w(\cdot,\tau)$. By construction, the self-similar solutions (1.3) of the original equation (1.1) correspond to the family of equilibria $\{\alpha G \,|\, \alpha \in \mathbb{R}\}$ of the rescaled equation (1.7), where $\alpha = \Gamma/\nu$ is the circulation Reynolds number. Our main purpose is to study the stability of these equilibria in the large Reynolds number limit $|\alpha| \to \infty$.

Our first task is to choose a suitable function space for the solutions of (1.7). There are in principle several possibilities, see [23, 21], but to obtain uniform stability results in the large circulation limit it seems necessary to use the Hilbert space $X = L^2(\mathbb{R}^2, G^{-1}\,\mathrm{d}\xi)$, equipped with the scalar product

$$\langle w_1, w_2\rangle_X = \int_{\mathbb{R}^2} G(\xi)^{-1}\,\overline{w_1(\xi)}\,w_2(\xi)\,\mathrm{d}\xi\,. \tag{1.8}$$

The associated norm will be denoted by $\|w\|_X$, or simply by $\|w\|$ when no confusion is possible. The solutions of (1.7) are of course real-valued, but the spectral analysis of the linearized operator at Oseen's vortex will be performed in the complexified space defined by the scalar product (1.8). Since $w \in X$ if and only if $G^{-1/2}w \in L^2(\mathbb{R}^2)$, it follows from (1.4) that all elements of X have a Gaussian decay, in the L^2 sense, as $|\xi| \to \infty$. In particular we have $X \hookrightarrow L^p(\mathbb{R}^2)$ for any $p \in [1,2]$. For later use we introduce the following closed subspaces:

$$X_0 = \Bigl\{ w \in X \,\Big|\, \int_{\mathbb{R}^2} w(\xi)\,\mathrm{d}\xi = 0 \Bigr\}\,, \tag{1.9}$$

$$X_1 = \Bigl\{ w \in X_0 \,\Big|\, \int_{\mathbb{R}^2} \xi_i\, w(\xi)\,\mathrm{d}\xi = 0 \text{ for } i=1,2 \Bigr\}\,. \tag{1.10}$$

We next recall that the Cauchy problem for equation (1.7) is globally well-posed in the space X, and that all solutions converge to the family of equilibria $\{\alpha G\}_{\alpha\in\mathbb{R}}$ as $\tau \to +\infty$.

Proposition 1.1. **[23]** *For any $w_0 \in X$, the rescaled vorticity equation* (1.7) *has a unique global (mild) solution $w \in C^0([0,\infty), X)$ such that $w(0) = w_0$. This solution satisfies $\|w(\tau) - \alpha G\|_X \to 0$ as $\tau \to +\infty$, where $\alpha = \int_{\mathbb{R}^2} w_0(\xi)\,\mathrm{d}\xi$.*

According to the usual terminology, a mild solution of (1.7) is a solution of the associated integral equation, which is Eq. (5.1) below. Proposition 1.1 is essentially taken from [23], except that we use here a different function space. In [23], the rescaled vorticity equation (1.7) is studied in the polynomially weighted space $L^2(m) = \{w \,|\, (1+|\xi|^2)^{m/2} w \in L^2(\mathbb{R}^2)\}$ with $m > 1$, and it is asserted in [25, 20, 21], without detailed justification, that the results of [23] remain valid in the smaller space X. Since the choice of the Gaussian space X seems essential in the present paper, we give a short proof of Proposition 1.1 in Section 5.1 below.

We now study the behavior of the solutions of (1.7) in the neighborhood of the family of Oseen vortices. The following preliminary result shows that the equilibrium αG is asymptotically stable for any value of the circulation parameter α, and provides a uniform estimate on the size of the basin of attraction.

Proposition 1.2. [25, 19] *There exists $\epsilon > 0$ such that, for all $\alpha \in \mathbb{R}$ and all $w_0 \in \alpha G + X_0$ such that $\|w_0 - \alpha G\|_X \le \epsilon$, the unique solution of (1.7) with initial data w_0 satisfies*

$$\|w(\cdot,\tau) - \alpha G\|_X \,\le\, \min(1, 2\,e^{-\tau/2})\|w_0 - \alpha G\|_X\,, \qquad \forall \tau \ge 0\,. \tag{1.11}$$

The assumption that the initial perturbation has zero average, namely that $w_0 - \alpha G \in X_0$, does not restrict the generality : as is shown in Section 4.1, the general case can be reduced to that particular situation by an elementary transformation. Estimate (1.11) is established in [25, Proposition 4.1] or in [19, Proposition 4.5], but the proof is quite simple and for the reader's convenience we reproduce it at the end of Section 5.1.

The limitation of Proposition 1.2 is that it does not take into account the enhanced dissipation effect due to the differential rotation, which is effective for large $|\alpha|$. When translated back into the original variables, estimate (1.11) simply asserts that small perturbations of the Lamb-Oseen vortex decay to zero on the diffusion time scale. Building on the recent work of Li, Wei, and Zhang [29], we now formulate an improved stability result which shows that the basin of attraction of Oseen's vortex becomes very large in the high Reynolds number limit $|\alpha| \to \infty$, and that perturbations relax to axisymmetry in a much shorter time. For simplicity, we restrict ourselves to solutions of (1.7) that satisfy $w(\cdot,\tau) - \alpha G \in X_1$ for all $\tau \ge 0$. This condition is preserved under the evolution defined by (1.7), and our assumption means that we consider perturbations that do not alter the total circulation α nor the first-order moments of the vorticity distribution.

Our main result can be stated as follows.

Theorem 1.3. *There exist positive constants C_1, C_2, and κ such that, for all $\alpha \in \mathbb{R}$ and all initial data $w_0 \in \alpha G + X_1$ such that*

$$\|w_0 - \alpha G\|_X \,\le\, \frac{C_1\,(1+|\alpha|)^{1/6}}{\log(2+|\alpha|)}\,, \tag{1.12}$$

the unique solution of (1.7) in X given by Proposition 1.1 satisfies, for all $\tau \ge 0$,

$$\|w(\cdot,\tau) - \alpha G\|_X \,\le\, C_2\,e^{-\tau}\|w_0 - \alpha G\|_X\,, \tag{1.13}$$

$$\|(1-P_r)(w(\cdot,\tau) - \alpha G)\|_X \,\le\, C_2\|w_0 - \alpha G\|_X\,\exp\Bigl(-\frac{\kappa(1+|\alpha|)^{1/3}\tau}{\log(2+|\alpha|)}\Bigr)\,, \tag{1.14}$$

where P_r is the orthogonal projection in X onto the subspace of all radially symmetric functions.

Remarks 1.4.

1. Theorem 1.3 improves Proposition 1.2 only when the circulation parameter $|\alpha|$ is sufficiently large. In the proof, we shall therefore assume that $|\alpha| \ge \alpha_0 \gg 1$, in which case we can replace $1+|\alpha|$ and $2+|\alpha|$ by $|\alpha|$ in estimates (1.12), (1.14). Note that, when $w_0 - \alpha G \in X_1$, the bound

(1.11) still holds if we replace $e^{-\tau/2}$ by $e^{-\tau}$ in the right-hand side [19], and this is why we also have the overall decay rate $e^{-\tau}$ in (1.13).

2. As is explained in Section 4.1, the assumption that the initial perturbation $w_0 - \alpha G$ belongs to the subspace X_1 does not restrict the generality if $\alpha \neq 0$, because the general case can be reduced to that situation by a simple change of variables. From a more conceptual point of view, if the initial vorticity distribution ω_0 has a nonzero total circulation $\Gamma = \alpha\nu$, and if we choose the parameter x_0 in (1.5) to be the *center of vorticity*, then by construction the rescaled vorticity satisfies $w - \alpha G \in X_1$ at initial time $\tau = 0$, hence for all subsequent times since both the total circulation and the center of vorticity are conserved quantities for the two-dimensional Navier-Stokes equations.

3. Estimate (1.12) shows that the size of the immediate basin of attraction of Oseen's vortex αG grows at least like $|\alpha|^{1/6}$ as $|\alpha| \to \infty$, up to a logarithmic factor. By "immediate basin of attraction", we mean here the set of initial data for which convergence to Oseen's vortex can be proved using only the decay properties of the linearized equation and, for instance, Duhamel's formula. We recall that, according to Proposition 1.1, all solutions of (1.7) with initial data $w_0 \in \alpha G + X_0$ converge to αG as $\tau \to +\infty$, so in this sense the basin of attraction of Oseen's vortex αG has infinite size for any fixed α. But Proposition 1.1 is nonconstructive, and general solutions of (1.7) can go through all the stages of two-dimensional freely decaying turbulence before reaching the asymptotic regime described by Oseen's vortex, whereas Theorem 1.3 provides explicit control on the solutions for all times, as illustrated in estimates (1.13), (1.14).

4. The decay rate in (1.14) is a direct consequence of the resolvent estimate obtained in [29], which is known to be sharp, hence there are reasons to believe that the bound (1.14) is close to optimal. As for the size of the basin of attraction, although the proof of Theorem 1.3 naturally leads to (1.12), we do not know if that estimate is optimal in any sense.

5. Using the spectral estimate established in [29, Section 6], it is possible to show that the solutions of (1.7) considered in Theorem 1.3 satisfy

$$\limsup_{\tau\to+\infty} \frac{1}{\tau} \log \|(1 - P_r)(w(\cdot,\tau) - \alpha G)\|_X \;\le\; -\kappa'(1 + |\alpha|)^{1/2}\,, \tag{1.15}$$

for some constant κ' independent of α, see also inequality (3.10) below. However the asymptotic regime described in (1.15) is only reached at relatively large times, and may not be observable in real flows.

It is instructive to compare the conclusions of Theorem 1.3 with predictions that can be found in the physical literature [9, 3], and with rigorous results describing the asymptotic stability of the two-dimensional Couette flow and other viscous shear flows [7, 8, 5]. To do that, it is convenient to fix the circulation and the spatial extent of the vortex, while choosing the viscosity parameter small enough to reach the high Reynolds number regime. We thus consider the following initial data for the original vorticity equation (1.1) :

$$\omega_0(x) \;=\; \frac{\Gamma}{R^2}\, G\Big(\frac{x}{R}\Big) + \tilde\omega_0(x)\,, \qquad x \in \mathbb{R}^2\,,$$

where the circulation $\Gamma > 0$ and the vortex radius $R > 0$ are fixed parameters. We assume that the perturbation $\tilde\omega_0$ decays rapidly enough at infinity so that $\exp(|x|^2/(8R^2))\tilde\omega_0 \in L^2(\mathbb{R}^2)$, and satisfies

$$\int_{\mathbb{R}^2} \tilde\omega_0(x)\,\mathrm{d}x \;=\; 0\,, \qquad \int_{\mathbb{R}^2} x_1\,\tilde\omega_0(x)\,\mathrm{d}x \;=\; \int_{\mathbb{R}^2} x_2\,\tilde\omega_0(x)\,\mathrm{d}x \;=\; 0\,.$$

We introduce the turnover time $T = R^2/\Gamma$, and the diffusion time $t_0 = R^2/\nu$ which depends on the viscosity parameter. As $t_0/T = \Gamma/\nu$ is the circulation Reynolds number, henceforth denoted by α, we are interested in the regime where $t_0 \gg T$. If we use the change of variables (1.5) with $x_0 = 0$ and t_0 as above, the rescaled vorticity at initial time takes the form $w_0 = \alpha G + \tilde{w}_0$, where $\tilde{w}_0(\xi) = t_0 \tilde{\omega}_0(R\xi)$. Thus $\tilde{w}_0 \in X_1$ by construction, and we can apply Theorem 1.3 to control the solution of (1.1) in the small viscosity regime. First, we infer from (1.12) that

$$\frac{1}{\Gamma} \int_{\mathbb{R}^2} |\tilde{\omega}_0(x)| \,\mathrm{d}x \,=\, \frac{\nu}{\Gamma} \int_{\mathbb{R}^2} |\tilde{w}_0(\xi)| \,\mathrm{d}\xi \,\le\, C\,\frac{\nu}{\Gamma}\,\|\tilde{w}_0\|_X \,\le\, C\,\frac{\nu}{\Gamma}\,|\alpha|^{1/6} \,=\, C\Big(\frac{\nu}{\Gamma}\Big)^{5/6},$$

and this indicates that we can consider perturbations $\tilde{\omega}_0$ whose size shrinks to zero like ν^γ as $\nu \to 0$, for any $\gamma > 5/6$. In the terminology of [8], we have shown that the *stability threshold* for the Oseen vortex in the space X is not larger than $5/6$. Next, we deduce from (1.14) that the non-axisymmetric part of the perturbation $\tilde{w}$ disappears in a time τ of the order of $\alpha^{-1/3}$, and since $\tau = \log(1+t/t_0)$ we conclude that the vortex relaxes to axisymmetry in a time proportional to

$$t_{relax} \,=\, C\,\frac{t_0}{\alpha^{1/3}} \,=\, C\,T\,\frac{t_0}{T}\,\Big(\frac{\nu}{\Gamma}\Big)^{1/3} \,=\, C\,T\,\Big(\frac{\Gamma}{\nu}\Big)^{2/3}.$$

The relaxation time predicted by Theorem 1.3 is thus proportional to $T\alpha^{2/3}$, where T is the turnover time of the unperturbed vortex, and $\alpha = \Gamma/\nu$ is the circulation Reynolds number.

In contrast, the calculations performed in [9, 2] indicate that initial perturbations located near the vortex core, where the differential rotation is maximal, relax to axisymmetry in a time proportional to $T\alpha^{1/3}$. Similarly, perturbations of the two-dimensional Couette flow relax to a shear flow in a time of the order of $T Re^{1/3}$ [7, 8], where Re is the Reynolds number and T is again an appropriate turnover time. The apparent discrepancy with the conclusions of Theorem 1.3 is entirely due to the fact that the differential rotation of Oseen's vortex vanishes at the origin and at infinity. Such a degeneracy certainly does not exist for the shear flows considered in [7, 8], which are close to Couette. For the vortex problem, the relaxation time is known to be proportional to $T\alpha^{1/2}$ for perturbations initially located at the vortex center [1], and to the diffusive time $t_0 = T\alpha$ for perturbations very far away from the center [40]. From a mathematical point of view, these various regimes cannot be considered separately, because they are coupled even at the level of the linearized equation. If translation invariant norms are used, this means that for perturbations of the Oseen vortex the diffusive decay rate is optimal in general. To obtain a stability result that takes advantage of the differential rotation, we use in Theorem 1.3 the Gaussian space X whose weight creates an artificial damping of the perturbations far away from the origin, see Section 2.5 below for a more precise description of that effect. This is why we can obtain, not only for the linearized equation but even for the full nonlinear problem, a uniform relaxation time $T\alpha^{2/3}$ that is substantially smaller than the diffusion time scale.

To conclude this introduction, we briefly mention that phenomena such as relaxation of vortices to axisymmetry or stability of shear flows can also be studied for perfect fluids, where the situation is quite different (and in some sense more subtle). The interested reader is referred to [2, 4, 10] for a physical analysis and to [6, 44] for recent mathematical results. In a different perspective, it is interesting to draw a comparison with the convergence results available for rotating geophysical fluids, see e.g. the monograph [14] and the references therein. To take into account Earth's rotation, a Coriolis term of the form $\Omega(e_3 \wedge u)$ is included in the velocity formulation of the three-dimensional Euler or Navier-Stokes equations, and the problem is to describe the dynamics in the limit where the angular frequency Ω is large. Due to the dispersive effects of the Rossby operator, the asymptotic motion occurs in horizontal planes, but depending on the precise setting the limiting equation may include contributions from wave interactions or from boundary conditions. If viscosity is taken into account, the linearized operator at $u = 0$ is of the form $L = A - \Omega B$, where $A = \nu\Delta$ is selfadjoint and $B = \mathbb{P}(e_3 \wedge \cdot)$ is skew-symmetric, in close analogy with the operator $\mathcal{L} - \alpha\Lambda$ introduced in (2.1) below. In both situations, the effect of fast rotation is to project the evolution on the kernel of the skew-symmetric part, which consists of horizontal motions for rotating fluids and of radially symmetric flows for two-dimensional vortices. However, it should be emphasized that the enhanced dissipation effect considered in the present work relies on the fact that the operators $\mathcal{L}$, Λ in (2.2), (2.3) *do not commute*, whereas the dispersive effects exploited in rotating fluids are entirely due to the skew-symmetric operator B, which actually commutes with the diffusion term A except for the boundary conditions. The analogy between the mathematical results obtained in both situations is therefore quite superficial.

