Long-time asymptotics for two-dimensional exterior flows with small circulation at infinity

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Abstract

We consider the incompressible Navier-Stokes equations in a two-dimensional exterior domain Ω, with no-slip boundary conditions. Our initial data are of the form $u_0 = \alpha\Theta_0 + v_0$, where Θ_0 is the Oseen vortex with unit circulation at infinity and v_0 is a solenoidal perturbation belonging to $L^2(\Omega)^2 \cap L^q(\Omega)^2$ for some $q \in (1,2)$. If $\alpha \in \mathbb{R}$ is sufficiently small, we show that the solution behaves asymptotically in time like the self-similar Oseen vortex with circulation α. This is a global stability result, in the sense that the perturbation v_0 can be arbitrarily large, and our smallness assumption on the circulation α is independent of the domain Ω.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a smooth exterior domain, namely an unbounded connected open subset of the Euclidean plane with a smooth compact boundary $\partial\Omega$. We consider the free motion of an incompressible viscous fluid in Ω, with no-slip boundary conditions on $\partial\Omega$. The evolution is governed by the Navier-Stokes equations

$$\left\{\begin{array}{lll} \partial_t u + (u\cdot\nabla)u \,=\, \Delta u - \nabla p\,, \quad \operatorname{div} u \,=\, 0\,, & \text{for } x \in \Omega\,, & t > 0\,, \\ u(x,t) \,=\, 0\,, & \text{for } x \in \partial\Omega\,, & t > 0\,, \\ u(x,0) \,=\, u_0(x)\,, & \text{for } x \in \Omega\,, & \end{array}\right. \tag{1}$$

where $u(x,t) \in \mathbb{R}^2$ denotes the velocity of a fluid particle at point $x \in \Omega$ and time $t > 0$, and $p(x,t)$ is the pressure in the fluid at the same point. For simplicity, both the kinematic viscosity and the density of the fluid have been normalized to 1. The initial velocity field $u_0 : \Omega \to \mathbb{R}^2$ is assumed to be divergence-free and tangent to the boundary on $\partial\Omega$.

If the initial velocity u_0 belongs to the energy space

$$L^2_\sigma(\Omega) \,=\, \{u \in L^2(\Omega)^2 \mid \operatorname{div} u = 0 \text{ in } \Omega\,,\ u\cdot n = 0 \text{ on } \partial\Omega\}\,,$$

where n denotes the unit normal on $\partial\Omega$, then it is known that system (1) has a unique global solution $u \in C^0([0,\infty); L^2_\sigma(\Omega)) \cap C^1((0,\infty); L^2_\sigma(\Omega)) \cap C^0((0,\infty); H^1_0(\Omega)^2 \cap H^2(\Omega)^2)$, which satisfies the energy equality

$$\frac{1}{2}\|u(\cdot,t)\|_{L^2(\Omega)}^2 + \int_0^t \|\nabla u(\cdot,s)\|_{L^2(\Omega)}^2 \,\mathrm{d}s \;=\; \frac{1}{2}\|u_0\|_{L^2(\Omega)}^2 \;, \qquad \text{for all } t > 0 \;.$$

This global well-posedness result was first established by Leray [23] in the particular case where $\Omega = \mathbb{R}^2$, and subsequently extended to more general domains, including exterior domains, by various authors [24, 22, 25, 18, 11, 19]. It is also known that the kinetic energy $\frac{1}{2}\|u(\cdot,t)\|_{L^2(\Omega)}^2$ converges to zero as $t \to \infty$ [28, 3, 19], and precise decay rates can be obtained under additional assumptions on the initial data [20, 7, 1].

In two-dimensional fluid mechanics, however, the assumption that the velocity field u be square integrable is quite restrictive, because it implies (if $u = 0$ on $\partial\Omega$) that the associated vorticity field $\omega = \partial_1 u_2 - \partial_2 u_1$ has zero mean over Ω, see [26, Section 3.1.3]. In many important examples, this condition is not satisfied and the kinetic energy of the flow is therefore infinite. For instance, when $\Omega = \mathbb{R}^2$, the Navier-Stokes equations (1) have a family of explicit self-similar solutions of the form $u(x,t) = \alpha\Theta(x,t)$, $p(x,t) = \alpha^2\Pi(x,t)$, where $\alpha \in \mathbb{R}$ is a parameter and

$$\Theta(x,t) \;=\; \frac{1}{2\pi}\,\frac{x^\perp}{|x|^2}\Bigl(1 - e^{-\frac{|x|^2}{4(1+t)}}\Bigr)\;, \qquad \nabla\Pi(x,t) \;=\; \frac{x}{|x|^2}|\Theta(x,t)|^2 \;. \tag{2}$$

Here and in the sequel, if $x = (x_1, x_2) \in \mathbb{R}^2$, we denote $x^\perp = (-x_2, x_1)$ and $|x|^2 = x_1^2 + x_2^2$. The solution (2) is called the *Lamb-Oseen vortex* with circulation α. Remark that $|\Theta(x,t)| = \mathcal{O}(|x|^{-1})$ as $|x| \to \infty$, so that $\Theta(\cdot,t) \notin L^2(\mathbb{R}^2)^2$, and that the circulation at infinity of the vector field Θ is equal to 1, in the sense that $\oint_{|x|=R} \Theta_1 \,\mathrm{d}x_1 + \Theta_2 \,\mathrm{d}x_2 \to 1$ as $R \to \infty$. The corresponding vorticity distribution

$$\Xi(x,t) \;=\; \partial_1\Theta_2(x,t) - \partial_2\Theta_1(x,t) \;=\; \frac{1}{4\pi(1+t)}\, e^{-\frac{|x|^2}{4(1+t)}} \;, \tag{3}$$

has a constant sign and satisfies $\int_{\mathbb{R}^2} \Xi(x,t)\,\mathrm{d}x = 1$ for all $t \ge 0$. Oseen's vortex plays an important role in the dynamics of the Navier-Stokes equations in $\mathbb{R}^2$, because it describes the long-time asymptotics of all solutions whose vorticity distribution is integrable. This result was first proved by Giga and Kambe for small solutions [15], and subsequently by Carpio for large solutions with small circulation [5]. The general case was finally settled by Wayne and the first named author [14]. It is worth mentioning that all these results were obtained using the vorticity formulation of the Navier-Stokes equations.

In the case of an exterior domain $\Omega \subset \mathbb{R}^2$, much less is known about infinite-energy solutions, mainly because the vorticity formulation is not convenient anymore due to the boundary conditions. A general existence result was established by Kozono and Yamazaki, who proved that system (1) is globally well-posed for initial data u_0 in the weak L^2 space $L^{2,\infty}_\sigma(\Omega)$, provided that the local singularity of u_0 in $L^{2,\infty}$ is sufficiently small [21]. In what follows, we consider initial data of the form

$$u_0 \;=\; \alpha\chi\Theta_0 + v_0 \;, \tag{4}$$

where $\Theta_0(x) = \Theta(x,0)$ is Oseen's vortex at time $t = 0$, and $\chi : \mathbb{R}^2 \to [0,1]$ is a smooth, radially symmetric cut-off function such that $\chi = 0$ on a neighborhood of $\mathbb{R}^2 \setminus \Omega$ and $\chi(x) = 1$ when $|x|$ is sufficiently large. For any $\alpha \in \mathbb{R}$ and any $v_0 \in L^2_\sigma(\Omega)$, Theorem 4 in [21] asserts that the Navier-Stokes equation (1) has a global solution with initial data (4), which is unique in an appropriate class. However, little is known about the long-time behavior of this solution, and in particular there is no a priori estimate which guarantees that the $L^{2,\infty}$ norm of u remains bounded for all times.

Very recently, a first result concerning the long-time behavior of solutions of (1) with initial data of the form (4) was obtained by Iftimie, Karch, and Lacave :

Theorem 1.1 [16] *Let $\Omega \subset \mathbb{R}^2$ be a smooth exterior domain whose complement $\mathbb{R}^2 \setminus \Omega$ is a connected set in $\mathbb{R}^2$. For any $v_0 \in L^2_\sigma(\Omega)$, there exists a constant $\epsilon = \epsilon(v_0, \Omega) > 0$ such that, for all $\alpha \in [-\epsilon, \epsilon]$, the solution of* (1) *with initial data* (4) *satisfies*

$$\lim_{t\to\infty} t^{\frac{1}{2}-\frac{1}{p}} \|u(\cdot,t) - \alpha\Theta(\cdot,t)\|_{L^p(\Omega)} \;=\; 0 \;, \qquad \textit{for all} \quad p \in (2,\infty) \;. \tag{5}$$

Moreover, there exists $\epsilon_0 = \epsilon_0(\Omega) > 0$ such that $\epsilon \ge \epsilon_0$ if $\|v_0\|_{L^2} \le \epsilon_0$.

Theorem 1.1 shows that solutions of (1) which are finite-energy perturbations of Oseen's vortex $\alpha\Theta_0$ behave asymptotically in time like the self-similar Oseen vortex $\alpha\Theta(x,t)$, provided that the circulation at infinity α is sufficiently small, depending on the size of the initial perturbation. The conclusion holds in particular when both the circulation α and the finite-energy perturbation v_0 are small, so that Theorem 1.1 extends to exterior domains the result of Giga and Kambe [15]. For large solutions, however, the assumption that α be small depending on v_0 is very restrictive. The goal of the present paper is to prove the following result, which reaches a conclusion similar to that of Theorem 1.1 under different assumptions on the initial data:

Theorem 1.2 *Fix $q \in (1,2)$, and let $\mu = 1/q - 1/2$. There exists a constant $\epsilon = \epsilon(q) > 0$ such that, for any smooth exterior domain $\Omega \subset \mathbb{R}^2$ and for all initial data of the form* (4) *with $|\alpha| \le \epsilon$ and $v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$, the solution of the Navier-Stokes equations* (1) *satisfies*

$$\|u(\cdot,t) - \alpha\Theta(\cdot,t)\|_{L^2(\Omega)} + t^{1/2}\|\nabla u(\cdot,t) - \alpha\nabla\Theta(\cdot,t)\|_{L^2(\Omega)} \;=\; \mathcal{O}(t^{-\mu}) \;, \tag{6}$$

as $t \to +\infty$.

