Distribution of Energy and Convergence to Equilibria in Extended Dissipative Systems

Thierry Gallay
Institut Fourier
Université de Grenoble & CNRS
BP 74
38402 Saint-Martin-d'Hères, France
`Thierry.Gallay@ujf-grenoble.fr`

Siniša Slijepčević
Department of Mathematics
University of Zagreb
Bijenička 30
10000 Zagreb, Croatia
`slijepce@math.hr`

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Abstract

We are interested in understanding the dynamics of dissipative partial differential equations on unbounded spatial domains. We consider systems for which the energy density $e \geq 0$ satisfies an evolution law of the form $\partial_t e = \mathrm{div}_x f - d$, where $-f$ is the energy flux and $d \geq 0$ the energy dissipation rate. We also suppose that $|f|^2 \leq b(e)d$ for some nonnegative function b. Under these assumptions we establish simple and universal bounds on the time-integrated energy flux, which in turn allow us to estimate the amount of energy that is dissipated in a given domain over a long interval of time. In low space dimensions $N \leq 2$, we deduce that any relatively compact trajectory converges on average to the set of equilibria, in a sense that we quantify precisely. As an application, we consider the incompressible Navier-Stokes equation in the infinite cylinder $\mathbb{R} \times \mathbb{T}$, and for solutions that are merely bounded we prove that the vorticity converges uniformly to zero on large subdomains, if we disregard a small subset of the time interval.

1 Introduction

Many time-dependent partial differential equations arising in Mathematical Physics are dissipative in the sense that there exists a nonnegative *energy density* $e(x,t)$, depending on the space variable $x \in \mathbb{R}^N$ and the time $t \in \mathbb{R}$, which is locally dissipated under the evolution defined by the system. By this we mean that $e(x,t)$ satisfies an equation of the form

$$\partial_t e(x,t) \;=\; \mathrm{div}_x f(x,t) - d(x,t)\ , \tag{1}$$

where $-f(x,t) \in \mathbb{R}^N$ denotes the *energy flux* in the system and $d(x,t) \geq 0$ is the *energy dissipation rate*. Equivalently, integrating (1) with respect to both variables x, t and applying the divergence theorem, we obtain the energy balance equation

$$\int_\Omega e(x,T_2)\,\mathrm{d}x - \int_\Omega e(x,T_1)\,\mathrm{d}x \;=\; \int_{T_1}^{T_2}\!\int_{\partial\Omega} f(x,t)\cdot\nu\,\mathrm{d}\sigma\,\mathrm{d}t - \int_{T_1}^{T_2}\!\int_\Omega d(x,t)\,\mathrm{d}x\,\mathrm{d}t\ , \tag{2}$$

which holds for any time interval $[T_1, T_2] \subset \mathbb{R}_+$ and any admissible domain $\Omega \subset \mathbb{R}^N$. Here ν denotes the outward pointing unit normal on $\partial\Omega$, and $\mathrm{d}\sigma$ is the elementary surface area.

As a typical example, consider the reaction-diffusion equation

$$\partial_t u(x,t) \;=\; \Delta_x u(x,t) - V'(u(x,t)) \;, \qquad x \in \mathbb{R}^N \;, \quad t \ge 0 \;, \tag{3}$$

where $u : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}$ is the unknown function and $V : \mathbb{R} \to \mathbb{R}_+$ is a smooth potential. This equation appears for instance in the theory of phase transitions [3] and in population genetics [5]. In the particular case where $V(u) = \frac{1}{4}(1-u^2)^2$, Eq. (3) is often referred to as the Allen-Cahn equation or the real Ginzburg-Landau equation. If $u(x,t)$ is any smooth solution of (3), we define the energy density, the (backward) energy flux, and the energy dissipation rate by the formulas

$$e \;=\; \frac{1}{2}|\nabla u|^2 + V(u) \;, \qquad f \;=\; u_t \nabla u \;, \qquad d \;=\; u_t^2 \;, \tag{4}$$

where $u_t = \partial_t u$. It is then straightforward to verify that (1) holds, which means that energy is locally dissipated under the evolution defined by (3). Since V is nonnegative, we also deduce from (4) that $|f|^2 \le 2ed$. We shall list in Section 2 several other examples of classical PDE's which define dissipative dynamical systems in the same sense. In most of these examples, the energy flux happens to satisfy an inequality of the form

$$|f|^2 \;\le\; Ced \;, \tag{5}$$

for some positive constant C.

If a dissipative PDE such as (3) is considered in a bounded domain $\Omega \subset \mathbb{R}^N$, with boundary conditions ensuring that $f \cdot \nu \le 0$ on $\partial\Omega$, then (2) shows that the total energy

$$E(t) \;=\; \int_\Omega e(x,t)\,\mathrm{d}x$$

is a *Lyapunov function* of the system, namely $E(t)$ is a decreasing function of time for all solutions of (3) which are not equilibria. Under natural coercivity assumptions on the potential V, this Lyapunov structure implies that all finite-energy solutions of (3) in a bounded domain Ω converge to the set of equilibria as $t \to \infty$ [19, 20].

The situation is rather different if we work in an unbounded domain such as $\Omega = \mathbb{R}^N$. In that case, Eq. (3) may have travelling wave solutions of the form $u(x,t) = v(x - ct)$ for some nonzero $c \in \mathbb{R}^N$ [5], and such solutions do not converge uniformly to equilibria as $t \to \infty$. One may object that travelling waves do converge to equilibria uniformly on compact sets, but it is possible to construct more complicated solutions for which convergence to equilibria does not hold even in that weaker sense, see Example 5.7 below. Moreover, if $N > 2$, one can exhibit scalar reaction-diffusion equations of the form $\partial_t u = \Delta u + F(x,u)$ which have nontrivial time-periodic solutions [15]. This is in sharp contrast with what happens for gradient systems, but one should keep in mind that all counter-examples above involve infinite-energy solutions.

When the total energy cannot be used as a Lyapunov function, a natural idea is to exploit the energy balance equation (2) or its differential version (1) to obtain relevant information on the dynamics. In the context of extended dissipative systems, this approach was initiated in a previous paper by the authors [15], the main conclusions of which can be summarized as follows. If $N \le 2$, the reaction-diffusion equation (3) on $\mathbb{R}^N$ cannot have any nontrivial solution such that $e(x,T_2) \ge e(x,T_1)$ for some $T_2 > T_1$ and all $x \in \mathbb{R}^N$; in particular, nontrivial time-periodic solutions are excluded. Furthermore, all bounded solutions converge on average (in time), uniformly on compact sets (in space), toward the set of equilibria as $t \to +\infty$. In other words, due to the local energy dissipation law (1), Eq. (3) retains some dynamical properties of usual gradient systems, provided $N \le 2$. In contrast, if $N > 2$, highly non-gradient behaviors such as nontrivial periodic orbits can occur. The conclusions of [15] also apply to a damped hyperbolic equation which reduces to (3) in the limit of strong damping.

The aim of the present paper is to strengthen and generalize the results of [15]. Instead of considering a particular equation, we work in an abstract setting, assuming only the local energy dissipation law (1) and an estimate of the form (5) for the energy flux. As a consequence, our results apply to a much larger class of systems, some of which are listed in Section 2. Another substantial progress with respect to [15] is a new estimate on the time-integrated energy flux through a closed hypersurface, which we derive in Section 3. This bound allows us to obtain *quantitative* versions of the main results in [15]. For instance, in Section 4 we give an explicit estimate of the energy dissipated in a given domain over a long time interval, and in Section 5 we measure the fraction of time spent by any relatively compact trajectory outside a neighborhood of the set of equilibria. As can be expected from [15], our results depend strongly on the space dimension N, and some of them fail if $N > 2$. As a final application, we consider in Section 6 the incompressible Navier-Stokes equation in the two-dimensional cylinder $\mathbb{R} \times \mathbb{T}$, and for solutions that are merely bounded we prove some convergence results for the vorticity which are apparently new in this context.

2 Extended Dissipative Systems

To treat in a unified way various dissipative PDE's on unbounded domains, we introduce in this section the notion of an *extended dissipative system*, which will be studied in the rest of this paper. We also list a few classical examples which fit into our abstract framework.

Let X be a metrizable topological space. We say that a family $(\Phi(t))_{t\geq 0}$ of continuous maps in X is a *continuous semiflow* on X if

- $\Phi(0) = \mathbf{1}$ (the identity map);
- $\Phi(t_1 + t_2) = \Phi(t_1) \circ \Phi(t_2)$ for all $t_1, t_2 \geq 0$;
- For any $T > 0$, the map $(t, u) \mapsto \Phi(t)u$ is continuous from $[0, T] \times X$ to X.

In particular, if $u_0 \in X$, the trajectory $u : \mathbb{R}_+ \to X$ defined by $u(t) = \Phi(t)u_0$ for all $t \in \mathbb{R}_+ = [0, \infty)$ is continuous, and $u(t)$ depends continuously on the initial data u_0, uniformly in time on compact intervals. If $\Phi(t)u_0 = u_0$ for all $t \geq 0$, we say that $u_0 \in X$ is an *equilibrium* of the system.

As an example, if $V(u) = \frac{1}{4}(1-u^2)^2$, the reaction-diffusion equation (3) defines a continuous evolution semiflow on the space $X = C^0(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, if X is equipped with the topology of uniform convergence on compact sets of $\mathbb{R}^N$. More generally, the systems we are interested in are those for which one can define an energy density e, an energy flux f, and an energy dissipation rate d with the same properties as in the example (3). This leads to the following definition :

Definition 2.1. *Let $N \in \mathbb{N}^*$. We say that a continuous semiflow $(\Phi(t))_{t\geq 0}$ on a metrizable space X is an* extended dissipative system *on $\mathbb{R}^N$ if one can associate to each $u \in X$ a triple (e, f, d) with $e, d \in C^0(\mathbb{R}^N, \mathbb{R}_+)$ and $f \in C^0(\mathbb{R}^N, \mathbb{R}^N)$ such that :*

(A1) *The functions e, f, d depend continuously on $u \in X$, uniformly on compact sets of $\mathbb{R}^N$;*

(A2) *$|f|^2 \leq b(e)d$ for some nondecreasing function $b : \mathbb{R}_+ \to \mathbb{R}_+$;*

and such that, if the quantities e, f, d are evolved according to the semiflow $(\Phi(t))_{t\geq 0}$, the following properties hold :

(A3) *If $d(x, t) = 0$ for all $(x, t) \in \mathbb{R}^N \times [0, t_0]$, where $t_0 > 0$, then u is an equilibrium;*

(A4) *The energy balance $\partial_t e = \mathrm{div} f - d$ holds in the sense of distributions on $\mathbb{R}^N \times \mathbb{R}_+$.*

Remarks 2.2.

1. In Definition 2.1, assumptions (A1) and (A2) concern the densities $e(x), f(x), d(x)$ associated to a given state $u \in X$, whereas (A3) and (A4) involve the time-dependent quantities $e(x,t), f(x,t), d(x,t)$ corresponding to the evolved state $\Phi(t)u$, with $t \ge 0$.

2. More generally, one can define extended dissipative systems on any (unbounded) domain $\Omega \subset \mathbb{R}^N$ by substituting Ω for $\mathbb{R}^N$ everywhere in Definition 2.1. In that case, one also has to assume that the energy flux $f \in C^0(\Omega, \mathbb{R}^N)$ satisfies $f \cdot \nu \le 0$ on $\partial\Omega$, where ν is the outward unit normal.

3. We emphasize that, in Definition 2.1, both the energy density e and the energy dissipation rate d are supposed to be nonnegative. The first condition ensures that the energy density is bounded from below, and the positivity of d together with the energy balance (1) imply that energy is locally *dissipated* (and never created) in the system.

4. In assumption (A2), it is understood that the function $b : \mathbb{R}_+ \to \mathbb{R}_+$ is independent of $u \in X$. In many examples one can take $b(e) = Ce$ for some positive constant C, as in (5), but the generalization proposed here is necessary if one considers systems such as the nonlinear diffusion equation (see below) or the two-dimensional vorticity equation (see Section 6).

5. Assumption (A3) means that all trajectories, except equilibria, dissipate some energy. Remark that, if $d \equiv 0$ on some time interval, then $f \equiv 0$ by (A2) hence $\partial_t e \equiv 0$ by (A4), but we still need (A3) to conclude that the system is at equilibrium. Note also that extended dissipative systems can have equilibria for which d is not identically zero; these may be called *nonequilibrium steady states*, in the terminology of Statistical Mechanics. Finally, we mention that, if one considers systems with a continuous group of symmetries, it may be useful to relax assumption (A3) so as to allow for a vanishing energy dissipation on *relative equilibria* of the system; these are equilibria up to the action of the symmetry group, see the example of the complex Ginzburg-Landau equation below.

6. To avoid technicalities, we supposed that, for each $u \in X$, the densities e, f, d are continuous functions on $\mathbb{R}^N$. In view of (A1), the time-dependent quantities $e(x,t), f(x,t), d(x,t)$ associated to the evolved state $\Phi(t)u$ are thus jointly continuous in space and time. This implies that the integrated energy balance equation (2) holds for any time interval $[T_1, T_2] \subset \mathbb{R}_+$ and any smooth bounded domain $\Omega \subset \mathbb{R}^N$, and that all four terms in (2) depend continuously on the initial state $u \in X$. These comfortable assumptions are not unrealistic, and can be verified in all systems listed below if we choose functions spaces of sufficiently high regularity. However, especially in nonparabolic PDE's, it is often more convenient to use larger function spaces, in which (for instance) the energy density is locally integrable but not continuous. In that case, instead of (A1) and (A4), it is sufficient to require that the energy balance equation (2) be satisfied for any time interval $[T_1, T_2] \subset \mathbb{R}_+$ and any smooth bounded domain $\Omega \subset \mathbb{R}^N$, and that the various quantities in (2) depend continuously on the initial data.