The rest of this paper is organized as follows. In Section 2 we summarize what is known about the linearized operator at Oseen's vortex, and we recall the beautiful resolvent estimate recently obtained by Li, Wei, and Zhang [29], which allows us to derive semigroup estimates in the subsequent Section 3. The analysis of the nonlinear problem is postponed to Section 4, which contains the proof of Theorem 1.3. The final section is an appendix, where we collect some known results on the rescaled vorticity equation (1.7), and derive accurate estimates on the Biot-Savart law (1.2) that are needed in our analysis.

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2 Resolvent estimates for the linearized operator

In this section we study the linearization of equation (1.7) at the equilibrium point $w = \alpha G$, for a given $\alpha \in \mathbb{R}$. Setting $w = \alpha G + \tilde{w}$ and $v = \alpha v^G + \tilde{v}$, where $\tilde{v} = K_{BS} * \tilde{w}$, we obtain for the perturbation $\tilde{w}$ the evolution equation

$$\partial_\tau \tilde{w} + \tilde{v} \cdot \nabla \tilde{w} \;=\; (\mathcal{L} - \alpha\Lambda)\tilde{w}\,, \tag{2.1}$$

where $\mathcal{L}$ is the linear operator in the right-hand side of (1.7) :

$$\mathcal{L}w \;=\; \Delta w + \frac{1}{2}\,\xi \cdot \nabla w + w\,, \tag{2.2}$$

and Λ is the linearization at $w = G$ of the quadratic term $v \cdot \nabla w = (K_{BS} * w) \cdot \nabla w$:

$$\Lambda w \;=\; v^G \cdot \nabla w + (K_{BS} * w) \cdot \nabla G \;\equiv\; \Lambda_{\mathrm{ad}} w + \Lambda_{\mathrm{nl}} w\,. \tag{2.3}$$

Here and in what follows, it is understood that all differential operators act on the space variable $\xi \in \mathbb{R}^2$, except for the time derivative ∂_τ which is always explicitly indicated.

We first recall a few classical properties of the operators $\mathcal{L}$ and Λ, which can be found e.g. in [22, 23, 32, 19, 21]. We only consider the situation where these operators act on the Hilbert space $X = L^2(\mathbb{R}^2, G^{-1}\,\mathrm{d}\xi)$, equipped with the scalar product (1.8), but similar results in larger function spaces can be found in [23, 21]. Our goal is to present the optimal resolvent bounds obtained by Li, Wei, and Zhang [29] for the linearized operator $\mathcal{L} - \alpha\Lambda$ in the fast rotation limit $|\alpha| \to +\infty$. These estimates will serve as a basis for all developments in Sections 3 and 4.

2.1 Fundamental properties of $\mathcal{L}$ and Λ

The first observation is that the operator $\mathcal{L}$ is *selfadjoint* in the space X, with compact resolvent and purely discrete spectrum

$$\sigma(\mathcal{L}) \,=\, \Bigl\{-\frac{n}{2}\,\Big|\, n = 0,1,2,\dots\Bigr\}\,. \tag{2.4}$$

Indeed, a formal calculation shows that $\mathcal{L}$ is conjugated to the Hamiltonian of the harmonic oscillator in $\mathbb{R}^2$:

$$L \,:=\, G^{-1/2}\,\mathcal{L}\,G^{1/2} \,=\, \Delta - \frac{|\xi|^2}{16} + \frac{1}{2}\,. \tag{2.5}$$

As is well known (see e.g. [26]), the operator L, when defined on its maximal domain, is selfadjoint in $L^2(\mathbb{R}^2)$ with compact resolvent and spectrum given by (2.4). This implies the desired properties of the operator $\mathcal{L}$ in $X = L^2(\mathbb{R}^2, G^{-1}\,\mathrm{d}\xi)$, and we also obtain in this way the following characterization of its domain:

$$D(\mathcal{L}) \,=\, \bigl\{w \in X \,|\, \mathcal{L}w \in X\bigr\} \,=\, \bigl\{w \in X \,|\, \Delta w \in X,\ (1+|\xi|)\nabla w \in X,\ (1+|\xi|)^2 w \in X\bigr\}\,.$$

Concerning the eigenproperties of $\mathcal{L}$, we mention that the kernel $\ker(\mathcal{L})$ is the one-dimensional subspace spanned by the Gaussian vorticity profile G, and is orthogonal in X to the hyperplane X_0 defined by (1.9). The second eigenvalue $-1/2$ has multiplicity two, with eigenfunctions given by the first order derivatives $\partial_i G = -\frac{1}{2}\xi_i G$ for $i = 1,2$, and for later use we observe that the orthogonal complement of the spectral subspace spanned by $G, \partial_1 G, \partial_2 G$ is precisely the subspace X_1 defined by (1.10). More generally, for any $k \in \mathbb{N}$, the eigenvalue $-k/2$ has multiplicity $k+1$ and the corresponding eigenspace is spanned by Hermite functions of order k, namely homogeneous k^{th} order derivatives of the Gaussian profile G [22].

The second key observation is that the operator Λ is a *relatively compact* perturbation of $\mathcal{L}$, which is *skew-symmetric* in X:

$$\langle \Lambda w_1\,,\, w_2\rangle_X + \langle w_1\,,\, \Lambda w_2\rangle_X \,=\, 0\,, \qquad \text{for all } w_1, w_2 \in D(\mathcal{L})\,. \tag{2.6}$$

Indeed, if Λ is decomposed as in (2.3), the advection term $\Lambda_{\text{ad}} = v^G\cdot\nabla$ is a first order differential operator with smooth coefficients decaying to zero at infinity, hence is a relatively compact perturbation of the second order elliptic operator $\mathcal{L}$. Similarly, it is straightforward to verify that the nonlocal operator Λ_{nl} in (2.3) is compact in X, hence also relatively compact with respect to $\mathcal{L}$. On the other hand, since the velocity field v^G is divergence-free and satisfies $\xi\cdot v^G(\xi) = 0$ for all $\xi \in \mathbb{R}^2$, we have $\mathrm{div}(G^{-1}v^G) = 0$, and this implies that the operator Λ_{ad} is skew-symmetric in X. The same property holds for Λ_{nl} too, and can be established by a direct calculation which takes into account the structure of the Biot-Savart kernel, see [23, 19]. In fact, it is shown in [32] that the operator Λ is even *skew-adjoint* in X when defined on its maximal domain $D(\Lambda) = \{w \in X \,|\, \Lambda w \in X\}$.

Another useful result of [32] is the following characterization of the kernel of Λ in X:

$$\ker(\Lambda) \,=\, Y_0 \oplus \bigl\{\beta_1\partial_1 G + \beta_2\partial_2 G \,|\, \beta_1,\beta_2 \in \mathbb{R}\bigr\}\,, \tag{2.7}$$

where $Y_0 \subset X$ denotes the subspace of all radially symmetric functions. Indeed, it is clear by symmetry that Λ vanishes on any radially symmetric function in X, so that $Y_0 \subset \ker(\Lambda)$. Moreover, if we differentiate the identity $v^G\cdot\nabla G = (K_{BS}*G)\cdot\nabla G = 0$ with respect to ξ_1 and ξ_2, we see that $\Lambda(\partial_i G) = 0$ for $i = 1,2$. Thus $\ker(\Lambda)$ contains the right-hand side of (2.7), and the converse inclusion is established in [32], see also [19], using the Fourier decomposition presented in Section 2.4 below and the explicit form of the one-dimensional operator Λ_n in (2.13).

2.2 Dissipativity and linear stability

We recall that an operator $A : D(A) \to X$ is *dissipative* if $\mathrm{Re}\,\langle Aw, w\rangle_X \le 0$ for all $w \in D(A)$, or equivalently if $\|(\lambda - A)w\| \ge \lambda\|w\|$ for all $w \in D(A)$ and all $\lambda > 0$ [37]. The operator A is called m-dissipative if in addition any $\lambda > 0$ belongs to the resolvent set of A [28]. By the Lumer-Phillips theorem, an operator A is m-dissipative if and only if it generates a strongly continuous semigroup of contractions in X [37].

The properties collected in Section 2.1 readily imply the following important result :

Proposition 2.1. [23, 19] *For any $\alpha \in \mathbb{R}$:*

a) the operator $\mathcal{L} - \alpha\Lambda$ is m-dissipative in X;

b) the operator $\mathcal{L} - \alpha\Lambda + \frac{1}{2}$ is m-dissipative in X_0;

c) the operator $\mathcal{L} - \alpha\Lambda + 1$ is m-dissipative in X_1.

Proposition 2.1 shows in particular that the Oseen vortex αG is a *linearly stable* equilibrium of the rescaled vorticity equation (1.7), for any value of the circulation Reynolds number $\alpha \in \mathbb{R}$. In addition, if we restrict ourselves to perturbations in the invariant subspace X_0, the linearized operator $\mathcal{L} - \alpha\Lambda$ has a *uniform spectral gap* (of size 1/2) for all $\alpha \in \mathbb{R}$. As the nonlinearity in (2.1) does not involve the parameter α, this implies a uniform lower bound on the size of the (immediate) basin of attraction of the vortex, as asserted in Proposition 1.2. In the invariant subspace $X_1 \subset X_0$, the spectral gap is even larger (of size 1), and the perturbations therefore decay to zero like $e^{-\tau}$ as $\tau \to +\infty$. More details can be found in Section 5.1, which contains in particular a short proof of Proposition 1.2.

2.3 Enhanced dissipation for large circulation α

The main purpose of the present paper is to investigate what can be said beyond Proposition 2.1, using the enhanced dissipation properties of the linearized operator $\mathcal{L} - \alpha\Lambda$ for large values of α. To do that, it is obviously necessary to restrict ourselves to perturbations in the orthogonal complement $\ker(\Lambda)^\perp$, because on $\ker(\Lambda)$ the linearized operator $\mathcal{L} - \alpha\Lambda$ reduces to $\mathcal{L}$ and, therefore, does not depend on α. Using (2.7) is it easy to verify that the subspace $\ker(\Lambda)^\perp$ is invariant under the actions of both $\mathcal{L}$ and Λ, and that $\ker(\Lambda)^\perp \subset X_1$ where X_1 is defined in (1.10). Proposition 2.1 thus shows that $\mathcal{L} - \alpha\Lambda + 1$ is m-dissipative in $\ker(\Lambda)^\perp$ for all $\alpha \in \mathbb{R}$, but much more is known for large values of $|\alpha|$. The following resolvent estimate is the main result of the paper by Li, Wei, and Zhang :

Proposition 2.2. [29] *There exist positive constants c_1, c_2 such that, for all $\alpha \in \mathbb{R}$,*

$$c_1(1 + |\alpha|)^{-1/3} \le \sup_{\lambda\in\mathbb{R}} \|(\mathcal{L} - \alpha\Lambda - i\lambda)^{-1}\|_{X_\perp \to X_\perp} \le c_2(1 + |\alpha|)^{-1/3}\,, \tag{2.8}$$

where $X_\perp = \ker(\Lambda)^\perp \subset X$.

Remark 2.3. Here and in what follows, if Y is a Banach space, we denote by $\|B\|_{Y\to Y}$ the operator norm of any bounded linear map $B : Y \to Y$.

Since Λ is a relatively compact perturbation of the operator $\mathcal{L}$, which itself has compact resolvent, it is clear that the linearized operator $\mathcal{L} - \alpha\Lambda$ has compact resolvent in X for any $\alpha \in \mathbb{R}$. In particular, the spectrum $\sigma(\mathcal{L} - \alpha\Lambda)$ is a sequence of complex eigenvalues $\lambda_k(\alpha)$, where $k \in \mathbb{N}$, and it is not difficult to verify that $\mathrm{Re}(\lambda_k(\alpha)) \to -\infty$ as $k \to \infty$. Moreover, Proposition 2.1 shows that $\mathrm{Re}\,\lambda_k(\alpha) \le 0$ for all $k \in \mathbb{N}$, and that $\mathrm{Re}\,\lambda_k(\alpha) \le -1$ if we only consider eigenvalues corresponding to eigenfunctions in the invariant subspace X_1. Again, much more is known if we restrict ourselves to the smaller subspace $\ker(\Lambda)^\perp \subset X_1$. To formulate that, we define for any $\alpha \in \mathbb{R}$ the spectral lower bound

$$\Sigma(\alpha) \,=\, \inf\bigl\{\mathrm{Re}(z) \,\big|\, z \in \sigma(-\mathcal{L}_\perp + \alpha\Lambda_\perp)\bigr\}\,, \tag{2.9}$$

where $\mathcal{L}_\perp, \Lambda_\perp$ denote the restrictions of $\mathcal{L}, \Lambda$ to $X_\perp = \ker(\Lambda)^\perp$. Then Proposition 2.2 implies that $\Sigma(\alpha) \ge c_2^{-1}(1+|\alpha|)^{1/3}$ for all $\alpha \in \mathbb{R}$, because for any linear operator A in X one has the inequality

$$\|(A-z)^{-1}\| \,\ge\, \frac{1}{\mathrm{dist}(z,\sigma(A))}\,, \qquad \text{for all } z \in \mathbb{C}\setminus\sigma(A)\,. \tag{2.10}$$

In fact, another result of Li, Wei, and Zhang provides an improved lower bound on $\Sigma(\alpha)$:

Proposition 2.4. [29] *There exists a positive constant c_3 such that $\Sigma(\alpha) \ge c_3(1+|\alpha|)^{1/2}$ for all $\alpha \in \mathbb{R}$.*

According to Proposition 2.4, the eigenvalues $\lambda_k(\alpha)$ of the linearized operator $\mathcal{L} - \alpha\Lambda$ are either independent of α, because the corresponding eigenfunctions lie in the kernel of Λ, or have real parts that converge to $-\infty$ at least as fast as $-c_3|\alpha|^{1/2}$ when $|\alpha| \to \infty$. This is in full agreement with the numerical calculations of Prochazka and Pullin [38, 39], which indicate that the rate $\mathcal{O}(|\alpha|^{1/2})$ is indeed optimal. Note also that the spectral lower bound $\Sigma(\alpha)$ is much larger, when $|\alpha| \gg 1$, than what can be predicted from the pseudospectral estimate (2.8), and this is due to the fact that the linearized operator $\mathcal{L}-\alpha\Lambda$ is highly non-selfadjoint in that regime. Indeed, it is well-known that equality holds in (2.10) if A is a selfadjoint (or normal) operator in X [28].