Here, we also suppose that the circulation at infinity is small, and we assume in addition that the initial perturbation belongs to $L^2_\sigma(\Omega) \cap L^q(\Omega)^2$ for some $q < 2$. Unlike in Theorem 1.1, the limiting case $q = 2$ is not included, and the proof shows that $\epsilon(q) = \mathcal{O}(\sqrt{2-q})$ as $q \to 2$. However, there is absolutely no restriction on the size of the perturbation v_0, hence Theorem 1.2 establishes a *global stability property* for the Lamb-Oseen vortices (with small circulation) in two-dimensional exterior domains. In this sense, our result can be considered as a generalization to exterior domains of the work of Carpio [5], although our proof relies on completely different ideas. On the other hand, since our perturbations decay faster at infinity (in space) than those considered by Iftimie, Karch, and Lacave, we are able to show that the difference $u(x,t) - \alpha\Theta(x,t)$ converges rapidly to zero, like an inverse power of time, as $t \to \infty$. In particular, using (6) and elementary interpolation, we obtain the estimate

$$\sup_{t>0} t^{\frac{1}{q}-\frac{1}{p}} \|u(\cdot,t) - \alpha\Theta(\cdot,t)\|_{L^p(\Omega)} \;<\; \infty \;, \qquad \text{for all} \quad p \in [2,\infty) \;,$$

which improves (5) since $q < 2$.

At this point, it is useful to mention that the assumption that u_0 can be decomposed as in (4) for some $\alpha \in \mathbb{R}$ and some $v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$ is automatically satisfied if we suppose that the initial vorticity $\omega_0 = \operatorname{curl} u_0$ is sufficiently localized. Indeed, let us assume for simplicity that u_0 vanishes on the boundary $\partial\Omega$. For $1 \le p < \infty$, we denote by $\dot W^{1,p}_{0,\sigma}(\Omega)$ the completion with respect to the norm $u \mapsto \|\nabla u\|_{L^p}$ of the space of all smooth, divergence-free vector fields with compact support in Ω. Using this notation, we have the following result:

Proposition 1.3 *Fix $q \in (1,2)$. Assume that u_0 belongs to $\dot W^{1,p}_{0,\sigma}(\Omega)$ for some $p \in [1,2)$, and that the associated vorticity $\omega_0 = \operatorname{curl}\, u_0$ satisfies*

$$\int_\Omega (1+|x|^2)^m |\omega_0(x)|^2 \,\mathrm{d}x \;<\; \infty \;, \tag{7}$$

for some $m > 2/q$. If we denote $\alpha = \int_\Omega \omega_0(x)\,\mathrm{d}x$, then u_0 can be decomposed as in (4) *for some $v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$. In particular, if $|\alpha| \le \epsilon$, the conclusion of Theorem 1.2 holds.*

For completeness, we give a short proof of Proposition 1.3 in the Appendix. Returning to the discussion of Theorem 1.2, we emphasize that the smallness condition on the circulation α is independent of the domain Ω, which can be an arbitrary multiply connected exterior domain. In fact, the proof will show that the optimal constant $\epsilon(q)$ is entirely determined by quantities that appear in the evolution equation for the perturbation of Oseen's vortex in the whole plane $\mathbb{R}^2$. Note that Oseen vortices are known to be globally stable for all values of the circulation α when $\Omega = \mathbb{R}^2$ [14], but in that particular case one can use the vorticity equation to obtain precise informations on the solutions of (1). The reader who is not interested in precise convergence rates could consider the following variant of Theorem 1.2, where the condition on the circulation is totally explicit :

Corollary 1.4 *There exists a universal constant $\epsilon_* \ge 4.956$ such that, if $|\alpha| < \epsilon_*$ and if $v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$ for all $q \in (1,2)$, the solution of the Navier-Stokes equations* (1) *with initial data* (4) *satisfies $\|u(\cdot,t) - \alpha\Theta(\cdot,t)\|_{L^2(\Omega)} \to 0$ as $t \to \infty$.*

The rest of this paper is devoted to the proof of Theorem 1.2, which is quite different from that of Theorem 1.1 in [16]. In the preliminary Section 2, we collect various estimates on the truncated Oseen vortex $\chi\Theta$, which can be verified by direct calculations. In Section 3, following the classical approach of Fujita and Kato [11], we prove the existence of a unique global solution of (1) for small initial data of the form (4), and we obtain the asymptotics (6) for small solutions. To deal with large solutions, we derive in Section 4 a "logarithmic energy estimate", which shows that the energy norm of the perturbation v has at most a logarithmic growth as $t \to \infty$. This is the key new ingredient, which we use as a substitute for the classical energy inequality when $\alpha \neq 0$. Exploiting this estimate and our assumption that $v_0 \in L^q(\Omega)^2$, we control in Section 5 the evolution of a fractional primitive of v, and we deduce that the perturbation $v(\cdot,t)$ converges to zero in energy norm, at least along a sequence of times. Thus we can eventually use the results of Section 3, and the conclusion follows.

2 The truncated Oseen vortex

Fix $\rho \ge 1$ large enough so that $\{x \in \mathbb{R}^2 \,|\, |x| \ge \rho\} \subset \Omega$. Let $\chi(x) = \tilde\chi(x/\rho)$, where $\tilde\chi \in C^\infty(\mathbb{R}^2)$ is a radially symmetric cut-off function satisfying $\tilde\chi(x) = 0$ when $|x| \le 1$, $\tilde\chi(x) = 1$ when $|x| \ge 2$, and $0 \le \tilde\chi(x) \le 1$ for all $x \in \mathbb{R}^2$. We define the *truncated Oseen vortex* (with unit circulation) as follows :

$$u^\chi(x,t) \,=\, \chi(x)\Theta(x,t) \,=\, \frac{1}{2\pi}\frac{x^\perp}{|x|^2}\Big(1 - e^{-\frac{|x|^2}{4(1+t)}}\Big)\chi(x) \;, \qquad x \in \mathbb{R}^2, \quad t \ge 0 \;. \tag{8}$$

Since χ is radially symmetric and $\operatorname{supp}\chi \subset \{x \in \mathbb{R}^2 \,|\, |x| \ge \rho\} \subset \Omega$, it is clear that $u^\chi(x,t)$ is a smooth divergence-free vector field which vanishes in a neighborhood of $\mathbb{R}^2 \setminus \Omega$. Let $\omega^\chi = \partial_1 u^\chi_2 - \partial_2 u^\chi_1$ be the corresponding vorticity field, namely

$$\omega^\chi(x,t) \,=\, \chi(x)\Xi(x,t) + \frac{1}{2\pi}\frac{1}{|x|^2}\Big(1 - e^{-\frac{|x|^2}{4(1+t)}}\Big)x\cdot\nabla\chi(x) \;, \tag{9}$$

where $\Xi(x,t)$ is defined in (3). Since $u^\chi(x,t) = \Theta(x,t)$ whenever $|x| \ge 2\rho$, the circulation of u^χ at infinity is equal to 1, so that $\int_{\mathbb{R}^2}\omega^\chi\,\mathrm{d}x = 1$. Moreover, a direct calculation shows that

$$(u^\chi\cdot\nabla)u^\chi \,=\, \frac{1}{2}\nabla|u^\chi|^2 + (u^\chi)^\perp\omega^\chi \,=\, -\frac{x}{|x|^2}|u^\chi|^2 \;, \tag{10}$$

hence there exists a radially symmetric function $p^\chi(x,t)$ such that $-\nabla p^\chi = (u^\chi\cdot\nabla)u^\chi$. This shows that $P(u^\chi\cdot\nabla)u^\chi = 0$, where P denotes the Leray-Hopf projection in Ω, namely the orthogonal projection in $L^2(\Omega)^2$ onto the subspace $L^2_\sigma(\Omega)$.

The following elementary estimates will be useful:

Lemma 2.1
1. *For any $p\in(2,\infty]$, there exists a constant $a_p>0$ such that*

$$\|u^\chi(\cdot,t)\|_{L^p(\mathbb{R}^2)} \le \frac{a_p}{(1+t)^{\frac12-\frac1p}}\ , \qquad t\ge 0\ . \tag{11}$$

2. *For any $p\in(1,\infty]$, there exists a constant $b_p>0$ such that*

$$\|\nabla u^\chi(\cdot,t)\|_{L^p(\mathbb{R}^2)} \le \frac{b_p}{(1+t)^{1-\frac1p}}\ , \qquad t\ge 0\ . \tag{12}$$

3. *For all $t,s\ge 0$, we have*

$$\|u^\chi(\cdot,t)-u^\chi(\cdot,s)\|^2_{L^2(\mathbb{R}^2)} \le \frac{1}{4\pi}\left|\log\frac{1+t}{1+s}\right|\ . \tag{13}$$

4. *There exists a constant $\kappa_1>0$ such that, for all $t,s\ge 0$,*

$$\|\nabla u^\chi(\cdot,t)-\nabla u^\chi(\cdot,s)\|^2_{L^2(\mathbb{R}^2)} \le \kappa_1\left|\frac{1}{1+t}-\frac{1}{1+s}\right|\ . \tag{14}$$

Moreover all constants a_p, b_p, and κ_1 are independent of ρ, hence of the domain Ω.

Proof. By (8) we have

$$u^\chi(x,t) = \chi(x)\Theta(x,t) = \frac{\chi(x)}{\sqrt{1+t}}\,\Theta_0\Big(\frac{x}{\sqrt{1+t}}\Big)\ ,$$

where $\Theta_0(x)=\Theta(x,0)$. Since $0\le\chi\le 1$ and $\Theta_0\in L^p(\mathbb{R}^2)^2$ for all $p>2$, we find

$$\|u^\chi(\cdot,t)\|_{L^p(\mathbb{R}^2)} \le \frac{1}{\sqrt{1+t}}\Big\|\Theta_0\Big(\frac{\cdot}{\sqrt{1+t}}\Big)\Big\|_{L^p(\mathbb{R}^2)} = \frac{\|\Theta_0\|_{L^p(\mathbb{R}^2)}}{(1+t)^{\frac12-\frac1p}}\ , \qquad t\ge 0\ .$$

This proves (11).

