Invariant manifolds and the long-time asymptotics of the Navier-Stokes and vorticity equations on $\mathbf{R}^2$

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Abstract

We construct finite-dimensional invariant manifolds in the phase space of the Navier-Stokes equation on $\mathbf{R}^2$ and show that these manifolds control the long-time behavior of the solutions. This gives geometric insight into the existing results on the asymptotics of such solutions and also allows one to extend those results in a number of ways.

1 Introduction

In the last decade and a half, starting with the work of T. Kato, K. Masuda, M. Schonbek, and M. Wiegner, the long-time behavior of solutions of the Navier-Stokes equation (and the related vorticity equation) on unbounded spatial domains has been extensively studied. (See [20], [22], [26], [19], [31], [1] [15], [27] [5], [6], [11], [28], [24] and [23] for a small sampling of this literature.) This prior work used a variety of techniques including energy estimates, the Fourier splitting method, and a detailed analysis of the semigroup of the linear part of the equation. In the present paper we introduce another approach, based on ideas from the theory of dynamical systems, to compute these asymptotics. We prove that there exist finite-dimensional invariant manifolds in the phase space of these equations, and that all solutions in a neighborhood of the origin approach one of these manifolds with a rate that can be easily computed. Thus, computing the asymptotics of solutions up to that order is reduced to the simpler task of determining the asymptotics of the system of *ordinary* differential equations that result when the original partial differential equation is restricted to the invariant manifold. Although it is technically quite different from our work, Foias and Saut, [10], have also used invariant manifold theory to study the long-time behavior of solutions of the Navier-Stokes equation in a *bounded* domain.

As we shall see, if one specifies some given order of decay in advance, say $\mathcal{O}(t^{-k})$, one can find a (finite-dimensional) invariant manifold such that all small solutions approach the manifold with at least that rate. Therefore, one can in principle compute the asymptotics of solutions to any order with this method. In Subsection 4.1, for instance, we illustrate how these ideas can be used to extend known results about the stability of the Oseen vortex. In particular, we show that as solutions approach the vortex their velocity fields have a universal profile, see Corollary 4.8.

If one adopts a dynamical systems point of view towards the question of the long-time behavior of the solutions, many phenomena which have been investigated in finite-dimensional dynamics immediately suggest possible effects in the Navier-Stokes equation. As an example, one knows that in ordinary differential equations the presence of resonances between the eigenvalues of the linearized equations may produce dramatic changes in the asymptotic behavior of solutions. Exploiting this observation in Subsection 4.2 we construct solutions of the Navier-Stokes and vorticity equations whose asymptotic expansion contains logarithmic terms in time, see Corollary 4.15.

Finally, we feel that the geometric insights that the invariant manifold method provides are very valuable. As an example of the sorts of insights this method provides, we reexamine the results of Miyakawa and Schonbek [23] on optimal decay rates in Subsection 4.3. We find that the moment conditions derived in [23] have a very simple geometric interpretation – they are the analytic expression of the requirement that the solutions lie on certain invariant manifolds.

To explain our results and methods in somewhat more detail, we recall that the Navier-Stokes equation in $\mathbf{R}^2$ has the form

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \Delta\mathbf{u} - \nabla p \;, \quad \nabla\cdot\mathbf{u} = 0 \;, \tag{1}$$

where $\mathbf{u} = \mathbf{u}(x,t) \in \mathbf{R}^2$ is the velocity field, $p = p(x,t) \in \mathbf{R}$ is the pressure field, and $x \in \mathbf{R}^2$, $t \ge 0$. For simplicity, the kinematic viscosity has been rescaled to 1 in Eq.(1).

Our results are based on two ideas. The first observation is that it is easier to derive the asymptotics of (1) by working with the vorticity formulation of the problem. That is, we set $\omega = \mathrm{rot}\,\mathbf{u} = \partial_1 u_2 - \partial_2 u_1$ and study the equation (2) for ω rather than (1) itself. One can then recover the solution of the Navier-Stokes equation via the Biot-Savart law (3) (see Lemma 2.1 and Appendix B for estimates of various L^p norms of the velocity field in terms of the vorticity.) While there have been some studies of the asymptotics of solutions of the vorticity equation (notably [5] and [15]), the relationship between the vorticity and velocity does not seem to have been systematically exploited to study the asymptotics of the Navier-Stokes equations. We feel that approaching the problem through the vorticity equation has a significant advantage. It has been realized for some time that the decay rate in time of solutions of (1) is affected by the decay rate in space of the initial data. For instance, general solutions of the Navier-Stokes equation in $L^2(\mathbf{R}^2)$ with initial data in $L^2 \cap L^1$ have L^2 norm decaying like $Ct^{-1/2}$ as $t \to +\infty$. If in addition the initial data satisfy $\int_{\mathbf{R}^2}(1+|x|)|\mathbf{u}_0(x)|\,\mathrm{d}x < \infty$, it follows from Wiegner's result [31] that this decay rate can be improved to Ct^{-1}. However, the semiflow generated by the Navier-Stokes equation does not preserve such a condition. For example, in Subsection 4.3, we prove that there exist solutions of (1) which satisfy $\int_{\mathbf{R}^2}(1+|x|)|\mathbf{u}(x,t)|\,\mathrm{d}x < \infty$ when $t = 0$, but not for later times. The vorticity equation does not suffer from this drawback –

as we demonstrate in Section 3, solutions of the vorticity equation in $\mathbf{R}^2$ which lie in weighted Sobolev spaces at time zero remain in those spaces for all time. A similar result holds for small solutions of the vorticity equation in $\mathbf{R}^3$, see [14]. As we will see in later sections, these decay conditions are crucial for determining the asymptotic behavior of the solutions, and the fact that the vorticity equation preserves them makes it natural to work in the vorticity formulation. It should be noted, however, that this approach requires the vorticity to decay sufficiently rapidly as $|x| \to \infty$ and this is not assumed in [31], for instance. On the other hand, assuming that the vorticity decreases rapidly at infinity is very reasonable from a physical point of view. This is the case, for instance, if the initial data are created by stirring the fluid with a (finite size) tool. Note that in contrast, even very localized stirring may not result in a velocity field that decays rapidly as x tends to infinity. As an example, in Subsection 4.1 we discuss the Oseen vortex which is a solution of (1) whose vorticity is Gaussian, but whose velocity field is not even in L^2.

The second idea that helps to understand the long-time asymptotics of (1) is the introduction of scaling variables (see (12) and (13) below). It has, of course, been realized for a long-time that in studying the long-time asymptotics of parabolic equations it is natural to work with rescaled spatial variables. However, these variables do not seem to have been used very much in the context of the Navier-Stokes equation. (Though they are exploited in a slightly different context in [4].) For our work they offer a special advantage. If one linearizes the Navier-Stokes equation or vorticity equation around the zero solution, the resulting linear equation has continuous spectrum that extends all the way from minus infinity to the origin. If one wishes to construct finite-dimensional invariant manifolds in the phase space of these equations which control the asymptotics of solutions, it is not obvious how to do this even for the linearized equation – let alone for the full nonlinear problem. However, building on ideas of [30] we show that, when rewritten in terms of the scaling variables, the linearized operator has an infinite set of eigenvalues with explicitly computable eigenfunctions, and the continuous spectrum can be pushed arbitrarily far into the left half plane by choosing the weighted Sobolev spaces in which we work appropriately. (See Appendix A for more details.) We are then able to construct invariant manifolds tangent at the origin to the eigenspaces of the point spectrum of this linearized equation and exploit the ideas that have been developed in finite-dimensional dynamics to analyze the asymptotic behavior of solutions of these partial differential equations.

We conclude this introduction with a short survey of the remainder of the paper. Although our ideas are applicable in all dimensions we consider here the case of fluids in two dimensions. Because the vorticity is a scalar in this case many calculations are simplified, and we feel that the central ideas of our method can be better seen without being obscured by technical details. Thus, in Section 2 we begin by surveying the (well developed) existence and uniqueness theory for the two-dimensional vorticity equation. In Section 3 we introduce scaling variables and the weighted Sobolev spaces that we use in our analysis. We then show (see Theorem 3.5) that in this formulation there exist families of finite-dimensional invariant manifolds in the phase space of the problem, and that all solutions near the origin either lie in, or approach these manifolds at a computable rate. Furthermore, we obtain a geometrical characterization of solutions that approach the origin "faster than expected", see Theorem 3.8. In Section 4 we apply the invariant manifolds constructed in the previous section to derive the above mentioned results about the long-time behavior of the vorticity and Navier-Stokes equations. In a companion paper

[14], we obtain similar results for the small solutions of the Navier-Stokes equation in three dimensions by a slightly different method. Finally, there are three appendices which derive a number of facts we need in the previous sections. In Appendix A, we study the spectrum of the linear operator $\mathcal{L}$ appearing in the rescaled vorticity equation (14), and we obtain sharp estimates on the semigroup it generates. In Appendix B, we study in detail the relationship between the velocity field $\mathbf{u}$ and the associated vorticity ω. In particular, we show how the spatial decay of $\mathbf{u}$ is related to the moments of the vorticity ω. Appendix C derives a technical estimate on the invariant manifold constructed in Subsection 4.2.

Notation: Throughout the paper we use boldface letters for vector-valued functions, such as $\mathbf{u}(x,t)$. However, to avoid a proliferation of boldface symbols we use standard italic characters for points in $\mathbf{R}^2$, such as $x = (x_1, x_2)$. In both cases, $|\cdot|$ denotes the Euclidean norm in $\mathbf{R}^2$. For any $p \in [1,\infty]$ we denote by $|f|_p$ the norm of a function in the Lebesgue space $L^p(\mathbf{R}^2)$. If $\mathbf{f} \in (L^p(\mathbf{R}^2))^2$, we set $|\mathbf{f}|_p = |\ |\mathbf{f}|\ |_p$. Weighted norms play an important role in this paper. We introduce the weight function $b : \mathbf{R}^2 \to \mathbf{R}$ defined by $b(x) = (1+|x|^2)^{\frac{1}{2}}$. For any $m \geq 0$, we set $\|f\|_m = |b^m f|_2$, and denote the resulting Hilbert space by $L^2(m)$. If $f \in C^0([0,T], L^p(\mathbf{R}^2))$, we often write $f(\cdot,t)$ or simply $f(t)$ to denote the map $x \to f(x,t)$. Finally, we denote by C a generic positive constant, which may differ from place to place, even in the same chain of inequalities.

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2 The Cauchy problem for the vorticity equation

In this section we describe existence and uniqueness results for solutions of the vorticity equation. As stressed in the introduction, our approach is to study in detail the behavior of solutions of the vorticity equation, and then to derive information about the solutions of the Navier-Stokes equation as a corollary. The results in this section are not new and are reproduced here for easy reference.

In two dimensions, the vorticity equation is

$$\omega_t + (\mathbf{u}\cdot\nabla)\omega = \Delta\omega \ , \tag{2}$$

where $\omega = \omega(x,t) \in \mathbf{R}$, $x = (x_1,x_2) \in \mathbf{R}^2$, $t \geq 0$. The velocity field $\mathbf{u}$ is defined in terms of the vorticity via the Biot-Savart law

$$\mathbf{u}(x) = \frac{1}{2\pi}\int_{\mathbf{R}^2} \frac{(\mathbf{x}-\mathbf{y})^\perp}{|x-y|^2}\omega(y)\,\mathrm{d}y \ , \quad x \in \mathbf{R}^2 \ . \tag{3}$$

Here and in the sequel, if $x = (x_1,x_2) \in \mathbf{R}^2$, we denote $\mathbf{x} = (x_1,x_2)^{\mathrm{T}}$ and $\mathbf{x}^\perp = (-x_2,x_1)^{\mathrm{T}}$.

The following lemma collects useful estimates for the velocity $\mathbf{u}$ in terms of ω.

Lemma 2.1 *Let $\mathbf{u}$ be the velocity field obtained from ω via the Biot-Savart law (3).*
(a) *Assume that $1 < p < 2 < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$. If $\omega \in L^p(\mathbf{R}^2)$, then $\mathbf{u} \in L^q(\mathbf{R}^2)^2$, and there exists $C > 0$ such that*

$$|\mathbf{u}|_q \le C|\omega|_p \ . \tag{4}$$

(b) *Assume that $1 \le p < 2 < q \le \infty$, and define $\alpha \in (0,1)$ by the relation $\frac{1}{2} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$. If $\omega \in L^p(\mathbf{R}^2) \cap L^q(\mathbf{R}^2)$, then $\mathbf{u} \in L^\infty(\mathbf{R}^2)^2$, and there exists $C > 0$ such that*

$$|\mathbf{u}|_\infty \le C|\omega|_p^\alpha |\omega|_q^{1-\alpha} \ . \tag{5}$$

(c) *Assume that $1 < p < \infty$. If $\omega \in L^p(\mathbf{R}^2)$, then $\nabla \mathbf{u} \in L^p(\mathbf{R}^2)^4$ and there exists $C > 0$ such that*

$$|\nabla \mathbf{u}|_p \le C|\omega|_p \ . \tag{6}$$

In addition, $\mathrm{div}\, \mathbf{u} = 0$ *and* $\mathrm{rot}\, \mathbf{u} \equiv \partial_1 u_2 - \partial_2 u_1 = \omega$.

Proof: **(a)** follows from the Hardy-Littlewood-Sobolev inequality, see for instance Stein [29], Chapter V, Theorem 1. To prove **(b)**, we remark that, for all $R > 0$,

$$\begin{aligned} |\mathbf{u}(x)| &\le \frac{1}{2\pi} \int_{|y| \le R} |\omega(x-y)| \frac{1}{|y|} \, \mathrm{d}y + \frac{1}{2\pi} \int_{|y| \ge R} |\omega(x-y)| \frac{1}{|y|} \, \mathrm{d}y \\ &\le C|\omega|_q R^{1-\frac{2}{q}} + C|\omega|_p \frac{1}{R^{\frac{2}{p}-1}} \ , \end{aligned}$$

by Hölder's inequality. Choosing $R = (|\omega|_p / |\omega|_q)^\beta$, where $\beta = \frac{\alpha}{1-2/q} = \frac{1-\alpha}{2/p-1}$, we obtain (5). Finally, (6) holds because $\nabla \mathbf{u}$ is obtained from ω via a singular integral kernel of Calderón-Zygmund type, see [29], Chapter II, Theorem 3. □

The next result shows that the Cauchy problem for (2) is globally well-posed in the space $L^1(\mathbf{R}^2)$.

Theorem 2.2 *For all initial data $\omega_0 \in L^1(\mathbf{R}^2)$, equation (2) has a unique global solution $\omega \in C^0([0,\infty), L^1(\mathbf{R}^2)) \cap C^0((0,\infty), L^\infty(\mathbf{R}^2))$ such that $\omega(0) = \omega_0$. Moreover, for all $p \in [1, +\infty]$, there exists $C_p > 0$ such that*

$$|\omega(t)|_p \le \frac{C_p |\omega_0|_1}{t^{1-\frac{1}{p}}} \ , \quad t > 0 \ . \tag{7}$$

Finally, the total mass of ω is preserved under the evolution:

$$\int_{\mathbf{R}^2} \omega(x,t) \, \mathrm{d}x = \int_{\mathbf{R}^2} \omega_0(x) \, \mathrm{d}x \ , \quad t \ge 0 \ .$$

Proof: The strategy of the proof is to rewrite (2) as an integral equation (see (11) below), and then to solve this equation using a fixed point argument in some appropriate function space. We refer to Ben-Artzi [2] and Brezis [3] for details. This result was also proved independently by Kato in [21]. □

Remark 2.3 *$L^1(\mathbf{R}^2)$ is not the largest space in which the Cauchy problem for equation (2) can be solved. For instance, it is shown in [16], [15], [21] that (2) has a global solution for any $\omega_0 \in \mathcal{M}(\mathbf{R}^2)$, the set of all finite measures on $\mathbf{R}^2$. However, uniqueness of this solution is known only if the atomic part of ω_0 is sufficiently small. Moreover, such a solution is smooth and belongs to $L^1(\mathbf{R}^2)$ for any $t > 0$. Thus, since we are interested in the long-time behavior of the solutions, there is no loss of generality in assuming that $\omega_0 \in L^1(\mathbf{R}^2)$.*

If $\omega(t)$ is the solution of (2) given by Theorem 2.2, it follows from Lemma 2.1 that the velocity field $\mathbf{u}(t)$ constructed from $\omega(t)$ satisfies $\mathbf{u} \in C^0((0,\infty), L^q(\mathbf{R}^2)^2)$ for all $q \in (2,\infty]$, and that there exist constants $C_q > 0$ such that

$$|\mathbf{u}(t)|_q \le \frac{C_q |\omega_0|_1}{t^{\frac{1}{2}-\frac{1}{q}}} \;, \quad t > 0 \;. \tag{8}$$

Moreover, one can show that $\mathbf{u}(t)$ is a solution of the integral equation associated to (1).

In (8), the fact that we cannot bound the L^2 norm of the velocity field is not a technical restriction. As we will see, even if $\omega(x)$ is smooth and rapidly decreasing, the velocity field $\mathbf{u}(x)$ given by (3) may not be in $L^2(\mathbf{R}^2)^2$. A typical example is the so-called *Oseen vortex*, which will be studied in Section 4.1 below. In fact, it is not difficult to verify that, if $\mathbf{u} \in L^2(\mathbf{R}^2)^2$ and $\omega = \operatorname{rot} \mathbf{u} \in L^1(\mathbf{R}^2)$, then necessarily $\int_{\mathbf{R}^2} \omega(x)\,\mathrm{d}x = 0$. In this situation, the decay estimates (7), (8) can be improved as follows:

Theorem 2.4 *Assume that $\mathbf{u}_0 \in L^2(\mathbf{R}^2)^2$ and that $\omega_0 = \operatorname{rot} \mathbf{u}_0 \in L^1(\mathbf{R}^2)$. Let $\omega(t)$ be the solution of (2) given by Theorem 2.2. Then*

$$\lim_{t\to\infty} t^{1-\frac{1}{p}} |\omega(t)|_p = 0 \;, \quad 1 \le p \le \infty \;. \tag{9}$$

If $\mathbf{u}(t)$ is the velocity field obtained from $\omega(t)$ via the Biot-Savart law (3), then

$$\lim_{t\to\infty} t^{\frac{1}{2}-\frac{1}{q}} |\mathbf{u}(t)|_q = 0 \;, \quad 2 \le q \le \infty \;. \tag{10}$$

Proof: The decay estimate (9) is a particular case of Theorem 1.2 in [5]. However, Theorem 2.4 can also be proved by the following simple argument. Since $\mathbf{u}(t)$ is a solution of the Navier-Stokes equation in $L^2(\mathbf{R}^2)^2$, it is well-known that $\nabla \mathbf{u} \in L^2((0,+\infty), L^2(\mathbf{R}^2)^4)$. In particular, $\int_0^\infty |\omega(t)|_2^2\,\mathrm{d}t < \infty$. On the other hand, since $\frac{\mathrm{d}}{\mathrm{d}t}|\omega(t)|_2^2 = -2|\nabla\omega(t)|_2^2$, the function $t \mapsto |\omega(t)|_2$ is non-increasing. Therefore,

$$t|\omega(t)|_2^2 \le 2\int_{\frac{t}{2}}^t |\omega(s)|_2^2\,\mathrm{d}s \stackrel{\text{def}}{=} \epsilon(t)^2 \xrightarrow[t\to+\infty]{} 0 \;,$$

which proves (9) for $p = 2$.

We now use the integral equation satisfied by $\omega(t)$, namely

$$\omega(t) = \mathrm{e}^{t\Delta}\omega_0 - \int_0^t \nabla\cdot \mathrm{e}^{(t-s)\Delta}(\mathbf{u}(s)\omega(s))\,\mathrm{d}s = \omega_1(t) + \omega_2(t) \;. \tag{11}$$

Since $\omega_0 \in L^1(\mathbf{R}^2)$ and $\int_{\mathbf{R}^2} \omega_0(x)\,\mathrm{d}x = 0$, a direct calculation shows that $|\omega_1(t)|_1 \to 0$ as $t \to +\infty$. On the other hand,

$$
\begin{aligned}
|\omega_2(t)|_1 &\leq \int_0^t \frac{C}{\sqrt{t-s}} |\mathbf{u}(s)\omega(s)|_1 \,\mathrm{d}s \leq \int_0^t \frac{C}{\sqrt{t-s}} |\mathbf{u}(s)|_2 |\omega(s)|_2 \,\mathrm{d}s \\
&\leq \int_0^t \frac{C|\mathbf{u}_0|_2}{\sqrt{t-s}} \frac{\epsilon(s)}{\sqrt{s}} \,\mathrm{d}s = C|\mathbf{u}_0|_2 \int_0^1 \frac{\epsilon(tx)}{\sqrt{x(1-x)}} \,\mathrm{d}x \xrightarrow[t\to+\infty]{} 0 \;,
\end{aligned}
$$

by Lebesgue's dominated convergence theorem. Thus, (9) holds for $p = 1$, hence for $1 \leq p \leq 2$ by interpolation. Next, using the integral equation again and proceeding as above, it is straightforward to verify that (9) holds for all $p \in [1, +\infty]$. From Lemma 2.1, we then obtain (10) for $2 < q \leq \infty$. Finally, the fact that $|\mathbf{u}(t)|_2 \to 0$ as $t \to +\infty$ is established in [20]. $\square$

3 Scaling Variables and Invariant Manifolds

Our analysis of the long-time asymptotics of (2) depends on rewriting the equation in terms of "scaling variables" or "similarity variables":

$$
\xi = \frac{x}{\sqrt{1+t}} \;, \qquad \tau = \log(1+t) \;.
$$

If $\omega(x,t)$ is a solution of (2) and if $\mathbf{u}(t)$ is the corresponding velocity field, we define new functions $w(\xi,\tau)$, $\mathbf{v}(\xi,\tau)$ by

$$
\omega(x,t) = \frac{1}{1+t}\, w\Big(\frac{x}{\sqrt{1+t}}, \log(1+t)\Big) \;, \tag{12}
$$

$$
\mathbf{u}(x,t) = \frac{1}{\sqrt{1+t}}\, \mathbf{v}\Big(\frac{x}{\sqrt{1+t}}, \log(1+t)\Big) \;. \tag{13}
$$

Then $w(\xi,\tau)$ satisfies the equation

$$
\partial_\tau w = \mathcal{L} w - (\mathbf{v} \cdot \nabla_\xi) w \;, \tag{14}
$$

where

$$
\mathcal{L} w = \Delta_\xi w + \frac{1}{2}(\boldsymbol{\xi} \cdot \nabla_\xi) w + w \tag{15}
$$

and

$$
\mathbf{v}(\xi,\tau) = \frac{1}{2\pi} \int_{\mathbf{R}^2} \frac{(\boldsymbol{\xi} - \boldsymbol{\eta})^\perp}{|\xi - \eta|^2} w(\eta,\tau)\,\mathrm{d}\eta \;. \tag{16}
$$

Scaling variables have been previously used to study the evolution of the vorticity by Giga and Kambe [15] and Carpio [5].

The results of the previous section already provide us with some information about solutions of (14). For example, for all $w_0 \in L^1(\mathbf{R}^2)$, there exists a unique solution $w \in C^0([0,\infty), L^1(\mathbf{R}^2)) \cap C^0((0,\infty), L^\infty(\mathbf{R}^2))$ of (14) such that $w(0) = w_0$. Translating (7) and (8) into rescaled variables, we see that, for all $\tau > 0$,

$$
|w(\tau)|_p \leq \frac{C_p |w_0|_1}{a(\tau)^{(1-\frac{1}{p})}} \;, \quad 1 \leq p \leq \infty \;, \tag{17}
$$

$$
|\mathbf{v}(\tau)|_q \leq \frac{C_q |w_0|_1}{a(\tau)^{(\frac{1}{2}-\frac{1}{q})}} \;, \quad 2 < q \leq \infty \;, \tag{18}
$$

where $a(\tau) = 1 - \mathrm{e}^{-\tau}$. Moreover, if $\mathbf{v}_0 = \mathbf{v}(\cdot,0) \in L^2(\mathbf{R}^2)^2$, then $\int_{\mathbf{R}^2} w(\xi,\tau)\,\mathrm{d}\xi = 0$ for all $\tau \ge 0$, and it follows from (9) that $|w(\tau)|_p \to 0$ as $\tau \to +\infty$ for all $p \in [1,+\infty]$.

