Vanishing viscosity limit for axisymmetric vortex rings

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Abstract

For the incompressible Navier-Stokes equations in $\mathbb{R}^3$ with low viscosity $\nu > 0$, we consider the Cauchy problem with initial vorticity ω_0 that represents an infinitely thin vortex filament of arbitrary given strength Γ supported on a circle. The vorticity field $\omega(x,t)$ of the solution is smooth at any positive time and corresponds to a vortex ring of thickness $\sqrt{\nu t}$ that is translated along its symmetry axis due to self-induction, an effect anticipated by Helmholtz in 1858 and quantified by Kelvin in 1867. For small viscosities, we show that $\omega(x,t)$ is well-approximated on a large time interval by $\omega_{\rm lin}(x-a(t),t)$, where $\omega_{\rm lin}(\cdot,t) = \exp(\nu t\Delta)\omega_0$ is the solution of the heat equation with initial data ω_0, and $\dot a(t)$ is the instantaneous velocity given by Kelvin's formula. This gives a rigorous justification of the binormal motion for circular vortex filaments in weakly viscous fluids. The proof relies on the construction of a precise approximate solution, using a perturbative expansion in self-similar variables. To verify the stability of this approximation, one needs to rule out potential instabilities coming from very large advection terms in the linearized operator. This is done by adapting V. I. Arnold's geometric stability methods developed in the inviscid case $\nu = 0$ to the slightly viscous situation. It turns out that although the geometric structures behind Arnold's approach are no longer preserved by the equation for $\nu > 0$, the relevant quadratic forms behave well on larger subspaces than those originally used in Arnold's theory and interact favorably with the viscous terms.

1 Introduction and main result

We consider the Cauchy problem for the 3d incompressible Navier-Stokes equations in the vorticity form

$$\partial_t\omega + u\cdot\nabla\omega - \omega\cdot\nabla u \,=\, \nu\Delta\omega \quad \text{in } \mathbb{R}^3\times(0,\infty)\,, \tag{1.1}$$

$$\omega|_{t=0} \,=\, \omega_0 \qquad \text{in } \mathbb{R}^3\,, \tag{1.2}$$

where we use the familiar notation $\omega(x,t)$ for the vorticity of the fluid, and the velocity $u(x,t)$ is given by the Biot-Savart law $u(x,t) = (4\pi)^{-1}\int_{\mathbb{R}^3}\omega(y,t)\wedge(x-y)\,|x-y|^{-3}\,\mathrm{d}y$. Our focus is on the special case where the initial vorticity $\omega_0 = \Gamma\delta_{\mathsf{C}}$ is an idealized vortex filament given by a current[1] of strength Γ concentrated on an oriented circle $\mathsf{C}\subset\mathbb{R}^3$. More precisely, ω_0 is the vector-valued measure on $\mathbb{R}^3$ defined by the identity

$$\langle\omega_0\,,\,\varphi\rangle \,=\, \Gamma\sum_{i=1}^3\int_{\mathsf{C}}\varphi_i\,\mathrm{d}x_i\,, \tag{1.3}$$

which is assumed to hold for any continuous test function $\varphi = (\varphi_1,\varphi_2,\varphi_3)$. In the well-known analogy between fluid mechanics and electromagnetism, ω_0 can be thought of as an electric

[1] Here the term *current* can be understood in its heuristic meaning but also in the technical meaning of the geometric measure theory, such as in [24].

current of intensity Γ flowing through an infinitely thin wire represented by the circle C; the direction of the current is then given by the orientation of the circle and the sign of Γ. Vortex filaments were already considered in the classical 1858 paper of Helmholtz [40] which deals with the inviscid case $\nu = 0$ corresponding to the Euler equation. Helmholtz argued that a circular vortex filament of zero thickness would move with infinite speed along its symmetry axis. In 1867 Kelvin [44] established the following formula for the speed of a vortex ring of small but finite thickness $d > 0$ and radius $r_0 \gg d$:

$$V \approx \frac{\Gamma}{4\pi r_0}\Bigl(\log\frac{8r_0}{d} - C\Bigr), \tag{1.4}$$

where $C \in \mathbb{R}$ is a dimensionless constant that depends on the distribution of vorticity inside a cross section of the ring. If the distribution is uniform, which is probably the assumption made by Kelvin, the relevant value is $C = \frac{1}{4}$, see [46, §163].