Similarly, we have $\partial_i u^\chi = \chi\partial_i\Theta + (\partial_i\chi)\Theta$ for $i=1,2$. As $\partial_i\Theta_0\in L^p(\mathbb{R}^2)^2$ for all $p>1$, we obtain as before

$$\|\chi\partial_i\Theta(\cdot,t)\|_{L^p(\mathbb{R}^2)} \le \frac{1}{1+t}\Big\|\partial_i\Theta_0\Big(\frac{\cdot}{\sqrt{1+t}}\Big)\Big\|_{L^p(\mathbb{R}^2)} = \frac{\|\partial_i\Theta_0\|_{L^p(\mathbb{R}^2)}}{(1+t)^{1-\frac1p}}\ , \qquad t\ge 0\ . \tag{15}$$

On the other hand, the function $\partial_i\chi$ is supported in the annulus $D=\{x\in\mathbb{R}^2\,|\,\rho\le|x|\le 2\rho\}$, and satisfies $|\partial_i\chi(x)|\le C\rho^{-1}$ for some $C>0$ independent of ρ. Moreover, it follows from (2) that

$$|\Theta(x,t)| \le \frac{1}{2\pi}\min\Big(\frac{1}{|x|}\,,\,\frac{|x|}{4(1+t)}\Big)\ , \qquad x\in\mathbb{R}^2\ , \quad t\ge 0\ ,$$

hence

$$|(\partial_i\chi(x))\Theta(x,t)| \le C\min\Big(\frac{1}{\rho^2}\,,\,\frac{1}{1+t}\Big)\mathbf{1}_D(x)\ , \qquad x\in\mathbb{R}^2\ , \quad t\ge 0\ ,$$

where $\mathbf{1}_D$ is the characteristic function of D. Taking the L^p norm of both sides, we thus obtain

$$\|(\partial_i\chi)\Theta(\cdot,t)\|_{L^p(\mathbb{R}^2)} \;\le\; C\rho^{2/p}\min\Big(\frac{1}{\rho^2}\,,\,\frac{1}{1+t}\Big) \;\le\; \frac{C}{(1+t)^{1-\frac{1}{p}}}\;, \qquad t\ge 0\;. \tag{16}$$

Combining (15) and (16), we arrive at (12).

To prove (13), we observe that

$$\begin{aligned}\|u^\chi(\cdot,t)-u^\chi(\cdot,s)\|^2_{L^2(\mathbb{R}^2)} \;&\le\; \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\frac{1}{|x|^2}\Big(e^{-\frac{|x|^2}{4(1+t)}}-e^{-\frac{|x|^2}{4(1+s)}}\Big)^2\,\mathrm{d}x\\ &=\; \frac{1}{2\pi}\log\left\{\frac{1}{2}\sqrt{\frac{1+t}{1+s}}+\frac{1}{2}\sqrt{\frac{1+s}{1+t}}\right\} \;\le\; \frac{1}{4\pi}\Big|\log\frac{1+t}{1+s}\Big|\;,\end{aligned}$$

for all $t,s\ge 0$. Finally, using (9), we find

$$\omega^\chi(x,t)-\omega^\chi(x,s) \;=\; \chi(x)\Big(\Xi(x,t)-\Xi(x,s)\Big) - \frac{x\cdot\nabla\chi(x)}{2\pi|x|^2}\Big(e^{-\frac{|x|^2}{4(1+t)}}-e^{-\frac{|x|^2}{4(1+s)}}\Big)\;.$$

Thus $\|\nabla u^\chi(\cdot,t)-\nabla u^\chi(\cdot,s)\|^2_{L^2(\mathbb{R}^2)} = \|\omega^\chi(\cdot,t)-\omega^\chi(\cdot,s)\|^2_{L^2(\mathbb{R}^2)} \le \big(J_1(t,s)^{1/2}+J_2(t,s)^{1/2}\big)^2$, where

$$\begin{aligned}J_1(t,s) \;&=\; \int_{\mathbb{R}^2}\chi(x)^2\Big(\Xi(x,t)-\Xi(x,s)\Big)^2\,\mathrm{d}x \;\le\; \int_{\mathbb{R}^2}\Big(\Xi(x,t)-\Xi(x,s)\Big)^2\,\mathrm{d}x\\ &=\; \frac{1}{8\pi}\left\{\frac{1}{1+t}+\frac{1}{1+s}-\frac{4}{t+s+2}\right\} \;\le\; \frac{1}{8\pi}\Big|\frac{1}{1+t}-\frac{1}{1+s}\Big|\;,\end{aligned}$$

and

$$\begin{aligned}J_2(t,s) \;&=\; \int_{\mathbb{R}^2}\frac{|\nabla\chi(x)|^2}{4\pi^2|x|^2}\Big(e^{-\frac{|x|^2}{4(1+t)}}-e^{-\frac{|x|^2}{4(1+s)}}\Big)^2\,\mathrm{d}x \;\le\; C\rho^{-4}\int_D\Big(e^{-\frac{|x|^2}{4(1+t)}}-e^{-\frac{|x|^2}{4(1+s)}}\Big)^2\,\mathrm{d}x\\ &\le\; C\rho^{-2}\sup_{x\in D}\Big|e^{-\frac{|x|^2}{4(1+t)}}-e^{-\frac{|x|^2}{4(1+s)}}\Big| \;\le\; C\Big|\frac{1}{1+t}-\frac{1}{1+s}\Big|\;.\end{aligned}$$

We thus obtain (14), which is the desired estimate. For later use, we also observe that $J_2(t,s)$ can be bounded by $C\rho^2(\frac{1}{1+t}-\frac{1}{1+s})^2$, for some $C>0$ independent of ρ. Since $\rho\ge 1$, this gives the alternative estimate

$$\|\nabla u^\chi(\cdot,t)-\nabla u^\chi(\cdot,s)\|^2_{L^2(\mathbb{R}^2)} \;\le\; \frac{1}{8\pi}\Big|\frac{1}{1+t}-\frac{1}{1+s}\Big| + C\rho^2\Big|\frac{1}{1+t}-\frac{1}{1+s}\Big|^{3/2}\;, \tag{17}$$

which will be used in Section 4. This concludes the proof of Lemma 2.1. $\square$

The truncated Oseen vortex is not a solution of the Navier-Stokes equation, and therefore we need to control the remainder term $R^\chi = \Delta u^\chi - \partial_t u^\chi = (\Delta\chi)\Theta + 2(\nabla\chi\cdot\nabla)\Theta$, which has the explicit expression

$$R^\chi(x,t) \;=\; \Theta(x,t)\Delta\chi(x) + 2\frac{x\cdot\nabla\chi(x)}{|x|^2}\Big(x^\perp\Xi(x,t)-\Theta(x,t)\Big)\;. \tag{18}$$

Lemma 2.2 *There exists a constant $\kappa_2>0$ (independent of ρ) such that, for any $p\in[1,\infty]$,*

$$\|R^\chi(\cdot,t)\|_{L^p(\mathbb{R}^2)} \;\le\; \frac{\kappa_2\,\rho^{\frac{2}{p}-1}}{1+t}\;, \qquad t\ge 0\;. \tag{19}$$

Moreover, for any vector field $u \in H^1_{\rm loc}(\mathbb{R}^2)^2$, we have

$$\Big|\int_{\mathbb{R}^2} R^\chi(x,t)\cdot u(x)\,\mathrm{d}x\Big| \;\le\; \frac{\kappa_2\,\rho}{1+t}\,\|\nabla u\|_{L^2(D)}~, \qquad t \ge 0~, \tag{20}$$

where $D = \{x \in \mathbb{R}^2\,|\,\rho \le |x| \le 2\rho\}$.

Proof. It is clear from (18) that $|R^\chi(x,t)| \le C\rho^{-1}(1+t)^{-1}\mathbf{1}_D(x)$ for all $x \in \mathbb{R}^2$ and all $t \ge 0$, and (19) follows immediately. Moreover, we have $R^\chi(x,t) = x^\perp Q^\chi(x,t)$ for some radially symmetric scalar function $Q(x,t)$, hence $R^\chi(\cdot,t)$ has zero mean over the annulus D. If $u \in H^1_{\rm loc}(\mathbb{R}^2)^2$ and if we denote by $\bar u$ the average of u over D, the Poincaré-Wirtinger inequality implies

$$\Big|\int_{\mathbb{R}^2} R^\chi(x,t)\cdot u(x)\,\mathrm{d}x\Big| \;=\; \Big|\int_D R^\chi(x,t)\cdot(u(x)-\bar u)\,\mathrm{d}x\Big| \;\le\; C\rho\|R^\chi(\cdot,t)\|_{L^2(\mathbb{R}^2)}\|\nabla u\|_{L^2(D)}~,$$

and using (19) with $p = 2$ we obtain (20). □

3 Asymptotic behavior of small solutions

Given $\alpha \in \mathbb{R}$, we consider solutions of (1) of the form

$$u(x,t) \;=\; \alpha u^\chi(x,t) + v(x,t)~, \qquad p(x,t) \;=\; \alpha^2 p^\chi(x,t) + q(x,t)~, \tag{21}$$

where $u^\chi(x,t)$ is the truncated Oseen vortex (8) and p^χ is the associated pressure. The perturbation $v(x,t)$ satisfies the no-slip boundary condition and the equation

$$\partial_t v + \alpha(u^\chi\cdot\nabla)v + \alpha(v\cdot\nabla)u^\chi + (v\cdot\nabla)v \;=\; \Delta v + \alpha R^\chi - \nabla q~, \qquad \operatorname{div} v \;=\; 0~, \tag{22}$$

where R^χ is given by (18). If we apply the Leray-Hopf projection P and use the fact that $PR^\chi = R^\chi$, we obtain the equivalent system

$$\partial_t v + \alpha P\Big((u^\chi\cdot\nabla)v + (v\cdot\nabla)u^\chi\Big) + P(v\cdot\nabla)v \;=\; -Av + \alpha R^\chi~, \tag{23}$$

where $A = -P\Delta$ is the Stokes operator, which is selfadjoint and nonnegative in $L^2_\sigma(\Omega)$ with domain $D(A) = L^2_\sigma(\Omega)\cap H^1_0(\Omega)^2\cap H^2(\Omega)^2$, see [8].

In this section, we fix some initial time $t_0 \ge 0$ and prove the existence of global solutions to (23) with small initial data $v_0 = v(\cdot,t_0)$ in the energy space. The integral equation associated with (23) is

$$\begin{aligned} v(t) \;=\; S(t-t_0)v_0 + \int_{t_0}^t S(t-s)\Big\{&\alpha R^\chi(s) - P(v(s)\cdot\nabla)v(s) \\ &- \alpha P\Big((u^\chi(s)\cdot\nabla)v(s) + (v(s)\cdot\nabla)u^\chi(s)\Big)\Big\}\,\mathrm{d}s~, \end{aligned} \tag{24}$$

where $v(t) \equiv v(\cdot,t)$ and $S(t) = \exp(-tA)$ is the Stokes semigroup. For $p \in (1,\infty)$, we denote by $L^p_\sigma(\Omega)$ the closure in $L^p(\Omega)^2$ of the set of all smooth divergence-free vector fields with compact support in Ω. We then have the following standard estimates :

Proposition 3.1 *The Stokes operator $-A$ generates an analytic semigroup of contractions in $L^2_\sigma(\Omega)$. Moreover, for each $t > 0$ the operator $S(t) = \exp(-tA)$ extends to a bounded linear operator from $L^q_\sigma(\Omega)$ into $L^2_\sigma(\Omega)$ for $1 < q \le 2$, and there exists a constant $C = C(q) > 0$ (independent of Ω) such that*

$$t^{\frac{1}{q}-\frac{1}{2}}\|S(t)v_0\|_{L^2(\Omega)} + t^{\frac{1}{q}}\|\nabla S(t)v_0\|_{L^2(\Omega)} \;\le\; C\|v_0\|_{L^q(\Omega)}~, \qquad t > 0~, \tag{25}$$

for all $v_0 \in L^q_\sigma(\Omega)$. In particular, we can take $C = 2$ in (25) if $q = 2$.