However, in order to derive the long-time asymptotics of these solutions we need to work not only in "ordinary" L^p spaces, but also in *weighted* L^2 spaces. For any $m \ge 0$, we define the Hilbert space $L^2(m)$ by

$$\begin{aligned} L^2(m) &= \{f \in L^2(\mathbf{R}^2) \,|\, \|f\|_m < \infty\} ~, \\ \|f\|_m &= \left(\int_{\mathbf{R}^2} (1+|\xi|^2)^m |f(\xi)|^2 \,\mathrm{d}\xi\right)^{1/2} = |b^m f|_2 ~, \end{aligned}$$

where $b(\xi) = (1+|\xi|^2)^{1/2}$. If $m > 1$, then $L^2(m) \hookrightarrow L^1(\mathbf{R}^2)$. In this case, we denote by $L^2_0(m)$ the closed subspace of $L^2(m)$ given by

$$L^2_0(m) = \left\{ w \in L^2(m) \,\Big|\, \int_{\mathbf{R}^2} w(\xi)\,\mathrm{d}\xi = 0 \right\} .$$

Weighted Sobolev spaces can be defined in a similar way. For instance,

$$H^1(m) = \{w \in L^2(m) \,|\, \partial_i f \in L^2(m) \text{ for } i = 1,2\} ~.$$

As is well known, the operator $\mathcal{L}$ is the generator of a strongly continuous semigroup $\mathrm{e}^{\tau\mathcal{L}}$ in $L^2(m)$, see Appendix A. Note that, unlike $\mathrm{e}^{t\Delta}$, the semigroup $\mathrm{e}^{\tau\mathcal{L}}$ does not commute with space derivatives. Indeed, since $\partial_i \mathcal{L} = (\mathcal{L}+\frac{1}{2})\partial_i$ for $i = 1,2$, we have $\partial_i \mathrm{e}^{\tau\mathcal{L}} = \mathrm{e}^{\frac{\tau}{2}} \mathrm{e}^{\tau\mathcal{L}} \partial_i$ for all $\tau \ge 0$. Using this remark and the fact that $\nabla \cdot \mathbf{v} = 0$, we can rewrite equation (14) in integral form:

$$w(\tau) = \mathrm{e}^{\tau\mathcal{L}} w_0 - \int_0^\tau \mathrm{e}^{-\frac{1}{2}(\tau-s)} \nabla \cdot \mathrm{e}^{(\tau-s)\mathcal{L}} (\mathbf{v}(s) w(s))\,\mathrm{d}s ~, \tag{19}$$

where $w_0 = w(0) \in L^2(m)$. The following lemma shows that the quadratic nonlinear term in (19) is bounded (hence smooth) in $L^2(m)$ if $m > 0$.

Lemma 3.1 *Fix $m > 0$ and $T > 0$. Given $w_1, w_2 \in C^0([0,T], L^2(m))$, define*

$$R(\tau) = \int_0^\tau \nabla \cdot \mathrm{e}^{(\tau-s)\mathcal{L}} (\mathbf{v}_1(s) w_2(s))\,\mathrm{d}s ~, \quad 0 \le \tau \le T ~,$$

where $\mathbf{v}_1$ is obtained from w_1 via the Biot-Savart law (16). Then $R \in C^0([0,T], L^2(m))$, and there exists $C_0 = C_0(m,T) > 0$ such that

$$\sup_{0\le\tau\le T} \|R(\tau)\|_m \le C_0 \Big(\sup_{0\le\tau\le T} \|w_1(\tau)\|_m \Big) \Big(\sup_{0\le\tau\le T} \|w_2(\tau)\|_m \Big) ~.$$

Moreover, $C_0(m,T) \to 0$ as $T \to 0$.

Proof: Choose $q \in (1,2)$ such that $q > \frac{2}{m+1}$. Using Proposition A.5 (with $N = p = 2$), we obtain

$$|b^m R(\tau)|_2 \le C \int_0^\tau \frac{1}{a(\tau-s)^{(\frac{1}{q}-\frac{1}{2})+\frac{1}{2}}} \,|b^m \mathbf{v}_1(s) w_2(s)|_q \,\mathrm{d}s ~,$$

where $a(\tau) = 1 - \mathrm{e}^{-\tau}$ and $b(\xi) = (1 + |\xi|^2)^{1/2}$. Moreover, using Hölder's inequality and Lemma 2.1, we have

$$|b^m \mathbf{v}_1 w_2|_q \le |b^m w_2|_2 |\mathbf{v}_1|_{\frac{2q}{2-q}} \le C\|w_2\|_m |w_1|_q \le C\|w_2\|_m \|w_1\|_m \ ,$$

since $L^2(m) \hookrightarrow L^q(\mathbf{R}^2)$. Therefore, for all $\tau \in [0,T]$, we find

$$\|R(\tau)\|_m \le C\Big(\int_0^\tau \frac{1}{a(s)^{\frac{1}{q}}}\,\mathrm{d}s\Big)\Big(\sup_{0\le s\le T}\|w_1(s)\|_m\Big)\Big(\sup_{0\le s\le T}\|w_2(s)\|_m\Big) \ .$$

This concludes the proof of Lemma 3.1. □

We now show that (14) has global solutions in $L^2(m)$ if $m > 1$.

Theorem 3.2 *Suppose that $w_0 \in L^2(m)$ for some $m > 1$. Then (14) has a unique global solution $w \in C^0([0,\infty), L^2(m))$ with $w(0) = w_0$, and there exists $C_1 = C_1(\|w_0\|_m) > 0$ such that*

$$\|w(\tau)\|_m \le C_1 \ , \quad \tau \ge 0 \ . \tag{20}$$

Moreover, $C_1(\|w_0\|_m) \to 0$ as $\|w_0\|_m \to 0$. Finally, if $w_0 \in L^2_0(m)$, then $\int_{\mathbf{R}^2} w(\xi,\tau)\,\mathrm{d}\xi = 0$ for all $\tau \ge 0$, and $\lim_{\tau\to\infty} \|w(\tau)\|_m = 0$.

Proof: Using Lemma 3.1 and a fixed point argument, it is easy to to show that, for any $K > 0$, there exists $\tilde{T} = \tilde{T}(K) > 0$ such that, for all $w_0 \in L^2(m)$ with $\|w_0\|_m \le K$, equation (19) has a unique local solution $w \in C^0([0,\tilde{T}], L^2(m))$. This solution $w(\tau)$ depends continuously on the initial data w_0, uniformly in $\tau \in [0,\tilde{T}]$. Moreover, $\tilde{T}$ can be chosen so that $\|w(\tau)\|_m \le 2\|w_0\|_m$ for all $\tau \in [0,\tilde{T}]$. Thus, to prove global existence, it is sufficient to show that any solution $w \in C^0([0,T], L^2(m))$ of (19) satisfies the bound (20) for some $C_1 > 0$ (independent of T.)

Let $w_0 \in L^2(m)$, $T > 0$, and assume that $w \in C^0([0,T], L^2(m))$ is a solution of (19). Without loss of generality, we suppose that $T \ge \tilde{T} \equiv \tilde{T}(\|w_0\|_m)$. Since $L^2(m) \hookrightarrow L^p(\mathbf{R}^2)$ for all $p \in [1,2]$, there exists $C > 0$ such that $|w(\tau)|_p \le C\|w(\tau)\|_m \le 2C\|w_0\|_m$ for all $\tau \in [0,\tilde{T}]$. By (17), we also have $|w(\tau)|_p \le C_p a(\tilde{T})^{-1+1/p}|w_0|_1 \le C\|w_0\|_m$ for all $\tau \in [\tilde{T},T]$. Thus, there exists $C_2 = C_2(\|w_0\|_m) > 0$ such that $|w(\tau)|_p \le C_2$ for all $p \in [1,2]$ and all $\tau \in [0,T]$. Moreover, $C_2(\|w_0\|_m) \to 0$ as $\|w_0\|_m \to 0$. To bound $||\xi|^m w(\tau)|_2$, we compute

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\mathbf{R}^2} |\xi|^{2m} w(\xi,\tau)^2 d\xi &= \int |\xi|^{2m} w(\partial_\tau w)\,\mathrm{d}\xi \\ &= \int |\xi|^{2m}\{w\Delta w + \frac{w}{2}(\boldsymbol{\xi}\cdot\nabla)w + w^2 - w(\mathbf{v}\cdot\nabla)w\}\,\mathrm{d}\xi \ .\end{aligned} \tag{21}$$

Integrating by parts and using the fact that $\operatorname{div}\mathbf{v} = 0$, we can rewrite

$$\begin{aligned}\int |\xi|^{2m} w(\Delta w)\,\mathrm{d}\xi &= -\int |\xi|^{2m}|\nabla w|^2\,\mathrm{d}\xi + 2m^2\int |\xi|^{2m-2}w^2\,\mathrm{d}\xi \ , \\ \int |\xi|^{2m}\frac{w}{2}(\boldsymbol{\xi}\cdot\nabla)w\,\mathrm{d}\xi &= -\frac{m+1}{2}\int |\xi|^{2m}w^2\,\mathrm{d}\xi \ , \\ \int |\xi|^{2m} w(\mathbf{v}\cdot\nabla)w\,\mathrm{d}\xi &= \frac{1}{2}\int |\xi|^{2m}(\mathbf{v}\cdot\nabla)w^2\,\mathrm{d}\xi = \frac{1}{2}\int |\xi|^{2m}\nabla\cdot(\mathbf{v}w^2)\,\mathrm{d}\xi \\ &= -m\int |\xi|^{2m-2}(\boldsymbol{\xi}\cdot\mathbf{v})w^2\,\mathrm{d}\xi \ .\end{aligned}$$

Inserting these expressions into (21), we find:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\int |\xi|^{2m} w^2 d\xi &= -\int |\xi|^{2m}|\nabla w|^2\,\mathrm{d}\xi - \frac{m-1}{2}\int |\xi|^{2m} w^2\,\mathrm{d}\xi \\ &\quad + 2m^2\int |\xi|^{2m-2} w^2\,\mathrm{d}\xi + m\int |\xi|^{2m-2}(\boldsymbol{\xi}\cdot\mathbf{v}) w^2\,\mathrm{d}\xi\ .\end{aligned}$$

We next remark that, for all $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\begin{aligned}\int |\xi|^{2m-2} w^2\,\mathrm{d}\xi &\le \epsilon\int |\xi|^{2m} w^2\,\mathrm{d}\xi + C_\epsilon \int w^2\,\mathrm{d}\xi\ , \\ \Big|\int |\xi|^{2m-2}(\boldsymbol{\xi}\cdot\mathbf{v}) w^2\,\mathrm{d}\xi\Big| &\le \epsilon\int |\xi|^{2m} w^2\,\mathrm{d}\xi + C_\epsilon |\mathbf{v}|_\infty^{2m}\int w^2\,\mathrm{d}\xi\ .\end{aligned}$$

Note that, just as in (5) of Lemma 2.1, $|\mathbf{v}(\tau)|_\infty \le C|w|_p^\alpha |w|_q^{1-\alpha}$ where $1 < p < 2 < q < \infty$ and $\frac{\alpha}{p} + \frac{1-\alpha}{q} = \frac{1}{2}$. We already know that $|w(\tau)|_p \le C_2$ for all $\tau \in [0,T]$. Choosing $\alpha = 1 - \frac{1}{8m}$, $q = 4 + \frac{1}{2m}$, and using (17) to bound $|w(\tau)|_q$, we obtain

$$|\mathbf{v}(\tau)|_\infty^{2m} \le C C_2^{2m\alpha}\Big(\frac{C_q |w_0|_1}{a(\tau)^{(1-\frac{1}{q})}}\Big)^{2m(1-\alpha)} \le C_3(1+\tau^{-1/4})\ , \quad 0 < \tau \le T\ ,$$

where $C_3 = C_3(m, \|w_0\|_m)$. Thus, choosing $\epsilon > 0$ sufficiently small, we see that for any $\delta \in (0, m-1)$, there exists $C_4 = C_4(m, \|w_0\|_m)$ such that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\int |\xi|^{2m} w^2 d\xi &\le -2\int |\xi|^{2m}|\nabla w|^2\,\mathrm{d}\xi - \delta\int |\xi|^{2m} w^2\,\mathrm{d}\xi \\ &\quad + C_4(1+\tau^{-1/4})\int w^2\,\mathrm{d}\xi\ .\end{aligned} \tag{22}$$

Integrating this inequality and using the fact that $|w(\tau)|_2 \le C_2$ for all $\tau \in [0,T]$, we obtain that $\int |\xi|^{2m} w^2\,\mathrm{d}\xi \le C_5$ for all $\tau \in [0,T]$, where $C_5 = C_5(m, \|w_0\|_m) \to 0$ as $\|w_0\|_m \to 0$. This proves that all solutions of (19) in $L^2(m)$ are global and satisfy (20). Finally, if $w_0 \in L^2_0(m)$ and if $\mathbf{v}_0$ is the velocity field obtained from w_0 via the Biot-Savart law (16), then $\mathbf{v}_0 \in L^2(\mathbf{R}^2)^2$ by Corollary B.3. In this case, we already observed that $|w(\tau)|_2$ converges to zero as $\tau \to +\infty$, and so does $||\xi|^m w(\tau)|_2$ by (22). This concludes the proof of Theorem 3.2. □

We now explain the motivation for introducing the scaling variables and discuss briefly how we will proceed. The spectrum of the operator $\mathcal{L}$ acting on $L^2(m)$ consists of a sequence of eigenvalues $\sigma_d = \{-\frac{k}{2} \mid k = 0,1,2,\ldots\}$, plus continuous spectrum $\sigma_c = \{\lambda \in \mathbf{C} \mid \Re\lambda \le -(\frac{m-1}{2})\}$, see Appendix A. Ignoring the continuous spectrum for a moment (since we can "push it out of the way" by choosing m appropriately) we choose coordinates for $L^2(m)$ whose basis vectors are the eigenvectors of $\mathcal{L}$. These eigenvectors can be computed explicitly. Expressed in these coordinates (14) becomes an infinite set of ordinary differential equations with a very simple linear part and a quadratic nonlinear term. In the study of dynamical systems, powerful tools such as invariant manifold theory and the theory of normal forms have been developed to investigate such systems of equations. We will apply these tools to study (14). In particular, we will show that given any $\mu > 0$ such that $2\mu \notin \mathbf{N}$, the long time behavior of solutions of (14) up to terms of order $\mathcal{O}(\mathrm{e}^{-\mu\tau})$ (which corresponds to the behavior of solutions of (2) up to terms of order $\mathcal{O}(t^{-(\mu+1)})$) is given by a *finite* system of ordinary differential equations which results from restricting (14) to a finite-dimensional, invariant manifold in $L^2(m)$. This manifold is tangent at the origin to the eigenspace spanned by the eigenvalues in σ_d with eigenvalues bigger than $-\mu$. Using these manifolds we will see that we can, at least in principle, calculate the long-time asymptotics to any order. As an example we will show that in computing the asymptotics of the velocity field we obtain additional terms beyond those calculated in [6] and [11] due to the fact that we work with the vorticity formulation. We also exhibit an example that shows that in general the asymptotic behavior of the solution of (2) contains terms logarithmic in t.

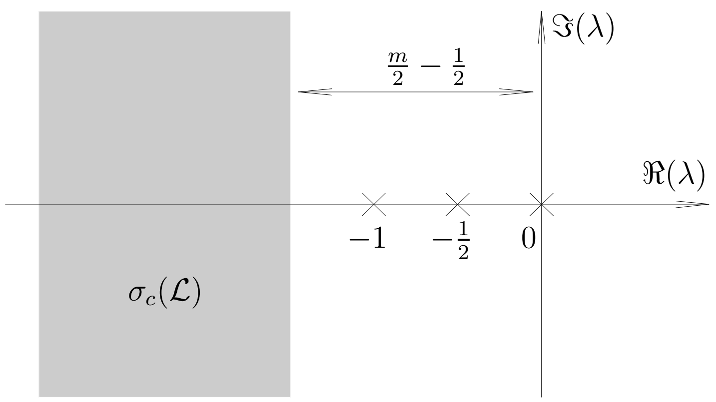

Fig. 1. The spectrum of the linear operator $\mathcal{L}$ in $L^2(m)$, when $m = 4$.

There are some technical difficulties which have to be overcome in order to implement this picture. The two most important ones are the presence of the continuous spectrum of $\mathcal{L}$, and the fact that the nonlinear term in (14) is not a smooth function from $L^2(m)$ to itself. We circumvent these problems by working not with the differential equation itself, but rather with the semiflow defined by it. For this semiflow we can easily develop estimates which allow us to apply the invariant manifold theorem of Chen, Hale and Tan [7].

In order to apply the invariant manifold theorem we consider the behavior of solutions of (14) in a neighborhood of the fixed point $w = 0$. Since we wish for the moment to concentrate only on this neighborhood we cut-off the nonlinearity outside of a neighborhood of size $r_0 > 0$. More precisely, let $\chi : L^2(m) \to \mathbf{R}$ be a bounded C^∞ function such that $\chi(w) = 1$ if $\|w\|_m \leq 1$ and $\chi(w) = 0$ if $\|w\|_m \geq 2$. For any $r_0 > 0$, we define $\chi_{r_0}(w) = \chi(w/r_0)$ for all $w \in L^2(m)$.

Remark 3.3 *Such a cut-off function χ may not exist for all Banach spaces. However, for a Hilbert space like $L^2(m)$ one can always find such a function.*

We now replace (14) by

$$\partial_\tau w = \mathcal{L}w - (\mathbf{v} \cdot \nabla)(\chi_{r_0}(w)w) \ . \tag{23}$$

Note that (23) and (14) are equivalent whenever $\|w\|_m \le r_0$, but (23) becomes a linear equation for $\|w\|_m \ge 2r_0$. Using the analogue of Lemma 3.1, it is easy to show that all solutions of (23) in $L^2(m)$ are global if $m > 0$, and stay bounded if $m > 1$. In the sequel, we denote by $\Phi_\tau^{r_0,m}$ the semiflow on $L^2(m)$ defined by (23), and by Φ_τ^m the semiflow associated with (14). The following properties of $\Phi_\tau^{r_0,m}$ will be useful:

Proposition 3.4 *Fix $m > 1$, $r_0 > 0$, and let $\Phi_\tau^{r_0,m}$ be the semiflow on $L^2(m)$ defined by (23). If $r_0 > 0$ is sufficiently small, then for $\tau = 1$ we can decompose*

$$\Phi_1^{r_0,m} = \Lambda + \mathcal{R} \;, \tag{24}$$

where $\Lambda \equiv \mathrm{e}^{\mathcal{L}}$ is a bounded linear operator on $L^2(m)$, and $\mathcal{R} : L^2(m) \to L^2(m)$ is a C^∞ map satisfying $\mathcal{R}(0) = 0$, $D\mathcal{R}(0) = 0$. Moreover, $\mathcal{R}$ is globally Lipschitz, and there exists $L > 0$ (independent of r_0) such that $\mathrm{Lip}(\mathcal{R}) \le Lr_0$.

Proof: Let $w_1, w_2 \in L^2(m)$, and define $w_i(\tau) = \Phi_\tau^{r_0,m} w_i$ for $i = 1, 2$ and $0 \le \tau \le 1$. Then

$$w_i(\tau) = \mathrm{e}^{\tau\mathcal{L}} w_i - \int_0^\tau \mathrm{e}^{-\frac{1}{2}(\tau-s)} \nabla \cdot \mathrm{e}^{(\tau-s)\mathcal{L}} (\mathbf{v}_i(s) \chi_{r_0}(w_i(s)) w_i(s)) \,\mathrm{d}s \;, \quad i = 1, 2 \;,$$

where $\mathbf{v}_i(\tau)$ is the velocity field obtained from $w_i(\tau)$ via the Biot-Savart law (16). Proceeding as in the proof of Lemma 3.1, we find that there exists $C \ge 1$ and $K > 0$ such that

$$\sup_{0\le\tau\le1} \|w_1(\tau) - w_2(\tau)\|_m \le C\|w_1 - w_2\|_m + Kr_0 \sup_{0\le\tau\le1} \|w_1(\tau) - w_2(\tau)\|_m \;. \tag{25}$$

Thus, assuming $Kr_0 \le 1/2$, we obtain $\|\Phi_\tau^{r_0,m} w_1 - \Phi_\tau^{r_0,m} w_2\|_m \le 2C\|w_1 - w_2\|_m$ for all $\tau \in [0, 1]$. We now define $\mathcal{R} = \Phi_1^{r_0,m} - \Lambda$, so that

$$\mathcal{R}(w_i) = -\int_0^1 \mathrm{e}^{-\frac{1}{2}(1-\tau)} \nabla \cdot \mathrm{e}^{(1-\tau)\mathcal{L}} (\mathbf{v}_i(\tau) \chi_{r_0}(w_i(\tau)) w_i(\tau)) \,\mathrm{d}\tau \;, \quad i = 1, 2 \;. \tag{26}$$

In view of (25), we have $\|\mathcal{R}(w_1) - \mathcal{R}(w_2)\|_m \le Lr_0\|w_1 - w_2\|_m$, where $L = 2CK$. Moreover, using (26), it is not difficult to show that $\mathcal{R} : L^2(m) \to L^2(m)$ is smooth and satisfies $D\mathcal{R}(0) = 0$. $\square$

In the rest of this section, we fix $k \in \mathbf{N}$ and we assume that $m \ge k + 2$. As is shown in Appendix A, the spectrum of $\Lambda = \mathrm{e}^{\mathcal{L}}$ in $L^2(m)$ is $\sigma(\Lambda) = \Sigma_c \cup \Sigma_d$, where $\Sigma_d = \{\mathrm{e}^{-n/2} \,|\, n = 0, 1, 2 \ldots\}$ and $\Sigma_c = \{\lambda \in \mathbf{C} \,|\, |\lambda| \le \mathrm{e}^{-(\frac{m-1}{2})}\}$. Since $m - 1 > k$, this means that Λ has at least $k + 1$ isolated eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_k$, where $\lambda_j = \mathrm{e}^{-j/2}$. Let P_k be the spectral projection onto the (finite-dimensional) subspace of $L^2(m)$ spanned by the eigenvectors of Λ corresponding to the eigenvalues $\lambda_0, \ldots, \lambda_k$, and let $Q_k = \mathbf{1} - P_k$. Applying to the semiflow $\Phi_\tau^{r_0,m}$ the invariant manifold theorem as stated in Chen, Hale, and Tan [7], we obtain the following result:

Theorem 3.5 *Fix $k \in \mathbf{N}$, $m \ge k + 2$, and choose $\mu_1, \mu_2 \in \mathbf{R}$ such that $\frac{k}{2} < \mu_1 < \mu_2 < \frac{k+1}{2}$. Let P_k, Q_k be the spectral projections defined above, and let $E_c = P_k L^2(m)$,*

$E_s = Q_k L^2(m)$. *Then, for* $r_0 > 0$ *sufficiently small, there exists a* C^1 *and globally Lipschitz map* $g : E_c \to E_s$ *with* $g(0) = 0$, $Dg(0) = 0$, *such that the submanifold*

$$W_c = \{w_c + g(w_c) \mid w_c \in E_c\}$$

has the following properties:
1. (Invariance) *The restriction to* W_c *of the semiflow* $\Phi_\tau^{r_0,m}$ *can be extended to a Lipschitz flow on* W_c. *In particular,* $\Phi_\tau^{r_0,m}(W_c) = W_c$ *for all* $\tau \geq 0$, *and for any* $w_0 \in W_c$, *there exists a unique negative semi-orbit* $\{w(\tau)\}_{\tau \leq 0}$ *in* W_c *with* $w(0) = w_0$. *If* $\{w(\tau)\}_{\tau \leq 0}$ *is a negative semi-orbit contained in* W_c, *then*

$$\limsup_{\tau \to -\infty} \frac{1}{|\tau|} \ln \|w(\tau)\|_m \leq \mu_1 \ . \tag{27}$$

Conversely, if a negative semi-orbit of $\Phi_\tau^{r_0,m}$ *satisfies*

$$\limsup_{\tau \to -\infty} \frac{1}{|\tau|} \ln \|w(\tau)\|_m \leq \mu_2 \ , \tag{28}$$

then it lies in W_c.
2. (Invariant Foliation) *There is a continuous map* $h : L^2(m) \times E_s \to E_c$ *such that, for each* $w \in W_c$, $h(w, Q_k(w)) = P_k(w)$ *and the manifold* $M_w = \{h(w, w_s) + w_s \mid w_s \in E_s\}$ *passing through* w *satisfies* $\Phi_\tau^{r_0,m}(M_w) \subset M_{\Phi_\tau^{r_0,m}(w)}$ *and*

$$M_w = \Big\{\tilde{w} \in L^2(m) \ \Big| \ \limsup_{\tau \to +\infty} \frac{1}{\tau} \ln \|\Phi_\tau^{r_0,m}(\tilde{w}) - \Phi_\tau^{r_0,m}(w)\|_m \leq -\mu_2\Big\} \ . \tag{29}$$

Moreover, $h : L^2(m) \times E_s \to E_c$ *is* C^1 *in the* E_s *direction.*
3. (Completeness) *For every* $w \in L^2(m)$, $M_w \cap W_c$ *is exactly a single point. In particular,* $\{M_w\}_{w \in W_c}$ *is a foliation of* $L^2(m)$ *over* W_c.