2.4 Fourier decomposition and reduction to one-dimensional operators

For later use, we briefly describe one important step in the proof of Proposition 2.2. Our starting point is the observation that both operators $\mathcal{L}$ and Λ are invariant under rotations about the origin in $\mathbb{R}^2$. To fully exploit this symmetry, it is useful to introduce polar coordinates (r,θ) in the plane and to expand the vorticity and the velocity field in Fourier series with respect to the angular variable $\theta \in \mathbb{S}^1$. In this way, our function space X is decomposed into a direct sum:

$$X \,=\, \bigoplus_{n\in\mathbb{Z}} Y_n\,, \tag{2.11}$$

where $Y_n = \{w \in X \,|\, e^{-in\theta}w \text{ is radially symmetric}\}$. The crucial point is that, for each $n \in \mathbb{Z}$, the closed subspace Y_n is invariant under the action of both linear operators $\mathcal{L}$ and Λ. As is shown in [23], the restriction $\mathcal{L}_n$ of $\mathcal{L}$ to Y_n is the one-dimensional operator

$$\mathcal{L}_n \,=\, \partial_r^2 + \Bigl(\frac{r}{2}+\frac{1}{r}\Bigr)\partial_r + \Bigl(1-\frac{n^2}{r^2}\Bigr)\,, \tag{2.12}$$

which is defined on the positive half-line $\{r > 0\}$, with homogeneous Dirichlet condition at the origin if $n = 0$ or $|n| \ge 2$, and homogeneous Neumann condition if $|n| = 1$. Similarly, the restriction Λ_n of Λ to Y_n vanishes for $n = 0$ and is given by

$$\Lambda_n w \,=\, in\bigl(\phi w - g\Omega_n[w]\bigr)\,, \qquad \text{for } n \neq 0\,, \tag{2.13}$$

where ϕ, g are the functions on $\mathbb{R}_+$ defined by

$$\phi(r) \,=\, \frac{1}{2\pi r^2}(1-e^{-r^2/4})\,, \qquad g(r) \,=\, \frac{1}{4\pi}\,e^{-r^2/4}\,, \qquad r > 0\,, \tag{2.14}$$

and $\Omega_n = \Omega_n[w]$ is the unique solution of the differential equation $-\Omega_n'' - \frac{1}{r}\Omega_n' + \frac{n^2}{r^2}\Omega_n = \frac{1}{2}w$ on $\mathbb{R}_+$ that is regular at the origin and at infinity, namely

$$\Omega_n(r) \,=\, \frac{1}{4|n|}\left(\int_0^r \Bigl(\frac{r'}{r}\Bigr)^{|n|} r'w(r')\,\mathrm{d}r' + \int_r^\infty \Bigl(\frac{r}{r'}\Bigr)^{|n|} r'w(r')\,\mathrm{d}r'\right)\,, \qquad r > 0\,. \tag{2.15}$$

Thanks to the decomposition (2.11), to prove Proposition 2.2 it is sufficient to study the family of one-dimensional operators

$$H_{n,\beta} \,=\, -\mathcal{L}_n + \alpha\Lambda_n \,\equiv\, -\mathcal{L}_n + i\beta M_n\,, \qquad n \neq 0\,, \tag{2.16}$$

where $\beta = n\alpha \in \mathbb{R}$ and $M_n w = \phi w - g\Omega_n[w]$. When $|n| \ge 2$, these operators act on the Hilbert space $Z = L^2(\mathbb{R}_+, g^{-1} r\,\mathrm{d}r)$, which is the analog of the original space X in polar coordinates. When $n = \pm 1$, the operator M_n has a one-dimensional kernel spanned by the function rg, because $\Omega_n[rg] = r\phi$ if $|n| = 1$. In that case, to obtain enhanced dissipation estimates, it is necessary to consider $H_{n,\beta}$ as acting on the orthogonal complement of the kernel, namely on the hyperplane

$$Z_0 \,=\, \Bigl\{w \in Z \,\Big|\, \int_0^\infty r^2 w(r)\,\mathrm{d}r = 0\Bigr\}\,. \tag{2.17}$$

As in Section 2.1 above, one can verify that the operator $\mathcal{L}_n$ is selfadjoint in Z, with $\mathcal{L}_n \ge |n|/2$ for any $n \in \mathbb{Z}$. Moreover, if $n = \pm 1$, then $\mathcal{L}_n \ge 3/2$ on Z_0. Finally, the bounded operator M_n is symmetric in Z for any $n \neq 0$.

Proposition 2.2 is a direct consequence of the following optimal resolvent estimate for the family of one-dimensional operators $H_{n,\beta}$ with $|n| \ge 1$ and $\beta \in \mathbb{R}$.

Proposition 2.5. **[29]** *There exist positive constants c_1, c_2 such that, for any $\beta \in \mathbb{R}$ and any $n \in \mathbb{Z}$ with $|n| \ge 2$, the following estimate holds:*

$$c_1(1+|\beta|)^{-1/3} \,\le\, \sup_{\lambda\in\mathbb{R}} \|(H_{n,\beta} - i\lambda)^{-1}\|_{Z\to Z} \,\le\, c_2(1+|\beta|)^{-1/3}\,. \tag{2.18}$$

Moreover, if $n = \pm 1$, we have the same estimate in the subspace Z_0:

$$c_1(1+|\beta|)^{-1/3} \,\le\, \sup_{\lambda\in\mathbb{R}} \|(H_{n,\beta} - i\lambda)^{-1}\|_{Z_0\to Z_0} \,\le\, c_2(1+|\beta|)^{-1/3}\,. \tag{2.19}$$

2.5 Historical remarks and discussion

Proposition 2.2 is the culmination of a series of works where resolvent estimates similar to (2.8) were obtained for simplified models. It was first realized that, in the stability analysis of the Lamb-Oseen vortex, the enhanced dissipation effect for large values of $|\alpha|$ is due to the interplay of the diffusion operator $\mathcal{L}$ and the advection term $\Lambda_{\mathrm{ad}} = v^G\cdot\nabla$, whereas the nonlocal correction Λ_{nl} plays a relatively minor role. As in (2.5), we observe that

$$G^{-1/2}\,(\mathcal{L} - \alpha\Lambda_{\mathrm{ad}})\,G^{1/2} \,=\, L - \alpha\Lambda_{\mathrm{ad}}\,,$$

because the operator Λ_{ad} commutes with the radially symmetric weight $G^{1/2}$. In this way, we are led to study a large, skew-symmetric perturbation of the harmonic oscillator L in $L^2(\mathbb{R}^2)$. In [17], I. Gallagher, F. Nier and the author analyzed the following complex Schrödinger operator in $L^2(\mathbb{R})$:

$$H_\alpha \,=\, -\partial_x^2 + x^2 + i\alpha\phi(x)\,, \qquad x \in \mathbb{R}\,,$$

which can be considered as a one-dimensional analog of $L - \alpha\Lambda_{\rm ad}$. If ϕ is given by (2.14), they proved that

$$\sup_{\lambda\in\mathbb{R}} \|(H_\alpha - i\lambda)^{-1}\| \,=\, \mathcal{O}(|\alpha|^{-1/3})\,, \quad \text{and} \quad \inf\bigl\{\mathrm{Re}(z)\,\big|\, z\in\sigma(H_\alpha)\bigr\} \,\ge\, \mathcal{O}(|\alpha|^{1/2})\,, \tag{2.20}$$

as $|\alpha| \to \infty$, and they explained the origin of the exponents 1/3 and 1/2 appearing in (2.20). Using similar techniques, Wen Deng [15] obtained for the simplified operator $\mathcal{L} - \alpha\Lambda_{\rm ad}$ the estimate

$$c_1(1+|\alpha|)^{-1/3} \,\le\, \sup_{\lambda\in\mathbb{R}} \|(\mathcal{L} - \alpha\Lambda_{\rm ad} - i\lambda)^{-1}\|_{Y_0^\perp\to Y_0^\perp} \,\le\, c_2(1+|\alpha|)^{-1/3}\,,$$

where $Y_0^\perp$ is the orthogonal complement in X of the radially symmetric subspace. Subsequently, she also proved that the bound (2.18) holds for the full operator $H_{n,\beta}$ provided the azimuthal wavenumber $|n|$ is sufficiently large [16]. That restriction was completely removed by Li, Wei, and Zhang in [29], using careful estimates which show (roughly speaking) that the nonlocal term $\Lambda_{\rm nl}$ in the skew-symmetric operator Λ can be considered as a perturbation of the local differential operator $\Lambda_{\rm ad}$. This argument fails when $n = \pm 1$, as is attested by the existence of a nontrivial element in the kernel of Λ_n, but in that particular case the authors of [29] were able to eliminate completely the nonlocal term $\Lambda_{\rm nl}$ using a beautiful transformation inspired from scattering theory.

It is important to emphasize here the crucial role played by the Gaussian weight $G^{-1}(\xi)$ in all resolvent estimates presented in this section. The analysis of simplified one-dimensional models in [17, 15] shows that, for large values of the circulation parameter, the resolvent on the imaginary axis is bounded by $C|\alpha|^{-2/3}$ when we consider perturbations located in the vortex core, where the differential rotation is maximal, and by $C|\alpha|^{-1/2}$ for perturbations located at the origin, where the differential rotation degenerates. When translated back into the original variables, these partial results indicate that axisymmetrization occurs in a time proportional to $|\alpha|^{1/3}$ and $|\alpha|^{1/2}$, respectively, in full agreement with the predictions made in [9, 3, 1]. For arbitrary perturbations, however, it is clear that no dependence on $|\alpha|$ can be obtained if one estimates the resolvent using translation invariant norms, because the differential rotation of the vortex vanishes at infinity. The situation is different in the weighted space X, where the diffusion operator is replaced by the harmonic oscillator, see (2.5). In that case, due to the quadratic potential, the resolvent is small also for perturbations located far away from the origin. As a consequence, there is a critical distance to the origin, of the order of $|\alpha|^{1/6}$ if $|\alpha| \gg 1$, where the enhanced dissipation due to the differential rotation is of the same order as the artificial damping due to the quadratic potential in the harmonic oscillator, and this is what determines the overall size of the resolvent in the weighted space X, which is $\mathcal{O}(|\alpha|^{-1/3})$ according to Proposition 2.2. We insist on saying that this new exponent $-1/3$, which predicts axisymmetrization in a time proportional to $|\alpha|^{2/3}$, may not be directly related to physical phenomena : it is rather a consequence of our choice of measuring perturbations in the Gaussian weighted space X.

3 Semigroup estimates

Applying the resolvent bounds established in Propositions 2.2 and 2.5, we now obtain sharp decay estimates for the semigroup generated by the linearized operator $\mathcal{L} - \alpha\Lambda$ in the subspace $X_\perp = \ker(\Lambda)^\perp$. We use the Fourier decomposition introduced in Section 2.4 and first consider the semigroup defined by the one-dimensional operator (2.16) in the space $Z = L^2(\mathbb{R}_+, g^{-1}r\,\mathrm{d}r)$, equipped with the scalar product

$$\langle w_1, w_2\rangle_Z \;=\; \int_0^\infty rg(r)^{-1}\,\overline{w_1(r)}\,w_2(r)\,\mathrm{d}r\,. \tag{3.1}$$

Proposition 3.1. *For any $n \in \mathbb{Z}$, $n \neq 0$, and any $\beta \in \mathbb{R}$, the linear operator $-H_{n,\beta}$ defined in* (2.16) *is the generator of an analytic semigroup in the space $Z = L^2(\mathbb{R}_+, g^{-1}r\,\mathrm{d}r)$. Moreover, there exist positive constants c_4, c_5 such that, for any $n \in \mathbb{Z}$ with $|n| \geq 2$ and any $\beta \in \mathbb{R}$ with $|\beta| \geq 1$, the following estimate holds:*

$$\|e^{-\tau H_{n,\beta}}\|_{Z\to Z} \;\leq\; \min\Bigl(e^{-|n|\tau/2}\,,\, c_4|\beta|^{2/3}\,e^{-c_5|\beta|^{1/3}\tau}\Bigr)\,, \qquad \tau \geq 0\,. \tag{3.2}$$

If $n = \pm 1$, we have a similar estimate in the subspace Z_0 defined in (2.17)*:*

$$\|e^{-\tau H_{n,\beta}}\|_{Z_0\to Z_0} \;\leq\; \min\Bigl(e^{-3\tau/2}\,,\, c_4|\beta|^{2/3}\,e^{-c_5|\beta|^{1/3}\tau}\Bigr)\,, \qquad \tau \geq 0\,. \tag{3.3}$$

Proof. We recall that $-H_{n,\beta} = \mathcal{L}_n - i\beta M_n$, where $\mathcal{L}_n$ is a selfadjoint operator in Z satisfying $\mathcal{L}_n \leq -|n|/2$, and M_n is a bounded symmetric operator. By classical perturbation theory [37, Section 3.2], it follows that $-H_{n,\beta}$ generates an analytic semigroup in Z. Moreover, since the operator $-H_{n,\beta} + |n|/2$ is m-dissipative, the Lumer-Phillips theorem [37, Section 1.4] implies that $\|e^{-\tau H_{n,\beta}}\|_{Z\to Z} \leq e^{-|n|\tau/2}$ for all $\tau \geq 0$. For later use, we observe that there exists a constant $c_6 > 0$ such that $\|M_n\|_{Z\to Z} \leq c_6$ for all nonzero $n \in \mathbb{Z}$. Indeed $M_n w = \phi w - g\Omega_n[w]$ where $|\phi| \leq (8\pi)^{-1}$, and it follows from (2.15) that $|\Omega_n[w]| \leq \frac{1}{4}\int_0^\infty r|w(r)|\,\mathrm{d}r \leq C\|w\|_Z$ for all $n \in \mathbb{Z}$, $n \neq 0$. This shows that the *numerical range*

$$\mathcal{N}(H_{n,\beta}) \;=\; \bigl\{\langle w\,,\, H_{n,\beta}w\rangle_Z \,\big|\, w \in D(\mathcal{L}_n)\,,\ \|w\|_Z = 1\bigr\}$$

satisfies

$$\mathcal{N}(H_{n,\beta}) \;\subset\; \bigl\{z \in \mathbb{C} \,\big|\, \mathrm{Re}(z) \geq |n|/2\,,\ |\mathrm{Im}(z)| \leq c_6|\beta|\bigr\}\,. \tag{3.4}$$

If $|n| = 1$, then $-H_{n,\beta} + \frac{3}{2}$ is m-dissipative and $\mathcal{N}(H_{n,\beta}) \subset \bigl\{z \in \mathbb{C} \,\big|\, \mathrm{Re}(z) \geq \frac{3}{2}\,,\ |\mathrm{Im}(z)| \leq c_6|\beta|\bigr\}$.

To estimate the semigroup $e^{-\tau H_{n,\beta}}$ for $\tau > 0$ and $|\beta| \geq 1$, we use the inverse Laplace formula

$$e^{-\tau H_{n,\beta}} \;=\; \frac{1}{2\pi i}\int_\Gamma (H_{n,\beta} - z)^{-1}\,e^{-z\tau}\,\mathrm{d}z\,, \tag{3.5}$$

where Γ is the integration path in the complex plane depicted in Fig. 1. More precisely, we define $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ where

$$\begin{aligned} \Gamma_1 \;&=\; \Bigl\{x_0 - iy_0 - (1-i)s \,\Big|\, -\infty \leq s \leq 0\Bigr\}\,, \\ \Gamma_2 \;&=\; \Bigl\{x_0 + iy \,\Big|\, -y_0 \leq y \leq y_0\Bigr\}\,, \\ \Gamma_3 \;&=\; \Bigl\{x_0 + iy_0 + (1+i)s \,\Big|\, 0 \leq s \leq \infty\Bigr\}\,. \end{aligned} \tag{3.6}$$

Here $x_0 = |\beta|^{1/3}/(2c_2)$ and $y_0 = 2c_6|\beta|$, where the constants c_2, c_6 are as in (2.18), (3.4).