Since A is selfadjoint and nonnegative, it is clear that $\{S(t)\}_{t\geq 0}$ is an analytic semigroup of contractions in $L^2_\sigma(\Omega)$. In particular, we have $\|S(t)v_0\|_{L^2} \leq \|v_0\|_{L^2}$ and $t^{1/2}\|\nabla S(t)v_0\|_{L^2} = t^{1/2}\|A^{1/2}S(t)v_0\|_{L^2} \leq \|v_0\|_{L^2}$ for all $t > 0$ if $v_0 \in L^2_\sigma(\Omega)$. On the other hand, general $L^q - L^p$ estimates for $S(t)$ were established in [4, 9, 10, 21, 27], but the corresponding constants depend a priori on the domain Ω. The fact that (25) holds with C independent of Ω was already observed in [3, 20]. For the reader's convenience, we reproduce the proof of (25) in Section 5 below.

The main result of this section is:

Proposition 3.2 *Fix $\mu \in (0, 1/2)$. There exist positive constants K_0, δ, V_Ω, and T_Ω such that, if $t_0 \geq T_\Omega$, if $|\alpha| \leq \delta$, and if $\|v_0\|_{L^2(\Omega)} \leq V_\Omega$, then the perturbation equation* (23) *has a unique global solution $v \in C^0([t_0,\infty); L^2_\sigma(\Omega))$ such that*

$$\sup_{t\geq t_0}\|v(t)\|_{L^2(\Omega)} + \sup_{t>t_0}(t-t_0)^{\frac{1}{2}}\|\nabla v(t)\|_{L^2(\Omega)} \;\leq\; 4\|v_0\|_{L^2(\Omega)} + K_0\,\rho^{\frac{1}{2}}|\alpha|(1+t_0)^{-\frac{1}{4}}\;. \tag{26}$$

Here K_0 and δ are independent of Ω. In addition, if

$$M \;:=\; \sup_{\tau>0}\tau^{\mu}\|S(\tau)v_0\|_{L^2(\Omega)} + \sup_{\tau>0}\tau^{\mu+\frac{1}{2}}\|\nabla S(\tau)v_0\|_{L^2(\Omega)} \;<\; \infty\;, \tag{27}$$

then

$$\sup_{t>t_0}(t-t_0)^{\mu}\|v(t)\|_{L^2(\Omega)} + \sup_{t>t_0}(t-t_0)^{\mu+\frac{1}{2}}\|\nabla v(t)\|_{L^2(\Omega)} \;\leq\; 2M + C_\Omega|\alpha|\;, \tag{28}$$

for some $C_\Omega > 0$ depending on Ω.

Proof. We follow the classical approach of Fujita and Kato [11]. Given $t_0 \geq 0$, we introduce the Banach space $X = \{v \in C^0([t_0,\infty); L^2_\sigma(\Omega)) \cap C^0((t_0,\infty); H^1_0(\Omega)^2)\,|\,\|v\|_X < \infty\}$, equipped with the norm

$$\|v\|_X \;=\; \sup_{t\geq t_0}\|v(t)\|_{L^2} + \sup_{t>t_0}(t-t_0)^{\frac{1}{2}}\|\nabla v(t)\|_{L^2}\;.$$

If $v_0 \in L^2_\sigma(\Omega)$, we denote $\bar v(t) = S(t-t_0)v_0$ for $t \geq t_0$. In view of (25), we have $\bar v \in X$ and $\|\bar v\|_X \leq 2\|v_0\|_{L^2}$. On the other hand, given any $v \in X$ we denote, for $t \geq t_0$,

$$(Fv)(t) \;=\; \int_{t_0}^t S(t-s)\big(\alpha R^\chi(s) + \alpha G_1^v(s) + G_2^v(s)\big)\,\mathrm{d}s \;=\; \alpha F_0(t) + \alpha(F_1v)(t) + (F_2v)(t)\;,$$

where $G_1^v(s) = -P(u^\chi(s)\cdot\nabla)v(s) - P(v(s)\cdot\nabla)u^\chi(s)$ and $G_2^v(s) = -P(v(s)\cdot\nabla)v(s)$. We shall show that F maps X into X, and that there exist positive constants $C_1, C_2, C_{3,\Omega}$ (independent of t_0) such that

$$\|Fv\|_X \;\leq\; C_1\rho^{\frac{1}{2}}|\alpha|(1+t_0)^{-\frac{1}{4}} + |\alpha|C_2\|v\|_X + C_{3,\Omega}\|v\|_X^2\;, \tag{29}$$

$$\|Fv - F\tilde v\|_X \;\leq\; |\alpha|C_2\|v-\tilde v\|_X + C_{3,\Omega}(\|v\|_X + \|\tilde v\|_X)\|v-\tilde v\|_X\;, \tag{30}$$

for all $v, \tilde v \in X$.

To prove (29), we estimate separately the contributions of F_0, F_1, and F_2. First, using (25) with $q = 4/3$, we obtain for $t > t_0$:

$$\|F_0(t)\|_{L^2} + (t-t_0)^{\frac{1}{2}}\|\nabla F_0(t)\|_{L^2} \;\leq\; C\int_{t_0}^t\left(\frac{1}{(t-s)^{\frac{1}{4}}} + \frac{(t-t_0)^{\frac{1}{2}}}{(t-s)^{\frac{3}{4}}}\right)\|R^\chi(s)\|_{L^{\frac{4}{3}}}\,\mathrm{d}s\;, \tag{31}$$

and from Lemma 2.2 we know that $\|R^\chi(s)\|_{L^{4/3}} \leq C\rho^{1/2}(1+s)^{-1}$ for all $s \geq 0$. It follows that $\|F_0\|_X \leq C_1\rho^{1/2}(1+t_0)^{-1/4}$ for some $C_1 > 0$ independent of t_0 and Ω. In a similar way, we find

$$\|(F_2v)(t)\|_{L^2} + (t-t_0)^{\frac{1}{2}}\|\nabla(F_2v)(t)\|_{L^2} \;\leq\; C\int_{t_0}^t\left(\frac{1}{(t-s)^{\frac{1}{4}}} + \frac{(t-t_0)^{\frac{1}{2}}}{(t-s)^{\frac{3}{4}}}\right)\|G_2^v(s)\|_{L^{\frac{4}{3}}}\,\mathrm{d}s\;. \tag{32}$$

Using the fact that the Leray-Hopf projection is a bounded operator in $L^{4/3}(\Omega)^2$, whose norm depends a priori on Ω, we estimate

$$\|G_2^v(s)\|_{L^{\frac{4}{3}}} \le C_\Omega \|v(s)\|_{L^4} \|\nabla v(s)\|_{L^2} \le C_\Omega \|v(s)\|_{L^2}^{\frac{1}{2}} \|\nabla v(s)\|_{L^2}^{\frac{3}{2}} \le \frac{C_\Omega \|v\|_X^2}{(s-t_0)^{\frac{3}{4}}} ,$$

for all $s > t_0$. It follows that $\|F_2 v\|_X \le C_{3,\Omega} \|v\|_X^2$, where $C_{3,\Omega} > 0$ is independent of t_0. Finally, to bound F_1, we proceed in a slightly different way in order to obtain a constant C_2 that does not depend on Ω. Observing that $G_1^v(s) = -A^{1/2} A^{-1/2} P \operatorname{div}(u^\chi \otimes v + v \otimes u^\chi)(s)$, and that $\|A^{1/2} v\|_{L^2} = \|\nabla v\|_{L^2}$ for all $v \in L^2_\sigma(\Omega)) \cap H^1_0(\Omega)^2$, we can use (25) with $q = 2$ to obtain

$$\|(F_1 v)(t)\|_{L^2} \le \int_{t_0}^t (t-s)^{-\frac{1}{2}} \|A^{-1/2} P \operatorname{div}(u^\chi \otimes v + v \otimes u^\chi)(s)\|_{L^2} \,\mathrm{d}s . \tag{33}$$

Similarly, the quantity $(t-t_0)^{\frac{1}{2}} \|\nabla (F_1 v)(t)\|_{L^2}$ can be bounded by

$$\int_{t_0}^{\frac{t+t_0}{2}} \frac{(t-t_0)^{\frac{1}{2}}}{t-s} \|A^{-1/2} P \operatorname{div}(u^\chi \otimes v + v \otimes u^\chi)(s)\|_{L^2} \,\mathrm{d}s + \int_{\frac{t+t_0}{2}}^t \frac{(t-t_0)^{\frac{1}{2}}}{(t-s)^{\frac{1}{2}}} \|G_1^v(s)\|_{L^2} \,\mathrm{d}s . \tag{34}$$

Since $A^{-1/2} P \operatorname{div}$ defines a bounded operator from $L^2(\Omega)^4$ into $L^2_\sigma(\Omega)$ whose norm is less than or equal to 1 (see [29, Lemma III-2-6-1]), we have from (11)

$$\|A^{-1/2} P \operatorname{div}(u^\chi \otimes v + v \otimes u^\chi)(s)\|_{L^2} \le 2 \|u^\chi(s) v(s)\|_{L^2} \le 2 a_\infty (1+s)^{-\frac{1}{2}} \|v\|_X .$$

Moreover, using (11) and (12) we find

$$\|G_1^v(s)\|_{L^2} \le \|u^\chi(s) \nabla v(s)\|_{L^2} + \|v(s) \nabla u^\chi(s)\|_{L^2} \le \frac{a_\infty \|v\|_X}{(1+s)^{\frac{1}{2}} (s-t_0)^{\frac{1}{2}}} + \frac{b_\infty \|v\|_X}{1+s} .$$

Inserting these estimates into (33) and (34), we obtain $\|F_1 v\|_X \le C_2 \|v\|_X$ for some $C_2 > 0$ independent of t_0 and Ω. Since $Fv = \alpha F_0 + \alpha F_1 v + F_2 v$, this concludes the proof of (29), and the Lipschitz bound (30) is established in exactly the same way.