Remark 3.6 *Although the nonlinearity* $\mathcal{R}$ *is* C^∞, *the invariant manifold* W_c *is in general not smooth. However, for any* $\rho > 1$ *such that* $k\rho < k + 1$, *one can prove that the map* $g : E_c \to E_s$ *is of class* C^ρ *if* r_0 *is sufficiently small. (See [18], Section 6.1.) Note also that the manifold* W_c *is not unique, since it depends on the choice of the cut-off function* χ. *This is the only source of non-uniqueness however – once the cutoff function has been fixed, the construction of [7] produces a unique global manifold.*

Proof: All we need to do is verify that hypotheses (**H.1**)–(**H.4**) of Theorem 1.1 in [7] hold for the semiflow $\Phi_\tau^{r_0,m}$. If $r_0 > 0$ is small enough, assumption (**H.1**) is satisfied because $w \mapsto \Phi_\tau^{r_0,m} w$ is globally Lipschitz, uniformly in $\tau \in [0, 1]$, see (25). Hypothesis (**H.2**) is nothing but the decomposition $\Phi_\tau^{r_0,m} = \Lambda + \mathcal{R}$ obtained in Proposition 3.4. Assumption (**H.4**) is a smallness condition on $\mathrm{Lip}(\mathcal{R})$, which is easily achieved by taking $r_0 > 0$ sufficiently small. To verify (**H.3**), we recall that $L^2(m) = E_c \oplus E_s$, and we set $\Lambda_c = P_k \Lambda P_k$, $\Lambda_s = Q_k \Lambda Q_k$. Since $\sigma(\Lambda_c) = \{1, \mathrm{e}^{-\frac{1}{2}}, \ldots, \mathrm{e}^{-\frac{k}{2}}\}$ and all eigenvalues of Λ_c are semisimple, it is clear that Λ_c has a bounded inverse, and that there exists $C_c \geq 1$ such that

$$\|\Lambda_c^{-j} w\|_m \leq C_c \, \mathrm{e}^{j\frac{k}{2}} \|w\|_m \ , \quad \text{for all } j \in \mathbf{N} \text{ and all } w \in E_c \ .$$

On the other hand, take $\epsilon = 0$ if $m > k+2$ or $\epsilon > 0$ arbitrarily small if $m = k+2$. By Proposition A.2, there exists $C_s \geq 1$ such that

$$\|\Lambda_s^j w\|_m \leq C_s \, \mathrm{e}^{-j(\frac{k+1}{2}-\epsilon)} \|w\|_m \;, \quad \text{for all } j \in \mathbf{N} \text{ and all } w \in E_s \;.$$

These estimates are exactly what is required in (**H.3**). $\square$

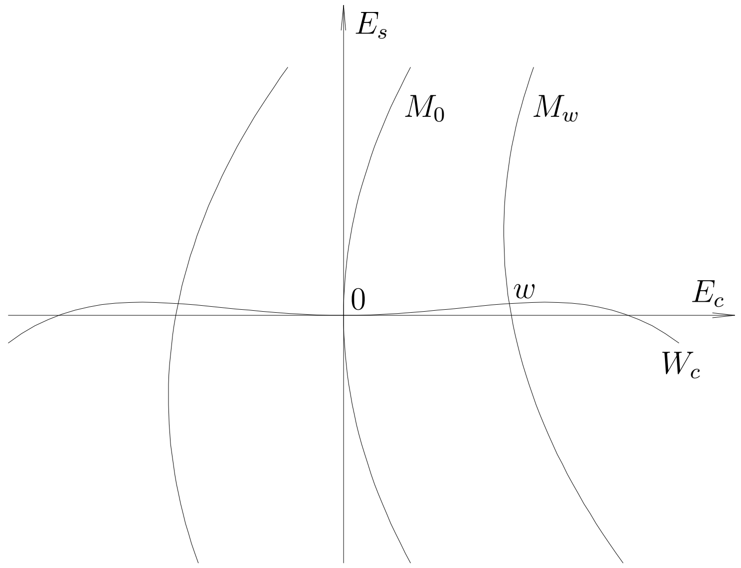

Fig. 2. A picture illustrating the invariant manifold W_c and the foliation $\{M_w\}_{w \in W_c}$ of $L^2(m)$ over W_c. The leaf M_0 of the foliation passing though the origin coincides with the strong-stable manifold W_s^{loc} in a neighborhood of the origin. Note that $\dim(E_c) < \infty$, whereas the subspace E_s is infinite-dimensional.

From now on, we assume that $r_0 > 0$ is sufficiently small so that the conclusion of Theorem 3.5 holds. By Theorem 3.2, there exists $r_1 > 0$ such that all solutions of (14) with $\|w(0)\|_m \leq r_1$ satisfy $\|w(\tau)\|_m \leq r_0$ for all $\tau \geq 0$. For such solutions the semiflow Φ_τ^m defined by (14) coincides with $\Phi_\tau^{r_0,m}$, and thus the invariant manifolds in Theorem 3.5 are also locally invariant with respect to Φ_τ^m. As a consequence, there exist a multitude of finite-dimensional invariant manifolds in the phase space of the vorticity equation (14), hence in the phase space of the Navier-Stokes equation. Furthermore, points **2** and **3** in Theorem 3.5 give us an estimate of how rapidly solutions approach these invariant manifolds. Since this result will be crucial in the applications, we state it here as a Corollary.

Corollary 3.7 *Fix $k \in \mathbf{N}$, $m \geq k+2$, and let W_c be the submanifold of $L^2(m)$ constructed in Theorem 3.5. Define*

$$W_c^{loc} = W_c \cap \{w \in L^2(m) \,|\, \|w\|_m < r_0\} \;. \tag{30}$$

Then W_c^{loc} is locally invariant under the semiflow Φ_τ^m defined by (14). If $\{w(\tau)\}_{\tau \leq 0}$ is a negative semi-orbit of (14) such that $\|w(\tau)\|_m < r_0$ for all $\tau \leq 0$, then $w(\tau) \in W_c^{loc}$ for all $\tau \leq 0$. Moreover, for any $\mu < \mu_2$ (where μ_2 is as in Theorem 3.5), there exist $r_2 > 0$

and $C > 0$ with the following property: for all $\tilde{w}_0 \in L^2(m)$ with $\|\tilde{w}_0\|_m \le r_2$, there exists a unique $w_0 \in W_c^{loc}$ such that $\Phi_\tau^m(w_0) \in W_c^{loc}$ for all $\tau \ge 0$ and

$$\|\Phi_\tau^m(\tilde{w}_0) - \Phi_\tau^m(w_0)\|_m \le C\mathrm{e}^{-\mu\tau} \ , \quad \tau \ge 0 \ . \tag{31}$$

Proof: It follows immediately from point **1** in Theorem 3.5 that W_c^{loc} is locally invariant under the semiflow Φ_τ^m and contains the negative semi-orbits that stay in a neighborhood of the origin. Choose $r_1 > 0$ so that $\|w_0\|_m \le r_1$ implies $\|\Phi_\tau^m(w_0)\|_m < r_0$ for all $\tau \ge 0$. By points **2** and **3** in Theorem 3.5, given $\tilde{w}_0 \in L^2(m)$, there exists a unique $w_0 \in M_{\tilde{w}_0} \cap W_c$. Setting $w_0 = w_c + g(w_c)$, we see from the definitions that w_c solves the equation

$$w_c = h(\tilde{w}_0, g(w_c)) \ . \tag{32}$$

Since h is continuous and $w_c \mapsto h(\tilde{w}_0, g(w_c))$ is Lipschitz (with a small Lipschitz constant), it is clear that (32) has a unique solution w_c which depends continuously on $\tilde{w}_0$. Moreover, by (29), $w_c = 0$ if $\tilde{w}_0 = 0$. Therefore, by continuity, there exists $r_2 \in (0, r_1]$ such that, if $\|\tilde{w}_0\|_m \le r_2$, then $\|w_0\|_m = \|w_c + g(w_c)\|_m \le r_1$. In this case, $\max(\|\Phi_\tau^m(\tilde{w}_0)\|_m \, , \, \|\Phi_\tau^m(w_0)\|_m) < r_0$ for all $\tau \ge 0$, and (31) follows from (29). ☐

In particular, given $\mu < \mu_2$, it follows from Corollary 3.7 that a solution $w(\tau)$ of (14) on W_c^{loc} cannot converge to zero faster than $\mathrm{e}^{-\mu\tau}$ as $\tau \to +\infty$, unless $w(\tau) \equiv 0$. For this reason, W_c^{loc} is usually called the "weak-stable" manifold of the origin. In the applications, we will also be interested in solutions that approach the origin "rapidly", namely with a rate $\mathcal{O}(\mathrm{e}^{-\mu_2\tau})$ or faster. By Theorem 3.5, all such solutions lie, for τ sufficiently large, in the leaf of the foliation $\{M_w\}_{w\in W_c}$ which passes through the origin. We refer to this leaf M_0 as the "strong-stable" manifold of the origin, and we denote by W_s^{loc} its restriction to a neighborhood of zero. In contrast to W_c^{loc}, the local strong-stable manifold W_s^{loc} is smooth and unique. In particular, it does not depend on the way in which the cut-off function χ is chosen.

Theorem 3.8 *Fix $k \in \mathbf{N}$, $m \ge k+2$, and let E_c, E_s be as in Theorem 3.5. Then there exists $r_3 > 0$ and a unique C^∞ function $f : \{w_s \in E_s \,|\, \|w_s\|_m < r_3\} \to E_c$ with $f(0) = 0$, $Df(0) = 0$, such that the submanifold*

$$W_s^{loc} = \{w_s + f(w_s) \mid w_s \in E_s \ , \ \|w_s\|_m < r_3\}$$

satisfies, for any $\mu \in (\frac{k}{2}, \frac{k+1}{2}]$,

$$W_s^{loc} = \left\{ w \in L^2(m) \,\middle|\, |||w|||_m < r_3 \, , \ \limsup_{\tau\to+\infty} \tau^{-1} \ln \|\Phi_\tau^m w\|_m \le -\mu \right\} \ , \tag{33}$$

where $|||w|||_m = \max(\|P_k w\|_m, \|Q_k w\|_m)$. In particular, if $w_0 \in W_s^{loc}$, there exists $T \ge 0$ such that $\Phi_\tau^m w_0 \in W_s^{loc}$ for all $\tau \ge T$.

Proof: Choose $r_3 > 0$ sufficiently small so that $\|\Phi_\tau^m w_0\|_m \le r_0$ for all $\tau \ge 0$ whenever $\|w_0\|_m \le 2r_3$. Take the function $h(\cdot,\cdot)$ of point **2** in Theorem 3.5, and define

$$f(w_s) = h(0, w_s) \ , \quad \text{for all } w_s \in E_s \text{ with } \|w_s\|_m < r_3 \ .$$

Then f is of class C^1, $f(0) = 0$, and the characterization (33) with $\mu = \mu_2$ follows immediately from (29) (with $w = 0$). In particular, f is unique. Moreover, since any solution $w(\tau)$ on W_s^{loc} converges to zero as $\tau \to +\infty$, and since $\mu_2 \in (\frac{k}{2}, \frac{k+1}{2})$ was arbitrary, it is clear that (33) holds for any $\mu \in (\frac{k}{2}, \frac{k+1}{2})$, hence for $\mu = \frac{k+1}{2}$ also. Finally, the smoothness of f and the fact that $Df(0) = 0$ can be proved using the integral equation satisfied by f. (See [18], Section 5.2.) □

If W_s^{loc} is as in Theorem 3.8, we also define the global strong-stable manifold of the origin by

$$\begin{aligned} W_s &= \{w_0 \in L^2(m) \,|\, \Phi_\tau^m(w_0) \in W_s^{loc} \text{ for some } \tau \geq 0\} \\ &= \{w_0 \in L^2(m) \,|\, \limsup_{\tau \to +\infty} \tau^{-1} \ln \|w(\tau)\|_m \leq -\mu\} \ , \end{aligned} \tag{34}$$

where $\mu \in (\frac{k}{2}, \frac{k+1}{2}]$. Then W_s is a smooth embedded submanifold of $L^2(m)$ of finite codimension. (See Henry [18], Sections 6.1 and 7.3.) The proof of this statement is rather involved and uses two main ingredients. First, for any $\tau \geq 0$, the map $\Phi_\tau^m : L^2(m) \to L^2(m)$ is one-to-one. This property of the semiflow Φ_τ^m is usually called *backwards uniqueness*. Next, since Φ_τ^m preserves the integral of w, it is clear that W_s is contained in the subspace $L_0^2(m) = \{w \in L^2(m) \,|\, \int_{\mathbf{R}^2} w(\xi)\,\mathrm{d}\xi = 0\}$. If $w(\tau)$ is any solution of (14) in $L_0^2(m)$ and if $\mathbf{v}(\tau)$ is the corresponding velocity field, it follows from Corollary B.3 that $\mathbf{v}(\tau) \in L^2(\mathbf{R}^2)$. Then, the classical energy equality

$$|\mathbf{v}(\tau)|_2^2 = |\mathbf{v}(0)|_2^2 - 2\int_0^\tau |\nabla \mathbf{v}(s)|_2^2 \,\mathrm{d}s \ , \quad \tau \geq 0 \ ,$$

shows that $|\mathbf{v}(\tau)|_2$ is a strictly decreasing function of τ unless $\nabla \mathbf{v}(\tau) \equiv 0$, which is equivalent to $w(\tau) \equiv 0$. In other words, the semiflow Φ_τ^m is *strictly gradient* in the subspace $L_0^2(m)$.

4 The long-time asymptotics of solutions

In this section we give three applications of the results of the preceding section. In the first, we examine the long-time asymptotics of small solutions of (14) and show that all such solutions with non-zero total vorticity asymptotically approach the Oseen vortex, thereby recovering the results of Giga and Kambe [15]. The invariant manifold approach yields additional information and as an example, we show that all solutions approach the vortex in a "universal" way.

Prior investigations of the long-time asymptotics of the Navier-Stokes equations ([6], [11] [23]) have yielded expressions in which the terms were proportional to inverse powers of $\sqrt{t}$. As a second application of the invariant manifold approach we show that if one extends these calculations to higher order (and in principle the invariant manifold approach allows one to extend the asymptotics to any order) one must include terms in the asymptotics proportional to $(\log(t))^\alpha/(\sqrt{t})^\beta$, where $\alpha, \beta \in \mathbf{N}$. We also exhibit specific classes of solutions for which such logarithmic terms appear.

Finally, in a third application we extend some recent results of Miyakawa and Schonbek [23] on solutions of the Navier-Stokes equations that decay "faster than expected". The invariant manifold approach allows us both to give a more complete characterization of the set of such solutions and provides a natural geometrical interpretation of the conditions in [23].

4.1 Stability of the Oseen Vortex

We begin by considering the behavior of small solutions of (14) in the space $L^2(m)$ with $m = 2$. Acting on $L^2(2)$, the operator $\mathcal{L}$ has a simple, isolated eigenvalue $\lambda_0 = 0$, with eigenfunction

$$G(\xi) \;=\; \frac{1}{4\pi}\,\mathrm{e}^{-|\xi|^2/4}\;, \quad \xi \in \mathbf{R}^2\;.$$

(See Appendix B.) Let $\mathbf{v}^G$ denote the corresponding velocity field, satisfying $\operatorname{rot}\mathbf{v}^G = G$. From the explicit expression (103), it is clear that $\boldsymbol{\xi} \perp \mathbf{v}^G(\xi)$, hence

$$\mathbf{v}^G(\xi)\cdot\nabla G(\xi) = 0\;, \quad \xi \in \mathbf{R}^2\;. \tag{35}$$

As a consequence, for any $\alpha \in \mathbf{R}$, the function $w(\xi) = \alpha G(\xi)$ is a stationary solution of (14), called *Oseen's vortex.*

Let $E_c = \operatorname{span}\{G\}$, let E_s be the spectral subspace of $\mathcal{L}$ corresponding to the continuous spectrum $\sigma_c = \{\lambda \in \mathbf{C} \mid \Re(\lambda) \le -\frac{1}{2}\}$, and consider the local center manifold W_c^{loc} given by Corollary 3.7 (with $k = 0$, $m = 2$). We claim that

$$W_c^{loc} = \{\alpha G \,|\, \alpha \in \mathbf{R}\;,\; |\alpha|\|G\|_2 < r_0\}\;. \tag{36}$$

Indeed, if $|\alpha|\|G\|_2 < r_0$, then $w(\xi,\tau) = \alpha G(\xi)$ is a solution of (14) such that $\|w(\tau)\|_2 < r_0$ for all $\tau \le 0$, hence $w(\tau) \in W_c^{loc}$ by Corollary 3.7. Since $W_c^{loc} \subset \{\alpha G + g(\alpha G) \,|\, \alpha \in \mathbf{R}\}$ for some $g : E_c \to E_s$, it follows that $g \equiv 0$ and that (36) holds. (Remark that, in this particular case, the local center manifold is *unique.*) Applying Corollary 3.7 we conclude:

Proposition 4.1 *Fix $0 < \mu < \frac{1}{2}$. There exist positive constants r_2 and C such that, for any initial data w_0 with $\|w_0\|_2 \le r_2$, the solution $w(\cdot,\tau)$ of (14) satisfies:*

$$\|w(\cdot,\tau) - AG(\cdot)\|_2 \le C\mathrm{e}^{-\mu\tau}\;, \quad \tau \ge 0\;, \tag{37}$$

where $A = \int_{\mathbf{R}^2} w_0(\xi)\,\mathrm{d}\xi$.

Proof: If $r_2 > 0$ is sufficiently small, it follows from (31) and (36) that (37) holds for some $A \in \mathbf{R}$. Now, an important property of (14) is the *conservation of mass:* if $\alpha(\tau) = \int_{\mathbf{R}^2} w(\xi,\tau)\,\mathrm{d}\xi$, then

$$\dot\alpha = \int_{\mathbf{R}^2} (\mathcal{L}w - \mathbf{v}\cdot\nabla w)\,\mathrm{d}\xi = \int_{\mathbf{R}^2} \nabla\cdot(\nabla w + \frac{1}{2}\boldsymbol{\xi} w - \mathbf{v}w)\,\mathrm{d}\xi = 0\;. \tag{38}$$

Since $\int_{\mathbf{R}^2} G(\xi)\,\mathrm{d}\xi = 1$, the conservation law (38) implies that $A = \int_{\mathbf{R}^2} w_0(\xi)\,\mathrm{d}\xi$ in (37). $\square$

To facilitate the comparison of our results with those of [15] we revert to the unscaled variables (x,t). Let

$$\Omega(x,t) = \frac{1}{1+t}\,G\left(\frac{x}{\sqrt{1+t}}\right)\;, \quad \mathbf{u}^\Omega(x,t) = \frac{1}{\sqrt{1+t}}\,\mathbf{v}^G\left(\frac{x}{\sqrt{1+t}}\right)\;.$$

Thus Ω is the solution of (2) corresponding, via the change of variables (12), to the solution G of (14), and $\mathbf{u}^\Omega$ is the associated velocity field. From Proposition 4.1, we obtain:

Corollary 4.2 *Fix $0 < \mu < \frac{1}{2}$. There exists $r_2 > 0$ such that, for all initial data $\omega_0 \in L^2(2)$ with $\|\omega_0\|_2 \le r_2$, the solution $\omega(x,t)$ of (2) satisfies*

$$|\omega(\cdot,t) - A\Omega(\cdot,t)|_p \le \frac{C_p}{(1+t)^{1+\mu-\frac{1}{p}}} \ , \quad 1 \le p \le 2 \ , \quad t \ge 0 \ , \tag{39}$$

where $A = \int_{\mathbf{R}^2} \omega_0(x)\, \mathrm{d}x$. If $\mathbf{u}(x,t)$ is the velocity field obtained from $\omega(x,t)$ via the Biot-Savart law (3), then

$$|\mathbf{u}(\cdot,t) - A\mathbf{v}^\Omega(\cdot,t)|_q \le \frac{C_q}{(1+t)^{\frac{1}{2}+\mu-\frac{1}{q}}} \ , \quad 1 < q < \infty \ , \quad t \ge 0 \ . \tag{40}$$

Proof: Let $\omega(x,t)$ be the solution of (2) with $\omega(\cdot,0) = \omega_0$, and let $w(\xi,\tau)$ be the solution of (14) with the same initial data. If $1 \le p \le 2$, then $L^2(2) \hookrightarrow L^p(\mathbf{R}^2)$. Using (12) and (37), we thus obtain

$$\begin{aligned} |\omega(\cdot,t) - A\Omega(\cdot,t)|_p &= (1+t)^{-1+\frac{1}{p}} |w(\cdot, \log(1+t)) - AG(\cdot)|_p \\ &\le C(1+t)^{-1+\frac{1}{p}} \|w(\cdot,\log(1+t)) - AG(\cdot)\|_2 \le C(1+t)^{-1-\mu+\frac{1}{p}} \ . \end{aligned}$$

It then follows from Lemma 2.1 that (40) holds for all $q \in (2,\infty)$. Finally, assume that $1 < q \le 2$, and fix $m \in (2/q, 2)$. If $\tilde{w}(\tau) = w(\tau) - AG$ and if $\tilde{\mathbf{v}}(\tau)$ denotes the corresponding velocity field, it follows from Proposition B.1 and Hölder's inequality that

$$|\tilde{\mathbf{v}}(\tau)|_q \le C|b^{m-\frac{1}{2}}\tilde{\mathbf{v}}(\tau)|_4 \le C|b^m \tilde{w}(\tau)|_2 \le C\|\tilde{w}(\tau)\|_2 \le C\mathrm{e}^{-\mu\tau} \ , \quad \tau \ge 0 \ ,$$

where $b(\xi) = (1+|\xi|^2)^{1/2}$. Using the change of variables (13), we thus obtain (40) for $1 < q \le 2$. ☐

Remark 4.3 *Once (39) is known for $p \in [1,2]$, a bootstrap argument using the integral equation satisfied by $\omega(x,t)$ gives the same estimate for all $p \in [1,\infty]$ if $t \ge 1$ (see [15], Proposition 5.3.) Similarly, (40) holds for $q \in (1,+\infty]$ if $t \ge 1$. However, as we shall see below, the difference $\mathbf{u}(t) - A\mathbf{v}^\Omega(t)$ is not in $L^1(\mathbf{R}^2)^2$ in general.*

Remark 4.4 *In [15], Giga and Kambe show that (39) remains true if ω_0 is any finite measure on $\mathbf{R}^2$ satisfying $\int_{\mathbf{R}^2}(1+|x|^2)|\omega_0|(\mathrm{d}x) < \infty$ and $\int_{\mathbf{R}^2}|\omega_0|(\mathrm{d}x) \ll 1$. Note that, in this case, the solution $\omega(x,t)$ satisfies $\|\omega(\cdot,t)\|_2 \le r_2$ for all $t \ge 1$, so that Corollary 4.2 applies to $\omega(x,t)$ restricted to $t \ge 1$. In [5], Carpio also shows that solutions of (2) with finite measures as initial data satisfy estimate (39). Instead of requiring ω_0 to be small in some norm, she assumes that the Cauchy problem for the vorticity equation (2) with initial data $A\delta$ has a unique solution, where $A = \int_{\mathbf{R}^2}\omega_0(\mathrm{d}x)$ and δ is the Dirac mass at the origin. However, to the best of our knowledge, the uniqueness of solutions for this problem has only been established if $|A|$ is sufficiently small.*

We next note that not only does the invariant manifold approach yield the stability of the Oseen vortex but it also allows us to systematically compute the way in which the vortex is approached. Somewhat surprisingly, the approach to the vortex solution has a universal form if the norm of the initial data is not too large. More precisely, we prove:

Theorem 4.5 *Fix $\frac{1}{2} < \mu < 1$. There exist $r_2 > 0$ and $C > 0$ such that, for all initial data $w_0 \in L^2(3)$ with $\|w_0\|_3 \le r_2$, the solution $w(\cdot,\tau)$ of (14) satisfies*

$$\|w(\xi,\tau) - AG(\xi) + \frac{1}{2}(B_1\xi_1 + B_2\xi_2)G(\xi)\mathrm{e}^{-\tau/2}\|_3 \le C\mathrm{e}^{-\mu\tau}\;, \quad \tau \ge 0\;, \tag{41}$$

where $A = \int w_0(\xi)\,\mathrm{d}\xi$, $B_1 = -\int \xi_1 w_0(\xi)\,\mathrm{d}\xi$, and $B_2 = -\int \xi_2 w_0(\xi)\,\mathrm{d}\xi$.

Proof: Acting on $L^2(3)$, the operator $\mathcal{L}$ has, in addition to the simple eigenvalue $\lambda_0 = 0$, a double eigenvalue $\lambda_1 = -\frac{1}{2}$, with eigenfunctions $F_1(\xi) = -\frac{\xi_1}{2}G(\xi)$ and $F_2(\xi) = -\frac{\xi_2}{2}G(\xi)$. (See Appendix B.) Let $E_c = \mathrm{span}\{G, F_1, F_2\}$, let E_s be the spectral subspace of $\mathcal{L}$ corresponding to the continuous spectrum $\sigma_c = \{\lambda \in \mathbf{C} \mid \Re(\lambda) \le 1\}$, and consider the local invariant manifold W_c^{loc} given by Corollary 3.7 (with $k = 1$, $m = 3$).

By construction, a point $w \in W_c^{loc}$ can be written, in a unique way, as $w = \alpha G + \beta_1 F_1 + \beta_2 F_2 + g(\alpha, \boldsymbol{\beta})$, where $\boldsymbol{\beta} = (\beta_1, \beta_2)$, and $g(\alpha,\boldsymbol{\beta}) \in E_s$. The coefficients $\alpha, \boldsymbol{\beta}$ are given by the formulas

$$\alpha = \int_{\mathbf{R}^2} w(\xi)\,\mathrm{d}\xi\;, \quad \beta_i = -\int_{\mathbf{R}^2} \xi_i w(\xi)\,\mathrm{d}\xi\;, \quad i = 1, 2\;. \tag{42}$$

Now, if $w(\cdot,\tau)$ evolves according to (14), we know from (38) that $\dot{\alpha} = 0$, *i.e.* α does not change with time. In an analogous way, we find

$$\dot{\beta}_i = -\int_{\mathbf{R}^2} \xi_i(\mathcal{L}w - \mathbf{v}\cdot\nabla w)\,\mathrm{d}\xi = -\frac{1}{2}\beta_i\;, \quad i = 1, 2\;. \tag{43}$$

Indeed, since $\nabla\cdot\mathbf{v} = 0$ and $\mathrm{rot}\,\mathbf{v} = \partial_1 v_2 - \partial_2 v_1 = w$, we have the identities

$$\begin{aligned} \xi_1\mathcal{L}w + \frac{1}{2}\xi_1 w &= \partial_1\Big(\xi_1\partial_1 w + \frac{1}{2}\xi_1^2 w - w\Big) + \partial_2\Big(\xi_1\partial_2 w + \frac{1}{2}\xi_1\xi_2 w\Big)\;, \\ \xi_1\mathbf{v}\cdot\nabla w &= \partial_1\Big(\xi_1 v_1 w - v_1 v_2\Big) + \partial_2\Big(\xi_1 v_2 w + \frac{1}{2}(v_1^2 - v_2^2)\Big)\;, \end{aligned}$$

which prove (43) for $i = 1$. The case $i = 2$ is similar.