Before giving some concrete examples, we emphasize that most of our analysis in the subsequent sections uses additional properties of extended dissipative systems that were not included in Definition 2.1. Two of them are worth mentioning here:

a) *Boundedness of the energy density.* Our approach relies in an essential way on a lower bound on the energy dissipation rate in terms of (the square of) the energy flux. Such an estimate follows from assumption (A2) if we suppose, for instance, that the energy density is uniformly bounded. Indeed, if β is any positive constant larger than $b(e(x,t))$ for all $x \in \mathbb{R}^N$ and all $t \ge 0$, we have $d \ge |f|^2/\beta$ by (A2), and that inequality allows us to quantify how much energy is dissipated in the system when the energy flux is not identically zero. Most of the results in

Sections 3 and 4 below apply to extended dissipative systems whose trajectories have uniformly bounded energy density.

b) *Compactness of the trajectories.* In Section 5 below, to investigate the long-time behavior of low-dimensional extended dissipative systems, we shall assume that the trajectories are relatively compact. In the applications, this requirement can often be fulfilled by endowing the space X with a sufficiently weak topology. The idea of introducing a weak (or localized) topology to restore compactness plays an important role in the study of dissipative PDE's on unbounded domains, in particular when constructing global attractors [7, 14, 23].

Examples 2.3.
1. *A reaction-diffusion equation*
We consider again the reaction-diffusion equation (3), and specify in which function spaces it defines an extended dissipative system in the sense of Definition 2.1. There are of course many possibilities, and we just mention here two reasonable ones. Since we want global solutions of (3), it is natural to assume that the potential $V : \mathbb{R} \to \mathbb{R}_+$ is coercive in some sense. For instance, we can suppose that $uV'(u) \geq 0$ whenever $|u|$ is sufficiently large. Then it is known that the Cauchy problem for (3) is globally well-posed in $C^k_{\rm bu}(\mathbb{R}^N)$, the Banach space of all functions $u : \mathbb{R}^N \to \mathbb{R}$ that are bounded and uniformly continuous together with their derivatives up to order $k \in \mathbb{N}$. This means that (3) defines a continuous semiflow $(\Phi(t))_{t\geq 0}$ on $X = C^k_{\rm bu}(\mathbb{R}^N)$. Moreover, if $k \geq 2$, the quantities e, f, d introduced in (4) belong to $C^0(\mathbb{R}^N)$ and depend continuously on $u \in X$, uniformly on compact sets of $\mathbb{R}^N$. Together with (1) and (5), this shows that, if $k \geq 2$, the semiflow of (3) on $X = C^k_{\rm bu}(\mathbb{R}^N)$ is an extended dissipative system. In addition, our assumption on the potential V ensures that all trajectories have uniformly bounded energy density. Finally, if $k > 2$ and if we endow X with the topology of $C^0(\mathbb{R}^N)$, namely the topology of uniform convergence on compact sets, then all trajectories are relatively compact and the quantities e, f, d are still continuous functions of $u \in X$.

Instead of $C^k_{\rm bu}(\mathbb{R}^N)$, another possible choice is the uniformly local Sobolev space $H^s_{\rm ul}(\mathbb{R}^N)$, on which (3) also defines a continuous semiflow if $s > N/2$, see [6, 15]. If moreover $s > 2 + N/2$, the densities (4) are continuous and we again obtain an extended dissipative system whose trajectories have uniformly bounded energy density, and are relatively compact if we endow X with the topology of $L^2_{\rm loc}(\mathbb{R}^N)$.

2. *A strongly damped wave equation* [21, 24]
Given $\alpha \geq 0$ and a smooth potential $V : \mathbb{R} \to \mathbb{R}_+$, we consider the equation

$$u_{tt} + u_t - \alpha\Delta u_t \;=\; \Delta u - V'(u)\ , \qquad x \in \mathbb{R}^N\ , \quad t \geq 0\ , \tag{6}$$

where $u : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}$. As usual, this second-order equation can be written as a first-order system for the pair (u, u_t). For simplicity, we assume that the potential V is quadratic near infinity, namely $V''(u) = m > 0$ for all sufficiently large $u \in \mathbb{R}$. Then the initial value problem for Eq. (6) is globally well-posed in the uniformly local space $X = H^s_{\rm ul}(\mathbb{R}^N) \times H^{s-1}_{\rm ul}(\mathbb{R}^N)$ if $s > N/2$. For any pair $(u, u_t) \in X$, we introduce the densities

$$e \;=\; \frac{1}{2}u_t^2 + \frac{1}{2}|\nabla u|^2 + V(u)\ , \qquad f \;=\; u_t(\nabla u + \alpha\nabla u_t)\ , \qquad d \;=\; u_t^2 + \alpha|\nabla u_t|^2\ ,$$

which are well-defined and continuous provided $s > 2 + N/2$, or $s > 1 + N/2$ if $\alpha = 0$. Then (6) implies that the local energy dissipation law (1) is satisfied, and a direct calculation shows that (5) holds with $C = 2\max(1, \alpha)$. Finally $d \equiv 0$ implies $u_t \equiv 0$. Thus the semiflow of the strongly damped wave equation (6) in X is an extended dissipative system in the sense of Definition 2.1. In the particular case where $\alpha = 0$, the local dissipation of energy for Eq. (6) was studied in [15].

3. *A complex Ginzburg-Landau equation* [4, 9, 22]
Our next example originates from the complex Ginzburg-Landau equation

$$u_t \;=\; (1+i\alpha)\Delta u + u - (1+i\beta)|u|^2 u \;, \qquad x \in \mathbb{R}^N \;, \quad t \ge 0 \;, \tag{7}$$

where $u : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{C}$ and α, β are real parameters. To have a gradient structure, we assume that $\beta = \alpha$, and we introduce the auxiliary function $v(x,t) = u(x,t)e^{i\alpha t}$, which satisfies the equation

$$v_t \;=\; (1+i\alpha)\Big(\Delta v + v - |v|^2 v\Big) \;, \qquad x \in \mathbb{R}^N \;, \quad t \ge 0 \;. \tag{8}$$

The Cauchy problem for (8) is globally well-posed in the function space $X = C^k_{\rm bu}(\mathbb{R}^N, \mathbb{C})$ for $k \ge 0$ or $X = H^s_{\rm ul}(\mathbb{R}^N, \mathbb{C})$ for $s > N/2$. If in addition $k \ge 2$ or $s > 2 + N/2$, then for any $v \in X$ the densities

$$e \;=\; \frac{1}{2}|\nabla v|^2 + \frac{1}{4}(1-|v|^2)^2 \;, \qquad f \;=\; \mathrm{Re}(v_t \nabla \bar v) \;, \qquad d \;=\; \frac{|v_t|^2}{1+\alpha^2} \;,$$

are well-defined and continuous. A direct calculation also shows that (1) holds, as well as (5) with $C = 2(1+\alpha^2)$. Thus the semiflow of (8) in X is an extended dissipative system in the sense of Definition 2.1. We also remark that $d \equiv 0$ if and only if $u(x,t) = v(x)e^{-i\alpha t}$ for some $v \in X$, which means that $u(\cdot,t)$ is a *relative equilibrium* of (7) : $u(\cdot,t)$ moves without dissipation along an orbit of the symmetry group $U(1)$. Thus the semiflow of (7) in X is an extended dissipative system only if assumption (A3) is relaxed as suggested in Remark 2.2.5.

4. *The Landau-Lifshitz-Gilbert equation* [18]
We now consider a vector-valued PDE appearing in micromagnetism. Given $\alpha \in \mathbb{R}$, the Landau-Lifshitz equation reads

$$u_t \;=\; -u \wedge (u \wedge \Delta u) + \alpha u \wedge \Delta u \;, \qquad x \in \mathbb{R}^N \;, \quad t \ge 0 \;, \tag{9}$$

where $u : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{S}^2 = \{v \in \mathbb{R}^3 \,|\, |v| = 1\}$. Here $\wedge$ denotes the usual cross product in $\mathbb{R}^3$. In particular $-u \wedge (u \wedge \Delta u) = \Delta u - u(u \cdot \Delta u) = \Delta u + |\nabla u|^2 u$ is the orthogonal projection of Δu onto the plane orthogonal to the direction $u \in \mathbb{S}^2$, and $u \wedge \Delta u$ is the same vector rotated by $\pi/2$ in the orthogonal plane. The initial value problem for (9) is *locally* well-posed in the space $X = C^k_{\rm bu}(\mathbb{R}^N)$ for $k > 0$ or $X = H^s_{\rm ul}(\mathbb{R}^N)$ for $s > N/2$, but in general finite-time singularities are expected to occur, unlike in the previous examples. To obtain a continuous semiflow $(\Phi(t))_{t \ge 0}$, it is therefore necessary to restrict our space X to a family of global trajectories. Now, if $k \ge 2$ or $s > 2 + N/2$, the densities

$$e \;=\; \frac{1}{2}|\nabla u|^2 \;, \qquad f \;=\; u_t \nabla u \;\equiv\; \sum_{k=1}^{3} (\partial_t u_k) \nabla u_k \;, \qquad d \;=\; |u \wedge \Delta u|^2 \;,$$

are well-defined and continuous for any $u \in X$, and it is again straightforward to verify that (1) and (5) hold with $C = 2(1+\alpha^2)$. Moreover $d \equiv 0$ implies $u \wedge \Delta u \equiv 0$, hence $u_t \equiv 0$. Thus Eq. (9) also defines an extended dissipative system in the sense of Definition 2.1, provided we restrict the space X to a suitable family of global solutions.

5. *A nonlinear diffusion equation*
To motivate assumption (A2) in Definition 2.1, we also give an example where the relation between the energy flux and the energy dissipation is more complex than in (5). Given a smooth function $a : \mathbb{R} \to (0, \infty)$, we consider the nonlinear diffusion equation

$$u_t \;=\; \mathrm{div}(a(u)\nabla u) \;, \qquad x \in \mathbb{R}^N \;, \quad t \ge 0 \;, \tag{10}$$

which is globally well-posed in the space $X = C^k_{\rm bu}(\mathbb{R}^N)$ for $k \ge 0$ or $X = H^s_{\rm ul}(\mathbb{R}^N)$ for $s > N/2$. If $k \ge 1$ or $s > 1 + N/2$, we denote for all $u \in X$:

$$e \,=\, \frac{1}{2}u^2\;, \qquad f \,=\, ua(u)\nabla u, \qquad d \,=\, a(u)|\nabla u|^2\;.$$

Then (1) holds, and it is clear that $d \equiv 0$ implies $u_t \equiv 0$. Moreover $|f|^2 \le 2a(u)ed$. Thus, if we define

$$b(e) \,=\, 2e\sup\{a(u)\,|\,u^2 \le 2e\}\;, \qquad e \ge 0\;,$$

then $e \mapsto b(e)$ is increasing and $|f|^2 \le b(e)d$ by construction. Thus (10) defines an extended dissipative system in X in the sense of Definition 2.1.

6. *The two-dimensional vorticity equation*
As a final example, we consider the vorticity equation associated to the two-dimensional incompressible Navier-Stokes system. In this model, the velocity of the fluid, which is denoted by $u(x,t) \in \mathbb{R}^2$, satisfies the incompressibility condition $\partial_1 u_1 + \partial_2 u_2 = 0$, and the corresponding vorticity field $\omega = \partial_1 u_2 - \partial_2 u_1$ evolves according to the advection-diffusion equation

$$\partial_t \omega + u \cdot \nabla \omega \,=\, \Delta \omega\;, \qquad x \in \mathbb{R}^2\;, \quad t \ge 0\;. \tag{11}$$

If we define the enstrophy density e, the enstrophy flux f, and the enstrophy dissipation rate d by the formulas

$$e \,=\, \frac{1}{2}\omega^2\;, \qquad f \,=\, \omega\nabla\omega - \frac{1}{2}u\,\omega^2\;, \qquad d \,=\, |\nabla\omega|^2\;, \tag{12}$$

it is easy to verify that (1) is satisfied. Moreover, $d \equiv 0$ clearly implies that $\partial_t\omega \equiv 0$. However, due to the term $\frac{1}{2}u\,\omega^2$ in the enstrophy flux, it is not possible to obtain here an inequality of the form $f^2 \le b(e)d$. Indeed, the velocity u is not a local function of ω, and is not known to be uniformly bounded in time if the initial data u_0 are merely bounded, see [17, 26, 27]. In addition, the term $\frac{1}{2}u\,\omega^2$ does not contain any derivative of ω, hence does not necessarily vanish when $d = 0$. This means that enstrophy can (a priori) be transported without any dissipation, whereas it is essential in our approach that the energy dissipation be bounded from below in terms of the energy flux. Surprisingly enough, we shall show in Section 6 that these difficulties essentially disappear if we consider the vorticity equation (11) in the infinite cylinder $\mathbb{R} \times \mathbb{T}$ instead of the whole plane $\mathbb{R}^2$. Thus, if we assume periodicity in one space direction, the semiflow of (11) defines a one-dimensional extended dissipative system which (nearly) satisfies the assumptions in Definition 2.1.