Now let $B_r = \{v \in X \,|\, \|v\|_X \le r\}$, where $r > 0$ is small enough so that $4 r C_{3,\Omega} \le 1$. If we assume that $4|\alpha| C_2 \le 1$, $8\|v_0\|_{L^2} \le r$, and $4 C_1 \rho^{1/2} |\alpha| (1+t_0)^{-1/4} \le r$, the estimates above imply that the map $v \mapsto \bar{v} + Fv$ leaves the closed ball B_r invariant and is a strict contraction in B_r. By construction, the unique fixed point of that map in B_r is the desired solution of (24). This proves the existence part of Proposition 3.2 with

$$K_0 = 2C_1 , \quad \delta = \frac{1}{4C_2} , \quad V_\Omega = \frac{1}{32 C_{3,\Omega}} , \quad T_\Omega = \Big(\frac{4 C_1 C_{3,\Omega} \rho^{\frac{1}{2}}}{C_2} \Big)^4 .$$

In a second step, we assume that (27) holds for some $\mu \in (0, 1/2)$. Given any $T > t_0$, we denote

$$\mathcal{E}_T = \sup_{t_0 \le t \le T} (t-t_0)^\mu \|v(t)\|_{L^2} + \sup_{t_0 < t \le T} (t-t_0)^{\mu + \frac{1}{2}} \|\nabla v(t)\|_{L^2} ,$$

where v is the solution of (24) constructed in the previous step. Our goal is to show that $\mathcal{E}_T$ is uniformly bounded by a constant which does not depend on T. Since $v(t) = S(t-t_0) v_0 + (Fv)(t)$, we have

$$\mathcal{E}_T \le M + \sup_{t_0 \le t \le T} (t-t_0)^\mu \|(Fv)(t)\|_{L^2} + \sup_{t_0 < t \le T} (t-t_0)^{\mu + \frac{1}{2}} \|\nabla (Fv)(t)\|_{L^2} , \tag{35}$$

where M is defined in (27). To estimate the last two terms, we proceed as above. Let $p \in (1,2)$ be such that $1/p > \mu + 1/2$, and define $q \in (2,\infty)$ by the relation $1/q = 1/p - 1/2$. As in (31) and (32), we have

$$(t-t_0)^\mu \|F_0(t)\|_{L^2} + (t-t_0)^{\mu+\frac{1}{2}} \|\nabla F_0(t)\|_{L^2} \le C \int_{t_0}^t \left(\frac{(t-t_0)^\mu}{(t-s)^{\frac{1}{q}}} + \frac{(t-t_0)^{\mu+\frac{1}{2}}}{(t-s)^{\frac{1}{p}}} \right) \|R^\chi(s)\|_{L^p} \,\mathrm{d}s \;,$$

$$(t-t_0)^\mu \|(F_2 v)(t)\|_{L^2} + (t-t_0)^{\mu+\frac{1}{2}} \|\nabla (F_2 v)(t)\|_{L^2} \le C \int_{t_0}^t \left(\frac{(t-t_0)^\mu}{(t-s)^{\frac{1}{q}}} + \frac{(t-t_0)^{\mu+\frac{1}{2}}}{(t-s)^{\frac{1}{p}}} \right) \|G_2^v(s)\|_{L^p} \,\mathrm{d}s \;,$$

for $t \in (t_0, T]$. Moreover $\|R^\chi(s)\|_{L^p} \le C\rho^{\frac{2}{p}-1}(1+s)^{-1}$ and

$$\|P(v(s)\cdot\nabla)v(s)\|_{L^p} \le C_\Omega \|v(s)\|_{L^q} \|\nabla v(s)\|_{L^2} \le C_\Omega \|v(s)\|_{L^2}^{\frac{2}{q}} \|\nabla v(s)\|_{L^2}^{2-\frac{2}{q}} \le \frac{C_\Omega \|v\|_X \mathcal{E}_T}{(s-t_0)^{\mu+1-\frac{1}{q}}} \;,$$

for all $s \in (t_0, T]$. The term involving $F_1 v$ is estimated as in (33) and (34), and we find

$$(t-t_0)^\mu \|(F_1 v)(t)\|_{L^2} \le C \int_{t_0}^t \frac{(t-t_0)^\mu \mathcal{E}_T}{(t-s)^{\frac{1}{2}}(1+s)^{\frac{1}{2}}(s-t_0)^\mu} \,\mathrm{d}s \;,$$

$$\begin{aligned} (t-t_0)^{\mu+\frac{1}{2}} \|\nabla (F_1 v)(t)\|_{L^2} &\le C \int_{t_0}^{\frac{t+t_0}{2}} \frac{(t-t_0)^{\mu+\frac{1}{2}} \mathcal{E}_T}{(t-s)(1+s)^{\frac{1}{2}}(s-t_0)^\mu} \,\mathrm{d}s \\ &\quad + C \int_{\frac{t+t_0}{2}}^t \frac{(t-t_0)^{\mu+\frac{1}{2}}}{(t-s)^{\frac{1}{2}}} \Big(\frac{\mathcal{E}_T}{(1+s)^{\frac{1}{2}}(s-t_0)^{\mu+\frac{1}{2}}} + \frac{\mathcal{E}_T}{(1+s)(s-t_0)^\mu} \Big) \,\mathrm{d}s \;. \end{aligned}$$

If we insert these estimates into (35), we obtain after elementary calculations

$$\mathcal{E}_T \le M + \tilde{C}_1 \rho^{\frac{2}{p}-1} |\alpha| (1+t_0)^{-\frac{1}{p}+\mu+\frac{1}{2}} + \tilde{C}_2 |\alpha| \mathcal{E}_T + \tilde{C}_{3,\Omega} \|v\|_X \mathcal{E}_T \;, \tag{36}$$

for some positive constants $\tilde{C}_1, \tilde{C}_2, \tilde{C}_{3,\Omega}$ independent of T and t_0. Now, taking δ and V_Ω smaller and T_Ω larger if needed, we can ensure that $\tilde{C}_2|\alpha| + \tilde{C}_{3,\Omega}\|v\|_X \le 1/2$. Then (36) implies that

$$\mathcal{E}_T \le 2M + 2\frac{\tilde{C}_1 \rho^{\frac{2}{p}-1} |\alpha|}{(1+t_0)^{\frac{1}{p}-\mu-\frac{1}{2}}} \;,$$

for all $T > t_0$, and (28) follows. This concludes the proof. □

Remark 3.3 The proof of Proposition 3.2 can be modified in a classical way [11, 2] to yield the following local existence result. For any $\alpha \in \mathbb{R}$, any $t_0 \ge 0$, and any $v_0 \in L^2_\sigma(\Omega)$, there exists $T = T(\alpha, v_0, \Omega) > 0$ such that Eq. (23) has a unique solution $v \in C^0([t_0, t_0+T]; L^2_\sigma(\Omega)) \cap C^0((t_0, t_0+T]; H^1_0(\Omega)^2)$ satisfying $v(t_0) = v_0$; moreover, any upper bound on $|\alpha| + \|v_0\|_{H^1}$ gives a lower bound on the local existence time T. In our formulation of Proposition 3.2, smallness conditions were imposed on α and v_0 to ensure global existence, and the assumption on the intial time t_0 guarantees that the smallness condition on α is independent of the domain Ω.

4 A logarithmic energy estimate

In this section, we establish our key estimate for large solutions of (23) in the energy space. Fix $\alpha \in \mathbb{R}$, $v_0 \in L^2_\sigma(\Omega)$, and let $v \in C^0([0,T]; L^2_\sigma(\Omega)) \cap C^0((0,T]; H^1_0(\Omega)^2)$ be a solution of (23) with initial data $v(0) = v_0$, see Remark 3.3. We first derive a crude bound on v using a classical energy estimate. Multiplying both sides of (23) by v and integrating by parts over Ω, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2 \;=\; \alpha\langle v(t), R^\chi(t)\rangle - \alpha\langle v(t), (v(t)\cdot\nabla)u^\chi(t)\rangle\;, \tag{37}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product in $L^2_\sigma(\Omega)$, so that $\|\cdot\|_{L^2} = \langle\cdot\,,\cdot\rangle^{1/2}$. Using (20), we easily obtain

$$|\alpha\langle v(t), R^\chi(t)\rangle| \;\le\; \frac{\kappa_2\,\rho|\alpha|}{1+t}\|\nabla v(t)\|_{L^2} \;\le\; \frac{\eta}{2}\|\nabla v(t)\|_{L^2}^2 + \frac{\kappa_2^2\rho^2\alpha^2}{2\eta(1+t)^2}\;,$$

for any $\eta \in (0,1]$. Moreover, applying (12) with $p = \infty$, we see that

$$|\langle v(t), (v(t)\cdot\nabla)u^\chi(t)\rangle| \;\le\; \frac{b_\infty}{1+t}\,\|v(t)\|_{L^2}^2\;.$$

We thus obtain the energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_{L^2}^2 + (2-\eta)\|\nabla v(t)\|_{L^2}^2 \;\le\; \frac{2b_\infty|\alpha|}{1+t}\|v(t)\|_{L^2}^2 + \frac{\kappa_2^2\rho^2\alpha^2}{\eta(1+t)^2}\;, \qquad 0 < t \le T\;.$$

Using Gronwall's lemma, we deduce that

$$\|v(t)\|_{L^2}^2 + (2-\eta)\int_{t_0}^t \|\nabla v(s)\|_{L^2}^2\,\mathrm{d}s \;\le\; \Bigl(\frac{1+t}{1+t_0}\Bigr)^{2b_\infty|\alpha|}\Bigl(\|v(t_0)\|_{L^2}^2 + \frac{\kappa_2^2\rho^2\alpha^2}{\eta(1+t_0)}\Bigr)\;, \tag{38}$$

for $0 \le t_0 < t \le T$.

We shall see that estimate (38) is pessimistic for large times, but it already implies that the solutions of (23) in the energy space $L^2_\sigma(\Omega)$ are global. Indeed, (38) shows that the norm $\|v(t)\|_{L^2}$ grows at most polynomially in time, and it is then straightforward to establish a similar result for $\|\nabla v(t)\|_{L^2}$. In particular, the H^1 norm of $v(t)$ cannot blow up in finite time, and using Remark 3.3 we conclude that all solutions of (23) in $L^2_\sigma(\Omega)$ are global.

The aim of this section is to establish the following "logarithmic energy estimate", which improves (38) for large times.