We first assume that $|n| \geq 2$ and compute the norm of the semigroup in the whole space Z using formula (3.5):

$$\|e^{-\tau H_{n,\beta}}\|_{Z\to Z} \;\leq\; \frac{1}{2\pi}\sum_{j=1}^{3}\int_{\Gamma_j} \|(H_{n,\beta} - z)^{-1}\|_{Z\to Z}\,e^{-\mathrm{Re}(z)\tau}\,|\mathrm{d}z| \;=\; I_1(\tau) + I_2(\tau) + I_3(\tau)\,.$$

If $z = x_0 + iy \in \Gamma_2$, the standard factorization $H_{n,\beta} - z = (H_{n,\beta} - iy)(1 - x_0(H_{n,\beta} - iy)^{-1})$ combined with estimate (2.18) yields the bound

$$\|(H_{n,\beta} - z)^{-1}\| \le \frac{\|(H_{n,\beta} - iy)^{-1}\|}{1 - x_0 \|(H_{n,\beta} - iy)^{-1}\|} \le 2c_2|\beta|^{-1/3}, \tag{3.7}$$

because $\|(H_{n,\beta} - iy)^{-1}\| \le c_2|\beta|^{-1/3} = 1/(2x_0)$ for any $y \in \mathbb{R}$. It follows that

$$I_2(\tau) = \frac{1}{2\pi}\int_{-y_0}^{y_0} \|(H_{n,\beta} - z)^{-1}\| \, e^{-x_0\tau} \, \mathrm{d}y \le \frac{2y_0c_2}{\pi} |\beta|^{-1/3} e^{-x_0\tau} = \frac{4c_2c_6}{\pi} |\beta|^{2/3} e^{-\tau|\beta|^{1/3}/(2c_2)}.$$

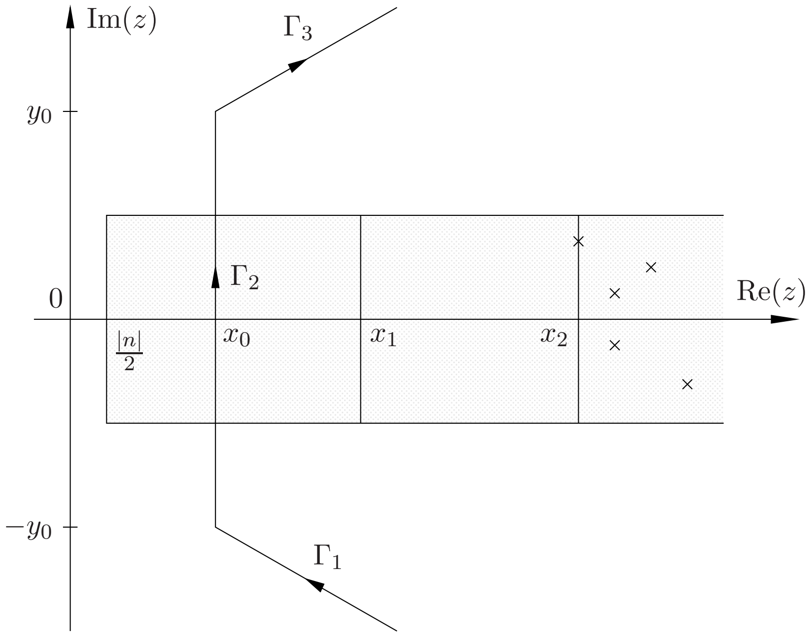

Fig. 1: The integration path (3.6) surrounds the spectrum of $H_{n,\beta}$, which is discrete and entirely contained in the shaded region defined by (3.4). The pseudospectral abscissa $x_1 = 2x_0 = |\beta|^{1/3}/c_2$ is given by Proposition 2.5, and the spectral lower bound $x_2 = c_3|\beta|^{1/2}$ by [29, Theorem 6.1], see Remark 3.2 below.

On the other hand, if $z = x_0 + iy_0 + (1 + i)s \in \Gamma_3$, then $z \notin \mathcal{N}(H_{n,\beta})$ because $\mathrm{Im}(z) = 2c_6|\beta| + s > c_6|\beta|$. It follows that

$$\|(H_{n,\beta} - z)^{-1}\| \le \frac{1}{\mathrm{dist}(z, \mathcal{N}(H_{n,\beta}))} \le \frac{1}{c_6|\beta| + s},$$

and we deduce that

$$I_3(\tau) = \frac{1}{2\pi}\int_0^{\infty} \|(H_{n,\beta} - z)^{-1}\| \, e^{-(x_0+s)\tau}\sqrt{2}\,\mathrm{d}s \le \frac{1}{\sqrt{2}\pi} e^{-x_0\tau}\Psi(c_6|\beta|\tau),$$

where $\Psi : (0, \infty) \to (0, \infty)$ is the decreasing function defined by

$$\Psi(\rho) = \int_0^{\infty} \frac{e^{-s}}{\rho + s} \, \mathrm{d}s \sim \begin{cases} \log(\rho^{-1}) & \text{as } \rho \to 0, \\ \rho^{-1} & \text{as } \rho \to \infty. \end{cases}$$

If we assume that $|\beta|\tau \ge 1$, we thus obtain

$$I_3(\tau) \le \frac{\Psi(c_6)}{\sqrt{2}\pi} e^{-x_0\tau} = \frac{\Psi(c_6)}{\sqrt{2}\pi} e^{-\tau|\beta|^{1/3}/(2c_2)}.$$

The same bound also holds for the integral $I_1(\tau)$, because $\Gamma_1 = \overline{\Gamma}_3$ and all estimates we used for the resolvent $(H_{n,\beta} - z)^{-1}$ are unchanged if z is replaced by the complex conjugate $\bar{z}$.

Summarizing we have shown that, if $|\beta| \ge 1$ and $|\beta|\tau \ge 1$, then

$$\|e^{-\tau H_{n,\beta}}\|_{Z\to Z} \;\le\; I_1(\tau) + I_2(\tau) + I_3(\tau) \;\le\; c_4|\beta|^{2/3}\, e^{-c_5|\beta|^{1/3}\tau}\,, \tag{3.8}$$

for some positive constants c_4, c_5. In fact, we can assume without loss of generality that $c_4 \ge e^{c_5}$, in which case inequality (3.8) also holds for $0 \le \tau < |\beta|^{-1}$ because we already know that $\|e^{-\tau H_{n,\beta}}\|_{Z\to Z} \le e^{-|n|\tau/2} \le 1$ for all $\tau \ge 0$. This proves estimate (3.2), and exactly the same argument gives (3.3) for $n = \pm 1$ if we restrict the operator $H_{n,\beta}$ to the subspace Z_0 where $-H_{n,\beta} + \frac{3}{2}$ is m-dissipative and inequality (2.19) holds. □

Remark 3.2. We emphasize that there is a lot of freedom in the choice of the abscissa x_0 in the proof of Proposition 3.1. In fact, the only real constraint is that the spectrum of $H_{n,\beta}$ be entirely contained in the half-plane $\{\mathrm{Re}(z) > x_0\}$. In view of estimates (2.18), (2.19) this is certainly the case if $x_0 = \kappa|\beta|^{1/3}/c_2$ for some $\kappa \in [0,1)$, in which case a slight modification of the argument above gives the bounds (3.2), (3.3) with $c_5 = \kappa/c_2$ and $c_4 > 0$ depending only on κ. However, we know from [29, Theorem 6.1] that all eigenvalues of the operator $H_{n,\beta}$ (restricted to Z_0 if $n = \pm 1$) have real parts larger than $c_3|\beta|^{1/2}$ if $|\beta| \ge 1$, see also Proposition 2.4. If we take x_0 such that $|\beta|^{1/3}/c_2 < x_0 < c_3|\beta|^{1/2}$, we obtain for $|n| \ge 2$ an estimate of the form

$$\|e^{-\tau H_{n,\beta}}\|_{Z\to Z} \;\le\; C(x_0, |\beta|)\, e^{-x_0\tau}\,, \qquad \tau \ge 0\,, \tag{3.9}$$

which is clearly superior to (3.2) for large times, as it implies that

$$\limsup_{\tau\to\infty} \frac{1}{\tau} \log \|e^{-\tau H_{n,\beta}}\|_{Z\to Z} \;\le\; -c_3|\beta|^{1/2}\,. \tag{3.10}$$

However, we have no control anymore on the constant $C(x_0, |\beta|)$, because the resolvent norm $\|(H_{n,\beta} - z)^{-1}\|$ can be extremely large on the vertical line $\{\mathrm{Re}(z) = x_0\}$ which meets the pseudospectrum of $H_{n,\beta}$, see [17, Lemma 1.3] for a detailed discussion. In the applications to nonlinear stability in Section 4, we need to control the size of the perturbations not only in the limit $\tau \to \infty$, but also for intermediate times, and this is why we cannot use estimate (3.9) for $x_0 > |\beta|^{1/3}/c_2$.

As a corollary of Proposition 3.1, we deduce the following decay estimate for the semigroup generated by the linearized operator $\mathcal{L} - \alpha\Lambda$ in the subspace $X_\perp = \ker(\Lambda)^\perp$.

Proposition 3.3. *For any $\alpha \in \mathbb{R}$ with $|\alpha| \ge 1$, we have*

$$\|e^{\tau(\mathcal{L}-\alpha\Lambda)}\|_{X_\perp\to X_\perp} \;\le\; \min\Bigl(e^{-\tau}\,,\; c_7|\alpha|^{2/3}\, e^{-c_5|\alpha|^{1/3}\tau}\Bigr)\,, \qquad \tau \ge 0\,, \tag{3.11}$$

where $c_7 = \max(c_4, e^2)$ and c_4, c_5 are as in Proposition 3.1.

Proof. According to (2.11), any $w \in X_\perp$ can be represented in polar coordinates as

$$w \;=\; \sum_{n\neq 0} w_n(r)\, e^{in\theta}\,, \qquad \|w\|_X^2 \;=\; 2\pi \sum_{n\neq 0} \|w_n\|_Z^2\,,$$

where $w_{\pm 1} \in Z_0$ and $w_n \in Z$ for $|n| \ge 2$. In particular, we have by definition

$$\|e^{\tau(\mathcal{L}-\alpha\Lambda)} w\|_X^2 \;=\; 2\pi \sum_{n\neq 0} \|e^{\tau(\mathcal{L}_n - \alpha\Lambda_n)} w_n\|_Z^2 \;=\; 2\pi \sum_{n\neq 0} \|e^{-\tau H_{n,n\alpha}} w_n\|_Z^2\,, \qquad \tau \ge 0\,.$$

In view of Proposition 3.1, to prove (3.11) we only need to verify that

$$\sup_{n\neq 0} \min\Bigl(e^{-\tau}\,,\, c_4|n|^{2/3}|\alpha|^{2/3}\,e^{-c_5|n|^{1/3}|\alpha|^{1/3}\tau}\Bigr) \,\le\, \min\Bigl(e^{-\tau}\,,\, c_7|\alpha|^{2/3}\,e^{-c_5|\alpha|^{1/3}\tau}\Bigr)\,, \tag{3.12}$$

for any $\tau \ge 0$ and any $\alpha \in \mathbb{R}$ with $|\alpha| \ge 1$.

Let $T = c_5|\alpha|^{1/3}\tau$. If $T \ge 2$, the quantity $|n|^{2/3}\,e^{-|n|^{1/3}T}$ reaches its maximum (over all nonzero integers) when $|n| = 1$, and (3.12) follows immediately since $c_7 \ge c_4$. If $0 \le T < 2$, we observe that

$$e^{-\tau} \,\le\, 1 \,\le\, |\alpha|^{2/3}\,e^{2-T} \,\le\, c_7|\alpha|^{2/3}\,e^{-c_5|\alpha|^{1/3}\tau}\,,$$

because $|\alpha| \ge 1$ and $c_7 \ge e^2$, hence (3.12) holds in that case too. □

Remark 3.4. It is also possible to establish (3.11) using the inverse Laplace formula for the semigroup $e^{\tau(\mathcal{L}-\alpha\Lambda)}$ in $X_\perp$ and the resolvent estimate given by Proposition 2.2. In that alternative approach, one needs to locate the numerical range of the operator $-\mathcal{L}+\alpha\Lambda$ in $X_\perp$, in order to choose an appropriate integration path. From (3.4) it is easy to deduce that

$$\mathcal{N}(-\mathcal{L}_\perp + \alpha\Lambda_\perp) \,\subset\, \bigl\{z \in \mathbb{C}\,\big|\, \mathrm{Re}(z) \ge 1\,,\, |\mathrm{Im}(z)| \le 2c_6|\alpha|\,\mathrm{Re}(z)\bigr\}\,,$$

but this "sectorial" estimate can be improved into a "parabolic" estimate as follows.

Lemma 3.5. *There exist positive constants c_8, c_9 such that*

$$\mathcal{N}(-\mathcal{L}_\perp + \alpha\Lambda_\perp) \,\subset\, \bigl\{z \in \mathbb{C}\,\big|\, \mathrm{Re}(z) \ge 1\,,\, |\mathrm{Im}(z)| \le |\alpha|(c_8\,\mathrm{Re}(z)^{1/2} + c_9)\bigr\}\,.$$

Proof. Let $w \in D(\mathcal{L}) \subset X$. If $w \in X_1$, inequality (5.12) in Lemma 5.1 below shows that

$$\langle w, \mathcal{L}w\rangle_X \,\le\, -\frac{1}{4}\|\nabla w\|_X^2 - \frac{1}{64}\|\xi w\|_X^2 - \frac{1}{8}\|w\|_X^2\,. \tag{3.13}$$

To bound the quantity $\langle w, \Lambda w\rangle_X$, we decompose $\Lambda = \Lambda_{\mathrm{ad}} + \Lambda_{\mathrm{nl}}$ as in (2.3). We first observe that

$$|\langle w, \Lambda_{\mathrm{ad}} w\rangle_X| \,=\, \Bigl|\int_{\mathbb{R}^2} G^{-1}\bar{w}\,v^G\cdot\nabla w\,\mathrm{d}\xi\Bigr| \,\le\, \|v^G\|_{L^\infty}\|w\|_X\|\nabla w\|_X\,. \tag{3.14}$$

On the other hand, since $\nabla G = -\frac{1}{2}\xi G$, we have

$$\langle w, \Lambda_{\mathrm{nl}} w\rangle_X \,=\, \int_{\mathbb{R}^2} G^{-1}\bar{w}\,(K_{BS}*w)\cdot\nabla G\,\mathrm{d}\xi \,=\, -\frac{1}{2}\int_{\mathbb{R}^2}\bar{w}\,(K_{BS}*w)\cdot\xi\,\mathrm{d}\xi\,.$$

In view of Lemma 5.2 below, we have $\|K_{BS}*w\|_{L^4} \le C\|w\|_{L^{4/3}}$, hence

$$|\langle w, \Lambda_{\mathrm{nl}} w\rangle_X| \,\le\, \frac{1}{2}\|K_{BS}*w\|_{L^4}\||\xi|w\|_{L^{4/3}} \,\le\, C\|w\|_{L^{4/3}}\||\xi|w\|_{L^{4/3}} \,\le\, C\|w\|_X^2\,. \tag{3.15}$$

Now, we fix $\alpha \in \mathbb{R}$, and we assume that $w \in D(\mathcal{L}) \cap X_\perp$ is normalized so that $\|w\|_X = 1$. We consider the complex number

$$z \,=\, \langle w, (-\mathcal{L}+\alpha\Lambda)w\rangle_X \,\in\, \mathcal{N}(-\mathcal{L}_\perp + \alpha\Lambda_\perp)\,,$$

which satisfies $\mathrm{Re}(z) = -\langle w, \mathcal{L}w\rangle_X \ge 1$ and $\mathrm{Im}(z) = -i\alpha\langle w, \Lambda w\rangle_X$. We know from (3.13), (3.14), (3.15) that

$$\mathrm{Re}(z) \ge \frac{1}{4}\|\nabla w\|_X^2 + \frac{1}{8}, \qquad |\mathrm{Im}(z)| \,\le\, C|\alpha|\,(\|\nabla w\|_X + 1)\,,$$

and this implies that $|\mathrm{Im}(z)| \le |\alpha|(c_8\,\mathrm{Re}(z)^{1/2} + c_9)$ for some $c_8, c_9 > 0$. □

The semigroup estimates given by Propositions 3.1 and 3.3 are not sufficient by themselves to prove Theorem 1.3, mainly because the nonlinear term in equation (2.1) involves a derivative. Using the inverse Laplace formula, it is possible to obtain accurate bounds on the first order derivative $\nabla e^{\tau(\mathcal{L}-\alpha\Lambda)}$ for large times, but the problem is that we also need an estimate for short times that does not blow up in the large circulation limit $|\alpha| \to \infty$. For the semigroup itself (without derivative), such an estimate was readily available thanks to the dissipativity properties of the generator $\mathcal{L} - \alpha\Lambda$. In a similar spirit, the following elementary result will allow us in Section 4 to control the nonlinearities for short times without loosing any power of the circulation parameter $|\alpha|$.