Remark 2.4. The above list of examples can certainly be made longer, but all extended dissipative systems we are aware of are related somehow to a parabolic equation involving a second order differential operator. Higher-order systems, such as the Cahn-Hilliard equation, do not fit into our framework since they require a radical modification of the bound (5), which would affect our results in an essential way.

3 Bounds on the Energy Flux

We now begin our study of the dynamics of extended dissipative systems. Given a continuous semiflow $(\Phi(t))_{t\ge 0}$ on a metrizable space X satisfying the assumptions of Definition 2.1 for some $N \in \mathbb{N}^*$, we consider a trajectory $u(t) = \Phi(t)u_0$ for which the energy density $e(x,t)$ is uniformly bounded. If we denote

$$e_0 \,=\, \sup_{x\in\mathbb{R}^N} e(x,0) \,<\, \infty\;, \qquad \beta \,=\, \sup_{x\in\mathbb{R}^N}\sup_{t\ge 0} b(e(x,t)) \,<\, \infty\;, \tag{13}$$

where $e \mapsto b(e)$ is the nondecreasing function appearing in Definition 2.1, assumption (A2) then implies

$$|f(x,t)|^2 \le \beta\, d(x,t)~, \qquad x \in \mathbb{R}^N~, \quad t \ge 0~. \tag{14}$$

Using only (1), (14), and the positivity of $e(x,t)$ and $d(x,t)$, we shall derive a universal bound on the total energy flux through a given hypersurface in $\mathbb{R}^N$ over the time interval $[0,T]$.

We first consider the one-dimensional case $N = 1$, where our hypersurface is reduced to a single point. Our main result in this case is:

Proposition 3.1. *Assume that $N = 1$, and let $u(t) = \Phi(t)u_0$ be a trajectory for which the energy density $e(x,t)$ satisfies* (13)*. Then for any $x \in \mathbb{R}$ and any $T > 0$, we have*

$$\Bigl|\int_0^T f(x,t)\,\mathrm{d}t\Bigr| \le \sqrt{\beta T e_0}~. \tag{15}$$

Proof. Without loss of generality, we can assume that $\beta > 0$. Given any $T > 0$, we introduce the integrated energy flux

$$F_1(x,T) = \int_0^T f(x,t)\,\mathrm{d}t~, \qquad x \in \mathbb{R}~, \tag{16}$$

which is a continuous function of $x \in \mathbb{R}$. Since $\partial_x f = \partial_t e + d$ by (1), we see that F_1 is differentiable with respect to x, and we easily obtain

$$\partial_x F_1(x,T) = \int_0^T \partial_x f(x,t)\,\mathrm{d}t = e(x,T) - e(x,0) + \int_0^T d(x,t)\,\mathrm{d}t~. \tag{17}$$

To estimate the right-hand side from below, we observe that $e(x,T) \ge 0$ and $e(x,0) \le e_0$. In addition, using (14) and the Cauchy-Schwarz inequality, we find

$$\int_0^T d(x,t)\,\mathrm{d}t \ge \frac{1}{\beta}\int_0^T f(x,t)^2\,\mathrm{d}t \ge \frac{1}{\beta T}F_1(x,T)^2~.$$

Thus F_1 satisfies the differential inequality

$$\partial_x F_1(x,T) \ge -e_0 + \frac{1}{\beta T}F_1(x,T)^2~, \qquad x \in \mathbb{R}~. \tag{18}$$

If $F_1(x_0,T) > (\beta T e_0)^{1/2}$ for some $x_0 \in \mathbb{R}$, it follows from (18) that the function $x \mapsto F_1(x,T)$ is strictly increasing for $x > x_0$ and blows up at some point $x_1 > x_0$, which contradicts our assumptions. Similarly, if $F_1(x_0,T) < -(\beta T e_0)^{1/2}$, then $F_1(x,T)$ blows up to $-\infty$ at some point $x_2 < x_0$, which is again impossible. Thus we necessarily have $|F_1(x,T)| \le (\beta T e_0)^{1/2}$ for all $x \in \mathbb{R}$. □

Remark 3.2. Albeit elementary, Proposition 3.1 has interesting dynamical consequences. For instance, it immediately implies that an extended dissipative system on $\mathbb{R}$ cannot have any nontrivial time-periodic orbit with uniformly bounded energy density, see [15]. Indeed, for such a periodic orbit, we can take an interval $I = [x_1, x_2] \subset \mathbb{R}$ large enough so that the energy dissipation $d(x,t)$ is not identically zero for $x \in I$. Integrating (17) over I, we obtain the energy balance equation

$$F_1(x_2,T) = F_1(x_1,T) + \int_{x_1}^{x_2}\Bigl(e(x,T) - e(x,0)\Bigr)\,\mathrm{d}x + \int_{x_1}^{x_2}\int_0^T d(x,t)\,\mathrm{d}t\,\mathrm{d}x~. \tag{19}$$

By assumption, the last integral in the right-hand side grows linearly in T as $T \to \infty$, whereas the first integral is uniformly bounded by periodicity and the flux terms are $\mathcal{O}(T^{1/2})$ by (15). Thus (19) cannot hold for sufficiently large times.

We next investigate the analog of Proposition 3.1 in the higher-dimensional case $N \geq 2$. Here we consider the energy flux through the boundary of the ball $B_R = \{x \in \mathbb{R}^N \,|\, |x| < R\}$, for various values of the radius R. We recall that the Euclidean measure of the sphere ∂B_R is $\omega_N R^{N-1}$, where

$$\omega_N \;=\; \frac{2\pi^{N/2}}{\Gamma(N/2)}\;, \qquad \text{and} \quad \Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\,\mathrm{d}t\;, \quad \alpha > 0\;.$$

Given $R > 0$ and $T > 0$, we thus define the integrated flux

$$F(R,T) \;=\; \int_0^T \int_{|x|=R} f(x,t)\cdot \frac{x}{|x|}\,\mathrm{d}\sigma\,\mathrm{d}t\;, \tag{20}$$

which represents the total energy entering the ball B_R through the boundary over the time interval $[0,T]$ (the energy leaving the ball is of course counted negatively).

Before stating our result, we introduce the higher-dimensional analog of the differential equation (18), which (after suitable normalization) becomes

$$h'(r) + \frac{N{-}1}{r}\,h(r) \;=\; -1 + h(r)^2\;, \qquad r > 0\;. \tag{21}$$

The following elementary result will be established in Section 7.

Lemma 3.3. *For any $N \in \mathbb{N}^*$ the differential equation (21) has a unique positive solution $h_N : (0,+\infty) \to (0,+\infty)$. If $N \geq 2$, this solution is strictly decreasing and satisfies*

$$h_N(r) \;=\; 1 + \frac{N{-}1}{2r} + \mathcal{O}\Big(\frac{1}{r^2}\Big) \qquad \textit{as} \quad r \to +\infty\;, \tag{22}$$

and

$$h_N(r) \;\sim\; \begin{cases} \dfrac{1}{r\log(1/r)} & \textit{if} \quad N = 2\;, \\[2mm] \dfrac{N{-}2}{r} & \textit{if} \quad N \geq 3\;, \end{cases} \qquad \textit{as} \quad r \to 0\;. \tag{23}$$

Moreover, any solution of (21) above h_N blows up in finite time, and any solution below h_N cannot stay positive. Finally, h_N is given by the explicit formula

$$h_N(r) \;=\; \frac{K_{\frac{N}{2}}(r)}{K_{\frac{N}{2}-1}(r)}\;, \qquad r > 0\;, \tag{24}$$

where K_ν denotes the modified Bessel function as defined in [1, Section 9.6]. In particular $h_1(r) = 1$ and $h_3(r) = 1 + 1/r$ for all $r > 0$.

We are now able to state the main result of this section:

Proposition 3.4. *Assume that $N \geq 2$, and let $u(t) = \Phi(t)u_0$ be a trajectory for which the energy density $e(x,t)$ satisfies (13) for some $e_0 > 0$ and $\beta > 0$. Then, for any $R > 0$ and any $T > 0$, the integrated energy flux (20) satisfies*

$$\frac{F(R,T)}{\omega_N R^{N-1}} \;\leq\; \sqrt{\beta T e_0}\; h_N\Big(R\sqrt{\frac{e_0}{\beta T}}\Big)\;, \tag{25}$$

where h_N is given by (24).

Proof. Given $T > 0$ and $R > R_0 > 0$, we consider the energy balance equation (2) in the spherical shell $\Omega = \{x \in \mathbb{R}^N \,|\, R_0 < |x| < R\}$ over the time interval $[0, T]$. Using the notation (20) we have

$$F(R,T) \;=\; F(R_0,T) + \int_\Omega \Big(e(x,T) - e(x,0)\Big)\,\mathrm{d}x + \int_0^T\!\!\int_\Omega d(x,t)\,\mathrm{d}x\,\mathrm{d}t\;. \tag{26}$$

In particular, if we differentiate both sides with respect to R, we find

$$\partial_R F(R,T) \;=\; \int_{|x|=R} \Big(e(x,T) - e(x,0)\Big)\,\mathrm{d}\sigma + \int_0^T\!\!\int_{|x|=R} d(x,t)\,\mathrm{d}\sigma\,\mathrm{d}t\;. \tag{27}$$

To bound the right-hand side from below, we proceed as in the proof of Proposition 3.1. We first observe that

$$\int_{|x|=R} \Big(e(x,T) - e(x,0)\Big)\,\mathrm{d}\sigma \;\ge\; -\int_{|x|=R} e_0\,\mathrm{d}\sigma \;=\; -e_0\omega_N R^{N-1}\;.$$

Next, applying the Cauchy-Schwarz inequality to (20) and using (14), we obtain

$$|F(R,T)|^2 \;\le\; \omega_N R^{N-1} T \int_0^T\!\!\int_{|x|=R} |f(x,t)|^2\,\mathrm{d}\sigma\,\mathrm{d}t \;\le\; \beta T\omega_N R^{N-1} \int_0^T\!\!\int_{|x|=R} d(x,t)\,\mathrm{d}\sigma\,\mathrm{d}t\;. \tag{28}$$

Thus it follows from (27) that $F(R,T)$ satisfies the differential inequality

$$\partial_R F(R,T) \;\ge\; -e_0\omega_N R^{N-1} + \frac{1}{\beta T\omega_N}\,\frac{F(R,T)^2}{R^{N-1}}\;, \qquad R > 0\;. \tag{29}$$

To eliminate all parameters in (29), we set

$$\frac{F(R,T)}{\omega_N R^{N-1}} \;=\; \sqrt{\beta T e_0}\; H\left(R\sqrt{\frac{e_0}{\beta T}}\right)\;, \qquad R > 0\;.$$

Then the rescaled function H is a solution of the normalized inequality

$$H'(r) + \frac{N-1}{r}H(r) \;\ge\; -1 + H(r)^2\;, \qquad r > 0\;,$$

and can therefore be compared to the solutions of (21). In particular, if $H(r_0) > h_N(r_0)$ for some $r_0 > 0$, where h_N is given by (24), then Lemma 3.3 shows that $H(r)$ blows up at some point $r_1 > r_0$, which contradicts our assumption that the integrated flux (20) is well-defined and continuous for all $R > 0$. Thus we necessarily have $H(r) \le h_N(r)$ for all $r > 0$, and going back to the original variables this gives inequality (25) for all $R > 0$. □

Remarks 3.5.

1. The bound (25) also holds for $N = 1$ if we set $\omega_1 = 2$ and $h_1 \equiv 1$. It then asserts that the total energy entering the segment $[-R, R]$ over the time interval $[0, T]$ is bounded from above by $2(\beta T e_0)^{1/2}$. This is of course an immediate consequence of Proposition 3.1.

2. It is also possible to obtain a bound on the energy flux leaving the ball B_R over the time interval $[0, T]$. For instance, using the energy balance equation (2) with $\Omega = B_R$ and $[T_1, T_2] = [0, T]$, we easily obtain the estimate $F(R,T) \ge -\omega_N R^N e_0/N$, which is uniform in T.

3. If $N = 2$, it follows from (23) that

$$\sqrt{\beta T e_0}\; h_N\Big(R\sqrt{\frac{e_0}{\beta T}}\Big) \;\sim\; \frac{2\beta T}{R\log\Big(\frac{\beta T}{e_0 R^2}\Big)} \qquad \text{as} \quad T \to +\infty\;. \tag{30}$$

In view of Proposition 3.4, it follows that, for any given $R > 0$, the integrated flux $F(R,T)$ can grow at most sub-linearly (like $T/\log(T)$) as $T \to +\infty$. This is enough to preclude the existence of nontrivial time-periodic solutions in two-dimensional extended dissipative systems, using the same argument as in Remark 3.2, see also [15]. In contrast, if $N \ge 3$, we have

$$\sqrt{\beta T e_0}\, h_N\Big(R\sqrt{\frac{e_0}{\beta T}}\Big) \;\sim\; (N-2)\frac{\beta T}{R} \qquad \text{as} \quad T \to +\infty\ . \tag{31}$$

In that case $F(R,T)$ may grow linearly in time as $T \to \infty$, which is compatible with the existence of nontrivial time-periodic orbits (see [15] for explicit examples). But such solutions must be spatially localized, because estimate (25) shows that the flux per unit area $F(R,T)/(\omega_N R^{N-1})$ decreases like $1/R$ as the radius R of the sphere increases to infinity.