Proposition 4.1 *There exists a constant $K_1 > 0$ (independent of Ω) such that, for any $\alpha \in \mathbb{R}$ and any $v_0 \in L^2_\sigma(\Omega)$, the solution of (23) with initial data v_0 satisfies, for all $t \ge 1$,*

$$\|v(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\nabla v(s)\|_{L^2(\Omega)}^2\,\mathrm{d}s \;\le\; K_1\Bigl(\|v_0\|_{L^2(\Omega)}^2 + \alpha^2\log(1+t) + D_{\alpha,\rho}\Bigr)\;, \tag{39}$$

where $D_{\alpha,\rho} = \alpha^2\log(1+|\alpha|) + \alpha^2\rho^2$.

Proof. As in (38), we introduce here a parameter $\eta \in (0,1]$, which will be used in Section 5 below to specify the optimal smallness condition on the circulation α and prove Corollary 1.4. The reader who is not interested in optimal constants should set $\eta = 1$ everywhere.

Given any $\tau \ge 0$, we denote

$$\tilde v(x,t) \;=\; u(x,t) - \alpha u^\chi(x,t+\tau) \;=\; v(x,t) + \alpha\Bigl(u^\chi(x,t) - u^\chi(x,t+\tau)\Bigr)\;, \tag{40}$$

for all $x \in \Omega$ and all $t > 0$. Then $\tilde v$ satisfies (23) where $u^\chi(x,t)$ and $R^\chi(x,t)$ are replaced by $u^\chi(x,t+\tau)$ and $R^\chi(x,t+\tau)$, respectively. Proceeding exactly as above, we thus obtain the following energy estimate:

$$\|\tilde v(t)\|_{L^2}^2 + (2-\eta)\int_0^t \|\nabla\tilde v(s)\|_{L^2}^2\,\mathrm{d}s \;\le\; \Bigl(\frac{1+t+\tau}{1+\tau}\Bigr)^{2b_\infty|\alpha|}\Bigl(\|\tilde v(0)\|_{L^2}^2 + \frac{\kappa_2^2\rho^2\alpha^2}{\eta(1+\tau)}\Bigr)\;, \tag{41}$$

for all $t > 0$. Now, we fix $t \geq 1$ and choose $\tau = Nt - 1$, where

$$N \,=\, N_{\alpha,\eta} \,=\, \max\Big(1\,,\, \frac{2b_\infty|\alpha|}{\log(1+\eta)}\Big)\ .$$

This choice implies that

$$\Big(\frac{1+t+\tau}{1+\tau}\Big)^{2b_\infty|\alpha|} \,=\, \Big(1+\frac{1}{N}\Big)^{2b_\infty|\alpha|} \,\leq\, 1+\eta\ .$$

On the other hand, using (13), (40), we find

$$\|v(t)\|_{L^2}^2 \,\leq\, (1{+}\eta)\|\tilde v(t)\|_{L^2}^2 + \frac{1{+}\eta}{\eta}\alpha^2\|u^\chi(t) - u^\chi(t{+}\tau)\|_{L^2}^2 \,\leq\, (1{+}\eta)\|\tilde v(t)\|_{L^2}^2 + \frac{\alpha^2}{2\pi\eta}\log(N{+}1)\ ,$$

$$\|\tilde v(0)\|_{L^2}^2 \,\leq\, \frac{1{+}\eta}{\eta}\|v_0\|_{L^2}^2 + (1{+}\eta)\alpha^2\|u^\chi(0) - u^\chi(\tau)\|_{L^2}^2 \,\leq\, \frac{2}{\eta}\|v_0\|_{L^2}^2 + \frac{(1{+}\eta)\alpha^2}{4\pi}\log(Nt)\ .$$

Similarly, using (17), we find

$$\begin{aligned}\int_0^t \|\nabla v(s)\|_{L^2}^2\,\mathrm{d}s \,&\leq\, 2\int_0^t \|\nabla\tilde v(s)\|_{L^2}^2\,\mathrm{d}s + 2\alpha^2\int_0^t \|\nabla u^\chi(s) - \nabla u^\chi(s+\tau)\|_{L^2}^2\,\mathrm{d}s \\ &\leq\, 2\int_0^t \|\nabla\tilde v(s)\|_{L^2}^2\,\mathrm{d}s + \frac{\alpha^2}{4\pi}\log(1+t) + C\rho^2\alpha^2\ .\end{aligned}$$

Thus, it follows from (41) that

$$\|v(t)\|_{L^2}^2 \,\leq\, \frac{(1{+}\eta)^3\alpha^2}{4\pi}\log t + \frac{C}{\eta}\Big(\|v_0\|_{L^2}^2 + \alpha^2\log(N+1) + \alpha^2\rho^2\Big)\ , \tag{42}$$

$$\int_0^t \|\nabla v(s)\|_{L^2}^2\,\mathrm{d}s \,\leq\, \frac{(1{+}\eta)^3\alpha^2}{2\pi}\log(1+t) + \frac{C}{\eta}\Big(\|v_0\|_{L^2}^2 + \alpha^2\rho^2\Big) + C\alpha^2\log N\ , \tag{43}$$

for some universal constant $C > 0$. Setting $\eta = 1$ and using the definition of N, we see that (39) follows from (42), (43). $\square$

5 Estimate for a fractional primitive of the velocity field

In this final section, we consider the solution of (23) with initial data $v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$, for some fixed $q \in (1,2)$, and we denote $\mu = 1/q - 1/2 \in (0,1/2)$. If A is the Stokes operator in $L^2_\sigma(\Omega)$, we recall that A is selfadjoint and nonnegative in $L^2_\sigma(\Omega)$, so that the fractional power A^β can be defined for all $\beta > 0$. The following result shows that the range of A^μ contains the (dense) subspace $L^2_\sigma(\Omega) \cap L^q(\Omega)^2$.

Lemma 5.1 [3, 20] *Let $q \in (1,2)$ and $\mu = 1/q - 1/2$. For all $v \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$, there exists a unique $w \in D(A^\mu) \subset L^2_\sigma(\Omega)$ such that $v = A^\mu w$. Moreover, there exists a constant $C = C(q) > 0$ (independent of v and Ω) such that $\|w\|_{L^2(\Omega)} \leq C\|v\|_{L^q(\Omega)}$.*

Remark 5.2 *If v, w are as in Lemma 5.1, we denote $w = A^{-\mu}v$. The fact that inequality $\|w\|_{L^2(\Omega)} \leq C\|v\|_{L^q(\Omega)}$ holds with a constant C independent of the domain Ω follows directly from the proof given in [20, Lemmas 2.1 and 2.2].*

As a first application of Lemma 5.1, we give a short proof of inequality (25), which was used in Section 3.

Proof of Proposition 3.1. It is sufficient to prove (25) for $1 < q < 2$. Let $\mu = 1/q - 1/2$, and let $v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$. By Lemma 5.1, there exists a unique $w_0 \in D(A^\mu)$ such that $v_0 = A^\mu w_0$. Thus

$$\|S(t)v_0\|_{L^2(\Omega)} \;=\; \|A^\mu S(t)w_0\|_{L^2(\Omega)} \;\le\; t^{-\mu}\|w_0\|_{L^2(\Omega)} \;\le\; Ct^{-\mu}\|v_0\|_{L^q(\Omega)}\;,$$

with C depending only on q. The estimate for the first derivative is proved in the same way, since $\|\nabla S(t)v_0\|_{L^2(\Omega)} = \|A^{\mu+1/2}S(t)w_0\|_{L^2(\Omega)}$. This proves (25) for all $v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$, and the general case follows by a density argument. □

Let $v \in C^0([0,\infty); L^2_\sigma(\Omega)) \cap C^0((0,\infty); H^1_0(\Omega)^2)$ be the solution of (23) with initial data v_0, which was constructed in Sections 3 and 4. Since $v_0 \in L^q_\sigma(\Omega)$ by assumption, it is rather straightforward to verify that $v(t) \in L^q_\sigma(\Omega)$ for all $t > 0$. Thus, by Lemma 5.1, we can define $w(t) = A^{-\mu}v(t)$ for all $t > 0$. This quantity solves the equation

$$\partial_t w + Aw + \alpha F_\mu(u^\chi, v) + \alpha F_\mu(v, u^\chi) + F_\mu(v,v) \;=\; \alpha A^{-\mu}R^\chi\;, \tag{44}$$

where $F_\mu(u,v)$ is the bilinear term formally defined by

$$F_\mu(u,v) \;=\; A^{-\mu}P(u\cdot\nabla)v\;. \tag{45}$$

We refer to [20, Section 2] for a rigorous definition and a list of properties of the bilinear map F_μ. Our goal here is to establish the following estimate:

Proposition 5.3 *There exist positive constants K_2 and c (independent of Ω) such that, for any $\alpha \in \mathbb{R}$ and any solution v of (23) with initial data $v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$, the function $w(t) = A^{-\mu}v(t)$ satisfies, for all $t \ge 1$,*

$$\|w(t)\|^2_{L^2} + \int_0^t \|\nabla w(s)\|^2_{L^2}\,\mathrm{d}s \;\le\; K_2(1+t)^{c\alpha^2}\exp\Big(K_2(\|v_0\|^2_{L^2} + D_{\alpha,\rho})\Big)\big(\|v_0\|^2_{L^q} + \rho^2\alpha^2\big)\;, \tag{46}$$

where $D_{\alpha,\rho} = \alpha^2\log(1+|\alpha|) + \alpha^2\rho^2$.