Remark 4.6 *In the unscaled variables, (43) means that equation (2) conserves the first moments of the vorticity $\omega(x,t)$.*

Thus, remarkably, the semiflow induced by (14) on the three-dimensional invariant manifold is described by the *linear* equations (38), (43)! To complete the investigation of how the solutions evolve on the center manifold, we must estimate the nonlinear term $g(\alpha,\boldsymbol{\beta})$. Given $\frac{1}{2} < \mu < 1$, we claim that there exist $\epsilon > 0$ and $C_g > 0$ such that

$$\|g(\alpha,\boldsymbol{\beta})\|_3 \le C_g|\boldsymbol{\beta}|^{2\mu}\;, \tag{44}$$

for all $(\alpha,\boldsymbol{\beta}) \in \mathbf{R}^3$ with $|\alpha| + |\boldsymbol{\beta}| \le \epsilon$. Indeed, we know from Theorem 3.5 (see also Remark 3.6) that $g : \mathbf{R}^3 \to E_s$ is of class $C^{2\mu}$, and that $g(0,0) = 0$, $Dg(0,0) = 0$. Moreover, since αG is a fixed point for (14), it must lie in W_c^{loc} when α is sufficiently small, so we have $g(\alpha, 0) = 0$ for such values of α. Thus, to prove (44), it is sufficient to show that $\partial_\beta g(\alpha,0) = 0$ when $|\alpha|$ is small.

To prove this, we linearize the map $\Phi_1^{r_0,m}$ in (24) about the fixed point $\alpha^* G$ obtaining

$$D_{\alpha^* G}\Phi_1^{r_0,m} = \Lambda + D_{\alpha^* G}\mathcal{R} \; . \tag{45}$$

Since $\mathcal{R}$ is smooth, and $\mathcal{R} = 0$, $D_0\mathcal{R} = 0$, $D_{\alpha^* G}\mathcal{R}$ can be made arbitrarily small in norm by taking α^* sufficiently small. In particular, since we know the spectrum of Λ explicitly, we can choose α^* sufficiently small so that the spectrum of $D_{\alpha^* G}\Phi_1^{r_0,m}$ is "close" to the spectrum of Λ. More precisely, if $0 < \delta < \min(\frac{1}{4}, 1-\mu)$, there exists $\alpha_0 > 0$ such that if $|\alpha^*| \le \alpha_0$, $D_{\alpha^* G}\Phi_1^{r_0,m}$ has a three-dimensional spectral subspace $E_c^{\alpha^*}$ with the spectrum of $D_{\alpha^* G}\Phi_1^{r_0,m}|_{E_c^{\alpha^*}}$ consisting of eigenvalues with absolute value greater than $\exp(-(\frac{1}{2}+\delta))$ (these are the perturbations of the eigenvalues 1 and $\mathrm{e}^{-1/2}$ of Λ) and the remainder of the spectrum of $D_{\alpha^* G}\Phi_1^{r_0,m}$ contained in a disk in the complex plane of radius $\exp(-(1-\delta))$. Applying the invariant manifold theorem of [7] to the semiflow $\Phi_\tau^{r_0,m}$ about the fixed point $\alpha^* G$ (instead of the origin), we find just as in Theorem 3.5 that $\Phi_\tau^{r_0,m}$ has a three-dimensional invariant manifold $W_c^{\alpha^*}$, which in is tangent to $E_c^{\alpha^*}$ at $\alpha^* G$. Furthermore, as we noted in Remark 3.6, once the cutoff function χ_{r_0} in (23) is fixed, the invariant manifold constructed in [7] is unique. Therefore, it follows from the characterizations (27), (28) that $W_c^{\alpha^*}$ actually coincides with W_c^{loc} in a neighborhood of $\alpha^* G$. We complete the argument by showing that $E_c^{\alpha^*} = \mathrm{span}\{G, F_1, F_2\}$. This implies that $\partial_\beta g(\alpha^*, 0) = 0$ for α^* sufficiently small, thus proving (44).

To compute $E_c^{\alpha^*}$, note that if we linearize (14) about the fixed point $w = \alpha^* G$, we obtain

$$\partial_\tau w = \mathcal{L}^{\alpha^*} w = \Delta_\xi w + \frac{1}{2}(\boldsymbol{\xi}\cdot\nabla_\xi)w + w - \alpha^*(\mathbf{v}^G\cdot\nabla_\xi w + \mathbf{v}\cdot\nabla_\xi G) \; , \tag{46}$$

where $\mathbf{v}^G$ is the velocity field associated with the Oseen vortex G and $\mathbf{v}$ is the velocity field constructed from w via (16). Differentiating identity (35) with respect to ξ_j ($j = 1,2$), we obtain

$$\begin{aligned} 0 &= \partial_j(\mathbf{v}^G\cdot\nabla_\xi G) = ((\partial_j\mathbf{v}^G)\cdot\nabla_\xi)G + (\mathbf{v}^G\cdot\nabla_\xi)(\partial_j G) \\ &= (\mathbf{v}^{F_j}\cdot\nabla_\xi)G + \mathbf{v}^G\cdot\nabla_\xi F_j \; . \end{aligned} \tag{47}$$

Combining (35) and (47) we see immediately that

$$\begin{aligned} \mathcal{L}^{\alpha^*}G &= \mathcal{L}G - \alpha^*(\mathbf{v}^G\cdot\nabla_\xi G + \mathbf{v}^G\cdot\nabla_\xi G) = \mathcal{L}G = 0 \\ \mathcal{L}^{\alpha^*}F_j &= \mathcal{L}F_j - \alpha^*(\mathbf{v}^G\cdot\nabla_\xi F_j + \mathbf{v}^{F_j}\cdot\nabla_\xi G) = \mathcal{L}F_j = -\frac{1}{2}F_j \; , \quad j = 1,2 \; . \end{aligned}$$

Exponentiating these results we find that $D_{\alpha^* G}\Phi_1^{r_0,m} = \exp(\mathcal{L}^{\alpha^*})$ has eigenvalues 1 and $\mathrm{e}^{-\frac{1}{2}}$ with eigenspaces $\{G\}$ and $\{F_1, F_2\}$ respectively. This concludes the proof of (44).

Assume now that $w(\cdot,\tau)$ is a solution of (14) on W_c^{loc}, and let $\alpha(\tau) = A$, $\beta_1(\tau) = B_1\mathrm{e}^{-\tau/2}$, $\beta_2(\tau) = B_2\mathrm{e}^{-\tau/2}$, where A, B_1, B_2 are as in Theorem 4.5. Then

$$w(\cdot,\tau) = \alpha(\tau)G + \beta_1(\tau)F_1 + \beta_2(\tau)F_2 + g(\alpha(\tau), \boldsymbol{\beta}(\tau)) \; ,$$

hence (41) follows directly from (44).

On the other hand, if $\|w_0\|_3 \le r_2$ and if $w(\tau)$ is the solution of (14) with initial data w_0, Corollary 3.7 shows that there exists a solution $\tilde{w}(\tau)$ on W_c^{loc} such that $\|w(\tau) - \tilde{w}(\tau)\|_3 \le C\mathrm{e}^{-\mu\tau}$ for all $\tau \ge 0$. In view of the previous result, this means that (41) holds for some

$A, B_1, B_2 \in \mathbf{R}$. But equations (38) and (43), which hold for any solution of (14), imply that $A = \int w_0(\xi)\,\mathrm{d}\xi$ and $B_i = -\int \xi_i w_0(\xi)\,\mathrm{d}\xi$, $i = 1, 2$. This concludes the proof of Theorem 4.5. $\square$

Remark 4.7 *Note that although the preceding computation of the asymptotics of the solutions near the Oseen vortex applies only if α^* is small, 0 and $-\frac{1}{2}$ are eigenvalues of $\mathcal{L}^{\alpha^*}$, with eigenfunctions G and F_1, F_2, for all values of α^*.*

One can also rewrite the result of Theorem 4.5 in terms of the unscaled variables as we did in Corollary 4.2. Denote

$$\begin{aligned}
\omega_{\mathrm{app}}(x,t) &= \frac{A}{1+t}\, G\Big(\frac{x}{\sqrt{1+t}}\Big) + \sum_{i=1}^{2} \frac{B_i}{(1+t)^{3/2}}\, F_i\Big(\frac{x}{\sqrt{1+t}}\Big)\ , \\
\mathbf{u}_{\mathrm{app}}(x,t) &= \frac{A}{\sqrt{1+t}}\, \mathbf{v}^G\Big(\frac{x}{\sqrt{1+t}}\Big) + \sum_{i=1}^{2} \frac{B_i}{1+t}\, \mathbf{v}^{F_i}\Big(\frac{x}{\sqrt{1+t}}\Big)\ .
\end{aligned}$$

Corollary 4.8 *Fix $\frac{1}{2} < \mu < 1$. There exists $r_2 > 0$ such that, for all initial data $\omega_0 \in L^2(3)$ with $\|w_0\|_3 \le r_2$, the solution $\omega(\cdot,\tau)$ of (14) satisfies*

$$|\omega(\cdot,t) - \omega_{\mathrm{app}}(\cdot,t)|_p \le \frac{C_p}{(1+t)^{1+\mu-\frac{1}{p}}}\ , \quad 1 \le p \le 2\ , \quad t \ge 0\ ,$$

where $A = \int \omega_0(x)\,\mathrm{d}x$ and $B_i = -\int x_i \omega_0(x)\,\mathrm{d}x$, $i = 1, 2$. If $\mathbf{u}(x,t)$ is the velocity field obtained from $\omega(x,t)$ via the Biot-Savart law (3), then

$$|\mathbf{u}(\cdot,t) - \mathbf{u}_{\mathrm{app}}(\cdot,t)|_q \le \frac{C_q}{(1+t)^{\frac{1}{2}+\mu-\frac{1}{q}}}\ , \quad 1 \le q < \infty\ , \quad t \ge 0\ .$$

4.2 Secular terms in the asymptotics

We next show that, in contrast to prior investigations of the long-time asymptotics of solutions of (2) which yielded expansions in inverse powers of the time, one will in general encounter terms in the asymptotics with contain factors of $\log(t)$. These terms arise from resonances between the eigenvalues of the linear operator $\mathcal{L}$, and as we will see, computing them is straightforward using ideas from the theory of dynamical systems.

In the previous subsection we saw that, if $w_0 \in L^2(2)$ is sufficiently small, the solution $w(\xi,\tau)$ of (14) with initial data w_0 approaches the Oseen vortex $AG(\xi)$ as time goes to infinity, where $A = \int w_0(\xi)\,\mathrm{d}\xi$. We now examine in more detail what happens if $A = 0$, namely $w_0 \in L^2_0(m)$. In this invariant subspace, we know from Theorem 3.2 that all solutions converge to zero as $\tau \to +\infty$, without any restriction on the size of the initial data. Assuming that $w_0 \in L^2(4)$, we will compute the long-time asymptotics of the solution up to terms of order $\mathcal{O}(\mathrm{e}^{-\mu\tau})$, for any μ in the interval $(1, \frac{3}{2})$.

We first note that the operator $\mathcal{L}$ acting on $L^2_0(4)$ has, in addition to $\lambda_1 = -\frac{1}{2}$, a triple eigenvalue $\lambda_1 = -1$, with eigenfunctions

$$H_1(\xi) = \frac{1}{4}(|\xi|^2 - 4)G(\xi)\ , \quad H_2(\xi) = \frac{1}{4}(\xi_1^2 - \xi_2^2)G(\xi)\ , \quad H_3(\xi) = \frac{1}{4}\xi_1\xi_2 G(\xi)\ . \qquad (48)$$

Of course, these eigenfunctions are not unique, but the choices above are convenient ones for computation.

Let $E_c = \mathrm{span}\{F_1, F_2, H_1, H_2, H_3\}$, and let $E_s \subset L^2_0(4)$ be the spectral subspace of $\mathcal{L}$ corresponding to the continuous spectrum $\sigma_c = \{\lambda \in \mathbf{C} \mid \Re(\lambda) \le -\frac{3}{2}\}$. Any function $w \in L^2_0(4)$ can be written as

$$w(\xi) = \sum_{i=1}^{2} \beta_i F_i(\xi) + \sum_{j=1}^{3} \gamma_j H_j(\xi) + R(\xi) \ , \qquad (49)$$

where $R \in E_s$. The velocity field $\mathbf{v}(\xi)$ associated to w has a similar decomposition:

$$\mathbf{v}(\xi) = \sum_{i=1}^{2} \beta_i \mathbf{v}^{F_i}(\xi) + \sum_{j=1}^{3} \gamma_j \mathbf{v}^{H_j}(\xi) + \mathbf{v}^R(\xi) \ , \qquad (50)$$

see Appendix B. The coefficients β_i are given by (42), and the corresponding formulas for γ_j read:

$$\gamma_j = \int_{\mathbf{R}^2} p_j(\xi) w(\xi)\, \mathrm{d}\xi \ , \quad j = 1, 2, 3 \ , \qquad (51)$$

where $p_1(\xi) = \frac{1}{4}(|\xi|^2 - 4)$, $p_2(\xi) = \frac{1}{4}(\xi_1^2 - \xi_2^2)$ and $p_3(\xi) = \xi_1 \xi_2$.

Assume now that $w(\xi, \tau)$ is a solution of (14) in $L^2_0(4)$, and consider the evolution equations for the coefficients γ_j and the remainder R in (49). Proceeding as in (43), we find

$$\dot{\gamma}_j = -\gamma_j - \int_{\mathbf{R}^2} p_j(\xi)(\mathbf{v} \cdot \nabla) w\, \mathrm{d}\xi \ , \quad j = 1, 2, 3 \ . \qquad (52)$$

The following elementary result will be useful:

Lemma 4.9 *Assume that $w \in L^2_0(2) \cap H^1(2)$, and let $\mathbf{v}$ be the velocity field obtained from w via the Biot-Savart law (16). Then, for any quadratic polynomial $p(\xi)$,*

$$\int_{\mathbf{R}^2} p(\xi)(\mathbf{v} \cdot \nabla) w\, \mathrm{d}\xi = \int_{\mathbf{R}^2} (v_1 v_2 (\partial_1^2 p - \partial_2^2 p) - (v_1^2 - v_2^2) \partial_1 \partial_2 p)\, \mathrm{d}\xi \ . \qquad (53)$$

Proof: Since $\nabla \cdot \mathbf{v} = 0$ and $w = \partial_1 v_2 - \partial_2 v_1$, we have the identity

$$p(\mathbf{v} \cdot \nabla) w = v_1 v_2 (\partial_1^2 p - \partial_2^2 p) - (v_1^2 - v_2^2) \partial_1 \partial_2 p + \partial_1 E_1 + \partial_2 E_2 \ ,$$

where $E_1 = p v_1 w - v_1 v_2 \partial_1 p + \frac{1}{2}(v_1^2 - v_2^2) \partial_2 p$ and $E_2 = p v_2 w + v_1 v_2 \partial_2 p + \frac{1}{2}(v_1^2 - v_2^2) \partial_1 p$. Applying Proposition B.1, we see that $(1 + |\xi|)\mathbf{v} \in L^q(\mathbf{R}^2)^2$ for all $q > 2$ (this follows from (111) with $m = 1 + 2/q$ if $q < \infty$, and from (112) with $m = 1$ if $q = \infty$.) Therefore, $E_i \in L^2(\mathbf{R}^2)$ for $i = 1, 2$. In addition, since $\nabla \mathbf{v} \in L^p(\mathbf{R}^2)^4$ for all $p \in (1, \infty)$ by Lemma 2.1, we have $\partial_i E_i \in L^1(\mathbf{R}^2)$ for $i = 1, 2$, and (53) follows. $\square$

It follows in particular from Lemma 4.9 that $\int_{\mathbf{R}^2} p_1(\xi)(\mathbf{v} \cdot \nabla) w\, \mathrm{d}\xi \equiv 0$. Thus, surprisingly enough, the equation for γ_1 is *linear:* $\dot{\gamma}_1 = -\gamma_1$. In contrast, the equations for γ_2, γ_3, and R contain nonlinear terms. For the purposes of this subsection, it will be useful to write out separately the terms that are quadratic in $\boldsymbol{\beta} = (\beta_1, \beta_2)$. A direct calculation shows that

$$(\beta_1 \mathbf{v}^{F_1} + \beta_2 \mathbf{v}^{F_2}) \cdot \nabla(\beta_1 F_1 + \beta_2 F_2) = ((\beta_1^2 - \beta_2^2)\xi_1 \xi_2 - \beta_1 \beta_2 (\xi_1^2 - \xi_2^2)) \Phi(\xi) \ , \qquad (54)$$

where $\Phi(\xi) = (8\pi^2|\xi|^4)^{-1}\mathrm{e}^{-|\xi|^2/4}(\mathrm{e}^{-|\xi|^2/4} - 1 + |\xi|^2/4)$. Thus, defining

$$\kappa = \int_{\mathbf{R}^2} \xi_1^2\xi_2^2\Phi(\xi)\,\mathrm{d}\xi = \frac{1}{4}\int_{\mathbf{R}^2} (\xi_1^2 - \xi_2^2)^2\Phi(\xi)\,\mathrm{d}\xi = \frac{1}{32\pi}\ , \tag{55}$$

we see that the quadratic terms in the equations for γ_2, γ_3 are respectively $\kappa\beta_1\beta_2$ and $-\kappa(\beta_1^2 - \beta_2^2)$. Therefore, the equations for $\boldsymbol{\beta}, \boldsymbol{\gamma}, R$ (where $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_2)$) have the following form:

$$\begin{aligned}
\dot{\beta}_1 &= -\frac{1}{2}\beta_1\ , \quad \dot{\beta}_2 = -\frac{1}{2}\beta_2\ , \quad \dot{\gamma}_1 = -\gamma_1\ , \\
\dot{\gamma}_2 &= -\gamma_2 + \kappa\beta_1\beta_2 + f_2(\boldsymbol{\beta}, \boldsymbol{\gamma}, R)\ , \\
\dot{\gamma}_3 &= -\gamma_3 - \kappa(\beta_1^2 - \beta_2^2) + f_3(\boldsymbol{\beta}, \boldsymbol{\gamma}, R)\ , \\
R_\tau &= \mathcal{L}R + \beta_1\beta_2\Psi_2 - (\beta_1^2 - \beta_2^2)\Psi_3 + \nabla\cdot\mathbf{F}(\boldsymbol{\beta}, \boldsymbol{\gamma}, R) - \mathbf{v}^R\cdot\nabla R\ ,
\end{aligned} \tag{56}$$

where $\Psi_2 = (\xi_1^2 - \xi_2^2)\Phi - \kappa H_2$ and $\Psi_3 = \xi_1\xi_2\Phi - \kappa H_3$. Moreover, since we have written out explicitly the quadratic terms in $\boldsymbol{\beta}$, the remainder terms f_j, $\mathbf{F}$ in (56) satisfy the estimates

$$|f_j| + \|\mathbf{F}\|_4 \le C(|\boldsymbol{\beta}| + |\boldsymbol{\gamma}|)(|\boldsymbol{\gamma}| + \|R\|_4) + C\|R\|_4^2\ . \tag{57}$$

To compute the long-time asymptotics of the system (56), a natural idea is first to study the behavior of the solutions on the five-dimensional invariant manifold W_c^{loc} tangent to E_c at the origin, and then to use the fact that all solutions of (56) approach W_c^{loc} faster than $\mathcal{O}(\mathrm{e}^{-\mu\tau})$ for any $\mu < \frac{3}{2}$, see Corollary 3.7. This is the approach we follow, but we begin with a simplifying observation. Note that if one drops the terms $\nabla\cdot\mathbf{F} - \mathbf{v}^R\cdot\nabla R$ from the last equation in (56), the terms $\beta_1\beta_2\Psi_2 - (\beta_1^2 - \beta_2^2)\Psi_3$ act as a simple "forcing term" in the equation for R. As a result, one can find an explicit invariant manifold for the simplified equations, namely:

$$R = g(\boldsymbol{\beta}) = -\beta_1\beta_2(\mathcal{L}+1)^{-1}\Psi_2 + (\beta_1^2 - \beta_2^2)(\mathcal{L}+1)^{-1}\Psi_3\ . \tag{58}$$

Remark that $\Psi_2, \Psi_3 \in E_s$, and that the restriction of $\mathcal{L}+1$ to E_s is invertible. Straightforward calculations, which are not reproduced here, show that $(\mathcal{L}+1)^{-1}\Psi_2 = \Omega_2$, $(\mathcal{L}+1)^{-1}\Psi_3 = \Omega_3$, where

$$\Omega_2(\xi) = \kappa(\xi_1^2 - \xi_2^2)G(\xi)\zeta(|\xi|^2/4)\ , \quad \Omega_3(\xi) = \kappa\xi_1\xi_2 G(\xi)\zeta(|\xi|^2/4)\ , \tag{59}$$

and

$$\zeta(r) = -\frac{\log 2}{4} + \int_0^r \frac{1}{2x^3}\Big(1 - x + \frac{x^2}{2} - e^{-x}\Big)\,\mathrm{d}x\ .$$

As we will see, the manifold (58) is a good approximation to an invariant manifold for the full equations (56). We thus introduce the new variable

$$\rho = R - g(\boldsymbol{\beta})\ , \tag{60}$$

and find that (56) can be rewritten as

$$\begin{aligned}
\dot{\beta}_1 &= -\frac{1}{2}\beta_1\ , \quad \dot{\beta}_2 = -\frac{1}{2}\beta_2\ , \quad \dot{\gamma}_1 = -\gamma_1\ , \\
\dot{\gamma}_2 &= -\gamma_2 + \kappa\beta_1\beta_2 + \tilde{f}_2(\boldsymbol{\beta}, \boldsymbol{\gamma}, \rho)\ , \\
\dot{\gamma}_3 &= -\gamma_3 - \kappa(\beta_1^2 - \beta_2^2) + \tilde{f}_3(\boldsymbol{\beta}, \boldsymbol{\gamma}, \rho)\ , \\
\rho_\tau &= \mathcal{L}\rho + \nabla\cdot\tilde{\mathbf{F}}(\boldsymbol{\beta}, \boldsymbol{\gamma}, \rho) - \mathbf{v}^\rho\cdot\nabla\rho\ ,
\end{aligned} \tag{61}$$

where $\tilde{f}_j$ and $\tilde{\mathbf{F}}$ obey the same estimates as f_j and $\mathbf{F}$, but with $\|\rho\|_4 + |\boldsymbol{\beta}|^2$ substituted for $\|R\|_4$. Note further that, if $R \in E_s$, then $\rho \in E_s$ as well.

If we now apply the results of [7] to the system (61), we find just as in Theorem 3.5 that (61) has a five-dimensional invariant manifold tangent at the origin to E_s. The next proposition provides an estimate of the behavior of this manifold. Let X be the Banach space $\mathbf{R}^2 \times \mathbf{R}^3 \times E_s$, equipped with the norm $\|(\boldsymbol{\beta}, \boldsymbol{\gamma}, R)\|_X = |\boldsymbol{\beta}| + |\boldsymbol{\gamma}| + \|R\|_4$, and set $E_c = \mathbf{R}^2 \times \mathbf{R}^3$, with coordinates $(\boldsymbol{\beta}, \boldsymbol{\gamma})$.

Proposition 4.10 *Fix $\mu \in (1, \frac{3}{2})$ and $\delta \in (0, \frac{3}{2} - \mu)$. There exists a C^1 map $\mathcal{G} : E_c \to E_s$ satisfying $\mathcal{G}(0) = 0$, $D\mathcal{G}(0) = 0$, such that in a neighborhood of the origin the graph of $\mathcal{G}$ is left invariant by the semiflow defined by (61). Moreover, there exists $r_4 > 0$ and $C > 0$ such that, for any solution $(\boldsymbol{\beta}(\tau), \boldsymbol{\gamma}(\tau), \rho(\tau))$ of (61) with initial data $(\boldsymbol{\beta}_0, \boldsymbol{\gamma}_0, \rho_0)$ satisfying $\|(\boldsymbol{\beta}_0, \boldsymbol{\gamma}_0, \rho_0)\|_X \le r_4$, there exists a solution $(\boldsymbol{\beta}^c(\tau), \boldsymbol{\gamma}^c(\tau), \mathcal{G}(\boldsymbol{\beta}^c(\tau), \boldsymbol{\gamma}^c(\tau))$ on the invariant manifold with*

$$|\boldsymbol{\beta}(\tau) - \boldsymbol{\beta}^c(\tau)| + |\boldsymbol{\gamma}(\tau) - \boldsymbol{\gamma}^c(\tau)| + \|\rho(\tau) - \mathcal{G}(\boldsymbol{\beta}^c(\tau), \boldsymbol{\gamma}^c(\tau))\|_4 \le C e^{-\mu\tau} . \tag{62}$$

Finally, there exists $C_\delta > 0$ such that

$$\|\mathcal{G}(\boldsymbol{\beta}, \boldsymbol{\gamma})\|_4 \le C_\delta (|\boldsymbol{\beta}|^{3-\delta} + |\boldsymbol{\gamma}|^{\frac{3}{2}-\delta}) . \tag{63}$$

With the exception of the last estimate, the proof of this statement is almost identical to the proof of Theorem 3.5 and Corollary 3.7. We prove (63) in Appendix C and concentrate here on showing how it can be used to determine the asymptotics of (61).