4. The right-hand side of (25) always decreases to zero as $\beta \to 0$. On the other hand, it is easy to verify that the asymptotics (30) and (31) also hold for a fixed $T > 0$ in the limit where $e_0 \to 0$. In particular, if $N \ge 3$, Proposition 3.4 does not preclude the existence of nontrivial solutions emerging from initial data with zero energy density.

5. As was mentioned in Remark 2.2.6, when solving a nonlinear PDE it is often convenient to use a function space for which the energy density is not continuous, but only locally integrable. Although we do not want to address such technicalities in the present paper, it is perhaps instructive to see how Proposition 3.4 is modified if we only suppose that the initial energy is bounded in the *uniformly local* sense, namely

$$\bar e_0 \;=\; \sup_{x\in\mathbb{R}^N} \int_{|y-x|\le 1} e(y,0)\,\mathrm{d}y \;<\; \infty\ .$$

For simplicity, we still assume that (14) holds for some $\beta > 0$. As in the proof of Proposition 3.4, we fix $R > R_0 \ge 1$ and we consider the energy balance equation (26) in the spherical shell $\Omega = \{x \in \mathbb{R}^N \,|\, R_0 < |x| < R\}$ over the time interval $[0,T]$. Since the energy density is not necessarily continuous, we do not differentiate with respect to R, but we bound from below the right-hand side of (26). The main new ingredient is an estimate of the energy initially contained in Ω. Using the definition of $\bar e_0$, we find

$$\int_\Omega e(x,0)\,\mathrm{d}x \;\le\; c_N \bar e_0 \frac{\omega_N}{N}(R^N - R_0^N) + d_N \bar e_0 \omega_N R_0^{N-1}\ ,$$

where c_N, d_N are positive constants related to the optimal covering of a (large) ball or sphere in $\mathbb{R}^N$ with balls of unit radius [11]. On the other hand, we know that $e(x,T) \ge 0$ and we have the lower bound

$$\int_0^T\!\!\int_\Omega d(x,t)\,\mathrm{d}x\,\mathrm{d}t \;=\; \int_0^T\!\!\int_{R_0}^R\!\int_{|x|=r} d(x,t)\,\mathrm{d}\sigma\,\mathrm{d}r\,\mathrm{d}t \;\ge\; \frac{1}{\beta T\omega_N}\int_{R_0}^R \frac{F(r,T)^2}{r^{N-1}}\,\mathrm{d}r\ , \tag{32}$$

which follows from (28). Summarizing, if we denote by $\tilde F(R)$ the solution of the ODE

$$\tilde F'(R) \;=\; -\bar e_0 c_N \omega_N R^{N-1} + \frac{1}{\beta T\omega_N}\frac{\tilde F(R)^2}{R^{N-1}}\ , \qquad R > R_0\ ,$$

with initial data $\tilde F(R_0) = F(R_0,T) - \bar e_0 d_N \omega_N R_0^{N-1}$, we deduce from (26) that $F(R,T) \ge \tilde F(R)$ as long as $\tilde F(R) \ge 0$. Arguing as before, this leads to the upper bound

$$\frac{F(R,T)}{\omega_N R^{N-1}} \;\le\; d_N \bar e_0 + \sqrt{c_N \beta T \bar e_0}\; h_N\Bigg(R\sqrt{\frac{c_N \bar e_0}{\beta T}}\Bigg)\ , \qquad R \ge 1\ , \tag{33}$$

which replaces (25). Note that the asymptotics as $T \to \infty$ are still given by (30), (31).

4 Bounds on the Energy Dissipation

As is clear from the balance equation (2), a bound on the amount of energy entering the ball $B_R = \{x \in \mathbb{R}^N \,|\, |x| < R\}$ over the time interval $[0,T]$ implies an estimate of the energy dissipated in B_R during the same time, provided the initial energy in B_R is under control. In this section, we derive various dissipation estimates using the flux bounds established in Section 3. We also show that, for nonequilibrium solutions of extended dissipative systems on $\mathbb{R}$ or $\mathbb{R}^2$, energy dissipation must occur "almost everywhere" in space.

4.1 Energy dissipation in fixed or increasing domains

As in Section 3, we consider a trajectory $u(t) = \Phi(t)u_0$ of an extended dissipative system satisfying the uniform bounds (13) for some $e_0 > 0$ and $\beta > 0$. Given $R > 0$ and $T > 0$, we denote by $F(R,T)$ the energy entering the ball B_R (through the boundary ∂B_R) over the time interval $[0,T]$. This quantity is defined by (20) for $N \ge 2$, and if $N = 1$ we set $F(R,T) = F_1(R,T) - F_1(-R,T)$, where F_1 is given by (16). Using the energy balance equation (2) and Proposition 3.4, we easily obtain

$$\int_0^T\!\!\int_{B_R} d(x,t)\,\mathrm{d}x\,\mathrm{d}t \;=\; F(R,T) + \int_{B_R}\Big(e(x,0)\,\mathrm{d}x - e(x,T)\Big)\,\mathrm{d}x \tag{34}$$

$$\le\; \omega_N R^{N-1}\sqrt{\beta T e_0}\; h_N\Big(R\sqrt{\frac{e_0}{\beta T}}\Big) + \frac{\omega_N}{N}R^N e_0\;, \tag{35}$$

where h_N is given by (24). Equivalently, if $\tilde{h}_N(r) = N h_N(r)/r$, we find

$$\frac{N}{\omega_N R^N}\int_0^T\!\!\int_{B_R} d(x,t)\,\mathrm{d}x\,\mathrm{d}t \;\le\; e_0\left(\tilde{h}_N\Big(R\sqrt{\frac{e_0}{\beta T}}\Big) + 1\right)\;. \tag{36}$$

We now investigate a few consequences of the general bound (35) or (36).

First, we fix $R > 0$ and consider the limit where $T \to +\infty$. Using the asymptotics (23) for $N \ge 2$ and the fact that $h_1 = 1$, we obtain the following result.

Corollary 4.1. *Under the assumptions of Proposition 3.1 or 3.4, the following inequalities hold for any $R > 0$.*

1) If $N = 1$,

$$\limsup_{T\to\infty}\frac{1}{\sqrt{T}}\int_0^T\!\!\int_{B_R} d(x,t)\,\mathrm{d}x\,\mathrm{d}t \;\le\; 2\sqrt{\beta e_0}\;. \tag{37}$$

2) If $N = 2$,

$$\limsup_{T\to\infty}\frac{\log(T)}{T}\int_0^T\!\!\int_{B_R} d(x,t)\,\mathrm{d}x\,\mathrm{d}t \;\le\; 4\pi\beta\;. \tag{38}$$

3) If $N \ge 3$,

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^T\!\!\int_{B_R} d(x,t)\,\mathrm{d}x\,\mathrm{d}t \;\le\; \beta(N-2)\omega_N R^{N-2}\;. \tag{39}$$

In particular, if $N \le 2$, it follows from (37), (38) that the energy dissipation in any fixed ball converges to zero "on average" as time goes to infinity. Since we assumed that energy dissipation vanishes only on equilibria of the system (see assumption (A3) in Definition 2.1), these estimates will imply that the trajectory $u(t)$ converges "on average" to the set of equilibria as $t \to \infty$, in a sense that will be specified in Section 5. Observe also that the bounds (37) and

(38) are independent of the radius R of the ball, whereas (38) and (39) do not depend on the initial energy density e_0.

It is also instructive to estimate the energy dissipation in a ball whose radius depends on the observation time T. In view of (36), it is natural to take $R = R_0\sqrt{T}$ for some $R_0 > 0$. We thus find :

Corollary 4.2. *Under the assumptions of Proposition 3.1 or 3.4, the following inequality holds for any $N \in \mathbb{N}^*$, any $R_0 > 0$, and any $T > 0$:*

$$\frac{N}{\omega_N R_0^N T^{N/2}} \int_0^T \int_{B_{R_0\sqrt{T}}} d(x,t)\,\mathrm{d}x\,\mathrm{d}t \;\le\; e_0\left(\tilde{h}_N\Big(R_0\sqrt{\frac{e_0}{\beta}}\Big) + 1\right) , \tag{40}$$

where $\tilde{h}_N(r) = N h_N(r)/r$.

Observe that the volume of the space-time cylinder $B_R \times [0,T]$ is $\omega_N N^{-1} R^N T$, hence (40) implies that the energy dissipation rate $d(x,t)$ is very small on average on $B_R \times [0,T]$ if $T \gg 1$. This remark will be exploited in Section 6, on a particular example, to prove convergence to equilibria uniformly on large domains (whose size increases with time).

Finally, in the two-dimensional case, it is also useful to consider the energy dissipation on a ball whose radius R has a slower growth than $T^{1/2}$ as $T \to \infty$. Obvious possibilities are $R = R_0 T^\gamma$ for $\gamma < 1/2$, or $R = R_0 T^{1/2}/\log(T)$. This gives the following estimates, which complement (38).

Corollary 4.3. *Assume that $N = 2$. Under the assumptions of Proposition 3.4, the following inequalities hold :*

1) If $R(T) = R_0 T^\gamma$ for some $R_0 > 0$ and some $\gamma \in [0, 1/2)$, then

$$\limsup_{T\to\infty} \frac{\log(T)}{T} \int_0^T \int_{B_{R(T)}} d(x,t)\,\mathrm{d}x\,\mathrm{d}t \;\le\; \frac{4\pi\beta}{1-2\gamma} .$$

2) If $R(T) = R_0 T^{1/2}/\log(T)$ for some $R_0 > 0$, then

$$\limsup_{T\to\infty} \frac{\log(\log(T))}{T} \int_0^T \int_{B_{R(T)}} d(x,t)\,\mathrm{d}x\,\mathrm{d}t \;\le\; 2\pi\beta .$$

4.2 Ubiquity of energy dissipation

In Section 3, we have seen a first way to exploit the energy relation (1) and the flux bound (14) to derive useful information on the dynamics of the system. We now consider the problem from a somewhat broader perspective. Let $u(t) = \Phi(t)u_0$ be a trajectory of an extended dissipative system in the sense of Definition 2.1, and suppose that the energy flux satisfies (14) for some $\beta > 0$. This is the case if the function $e \mapsto b(e)$ in assumption (A2) is bounded from above, or in more general situations if the energy density $e(x,t)$ is uniformly bounded. Given $R > 0$ and $T > 0$, we consider the integrated energy flux $F(R,T)$ defined by (20), and we also denote

$$E(R,T) \;=\; \int_{B_R} e(x,T)\,\mathrm{d}x , \qquad \delta E(R,T) \;=\; E(R,T) - E(R,0) ,$$

where as usual $B_R = \{x \in \mathbb{R}^N \,|\, |x| < R\}$. Then it follows from (27) and (28) that

$$F'(R,T) \;\ge\; \delta E'(R,T) + \frac{1}{\beta T \omega_N}\frac{F(R,T)^2}{R^{N-1}} , \qquad R > 0 , \tag{41}$$

where $'$ denotes here the differentiation with respect to R. Eq. (41) is a differential inequality of Riccati type, which imposes strong constraints on the integrated flux $F(R,T)$, especially because (41) has to hold for *all* values of $R > 0$.

Unfortunately, the solutions of the Riccati differential equation cannot be written in explicit form, so in general it is not easy to specify under which conditions (41) has global solutions. However, if simple assumptions are made on the source term $\delta E'(R,T)$, it is possible to extract useful information from (41). In Section 3, for instance, we assumed that $\delta E'(R,T) \geq -\omega_N R^{N-1} e_0$ for some $e_0 > 0$, and we obtained as a consequence the upper bound (25). Here we use the same strategy to prove that, if $N \leq 2$, the energy difference $\delta E(R,T)$ must be negative for most values of the radius $R > 0$. Given $T > 0$, denote

$$J_T = \Big\{ R > 0 \,\Big|\, E(R,T) \geq E(R,0) \Big\} \subset (0,\infty) \ . \qquad (42)$$

The main result of this section is :

Proposition 4.4. *Assume that $u_0 \in X$ is not an equilibrium, and that the trajectory $u(t) = \Phi(t)u_0$ satisfies* (14) *for some $\beta > 0$. Then for any $T > 0$ we have*

$$\int_1^\infty \frac{\mathbf{1}_{J_T}(r)}{r^{N-1}} \,\mathrm{d}r < \infty \ , \qquad (43)$$

where $\mathbf{1}_{J_T}$ is the characteristic function of the set J_T defined in (42)*.*

Remark 4.5. Of course, the conclusion of Proposition 4.4 is interesting only if $N \leq 2$. If $N = 1$ then (43) simply means that the Lebesgue measure of the set $J_T \subset (0,\infty)$ is finite. If $N = 2$ we have

$$\int_1^\infty \frac{\mathrm{meas}(J_T \cap [1,r])}{r^2} \,\mathrm{d}r = \int_1^\infty \frac{\mathbf{1}_{J_T}(r)}{r} \,\mathrm{d}r < \infty \ ,$$

which implies (roughly speaking) that $\mathrm{meas}(J_T \cap [1,R]) = o(R/\log(R))$ as $R \to \infty$. In both cases (43) shows that J_T is a very sparse subset of the half-line $(0,+\infty)$, so that $E(R,T) < E(0,T)$ for most values of $R > 0$. This considerably strengthens the results obtained (on a particular example) in [15, Section 2].