Lemma 3.6. *There exists a constant $C_0 > 0$ such that, for any $\alpha \in \mathbb{R}$, any $T > 0$, and any real-valued vector field $f \in C^0([0,T], X^2)$, we have the estimate*

$$\Big\|\int_0^\tau e^{(\tau-s)(\mathcal{L}-\alpha\Lambda)} \operatorname{div} f(s) \,\mathrm{d}s\Big\|_X^2 \,\le\, C_0 \int_0^\tau \|f(s)\|_X^2 \,\mathrm{d}s\,, \qquad \tau \in [0,T]\,. \tag{3.16}$$

Proof. We denote $w(\tau) = \int_0^\tau e^{(\tau-s)(\mathcal{L}-\alpha\Lambda)} \operatorname{div} f(s)\,\mathrm{d}s$ for $\tau \in [0,T]$. Then $w \in C^0([0,T], X)$ is the unique solution of the linear evolution equation

$$\partial_\tau w \,=\, (\mathcal{L} - \alpha\Lambda)w + \operatorname{div} f\,, \qquad \tau \in [0,T]\,, \tag{3.17}$$

with initial data $w(0) = 0$. Moreover it is clear that $w(\tau) \in X_0$ for any $\tau \in [0,T]$, because the source term in (3.17) has zero mean over $\mathbb{R}^2$. A direct calculation, using the skew-symmetry of the operator Λ in X and the assumption that f is real-valued, shows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\|w(\tau)\|_X^2 \,=\, \langle w, \mathcal{L}w\rangle_X + \langle w, \operatorname{div} f\rangle_X\,. \tag{3.18}$$

As $w \in X_0$, we know from Lemma 5.1 that

$$\langle w, \mathcal{L}w\rangle_X \,\le\, -\frac{1}{6}\|\nabla w\|_X^2 - \frac{1}{96}\|\xi w\|_X^2 - \frac{1}{12}\|w\|_X^2\,. \tag{3.19}$$

On the other hand, integrating by parts and using the fact that $\nabla G^{-1} = \frac{1}{2}\xi G^{-1}$, we obtain

$$\langle w, \operatorname{div} f\rangle_X \,=\, -\int_{\mathbb{R}^2} G^{-1} f \cdot \Big(\nabla w + \frac{\xi}{2}w\Big)\,\mathrm{d}\xi \,\le\, \epsilon\big(\|\nabla w\|_X^2 + \|\xi w\|_X^2\big) + \frac{C}{\epsilon}\,\|f\|_X^2\,. \tag{3.20}$$

If we take $\epsilon > 0$ small enough, we can combine (3.18), (3.19) and (3.20) to obtain the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\|w(\tau)\|_X^2 \,\le\, -\frac{1}{6}\|w(\tau)\|_X^2 + C_0\|f(\tau)\|_X^2 \,\le\, C_0\|f(\tau)\|_X^2\,, \qquad \tau \in [0,T]\,,$$

for some positive constant C_0, and (3.16) follows upon integrating over τ. □

4 Nonlinear stability and relaxation to axisymmetry

Equipped with the semigroups estimates derived in Section 3, we now study the nonlinear stability of the equilibrium $w = \alpha G$ for the rescaled vorticity equation (1.7). We assume that the circulation parameter $\alpha \in \mathbb{R}$ satisfies $|\alpha| \ge \alpha_0$, where $\alpha_0 > 0$ is large enough and will be determined later. Given any $T > 0$, we consider a solution $w \in C^0([0,T], X)$ of (1.7) with initial data $w_0 = \alpha G + \tilde{w}_0$, where $\tilde{w}_0 \in X$ satisfies $\|\tilde{w}_0\|_X \le C_3|\alpha|$ for some small constant $C_3 > 0$.

4.1 Preliminaries

We start with two elementary observations which allow us to concentrate on the situation where the initial perturbation $\tilde{w}_0$ belongs to the subspace X_1 defined by (1.10).

Observation 1 : Without loss of generality, we can assume that $\tilde{w}_0 \in X_0$, where $X_0 \subset X$ is the subspace defined in (1.9). Indeed, if $\tilde{\alpha} = \int_{\mathbb{R}^2} \tilde{w}_0 \,\mathrm{d}\xi \neq 0$, we decompose

$$w_0 \,=\, \hat{\alpha} G + \hat{w}_0\,, \qquad \text{where} \quad \hat{\alpha} \,=\, \alpha + \tilde{\alpha} \quad \text{and} \quad \hat{w}_0 \,=\, \tilde{w}_0 - \tilde{\alpha} G\,.$$

Then $\hat{w}_0 \in X_0$ by construction, and since $|\tilde{\alpha}| \le C\|\tilde{w}_0\|_X \le CC_3|\alpha|$ we can assume that $|\hat{\alpha}| \ge |\alpha|/2 \ge \alpha_0/2$ by taking the constant C_3 sufficiently small. The problem is thus reduced to the stability analysis of the modified vortex $\hat{\alpha}G$ with respect to perturbations $\hat{w}_0 \in X_0$, and we still have $\|\hat{w}_0\|_X \le C_3'|\hat{\alpha}|$ for some small constant C_3'. Of course, the subspace X_0 is invariant under the evolution defined by the perturbation equation (2.1).

Observation 2 : Without loss of generality, we can further assume that $\tilde{w}_0 \in X_1$, where $X_1 \subset X$ is defined in (1.10). Indeed, let w be the solution of (1.7) with initial data $w_0 = \alpha G + \tilde{w}_0$, where $\alpha \neq 0$ and $\tilde{w}_0 \in X_0$ satisfies $\|\tilde{w}_0\|_X \le C_3|\alpha|$. If $\tilde{w}_0 \notin X_1$, we introduce the first order moment

$$\eta \,=\, \frac{1}{\alpha}\int_{\mathbb{R}^2} \xi\, w_0(\xi)\,\mathrm{d}\xi \,=\, \frac{1}{\alpha}\int_{\mathbb{R}^2} \xi\, \tilde{w}_0(\xi)\,\mathrm{d}\xi \,\in\, \mathbb{R}^2\,,$$

and we consider the modified vorticity $\hat{w}$ and velocity $\hat{v}$ defined by

$$\hat{w}(\xi,\tau) \,=\, w(\xi + \eta\, e^{-\tau/2}, \tau)\,, \qquad \hat{v}(\xi,\tau) \,=\, v(\xi + \eta\, e^{-\tau/2}, \tau)\,. \tag{4.1}$$

It is straightforward to verify that the new functions $\hat{w}$, $\hat{v}$ satisfy the same equation (1.7), namely $\partial_\tau \hat{w} + \hat{v}\cdot\nabla\hat{w} = \mathcal{L}\hat{w}$. In addition, the explicit expression

$$\hat{w}(\xi,0) - \alpha G(\xi) \,=\, \alpha\big(G(\xi+\eta) - G(\xi)\big) + \tilde{w}_0(\xi+\eta)\,, \qquad \xi \in \mathbb{R}^2\,,$$

reveals that $\hat{w}(\cdot,0) - \alpha G \in X_1$ and $\|\hat{w}(\cdot,0) - \alpha G\|_X \le C\|\tilde{w}_0\|_X \le CC_3|\alpha|$. Thus the change of variables (4.1) allows us to reduce the stability analysis to perturbations in the subspace X_1. In terms of the original variables, that transformation is equivalent to choosing the parameter x_0 in (1.5) to be the *center of vorticity* of the distribution $\omega(\cdot,t)$, which is well defined as soon as $\alpha \neq 0$ and preserved under the evolution defined by (1.1). This is why, if $\tilde{w}_0 \in X_1$, the solution of (2.1) satisfies $\tilde{w}(\tau) \in X_1$ for all $\tau \ge 0$.

4.2 Decomposition of the perturbations

Taking into account the observations above, we assume henceforth that $\tilde{w} \in C^0([0,T], X_1)$ is a solution of (2.1) with initial data $\tilde{w}_0 \in X_1$ satisfying $\|\tilde{w}_0\|_X \le C_3|\alpha|$. According to the discussion in Section 2.1, we have the orthogonal decomposition $X_1 = X_r \oplus X_\perp$, where $X_r = X_0 \cap Y_0$ is the subset of X consisting of all radially symmetric functions with zero average, and $X_\perp = \ker(\Lambda)^\perp$ is the orthogonal complement of $\ker(\Lambda)$ in X. We thus decompose the perturbed vorticity as

$$\tilde{w} \,=\, \tilde{w}_r + \tilde{w}_\perp \,=\, P_r\tilde{w} + P_\perp\tilde{w}\,, \tag{4.2}$$

where $P_r = 1 - P_\perp$ is the orthogonal projection of X_1 onto X_r. If we introduce polar coordinates (r,θ) such that $\xi = (r\cos\theta, r\sin\theta)$, we have the explicit expression

$$(P_r\tilde{w})(r) \,=\, \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{w}(r,\theta)\,\mathrm{d}\theta\,, \qquad r > 0\,. \tag{4.3}$$

The corresponding decomposition of the velocity is

$$\tilde{v} \;=\; \tilde{v}_r + \tilde{v}_\perp \;=\; K_{BS} * \tilde{w}_r + K_{BS} * \tilde{w}_\perp\,. \tag{4.4}$$

Denoting $e_r = (\cos\theta, \sin\theta)$, $e_\theta = (-\sin\theta, \cos\theta)$, we have the following elementary result:

Lemma 4.1. *With the definitions* (4.2), (4.4), *we have* $\tilde{v}_r \cdot \nabla \tilde{w}_r = 0$ *and*

$$P_r(\tilde{v} \cdot \nabla \tilde{w}) = P_r(\tilde{v}_\perp \cdot \nabla \tilde{w}_\perp) = \mathrm{div}\, Z[\tilde{w}, \tilde{v}]\,, \tag{4.5}$$

where Z is the vector field defined by

$$Z[\tilde{w}, \tilde{v}] \;=\; P_r\big((\tilde{v}_\perp \cdot e_r)\tilde{w}_\perp\big) e_r\,. \tag{4.6}$$

Proof. Since $\tilde{w}_r$ is radially symmetric, the associated velocity $\tilde{v}_r = K_{BS} * \tilde{w}_r$ is purely azimuthal, namely $\tilde{v}_r \cdot e_r = 0$, and this implies that $\tilde{v}_r \cdot \nabla \tilde{w}_r = 0$. On the other hand, as $\tilde{v}$ is divergence free, we have

$$\tilde{v} \cdot \nabla \tilde{w} \;=\; \mathrm{div}\big(\tilde{v}\tilde{w}\big) \;=\; \frac{1}{r}\partial_r\big(r(\tilde{v} \cdot e_r)\tilde{w}\big) + \frac{1}{r}\partial_\theta\big((\tilde{v} \cdot e_\theta)\tilde{w}\big)\,.$$

If we apply the projection P_r, the last term in the right-hand side gives no contribution, and in the first term we have $\tilde{v} \cdot e_r = \tilde{v}_\perp \cdot e_r$, as already observed. That quantity has zero average over the angular variable θ, and this implies that $P_r\big((\tilde{v}_\perp \cdot e_r)\tilde{w}\big) = P_r\big((\tilde{v}_\perp \cdot e_r)\tilde{w}_\perp\big)$. Summarizing, we have shown that (4.5) holds if Z is defined by (4.6). □

In view of Lemma 4.1, the perturbation equation (2.1) is equivalent to the coupled system

$$\begin{aligned} \partial_\tau \tilde{w}_r + P_r(\tilde{v}_\perp \cdot \nabla \tilde{w}_\perp) \;&=\; \mathcal{L}\tilde{w}_r\,, \\ \partial_\tau \tilde{w}_\perp + \tilde{v}_r \cdot \nabla \tilde{w}_\perp + \tilde{v}_\perp \cdot \nabla \tilde{w}_r + P_\perp(\tilde{v}_\perp \cdot \nabla \tilde{w}_\perp) \;&=\; (\mathcal{L} - \alpha\Lambda)\tilde{w}_\perp\,, \end{aligned} \tag{4.7}$$

which is the starting point of our analysis. The integrated version of (4.7) is written in the form

$$\begin{aligned} \tilde{w}_r(\tau) \;&=\; e^{\tau\mathcal{L}}\tilde{w}_r(0) - \int_0^\tau e^{(\tau-s)\mathcal{L}}\, \mathrm{div}\, Z[\tilde{w}(s), \tilde{v}(s)]\,\mathrm{d}s\,, \\ \tilde{w}_\perp(\tau) \;&=\; e^{\tau(\mathcal{L}-\alpha\Lambda)}\tilde{w}_\perp(0) - \int_0^\tau e^{(\tau-s)(\mathcal{L}-\alpha\Lambda)}\, \mathrm{div}\, N[\tilde{w}(s), \tilde{v}(s)]\,\mathrm{d}s\,, \end{aligned} \tag{4.8}$$

where

$$N[\tilde{w}, \tilde{v}] \;=\; \tilde{v}_r\, \tilde{w}_\perp + \tilde{v}_\perp\, \tilde{w}_r + \tilde{v}_\perp\, \tilde{w}_\perp - Z[\tilde{w}, \tilde{v}]\,. \tag{4.9}$$

According to Proposition 3.3, the semigroups in (4.8) satisfy, for all $\tau \ge 0$,

$$\|e^{\tau\mathcal{L}}\tilde{w}_r\|_X \;\le\; e^{-\tau}\|\tilde{w}_r\|_X\,, \qquad \|e^{\tau(\mathcal{L}-\alpha\Lambda)}\tilde{w}_\perp\|_X \;\le\; \min(e^{-\tau}, e^{-\mu(\tau-\tau_0)})\|\tilde{w}_\perp\|_X\,, \tag{4.10}$$

where

$$\mu \;=\; c_5|\alpha|^{1/3}\,, \qquad \tau_0 \;=\; \frac{\log(c_7|\alpha|^{2/3})}{\mu}\,. \tag{4.11}$$

In what follows we assume that the constant $\alpha_0 \ge 2$ is large enough so that, if $|\alpha| \ge \alpha_0$, the quantities defined in (4.11) satisfy $\mu \ge 1$, $0 < \tau_0 \le 1$, and $\mu\tau_0 \ge 1 + \log(2)$.

4.3 Nonlinear estimates

Keeping the same notations as in the previous section, our goal is to control the solution of (4.8) using the following norm

$$\mathcal{M} \,=\, \mathcal{M}(T) \,=\, \sup_{0\le\tau\le T}\Bigl(\|\tilde w_r(\tau)\|_X + e^{\tau/\tau_0}\|\tilde w_\perp(\tau)\|_X\Bigr)\,. \tag{4.12}$$

The main technical result of this section is :

Lemma 4.2. *There exist positive constants C_4, C_5 such that, for any $\alpha\in\mathbb{R}$ with $|\alpha|\ge\alpha_0$, any $T>0$, and any solution $(\tilde w_r,\tilde w_\perp)\in C^0([0,T],X_r\oplus X_\perp)$ of (4.8) satisfying $\mathcal{M}\le|\alpha|$, the following estimate holds :*

$$\mathcal{M} \,\le\, C_4\|\tilde w_0\|_X + C_5\bigl(\tau_0\log|\alpha|\bigr)^{1/2}\mathcal{M}^2 + C_5\bigl(\tau_0\log_+\mathcal{M}^{-1}\bigr)^{1/2}\mathcal{M}^2\,, \tag{4.13}$$

where $\tilde w_0=\tilde w_r(0)+\tilde w_\perp(0)$. Here $\log_+(x)=\max\bigl(\log(x),0\bigr)$ for any $x>0$.