We begin by studying the behavior of the solutions on the invariant manifold. Such solutions satisfy the system of ordinary differential equations:

$$\begin{aligned}
\dot{\beta}_1 &= -\frac{1}{2}\beta_1 , \quad \dot{\beta}_2 = -\frac{1}{2}\beta_2 , \quad \dot{\gamma}_1 = -\gamma_1 , \\
\dot{\gamma}_2 &= -\gamma_2 + \kappa\beta_1\beta_2 + \tilde{f}_2(\boldsymbol{\beta}, \boldsymbol{\gamma}, \mathcal{G}(\boldsymbol{\beta}, \boldsymbol{\gamma})) \\
\dot{\gamma}_3 &= -\gamma_3 - \kappa(\beta_1^2 - \beta_2^2) + \tilde{f}_3(\boldsymbol{\beta}, \boldsymbol{\gamma}, \mathcal{G}(\boldsymbol{\beta}, \boldsymbol{\gamma})) .
\end{aligned} \tag{64}$$

There are various ways to analyze the asymptotics of solutions of (64), but perhaps the simplest is to make the change of variables:

$$\begin{aligned}
&\Gamma_1 = \gamma_1 , \quad \Gamma_2 = \gamma_2 + \kappa\beta_1\beta_2 \log|\beta_1\beta_2| , \\
&\Gamma_3 = \gamma_3 - \kappa(\beta_1^2 \log(\beta_1^2) - \beta_2^2 \log(\beta_2^2)) .
\end{aligned} \tag{65}$$

From a dynamical systems point of view, this is just a normal form transformation that eliminates the resonant terms in (64). Rewriting (64) in term of $\boldsymbol{\beta}$ and $\boldsymbol{\Gamma} = (\Gamma_1, \Gamma_2, \Gamma_3)$, we find

$$\begin{aligned}
\dot{\beta}_1 &= -\frac{1}{2}\beta_1 , \quad \dot{\beta}_2 = -\frac{1}{2}\beta_2 , \quad \dot{\Gamma}_1 = -\Gamma_1 , \\
\dot{\Gamma}_2 &= -\Gamma_2 + \tilde{g}_2(\boldsymbol{\beta}, \boldsymbol{\Gamma}) , \quad \dot{\Gamma}_3 = -\Gamma_3 + \tilde{g}_3(\boldsymbol{\beta}, \boldsymbol{\Gamma}) ,
\end{aligned} \tag{66}$$

where $\tilde{g}_j(\boldsymbol{\beta}, \boldsymbol{\Gamma})$ is just $\tilde{f}_j(\boldsymbol{\beta}, \boldsymbol{\gamma}, \mathcal{G}(\boldsymbol{\beta}, \boldsymbol{\gamma}))$ rewritten in terms of $(\boldsymbol{\beta}, \boldsymbol{\Gamma})$ instead of $(\boldsymbol{\beta}, \boldsymbol{\gamma})$.

Remark 4.11 *The change of variables $(\boldsymbol{\beta},\boldsymbol{\gamma}) \to (\boldsymbol{\beta},\boldsymbol{\Gamma})$ is not Lipschitz, so it might seem as if the standard existence and uniqueness theory for solutions does not apply to (66). However, due to the very simple form of the first equations in (66), we can first solve the equations for β_1 and β_2 explicitly and then insert these expressions into the equations for $\boldsymbol{\Gamma}$. The equations for $\boldsymbol{\Gamma}$ are then non-autonomous, but Lipschitz in Γ and hence standard theorems imply that solutions of (66) exist and are unique.*

It is clear from (66) that $\beta_i(\tau) = b_i \mathrm{e}^{-\tau/2}$ for $i = 1,2$ and $\Gamma_1(\tau) = c_1 \mathrm{e}^{-\tau}$, where $b_i = \beta_i(0)$ and $c_1 = \Gamma_1(0)$. To determine the long-time behavior of Γ_2 and Γ_3, we note that $\mathrm{e}^{\tau}\Gamma_j(\tau) = \Gamma_j(0) + \int_0^\tau \mathrm{e}^s \tilde{g}_j(\boldsymbol{\beta}(s),\boldsymbol{\Gamma}(s))\,\mathrm{d}s$. Thus, defining

$$c_j = \Gamma_j(0) + \int_0^\infty \mathrm{e}^s \tilde{g}_j(\boldsymbol{\beta}(s),\boldsymbol{\Gamma}(s))\,\mathrm{d}s\ , \quad j = 2,3\ , \tag{67}$$

and using the estimates on $\tilde{g}_j$ which come from (57), we find that for any $\mu \in (1,3/2)$ there exists $C > 0$ such that $|\Gamma_j(\tau) - \mathrm{e}^{-\tau} c_j| \le C\mathrm{e}^{-\mu\tau}$, for $j = 2,3$. Inverting the change of coordinates (65), we immediately find:

Lemma 4.12 *Fix $\mu \in (1,3/2)$. There exist $r_5 > 0$ and $C > 0$ such that, for any solution of (64) with initial data satisfying $|\boldsymbol{\beta}(0)| + |\boldsymbol{\gamma}(0)| \le r_5$, there exist constants b_1, b_2, c_1', c_2', and c_3' such that, for all $\tau \ge 0$,*

$$\begin{aligned}
&\beta_1(\tau) = b_1 \mathrm{e}^{-\tau/2}\ , \quad \beta_2(\tau) = b_2 \mathrm{e}^{-\tau/2}\ , \quad \gamma_1(\tau) = c_1' \mathrm{e}^{-\tau}\ , \\
&|\gamma_2(\tau) - c_2' \mathrm{e}^{-\tau} - \kappa\tau \mathrm{e}^{-\tau} b_1 b_2| \le C\mathrm{e}^{-\mu\tau}\ , \\
&|\gamma_3(\tau) - c_3' \mathrm{e}^{-\tau} + \kappa\tau \mathrm{e}^{-\tau}(b_1^2 - b_2^2)| \le C\mathrm{e}^{-\mu\tau}\ .
\end{aligned} \tag{68}$$

Remark 4.13 *The constants c_j' $(j = 1,2,3)$ in (68) are related to c_j by*

$$c_1' = c_1\ , \quad c_2' = c_2 - \kappa b_1 b_2 \log|b_1 b_2|\ , \quad c_3' = c_3 + \kappa(b_1^2 \log(b_1^2) - b_2^2 \log(b_2^2))\ .$$

We now return to the full system of equations (61). We know that any solution $(\boldsymbol{\beta}(\tau),\boldsymbol{\gamma}(\tau),\rho(\tau))$ of (61) with initial data in a sufficiently small neighborhood of the origin approaches a solution $(\boldsymbol{\beta}^c(\tau),\boldsymbol{\gamma}^c(\tau),\mathcal{G}(\boldsymbol{\beta}^c(\tau),\boldsymbol{\gamma}^c(\tau))$ on the invariant manifold with a rate $\mathcal{O}(\mathrm{e}^{-\mu\tau})$, see (62). On the other hand, from Lemma 4.12, we know that there exist constants b_i, c_j' such that estimates (68) hold with $\boldsymbol{\beta}^c, \boldsymbol{\gamma}^c$ substituted for $\boldsymbol{\beta}, \boldsymbol{\gamma}$. Finally, combining (68) with (63), we obtain the bound $\|\mathcal{G}(\boldsymbol{\beta}^c(\tau),\gamma^c(\tau))\|_4 \le C\mathrm{e}^{-\mu\tau}$. Thus, the following estimate holds for all solutions of (61) in a neighborhood of the origin:

$$\begin{aligned}
&|\beta_1(\tau) - b_1 \mathrm{e}^{-\tau/2}| + |\beta_2(\tau) - b_2 \mathrm{e}^{-\tau/2}| \\
&+ |\gamma_1(\tau) - c_1' \mathrm{e}^{-\tau}| + |\gamma_2(\tau) - c_2' \mathrm{e}^{-\tau} - \kappa\tau \mathrm{e}^{-\tau} b_1 b_2| \\
&+ |\gamma_3(\tau) - c_3' \mathrm{e}^{-\tau} + \kappa\tau \mathrm{e}^{-\tau}(b_1^2 - b_2^2)| + \|\rho(\tau)\|_4 \le C\mathrm{e}^{-\mu\tau}\ .
\end{aligned} \tag{69}$$

Finally, we undo the change of variables (60). The result is:

Theorem 4.14 *Fix $1 < \mu < \frac{3}{2}$. If $w(\xi,\tau)$ is any solution of (14) in $L_0^2(4)$, then there exist constants b_i, c_j', and C such that $\|w(\cdot,\tau) - w_{\mathrm{app}}(\cdot,\tau)\|_4 \le C\mathrm{e}^{-\mu\tau}$ for all $\tau \ge 0$, where*

$$\begin{aligned}
w_{\mathrm{app}}(\xi,\tau) \;=\;& \mathrm{e}^{-\frac{\tau}{2}}(b_1 F_1(\xi) + b_2 F_2(\xi)) + \mathrm{e}^{-\tau}(c_1' H_1(\xi) + c_2' H_2(\xi) + c_3' H_3(\xi)) \\
&+ \kappa\tau \mathrm{e}^{-\tau}(b_1 b_2 H_2(\xi) - (b_1^2 - b_2^2) H_3(\xi)) \\
&+ \mathrm{e}^{-\tau}(-b_1 b_2 \Omega_2(\xi) + (b_1^2 - b_2^2)\Omega_3(\xi))\ .
\end{aligned} \tag{70}$$

Moreover,

$$b_i = -\int_{\mathbf{R}^2} \xi_i w(\xi,0)\,\mathrm{d}\xi \;, \quad i=1,2 \;, \quad c_1' = \int_{\mathbf{R}^2} \frac{1}{4}(|\xi|^2-4) w(\xi,0)\,\mathrm{d}\xi \;. \tag{71}$$

Proof: Let $w(\xi,\tau)$ be a solution of (14) in $L^2_0(4)$, and define $\boldsymbol{\beta}(\tau), \gamma(\tau), R(\tau)$ by (49) and $\rho(\tau)$ by (60). Since $\|w(\xi,\tau)\|_4 \to 0$ as $\tau \to +\infty$, we can assume without loss of generality that $\boldsymbol{\beta}(0), \boldsymbol{\gamma}(0), \rho(0)$ satisfy the assumptions of Proposition 4.10 and Lemma 4.12. In particular, (69) holds for some constants $b_i, c_j \in \mathbf{R}$. The estimate $\|w(\cdot,\tau) - w_{\rm app}(\cdot,\tau)\|_4 \le C\mathrm{e}^{-\mu\tau}$ is then a direct consequence of (49), (58), and (69). Finally, (71) holds because the moments β_i and γ_1 satisfy simple linear evolution equations, see the first line in (56). $\square$

The terms in (70) we wish to call particular attention to are those proportional to $\tau\mathrm{e}^{-\tau}$. When we revert to the original, unscaled, variables (x,t), these yield terms of the form $(1+t)^{-1}\log(1+t)$. Denote

$$\begin{aligned}
\omega_{\rm app}(x,t) \;&=\; \sum_{i=1}^{2} \frac{b_i}{(1+t)^{3/2}} F_i\Big(\frac{x}{\sqrt{1+t}}\Big) + \sum_{j=1}^{3} \frac{c_j'}{(1+t)^2} H_j\Big(\frac{x}{\sqrt{1+t}}\Big) \\
&\quad + \frac{\kappa \log(1+t)}{(1+t)^2}\Big(b_1 b_2 H_2\Big(\frac{x}{\sqrt{1+t}}\Big) - (b_1^2-b_2^2) H_3\Big(\frac{x}{\sqrt{1+t}}\Big)\Big) \\
&\quad + \frac{1}{(1+t)^2}\Big(-b_1 b_2 \Omega_2\Big(\frac{x}{\sqrt{1+t}}\Big) + (b_1^2-b_2^2)\Omega_3\Big(\frac{x}{\sqrt{1+t}}\Big)\Big) \;, \\
\mathbf{u}_{\rm app}(x,t) \;&=\; \sum_{i=1}^{2} \frac{b_i}{1+t} \mathbf{v}^{F_i}\Big(\frac{x}{\sqrt{1+t}}\Big) + \sum_{j=1}^{3} \frac{c_j'}{(1+t)^{3/2}} \mathbf{v}^{H_j}\Big(\frac{x}{\sqrt{1+t}}\Big) \\
&\quad + \frac{\kappa \log(1+t)}{(1+t)^{3/2}}\Big(b_1 b_2 \mathbf{v}^{H_2}\Big(\frac{x}{\sqrt{1+t}}\Big) - (b_1^2-b_2^2) \mathbf{v}^{H_3}\Big(\frac{x}{\sqrt{1+t}}\Big)\Big) \\
&\quad + \frac{1}{(1+t)^{3/2}}\Big(-b_1 b_2 \mathbf{v}^{\Omega_2}\Big(\frac{x}{\sqrt{1+t}}\Big) + (b_1^2-b_2^2)\mathbf{v}^{\Omega_3}\Big(\frac{x}{\sqrt{1+t}}\Big)\Big) \;,
\end{aligned}$$

where Ω_2, Ω_3 are defined in (59), and $\mathbf{v}^{\Omega_2}, \mathbf{v}^{\Omega_3}$ are the corresponding velocity fields. Then, proceeding as in the proof of Corollary 4.2, we obtain the following result in the original variables:

Corollary 4.15 *Fix $1 < \mu < \frac{3}{2}$. If $\omega(x,t)$ is any solution of (2) in $L^2_0(4)$, then there exist constants b_i and c_j' such that*

$$|\omega(\cdot,t) - \omega_{\rm app}(\cdot,t)|_p \;\le\; \frac{C_p}{(1+t)^{1+\mu-\frac{1}{p}}} \;, \quad 1 \le p \le 2 \;, \quad t \ge 0 \;.$$

Similarly, if $\mathbf{u}(x,t)$ is the velocity field obtained from $\omega(x,t)$ via the Biot-Savart law (3), then

$$|\mathbf{u}(\cdot,t) - \mathbf{u}_{\rm app}(\cdot,t)|_q \;\le\; \frac{C_q}{(1+t)^{\frac{1}{2}+\mu-\frac{1}{q}}} \;, \quad 1 \le q < \infty \;, \quad t \ge 0 \;.$$

Remark 4.16 *Two other references which examine the asymptotics of solutions of (1) to this order are [6] and [11]. We provide a detailed comparison of our asymptotics with these two references in [14], but just mention here that since both of these references require that the initial velocity field satisfies* $(1+|x|)\mathbf{u}_0 \in L^1(\mathbf{R}^2)^2$ *in order to derive the asymptotics, the coefficients* b_1 *and* b_2 *are both zero for all the solutions they study. (This follows from Corollary B.4.) Thus many solutions of finite energy (i.e. of finite* L^2 *norm) are excluded from consideration by this hypothesis, and in particular one does not observe the logarithmic terms in the asymptotics to this order. This is a further reason that we feel it is preferable to impose the decay conditions on the vorticity field rather than on the velocity.*

4.3 Optimal Decay Rates

It has been known for a long time that there is a relationship between the spatial and temporal decay rates of solutions of the Navier-Stokes equation in $\mathbf{R}^N$, $N \geq 2$. For instance, for any initial data $\mathbf{u}_0 \in L^2(\mathbf{R}^N)^N \cap L^1(\mathbf{R}^N)^N$, there exists a global weak solution of the Navier-Stokes equation satisfying

$$|\mathbf{u}(t)|_2 \leq C(1+t)^{-N/4} \;, \quad t \geq 0 \;, \tag{72}$$

see [26], [19]. If, in addition, $(1+|x|)\mathbf{u}_0 \in L^1(\mathbf{R}^N)^N$, if follows from Wiegner's result [31] that

$$|\mathbf{u}(t)|_2 \leq C(1+t)^{-(N+2)/4} \;, \quad t \geq 0 \;. \tag{73}$$

In [23], T. Miyakawa and M.E. Schonbek investigate the optimality of the decay rate (73). More specifically, they prove:

Theorem 4.17 [23] *Assume that* $\mathbf{u}_0 \in L^2(\mathbf{R}^N)^N$, $\nabla\cdot\mathbf{u}_0 = 0$, *and* $(1+|x|)\mathbf{u}_0 \in L^1(\mathbf{R}^N)^N$, $N \geq 2$. *Let* $\mathbf{u}(t)$ *be a global weak solution of the Navier-Stokes equation with initial data* $\mathbf{u}_0$ *and satisfying (73). For all* $k, \ell \in \{1, \ldots, N\}$, *define*

$$b_{k\ell} = \int_{\mathbf{R}^N} x_\ell (\mathbf{u}_0)_k(x)\,\mathrm{d}x \;, \quad c_{k\ell} = \int_0^\infty \int_{\mathbf{R}^N} u_k(x,t) u_\ell(x,t)\,\mathrm{d}x\,\mathrm{d}t \;. \tag{74}$$

Then

$$\lim_{t\to\infty} t^{\frac{N+2}{4}} |\mathbf{u}(t)|_2 = 0 \tag{75}$$

if and only if there exists $c \geq 0$ *such that*

$$b_{k\ell} = 0 \quad \textit{and} \quad c_{k\ell} = c\delta_{k\ell} \;, \quad k, \ell \in \{1, \ldots, N\} \;. \tag{76}$$

The proofs in [23] are clear, but do not provide much intuition as to the meaning of the conditions (76). As the authors themselves remark, "We know nothing about the characterization of solutions satisfying $(c_{k\ell}) = (c\delta_{k\ell})$." As we will demonstrate below in the case $N = 2$ (see also [14] for the three-dimensional case), the invariant manifold approach provides a simple geometrical explanation of the meaning of both conditions in (76). In addition, it will allow us to construct additional solutions satisfying (75), but which do not fit into the setting of Theorem 4.17, because $(1+|x|)\mathbf{u}_0 \notin L^1(\mathbf{R}^2)^2$. Note however that our work does require that we consider solutions of (14) in the space $L^2(m)$

for some $m > 3$, and this decay requirement on the vorticity does not appear in the work of Miyakawa and Schonbek.

As before, we work with the vorticity equation rather than the Navier-Stokes equations themselves. This seems to us a particularly natural choice when examining the long-time behavior of solutions since, as we have noted above, the long-time asymptotics are influenced by assumptions about the spatial decay of the initial conditions, and such hypotheses are preserved by the evolution of the vorticity equation (see Theorem 3.2), but not by the Navier-Stokes evolution. Since our goal is to recover the results by Miyakawa and Schonbek, we restrict ourselves to velocity fields that belong to $L^1(\mathbf{R}^2)$. At the level of the vorticity, this condition is equivalent to

$$\int_{\mathbf{R}^2} w(\xi)\,\mathrm{d}\xi = 0\ , \quad \int_{\mathbf{R}^2} \xi_1 w(\xi)\,\mathrm{d}\xi = 0\ , \quad \int_{\mathbf{R}^2} \xi_2 w(\xi)\,\mathrm{d}\xi = 0\ , \tag{77}$$

see Corollary B.4. Thus, we shall study the solutions of the vorticity equation (14) in the invariant subspace of $L^2(4)$ defined by (77). If $w(\xi,\tau)$ is such a solution and if $\mathbf{v}(\xi,\tau)$ is the velocity field obtained from $w(\xi,\tau)$ via the Biot-Savart law (3), then w and $\mathbf{v}$ can be decomposed as follows

$$\begin{aligned} w(\xi,\tau) &= \gamma_1(\tau)H_1(\xi) + \gamma_2(\tau)H_2(\xi) + \gamma_3(\tau)H_3(\xi) + R(\xi,\tau)\ , \\ \mathbf{v}(\xi,\tau) &= \gamma_1(\tau)\mathbf{v}^{H_1}(\xi) + \gamma_2(\tau)\mathbf{v}^{H_2}(\xi) + \gamma_3(\tau)\mathbf{v}^{H_3}(\xi) + \mathbf{v}^R(\xi,\tau)\ , \end{aligned} \tag{78}$$

where the coefficients γ_j $(j = 1,2,3)$ are defined by (51). Setting $\boldsymbol{\beta} = 0$ in (56), we obtain the evolution system

$$\begin{aligned} \dot\gamma_1 &= -\gamma_1\ , \\ \dot\gamma_2 &= -\gamma_2 + f_2(0,\boldsymbol{\gamma},R)\ , \\ \dot\gamma_3 &= -\gamma_3 + f_3(0,\boldsymbol{\gamma},R)\ , \\ R_\tau &= \mathcal{L}R + \nabla\cdot\mathbf{F}(0,\boldsymbol{\gamma},R) - \mathbf{v}^R\cdot\nabla R\ . \end{aligned} \tag{79}$$

Also, using (52) and Lemma 4.9, we see that

$$f_2(0,\boldsymbol{\gamma},R) = -\int_{\mathbf{R}^2} v_1 v_2\,\mathrm{d}\xi\ , \quad f_3(0,\boldsymbol{\gamma},R) = \int_{\mathbf{R}^2} (v_1^2 - v_2^2)\,\mathrm{d}\xi\ , \tag{80}$$

where $\mathbf{v}$ is given by (78).

Now, let $E_c = \mathrm{span}\{H_1,H_2,H_3\}$, and let $E_s \subset L^2(4)$ be the spectral subspace of $\mathcal{L}$ corresponding to the continuous spectrum $\sigma_c = \{\lambda\in\mathbf{C} \mid \Re(\lambda) \le -\frac{3}{2}\}$. Given $1 < \mu < \frac{3}{2}$, Theorem 3.8 (with $k = 2$ and $m = 4$) shows that, for sufficiently small $r_1 > 0$, the set W_s^{loc} defined by (33) is a smooth (infinite-dimensional) manifold which is tangent to E_s at the origin. In view of (38), (43), it is clear that W_s^{loc} is contained in the subspace of $L^2(4)$ defined by (77), and so is the global strong-stable manifold W_s defined by (34). The following result gives various characterizations of W_s.

Proposition 4.18 *Assume that $w_0 \in L^2(4)$ satisfies (77). Let $w(\xi,\tau)$ be the solution of (14) with initial data w_0, and let $\mathbf{v}(\xi,\tau)$ be the velocity field obtained from $w(\xi,\tau)$ via the Biot-Savart law (3). Decompose $w(\xi,\tau)$ according to (78), and define the coefficients $c_{k\ell}$ by (74), where*

$$\mathbf{u}(x,t) = \frac{1}{\sqrt{1+t}}\,\mathbf{v}\Big(\frac{x}{\sqrt{1+t}}, \log(1+t)\Big)\ . \tag{81}$$

Then the following statements are equivalent:
1) w_0 *lies in the strong stable manifold* W_s*.*
2) $\lim_{\tau\to\infty} \mathrm{e}^{\tau}\|w(\cdot,\tau)\|_4 = 0$*.*
3) $\lim_{t\to\infty} t|\mathbf{u}(\cdot,t)|_2 = 0$*.*
4) $\gamma_1(0) = 0$*,* $\gamma_2(0) = c_{12}$*, and* $\gamma_3(0) = c_{22} - c_{11}$*.*

Remark 4.19 *Note that the global nature of the strong stable manifold means that Proposition 4.18 is not limited to solutions of small norm, but applies to solutions of arbitrary size.*

Proof: Fix $1 < \mu < \frac{3}{2}$. Applying Theorem 4.14 and noting that $b_1 = b_2 = 0$ due to (77), we obtain

$$\|w(\cdot,\tau) - (c_1H_1 + c_2H_2 + c_3H_3)\mathrm{e}^{-\tau}\|_4 \le C\mathrm{e}^{-\mu\tau} \ , \tag{82}$$

for some $c_1, c_2, c_3 \in \mathbf{R}$. We shall show that statements 1), 2), 3), 4) in Proposition 4.18 are all equivalent to
5) $c_1 = c_2 = c_3 = 0$.
Indeed, since the functions H_1, H_2, H_3 are linearly independent, it is clear from (34) and (82) that 1) $\Leftrightarrow$ 2) $\Leftrightarrow$ 5). On the other hand, it follows from (78) and (79) that the coefficients c_j in (82) are given by the formulas

$$c_1 = \gamma_1(0) \ , \quad c_j = \gamma_j(0) + \int_0^\infty \mathrm{e}^{\tau} f_j(0, \boldsymbol{\gamma}(\tau), R(\tau))\,\mathrm{d}\tau \ , \quad j = 2,3 \ ,$$

see also (67). Furthermore, using (80) and the change of variables (81), it is straightforward to verify that

$$\begin{aligned} c_2 &= \gamma_2(0) - \int_0^\infty \mathrm{e}^{\tau} \int_{\mathbf{R}^2} v_1(\xi,\tau)v_2(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau \ \equiv \ \gamma_2(0) - c_{12} \ , \\ c_3 &= \gamma_3(0) + \int_0^\infty \mathrm{e}^{\tau} \int_{\mathbf{R}^2} (v_1(\xi,\tau)^2 - v_2(\xi,\tau)^2)\,\mathrm{d}\xi\,\mathrm{d}\tau \ \equiv \ \gamma_3(0) + c_{11} - c_{22} \ , \end{aligned}$$

where $c_{k\ell}$ is defined in (74). Therefore, 4) $\Leftrightarrow$ 5). Finally, from Corollary 4.15, we have $|\mathbf{u}(\cdot,t) - \mathbf{u}_{\mathrm{app}}(\cdot,t)|_2 \le C(1+t)^{-\mu}$, where

$$\mathbf{u}_{\mathrm{app}}(x,t) = \frac{1}{(1+t)^{3/2}} \sum_{j=1}^{3} c_j \mathbf{v}^{H_j}\Big(\frac{x}{\sqrt{1+t}}\Big) \ .$$

Clearly, $|\mathbf{u}_{\mathrm{app}}(\cdot,t)|_2 = \frac{K}{1+t}$, where $K = 0$ if and only if $c_1 = c_2 = c_3 = 0$. Thus, 3) $\Leftrightarrow$ 5).
□

We are now able to give an alternative proof of Theorem 4.17 in the particular case where $N = 2$ and $w_0 = \partial_1(\mathbf{u}_0)_2 - \partial_2(\mathbf{u}_0)_1 \in L^2(4)$. Indeed, let $w(\xi,\tau)$ be the solution of (14) with initial data w_0. Since $\mathbf{u}_0 \in L^1(\mathbf{R}^2)^2$, it follows from Corollary B.4 that (77) holds, hence $w(\xi,\tau)$ can be decomposed according to (78). Moreover, by Corollary B.5, the assumption $(1+|x|)\mathbf{u}_0 \in L^1(\mathbf{R}^2)^2$ is equivalent to $\gamma_2(0) = \gamma_3(0) = 0$.

a) Assume first that (75) holds, namely $w_0 \in W_s$. Then point 4) in Proposition 4.18 shows that $c_{12} = 0$ and $c_{11} = c_{22}$, hence the matrix $(c_{k\ell})$ is scalar. In addition, $\gamma_1(0) = 0$, hence $b_{k\ell} = 0$ by Corollary B.5.

b) Conversely, assume that (76) holds. Then $\gamma_1(0) = 0$ by Corollary B.5, and since $\gamma_2(0) = 0 = c_{12}$, $\gamma_3(0) = 0 = c_{22} - c_{11}$ it follows from Proposition 4.18 that $w_0 \in W_s$. This concludes the proof. □

Note that as a consequence of these investigations, we see that there are many other solutions $\mathbf{u}(x,t)$ of the Navier-Stokes equation that lie in the strong stable manifold (and hence satisfy the decay estimate (75)), but which do not satisfy the moment condition $(1+|x|)\mathbf{u}_0 \in L^1(\mathbf{R}^2)^2$. Put another way, the conditions (76) are the necessary and sufficient conditions for (75) to hold within the class of solutions satisfying $(1+|x|)\mathbf{u}_0 \in L^1(\mathbf{R}^2)^2$, but if one looks at larger classes of initial conditions then one finds that (76) is no longer necessary, and that a better characterization of solutions satisfying (75) is that they lie in the strong stable manifold W_s. Indeed, as we observed above, the velocity fields $\mathbf{v}^{H_2}$ and $\mathbf{v}^{H_3}$ corresponding to the vorticities H_2 and H_3 do not satisfy $(1+|\xi|)\mathbf{v}^{H_j} \in L^1(\mathbf{R}^2)^2$. So if we choose as an initial condition any point in the strong stable manifold W_s which does not lie in the hyperplane E_s of functions of the form (78) with $\gamma_1(0) = \gamma_2(0) = \gamma_3(0) = 0$, we obtain a solution of the Navier-Stokes equation satisfying (75), but not (76).