Proof. Taking the scalar product of both sides of (44) with w, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2_{L^2} + \|A^{1/2}w(t)\|^2_{L^2} + \alpha\langle F_\mu(u^\chi(t), v(t)), w(t)\rangle + \alpha\langle F_\mu(v(t), u^\chi(t)), w(t)\rangle \\ + \langle F_\mu(v(t), v(t)), w(t)\rangle \;=\; \alpha\langle A^{-\mu}R^\chi(t), w(t)\rangle\;.\end{aligned} \tag{47}$$

We recall that $\|A^{1/2}w\|_{L^2} = \|\nabla w\|_{L^2}$ for all $w \in D(A^{1/2}) = L^2_\sigma(\Omega) \cap H^1_0(\Omega)^2$. To bound the other terms, we observe that

$$\begin{aligned}|\langle F_\mu(u^\chi, v), w\rangle| \;&=\; |\langle (u^\chi\cdot\nabla)v, A^{-\mu}w\rangle| \;=\; |\langle (u^\chi\cdot\nabla)A^{-\mu}w, v\rangle| \\ &\le\; \|u^\chi\|_{L^\infty}\|A^{\frac12-\mu}w\|_{L^2}\|v\|_{L^2} \;=\; \|u^\chi\|_{L^\infty}\|A^{\frac12-\mu}w\|_{L^2}\|A^\mu w\|_{L^2} \\ &\le\; \|u^\chi\|_{L^\infty}\|A^{1/2}w\|_{L^2}\|w\|_{L^2}\;,\end{aligned}$$

where in the last inequality we used the interpolation inequality for fractional powers of A. The same argument shows that $|\langle F_\mu(v, u^\chi), w\rangle| \le \|u^\chi\|_{L^\infty}\|A^{1/2}w\|_{L^2}\|w\|_{L^2}$. In a similar way, we find

$$\begin{aligned}|\langle F_\mu(v,v), w\rangle| \;&=\; |\langle (v\cdot\nabla)v, A^{-\mu}w\rangle| \;=\; |\langle (v\cdot\nabla)A^{-\mu}w, v\rangle| \\ &\le\; \|v\|^2_{L^4}\|A^{\frac12-\mu}w\|_{L^2} \;\le\; C_*^2\|\nabla v\|_{L^2}\|v\|_{L^2}\|A^{\frac12-\mu}w\|_{L^2} \\ &\le\; C_*^2\|\nabla v\|_{L^2}\|A^{1/2}w\|_{L^2}\|w\|_{L^2}\;,\end{aligned}$$

where $C_* > 0$ is the best constant of Gagliardo-Nirenberg's inequality

$$\|f\|_{L^4(\mathbb{R}^2)} \,\le\, C_* \|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \,. \tag{48}$$

Finally, since $|\langle A^{-\mu} R^\chi, w\rangle| \,=\, |\langle R^\chi, A^{-\mu} w\rangle| \,\le\, \kappa_2 \rho (1+t)^{-1} \|A^{\frac{1}{2}-\mu} w\|_{L^2}$ by (20), we can use interpolation and Young's inequality to obtain

$$|\alpha\langle A^{-\mu} R^\chi, w\rangle| \,\le\, \frac{\kappa_2 \rho|\alpha|}{1+t} \|A^{1/2} w\|_{L^2}^{1-2\mu} \|w\|_{L^2}^{2\mu} \,\le\, \frac{\eta}{4}\|A^{1/2} w\|_{L^2}^2 + \frac{\|w\|_{L^2}^2}{2(1+t)^{\gamma_1}} + \frac{C_\eta \rho^2 \alpha^2}{2(1+t)^{\gamma_2}} \,,$$

for some exponents $\gamma_1, \gamma_2 > 1$ satisfying $\gamma_2 + 2\mu\gamma_1 = 2$. Here $\eta \in (0,1]$ is as in the proof of Proposition 4.1, and $C_\eta > 0$ denotes a constant depending only on η. Inserting all these estimates into (47), we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 + 2\|\nabla w\|_{L^2}^2 \,\le\, 2H \|\nabla w\|_{L^2} \|w\|_{L^2} + \frac{\eta}{2}\|\nabla w\|_{L^2}^2 + \frac{\|w\|_{L^2}^2}{(1+t)^{\gamma_1}} + \frac{C_\eta \rho^2 \alpha^2}{(1+t)^{\gamma_2}} \,, \tag{49}$$

where $H = 2|\alpha| \|u^\chi\|_{L^\infty} + C_*^2 \|\nabla v\|_{L^2}$.

To exploit (49), we apply Young's inequality again and obtain the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 + \eta\|\nabla w\|_{L^2}^2 \,\le\, \Big(\frac{H^2}{2-3\eta/2} + \frac{1}{(1+t)^{\gamma_1}}\Big)\|w\|_{L^2}^2 + \frac{C_\eta \rho^2 \alpha^2}{(1+t)^{\gamma_2}} \,,$$

which can be integrated using Gronwall's lemma. The result is

$$\|w(t)\|_{L^2}^2 + \eta \int_0^t \|\nabla w(s)\|_{L^2}^2 \,\mathrm{d}s \,\le\, C \exp\Big(\frac{\Phi(t)}{1-3\eta/4}\Big)\Big(\|w_0\|_{L^2}^2 + C_\eta \rho^2 \alpha^2\Big) \,, \quad t \ge 0 \,, \tag{50}$$

where $\Phi(t) = \frac{1}{2}\int_0^t H(s)^2 \,\mathrm{d}s$ and C is a positive constant depending only on γ_1, γ_2. It remains to estimate the quantity $\Phi(t)$ in (50). Using (11) with $p = \infty$, the logarithmic energy estimate (43), and Minkowski's inequality, we find

$$\begin{aligned}
2\Phi(t) \,&=\, \int_0^t H(s)^2 \,\mathrm{d}s \,\le\, \int_0^t \Big\{\frac{2|\alpha| a_\infty}{(1+s)^{1/2}} + C_*^2 \|\nabla v(s)\|_{L^2}\Big\}^2 \mathrm{d}s \\
&\le\, \Big\{|\alpha| \log(1+t)^{1/2}\Big(2a_\infty + \frac{C_*^2 (1+\eta)^{\frac{3}{2}}}{\sqrt{2\pi}}\Big) + C_\eta(\|v_0\|_{L^2} + D_{\alpha,\rho}^{1/2})\Big\}^2 \\
&\le\, 2C_0 (1+\eta)^4 \alpha^2 \log(1+t) + C_\eta(\|v_0\|_{L^2}^2 + D_{\alpha,\rho}) \,, \qquad t \ge 1 \,,
\end{aligned} \tag{51}$$

where $D_{\alpha,\rho} = \alpha^2 \log(1+|\alpha|) + \alpha^2 \rho^2$ and

$$C_0 \,=\, \frac{1}{2}\Big(2a_\infty + \frac{C_*^2}{\sqrt{2\pi}}\Big)^2 \,. \tag{52}$$

If we now replace (51) into (50) and set $\eta = 1$, we obtain (46) since $\|w_0\|_{L^2} \le C\|v_0\|_{L^q}$ by Lemma 5.1. This concludes the proof. □

Corollary 5.4 *Under the assumptions of Proposition 5.3, there exists a positive constant K depending on Ω, α, and $\|v_0\|_{L^2 \cap L^q}$ such that, for any $T \ge 2$, there exists a time $t \in [T/2, T]$ for which*

$$\|v(t)\|_{L^2(\Omega)}^2 \,\le\, K(1+t)^{c\alpha^2 - 2\mu} \,. \tag{53}$$

Proof. Fix $T \geq 2$. In view of (46), there exists a time $t \in [T/2, T]$ such that

$$\|\nabla w(t)\|_{L^2}^2 \;\leq\; \frac{2}{T}\int_{T/2}^{T} \|\nabla w(s)\|_{L^2}^2 \,\mathrm{d}s \;\leq\; \frac{2}{T}C(1+T)^{c\alpha^2} \;\leq\; 2^{c\alpha^2+2}C(1+t)^{c\alpha^2-1}\;,$$

where C depends on ρ, α, and $\|v_0\|_{L^2\cap L^q}$. Moreover, $\|w(t)\|_{L^2}^2 \leq C(1+t)^{c\alpha^2}$ by (46). Thus, using the interpolation inequality $\|v(t)\|_{L^2} = \|A^\mu w(t)\|_{L^2} \leq \|\nabla w(t)\|_{L^2}^{2\mu}\,\|w(t)\|_{L^2}^{1-2\mu}$, we obtain (53). $\square$

Proof of Theorem 1.2. Fix $q \in (1,2)$, and assume that $\epsilon > 0$ is small enough so that $c\epsilon^2 < 2\mu$, where $\mu = 1/q - 1/2$ and c is as in Proposition 5.3. We also suppose that $\epsilon \leq \delta$, where $\delta > 0$ is as in Proposition 3.2. Given $\alpha \in [-\epsilon, \epsilon]$ and $v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$, let $v \in C^0([0,\infty); L^2_\sigma(\Omega)) \cap C^0((0,\infty); H^1_0(\Omega)^2)$ be the solution of (23) with initial data $v(0) = v_0$, which was constructed in Sections 3 and 4. In view of (53), since $c\alpha^2 < 2\mu$, we can take $t_0 > 0$ large enough (depending on Ω, α, and v_0) so that $\|v(t_0)\|_{L^2} \leq V_\Omega$, where V_Ω is as in Proposition 3.2. Moreover, since $v(t_0) = A^\mu w(t_0)$ for some $w(t_0) \in L^2_\sigma(\Omega)$, we have

$$\sup_{\tau>0} \tau^\mu \|S(\tau)v_0\|_{L^2} + \sup_{\tau>0} \tau^{\mu+\frac{1}{2}} \|\nabla S(\tau)v_0\|_{L^2} \;\leq\; C\|w(t_0)\|_{L^2} \;<\; \infty\;.$$

Applying Proposition 3.2, we conclude that the solution v of (23) satisfies (28), namely

$$\|u(\cdot,t) - \alpha u^\chi(\cdot,t)\|_{L^2} + t^{1/2}\|\nabla u(\cdot,t) - \alpha\nabla u^\chi(\cdot,t)\|_{L^2(\Omega)} \;=\; \mathcal{O}(t^{-\mu})\;, \tag{54}$$

as $t \to \infty$. But $\|u^\chi - \Theta\|_{L^2} + \|\nabla u^\chi - \nabla\Theta\|_{L^2} \leq C(1+t)^{-1}$ for all $t \geq 0$, hence (6) follows from (54). $\square$

Proof of Corollary 1.4. The proof of Proposition 5.3 shows that the constant c in (46), (53) satisfies $c \leq C_0(1 + \mathcal{O}(\eta))$, where C_0 is defined in (52) and $\eta \in (0,1]$ can be chosen arbitrarily small. On the other hand, since by assumption $v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$ for all $q \in (1,2)$, we can take $\mu = 1/q - 1/2$ arbitrarily close to $1/2$. Thus, if we assume that $|\alpha| < \epsilon_* = C_0^{-1/2}$, we see that the condition $c\alpha^2 < 2\mu$ can be fulfilled by an appropriate choice of η and μ. Now, take $t \geq 2$ and let $t_0 \in [t/2, t]$ be the time defined in Corollary 5.4, for which $\|v(t_0)\|_{L^2}^2 \leq K(1+t_0)^{c\alpha^2-2\mu}$. Using (38) with $\eta = 1$, we conclude

$$\|v(t)\|_{L^2}^2 \;\leq\; C\Big(\frac{1+t}{1+t_0}\Big)^{2b_\infty|\alpha|}\Big(\|v(t_0)\|_{L^2}^2 + (1+t_0)^{-1}\Big) \;\leq\; C(1+t)^{c\alpha^2-2\mu} \;\xrightarrow[t\to\infty]{}\; 0\;,$$

which is the desired result. Here the constant $C > 0$ depends on α, ρ, and v_0, but not on t. To estimate ϵ_*, we use (52) and observe that $a_\infty = \|\Theta_0\|_{L^\infty} \approx 0.050784$. Moreover, the optimal constant in the Gagliardo-Nirenberg inequality (48) satisfies $C_*^4 \leq 2/(3\pi)$, see [6]. Using these values, we find $C_0 \leq 0.0407108$, hence $\epsilon_* = C_0^{-1/2} \geq 4.95616$. Finally, it was kindly pointed to us by Jean Dolbeault that the optimal constant C_* can be computed numerically: $C_* \approx 0.6430$. This yields the approximate value $\epsilon_* \approx 5.306$. $\square$