Proof. We first establish a simple a priori estimate that will be useful later. By construction, the function $w(\tau)=\alpha G+\tilde w_r(\tau)+\tilde w_\perp(\tau)$ is a solution of the rescaled vorticity equation (1.7), hence of the integral equation (5.1) below. We also know that $\|w(\tau)\|_X\le C|\alpha|$ for all $\tau\in[0,T]$, because $\mathcal{M}\le|\alpha|$ by assumption. Our goal is to control the norm $\|G^{-1/2}w(\tau)\|_{L^3}$ for any $\tau\in[0,T]$ using the representation (5.1) and the semigroup bounds recalled in Section 5.1. Applying estimate (5.4) with $q=3$, $p=2$, inequality (5.5) with $q=3$, $p=4/3$, and finally estimate (5.6) with $w_1=w_2=w$, we obtain for any $\tau\in(0,T]$:

$$\begin{aligned}\|G^{-1/2}w(\tau)\|_{L^3} \,&\le\, \frac{C}{a(\tau)^{1/6}}\,\|w(0)\|_X + \int_0^\tau \frac{Ce^{-(\tau-s)/2}}{a(\tau-s)^{11/12}}\,\|G^{-1/2}v(s)w(s)\|_{L^{4/3}}\,\mathrm{d}s\,,\\ &\le\, \frac{C}{a(\tau)^{1/6}}\,\|w(0)\|_X + \int_0^\tau \frac{Ce^{-(\tau-s)/2}}{a(\tau-s)^{11/12}}\,\|w(s)\|_X^2\,\mathrm{d}s\,,\end{aligned}$$

where $a(\tau)=1-e^{-\tau}$. As $|\alpha|\ge 1$, it readily follows that

$$\|G^{-1/2}\tilde w_r(\tau)\|_{L^3} + \|G^{-1/2}\tilde w_\perp(\tau)\|_{L^3} \,\le\, \frac{C_6|\alpha|^2}{a(\tau)^{1/6}}\,, \qquad 0<\tau\le T\,, \tag{4.14}$$

for some positive constant C_6.

We now focus on the proof of (4.13). The overall strategy is to estimate the right-hand side of system (4.8) using the semigroup bounds (4.10) and the integral estimate (3.16). We start with the equation satisfied by the radially symmetric component $\tilde w_r$, which is simpler to handle. We know that $\|e^{\tau\mathcal{L}}\tilde w_r(0)\|_X\le\|\tilde w_r(0)\|_X$, and using Lemma 3.6 together with definition (4.6) we obtain

$$\begin{aligned}\Bigl\|\int_0^\tau e^{(\tau-s)\mathcal{L}}\,\mathrm{div}\,Z[\tilde w(s),\tilde v(s)]\,\mathrm{d}s\Bigr\|_X^2 \,&\le\, C_0\int_0^\tau \|\tilde w_\perp(s)\tilde v_\perp(s)\|_X^2\,\mathrm{d}s\\ &\le\, C_0\int_0^\tau \|\tilde w_\perp(s)\|_X^2\,\|\tilde v_\perp(s)\|_{L^\infty}^2\,\mathrm{d}s\,.\end{aligned}$$

To control the L^∞ norm of the velocity field $\tilde v_\perp(s)$, we apply the results of Section 5.2. Since $\|\tilde w_\perp(s)\|_{L^1\cap L^2}\le C\|\tilde w_\perp(s)\|_X$ and $\|\tilde w_\perp(s)\|_X\le\mathcal{M}$, it follows from Lemma 5.5 that

$$\begin{aligned}\|\tilde v_\perp(s)\|_{L^\infty}^2 \,&\le\, C\|\tilde w_\perp(s)\|_{L^1\cap L^2}^2\Bigl(1+\log_+\frac{\|\tilde w_\perp(s)\|_{L^3}}{\|\tilde w_\perp(s)\|_{L^1\cap L^2}}\Bigr)\\ &\le\, C\,\mathcal{M}^2\Bigl(1+\log_+\frac{\|\tilde w_\perp(s)\|_{L^3}}{\mathcal{M}}\Bigr) \,\le\, C\mathcal{M}^2\Bigl(1+\log_+\frac{C_6|\alpha|^2}{a(s)^{1/6}\mathcal{M}}\Bigr)\,,\end{aligned}$$

where in the second inequality we used the fact that the map $a \mapsto a^2\big(1 + \log_+(b/a)\big)$ is strictly increasing over $\mathbb{R}_+$ for any $b > 0$, and in the last inequality we invoked the a priori estimate (4.14). Observing that $\log_+(ab) \le \log_+(a) + \log_+(b)$ and $\|\tilde{w}_\perp(s)\|_X \le \mathcal{M}e^{-s/\tau_0}$, we obtain after integrating over time

$$\int_0^\tau \|\tilde{w}_\perp(s)\|_X^2 \, \|\tilde{v}_\perp(s)\|_{L^\infty}^2 \, \mathrm{d}s \,\le\, C\mathcal{M}^4\tau_0\Big(1 + \log\frac{|\alpha|}{\tau_0} + \log_+\frac{1}{\mathcal{M}}\Big)\,,$$

where $\log(|\alpha|/\tau_0) \le C\log|\alpha|$ in view of definition (4.11). Altogether we have shown that

$$\sup_{0\le\tau\le T} \|\tilde{w}_r(\tau)\|_X \,\le\, \|\tilde{w}_r(0)\|_X + C\big(\tau_0\log|\alpha|\big)^{1/2}\mathcal{M}^2 + C\big(\tau_0\log_+\mathcal{M}^{-1}\big)^{1/2}\mathcal{M}^2\,. \tag{4.15}$$

We next consider the second equation in (4.8). When $\tau \le \tau_0$ the spatial weight e^{τ/τ_0} does not play any role in definition (4.12), so repeating the arguments above we obtain an estimate of the form (4.15) for $\|\tilde{w}_\perp(\tau)\|_X$ if $\tau \in [0,\tau_0]$. In the rest of the proof, we thus assume that $\tau > \tau_0$, and we decompose $\tau = N\tau_0 + \tau_1$ where $N \in \mathbb{N}$ and $\tau_0 < \tau_1 \le 2\tau_0$. Since $\mu\tau_0 \ge 1$ by assumption, we deduce from (4.10) that

$$\|e^{\tau(\mathcal{L}-\alpha\Lambda)}\tilde{w}_\perp(0)\|_X \,\le\, e^{-\mu(\tau-\tau_0)}\|\tilde{w}_\perp(0)\|_X \,\le\, e^{1-\tau/\tau_0}\|\tilde{w}_\perp(0)\|_X\,. \tag{4.16}$$

Moreover, the integral in the second line of (4.8) can be decomposed in the following way

$$\int_0^\tau e^{(\tau-s)(\mathcal{L}-\alpha\Lambda)}\,\mathrm{div}\,N[\tilde{w}(s),\tilde{v}(s)]\,\mathrm{d}s \,=\, I_0(\tau) + \sum_{k=1}^N J_k(\tau)\,, \tag{4.17}$$

where

$$\begin{aligned} I_0(\tau) \,&=\, \int_{N\tau_0}^\tau e^{(\tau-s)(\mathcal{L}-\alpha\Lambda)}\,\mathrm{div}\,N[\tilde{w}(s),\tilde{v}(s)]\,\mathrm{d}s\,,\\ J_k(\tau) \,&=\, e^{(\tau-k\tau_0)(\mathcal{L}-\alpha\Lambda)}\int_{(k-1)\tau_0}^{k\tau_0} e^{(k\tau_0-s)(\mathcal{L}-\alpha\Lambda)}\,\mathrm{div}\,N[\tilde{w}(s),\tilde{v}(s)]\,\mathrm{d}s\,. \end{aligned}$$

To control the nonlinear term $N[\tilde{w},\tilde{v}]$ defined in (4.9), we again apply the results of Section 5.2. We first observe that $\|\tilde{v}_r\,\tilde{w}_\perp + \tilde{v}_\perp\,\tilde{w}_\perp - Z[\tilde{w},\tilde{v}]\|_X \le C\big(\|\tilde{v}_r\|_{L^\infty} + \|\tilde{v}_\perp\|_{L^\infty}\big)\|w_\perp\|_X$, and we use Lemma 5.5 to obtain

$$\begin{aligned} \|\tilde{v}_r\|_{L^\infty}^2 \,&\le\, C\,\|\tilde{w}_r\|_X^2\Big(1 + \log_+\frac{\|\tilde{w}_r\|_{L^3}}{\|\tilde{w}_r\|_X}\Big) \,\le\, C\mathcal{M}^2\Big(1 + \log_+\frac{\|\tilde{w}_r\|_{L^3}}{\mathcal{M}}\Big)\,,\\ \|\tilde{v}_\perp\|_{L^\infty}^2 \,&\le\, C\,\|\tilde{w}_\perp\|_X^2\Big(1 + \log_+\frac{\|\tilde{w}_\perp\|_{L^3}}{\|\tilde{w}_\perp\|_X}\Big) \,\le\, C\mathcal{M}^2\Big(1 + \log_+\frac{\|\tilde{w}_\perp\|_{L^3}}{\mathcal{M}}\Big)\,. \end{aligned}$$

The last term $\tilde{v}_\perp\tilde{w}_r$ in $N[\tilde{w},\tilde{v}]$ is estimated directly by applying Lemma 5.6 with $\omega_1 = G^{-1/2}\tilde{w}_r$ and $\omega_2 = \tilde{w}_\perp$. This gives

$$\begin{aligned} \|\tilde{v}_\perp\tilde{w}_r\|_X^2 \,&\le\, C\,\|\tilde{w}_r\|_X^2\,\|\tilde{w}_\perp\|_X^2\Big(1 + \log_+\frac{\|G^{-1/2}\tilde{w}_r\|_{L^3}}{\|\tilde{w}_r\|_X}\Big)\\ &\le\, C\mathcal{M}^2\,\|\tilde{w}_\perp\|_X^2\Big(1 + \log_+\frac{\|G^{-1/2}\tilde{w}_r\|_{L^3}}{\mathcal{M}}\Big)\,. \end{aligned}$$

Summarizing, and using the a priori bounds (4.14) and $\|\tilde{w}_\perp(s)\|_X \le \mathcal{M}e^{-s/\tau_0}$, we arrive at

$$\|N[\tilde{w}(s),\tilde{v}(s)]\|_X^2 \,\le\, C\mathcal{M}^4\,e^{-2s/\tau_0}\Big(1 + \log_+\frac{C_6|\alpha|^2}{a(s)^{1/6}\mathcal{M}}\Big)\,, \qquad 0 < s \le T\,. \tag{4.18}$$

We now estimate the various terms in (4.17). Using (4.18) and Lemma 3.6, we first obtain

$$\|I_0(\tau)\|_X^2 \,\le\, C_0 \int_{N\tau_0}^{\tau} \|N[\tilde w(s),\tilde v(s)]\|_X^2 \,\mathrm{d}s \,\le\, C\mathcal{M}^4 \tau_0\, e^{-2N}\Bigl(1+\log\frac{|\alpha|}{\tau_0}+\log_+\frac{1}{\mathcal{M}}\Bigr)\,.$$

Similarly, for $k\in\{1,\dots,N\}$, we find

$$\begin{aligned}\|J_k(\tau)\|_X^2 \,&\le\, C_0\, e^{-2\mu(\tau-(k+1)\tau_0)} \int_{(k-1)\tau_0}^{k\tau_0} \|N[\tilde w(s),\tilde v(s)]\|_X^2\,\mathrm{d}s \\ &\le\, C\mathcal{M}^4\tau_0\, e^{-2\mu(\tau-(k+1)\tau_0)}\, e^{-2k}\Bigl(1+\log\frac{|\alpha|}{\tau_0}+\log_+\frac{1}{\mathcal{M}}\Bigr)\,.\end{aligned}$$

We know that $e^{-N}\le e^{2-\tau/\tau_0}$ by definition of N, and since $\mu\tau_0-1\ge\log(2)$ by assumption we also have

$$e^{-\mu(\tau-\tau_0)}\sum_{k=1}^{N} e^{k(\mu\tau_0-1)} \,\le\, 2\,e^{-\mu(\tau-\tau_0)}\,e^{N(\mu\tau_0-1)} \,\le\, 2\,e^{-N} \,\le\, 2\,e^{2-\tau/\tau_0}\,.$$

Therefore the estimates above imply that

$$\|I_0(\tau)\|_X + \sum_{k=1}^{N}\|J_k(\tau)\|_X \,\le\, C\mathcal{M}^2\tau_0^{1/2}\,e^{-\tau/\tau_0}\Bigl(1+\log\frac{|\alpha|}{\tau_0}+\log_+\frac{1}{\mathcal{M}}\Bigr)^{1/2}\,. \tag{4.19}$$

Combining (4.16), (4.17), and (4.19), we thus obtain

$$\sup_{0\le\tau\le T} e^{\tau/\tau_0}\|\tilde w_\perp(\tau)\|_X \,\le\, e\|\tilde w_\perp(0)\|_X + C\bigl(\tau_0\log|\alpha|\bigr)^{1/2}\mathcal{M}^2 + C\bigl(\tau_0\log_+\mathcal{M}^{-1}\bigr)^{1/2}\mathcal{M}^2\,. \tag{4.20}$$

Estimate (4.13) is now a direct consequence of (4.15) and (4.20). □

Remark 4.3. Using the definition of τ_0 in (4.11), we see that, if $|\alpha|$ is large enough, estimate (4.13) can be written in the alternative form

$$\mathcal{M}(T) \,\le\, C_4\|\tilde w_0\|_X + \frac{C_7\mathcal{M}(T)^2}{|\alpha|^{1/6}}\Bigl(\log|\alpha|+\log_+\frac{1}{\mathcal{M}(T)}\Bigr)\,, \tag{4.21}$$

for some universal constants $C_4\ge 1$ and $C_7>0$.

4.4 End of the proof of Theorem 1.3.

Let $C_3>0$ be a small constant, that we fix once and for all as discussed in Section 4.1, and let C_4, C_7 be the constants involved in inequality (4.21). We choose $C_1>0$ such that $8C_1C_4C_7<1$, and we fix $\alpha_0\ge 2$ large enough so that, whenever $|\alpha|\ge\alpha_0$:

i) The quantities μ and τ_0 defined in (4.11) satisfy $\mu\ge 1$, $0<\tau_0\le 1$, and $\mu\tau_0\ge 1+\log(2)$;

ii) Estimate (4.21) in Remark 4.3 is valid;

iii) The following inequalities hold : $C_1|\alpha|^{1/6}\le C_3|\alpha|\log|\alpha|$, and $4C_7\le|\alpha|^{1/6}$.