Of course, this argument requires that there be some point in W_s for which either γ_2 or γ_3 is nonzero. (Proposition 4.18 implies that all points in W_s^{loc} must have $\gamma_1 = 0$.) This could fail to occur only if W_s coincided with the hyperplane E_s, namely if E_s was invariant under the semiflow defined by (56). This in turn would happen only if $f_2(0,0,R) = f_3(0,0,R) = 0$ for all $R \in E_s$. We now demonstrate explicitly that this is not the case. Choose

$$K(\xi) = \partial_1 H_1(\xi) = \xi_1(1 - |\xi|^2/8)G(\xi) \; .$$

Then $\mathcal{L}K = -\frac{3}{2}K$, so that $K \in E_s$. The velocity field corresponding to K is

$$\mathbf{v}^K(\xi) = \partial_1 \mathbf{v}^{H_1}(\xi) = \frac{G(\xi)}{4}\begin{pmatrix} -\xi_1\xi_2 \\ \xi_1^2 - 2 \end{pmatrix} \; ,$$

and a direct calculation gives:

$$f_3(0,0,K) = \int_{\mathbf{R}^2} ((v_1^K)^2 - (v_2^K)^2)\,\mathrm{d}\xi = -\frac{1}{64\pi} \neq 0 \; .$$

This shows that $W_s \not\subset E_s$. On the other hand, it is easy to see that any $w \in E_s$ which is *radially symmetric* actually lies in W_s, because (2) reduces to the heat equation for radially symmetric vorticities. Thus the intersection $W_s \cap E_s$ contains at least an infinite-dimensional subspace of $L^2(m)$.

Summing up, we see that for solutions of the Navier-Stokes equation in $\mathbf{R}^2$, whose vorticity lies in $L^2(4)$, the strong-stable manifold identifies exactly those which converge to zero faster than C/t in energy norm. If the velocity field at $t = 0$ satisfies $(1+|x|)\mathbf{u} \in L^1(\mathbf{R}^2)^2$, this gives a natural geometrical interpretation of the moment conditions of [23]. It also shows, however, that there are additional solutions which decay with a rate faster than C/t but which do not satisfy the decay condition on the initial velocity field.

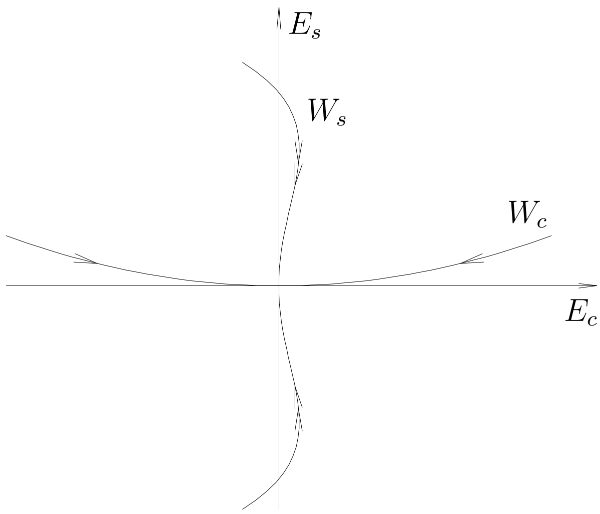

Fig. 3. A schematic picture of the dynamics defined by equations (79) in a neighborhood of the origin. All trajectories approach a three-dimensional invariant manifold W_c at a rate $\mathcal{O}(\mathrm{e}^{-\mu\tau})$ or faster. The strong-stable manifold W_s contains all solutions that converge to the origin faster than $\mathrm{e}^{-\tau}$. The intersection $W_s \cap E_s$, which is depicted here by two points only, contains in fact an infinite-dimensional subspace of $L^2(m)$.

We conclude this section by describing a somewhat surprising property of the strong-stable manifold. Although it was constructed to be invariant with respect to the dynamics of the vorticity equation in the rescaled variables, it is also invariant with respect to the dynamics expressed in the unscaled variables. Let Ψ_t^m be the semiflow on $L^2(m)$ defined by (2).

Proposition 4.20 *The semiflow Ψ_t^m leaves the manifold W_s invariant.*

Remark 4.21 *This should be contrasted with the behavior of W_c^{loc}. In order to obtain a manifold corresponding to W_c^{loc} that is invariant with respect to Ψ_t^m it is necessary to go the extended phase space $L^2(m) \times \mathbf{R}_+$, where $\mathbf{R}_+$ represents the time axis. (See the discussion in Section 3 of [30], in particular Theorem 3.6, where this question is studied in a related context.)*

Proof: The proposition follows from the characterization of W_s in point 3 of Proposition 4.18. Since $\mathbf{u}(\cdot, t)$ is the velocity field corresponding to the vorticity field $\Psi_t^m(w_0)$, and since this condition is clearly invariant with respect to time translation, the invariance of W_s with respect to Ψ_t^m is immediate. In other words, the fact that W_s is invariant with respect to both Ψ_t^m and Φ_t^m is related to the fact that solutions in W_s can be characterized by their asymptotics as $t \to +\infty$.

To be more explicit, note that combining point 2 in Proposition 4.18, with point 5 in the proof of that proposition, we see that w_0 lies in W_s if and only if the solution $w(\cdot, \tau)$ of (14) with initial data w_0 satisfies

$$\lim_{\tau \to +\infty} \mathrm{e}^{\tau} |w(\cdot, \tau)|_2 = 0 \ . \tag{83}$$

But letting $\omega(x,t) = \frac{1}{1+t} w(\frac{x}{\sqrt{1+t}}, \log(1+t))$, we see that (83) is equivalent to

$$\lim_{t\to+\infty} t^{3/2} |\omega(\cdot,t)|_2 = 0 \ . \tag{84}$$

Note also that $\omega(x,0) = w_0(x)$. Thus, W_s can also be characterized as the set of points in $L^2(4)$ satisfying (77) and for which

$$\lim_{t\to+\infty} t^{3/2} |\Psi_t^m(w_0)|_2 = 0 \ .$$

Since this characterization is invariant with respect to $\Psi_t^m(w_0)$, this shows that W_s is an invariant manifold in the original variables as well as in the rescaled variables. $\square$

A Spectrum of the linear operator

In this section, we study the spectrum of the linear operator $\mathcal{L}$ defined in (15). It involves essentially no additional effort to study the corresponding operator which results from studying the Navier-Stokes equations on $\mathbf{R}^N$ rather than just $\mathbf{R}^2$, and so we assume that $N \in \mathbf{N}$, $N \geq 1$. We consider the linear operator $\mathcal{L}$ given by

$$\mathcal{L} = \Delta_\xi + \frac{1}{2}\boldsymbol{\xi} \cdot \nabla_\xi + \frac{N}{2} \ , \quad \xi \in \mathbf{R}^N \ . \tag{85}$$

(The only change in considering our domain to have dimension N is in the constant term in the operator which changes from 1 to $N/2$.) As in Section 3, we shall work in the weighted space $L^2(m)$ defined by

$$\begin{aligned} L^2(m) &= \{ f \in L^2(\mathbf{R}^N) \,|\, \|f\|_m < \infty \} \ , \\ \|f\|_m &= \left(\int_{\mathbf{R}^N} (1+|\xi|^2)^m |f(\xi)|^2 \, \mathrm{d}\xi \right)^{1/2} \ . \end{aligned} \tag{86}$$

Since its coefficients depend linearly on the space variable ξ, the operator $\mathcal{L}$ becomes a first order differential operator when expressed in the Fourier variable p. Our convention for Fourier transformation is

$$\hat{f}(p) = \int_{\mathbf{R}^N} f(\xi) \exp(-\mathrm{i}\mathbf{p}\cdot\boldsymbol{\xi}) \, \mathrm{d}\xi \ , \tag{87}$$

$$f(\xi) = \frac{1}{(2\pi)^N} \int_{\mathbf{R}^N} \hat{f}(p) \exp(\mathrm{i}\mathbf{p}\cdot\boldsymbol{\xi}) \, \mathrm{d}p \ . \tag{88}$$

The expression of the operator $\mathcal{L}$ in Fourier space is:

$$(\widehat{\mathcal{L}f})(p) = -(|p|^2 + \frac{1}{2}\mathbf{p}\cdot\nabla_p)\hat{f}(p) \ . \tag{89}$$

The aim of this section is to prove the following result, which underlies our approach for computing the long-time asymptotics of the vorticity equation:

Theorem A.1 *Fix $m \ge 0$, and let $\mathcal{L}$ be the linear operator (85) in $L^2(m)$, defined on its maximal domain. Then the spectrum of $\mathcal{L}$ is*

$$\sigma(\mathcal{L}) = \Big\{\lambda \in \mathbf{C}\,\Big|\,\Re(\lambda) \le \frac{N}{4} - \frac{m}{2}\Big\} \cup \Big\{-\frac{k}{2}\,\Big|\,k \in \mathbf{N}\Big\}\ .$$

Moreover, if $m > \frac{N}{2}$ and if $k \in \mathbf{N}$ satisfies $k + \frac{N}{2} < m$, then $\lambda_k = -\frac{k}{2}$ is an isolated eigenvalue of $\mathcal{L}$, with multiplicity $\binom{N+k-1}{k}$.

In the one-dimensional case, this result is proved in [13], Appendix A. We give here a slightly different proof, which is valid for all $N \ge 1$. We begin with a few elementary observations.

(1) *The discrete spectrum of $\mathcal{L}$.* Fix $k \in \mathbf{N}$, and take $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbf{N}^N$ such that $|\alpha| = \alpha_1 + \ldots + \alpha_N = k$. Then the Hermite function $\phi_\alpha \in \mathcal{S}(\mathbf{R}^N)$ (where $\mathcal{S}$ is the Schwartz class) defined by

$$\begin{aligned} \hat{\phi}_\alpha(p) &= (\mathrm{i}p)^\alpha \mathrm{e}^{-|p|^2} \equiv \mathrm{i}^{|\alpha|} p_1^{\alpha_1} \dots p_N^{\alpha_N} \mathrm{e}^{-|p|^2}\ , \quad \text{or} \\ \phi_\alpha(\xi) &= (\partial_\xi^\alpha \phi_0)(\xi)\ , \quad \phi_0(\xi) = \frac{1}{(4\pi)^{N/2}}\, \mathrm{e}^{-|\xi|^2/4}\ , \end{aligned} \tag{90}$$

is an eigenfunction of $\mathcal{L}$ with eigenvalue $-\frac{k}{2}$. Thus, for any $m \ge 0$, we have $\sigma(\mathcal{L}) \supset \{-\frac{k}{2}\,|\,k \in \mathbf{N}\}$, and the multiplicity of the eigenvalue $\lambda_k = -\frac{k}{2}$ is greater or equal to $\binom{N+k-1}{k} = \#\{\alpha \in \mathbf{N}^N\,|\,|\alpha| = k\}$.

(2) *The "continuous" spectrum of $\mathcal{L}$.* Fix $\lambda \in \mathbf{C}$ such that $\Re(\lambda) < N/4$ and $-\lambda \notin \mathbf{N}$. The function $\psi_\lambda : \mathbf{R}^N \to \mathbf{R}$ defined (in Fourier variables) by

$$\hat{\psi}_\lambda(p) = |p|^{-2\lambda}\, \mathrm{e}^{-|p|^2}\ , \quad p \in \mathbf{R}^N \setminus \{0\}\ ,$$

is then an eigenfunction of $\mathcal{L}$ with eigenvalue λ. It is clear that $\psi_\lambda \in C^\infty(\mathbf{R}^N)$, and a standard calculation (see for instance [17], Section 2.3.3) shows that

$$\lim_{|\xi|\to\infty} |\xi|^{N-2\lambda} \psi_\lambda(\xi) = \frac{\Gamma(\frac{N}{2} - \lambda)}{2^{2\lambda}\pi^{\frac{N}{2}}\Gamma(\lambda)} \neq 0\ .$$

In particular, $\psi_\lambda \in L^2(m)$ if and only if $\Re(\lambda) < \frac{N}{4} - \frac{m}{2}$. Since the spectrum of $\mathcal{L}$ is closed, this shows that $\sigma(\mathcal{L}) \supset \{\lambda \in \mathbf{C}\,|\,\Re(\lambda) \le \frac{N}{4} - \frac{m}{2}\}$.

(3) *The spectral projections.* For all $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbf{N}^N$, we define the Hermite polynomial H_α by

$$H_\alpha(\xi) = \frac{2^{|\alpha|}}{\alpha!}\, \mathrm{e}^{|\xi|^2/4} \partial_\xi^\alpha \left(\mathrm{e}^{-|\xi|^2/4}\right)\ , \quad \xi \in \mathbf{R}^N\ ,$$

where $\alpha! = (\alpha_1!) \cdot \ldots \cdot (\alpha_N!)$. It is not difficult to verify that H_α is a polynomial of degree $|\alpha|$ which satisfies $\mathcal{L}^* H_\alpha = -\frac{|\alpha|}{2} H_\alpha$, where $\mathcal{L}^* = \Delta_\xi - \frac{1}{2}\boldsymbol{\xi} \cdot \nabla_\xi$ is the formal adjoint of $\mathcal{L}$. In addition, the following orthogonality relations hold for all $\alpha, \beta \in \mathbf{N}^N$:

$$\int_{\mathbf{R}^N} H_\alpha(\xi)\phi_\beta(\xi)\,\mathrm{d}\xi = \delta_{\alpha\beta}\ .$$

Finally, for any smooth function $\hat{f}(p)$, we have the relation

$$\left(H_\alpha(\mathrm{i}\nabla_p)\hat{f}\right)(0) = \frac{1}{\alpha!}\partial_p^\alpha\left(\hat{f}(p)\mathrm{e}^{|p|^2}\right)\Big|_{p=0} .$$

Assume now that $m > \frac{N}{2}$. For all $n \in \mathbf{Z}$ such that $n + \frac{N}{2} < m$, we define a continuous projection $P_n : L^2(m) \to L^2(m)$ by the formula

$$(P_n f)(\xi) = \sum_{|\alpha|\le n}\left(\int_{\mathbf{R}^N} H_\alpha(\xi')f(\xi')\,\mathrm{d}\xi'\right)^{1/2}\phi_\alpha(\xi) .$$

We also set $Q_n = \mathbf{1} - P_n$. Remark that $P_n = 0$ and $Q_n = \mathbf{1}$ if $n < 0$. If $n \ge 0$, it is clear from the definitions that P_n is the spectral projection onto the $\sum_{k=0}^n \binom{N+k-1}{k}$–dimensional subspace spanned by the eigenfunctions of $\mathcal{L}$ corresponding to the eigenvalues $\{-\frac{k}{2}\,|\, k = 0, 1, \ldots, n\}$. For later use, we note that the condition $P_n f = 0$ is equivalent to

$$\int_{\mathbf{R}^N} \xi^\alpha f(\xi)\,\mathrm{d}\xi = 0 \quad \text{for all } \alpha \in \mathbf{N}^N \text{ with } |\alpha| \le n .$$

(4) *The semigroup* $\mathrm{e}^{\tau\mathcal{L}}$. The operator $\mathcal{L}$ is the generator of a linear semigroup $S(\tau) = \mathrm{e}^{\tau\mathcal{L}}$ given by the following expressions:

$$\begin{aligned}
(\widehat{S(\tau)f})(p) &= \mathrm{e}^{-a(\tau)|p|^2}\hat{f}(p\,\mathrm{e}^{-\tau/2}) , \quad \text{or} \\
(S(\tau)f)(\xi) &= \frac{\mathrm{e}^{\frac{N\tau}{2}}}{(4\pi a(\tau))^{\frac{N}{2}}}\int_{\mathbf{R}^N}\exp\Bigl(-\frac{|\xi-\xi'|^2}{4a(\tau)}\Bigr)f(\xi'\mathrm{e}^{\tau/2})\,\mathrm{d}\xi' ,
\end{aligned}$$

where $a(\tau) = 1 - \mathrm{e}^{-\tau}$. It follows from these formulas that $S(\tau)$ is a strongly continuous (but not analytic) semigroup on $L^2(m)$ for any $m \ge 0$.

The main technical result of this section is the following estimate on the semigroup $S(\tau) = \mathrm{e}^{\tau\mathcal{L}}$:

Proposition A.2 **(a)** *Fix $m \ge 0$, and take $n \in \mathbf{Z}$ such that $n + \frac{N}{2} < m \le n + 1 + \frac{N}{2}$. For all $\alpha \in \mathbf{N}^N$ and all $\epsilon > 0$, there exists $C > 0$ such that*

$$\|\partial^\alpha S(\tau)Q_n f\|_m \le \frac{C}{a(\tau)^{|\alpha|/2}}\,\mathrm{e}^{\frac{\tau}{2}(\frac{N}{2}-m+\epsilon)}\|f\|_m , \tag{91}$$

for all $f \in L^2(m)$ and all $\tau > 0$.
(b) *Fix $n \in \mathbf{N}\cup\{-1\}$, and take $m \in \mathbf{R}$ such that $m > n + 1 + \frac{N}{2}$. For all $\alpha \in \mathbf{N}^N$, there exists $C > 0$ such that*

$$\|\partial^\alpha S(\tau)Q_n f\|_m \le \frac{C}{a(\tau)^{|\alpha|/2}}\,\mathrm{e}^{-\frac{n+1}{2}\tau}\|f\|_m , \tag{92}$$

for all $f \in L^2(m)$ and all $\tau > 0$.

Remark A.3 *Estimate (92) has been obtained in [30], [9], and [8] under the assumption that m is sufficiently large, depending on n. As is clear from Theorem A.1, the condition $m > n + 1 + \frac{N}{2}$ is optimal.*

Using Proposition A.2, it is easy to complete the proof of Theorem A.1. Indeed, if m and n are as in part **(a)** of Proposition A.2, then $\sigma(\mathcal{L}) = \sigma(\mathcal{L}P_n) \cup \sigma(\mathcal{L}Q_n)$. By construction, $\sigma(\mathcal{L}P_n) = \emptyset$ if $n < 0$ and $\sigma(\mathcal{L}P_n) = \{0; -\frac{1}{2}; \dots ; -\frac{n}{2}\}$ if $n \in \mathbf{N}$. On the other hand, by the Hille-Yosida theorem (see for instance [25]), the bound (91) implies that $\sigma(\mathcal{L}Q_n) \subset \{\lambda \in \mathbf{C} \,|\, \Re(\lambda) \le \frac{N}{4} - \frac{m}{2}\}$. Thus

$$\sigma(\mathcal{L}) \subset \left\{\lambda \in \mathbf{C} \,\middle|\, \Re(\lambda) \le \frac{N}{4} - \frac{m}{2}\right\} \cup \left\{-\frac{k}{2} \,\middle|\, k \in \mathbf{N}\right\} ,$$

and the reverse inclusion has already been established (see **(1)** and **(2)** above). Finally, since $\sigma(\mathcal{L}P_n) \cap \sigma(\mathcal{L}Q_n) = \emptyset$, we see that the multiplicity of the eigenvalue $\lambda_k = -\frac{k}{2}$ $(k = 0, \dots, n)$ is exactly $\binom{N+k-1}{k}$. This concludes the proof of Theorem A.1. $\square$

Proof of Proposition A.2. We first show that **(a)** $\Rightarrow$ **(b)**. Let $n \in \mathbf{N} \cup \{-1\}$ and $m > n + 1 + \frac{N}{2}$. Taking $\bar{n} \in \mathbf{N}$ such that $\bar{n} + \frac{N}{2} < m \le \bar{n} + 1 + \frac{N}{2}$, we can write, for any $f \in L^2(m)$,

$$Q_n f = (\mathbf{1} - P_n) f = (P_{\bar{n}} - P_n) f + Q_{\bar{n}} f \ .$$

Since $m - \frac{N}{2} > \bar{n} \ge n + 1$, we have by (91)

$$\|\partial^\alpha S(\tau) Q_{\bar{n}} f\|_m \le \frac{C}{a(\tau)^{|\alpha|/2}} \, \mathrm{e}^{-\frac{n+1}{2}\tau} \|f\|_m \ , \quad \tau > 0 \ .$$

On the other hand, since $S(\tau)\phi_\alpha = \mathrm{e}^{-|\alpha|\tau/2}\phi_\alpha$ and

$$(P_{\bar{n}} - P_n) f = \sum_{n < |\alpha| \le \bar{n}} \left(\int_{\mathbf{R}^N} H_\alpha(\xi) f(\xi) \, \mathrm{d}\xi \right)^{1/2} \phi_\alpha \ ,$$

we easily obtain $\|\partial^\alpha S(\tau)(P_{\bar{n}} - P_n) f\|_m \le C \mathrm{e}^{-\frac{n+1}{2}\tau} \|f\|_m$. This proves (92).