Proof of Proposition 4.4. Fix $T > 0$. We start from the energy balance equation (34), namely

$$F(R,T) = \delta E(R,T) + \int_0^T \int_{B_R} d(x,t) \,\mathrm{d}x \,\mathrm{d}t \ , \qquad R > 0 \ . \qquad (44)$$

Since $u_0 \in X$ is not an equilibrium, assumption (A3) in Definition 2.1 implies that the last term in (44) is positive when $R \geq R_1$, for some (sufficiently large) $R_1 > 0$. If $J_T \subset (0, R_1]$, then obviously (43) holds. If this is not the case, we choose $R_2 \in J_T \cap (R_1, +\infty)$ and (44) then implies that $F(R_2,T) > 0$. Using (32) with $R_0 = 0$, we also have

$$F(R,T) \geq \delta E(R,T) + \frac{1}{\beta T \omega_N} \int_0^R \frac{F(r,T)^2}{r^{N-1}} \,\mathrm{d}r \ , \qquad R > 0 \ . \qquad (45)$$

We now define

$$\mathcal{F}(R) = \frac{1}{(\beta T \omega_N)^2} \int_0^R \frac{F(r,T)^2}{r^{N-1}} \,\mathrm{d}r \ , \qquad R > 0 \ .$$

The function $\mathcal{F} : (0,\infty) \to \mathbb{R}_+$ is nondecreasing and $\mathcal{F}(R) > 0$ for all $R \geq R_2$. Moreover, using (45) and the definition (42) of J_T, we easily find

$$\mathcal{F}'(R) \geq \mathbf{1}_{J_T}(R) \frac{\mathcal{F}(R)^2}{R^{N-1}} \ , \qquad R > 0 \ .$$

Thus, for all $R > R_2$, we have

$$\int_{R_2}^{R} \frac{\mathbf{1}_{J_T}(r)}{r^{N-1}}\,\mathrm{d}r \;\le\; \int_{R_2}^{R} \frac{\mathcal{F}'(r)}{\mathcal{F}(r)^2}\,\mathrm{d}r \;=\; \frac{1}{\mathcal{F}(R_2)} - \frac{1}{\mathcal{F}(R)} \;\le\; \frac{1}{\mathcal{F}(R_2)}\;,$$

and (43) follows. □

5 Convergence to Equilibria

So far we considered a given trajectory $u(t) = \Phi(t)u_0$ of an extended dissipative system, and assuming that the energy density is uniformly bounded we established precise estimates on the flux and the dissipation of energy. Now, in the spirit of Remark 3.2, we want to show that these results impose nontrivial restrictions to the dynamics of the whole system, at least if the space dimension is not larger than 2. To do that, we often assume that the trajectories under consideration are relatively compact in our space X. As was already mentioned, compactness is generally easy to achieve by endowing X with a sufficiently weak topology.

If $d(x,t)$ is the energy dissipation rate for the trajectory $u(t) = \Phi(t)u_0$, we denote for all $R > 0$ and all $T > 0$:

$$D(R,T) \;=\; \int_0^T \Lambda_R(u(t))\,\mathrm{d}t\;, \qquad \text{where} \quad \Lambda_R(u(t)) \;=\; \int_{B_R} d(x,t)\,\mathrm{d}x\;. \tag{46}$$

Here, as usual, $B_R = \{x \in \mathbb{R}^N \,|\, |x| < R\}$. By assumption (A1) in Definition 2.1, we know that $D(R,T)$ depends continuously on the initial data u_0 in the topology of X.

As a consequence of Corollary 4.1, we first estimate the time spent by our trajectory in a neighborhood of a nonequilibrium point.

Proposition 5.1. *Consider an extended dissipative system on $\mathbb{R}^N$ with $N \le 2$. If $\bar{u} \in X$ is not an equilibrium point, then $\bar{u}$ has a neighborhood $\mathcal{V}$ in X such that any trajectory $u(t) = \Phi(t)u_0$ with uniformly bounded energy density satisfies*

$$\limsup_{T\to\infty} \frac{\Psi_N(T)}{T} \int_0^T \mathbf{1}_{\mathcal{V}}(u(t))\,\mathrm{d}t \;<\; \infty\;, \tag{47}$$

where $\Psi_1(T) = \sqrt{T}$, $\Psi_2(T) = \log(T)$, and $\mathbf{1}_{\mathcal{V}}$ denotes the characteristic function of $\mathcal{V} \subset X$.

Proof. We proceed as in [15, Section 5.1]. If $\bar{u} \in X$ is not an equilibrium point, then assumption (A3) in Definition 2.1 implies that the trajectory $\bar{u}(t) = \Phi(t)\bar{u}$ satisfies $\bar{D}(R, T_0) > 0$ for some $R > 0$ and some $T_0 > 0$, where $\bar{D}(R, T_0)$ denotes the energy dissipation (46) for the solution $\bar{u}(t)$. By continuity, there exists $\epsilon > 0$ and a neighborhood $\mathcal{V}$ of $\bar{u}$ in X such that, for any $u_0 \in \mathcal{V}$, the solution $u(t) = \Phi(t)u_0$ satisfies $D(R, T_0) \ge \epsilon > 0$, where $D(R, T_0)$ is given by (46).

Now, let $u(t) = \Phi(t)u_0$ be any trajectory of our system. Using the notation (46), we have for all $T > 0$:

$$\begin{aligned}\frac{1}{T}\int_0^{T+T_0} \Lambda_R(u(t))\,\mathrm{d}t &\ge \frac{1}{T}\int_0^T \left(\frac{1}{T_0}\int_t^{t+T_0} \Lambda_R(u(s))\,\mathrm{d}s\right)\mathrm{d}t \\ &\ge \frac{1}{T}\int_0^T \mathbf{1}_{\mathcal{V}}(u(t))\left(\frac{1}{T_0}\int_t^{t+T_0} \Lambda_R(u(s))\,\mathrm{d}s\right)\mathrm{d}t \;\ge\; \frac{\epsilon}{TT_0}\int_0^T \mathbf{1}_{\mathcal{V}}(u(t))\,\mathrm{d}t\;,\end{aligned}$$

hence

$$\int_0^T \mathbf{1}_{\mathcal{V}}(u(t))\,\mathrm{d}t \;\le\; \frac{T_0}{\epsilon}\int_0^{T+T_0}\int_{B_R} d(x,t)\,\mathrm{d}x\,\mathrm{d}t \;, \qquad T>0\;.$$

If we multiply both sides by $1/\sqrt{T}$ (if $N=1$) or $\log(T)/T$ (if $N=2$) and take the limit $T\to\infty$, we obtain (47) using Corollary 4.1. □

Remark 5.2. In [15], the following weaker result was obtained for a particular system: If $N\le 2$, any nonequilibrium point has a neighborhood $\mathcal{V}$ in X such that any trajectory $u(t)$ with uniformly bounded energy density satisfies

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbf{1}_{\mathcal{V}}(u(t))\,\mathrm{d}t \;=\; 0\;.$$

This of course follows from (47), which gives a more precise estimate of the fraction of time spent by the trajectory $u(t)$ in the neighborhood $\mathcal{V}$, depending on the space dimension N.

Proposition 5.1 was obtained without any compactness assumption, but we can use it to prove that all relatively compact trajectories of the system converge in some sense to the set equilibria. Indeed, given any such trajectory $u(t)=\Phi(t)u_0$, we define the omega-limit set

$$\omega \;=\; \Big\{u\in X\,\Big|\,\exists\, t_n\to+\infty \text{ such that } u(t_n)\xrightarrow[n\to\infty]{} u \text{ in } X\Big\}\;\subset\; X\;. \tag{48}$$

It is known [19] that ω is nonempty, compact, connected, fully invariant under the semiflow $\Phi(t)$, and that $\mathrm{dist}_X(u(t),\omega)\to 0$ as $t\to+\infty$. However, our assumptions *do not* imply that ω is contained in the set of equilibria. Counter-examples can indeed be constructed even for relatively simple systems such as the Allen-Cahn equation in one space dimension, see Example 5.7 below. Motivated by the conclusion of Proposition 5.1, we propose the following alternative definition:

Definition 5.3. *If $u(t)=\Phi(t)u_0$ is a trajectory of an extended dissipative system on $\mathbb{R}^N$ with $N\le 2$, we define*

$$\bar\omega \;=\; \Big\{\bar u\in X\,\Big|\,\limsup_{T\to\infty}\frac{\Psi_N(T)}{T}\int_0^T \mathbf{1}_{\mathcal{V}}(u(t))\,\mathrm{d}t=\infty \text{ for all neighborhoods } \mathcal{V} \text{ of } \bar u\Big\}\;, \tag{49}$$

where $\Psi_1(T)=\sqrt{T}$ and $\Psi_2(T)=\log(T)$.

In other words, $\bar\omega$ is the set of points in all neighborhoods of which the trajectory $u(t)$ spends a "substantial fraction of the total time". What is exactly meant by "substantial" depends on the space dimension N, and is specified by the function $\Psi_N(T)$. It is clear from the definition that $\bar\omega\subset\omega$, and Proposition 5.1 implies that $\bar\omega$ is contained in the set of equilibria of our system, provided the trajectory $u(t)$ has uniformly bounded energy density. More properties of $\bar\omega$ are collected in our final result:

Proposition 5.4. *Let $u(t)=\Phi(t)u_0$ be a relatively compact trajectory of an extended dissipative system on $\mathbb{R}^N$, $N\le 2$, with uniformly bounded energy density. Then the set $\bar\omega\subset X$ defined by (49) is nonempty, compact, and contained in the set of equilibria. Moreover, if $\mathcal{V}$ is any neighborhood of $\bar\omega$ in X, then*

$$\limsup_{T\to\infty}\frac{\Psi_N(T)}{T}\int_0^T \mathbf{1}_{\mathcal{V}^c}(u(t))\,\mathrm{d}t \;<\;\infty\;. \tag{50}$$

Proof. We proceed as in [15, Section 5.2]. We first observe that, if $\Gamma \subset X$ is compact and does not intersect $\bar\omega$, then there exists a neighborhood $\mathcal{V}$ of Γ such that

$$\limsup_{T\to\infty} \frac{\Psi_N(T)}{T} \int_0^T \mathbf{1}_\Gamma(u(t))\,\mathrm{d}t \;\le\; \limsup_{T\to\infty} \frac{\Psi_N(T)}{T} \int_0^T \mathbf{1}_\mathcal{V}(u(t))\,\mathrm{d}t \;<\; \infty\;. \tag{51}$$

Indeed, this property holds by definition if $\Gamma = \{u_1\}$ for some $u_1 \notin \bar\omega$, and the general case follows by a finite covering argument. Now, if we take for Γ the closure of the trajectory $\{u(t)\,|\,t \ge 0\}$, then Γ is compact and $T^{-1}\int_0^T \mathbf{1}_\Gamma(u(t))\,\mathrm{d}t = 1$ for all $T > 0$, which is incompatible with (51). Thus we must have $\Gamma \cap \bar\omega \neq \emptyset$, hence in particular $\bar\omega \neq \emptyset$. Moreover, it is clear from the definition that $\bar\omega$ is closed in X and contained in Γ, hence $\bar\omega$ is compact. On the other hand, Proposition 5.1 precisely means that $\bar\omega$ is contained in the set of equilibria. Finally, if $\mathcal{V}$ is any open neighborhood of $\bar\omega$ in X, then $\Gamma \cap \mathcal{V}^c$ is compact and does not intersect $\bar\omega$, hence by (51)

$$\limsup_{T\to\infty} \frac{\Psi_N(T)}{T} \int_0^T \mathbf{1}_{\mathcal{V}^c}(u(t))\,\mathrm{d}t \;=\; \limsup_{T\to\infty} \frac{\Psi_N(T)}{T} \int_0^T \mathbf{1}_{\mathcal{V}^c\cap\Gamma}(u(t))\,\mathrm{d}t \;<\; \infty\;,$$

which proves (50). □

Remark 5.5. Since $\bar\omega$ consists of equilibria, it is obvious that $\Phi(t)\bar\omega = \bar\omega$ for all $t \ge 0$. In fact, for any relatively compact trajectory of a continuous semiflow on a metrizable space X, one can prove that the set $\bar\omega$ defined by (49) is nonempty, compact, and fully invariant, see [15, Proposition 5.4]. These properties are therefore independent of the gradient structure. On the other hand, the set $\bar\omega$ (unlike ω) is not connected in general, as can be seen from Example 5.7.

Remark 5.6. Instead of $\bar\omega$, the following set was defined in [15] (for a particular system):

$$\tilde\omega \;=\; \Bigl\{\bar u \in X \;\Big|\; \limsup_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{1}_\mathcal{V}(u(t))\,\mathrm{d}t > 0 \text{ for all neighborhoods } \mathcal{V} \text{ of } \bar u\Bigr\}\;.$$

Clearly $\tilde\omega \subset \bar\omega$, hence Proposition 5.4 implies that $\tilde\omega$ is contained in the set of equilibria, as was proved in [15, Proposition 5.4]. It is also known that $\tilde\omega \neq \emptyset$, which implies that $\bar\omega \neq \emptyset$.