Given $\alpha\in\mathbb{R}$ with $|\alpha|\ge\alpha_0$, we consider initial perturbations $\tilde w_0\in X_1$ such that

$$\|\tilde w_0\|_X \,\le\, \frac{C_1|\alpha|^{1/6}}{\log|\alpha|}\,, \qquad \text{hence also} \quad \|\tilde w_0\|_X \,\le\, C_3|\alpha|\,. \tag{4.22}$$

By continuity, the solution of (2.1) with initial data $\tilde{w}_0$ satisfies, for $T > 0$ sufficiently small,

$$\mathcal{M}(T) \le 2C_4\|\tilde{w}_0\|_X < \frac{1}{4C_7}\frac{|\alpha|^{1/6}}{\log|\alpha|}, \tag{4.23}$$

where $\mathcal{M}(T)$ is defined in (4.12). But as long as (4.23) holds, we have by construction

$$\frac{C_7\mathcal{M}(T)^2}{|\alpha|^{1/6}}\log|\alpha| < \frac{\mathcal{M}(T)}{4}, \qquad \text{and} \qquad \frac{C_7\mathcal{M}(T)^2}{|\alpha|^{1/6}}\log_+\frac{1}{\mathcal{M}(T)} \le \frac{C_7\mathcal{M}(T)}{|\alpha|^{1/6}} \le \frac{\mathcal{M}(T)}{4},$$

and inequality (4.21) then shows that $\mathcal{M}(T) < C_4\|\tilde{w}_0\|_X + \mathcal{M}(T)/2$, which in turn implies that $\mathcal{M}(T) < 2C_4\|\tilde{w}_0\|_X$. So we conclude that (4.23) holds for any $T > 0$, namely

$$\sup_{\tau\ge 0}\Big(\|\tilde{w}_r(\tau)\|_X + e^{\tau/\tau_0}\|\tilde{w}_\perp(\tau)\|_X\Big) \le 2C_4\|\tilde{w}_0\|_X. \tag{4.24}$$

As τ_0 is given by (4.11) and $\tilde{w}_\perp = (1-P_r)\tilde{w} = (1-P_r)(w-\alpha G)$, we see that (4.24) implies in particular (1.14). To establish (1.13) for $\tau \ge 1$, we observe that the integral equation (4.8) satisfied by the radially symmetric component $\tilde{w}_r$ can be written in the form

$$\tilde{w}_r(\tau) = e^{\tau\mathcal{L}}\tilde{w}_r(0) - \int_0^\tau e^{-\frac{1}{2}(\tau-s)}\,\mathrm{div}\Big(e^{(\tau-s)\mathcal{L}}Z[\tilde{w}(s),\tilde{v}(s)]\Big)\,\mathrm{d}s, \tag{4.25}$$

see (5.1) below. As $\tilde{w}_0 \in X_1$, we have $\|e^{\tau\mathcal{L}}\tilde{w}_r(0)\|_X \le e^{-\tau}\|\tilde{w}_r(0)\|_X$ for all $\tau \ge 0$. Moreover, both components Z_1, Z_2 of the vector field $Z[\tilde{w},\tilde{v}]$ belong to the subspace X_0, on which the semigroup $e^{\tau\mathcal{L}}$ decays like $e^{-\tau/2}$ since $\mathcal{L} \le -1/2$. Thus using [20, estimate (2.10)] and inequality (5.6) below, we obtain the bound

$$\begin{aligned}\Big\|\mathrm{div}\Big(e^{(\tau-s)\mathcal{L}}Z[\tilde{w}(s),\tilde{v}(s)]\Big)\Big\|_X &\le C\,\frac{e^{-\frac{1}{2}(\tau-s)}}{a(\tau-s)^{3/4}}\,\|G^{-1/2}\tilde{v}_\perp(s)\tilde{w}_\perp(s)\|_{L^{4/3}} \\ &\le C\,\frac{e^{-\frac{1}{2}(\tau-s)}}{a(\tau-s)^{3/4}}\,\|\tilde{w}_\perp(s)\|_X^2, \end{aligned}\tag{4.26}$$

for $0 < s < \tau$, where $a(\tau) = 1 - e^{-\tau}$. Estimating the right-hand side of (4.25) for $\tau \ge 1$ with the help of (4.26) and (4.24), we easily find

$$\begin{aligned}\|\tilde{w}_r(\tau)\|_X &\le Ce^{-\tau}\|\tilde{w}_0\|_X + C\|\tilde{w}_0\|_X^2\int_0^\tau \frac{e^{-(\tau-s)}}{a(\tau-s)^{3/4}}\,e^{-2s/\tau_0}\,\mathrm{d}s \\ &\le Ce^{-\tau}\Big(\|\tilde{w}_0\|_X + \tau_0\|\tilde{w}_0\|_X^2\Big) \le Ce^{-\tau}\|\tilde{w}_0\|_X,\end{aligned}$$

where in the last inequality we used the fact that $\tau_0\|\tilde{w}_0\|_X$ is uniformly bounded if τ_0 is defined by (4.11) and $\tilde{w}_0$ satisfies (4.22). Together with (4.24), this concludes the proof of (1.13), hence of Theorem 1.3 when $|\alpha| \ge \alpha_0$. In the case where $|\alpha| < \alpha_0$, we already observed that Theorem 1.3 follows from Proposition 1.2. □

5 Appendix

In this final section, we collect for easy reference a few basic results concerning the rescaled vorticity equation (1.7) and the two-dimensional Biot-Savart law (1.2). In particular we give a short proof of Propositions 1.1 and 1.2.

5.1 Well-posedness and a priori estimates

We first recall that the Cauchy problem for the rescaled vorticity equation (1.7) is globally well-posed in the weighted space $X = L^2(\mathbb{R}^2, G^{-1}\,\mathrm{d}\xi)$ equipped with the scalar product (1.8). Indeed, the integral equation associated with (1.7) reads

$$\begin{aligned} w(\tau) &= e^{\tau\mathcal{L}}w_0 - \int_0^\tau e^{(\tau-s)\mathcal{L}}\,\mathrm{div}\big(v(s)w(s)\big)\,\mathrm{d}s \\ &= e^{\tau\mathcal{L}}w_0 - \int_0^\tau e^{-\frac{1}{2}(\tau-s)}\,\mathrm{div}\big(e^{(\tau-s)\mathcal{L}}v(s)w(s)\big)\,\mathrm{d}s\,, \qquad \tau \ge 0\,, \end{aligned} \tag{5.1}$$

where $\mathcal{L}$ is the differential operator (2.2) and $v(s) = K_{BS} * w(s)$ is the velocity field associated with the vorticity $w(s)$. In (5.1) we used the identity $\partial_i\, e^{\tau\mathcal{L}} = e^{\tau(\mathcal{L}+\frac{1}{2})}\partial_i$ for $i = 1, 2$, which itself follows from its infinitesimal version $\partial_i\mathcal{L} = (\mathcal{L} + \frac{1}{2})\partial_i$.

There is an explicit expression for the semigroup $e^{\tau\mathcal{L}}$ generated by $\mathcal{L}$, which can be found for instance in [22, Appendix A] :

$$\big(e^{\tau\mathcal{L}}w\big)(\xi) \,=\, \frac{1}{4\pi a(\tau)}\int_{\mathbb{R}^2} e^{-\frac{1}{4a(\tau)}|\xi-\eta e^{-\tau/2}|^2}\, w(\eta)\,\mathrm{d}\eta\,, \qquad \xi \in \mathbb{R}^2\,,\ \tau > 0\,, \tag{5.2}$$

where $a(\tau) = 1-e^{-\tau}$. Since we work in the space X with Gaussian weight $G^{-1/2}(\xi) = \sqrt{4\pi}\, e^{|\xi|^2/8}$, it is more convenient here to use Mehler's formula

$$\begin{aligned} \big(e^{\tau L}f\big)(\xi) &= \frac{1}{4\pi a(\tau)}\int_{\mathbb{R}^2} e^{|\xi|^2/8}\, e^{-\frac{1}{4a(\tau)}|\xi-\eta e^{-\tau/2}|^2}\, e^{-|\eta|^2/8}\, f(\eta)\,\mathrm{d}\eta\,, \\ &= \frac{1}{4\pi a(\tau)}\int_{\mathbb{R}^2} e^{-\frac{1}{8a(\tau)}\left(|\xi-\eta e^{-\tau/2}|^2+|\eta-\xi e^{-\tau/2}|^2\right)}\, f(\eta)\,\mathrm{d}\eta\,, \end{aligned} \tag{5.3}$$

which defines the semigroup generated by the operator $L = G^{-1/2}\mathcal{L}\,G^{1/2}$, see (2.5). A direct calculation based on (5.3) gives the L^p–L^q estimates $\|e^{\tau L}f\|_{L^q(\mathbb{R}^2)} \le Ca(\tau)^{\frac{1}{q}-\frac{1}{p}}\|f\|_{L^p(\mathbb{R}^2)}$ for $1 \le p \le q \le \infty$. Returning to the original operator $\mathcal{L}$, we conclude that

$$\|G^{-1/2}\, e^{\tau\mathcal{L}}w\|_{L^q(\mathbb{R}^2)} \,\le\, \frac{C}{a(\tau)^{\frac{1}{p}-\frac{1}{q}}}\,\|G^{-1/2}\, w\|_{L^p(\mathbb{R}^2)}\,, \qquad \tau > 0\,, \tag{5.4}$$

where $a(\tau) = 1 - e^{-\tau}$. Similarly, we have the corresponding estimate for the first derivatives :

$$\|G^{-1/2}\nabla\, e^{\tau\mathcal{L}}w\|_{L^q(\mathbb{R}^2)} \,\le\, \frac{C}{a(\tau)^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}}\,\|G^{-1/2}\, w\|_{L^p(\mathbb{R}^2)}\,, \qquad \tau > 0\,, \tag{5.5}$$

see [20, Proposition 2.1].

To establish local well-posedness for Eq. (1.7) in the space X it is sufficient to prove that the bilinear operator

$$B[w_1, w_2](\tau) \,=\, \int_0^\tau e^{-\frac{1}{2}(\tau-s)}\,\mathrm{div}\big(e^{(\tau-s)\mathcal{L}}v_1(s)w_2(s)\big)\,\mathrm{d}s\,, \quad \text{where } v_1 = K_{BS} * w_1\,,$$

is continuous in the space $C^0([0,T], X)$ for any $T > 0$, and has a small norm if $T \ll 1$. Proceeding as in [22, Lemma 3.1], we deduce from (5.5) with $q = 2$ and $p = 4/3$ that

$$\|B[w_1, w_2](\tau)\|_X \,\le\, \int_0^\tau \frac{C\, e^{-\frac{1}{2}(\tau-s)}}{a(\tau-s)^{3/4}}\,\|G^{-1/2}v_1(s)w_2(s)\|_{L^{4/3}}\,\mathrm{d}s\,.$$

Next, using Hölder's inequality and estimate (5.18) below (with $p = 4/3$, $q = 4$), we obtain

$$\|G^{-1/2}v_1 w_2\|_{L^{4/3}} \le \|v_1\|_{L^4}\|G^{-1/2}w_2\|_{L^2} \le C\|w_1\|_{L^{4/3}}\|w_2\|_X \le C\|w_1\|_X\|w_2\|_X\,. \tag{5.6}$$

Thus we conclude that

$$\sup_{0\le\tau\le T}\|B[w_1,w_2](\tau)\|_X \le \Bigl(\int_0^T \frac{C\,e^{-s/2}}{a(s)^{3/4}}\,\mathrm{d}s\Bigr)\Bigl(\sup_{0\le s\le T}\|w_1(s)\|_X\Bigr)\Bigl(\sup_{0\le s\le T}\|w_2(s)\|_X\Bigr)\,, \tag{5.7}$$

and this bilinear estimate implies local well-posedness in X by a standard fixed point argument.

To prove that all solutions of (1.7) in X are global we need to show that the norm $\|w(\tau)\|_X$ cannot blow-up in finite time. Sharp a priori estimates on the solutions of the original vorticity equation (1.1) have been obtained by Carlen and Loss in [13], and can be translated into useful bounds for the rescaled equation (1.7). In particular, it follows from [13, Theorem 3] that any solution of (1.7) with initial data $w_0 \in L^1(\mathbb{R}^2)$ satisfies the pointwise estimate

$$|w(\xi,\tau)| \le \frac{C_\beta(R)}{4\pi a(\tau)}\int_{\mathbb{R}^2} e^{-\frac{\beta}{4a(\tau)}|\xi-\eta e^{-\tau/2}|^2}\,|w_0(\eta)|\,\mathrm{d}\eta\,, \qquad \xi\in\mathbb{R}^2\,,\ \tau>0\,, \tag{5.8}$$

for any $\beta \in (0,1)$, where $R = \|w_0\|_{L^1(\mathbb{R}^2)}$ and $C_\beta(R) = \exp(\frac{\beta}{1-\beta}\frac{R^2}{2\pi^2})$, see also [25, Section 2]. If we assume that $w_0 \in X$ and $\beta > 1/2$, a direct calculation based on (5.8) shows that

$$\int_{\mathbb{R}^2} e^{|\xi|^2/4}|w(\xi,\tau)|^2\,\mathrm{d}\xi \le \frac{4C_\beta^2}{(2\beta-1+e^{-\tau})(1+(2\beta-1)e^{-\tau})}\int_{\mathbb{R}^2} e^{|\eta|^2/4}|w_0(\eta)|^2\,\mathrm{d}\eta\,, \tag{5.9}$$

for all $\tau > 0$. In particular, we have $\|w(\tau)\|_X \le 2C_\beta(2\beta-1)^{-1/2}\|w_0\|_X$ for all $\tau \ge 0$, and this implies that all solutions of (1.7) in X are global and uniformly bounded for positive times.

Finally, we prove that all solutions of (1.7) in X converge to αG as $\tau \to \infty$, where

$$\alpha = \int_{\mathbb{R}^2} w(\xi,\tau)\,\mathrm{d}\xi = \int_{\mathbb{R}^2} w_0(\xi)\,\mathrm{d}\xi\,.$$

Indeed, we decompose $w(\tau) = \alpha G + \tilde{w}(\tau)$ for all $\tau \ge 0$, and we consider the equation (2.1) satisfied by the perturbation $\tilde{w} \in X_0$. Using the skew-symmetry of the operator Λ and integrating by parts, we easily obtain the energy estimate

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\|\tilde{w}(\tau)\|_X^2 &= \langle\tilde{w}(\tau),\mathcal{L}\tilde{w}(\tau)\rangle_X - \int_{\mathbb{R}^2} G^{-1}\tilde{w}(\xi,\tau)\tilde{v}(\xi,\tau)\cdot\nabla\tilde{w}(\xi,\tau)\,\mathrm{d}\xi\\ &= \langle\tilde{w}(\tau),\mathcal{L}\tilde{w}(\tau)\rangle_X + \frac{1}{4}\int_{\mathbb{R}^2} G^{-1}(\xi\cdot\tilde{v}(\xi,\tau))\tilde{w}(\xi,\tau)^2\,\mathrm{d}\xi\,.\end{aligned} \tag{5.10}$$

The first term in the right-hand side is estimated using the following lemma.

Lemma 5.1. *Given $w \in D(\mathcal{L}) \subset X$, define $E[w] = -\langle w,\mathcal{L}w\rangle_X \ge 0$.*
i) If $w \in X_0$, then $E[w] \ge \frac{1}{2}\|w\|_X^2$ and

$$E[w] \ge \frac{1}{6}\|\nabla w\|_X^2 + \frac{1}{96}\|\xi w\|_X^2 + \frac{1}{12}\|w\|_X^2\,. \tag{5.11}$$

ii) If $w \in X_1$, then $E[w] \ge \|w\|_X^2$ and

$$E[w] \ge \frac{1}{4}\|\nabla w\|_X^2 + \frac{1}{64}\|\xi w\|_X^2 + \frac{1}{8}\|w\|_X^2\,. \tag{5.12}$$

Proof. Using definition (2.2) and integrating by parts, we obtain

$$E[w] \;=\; -\int_{\mathbb{R}^2} G^{-1} w(\mathcal{L}w)\,\mathrm{d}\xi \;=\; \int_{\mathbb{R}^2} G^{-1}|\nabla w|^2\,\mathrm{d}\xi - \int_{\mathbb{R}^2} G^{-1}|w|^2\,\mathrm{d}\xi\,. \tag{5.13}$$

Moreover it is easy to verify that

$$\int_{\mathbb{R}^2} G^{-1}|\nabla w|^2\,\mathrm{d}\xi \;=\; \int_{\mathbb{R}^2} |\nabla(G^{-1/2}w)|^2\,\mathrm{d}\xi + \frac{1}{16}\int_{\mathbb{R}^2} G^{-1}|\xi|^2|w|^2\,\mathrm{d}\xi + \frac{1}{2}\int_{\mathbb{R}^2} G^{-1}|w|^2\,\mathrm{d}\xi\,,$$

hence

$$E[w] \;\ge\; \frac{1}{16}\int_{\mathbb{R}^2} G^{-1}|\xi|^2|w|^2\,\mathrm{d}\xi - \frac{1}{2}\int_{\mathbb{R}^2} G^{-1}|w|^2\,\mathrm{d}\xi\,. \tag{5.14}$$

Finally, the dissipative properties of $\mathcal{L}$ recalled in Section 2.1 imply that $E[w] \ge 0$, and

$$E[w] \;\ge\; \frac{1}{2}\int_{\mathbb{R}^2} G^{-1}|w|^2\,\mathrm{d}\xi \quad \text{if } w \in X_0\,, \qquad E[w] \;\ge\; \int_{\mathbb{R}^2} G^{-1}|w|^2\,\mathrm{d}\xi \quad \text{if } w \in X_1\,. \tag{5.15}$$