To prove **(a)**, we first remark that, for all $\alpha \in \mathbf{N}^N$,

$$(\partial^\alpha S(\tau) f)(\xi) = \frac{\mathrm{e}^{\frac{N\tau}{2}}}{a(\tau)^{\frac{N+|\alpha|}{2}}} \int_{\mathbf{R}^N} \phi_\alpha\Big(\frac{\xi - \xi'}{a(\tau)^{1/2}}\Big) f(\xi' \mathrm{e}^{\frac{\tau}{2}}) \, \mathrm{d}\xi' \ , \tag{93}$$

where ϕ_α is defined in (90). Since $1 + |\xi|^m \le C(1 + |\xi - \xi'|^m)(1 + |\xi'|^m)$, we have

$$\begin{aligned}(1 + |\xi|^m) |(\partial^\alpha S(\tau) f)(\xi)| \ &\le \ C \frac{\mathrm{e}^{\frac{N\tau}{2}}}{a(\tau)^{\frac{N+|\alpha|}{2}}} \int_{\mathbf{R}^N} (1 + |\xi - \xi'|^m) \Big|\phi_\alpha\Big(\frac{\xi - \xi'}{a(\tau)^{1/2}}\Big)\Big| \\ &\qquad \times (1 + |\xi'|^m) |f(\xi' \mathrm{e}^{\frac{\tau}{2}})| \, \mathrm{d}\xi' \ .\end{aligned}$$

Observing that

$$\frac{1}{a(\tau)^{\frac{N}{2}}} \int_{\mathbf{R}^N} (1 + |\eta|^m) \Big|\phi_\alpha\Big(\frac{\eta}{a(\tau)^{1/2}}\Big)\Big| \, \mathrm{d}\eta \ \le \ C \ ,$$

and applying Young's inequality $|g * h|_2 \le |g|_1 |h|_2$, we obtain

$$\|\partial^\alpha S(\tau) f\|_m \le C \frac{\mathrm{e}^{\frac{N\tau}{4}}}{a(\tau)^{\frac{|\alpha|}{2}}} \|f\|_m \ , \quad \tau > 0 \ . \tag{94}$$

Replacing f by $Q_n f$ in (94), we see that (91) holds for $0 < \tau < 1$. Thus, it remains to prove (91) for $\tau \ge 1$. Since $1 - \mathrm{e}^{-1} \le a(\tau) \le 1$ when $\tau \ge 1$, we may drop for simplicity the factors $a(\tau)$ in (93). Thus, all we really need to prove is:

Lemma A.4 *Let $\phi \in \mathcal{S}(\mathbf{R}^N)$, $m \ge 0$, and $n \in \mathbf{Z}$ such that $n + \frac{N}{2} < m \le n+1+\frac{N}{2}$. For any $f \in L^2(m)$, define*

$$(\bar{S}(\tau)f)(\xi) = \int_{\mathbf{R}^N} \phi(\xi-\eta) f(\eta \mathrm{e}^{\frac{\tau}{2}})\,\mathrm{d}\eta\ , \quad \tau \ge 0\ .$$

Then, for all $\epsilon > 0$, there exists $C > 0$ such that, for all $f \in L^2(m)$,

$$\|\bar{S}(\tau) Q_n f\|_m \le C \mathrm{e}^{-\frac{\tau}{2}(m+\frac{N}{2}-\epsilon)} \|f\|_m\ , \quad \tau \ge 0. \tag{95}$$

Proof: Expanding ϕ in Taylor series, we obtain for all $\xi, \eta \in \mathbf{R}^N$:

$$\phi(\xi-\eta) = \sum_{|\alpha| \le n} \frac{(-1)^{|\alpha|}}{\alpha!} (\partial^\alpha \phi)(\xi) \eta^\alpha + \sum_{|\alpha|=n+1} \frac{(-1)^{n+1}}{\alpha!} \Phi_\alpha(\xi,\eta)\eta^\alpha\ ,$$

where

$$\Phi_\alpha(\xi,\eta) = (n+1)\int_0^1 (1-s)^n (\partial^\alpha\phi)(\xi - s\eta)\,\mathrm{d}s\ .$$

If $g = Q_n f$, then $P_n g = 0$, hence $\int_{\mathbf{R}^N} \eta^\alpha g(\eta)\,\mathrm{d}\eta = 0$ for all $\alpha \in \mathbf{N}^N$ with $|\alpha| \le n$. Therefore

$$\begin{aligned}
(\bar{S}(\tau)g)(\xi) &= \int_{\mathbf{R}^N} \Big(\phi(\xi-\eta) - \sum_{|\alpha|\le n} \frac{(-1)^{|\alpha|}}{\alpha!}(\partial^\alpha\phi)(\xi)\eta^\alpha\Big) g(\eta \mathrm{e}^{\frac{\tau}{2}})\,\mathrm{d}\eta \\
&= (\bar{S}_1(\tau)g)(\xi) + (\bar{S}_2(\tau)g)(\xi) + (\bar{S}_3(\tau)g)(\xi)\ ,
\end{aligned}$$

where

$$\begin{aligned}
(\bar{S}_1(\tau)g)(\xi) &= \int_{|\eta|\ge 1} \phi(\xi-\eta) g(\eta \mathrm{e}^{\frac{\tau}{2}})\,\mathrm{d}\eta\ , \\
(\bar{S}_2(\tau)g)(\xi) &= -\sum_{|\alpha|\le n} \frac{(-1)^{|\alpha|}}{\alpha!}(\partial^\alpha\phi)(\xi) \int_{|\eta|\ge 1} \eta^\alpha g(\eta \mathrm{e}^{\frac{\tau}{2}})\,\mathrm{d}\eta\ , \\
(\bar{S}_3(\tau)g)(\xi) &= \sum_{|\alpha|=n+1} \frac{(-1)^{n+1}}{\alpha!} \int_{|\eta|\le 1} \Phi_\alpha(\xi,\eta)\eta^\alpha g(\eta \mathrm{e}^{\frac{\tau}{2}})\,\mathrm{d}\eta\ .
\end{aligned}$$

Since $1 + |\xi|^m \le C(1 + |\xi-\eta|^m)|\eta|^m$ when $|\eta| \ge 1$, we have

$$(1+|\xi|^m)|(\bar{S}_1(\tau)g)(\xi)| \le C\int_{|\eta|\ge 1} (1+|\xi-\eta|^m)|\phi(\xi-\eta)||\eta|^m |g(\eta \mathrm{e}^{\frac{\tau}{2}})|\,\mathrm{d}\eta\ .$$

Applying Young's inequality, we conclude that

$$\|(\bar{S}_1(\tau)g)\|_m \le C \mathrm{e}^{-\frac{\tau}{2}(m+\frac{N}{2})}\|g\|_m\ , \quad \tau \ge 0\ . \tag{96}$$

On the other hand, if $|\alpha| \le n$, we have by Hölder's inequality

$$\begin{aligned}
\Big|\int_{|\eta|\ge 1} \eta^\alpha g(\eta \mathrm{e}^{\frac{\tau}{2}})\,\mathrm{d}\eta\Big| &\le \int_{|\eta|\ge 1} |\eta|^{n-m}(|\eta|^m |g(\eta \mathrm{e}^{\frac{\tau}{2}})|)\,\mathrm{d}\eta \\
&\le \Big(\int_{|\eta|\ge 1} |\eta|^{2(n-m)}\,\mathrm{d}\eta\Big)^{1/2} \mathrm{e}^{-\frac{\tau}{2}(m+\frac{N}{2})}\|g\|_m\ ,
\end{aligned}$$

hence

$$\|(\bar{S}_2(\tau)g)\|_m \le C\mathrm{e}^{-\frac{\tau}{2}(m+\frac{N}{2})}\|g\|_m\ , \quad \tau \ge 0\ . \tag{97}$$

To bound the last term $\bar{S}_3(\tau)g$, let $\alpha \in \mathbf{N}^N$ such that $|\alpha| = n+1$, and define $\Psi_\alpha(\xi,\eta) = (1+|\xi|^m)\Phi_\alpha(\xi,\eta)$. For any $r \ge 1$, it is easy to verify that

$$\sup_{\xi\in\mathbf{R}^N}\left(\int_{|\eta|\le 1}|\Psi_\alpha(\xi,\eta)|^r\,\mathrm{d}\eta\right)^{1/r} + \sup_{|\eta|\le 1}\left(\int_{\mathbf{R}^N}|\Psi_\alpha(\xi,\eta)|^r\,\mathrm{d}\xi\right)^{1/r} < \infty\ .$$

As is well-known, this implies that the linear operator defined by the integral kernel $\Psi_\alpha(\xi,\eta)$ is bounded from $L^q(\{|\eta|\le 1\})$ into $L^p(\mathbf{R}^N)$ for all $q \le p$. We now distinguish two cases:

Case 1. If $m \le n+1$ (this is possible only if $N=1$), we have

$$\begin{aligned}(1+|\xi|^m)|(\bar{S}_3(\tau)g)(\xi)| &\le \sum_{|\alpha|=n+1}\frac{1}{\alpha!}\int_{|\eta|\le 1}|\Psi_\alpha(\xi,\eta)||\eta|^{n+1}|g(\eta\mathrm{e}^{\frac{\tau}{2}})|\,\mathrm{d}\eta &(98)\\ &\le \sum_{|\alpha|=n+1}\frac{1}{\alpha!}\int_{|\eta|\le 1}|\Psi_\alpha(\xi,\eta)||\eta|^{m}|g(\eta\mathrm{e}^{\frac{\tau}{2}})|\,\mathrm{d}\eta\ . &(99)\end{aligned}$$

Using (99) and the remark above with $p=q=2$, we immediately obtain

$$\|(\bar{S}_3(\tau)g)\|_m \le C\mathrm{e}^{-\frac{\tau}{2}(m+\frac{N}{2})}\|g\|_m\ , \quad \tau \ge 0\ . \tag{100}$$

Case 2. If $n+1 < m \le n+1+\frac{N}{2}$, we define $q_* \in [1,2)$ by the relation $N(\frac{1}{q_*}-\frac{1}{2}) = m-(n+1)$. Given any $\epsilon > 0$, we choose $q \in (q_*,2)$ such that $N(\frac{1}{q_*}-\frac{1}{q}) \le \epsilon$. As is easily verified, there exists $C>0$ such that $||\eta|^{n+1}g(\eta)|_q \le \|g\|_m$ for all $g \in L^2(m)$. Using (98) and the remark above with $p=2$, we obtain

$$\begin{aligned}\|(\bar{S}_3(\tau)g)\|_m &\le C\left(\int_{|\eta|\le 1}(|\eta|^{n+1}|g(\eta\mathrm{e}^{\frac{\tau}{2}})|)^q\,\mathrm{d}\eta\right)^{1/q}\\ &\le C\mathrm{e}^{-\frac{\tau}{2}(n+1+\frac{N}{q})}||\eta|^{n+1}g(\eta)|_q \le C\mathrm{e}^{-\frac{\tau}{2}(m+\frac{N}{2}-\epsilon)}\|g\|_m &(101)\end{aligned}$$

since $n+1+\frac{N}{q} = m+\frac{N}{2}-N(\frac{1}{q_*}-\frac{1}{q}) \ge m+\frac{N}{2}-\epsilon$. Combining (96), (97), and (100) or (101), we obtain (95). This concludes the proof of Lemma A.4, hence of Proposition A.2. □

The results of this section can easily be generalized to weighted L^p spaces with $p \ne 2$. For instance, the following bounds are used in Section 3 to control the nonlinearity in (14).

Proposition A.5 *Let $1 \le q \le p \le \infty$, $m \ge 0$ and $T > 0$. For all $\alpha \in \mathbf{N}^N$, there exists $C > 0$ such that*

$$|b^m\partial^\alpha S(\tau)f|_p \le \frac{C}{a(\tau)^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p})+\frac{|\alpha|}{2}}}|b^mf|_q\ , \quad \tau > 0\ , \tag{102}$$

where $b(\xi) = (1+|\xi|^2)^{1/2}$.

Proof: This estimate follows easily from (93) and Young's inequality. □

B Bounds on the velocity field

In this section, we study in more detail the relationship between the vorticity $w(\xi)$ and the velocity field $\mathbf{v}(\xi)$ obtained from $w(\xi)$ via the Biot-Savart law (16). In particular, we obtain sharp estimates for the spatial decay of the velocity as $|\xi| \to \infty$. This information is systematically used in Section 4 to derive properties of the solutions of the Navier-Stokes equation from the results we have on the vorticity.

First, we give explicit formulas for the velocity fields corresponding to the first eigenfunctions of the linear operator $\mathcal{L}$ defined in (15). For our purposes in Section 4, it is sufficient to consider only the first three eigenvalues $\lambda_0 = 0$, $\lambda_1 = -1/2$, and $\lambda_2 = -1$.

1) The first eigenvalue $\lambda_0 = 0$ is simple, with Gaussian eigenfunction

$$G(\xi) = \frac{1}{4\pi}\, \mathrm{e}^{-|\xi|^2/4} \;.$$

The corresponding velocity field $\mathbf{v}^G(\xi)$ is given by

$$\mathbf{v}^G(\xi) = \frac{1}{2\pi} \frac{1 - \mathrm{e}^{-|\xi|^2/4}}{|\xi|^2} \begin{pmatrix} -\xi_2 \\ \xi_1 \end{pmatrix} \;. \tag{103}$$

Remark that $|\mathbf{v}^G(\xi)| \sim |\xi|^{-1}$ as $|\xi| \to \infty$, so that $\mathbf{v}^G \in L^q(\mathbf{R}^2)^2$ for all $q > 2$.

2) The second eigenvalue $\lambda_1 = -1/2$ is double, with eigenfunctions

$$F_i(\xi) = \partial_i G(\xi) = -\frac{\xi_i}{2} G(\xi) \;, \quad i = 1, 2 \;.$$

The corresponding velocity fields are

$$\mathbf{v}^{F_1}(\xi) = \partial_1 \mathbf{v}^G(\xi) \;, \quad \mathbf{v}^{F_2}(\xi) = \partial_2 \mathbf{v}^G(\xi) \;. \tag{104}$$

A direct calculation shows that

$$\mathbf{v}^{F_1}(\xi) = \frac{1}{2\pi|\xi|^4} \begin{pmatrix} 2\xi_1\xi_2 \\ \xi_2^2 - \xi_1^2 \end{pmatrix} + \tilde{\mathbf{v}}^{F_1}(\xi) \;, \quad \mathbf{v}^{F_2}(\xi) = \frac{1}{2\pi|\xi|^4} \begin{pmatrix} \xi_2^2 - \xi_1^2 \\ -2\xi_1\xi_2 \end{pmatrix} + \tilde{\mathbf{v}}^{F_2}(\xi) \;,$$

where $|\tilde{\mathbf{v}}^{F_i}(\xi)| = \mathcal{O}(\mathrm{e}^{-|\xi|^2/4})$ as $|\xi| \to \infty$. In particular, $|\mathbf{v}^{F_i}(\xi)| \sim |\xi|^{-2}$ as $|\xi| \to \infty$, so that $\mathbf{v}^{F_i} \in L^q(\mathbf{R}^2)^2$ for all $q > 1$.

3) The third eigenvalue $\lambda_1 = -1$ has multiplicity three. A convenient basis of eigenfunctions is

$$\begin{aligned} H_1(\xi) &= \Delta G(\xi) = (\tfrac{1}{4}|\xi|^2 - 1) G(\xi) \;, \\ H_2(\xi) &= (\partial_1^2 - \partial_2^2) G(\xi) = \tfrac{1}{4}(\xi_1^2 - \xi_2^2) G(\xi) \;, \\ H_3(\xi) &= \partial_1\partial_2 G(\xi) = \frac{1}{4}\xi_1\xi_2 G(\xi) \;, \end{aligned}$$

and the corresponding velocity fields read

$$\begin{aligned} \mathbf{v}^{H_1}(\xi) &= \Delta \mathbf{v}^G(\xi) = \frac{1}{2} G(\xi) \begin{pmatrix} \xi_2 \\ -\xi_1 \end{pmatrix} \;, \\ \mathbf{v}^{H_2}(\xi) &= (\partial_1^2 - \partial_2^2) \mathbf{v}^G(\xi) \;, \quad \mathbf{v}^{H_3}(\xi) \;=\; \partial_1\partial_2 \mathbf{v}^G(\xi) \;. \end{aligned} \tag{105}$$

Remark that $\mathbf{v}^{H_1}$ decreases rapidly as $|\xi| \to \infty$. In constrast, it is easy to verify that

$$\mathbf{v}^{H_2}(\xi) = \frac{2}{\pi|\xi|^6}\begin{pmatrix} \xi_2(\xi_2^2 - 3\xi_1^2) \\ \xi_1(\xi_1^2 - 3\xi_2^2)\end{pmatrix} + \tilde{\mathbf{v}}^{H_2}(\xi) \;, \quad \mathbf{v}^{H_3}(\xi) = \frac{1}{\pi|\xi|^6}\begin{pmatrix} \xi_1(\xi_1^2 - 3\xi_2^2) \\ -\xi_2(\xi_2^2 - 3\xi_1^2)\end{pmatrix} + \tilde{\mathbf{v}}^{H_3}(\xi) \;,$$

where $|\tilde{\mathbf{v}}^{H_j}(\xi)| = \mathcal{O}(|\xi|\mathrm{e}^{-|\xi|^2/4})$ as $|\xi| \to \infty$ for $j = 2, 3$. Thus, for $j = 2, 3$, $|\mathbf{v}^{H_j}(\xi)| \sim |\xi|^{-3}$ as $|\xi| \to \infty$, so that $\mathbf{v}^{H_j} \in L^1(\mathbf{R}^2)^2$, but $b\mathbf{v}^{H_j} \notin L^1(\mathbf{R}^2)^2$.

Next, assume that $w \in L^2(m)$ for some $m > 3$. Then w can be decomposed as follows:

$$w(\xi) = \alpha G(\xi) + \sum_{i=1}^{2} \beta_i F_i(\xi) + \sum_{j=1}^{3} \gamma_j H_j(\xi) + R(\xi) \;, \tag{106}$$

where α, β_i are given by (42) and γ_j by (51). The corresponding velocity field has a similar decomposition

$$\mathbf{v}(\xi) = \alpha \mathbf{v}^G(\xi) + \sum_{i=1}^{2} \beta_i \mathbf{v}^{F_i}(\xi) + \sum_{j=1}^{3} \gamma_j \mathbf{v}^{H_j}(\xi) + \mathbf{v}^R(\xi) \;, \tag{107}$$

where $\mathbf{v}^R$ is obtained from R via the Biot-Savart law (16). By construction, the remainder term R in (106) satisfies $\int_{\mathbf{R}^2} \xi_1^{n_1}\xi_2^{n_2} R(\xi)\,\mathrm{d}\xi = 0$ for all $n_1, n_2 \in \mathbf{N}$ such that $n_1 + n_2 \le 2$. Using this information, we shall show that $|\mathbf{v}^R(\xi)| \sim |\xi|^{-m}$ as $|\xi| \to \infty$, if $m \le 4$. In view of (106), it will follow that $\mathbf{v} \in L^2(\mathbf{R}^2)^2$ if and only if $\alpha = 0$, namely

$$\int_{\mathbf{R}^2} w(\xi)\,\mathrm{d}\xi = 0 \;. \tag{108}$$

Moreover, if (108) holds, then $\mathbf{v} \in L^1(\mathbf{R}^2)^2$ if and only if $\beta_1 = \beta_2 = 0$, namely

$$\int_{\mathbf{R}^2} \xi_1 w(\xi)\,\mathrm{d}\xi = \int_{\mathbf{R}^2} \xi_2 w(\xi)\,\mathrm{d}\xi = 0 \;. \tag{109}$$

Finally, if (108) and (109) hold, then $b\mathbf{v} \in L^1(\mathbf{R}^2)^2$ if and only if $\gamma_2 = \gamma_3 = 0$, namely

$$\int_{\mathbf{R}^2} (\xi_1^2 - \xi_2^2) w(\xi)\,\mathrm{d}\xi = \int_{\mathbf{R}^2} \xi_1\xi_2 w(\xi)\,\mathrm{d}\xi = 0 \;. \tag{110}$$

Remark that it is not necessary to assume here that $\gamma_1 = 0$, since the velocity field $\mathbf{v}^{H_1}$ (unlike $\mathbf{v}^{H_2}$ and $\mathbf{v}^{H_3}$) decreases rapidly at infinity.

Proposition B.1 *Let $w \in L^2(m)$ for some $m > 0$, and denote by $\mathbf{v}$ the velocity field obtained from w via the Biot-Savart law (16). Assume that either*
1) *$0 < m \le 1$, or*
2) *$1 < m \le 2$ and (108) holds, or*
3) *$2 < m \le 3$ and (108), (109) hold, or*
4) *$3 < m \le 4$ and (108), (109), (110) hold.*
If $m \notin \mathbf{N}$, then for all $q \in (2, +\infty)$, there exists $C > 0$ such that

$$|b^{m-2/q}\mathbf{v}|_q \le C|b^m w|_2 \;, \tag{111}$$

where $b(\xi) = (1 + |\xi|^2)^{1/2}$. If $m \in \mathbf{N}$ and $b^m w \in L^p(\mathbf{R}^2) \cap L^r(\mathbf{R}^2)$ for some $p < 2$, $r > 2$, then

$$|b^m \mathbf{v}|_\infty \le C(|b^m w|_p + |b^m w|_r) \;. \tag{112}$$

The proof of Proposition B.1 relies on the following weighted Hardy-Littlewood-Sobolev inequality:

Lemma B.2 *If* $0 < m < 1$ *and*

$$u(\xi) = \int_{\mathbf{R}^2} \frac{\omega(\eta)}{|\xi - \eta|} \,\mathrm{d}\eta \;, \quad \xi \in \mathbf{R}^2 \;,$$

then, for all $q \in (2, +\infty)$, $|b^{m-2/q} u|_q \le C |b^m \omega|_2$.

Proof: We use the dyadic decomposition

$$\mathbf{R}^2 = \bigcup_{j=0}^{\infty} B_j \;,$$

where $B_0 = \{\xi \in \mathbf{R}^2 \,|\, |\xi| \le 1\}$ and $B_j = \{\xi \in \mathbf{R}^2 \,|\, 2^{j-1} < |\xi| \le 2^j\}$ for $j \in \mathbf{N}^*$. Let $u_i = u\mathbf{1}_{B_i}$ and $\omega_i = \omega\mathbf{1}_{B_i}$, $i \in \mathbf{N}$. Clearly $u_i = \sum_{j\in\mathbf{N}} \Delta_{ij}$, where

$$\Delta_{ij}(\xi) = \mathbf{1}_{B_i}(\xi) \int_{B_j} \frac{\omega_j(\eta)}{|\xi - \eta|} \,\mathrm{d}\eta \;.$$

Fix $q \in (2, +\infty)$, and define $p \in (1, 2)$ by the relation $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$.
If $|i - j| \le 1$, it follows from (4) that $|\Delta_{ij}|_q \le C|\omega_j|_p$. By Hölder's inequality, $|\omega_j|_p \le \mathrm{mes}(B_j)^{1/q} |\omega_j|_2 \le C 2^{2j/q} |\omega_j|_2$. Thus, $|\Delta_{ij}|_q \le C 2^{2j/q} |\omega_j|_2$.
If $|i - j| \ge 2$, Young's inequality implies that $|\Delta_{ij}|_q \le M_1^{1-p/q} M_2^{p/q} |\omega_j|_2$, where where

$$M_1 = \sup_{\xi\in B_i} \Bigl(\int_{B_j} \frac{1}{|\xi - \eta|^p} \,\mathrm{d}\eta\Bigr)^{1/p} \;, \quad M_2 = \sup_{\eta\in B_j} \Bigl(\int_{B_i} \frac{1}{|\xi - \eta|^p} \,\mathrm{d}\xi\Bigr)^{1/p} \;.$$

If $i \ge j + 2$, then $|\xi - \eta| \ge |\xi| - |\eta| \ge 2^{i-1} - 2^j \ge 2^{i-2}$ for all $\xi \in B_i$, $\eta \in B_j$. Thus $M_1 \le C 2^{-i} \mathrm{mes}(B_j)^{1/p} \le C 2^{-i} 2^{2j/p}$ and $M_2 \le C 2^{-i} \mathrm{mes}(B_i)^{1/p} \le C 2^{-i} 2^{2i/p}$, for some $C > 0$ independent of i, j. It follows that $|\Delta_{ij}|_q \le C 2^{-(i-j)} 2^{2i/q} |\omega_j|_2$. If $j \ge i + 2$, then $|\xi - \eta| \ge 2^{j-2}$ for all $\xi \in B_i$, $\eta \in B_j$, and a similar calculation shows that $|\Delta_{ij}|_q \le C 2^{2i/q} |\omega_j|_2$. Summarizing, we have shown that

$$2^{-2i/q} |u_i|_q \le C \sum_{j\in\mathbf{N}} K_{ij} |\omega_j|_2 \;, \quad i \in \mathbf{N} \;,$$

where $K_{ij} = 2^{-\frac{1}{2}|i-j| - \frac{1}{2}(i-j)}$. Now, by definition of the sets B_i, there exists $C \ge 1$ such that $C^{-1} 2^i \le b(\xi) \le C 2^i$ for all $\xi \in B_i$ and all $i \in \mathbf{N}$. It follows that

$$|b^{m-2/q} u_i|_q \le C \sum_{j\in\mathbf{N}} K_{ij}^{(m)} |b^m \omega_j|_2 \;, \quad i \in \mathbf{N} \;,$$

where $K_{ij}^{(m)} = 2^{-\frac{1}{2}|i-j| + (m-\frac{1}{2})(i-j)}$. Since $0 < m < 1$, $|K_{ij}^{(m)}| \le 2^{-\alpha|i-j|}$ for some $\alpha > 0$, hence $K^{(m)}$ defines a bounded linear operator from $\ell^2(\mathbf{N})$ into $\ell^q(\mathbf{N})$. This concludes the proof. $\square$

Proof of Proposition B.1. The proof is naturally divided into four steps, according to the values of m.

1) If $\mathbf{v}$ and w are related via the Biot-Savart law (16), then

$$|\mathbf{v}(\xi)| \le \frac{1}{2\pi} \int_{\mathbf{R}^2} \frac{1}{|\xi - \eta|} |w(\eta)| \,\mathrm{d}\eta \;, \quad \xi \in \mathbf{R}^2 \;, \tag{113}$$

and (111) follows immediately from Lemma B.2. To prove (112), we remark that $b(\xi) \le 1 + |\xi| \le 1 + |\xi - \eta| + |\eta| \le |\xi - \eta| + 2b(\eta)$ for all $\xi, \eta \in \mathbf{R}^2$, hence

$$b(\xi)|\mathbf{v}(\xi)| \le \frac{1}{2\pi} \int_{\mathbf{R}^2} |w(\eta)| \,\mathrm{d}\eta + \frac{1}{\pi} \int_{\mathbf{R}^2} \frac{1}{|\xi - \eta|} b(\eta) w(\eta) \,\mathrm{d}\eta \;. \tag{114}$$

In view of (5), the second integral in (114) is uniformly bounded by $C(|bw|_p + |bw|_r)$ if $p < 2$ and $r > 2$. On the other hand, it is clear that $|w|_1 \le C|bw|_p$. This concludes the proof of case **1**.