6 Appendix: Proof of Proposition 1.3

We recall the following characterization of the space $\dot W^{1,p}_{0,\sigma}(\Omega)$ for $1 \leq p < 2$:

$$\dot W^{1,p}_{0,\sigma}(\Omega) \;=\; \big\{u \in L^{\frac{2p}{2-p}}(\Omega)^2 \;\big|\; \|\nabla u\|_{L^p} < \infty\,,\; u = 0 \text{ on } \partial\Omega\,,\; \operatorname{div} u = 0 \text{ in } \Omega\big\}\;, \tag{55}$$

see e.g. [12, Chapter III.5]. Here ∇u and $\operatorname{div} u$ denote weak derivatives of u, and the condition "$u = 0$ on $\partial\Omega$" means that the boundary trace of u, which is well defined because $\nabla u \in L^p(\Omega)^4$, vanishes.

Given $u_0 \in \dot{W}^{1,p}_{0,\sigma}(\Omega)$ satisfying (7), we define $u : \mathbb{R}^2 \to \mathbb{R}^2$ and $\omega : \mathbb{R} \to \mathbb{R}^2$ by

$$u(x) \;=\; \begin{cases} u_0(x) & \text{if} \quad x \in \Omega\ , \\ 0 & \text{if} \quad x \notin \Omega\ , \end{cases} \qquad \omega(x) \;=\; \begin{cases} \omega_0(x) & \text{if} \quad x \in \Omega\ , \\ 0 & \text{if} \quad x \notin \Omega\ . \end{cases}$$

Since $u = 0$ on $\partial\Omega$, we have $\nabla u \in L^p(\mathbb{R}^2)^4$ and $\partial_1 u_2 - \partial_2 u_1 = \omega \in L^p(\mathbb{R}^2)$. Moreover (7) implies that $\omega \in L^2(m)$ for some $m > 2/q > 1$, where

$$L^2(m) \;=\; \Big\{\omega \in L^2(\mathbb{R}^2) \,\Big|\, \int_{\mathbb{R}^2} (1+|x|^2)^m |\omega(x)|^2 \,\mathrm{d}x \;<\; \infty \Big\}\ .$$

Thus, using Hölder's inequality, it is easy to verify that $\omega \in L^1(\mathbb{R}^2)$, so that we can define

$$\alpha \;=\; \int_{\mathbb{R}^2} \omega(x)\,\mathrm{d}x \;=\; \int_\Omega \omega_0(x)\,\mathrm{d}x\ .$$

Moreover, using the the Biot-Savart formula in $\mathbb{R}^2$ and the fact that $u \in L^{2p/(p-2)}(\mathbb{R}^2)^2$, we obtain the equality

$$u(x) \;=\; \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\,\omega(y)\,\mathrm{d}y \;=\; \frac{1}{2\pi}\int_\Omega \frac{(x-y)^\perp}{|x-y|^2}\,\omega_0(y)\,\mathrm{d}y\ , \tag{56}$$

for almost all $x \in \mathbb{R}^2$. We emphasize at this point that the representation (56) is *not* what is usually called the Biot-Savart law in the domain Ω, because the velocity field defined by (56) for an arbitrary vorticity $\omega_0 \in L^1(\Omega)$ will not, in general, be tangent to the boundary on $\partial\Omega$. However, if we start from a velocity field u_0 that vanishes on $\partial\Omega$, the argument above shows that (56) holds with $\omega_0 = \operatorname{curl} u_0$. We refer to [17] for a more detailed discussion of the Biot-Savart law in a two-dimensional exterior domain.

Now, we decompose

$$u(x) \;=\; \alpha u^\chi(x,0) + v(x)\ , \qquad \omega(x) \;=\; \alpha\omega^\chi(x,0) + w(x)\ , \qquad x \in \mathbb{R}^2\ ,$$

where u^χ, ω^χ are defined in (8), (9). By construction, we have $w \in L^2(m)$ and $\int_{\mathbb{R}^2} w\,\mathrm{d}x = 0$. Applying [13, Proposition B.1], we deduce that the corresponding velocity field v, which is obtained from w via the Biot-Savart law in $\mathbb{R}^2$, satisfies

$$\int_{\mathbb{R}^2} (1+|x|^2)^{\frac{mr}{2}-1} |v(x)|^r \,\mathrm{d}x \;<\; \infty\ ,$$

for all $r > 2$. Using Hölder's inequality again, we conclude that $v \in L^s(\mathbb{R}^2)^2$ for all $s > 2/m$, hence in particular $v \in L^2(\mathbb{R}^2)^2 \cap L^q(\mathbb{R}^2)^2$. Clearly $v(x) = 0$ for all $x \notin \Omega$, hence denoting by v_0 the restriction of v to Ω we obtain (4) with $v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$. ☐

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References

[1] H. Bae and B. Jin, Asymptotic behavior for the Navier-Stokes equations in 2D exterior domains. J. Funct. Anal. **240** (2006), 508–529.

[2] H. Brezis, Remarks on the preceding paper by M. Ben-Artzi: "Global solutions of two-dimensional Navier-Stokes and Euler equations". Arch. Rational Mech. Anal. **128** (1994), 359–360.

[3] W. Borchers and T. Miyakawa, L^2-decay for Navier-Stokes flows in unbounded domains, with application to exterior stationary flows. Arch. Rational Mech. Anal. **118** (1992), 273–295.

[4] W. Borchers and W. Varnhorn, On the boundedness of the Stokes semigroup in two-dimensional exterior domains. Math. Z. **213** (1993), 275–299.

[5] A. Carpio, Asymptotic behavior for the vorticity equations in dimensions two and three. Commun. in PDE **19** (1994), 827–872.

[6] M. Del Pino and J. Dolbeault, Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions. J. Math. Pures Appl. **81** (2002), 847–875.

[7] C. He and T. Miyakawa, Nonstationary Navier-Stokes flows in a two-dimensional exterior domain with rotational symmetries. Indiana Univ. Math. J. **55** (2006), 1483–1555.

[8] P. Constantin and C. Foias, *Navier-Stokes equations.* University of Chicago Press, 1988.

[9] W. Dan and Y. Shibata, On the L_q–L_r estimates of the Stokes semigroup in a two-dimensional exterior domain. J. Math. Soc. Japan **51** (1999), 181–207.

[10] W. Dan and Y. Shibata, Remark on the L_q-L_∞ estimate of the Stokes semigroup in a 2-dimensional exterior domain. Pacific J. Math. **189** (1999), 223–239.

[11] H. Fujita and T. Kato, On the Navier-Stokes initial value problem. I. Arch. Rational Mech. Anal. **16** (1964), 269–315.

[12] G. P. Galdi, *An introduction to the Mathematical Theory of the Navier-Stokes equations, Vol.I.* Springer, New York, 1994.

[13] Th. Gallay and C. E. Wayne, Invariant manifolds and the long-time asymptotics of the Navier-Stokes and vorticity equations on $\mathbb{R}^2$. Arch. Ration. Mech. Anal. **163** (2002), 209–258.

[14] Th. Gallay and C. E. Wayne, Global Stability of vortex solutions of the two dimensional Navier-Stokes equation. Comm. Math. Phys. **255** (2005), 97–129.

[15] Y. Giga and T. Kambe, Large time behavior of the vorticity of two dimensional viscous flow and its application to vortex formation. Comm. Math. Phys. **117**, (1988) 549–568.

[16] D. Iftimie, G. Karch, and C. Lacave, Self-similar asymptotics of solutions to the Navier-Stokes system in two dimensional exterior domain. Preprint, arXiv:1107.2054.

[17] D. Iftimie, M. Lopes Filho, and H. Nussenzveig Lopes, Two dimensional incompressible ideal flow around a small obstacle. Comm. Partial Differential Equations **28** (2003), 349–379.

[18] T. Kato and H. Fujita, On the nonstationary Navier-Stokes system. Rend. Sem. Mat. Univ. Padova **32** (1962), 243–260.

[19] H. Kozono and T. Ogawa, Two-dimensional Navier-Stokes flow in unbounded domains. Math. Ann. **297** (1993), 1–31.

[20] H. Kozono and T. Ogawa, Decay properties of strong solutions for the Navier-Stokes equations in two-dimensional unbounded domains. Arch. Rational Mech. Anal. **122** (1993), 1–17.

[21] H. Kozono and M. Yamazaki, Local and global unique solvability of the Navier-Stokes exterior problem with Cauchy data in the space $L^{n,\infty}$. Houston J. Math. **21** (1995), 755–799.

[22] O. Ladyženskaja, Solution "in the large" of the nonstationary boundary value problem for the Navier-Stokes system with two space variables. Comm. Pure Appl. Math. **12** (1959), 427–433.

[23] J. Leray, Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique. J. Math. Pures Appliquées 9ème série **12** (1933), 1–82.

[24] J. Leray, Essai sur les mouvements plans dun fluide visqueux que limitent des parois. J. Maths Pures Applquées 9ème série **13** (1934), 331–418.

[25] J.-L. Lions and G. Prodi, Un théorème d'existence et unicité dans les équations de Navier-Stokes en dimension 2. C. R. Acad. Sci. Paris **248** (1959), 3519-3521.

[26] A. Majda and A. Bertozzi, *Vorticity and incompressible flow.* Cambridge Texts in Applied Mathematics **27**, Cambridge, 2002.

[27] P. Maremonti and V. Solonnikov, On nonstationary Stokes problem in exterior domains. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **24** (1997), 395–449.

[28] K. Masuda, Weak solutions of Navier-Stokes equations. Tohoku Math. J. **36** (1984), 623–646.

[29] H. Sohr, *The Navier-Stokes equations. An elementary functional analytic approach.* Birkhäuser, 2001.