Example 5.7. Under the assumptions of Proposition 5.4, it is not a priori obvious that the usual omega-limit set (48) is not necessarily contained in the set of equilibria. In this respect, the following example is instructive. We consider the one-dimensional reaction-diffusion equation (3) with the double-well potential $V(u) = \frac{1}{4}(1-u^2)^2$:

$$\partial_t u \;=\; \partial_x^2 u + u - u^3\;, \qquad x \in \mathbb{R}\;, \quad t \ge 0\;. \tag{52}$$

This system has three constant steady states: $u_0 = 0$ (which is unstable), and $u_\pm = \pm 1$ (which are stable). In addition, there is the "kink" solution

$$\psi(x) \;=\; \tanh(x/\sqrt{2})\;, \qquad x \in \mathbb{R}\;, \tag{53}$$

which connects u_- at $x = -\infty$ to u_+ at $x = +\infty$. It can be shown that $u_\pm$ and the translates of $\pm\psi$ are the only stable steady states of (52) in the space of bounded solutions.

Interesting nonequilibrium solutions of (52) can be constructed by gluing widely separated kinks. For instance, if $a \gg 1$, the function

$$V_a(x) \;=\; \psi(x-a) - \psi(x+a) + 1\;, \qquad x \in \mathbb{R}\;,$$

describes the superposition of a kink ψ located near $x = a$ and an "anti-kink" $-\psi$ near $x = -a$. This is not an equilibrium of (52), but it can be shown that the solution of (52) with initial data V_a stays very close to $V_{a(t)}$ for later times, provided the parameter a evolves according to the exponential law $\dot{a} \simeq -c_1 \exp(-c_2 a)$, for some $c_1, c_2 > 0$, see e.g. [8]. This approximation property remains valid as long as both kinks are widely separated, but when they get close to each other they "annihilate" and the solution converges uniformly to 1 as $t \to +\infty$.

Using these results and a general procedure that can be found e.g. in [13], one can show that there exists a unique eternal solution $u_\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of (52) such that $u_\psi(0,0) = 0$ and

$$\sup_{x\in\mathbb{R}} |u_\psi(x,t) - V_{a(t)}(x)| \xrightarrow[t\to-\infty]{} 0 \;, \qquad \text{where} \qquad a(t) \to +\infty \quad \text{as} \quad t \to -\infty \;. \tag{54}$$

In fact, one has $a(t) \sim c_2^{-1} \log(|t|)$ as $t \to -\infty$. This solution converges uniformly to $u_+ = 1$ as $t \to +\infty$, and uniformly on compact sets to $u_- = -1$ as $t \to -\infty$. In the topology $\mathcal{T}_{\mathrm{loc}}(\mathbb{R})$ of uniform convergence on compact sets of $\mathbb{R}$, it follows that $u_\psi(t)$ realizes a *heteroclinic connection* from u_- to u_+ through the symmetric collapse of a pair of kinks coming from infinity.

Now, using an idea taken from [10], we consider the solution $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ of (52) with initial data u_0 satisfying

$$u_0(x) \;=\; (-1)^{n+1} \qquad \text{if} \quad b_n \le |x| < b_{n+1} \;, \tag{55}$$

where $(b_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence satisfying $b_0 = 0$ and $b_{n+1} \gg b_n$ for all $n \in \mathbb{N}$. Under the evolution of the parabolic equation (52), the discontinuities of the initial data are rapidly smeared out, and replaced by smooth interfaces of the form (53), the positions of which slowly move according to exponential law specified above. In particular, since $b_1 \gg b_0$, the pair of kinks closest to the origin will hardly feel the presence of the other kinks, and will therefore evolve in time like the solution of (52) with initial data V_{b_0}. Once the first pair has disappeared, we are essentially back to the original configuration, with a central pair of kinks that is now located near $\pm b_1$. This pair evolves on a much slower time scale, but will eventually come close to the origin and annihilate, and the same process will continue forever since we started with infinitely many kinks. Such a coarsening dynamics was studied for instance in [12, 25].

These heuristic considerations lead to the following reasonable conjecture:

Conjecture 5.8. *Let $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ be the solution of* (52) *with initial data* (55)*. Then the omega-limit set of the trajectory $(u(t))_{t\ge0}$ in the topology $\mathcal{T}_{\mathrm{loc}}(\mathbb{R})$ is*

$$\omega \;=\; \{u_+\,,\, u_-\} \cup \{u_\psi(t)\,|\, t \in \mathbb{R}\} \cup \{-u_\psi(t)\,|\, t \in \mathbb{R}\} \;,$$

where $u_\pm = \pm 1$ and u_ψ is the eternal solution of (52) *defined by* (54)*.*

If this conjecture is true, then ω consists of two equilibria $u_\pm$ and two heteroclinic connections between them. Thus ω is a heteroclinic loop, which is not entirely contained in the set of equilibria. In contrast, for the same solution, the modified omega-limit set introduced in Definition 5.3 satisfies $\bar{\omega} = \{u_+\,,\, u_-\}$, hence is contained in the set of equilibria. Note that Proposition 5.1 implies that the number of annihilations of pairs of kinks that can occur in the time interval $[0, T]$ is bounded by $C\sqrt{T}$ for large T.

6 The Vorticity Equation in an Infinite Cylinder

In this section we analyze in some detail an interesting example which does not fit exactly into the framework of Definition 2.1, but can nevertheless be studied using the techniques developed in Sections 3 to 5. We consider the incompressible Navier-Stokes equation in the infinite cylinder

$\Omega = \mathbb{R} \times \mathbb{T}$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Points in Ω are denoted by $x = (x_1, x_2)$, where $x_1 \in \mathbb{R}$ is the "horizontal" and $x_2 \in \mathbb{T}$ the "vertical" variable. Our system reads

$$\partial_t u + (u \cdot \nabla) u \;=\; \Delta u - \nabla p \;, \qquad \operatorname{div} u \;=\; 0 \;, \tag{56}$$

where $u : \Omega \times \mathbb{R}_+ \to \mathbb{R}^2$ denotes the velocity field and $p : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ the pressure field. For each $t \ge 0$, both quantities $u(x_1, x_2, t)$, $p(x_1, x_2, t)$ are assumed to be bounded in Ω and 1-periodic with respect to x_2. Since u is divergence free, we have in particular

$$\partial_1 \int_{\mathbb{T}} u_1(x_1, x_2, t) \,\mathrm{d}x_2 \;=\; \int_{\mathbb{T}} \partial_1 u_1(x_1, x_2, t) \,\mathrm{d}x_2 \;=\; -\int_{\mathbb{T}} \partial_2 u_2(x_1, x_2, t) \,\mathrm{d}x_2 \;=\; 0 \;,$$

hence the vertical average of the horizontal speed, which we denote by $\langle u_1 \rangle$, does not depend on the horizontal variable x_1. As is explained in [2, 16], it then follows from (56) that $\partial_t \langle u_1 \rangle = 0$, so that $\langle u_1 \rangle$ is a constant which can be set to zero by an appropriate Galilean transformation. Thus, without loss of generality, we can assume that

$$u(x_1, x_2, t) \;=\; \begin{pmatrix} 0 \\ m(x_1, t) \end{pmatrix} + \begin{pmatrix} \hat{u}_1(x_1, x_2, t) \\ \hat{u}_2(x_1, x_2, t) \end{pmatrix} \;, \qquad (x_1, x_2) \in \Omega \;, \quad t > 0 \;, \tag{57}$$

where $m = \langle u_2 \rangle$. By construction, we then have $\langle \hat{u}_1 \rangle = \langle \hat{u}_2 \rangle = 0$.

In addition to (56), we shall study the evolution equation for the vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$. In view of (57), we have

$$\omega(x_1, x_2, t) \;=\; \partial_1 m(x_1, t) + \hat{\omega}(x_1, x_2, t) \;, \qquad (x_1, x_2) \in \Omega \;, \quad t > 0 \;,$$

where $\partial_1 m = \langle \omega \rangle$ and $\hat{\omega} = \partial_1 \hat{u}_2 - \partial_2 \hat{u}_1$. It is important to realize here that, since we want to consider solutions of (56) which do not necessarily decay to zero as $|x_1| \to \infty$, the velocity field u is not entirely determined by the vorticity ω. More precisely, one can show that the oscillating part $\hat{u}$ of the velocity field is given by a Biot-Savart formula:

$$\hat{u}(x_1, x_2, t) \;=\; \int_{\mathbb{R}} \int_{\mathbb{T}} \nabla^{\perp} K(x_1 - y_1, x_2 - y_2) \hat{\omega}(y_1, y_2, t) \,\mathrm{d}y_2 \,\mathrm{d}y_1 \;, \tag{58}$$

where $\nabla^{\perp} = (-\partial_2, \partial_1)$ and

$$K(x_1, x_2) \;=\; \frac{1}{4\pi} \log\Bigl(2\cosh(2\pi x_1) - 2\cos(2\pi x_2)\Bigr) - \frac{|x_1|}{2} \;, \qquad (x_1, x_2) \in \Omega \;, \tag{59}$$

see [2]. However, the vertical average $m = \langle u_2 \rangle$ cannot be completely expressed in terms of the vorticity, and we only know that $\partial_1 m = \langle \omega \rangle$. The following estimates will be useful.

Lemma 6.1. *There exists a constant $C_1 > 0$ such that, for any $\omega \in L^\infty(\Omega)$, the velocity field $\hat{u}$ defined by* (58) *satisfies*

$$\|\hat{u}\|_{L^\infty(\Omega)} \;\le\; C_1 \|\omega\|_{L^\infty(\Omega)} \;. \tag{60}$$

Moreover, we have $\hat{u}_1 = -\partial_2 v$ for some $v \in L^\infty(\Omega)$, and there exists $C_2 > 0$ such that

$$\|v\|_{L^\infty(\Omega)} \;\le\; C_2 \operatorname*{ess\,sup}_{x_1 \in \mathbb{R}} \left(\int_{\mathbb{T}} \omega(x_1, x_2)^2 \,\mathrm{d}x_2 \right)^{1/2} \;\le\; 2C_2 \|\omega\|_{L^\infty(\Omega)} \;. \tag{61}$$

Proof. Here and below we denote by $*$ the convolution on Ω (considered as an additive group). As is easily verified, the function K defined by (59) satisfies $\nabla K \in L^1(\Omega)$, hence

$$\|\hat{u}\|_{L^\infty(\Omega)} \;=\; \|\nabla^{\perp} K * \hat{\omega}\|_{L^\infty(\Omega)} \;\le\; \|\nabla K\|_{L^1(\Omega)} \|\hat{\omega}\|_{L^\infty(\Omega)} \;\le\; 2\|\nabla K\|_{L^1(\Omega)} \|\omega\|_{L^\infty(\Omega)} \;.$$

Similarly, one can check that $K \in L^1(\Omega)$ and

$$M_0 \;:=\; \int_{\mathbb{R}} \sup_{x_2 \in \mathbb{T}} |K(x_1,x_2)| \,\mathrm{d}x_1 \;<\; \infty \;.$$

Thus, if we define $v = K * \hat{\omega}$, we have $\hat{u}_1 = -\partial_2 v$ by (58) and a standard calculation shows that

$$\|v\|_{L^\infty(\Omega)}^2 \;\le\; M_0 \|K\|_{L^1(\Omega)} \operatorname*{ess\,sup}_{x_1 \in \mathbb{R}} \int_{\mathbb{T}} \hat{\omega}(x_1,x_2)^2 \,\mathrm{d}x_2 \;,$$

which gives the desired result since $\int_{\mathbb{T}} \hat{\omega}(x_1,x_2)^2 \,\mathrm{d}x_2 \le \int_{\mathbb{T}} \omega(x_1,x_2)^2 \,\mathrm{d}x_2$. □

Instead of the Navier-Stokes equation (56), we now consider the evolution system satisfied by the vorticity ω and the average speed $m = \langle u_2 \rangle$. As in [2] we obtain

$$\left\{ \begin{array}{rcll} \partial_t m + \partial_1 \langle \hat{u}_1 \hat{u}_2 \rangle &=& \partial_1^2 m \;, & \quad x_1 \in \mathbb{R} \;, \\ \partial_t \omega + u \cdot \nabla \omega &=& \Delta \omega \;, & \quad (x_1,x_2) \in \Omega \;. \end{array} \right. \tag{62}$$

Here it is understood that u is given by (57), where $\hat{u}_1, \hat{u}_2$ are obtained from ω via (58). Note that system (62) is somewhat redundant, because the horizontal derivative of the first equation is the vertical average of the second one, but as is explained above it is not possible to get rid completely of the first equation. Given a solution of (62), we define for all $x_1 \in \mathbb{R}$ and $t > 0$:

$$\begin{aligned} e(x_1,t) &= \frac{1}{2} \int_{\mathbb{T}} \omega(x_1,x_2,t)^2 \,\mathrm{d}x_2 \;, \\ f(x_1,t) &= \frac{1}{2} \int_{\mathbb{T}} \Big(\partial_1 \omega^2 - u_1 \omega^2 \Big)(x_1,x_2,t) \,\mathrm{d}x_2 \;, \\ d(x_1,t) &= \int_{\mathbb{T}} |\nabla \omega(x_1,x_2,t)|^2 \,\mathrm{d}x_2 \;. \end{aligned} \tag{63}$$

In agreement with the general terminology used in this paper, we shall call e the energy density, f the energy flux, and d the energy dissipation rate, although the term "enstrophy" would certainly be more appropriate than "energy" in the present context. Using (62), it is easy to verify that the quantities (63) satisfy $\partial_t e = \partial_1 f - d$, which is the one-dimensional version of (1). On the other hand, if $d \equiv 0$, then certainly $\partial_t \omega \equiv 0$ and $\hat{\omega} = \omega - \langle \omega \rangle \equiv 0$. Then $\hat{u} \equiv 0$ by (58), and since $\partial_1^2 m = \partial_1 \langle \omega \rangle \equiv 0$ it follows from (62) that $\partial_t m \equiv 0$ too. Thus $d \equiv 0$ only for equilibria of system (62). Finally, we have the following estimate for the energy flux:

Lemma 6.2. *There exists a constant $C_3 > 0$ such that*

$$|f(x_1)|^2 \;\le\; C_3 \Big(1 + \sup_{y_1 \in \mathbb{R}} e(y_1) \Big) e(x_1) d(x_1) \;, \qquad \text{for all} \quad x_1 \in \mathbb{R} \;. \tag{64}$$

Proof. We fix $x_1 \in \mathbb{R}$ and consider both terms in (63) separately. First, using the Cauchy-Schwarz inequality, we easily find

$$\frac{1}{2} \Big| \int_{\mathbb{T}} \partial_1 \omega^2 \,\mathrm{d}x_2 \Big| \;=\; \Big| \int_{\mathbb{T}} \omega \partial_1 \omega \,\mathrm{d}x_2 \Big| \;\le\; (2e)^{1/2} d^{1/2} \;.$$

On the other hand, since $u_1 = \hat{u}_1 = -\partial_2 v$ by Lemma 6.1, we have

$$\frac{1}{2} \int_{\mathbb{T}} u_1 \omega^2 \,\mathrm{d}x_2 \;=\; -\frac{1}{2} \int_{\mathbb{T}} (\partial_2 v) \omega^2 \,\mathrm{d}x_2 \;=\; \int_{\mathbb{T}} v \omega \partial_2 \omega \,\mathrm{d}x_2 \;,$$

hence using (61) we conclude

$$\left|\frac{1}{2}\int_{\mathbb{T}} u_1\omega^2 \,\mathrm{d}x_2\right| \;\le\; \|v\|_{L^\infty(\Omega)}(2e)^{1/2}d^{1/2} \;\le\; 2C_2 \sup_{y_1\in\mathbb{R}} e(y_1)^{1/2}\, e^{1/2}d^{1/2} \;.$$

Combining both estimates we obtain (64). □

The Cauchy problem for Eq. (56) is globally well-posed in the Banach space

$$X \;=\; \Bigl\{u \in C^0_{\mathrm{bu}}(\Omega)^2 \,\Big|\, \mathrm{div}\, u = 0\Bigr\} \;,$$

equipped with the L^∞ norm, see [2, 16, 17, 26, 27]. If $u(t)$ is the solution of (56) with initial data $u_0 \in X$, it is known that $\|u(t)\|_{L^\infty}$ cannot grow faster than $t^{1/2}$ as $t \to \infty$, see [2] and Eq. (65) below. Using the techniques developed in the present paper, it is possible to improve that estimate and to get some qualitative information on the long-time behavior of the velocity $u(t)$, in the spirit of the results of Sections 4 and 5, see [16]. Here our goal is to perform a similar study at the level of the vorticity $\omega(t)$, which is somewhat simpler to analyze. Without loss of generality, we assume that the initial vorticity $\omega_0 = \mathrm{curl}\, u_0$ is bounded, and we denote $M = \|\omega_0\|_{L^\infty}$. Since $\omega(t)$ evolves according to the advection-diffusion equation (62), the maximum principle implies that $\|\omega(t)\|_{L^\infty} \le M$ for all $t \ge 0$. It then follows from (60) that $\|\hat u(t)\|_{L^\infty} \le C_1 M$ for all $t \ge 0$, so that the oscillating part of the velocity is under control. On the other hand, if we apply Duhamel's formula to the first equation in (62), we obtain

$$m(t) \;=\; e^{t\partial_x^2}\, m_0 - \int_0^t \partial_1 e^{(t-s)\partial_x^2}\, \hat u_1(s)\hat u_2(s)\,\mathrm{d}s \;, \qquad t > 0 \;,$$

where $m_0 = m(0)$ is the vertical average of the initial speed u_0. The uniform bound on $\hat u(t)$ thus implies

$$\|m_1(t)\|_{L^\infty} \;\le\; \|m_0\|_{L^\infty} + \int_0^t \frac{\|\hat u(s)\|_{L^\infty}^2}{\sqrt{\pi(t-s)}}\,\mathrm{d}s \;\le\; \|m_0\|_{L^\infty} + \frac{2\sqrt{t}}{\sqrt{\pi}}\, C_1^2 M^2 \;, \qquad t > 0 \;, \tag{65}$$

hence $\|u(t)\|_{L^\infty} \le \|\hat u(t)\|_{L^\infty} + \|m(t)\|_{L^\infty} \le C(1+t)^{1/2}$, as already announced.

Under our assumptions, the energy density and the energy flux defined by (63) satisfy the uniform bounds $e(x_1,t) \le e_0$ and $f(x_1,t)^2 \le \beta d(x_1,t)$, where

$$e_0 \;=\; \frac{1}{2}\|\omega_0\|_{L^\infty}^2 \;, \qquad \text{and} \qquad \beta \;=\; C_3 e_0(1+e_0) \;, \tag{66}$$

see Lemma 6.2. Thus we are exactly in position to apply the results of Sections 3 and 4. In particular, using (35) with $N = 1$, we obtain :

Proposition 6.3. *If the initial vorticity is bounded, the solution of* (62) *satisfies, for all $T > 0$ and all $R > 0$,*

$$\int_0^T \int_{-R}^R \int_{\mathbb{T}} |\nabla\omega(x_1,x_2,t)|^2 \,\mathrm{d}x_2\,\mathrm{d}x_1\,\mathrm{d}t \;\le\; 2\sqrt{\beta T e_0} + 2Re_0 \;, \tag{67}$$

where e_0, β are given by (66).

Proceeding as in Section 5, one can then use (67) to show that the vorticity $\omega(x_1,x_2,t)$ converges uniformly on compact subdomains toward the set of equilibria

$$\mathcal{E} \;=\; \Bigl\{w \in C^0_{\mathrm{bu}}(\Omega) \,\Big|\, \nabla w \equiv 0\,,\; |w| \le \|\omega_0\|_{L^\infty}\Bigr\} \;.$$

More precisely, adapting Proposition 5.4 to the present situation, we infer that, if $\mathcal{V}$ is any neighborhood of $\mathcal{E} \subset X$ in the topology of $C^1(\Omega)$, the fraction of the time interval $[0,T]$ spent by the trajectory $\omega(t)$ outside $\mathcal{V}$ does not grow faster than $CT^{1/2}$ as $T \to \infty$. This already gives valuable information on the solutions of (62), but combining (67) with further a priori estimates one can obtain a stronger and more precise conclusion. In what follows, we assume that the solution of (62) under consideration satisfies

$$\sup_{x_1 \in \mathbb{R}} \int_{\mathbb{T}} |\partial_{12}^2 \omega(x_1,x_2,t)|^2 \, \mathrm{d}x_2 \;\le\; M_1^2 \;, \qquad \text{and} \qquad \sup_{x_1 \in \mathbb{R}} |m(x_1,t)| \;\le\; M_2(1+t)^\beta \;, \tag{68}$$

for all $t \ge 0$, where $M_1, M_2 > 0$ and $\beta \in [0,1/2]$. The first estimate in (68) is verified for all $t \ge 1$ by any bounded solution of the vorticity equation, due to parabolic regularization, and the second estimate with $\beta = 1/2$ is just (65). In fact, it is possible to show that (68) always holds with $\beta = 1/6$, see [16].

Proposition 6.4. *Consider a solution of* (62) *with bounded initial data such that* (68) *holds for some $\beta \in [0,1/2]$. If $\beta \le \alpha \le 1/2$, there exists a constant $K_0 > 0$ such that, for any $T \ge 1$,*

$$\mathrm{meas}\left\{ t \in [0,T] \;\middle|\; \sup_{|x_1| \le T^{(\alpha+2\beta)/3}} \; \sup_{x_2 \in \mathbb{T}} |\omega(x_1,x_2,t)| \;\ge\; \frac{K_0}{T^{(\alpha-\beta)/3}} \right\} \;\le\; K_0 T^{\alpha+1/2} \;. \tag{69}$$

Remark 6.5. Estimate (69) is useful especially when $\beta < \alpha < 1/2$. It then shows that the vorticity $\omega(x_1,x_2,t)$ converges uniformly to zero on subdomains of size $\mathcal{O}(t^{(\alpha+2\beta)/3})$, at a rate comparable to $t^{-(\alpha-\beta)/3}$, except for possible excursions whose probability density decays roughly like $t^{\alpha-1/2}$ as $t \to \infty$. The fact that ω converges to zero, and not to a nonzero constant w, can be understood as follows. If $\omega(x_1,x_2,t) = w \neq 0$ on a sufficiently large spatial domain, then $\partial_1 m = \langle \omega \rangle = w$ for x_1 in a large interval, and this is compatible with the assumed upper bound (68) only if w is small enough. Thus, in the particular case of equation (62) with bounded initial data for ω and m, the omega-limit set "on average" as defined in Remark 5.6 consists of a single point.

Proof of Proposition 6.4. Applying (67) with $R = \sqrt{T}$, we see that there exists $C_4 > 0$ such that

$$\int_0^T \int_{-\sqrt{T}}^{\sqrt{T}} \int_{\mathbb{T}} |\nabla \omega(x_1,x_2,t)|^2 \, \mathrm{d}x_2 \, \mathrm{d}x_1 \, \mathrm{d}t \;\le\; C_4 \sqrt{T} \;, \tag{70}$$

for all $T \ge 1$. Given $\alpha \in [0,1/2]$, we define

$$J_\alpha(T) \;=\; \left\{ t \in [0,T] \;\middle|\; \int_{-\sqrt{T}}^{\sqrt{T}} \int_{\mathbb{T}} |\nabla \omega(x_1,x_2,t)|^2 \, \mathrm{d}x_2 \, \mathrm{d}x_1 \;\ge\; \frac{1}{T^\alpha} \right\} \;\subset\; [0,T] \;.$$

It follows from (70) that $\mathrm{meas}(J_\alpha(T)) \le C_4 T^{\alpha+1/2}$, for all $T \ge 1$. Our goal is to give a uniform bound on the vorticity $\omega(x_1,x_2,t)$ on a large spatial domain for all $t \in [0,T] \setminus J_\alpha(T)$.

We observe that $|\omega(x_1,x_2,t)| \le |g(x_1,t)| + h(x_1,t)$ for all $(x_1,x_2) \in \Omega$ and all $t \in [0,T]$, where

$$g(x_1,t) \;=\; \int_{\mathbb{T}} \omega(x_1,x_2,t) \, \mathrm{d}x_2 \;, \qquad \text{and} \qquad h(x_1,t) \;=\; \Bigl(\int_{\mathbb{T}} |\partial_2 \omega(x_1,x_2,t)|^2 \, \mathrm{d}x_2 \Bigr)^{1/2} \;.$$

We first bound the average $g = \langle \omega \rangle$. If $L \le \sqrt{T}$ and $t \in [0,T] \setminus J_\alpha(T)$, we have

$$\int_{-L}^{L} |\partial_1 g(x_1,t)|^2 \, \mathrm{d}x_1 \;\le\; \int_{-L}^{L} \int_{\mathbb{T}} |\partial_1 \omega(x_1,x_2,t)|^2 \, \mathrm{d}x_2 \, \mathrm{d}x_1 \;\le\; \frac{1}{T^\alpha} \;. \tag{71}$$

Furthermore, we know that $g = \partial_1 m$, where $m(x_1,t)$ satisfies (68) for some $\beta \in [0,1/2]$. Using (68), (71) and Lemma 6.6 below, we thus obtain

$$\sup_{|x_1|\le L} |g(x_1,t)| \;\le\; \frac{C_5 T^\beta}{L} + \frac{(2L)^{1/2}}{T^{\alpha/2}}\;, \qquad t \in [0,T]\setminus J_\alpha(T)\;,$$

for some $C_5 > 0$. If we now choose $L = T^{(\alpha+2\beta)/3} \le T^{1/2}$, we arrive at

$$\sup\Big\{|g(x_1,t)|\,\Big|\, |x_1| \le T^{(\alpha+2\beta)/3}\Big\} \;\le\; \frac{C_6}{T^{(\alpha-\beta)/3}}\;, \qquad t \in [0,T]\setminus J_\alpha(T)\;, \tag{72}$$

for some $C_6 > 0$.

On the other hand, we know that $\int_{-L}^{L} h(x_1,t)^2 \,\mathrm{d}x_1 \le T^{-\alpha}$ when $t \in [0,T]\setminus J_\alpha(T)$. In addition, it follows from (68) that

$$|\partial_1 h(x_1,t)| \;\le\; \Big(\int_{\mathbb{T}} |\partial_{12}^2 \omega(x_1,x_2,t)|^2 \,\mathrm{d}x_2\Big)^{1/2} \;\le\; M_1\;, \qquad x_1 \in \mathbb{R}\;, \quad t \in [0,T]\;.$$

Thus Lemma 6.7 below implies that

$$\sup_{|x_1|\le L} |h(x_1,t)| \;\le\; C \max\Big(\frac{M_1^{1/3}}{T^{\alpha/3}}\,,\,\frac{1}{L^{1/2}T^{\alpha/2}}\Big) \;=\; \frac{C_7}{T^{\alpha/3}}\;, \qquad t \in [0,T]\setminus J_\alpha(T)\;, \tag{73}$$

for some $C_7 > 0$. Combining (72), (73) we obtain

$$\sup_{|x_1|\le L}\sup_{x_2\in\mathbb{T}} |\omega(x_1,x_2,t)| \;\le\; \sup_{|x_1|\le L}|g(x_1,t)| + \sup_{|x_1|\le L}|h(x_1,t)| \;\le\; \frac{C_6}{T^{(\alpha-\beta)/3}} + \frac{C_7}{T^{\alpha/3}} \;\le\; \frac{C_8}{T^{(\alpha-\beta)/3}}\;,$$

for all $t \in [0,T]\setminus J_\alpha(T)$. Since $L = T^{(\alpha+2\beta)/3}$ and $\mathrm{meas}(J_\alpha(T)) \le C_4 T^{\alpha+1/2}$, this gives (69). □

Finally, we state and prove two elementary interpolation lemmas which were used in the argument above.