2) For all $\xi, \eta \in \mathbf{R}^2$ with $\xi \neq 0$ and $\xi \neq \eta$, we have the identity

$$\frac{\xi_1 - \eta_1}{|\xi - \eta|^2} - \frac{\xi_1}{|\xi|^2} = \frac{1}{|\xi|^2 |\xi - \eta|^2} \Big((\xi_1 - \eta_1)(\boldsymbol{\xi} \cdot \boldsymbol{\eta}) + (\xi_2 - \eta_2)(\boldsymbol{\xi} \wedge \boldsymbol{\eta}) \Big) \;,$$

where $\boldsymbol{\xi} \cdot \boldsymbol{\eta} = \xi_1 \eta_1 + \xi_2 \eta_2$ and $\boldsymbol{\xi} \wedge \boldsymbol{\eta} = \xi_1 \eta_2 - \xi_2 \eta_1$. Therefore, for all $w \in L^2(m)$ satisfying (108), we find

$$\begin{aligned} |v_2(\xi)| &= \frac{1}{2\pi} \left| \int_{\mathbf{R}^2} \left(\frac{\xi_1 - \eta_1}{|\xi - \eta|^2} - \frac{\xi_1}{|\xi|^2} \right) w(\eta) \,\mathrm{d}\eta \right| \\ &\le C \int_{\mathbf{R}^2} \frac{1}{|\xi||\xi - \eta|} |\eta||w(\eta)| \,\mathrm{d}\eta \;. \end{aligned}$$

A similar bound holds for the first component $v_1(\xi)$ of the velocity field. Combining these estimates with (113), we obtain

$$|b(\xi)\mathbf{v}(\xi)| \le C \int_{\mathbf{R}^2} \frac{1}{|\xi - \eta|} |b(\eta) w(\eta)| \,\mathrm{d}\eta \;, \quad \xi \in \mathbf{R}^2 \;.$$

This inequality has exactly the same form as (113), with $\mathbf{v}, w$ replaced by $b\mathbf{v}, bw$. Therefore, applying the preceding result to $b\mathbf{v}, bw$ instead of $\mathbf{v}, w$, we obtain (111) and (112) in case **2**.

3) For all $\xi, \eta \in \mathbf{R}^2$ with $\xi \neq 0$ and $\xi \neq \eta$, we have the identity

$$\begin{aligned} &\frac{\xi_1 - \eta_1}{|\xi - \eta|^2} - \frac{\xi_1 - \eta_1}{|\xi|^2} - \frac{2\xi_1(\boldsymbol{\xi} \cdot \boldsymbol{\eta})}{|\xi|^4} \\ &\qquad = \frac{1}{|\xi|^4 |\xi - \eta|^2} \Big((\xi_1 - \eta_1)(2(\boldsymbol{\xi} \cdot \boldsymbol{\eta})^2 - |\xi|^2 |\eta|^2) + 2(\xi_2 - \eta_2)(\boldsymbol{\xi} \cdot \boldsymbol{\eta})(\boldsymbol{\xi} \wedge \boldsymbol{\eta}) \Big) \;. \end{aligned}$$

Therefore, for all $w \in L^2(m)$ satisfying (108) and (109), we find

$$\begin{aligned} |v_2(\xi)| &= \frac{1}{2\pi} \left| \int_{\mathbf{R}^2} \left(\frac{\xi_1 - \eta_1}{|\xi - \eta|^2} - \frac{\xi_1 - \eta_1}{|\xi|^2} - \frac{2\xi_1(\boldsymbol{\xi} \cdot \boldsymbol{\eta})}{|\xi|^4} \right) w(\eta) \,\mathrm{d}\eta \right| \\ &\le C \int_{\mathbf{R}^2} \frac{1}{|\xi|^2 |\xi - \eta|} |\eta|^2 |w(\eta)| \,\mathrm{d}\eta \;. \end{aligned}$$

A similar bound holds for the first component $v_1(\xi)$, hence

$$|b(\xi)^2\mathbf{v}(\xi)| \le C\int_{\mathbf{R}^2}\frac{1}{|\xi-\eta|}|b(\eta)^2 w(\eta)|\,\mathrm{d}\eta\;, \quad \xi\in\mathbf{R}^2\;.$$

This inequality has again the same form as (113), with $\mathbf{v}, w$ replaced by $b^2\mathbf{v}, b^2 w$. Therefore, proceeding as above, we obtain (111) and (112) in case **3**.

4) For all $\xi, \eta \in \mathbf{R}^2$ with $\xi \neq 0$ and $\xi \neq \eta$, we have the identity

$$\begin{aligned}\frac{\xi_1-\eta_1}{|\xi-\eta|^2} - \frac{\xi_1-\eta_1}{|\xi|^2} - \frac{2(\xi_1-\eta_1)(\boldsymbol{\xi}\cdot\boldsymbol{\eta})}{|\xi|^4} + \frac{\xi_1|\eta|^2}{|\xi|^4} - \frac{4\xi_1(\boldsymbol{\xi}\cdot\boldsymbol{\eta})^2}{|\xi|^6}&\\ = \frac{1}{|\xi|^6|\xi-\eta|^2}\Big((\xi_1-\eta_1)(\boldsymbol{\xi}\cdot\boldsymbol{\eta})(4(\boldsymbol{\xi}\cdot\boldsymbol{\eta})^2 - 3|\xi|^2|\eta|^2)&\\ + (\xi_2-\eta_2)(\boldsymbol{\xi}\wedge\boldsymbol{\eta})(4(\boldsymbol{\xi}\cdot\boldsymbol{\eta})^2 - |\xi|^2|\eta|^2)\Big)\;.&\end{aligned} \tag{115}$$

The left-hand side of (115) has the form $(\xi_1-\eta_1)/|\xi-\eta|^2 + \Lambda(\xi,\eta)$, where Λ is a polynomial of degree 2 in η with ξ-dependent coefficients. As is easily verified, the terms in Λ that are quadratic in η can be written as

$$\frac{1}{|\xi|^6}\Big(\xi_1(\xi_1^2-3\xi_2^2)(\eta_2^2-\eta_1^2) + 2\xi_2(\xi_2^2-3\xi_1^2)\eta_1\eta_2\Big)\;.$$

For all $w \in L^2(m)$ satisfying (108), (109) and (110), we thus have $\int_{\mathbf{R}^2}\Lambda(\xi,\eta)w(\eta)\,\mathrm{d}\eta \equiv 0$. Therefore, using (115), we obtain

$$|v_2(\xi)| \le C\int_{\mathbf{R}^2}\frac{1}{|\xi|^3|\xi-\eta|}|\eta|^3|w(\eta)|\,\mathrm{d}\eta\;.$$

A similar bound holds for the first component $v_1(\xi)$, hence

$$|b(\xi)^3\mathbf{v}(\xi)| \le C\int_{\mathbf{R}^2}\frac{1}{|\xi-\eta|}|b(\eta)^3 w(\eta)|\,\mathrm{d}\eta\;, \quad \xi\in\mathbf{R}^2\;.$$

This inequality has again the same form as (113), with $\mathbf{v}, w$ replaced by $b^3\mathbf{v}, b^3 w$. Therefore, proceeding as above, we obtain (111) and (112) in case **4**. This concludes the proof of Proposition B.1. □

Corollary B.3 *Assume that $w \in L^2(m)$ for some $m > 1$, and denote by $\mathbf{v}$ the velocity field obtained from w via the Biot-Savart law (16). Then $\mathbf{v} \in L^2(\mathbf{R}^2)^2$ if and only if (108) holds.*

Proof: Without loss of generality, we assume that $1 < m < 2$. For any $w \in L^2(m)$, we have the decompositions $w = \alpha G + \bar{w}$, $\mathbf{v} = \alpha\mathbf{v}^G + \bar{\mathbf{v}}$, where $\alpha = \int_{\mathbf{R}^2} w(\xi)\,\mathrm{d}\xi$ and $\bar{w}$ satisfies the assumptions of point 2 in Proposition B.1. Setting $q = 2/(m-1)$ in (111), we obtain $b\bar{\mathbf{v}} \in L^q(\mathbf{R}^2)^2$, which implies $\bar{\mathbf{v}} \in L^2(\mathbf{R}^2)^2$. Since $\mathbf{v}^G \notin L^2(\mathbf{R}^2)^2$, it is clear that $\mathbf{v} \in L^2(\mathbf{R}^2)^2$ if and only if $\alpha = 0$. □

Corollary B.4 *Assume that $w \in L^2(m)$ for some $m > 2$, and denote by $\mathbf{v}$ the velocity field obtained from w via the Biot-Savart law (16). Then $\mathbf{v} \in L^1(\mathbf{R}^2)^2$ if and only if (108) and (109) hold. In that case, $\int_{\mathbf{R}^2} v_i(\xi)\,\mathrm{d}\xi = 0$ for $i = 1, 2$.*

Proof: Without loss of generality, we assume that $2 < m < 3$. For any $w \in L^2(m)$, we have the decompositions

$$w = \alpha G + \beta_1 F_1 + \beta_2 F_2 + \tilde{w} \ , \quad \mathbf{v} = \alpha \mathbf{v}^G + \beta_1 \mathbf{v}^{F_1} + \beta_2 \mathbf{v}^{F_2} + \tilde{\mathbf{v}} \ ,$$

where α, β_1, β_2 are given by (42) and $\tilde{w}$ satisfies the assumptions of point 3 in Proposition B.1. Thus $b^2\tilde{\mathbf{v}} \in L^q(\mathbf{R}^2)^2$ with $q = 2/(m-2)$, hence $\tilde{\mathbf{v}} \in L^1(\mathbf{R}^2)^2$. On the other hand, it is easy to verify that $\mathbf{v} - \tilde{\mathbf{v}} \in L^1(\mathbf{R}^2)^2$ if and only if $\alpha = \beta_1 = \beta_2 = 0$. Thus, $\mathbf{v} \in L^1(\mathbf{R}^2)^2$ if and only if (108) and (109) hold. In that case, since $\operatorname{div} \mathbf{v} = 0$, we must have $\int_{\mathbf{R}^2} v_i(\xi) \, \mathrm{d}\xi = 0$ for $i = 1, 2$. $\square$

Corollary B.5 *Assume that $w \in L^2(m)$ for some $m > 3$, and denote by $\mathbf{v}$ the velocity field obtained from w via the Biot-Savart law (16). Then $b\mathbf{v} \in L^1(\mathbf{R}^2)^2$ if and only if (108), (109) and (110) hold. In that case,*

$$\begin{aligned} \int_{\mathbf{R}^2} \xi_1 v_1(\xi) \, \mathrm{d}\xi &= \int_{\mathbf{R}^2} \xi_2 v_2(\xi) \, \mathrm{d}\xi = 0 \ , \\ \int_{\mathbf{R}^2} \xi_2 v_1(\xi) \, \mathrm{d}\xi &= -\int_{\mathbf{R}^2} \xi_1 v_2(\xi) \, \mathrm{d}\xi = \gamma_1 \ . \end{aligned}$$

Proof: Without loss of generality, we assume that $3 < m < 4$. Let $w \in L^2(m)$, and consider the decomposition (106), (107). Then R satisfies the assumptions of point 4 in Proposition B.1. In particular, $b^3\mathbf{v}^R \in L^q(\mathbf{R}^2)^2$ for $q = 2/(m-3)$, hence $b\mathbf{v}^R \in L^1(\mathbf{R}^2)^2$. On the other hand, it is easy to verify that $b(\mathbf{v} - \mathbf{v}^R) \in L^1(\mathbf{R}^2)^2$ if and only if $\alpha = \beta_1 = \beta_2 = \gamma_2 = \gamma_3 = 0$. Thus, $b\mathbf{v} \in L^1(\mathbf{R}^2)^2$ if and only if (108), (109) and (110) hold.

Assume now that $b\mathbf{v} \in L^1(\mathbf{R}^2)$, and consider the vector field $\mathbf{A}(\xi) = \xi_1^2 \mathbf{v}(\xi)$. Then $\mathbf{A} \in L^2(\mathbf{R}^2)^2$ and $\operatorname{div} \mathbf{A} = 2\xi_1 v_1 \in L^1(\mathbf{R}^2)^2$, hence

$$0 = \int_{\mathbf{R}^2} \operatorname{div} \mathbf{A}(\xi) \, \mathrm{d}\xi = \int_{\mathbf{R}^2} 2\xi_1 v_1(\xi) \, \mathrm{d}\xi \ .$$

Similarly, using the vector fields $\xi_2^2 \mathbf{v}$ and $\xi_1\xi_2\mathbf{v}$, one obtains $\int_{\mathbf{R}^2} \xi_2 v_2 \, \mathrm{d}\xi = 0$ and $\int_{\mathbf{R}^2} (\xi_1 v_2 + \xi_2 v_1) \, \mathrm{d}\xi = 0$. Finally, if $\mathbf{B}(\xi) = |\xi|^2 \mathbf{v}^\perp = |\xi|^2(-v_2, v_1)$, then $\mathbf{B} \in L^2(\mathbf{R}^2)^2$ and $\operatorname{div} \mathbf{B} = -|\xi|^2 w + 2(\xi_2 v_1 - \xi_1 v_2) \in L^1(\mathbf{R}^2)^2$, hence $\int_{\mathbf{R}^2} \operatorname{div} \mathbf{B} \, \mathrm{d}\xi = 0$. Using (51), we thus find

$$4\gamma_1 = \int_{\mathbf{R}^2} |\xi|^2 w(\xi) \, \mathrm{d}\xi = 2 \int_{\mathbf{R}^2} (\xi_2 v_1(\xi) - \xi_1 v_2(\xi)) \, \mathrm{d}\xi = 4 \int_{\mathbf{R}^2} \xi_2 v_1(\xi) \, \mathrm{d}\xi \ .$$

This concludes the proof of Corollary B.5. $\square$

C Proof of Proposition 4.10

The existence and attractivity of the manifold follow from the results of [7] exactly as in the proof of Theorem 3.5 and Corollary 3.7. By construction, the function $\mathcal{G}$ whose graph gives the invariant manifold satisfies the integral equation (see, for instance, [12]):

$$\mathcal{G}(\boldsymbol{\beta}, \boldsymbol{\gamma}) = \int_0^\infty \mathrm{e}^{\tau\mathcal{L}} \mathcal{N}_0(\mathrm{e}^{\tau/2}\boldsymbol{\beta}, \phi^{-\tau}(\boldsymbol{\gamma}; \boldsymbol{\beta}), \mathcal{G}(\mathrm{e}^{\tau/2}\boldsymbol{\beta}, \phi^{-\tau}(\boldsymbol{\gamma}; \boldsymbol{\beta}))) \, \mathrm{d}\tau \ , \tag{116}$$

where $(\boldsymbol{\beta},\boldsymbol{\gamma}) \mapsto (\mathrm{e}^{-\tau/2}\boldsymbol{\beta}, \phi^{\tau}(\boldsymbol{\gamma};\boldsymbol{\beta}))$ is (the projection onto E_c of) the flow defined by equations (61) on the invariant manifold, and $\mathcal{N}_0$ is the nonlinear term in the equation for ρ in (61), cutoff outside a neighborhood of the origin as in (23). Note that in computing $\phi^{\tau}(\boldsymbol{\gamma};\boldsymbol{\beta})$, one can first solve the equations for $\boldsymbol{\beta}$ explicitly and then insert these solutions into the equations for γ_j, thus treating them as inhomogeneous terms.

Remark C.1 *Note that the nonlinear terms $\tilde{f}_j$ in the equations for γ_j are also cutoff so that they vanish outside a neighborhood of the origin of size r_0. It is an easy exercise to show that for any $\epsilon > 0$, there exists $r_0 > 0$ sufficiently small so that the resulting flow $\phi^{\tau}(\boldsymbol{\gamma};\boldsymbol{\beta})$ satisfies*

$$|\phi^{-\tau}(\boldsymbol{\gamma};\boldsymbol{\beta})| \leq C\mathrm{e}^{(1+\epsilon)\tau}(|\boldsymbol{\gamma}| + |\boldsymbol{\beta}|^2) \ , \tag{117}$$

for all $\tau \geq 0$.

Define a mapping

$$\mathcal{F}(\mathcal{G})(\boldsymbol{\beta},\boldsymbol{\gamma}) = \int_0^{\infty} \mathrm{e}^{\tau\mathcal{L}}\mathcal{N}_0(\mathrm{e}^{\tau/2}\boldsymbol{\beta}, \phi^{-\tau}(\boldsymbol{\gamma};\boldsymbol{\beta}), \mathcal{G}(\mathrm{e}^{\tau/2}\boldsymbol{\beta}, \phi^{-\tau}(\boldsymbol{\gamma};\boldsymbol{\beta})))\,\mathrm{d}\tau \ . \tag{118}$$

Equation (116) implies that $\mathcal{F}$ has a fixed point in the space of C^1 functions with globally bounded Lipschitz constant. Moreover, if one fixes the cutoff function χ_{r_0} in (23), then (for $r_0 > 0$ sufficiently small) $\mathcal{F}$ is a contraction in this function space, so that $\mathcal{G}$ is the unique fixed point of $\mathcal{F}$. Thus, the proof of the proposition is completed if we can show that for some $C_\delta > 0$, $\mathcal{F}$ maps the set of functions satisfying (63) into itself.

Let X be the Banach space $\mathbf{R}^2 \times \mathbf{R}^3 \times E_s$, equipped with the norm $\|(\boldsymbol{\beta},\boldsymbol{\gamma},\rho)\|_X = |\boldsymbol{\beta}| + |\boldsymbol{\gamma}| + \|\rho\|_4$, and let χ_{r_0} be a cutoff function like that introduced just prior to (23), but defined with respect to the norm $\|\cdot\|_X$ rather than $\|\cdot\|_m$. Then the nonlinear term in (116) can be written as

$$\mathcal{N}_0(\boldsymbol{\beta},\boldsymbol{\gamma},\rho) = Q_s\{\nabla\cdot(\chi_{r_0}(\boldsymbol{\beta},\boldsymbol{\gamma},\rho)\tilde{\mathbf{F}}(\boldsymbol{\beta},\boldsymbol{\gamma},\rho)) - \mathbf{v}^{\rho}\cdot\nabla(\chi_{r_0}(\boldsymbol{\beta},\boldsymbol{\gamma},\rho)\rho)\} \ , \tag{119}$$

where Q_s is the projection onto E_s. Commuting the derivatives in the nonlinear term through the factor of $\mathrm{e}^{\tau\mathcal{L}}$ as we did in (19), the right-hand side of (118) can be rewritten as

$$\begin{aligned}
&\int_0^1 Q_s\nabla\cdot\mathrm{e}^{\tau(\mathcal{L}-\frac{1}{2})}[(\chi_{r_0}\tilde{\mathbf{F}})(\boldsymbol{\beta}(\tau),\boldsymbol{\gamma}(\tau),\mathcal{G}(\boldsymbol{\beta}(\tau),\boldsymbol{\gamma}(\tau))) \\
&\qquad -\mathbf{v}^{\mathcal{G}(\beta(\tau),\gamma(\tau))}\chi_{r_0}(\boldsymbol{\beta}(\tau),\boldsymbol{\gamma}(\tau),\mathcal{G}(\boldsymbol{\beta}(\tau),\boldsymbol{\gamma}(\tau)))\mathcal{G}(\boldsymbol{\beta}(\tau),\boldsymbol{\gamma}(\tau))]\,\mathrm{d}\tau \\
&+\int_1^{\infty}\mathrm{e}^{(\tau-1)\mathcal{L}}Q_s\nabla\cdot\mathrm{e}^{\mathcal{L}-\frac{1}{2}}[(\chi_{r_0}\tilde{\mathbf{F}})(\boldsymbol{\beta}(\tau),\boldsymbol{\gamma}(\tau),\mathcal{G}(\boldsymbol{\beta}(\tau),\boldsymbol{\gamma}(\tau))) \\
&\qquad -\mathbf{v}^{\mathcal{G}(\beta(\tau),\gamma(\tau))}(\chi_{r_0}(\boldsymbol{\beta}(\tau),\boldsymbol{\gamma}(\tau),\mathcal{G}(\boldsymbol{\beta}(\tau),\boldsymbol{\gamma}(\tau)))\mathcal{G}(\boldsymbol{\beta}(\tau),\boldsymbol{\gamma}(\tau))]\,\mathrm{d}\tau \ ,
\end{aligned}$$

where in keeping with our usual notational convention, $\mathbf{v}^{\mathcal{G}}$ denotes the velocity field associated with the vorticity field $\mathcal{G}$. Also, to save space, we have used the short-hand notation $\boldsymbol{\beta}(\tau)$ for $\mathrm{e}^{\tau/2}\boldsymbol{\beta}$ and $\boldsymbol{\gamma}(\tau)$ for $\phi^{-\tau}(\boldsymbol{\gamma};\boldsymbol{\beta})$. Fixing $\epsilon > 0$, we can take the $\|\cdot\|_4$ norm of this expression and use the estimates on the semigroup from Appendix A to obtain

$$\begin{aligned}
\|\mathcal{F}(\mathcal{G})(\boldsymbol{\beta},\boldsymbol{\gamma})\|_4 \leq C\int_0^{\infty} &\mathrm{e}^{-(\frac{3}{2}-\epsilon)(\tau-1)}[(\chi_{r_0}\tilde{\mathbf{F}})(\boldsymbol{\beta}(\tau),\boldsymbol{\gamma}(\tau),\mathcal{G}(\boldsymbol{\beta}(\tau),\boldsymbol{\gamma}(\tau)))\|_4 \\
&+\chi_{r_0}(\boldsymbol{\beta}(\tau),\boldsymbol{\gamma}(\tau),\mathcal{G}(\boldsymbol{\beta}(\tau),\boldsymbol{\gamma}(\tau)))\|\mathcal{G}(\boldsymbol{\beta}(\tau),\boldsymbol{\gamma}(\tau))\|_4^2]\,\mathrm{d}\tau \ ,
\end{aligned} \tag{120}$$

where we estimated the term involving $\mathbf{v}^{\mathcal{G}}(\chi_{r_0}(\boldsymbol{\beta},\boldsymbol{\gamma},\mathcal{G}(\boldsymbol{\beta},\boldsymbol{\gamma}))\mathcal{G})$ in the same way as we estimated the term involving $\mathbf{v}(s)w(s)$ in Lemma 3.1. Using the estimate on $\tilde{\mathbf{F}}$ coming from (57), we see that

$$\|(\chi_{r_0}\tilde{\mathbf{F}})(\boldsymbol{\beta},\boldsymbol{\gamma},\mathcal{G})\|_4 \leq C((|\boldsymbol{\beta}|+|\boldsymbol{\gamma}|)(|\boldsymbol{\gamma}|+|\boldsymbol{\beta}|^2+\|\mathcal{G}\|_4)+(|\boldsymbol{\beta}|^2+\|\mathcal{G}\|_4)^2)\chi_{r_0}(\boldsymbol{\beta},\boldsymbol{\gamma},\mathcal{G}) \ .$$

If we now assume that $\|\mathcal{G}(\boldsymbol{\beta},\boldsymbol{\gamma})\|_4 \leq C_\delta(|\boldsymbol{\beta}|^{3-\delta}+|\boldsymbol{\gamma}|^{\frac{3}{2}-\delta})$, then, using the fact that $\chi_{r_0}(\boldsymbol{\beta},\boldsymbol{\gamma},\rho)=0$ if either $|\boldsymbol{\beta}|$ or $|\boldsymbol{\gamma}|$ is bigger than r_0, it is easy to verify that

$$\|\chi_{r_0}(\boldsymbol{\beta},\boldsymbol{\gamma},\mathcal{G})\tilde{\mathbf{F}}(\boldsymbol{\beta},\boldsymbol{\gamma},\mathcal{G})\|_4 \leq C r_0^\delta(|\boldsymbol{\beta}|^{3-\delta}+|\boldsymbol{\gamma}|^{\frac{3}{2}-\delta}) \ . \tag{121}$$

In a similar fashion we can bound

$$\chi_{r_0}(\boldsymbol{\beta},\boldsymbol{\gamma},\mathcal{G}(\boldsymbol{\beta},\boldsymbol{\gamma}))\|\mathcal{G}(\boldsymbol{\beta},\boldsymbol{\gamma}))\|_4^2 \leq C r_0^\delta(|\boldsymbol{\beta}|^{3-\delta}+|\boldsymbol{\gamma}|^{\frac{3}{2}-\delta}) \ .$$

Inserting these estimates into (120) and using (117) to bound $\gamma(\tau)=\phi^{-\tau}(\boldsymbol{\gamma};\boldsymbol{\beta})$, we find

$$\begin{aligned}\|\mathcal{F}(\mathcal{G})(\boldsymbol{\beta},\boldsymbol{\gamma})\|_4 &\leq C r_0^\delta \int_0^\infty \mathrm{e}^{-(\frac{3}{2}-\epsilon)(\tau-1)}[\,\mathrm{e}^{(3-\delta)(\tau/2)}|\boldsymbol{\beta}|^{3-\delta}+\mathrm{e}^{(\frac{3}{2}-\delta)(1+\epsilon)\tau}|\boldsymbol{\gamma}|^{\frac{3}{2}-\delta}]\,\mathrm{d}\tau \\ &\leq C(\epsilon,\delta) r_0^\delta(|\boldsymbol{\beta}|^{3-\delta}+|\boldsymbol{\gamma}|^{\frac{3}{2}-\delta}) \ , \end{aligned} \tag{122}$$

provided that $\epsilon < 2\delta/5$. For fixed ϵ and δ (satisfying $\epsilon < 2\delta/5$) we can always pick r_0 sufficiently small so that $C(\epsilon,\delta)r_0^\delta < C_\delta$. Thus $\mathcal{F}$ maps the set of functions satisfying (63) into itself (for r_0 sufficiently small) and hence the fixed point of $\mathcal{F}$ whose graph gives the invariant manifold must satisfy (63). This completes the proof of Proposition 4.10. $\square$

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