Taking a convex combination of (5.13), (5.14), (5.15) with coefficients 1/6, 1/6, 2/3 if $w \in X_0$, and with coefficients 1/4, 1/4, 1/2 if $w \in X_1$, we obtain estimates (5.11), (5.12), respectively. □

We now return to the analysis of (5.10). Using Hölder's inequality and estimate (5.11) in Lemma 5.1, we can control the cubic term as follows:

$$\Big|\int_{\mathbb{R}^2} G^{-1}(\xi\cdot\tilde v)\tilde w^2\,\mathrm{d}\xi\Big| \;\le\; \|G^{-1/2}|\xi|\tilde w\|_{L^2}\|G^{-1/2}\tilde w\|_{L^4}\|\tilde v\|_{L^4} \;\le\; CE[\tilde w]\,\|\tilde v\|_{L^4}\,, \tag{5.16}$$

because $\|G^{-1/2}|\xi|\tilde w\|_{L^2} = \||\xi|\tilde w\|_X \le CE[\tilde w]^{1/2}$ and

$$\|G^{-1/2}\tilde w\|_{L^4} \;\le\; C\|G^{-1/2}\tilde w\|_{L^2}^{1/2}\,\|\nabla(G^{-1/2}\tilde w)\|_{L^2}^{1/2} \;\le\; CE[\tilde w]^{1/2}\,.$$

We deduce from (5.10) and (5.16) that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\|\tilde w(\tau)\|_X^2 \;\le\; -2E[\tilde w(\tau)]\big(1 - C_8\|\tilde v(\tau)\|_{L^4}\big)\,, \tag{5.17}$$

for some positive constant $C_8 > 0$. Now, since our Gaussian space X is included in the polynomially weighted spaces $L^2(m)$ considered in [23], we know from Lemma 5.2 and [23, Proposition 1.5] that $\|\tilde v(\tau)\|_{L^4} \le C\|\tilde w(\tau)\|_{L^{4/3}} \to 0$ as $\tau \to +\infty$. In particular, we have $C_8\|\tilde v(\tau)\|_{L^4} \le 1/2$ for large times, and in this regime it follows from (5.17), (5.15) that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\|\tilde w(\tau)\|_X^2 \;\le\; -E[\tilde w(\tau)] \;\le\; -\frac{1}{2}\|\tilde w(\tau)\|_X^2\,.$$

This differential inequality implies that $\|\tilde w(\tau)\|_X = \|w(\tau) - \alpha G\|_X \to 0$ as $\tau \to \infty$, which concludes the proof of Proposition 1.1.

On the other hand, if we restrict ourselves to solutions in a neighborhood of αG, we can bound $C_8\|\tilde v(\tau)\|_{L^4} \le C_9\|\tilde w(\tau)\|_X$ for some constant $C_9 > 0$, and assume from the beginning that $C_9\|\tilde w(\tau)\|_X \le 1/2$. In that case (5.17) implies

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\|\tilde w(\tau)\|_X^2 \;\le\; -2E[\tilde w(\tau)]\big(1 - C_9\|\tilde w(\tau)\|_X\big) \;\le\; -\|\tilde w(\tau)\|_X^2\big(1 - C_9\|\tilde w(\tau)\|_X\big)\,,$$

hence

$$\frac{\|\tilde w(\tau)\|_X}{1 - C_9\|\tilde w(\tau)\|_X} \;\le\; \frac{\|\tilde w(0)\|_X}{1 - C_9\|\tilde w(0)\|_X}\,e^{-\tau/2}\,, \qquad \tau \ge 0\,.$$

We easily deduce that $\|\tilde w(\tau)\|_X$ is a nonincreasing function of time which satisfies $\|\tilde w(\tau)\|_X \le 2\,e^{-\tau/2}\|\tilde w(0)\|_X$ for all $\tau \ge 0$. This gives the conclusion of Proposition 1.2, with $\epsilon = 1/(2C_9)$.

5.2 The two-dimensional Biot-Savart law

Let $u = K_{BS} * \omega$ be the velocity field associated with the vorticity distribution ω via the Biot-Savart law (1.2). We first recall the following classical estimates.

Lemma 5.2.
1) Assume that $1 < p < 2 < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$. If $\omega \in L^p(\mathbb{R}^2)$, then $u \in L^q(\mathbb{R}^2)$ and

$$\|u\|_{L^q(\mathbb{R}^2)} \le C_q \|\omega\|_{L^p(\mathbb{R}^2)} . \tag{5.18}$$

2) Assume that $1 \le p < 2 < q \le \infty$. If $\omega \in L^p(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$, then $u \in L^\infty(\mathbb{R}^2)$ and

$$\|u\|_{L^\infty(\mathbb{R}^2)} \le C_{p,q} \|\omega\|^{\theta}_{L^p(\mathbb{R}^2)} \|\omega\|^{1-\theta}_{L^q(\mathbb{R}^2)} , \tag{5.19}$$

where $\theta \in (0,1)$ satisfies $\frac{\theta}{p} + \frac{1-\theta}{q} = \frac{1}{2}$.

Proof. Estimate (5.18) is a direct consequence of the Hardy-Littlewood-Sobolev inequality, see e.g. [30]. The optimal constant C_q in (5.18) is not explicitly known, but one can show that $C_q = \mathcal{O}(q^{1/2})$ as $q \to +\infty$. A possible expression for the constant $C_{p,q}$ in (5.19) is given in (5.22) below. □

Next we establish an estimate which strengthens (5.19) and shows that the L^∞ norm of the velocity field can be controlled by the L^2 norm of the vorticity up to a logarithmic correction.

Lemma 5.3. *Assume that $1 \le p < 2 < q \le \infty$. If $\omega \in L^p(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$, then $u \in L^\infty(\mathbb{R}^2)$ and*

$$\|u\|_{L^\infty} \le C \|\omega\|_{L^2} \Big(1 + \log \frac{\|\omega\|^{\theta}_{L^p} \|\omega\|^{1-\theta}_{L^q}}{\|\omega\|_{L^2}}\Big)^{1/2} , \tag{5.20}$$

where C depends only on p, q, and $\theta \in (0,1)$ satisfies $\frac{\theta}{p} + \frac{1-\theta}{q} = \frac{1}{2}$.

Remark 5.4. If $\omega \in L^2(\mathbb{R}^2)$, the velocity u belongs to the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^2)$, which has the same dimensional scaling as $L^\infty(\mathbb{R}^2)$ but fails to be included in $L^\infty(\mathbb{R}^2)$. In such a critical case, one can obtain various Sobolev-type inequalities with a logarithmic correction involving, for instance, a higher Sobolev norm, see Brézis-Gallouët [11] and Brézis-Wainger [12] for early results in this direction. A very general inequality can be found in [36, Theorem 3.1], a particular case of which gives an estimate that is similar to (5.20) but involves slightly different function spaces, i.e. homogeneous Besov spaces. For the reader's convenience, we provide here an elementary and self-contained proof of (5.20).

Proof. For any $x \in \mathbb{R}^2$ and any $R > 0$, we have

$$|u(x)| \le \frac{1}{2\pi} \int_{|y|<R} \frac{1}{|y|} |\omega(x-y)| \,\mathrm{d}y + \frac{1}{2\pi} \int_{|y|\ge R} \frac{1}{|y|} |\omega(x-y)| \,\mathrm{d}y .$$

Fix $1 < p < 2 < q < \infty$, and let p', q' be the conjugate exponents to p, q, respectively. Applying Hölder's inequality, we obtain

$$\begin{aligned} \|u\|_{L^\infty} &\le \frac{1}{2\pi} \Big(\int_{|y|<R} \frac{1}{|y|^{q'}} \,\mathrm{d}y \Big)^{1/q'} \|\omega\|_{L^q} + \frac{1}{2\pi} \Big(\int_{|y|\ge R} \frac{1}{|y|^{p'}} \,\mathrm{d}y \Big)^{1/p'} \|\omega\|_{L^p} \\ &\le C \Big(\frac{1}{2-q'} \Big)^{1/q'} R^{1-2/q} \|\omega\|_{L^q} + C \Big(\frac{1}{p'-2} \Big)^{1/p'} \frac{1}{R^{-1+2/p}} \|\omega\|_{L^p} \\ &= C \Big(\frac{q-1}{q-2} \Big)^{1-1/q} R^{1-2/q} \|\omega\|_{L^q} + C \Big(\frac{p-1}{2-p} \Big)^{1-1/p} \frac{1}{R^{-1+2/p}} \|\omega\|_{L^p} . \end{aligned} \tag{5.21}$$

If we choose $R > 0$ such that both terms in the right-hand side of (5.21) are equal, we arrive at inequality (5.19) with

$$C_{p,q} \,=\, C\Big(\frac{p-1}{2-p}\Big)^{\theta(1-\frac{1}{p})}\Big(\frac{q-1}{q-2}\Big)^{(1-\theta)(1-\frac{1}{q})}\,, \qquad \text{where} \quad \theta \,=\, \frac{p(q-2)}{2(q-p)}\,. \tag{5.22}$$

Remark that $C_{p,q}$ has a finite limit as $p \to 1$ or $q \to \infty$, which means that estimate (5.19) holds in these limiting cases too. In particular, we have $\|u\|_{L^\infty} \le C\|\omega\|_{L^1}^{1/2}\|\omega\|_{L^\infty}^{1/2}$.

We next assume that $p = 2-\epsilon$ and $q = 2+\epsilon$ for some $\epsilon \in (0, 1/2]$. With this choice, we have $\theta = (2-\epsilon)/4$ and

$$\Big(\frac{p-1}{2-p}\Big)^{\theta(1-\frac{1}{p})} \,=\, \Big(\frac{1-\epsilon}{\epsilon}\Big)^{\frac{1-\epsilon}{4}} \,\le\, \frac{C}{\epsilon^{1/4}}\,, \qquad \Big(\frac{q-1}{q-2}\Big)^{(1-\theta)(1-\frac{1}{q})} \,=\, \Big(\frac{1+\epsilon}{\epsilon}\Big)^{\frac{1+\epsilon}{4}} \,\le\, \frac{C}{\epsilon^{1/4}}\,.$$

so that inequality (5.19) reduces to

$$\|u\|_{L^\infty} \,\le\, \frac{C}{\epsilon^{1/2}}\,\|\omega\|_{L^{2-\epsilon}}^{\frac{2-\epsilon}{4}}\,\|\omega\|_{L^{2+\epsilon}}^{\frac{2+\epsilon}{4}}\,. \tag{5.23}$$

In the rest of the proof, we show how estimate (5.20) can be deduced from (5.23). Let again $1 \le p < 2 < q \le \infty$. If $0 < \epsilon \le \epsilon_0 := \min(2-p, q-2, 1/2)$, we can interpolate

$$\|\omega\|_{L^{2-\epsilon}} \,\le\, \|\omega\|_{L^p}^{\alpha}\,\|\omega\|_{L^2}^{1-\alpha}\,, \qquad \|\omega\|_{L^{2+\epsilon}} \,\le\, \|\omega\|_{L^q}^{\beta}\,\|\omega\|_{L^2}^{1-\beta}\,,$$

where $\frac{\alpha}{p}+\frac{1-\alpha}{2} = \frac{1}{2-\epsilon}$ and $\frac{\beta}{q}+\frac{1-\beta}{2} = \frac{1}{2+\epsilon}$. Substituting into (5.23), we obtain after straightforward calculations

$$\|u\|_{L^\infty} \,\le\, \frac{C}{\epsilon^{1/2}}\,\|\omega\|_{L^2}\,A^{\epsilon\gamma/2}\,, \qquad A \,=\, \frac{\|\omega\|_{L^p}^{\theta}\|\omega\|_{L^q}^{1-\theta}}{\|\omega\|_{L^2}} \,\ge\, 1\,, \tag{5.24}$$

where θ is as in the statement and $\gamma = \frac{1}{2-p}+\frac{1}{q-2}$. It remains to optimize the choice of $\epsilon \in (0, \epsilon_0]$. If $A \le \exp((\epsilon_0\gamma)^{-1})$, we take $\epsilon = \epsilon_0$ and (5.24) implies

$$\|u\|_{L^\infty} \,\le\, C\epsilon_0^{-1/2}\,A^{\epsilon_0\gamma/2}\,\|\omega\|_{L^2} \,\le\, C\epsilon_0^{-1/2}e^{1/2}\,\|\omega\|_{L^2}\,.$$

If $A > \exp((\epsilon_0\gamma)^{-1})$, we take $\epsilon = (\gamma\log(A))^{-1} < \epsilon_0$, and we obtain

$$\|u\|_{L^\infty} \,\le\, C(\gamma\log(A))^{1/2}A^{\frac{1}{2\log A}}\,\|\omega\|_{L^2} \,\le\, C(\log(A))^{1/2}\|\omega\|_{L^2}\,.$$

We conclude that estimate (5.20) holds in all cases. □

We conclude this section with two additional results in the spirit of Lemma 5.19, which are tailored for our purposes in Section 4.

Lemma 5.5. *Assume that $\omega \in L^1(\mathbb{R}^2) \cap L^3(\mathbb{R}^2)$, and let $u = K_{BS} * \omega$. Then*

$$\|u\|_{L^\infty} \,\le\, C\|\omega\|_{L^1\cap L^2}\Big(1 + \log_+\frac{\|\omega\|_{L^3}}{\|\omega\|_{L^1\cap L^2}}\Big)^{1/2}\,, \tag{5.25}$$

where $C > 0$ is a universal constant and $\|\omega\|_{L^1\cap L^2} = \max(\|\omega\|_{L^1}, \|\omega\|_{L^2})$.

Proof. We use inequality (5.24) with $p = 1$ and $q = 3$, so that $\gamma = 2$. This gives

$$\|u\|_{L^\infty} \,\le\, \frac{C}{\epsilon^{1/2}}\,\|\omega\|_{L^1}^{\epsilon/4}\,\|\omega\|_{L^2}^{1-\epsilon}\,\|\omega\|_{L^3}^{3\epsilon/4} \,\le\, \frac{C}{\epsilon^{1/2}}\,\|\omega\|_{L^1\cap L^2}^{1-3\epsilon/4}\,\|\omega\|_{L^3}^{3\epsilon/4}\,.$$

Optimizing the choice of $\epsilon > 0$ as in the proof of Lemma 5.3, we arrive at (5.25). □

Lemma 5.6. *Assume that* $\omega_1 \in L^2(\mathbb{R}^2) \cap L^3(\mathbb{R}^2)$, $\omega_2 \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, *and let* $u_2 = K_{BS} * \omega_2$. *Then*

$$\|\omega_1 u_2\|_{L^2} \le C\|\omega_1\|_{L^2}\|\omega_2\|_{L^1\cap L^2}\Big(1 + \log_+ \frac{\|\omega_1\|_{L^3}}{\|\omega_1\|_{L^2}}\Big)^{1/2}. \tag{5.26}$$

Proof. Applying Hölder's inequality and Lemma 5.2, we obtain for any $q \ge 6$:

$$\|\omega_1 u_2\|_{L^2} \le \|\omega_1\|_{L^r}\|u_2\|_{L^q} \le Cq^{1/2}\|\omega_1\|_{L^r}\|\omega_2\|_{L^p},$$

where $\frac{1}{r} + \frac{1}{q} = \frac{1}{2}$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$. By interpolation, we have $\|\omega_2\|_{L^p} \le \|\omega_2\|_{L^1\cap L^2}$ and

$$\|\omega_1\|_{L^r} \le \|\omega_1\|_{L^2}^{-2+6/r}\|\omega_1\|_{L^3}^{3-6/r} = C\|\omega_1\|_{L^2}^{1-6/q}\|\omega_1\|_{L^3}^{6/q},$$

hence

$$\|\omega_1 u_2\|_{L^2} \le Cq^{1/2}\|\omega_1\|_{L^2}^{1-6/q}\|\omega_1\|_{L^3}^{6/q}\|\omega_2\|_{L^1\cap L^2}.$$

Optimizing the choice of $q \in [6,\infty)$ as in the proof of Lemma 5.3, we arrive at (5.26). □

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