Lemma 6.6. *Assume that $g \in C^1([0,L])$ satisfies*

$$\int_0^L g'(x)^2 \,\mathrm{d}x \;\le\; \epsilon\;, \qquad \text{and} \qquad \Big|\int_0^L g(x)\,\mathrm{d}x\Big| \;\le\; M\;.$$

Then $\sup_{0\le x\le L} |g(x)| \;\le\; \dfrac{M}{L} + (L\epsilon)^{1/2}$.

Proof. We decompose $g(x) = \bar g + h(x)$, where $\bar g = L^{-1}\int_0^L g(y)\,\mathrm{d}y$. Since h has zero mean over $[0,L]$, there exists $x_0 \in [0,L]$ such that $h(x_0) = 0$. For all $x \in [0,L]$, we thus have

$$|h(x)| \;=\; \Big|\int_{x_0}^x h'(y)\,\mathrm{d}y\Big| \;\le\; |x-x_0|^{1/2}\Big(\int_{x_0}^x h'(y)^2\,\mathrm{d}y\Big)^{1/2} \;\le\; (L\epsilon)^{1/2}\;.$$

Since $|\bar g| \le M/L$ by assumption, we obtain the desired result. □

Lemma 6.7. *Assume that $h \in C^1([0,L])$ satisfies*

$$\int_0^L h(x)^2 \,\mathrm{d}x \;\le\; \epsilon\;, \qquad \text{and} \qquad \sup_{0\le x\le L} |h'(x)| \;\le\; M\;.$$

Then $\sup_{0\le x\le L} |h(x)| \;\le\; \max\Big((3M\epsilon)^{1/3}\,,\,(3\epsilon/L)^{1/2}\Big)$.

Proof. If $x_0 \in [0,L]$ is a point where $|h(x)|$ is maximal, we have

$$|h(x)| \;\ge\; \|h\|_{L^\infty} - M|x-x_0|\;, \qquad x \in [0,L]\;,$$

where $\|h\|_{L^\infty} = \sup\{|h(x)|\,|\,0 \le x \le L\}$. By straightforward calculations, we thus find

$$\epsilon \;\ge\; \int_0^L \Bigl(\|h\|_{L^\infty} - M|x-x_0|\Bigr)_+^2 \,\mathrm{d}x \;\ge\; \frac{1}{3}\min\Bigl(\frac{\|h\|_{L^\infty}^3}{M}\,,\,\|h\|_{L^\infty}^2 L\Bigr)\;.$$

This gives the desired result. □

7 Appendix: Proof of Lemma 3.3

In this final section, we study the positive solutions of the ordinary differential equation

$$h'(r) + \frac{N-1}{r}h(r) \;=\; h(r)^2 - 1\;, \qquad r > 0\;, \tag{74}$$

and we prove Lemma 3.3. All arguments are quite standard, and are reproduced here for the reader's convenience. Although the unique positive solution of (74) is given by an explicit formula which can be found using a Cole-Hopf transformation, we find it more instructive to prove the first part of Lemma 3.3, including the asymptotics (22) and (23), without using this explicit representation, which will be derived only at the end. We proceed in several steps:

1. *Construction of the stable manifold.* The nonautonomous ODE (74) has an asymptotic equilibrium $h = 1$ at $r = +\infty$, with a one-dimensional *stable manifold* which contains precisely the solution h_N we are looking for. To construct the stable manifold, we set

$$h(r) \;=\; 1 + \frac{N-1}{2r} + g(r)\;, \tag{75}$$

and obtain for g the ODE

$$g'(r) \;=\; 2g(r) + g(r)^2 - \frac{(N-1)(N-3)}{4r^2}\;. \tag{76}$$

As is easily seen, any solution of (76) that stays bounded as $r \to +\infty$ satisfies the integral equation

$$g(r) \;=\; \int_r^\infty e^{2(r-s)}\left(\frac{(N-1)(N-3)}{4s^2} - g(s)^2\right)\mathrm{d}s\;. \tag{77}$$

Now, fix $R \in (0,1)$ and take $r_0 > 0$ large enough so that $|(N-1)(N-3)| \le 4Rr_0^2$. It is then straightforward to verify that the right-hand side of (77) defines a strict contraction in the closed ball

$$B_{r_0}(R) \;=\; \Bigl\{g \in C^0([r_0,+\infty))\,\Big|\, \sup_{r\ge r_0}|g(r)| \le R\Bigr\}\;,$$

hence has a unique fixed point $g_N \in B_{r_0}(R)$ which, by construction, is a solution of (76) for $r > r_0$. Since g_N satisfies (77), it is clear that $g_N(r) \to 0$ as $r \to \infty$. Thus defining

$$h_N(r) \;=\; 1 + \frac{N-1}{2r} + g_N(r)\;, \qquad r > r_0\;, \tag{78}$$

we see that h_N satisfies (74) and $h_N(r) \to 1$ as $r \to \infty$. By construction h_N is the unique solution of (74) that converges to 1 as $r \to +\infty$.

Remark 7.1. If $N = 1$ or $N = 3$, it is clear from (77) that $g_N \equiv 0$, so that $h_1(r) = 1$ and $h_3(r) = 1 + 1/r$.

2. *Asymptotic behavior as* $r \to +\infty$. Using (77), we easily find

$$g_N(r) \;=\; \frac{(N-1)(N-3)}{8r^2} + \mathcal{O}\Big(\frac{1}{r^3}\Big)\;, \qquad \text{as} \quad r \to +\infty\;.$$

Thus (22) holds, and in view of (76) we also have $g_N'(r) = \mathcal{O}(r^{-3})$ as $r \to +\infty$, so that

$$h_N'(r) \;=\; -\frac{N-1}{2r^2} + \mathcal{O}\Big(\frac{1}{r^3}\Big)\;, \qquad \text{as} \quad r \to +\infty\;.$$

If $N \ge 2$, this shows that $h_N'(r) < 0$ for $r > 0$ sufficiently large.

3. *Global monotonicity.* We assume from now on that $N \ge 2$. Solving (74) backwards, we construct (for some $r_* \ge 0$) a maximal solution $h_N : (r_*, +\infty) \to \mathbb{R}$ which coincides with (78) for $r > r_0$. We claim that $h_N'(r) < 0$ for all $r > r_*$. Indeed, assume on the contrary that there exists $\bar r > r_*$ such that $h_n'(\bar r) = 0$ and $h_N'(r) < 0$ for all $r > \bar r$. Then

$$h_N''(\bar r) \;=\; \Big(2h_N(\bar r) - \frac{N-1}{\bar r}\Big)h_N'(\bar r) + \frac{N-1}{\bar r^2}h_N(\bar r) \;>\; 0\;,$$

because $h_N'(\bar r) = 0$ and $h_N(\bar r) > 1$. This implies that $h_N'(r) > 0$ for $r > \bar r$ close enough to $\bar r$, in contradiction with the definition of $\bar r$. Thus $h_N'(r) < 0$ for all $r > r_*$, and using (74) we deduce

$$1 \;<\; h_N(r) \;<\; \frac{N-1}{2r} + \sqrt{1 + \frac{(N-1)^2}{4r^2}} \qquad \text{for all} \quad r > r_*\;. \tag{79}$$

This estimate shows in particular that h_N cannot blow up at a finite point $r > 0$, hence we necessarily have $r_* = 0$ and $h_N'(r) < 0$ for all $r > 0$.

4. *Asymptotic behavior as* $r \to 0$. Setting $f_N = 1/h_N$ we obtain the ODE

$$f_N'(r) \;=\; \frac{N-1}{r}f_N(r) + f_N(r)^2 - 1\;, \qquad r > 0\;,$$

which is very similar to (74). In particular, we have

$$f_N(r) \;=\; r^{N-1}f_N(1) + \int_r^1 \Big(\frac{r}{s}\Big)^{N-1}(1 - f_N(s)^2)\,\mathrm{d}s\;, \qquad 0 < r < 1\;. \tag{80}$$

A direct study of (80) gives the following asymptotic expansion as $r \to 0$:

$$f_2(r) \;=\; r\log\frac{1}{r} + Cr + \mathcal{O}\Big(r^3\Big(\log\frac{1}{r}\Big)^2\Big) \quad \text{for some } C \in \mathbb{R}\;,$$

whereas $f_3(r) = r/(1+r)$ and

$$f_4(r) \;=\; \frac{r}{2} + \mathcal{O}\Big(r^3\log\frac{1}{r}\Big)\;, \qquad f_N(r) \;=\; \frac{r}{N-2} + \mathcal{O}(r^3) \quad \text{if } N \ge 5\;.$$

Since $h_N = 1/f_N$, this proves (23).

5. *Uniqueness and threshold behavior.* We first study the solutions of (74) that lie above h_N. Assume that h is a solution of (74) such that $h(r_1) > h_N(r_1)$ for some $r_1 > 0$. In particular, we have that $h(r) > h_N(r) \ge 1$ for all $r > r_1$. If $h'(r) \le 0$ for all $r \ge r_1$, then $h(r)$ converges to some limit $\ell \geq 1$ as $r \to +\infty$, and using (74) we easily see that $\ell^2 = 1$, hence $\ell = 1$. This implies that the function $g(r)$ defined by (75) is small for sufficiently large $r > 0$ and satisfies the integral equation (77), hence coincides with $g_N(r)$, which is of course impossible since $g(r_1) > g_N(r_1)$. Thus there must exist $r_2 \geq r_1$ such that $h(r_2) > h_N(r_2)$ and $h'(r_2) > 0$. If we now choose $r_3 > r_2$ so that $h'(r) > 0$ for all $r \in [r_2, r_3]$, we have on that interval $h(r) > (N-1)/(2r)$, hence

$$h''(r) \;=\; h'(r)\left(2h(r) - \frac{N-1}{r}\right) + \frac{N-1}{r^2}h(r) \;>\; 0\;, \qquad r_2 \leq r \leq r_3\;.$$

This argument shows that $h(r)$ is convex for $r \geq r_2$, and blows up at some finite point $r^* > r_2$. Indeed, if $h(r)$ was defined for all $r > r_2$, the convexity would imply that $h(r) \to +\infty$ as $r \to +\infty$, so that h would satisfy the differential inequality $h'(r) \geq \frac{1}{2}h(r)^2$ for all sufficiently large r, which is impossible because this inequality has no global positive solutions.

We next consider solutions of (74) that lie below h_N. Assume that h is a solution of (74) such that $0 < h(r_1) < h_N(r_1)$ for some $r_1 > 0$, so that $h(r) < h_N(r)$ for all $r \geq r_1$. If $h(r) \geq 0$ for all $r \geq r_1$, we have $h'(r) < 0$ for all $r \geq r_1$, hence $h(r)$ converges to some limit $\ell \in [0,1]$ as $r \to +\infty$. But the same arguments as above show that $\ell^2 = 1$ and $\ell \neq 1$, which is a contradiction. So the solution $h(r)$ must necessarily change sign for $r > r_1$.

It follows in particular that h_N is the unique positive solution of (74) that is defined for all $r > 0$.

6. *Explicit representation.* Let $u(r) = \exp(-\int_1^r h_N(s)\,\mathrm{d}s)$. Then u solves the second order *linear* ODE

$$u''(r) + \frac{N-1}{r}u'(r) \;=\; u(r)\;, \qquad r > 0\;,$$

and $u(r)$ decays exponentially to zero as $r \to +\infty$. Setting $\nu = 1 - N/2$ and $u(r) = r^\nu v(r)$, we obtain for v the differential equation

$$r^2 v''(r) + r v'(r) - (r^2 + \nu^2)v(r) \;=\; 0\;, \qquad r > 0\;,$$

which defines the modified Bessel functions, see [1, Eq. 9.6.1]. Since $v(r)$ decays exponentially as $r \to +\infty$, we must have $v(r) = CK_\nu(r)$ for some $C > 0$, see [1, Section 9.7]. Thus $u(r) = Cr^\nu K_\nu(r)$, and using [1, Eq. 9.6.28] we also find $u'(r) = -Cr^\nu K_{\nu-1}(r)$. Since $K_{-\nu}(r) = K_\nu(r)$ by [1, Eq. 9.6.6], we conclude that

$$h_N(r) \;=\; -\frac{u'(r)}{u(r)} \;=\; \frac{K_{\nu-1}(r)}{K_\nu(r)} \;=\; \frac{K_{\frac{N}{2}}(r)}{K_{\frac{N}{2}-1}(r)}\;, \qquad r > 0\;,$$

which proves (24). The proof of Lemma 3.3 is now complete. □

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