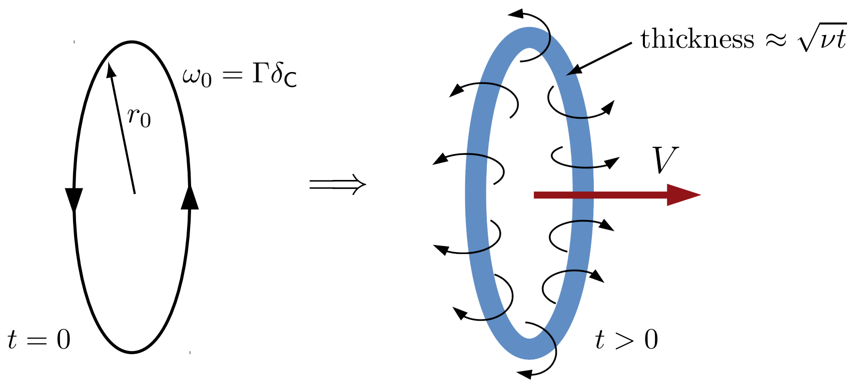

Figure 1: An illustration our main result. Starting from a vortex filament supported on an oriented circle C, the solution of the Navier-Stokes equation evolves into a viscous vortex ring of thickness proportional to $\sqrt{\nu t}$ which moves along the symmetry axis at a speed V given by the Kelvin-Saffman formula (1.5). In the right picture, the vortex lines are circles that fill the solid torus depicted in blue, whereas the black arrows denote the trajectories of the fluid particles.

In the viscous case $\nu > 0$, the solution originating from the singular filament $\omega_0 = \Gamma\delta_{\mathsf{C}}$ becomes smooth for any positive time $t > 0$ and is expected to represent a viscous vortex ring of thickness proportional to $\sqrt{\nu t}$, as long as that quantity is small compared to the radius r_0 of the ring. Based on Kelvin's formula one anticipates that the vortex ring will move at the (time-dependent) speed (1.4) with $d = \sqrt{\nu t}$ and C corresponding to a Gaussian distribution of vorticity inside the core. To the best of our knowledge, the relevant value of the constant C was first determined by Saffman in [55]. The computation gives $C = \frac{3}{2}\log(2) + \frac{1}{2}(1 - \gamma_E)$, where $\gamma_E \approx 0.5772...$ is Euler's constant.[2] We will refer to the formula

$$V(t) = \frac{\Gamma}{4\pi r_0}\left(\log\frac{8r_0}{\sqrt{\nu t}} - \frac{3}{2}\log 2 - \frac{1}{2}(1-\gamma_E)\right) \tag{1.5}$$

as the *Kelvin-Saffman formula* for the speed of a viscous vortex ring.

When the initial circle C is parametrized by $(r_0\cos\theta, r_0\sin\theta, 0)$ for $\theta \in [0, 2\pi]$, with the orientation in the direction of increasing θ, the translational motion will be in the positive direction along the x_3-axis if $\Gamma > 0$.

[2] Fraenkel's paper [26] contains formulae that can be used to obtain the same result. Tung and Ting in [56] also give a formula for C of a similar nature, which however needs a small correction.

It is proved in [33] that the Cauchy problem (1.1), (1.2) with $\omega_0 = \Gamma\delta_{\mathsf{C}}$ has a unique solution in natural classes of axisymmetric fields. The main result of the present paper, Theorem 1.1 below, describes the precise behavior of that solution in the low viscosity regime where the *circulation Reynolds number* $\mathrm{Re} := \Gamma/\nu$ is large. Our description is valid on a time interval whose length is intermediate between the *advection time* and the *diffusion time*, defined respectively as

$$T_{\mathrm{adv}} = \frac{r_0^2}{\Gamma}\,, \qquad T_{\mathrm{dif}} = \frac{r_0^2}{\nu}\,. \tag{1.6}$$

Note that $T_{\mathrm{adv}} \ll T_{\mathrm{dif}}$ when $\mathrm{Re} \gg 1$. The leading term in our approximation is exactly the one suggested by the Kelvin-Saffman formula together with the simplest diffusion heuristics: The ring diffuses according to the linear heat equation, and translates with speed (1.5) along its symmetry axis. Denoting by $\omega_{\mathrm{lin}}(x,t)$ the solution of the heat equation $\partial_t\omega = \nu\Delta\omega$ with initial condition $\omega|_{t=0} = \omega_0 = \Gamma\delta_{\mathsf{C}}$, and defining $\|\eta\| = \|\eta/r\|_{L^1(\mathbb{R}^3)}$, where $r = r(x)$ is the distance from x to the symmetry axis, we can state our main result as follows.

Theorem 1.1. *There exist dimensionless constants $K > 0$, $R_0 > 0$, and $\sigma \in (0, \frac{1}{3})$ such that, for all $\Gamma > 0$, all $r_0 > 0$, and all $\nu > 0$ satisfying $\mathrm{Re} := \Gamma/\nu \ge R_0$, the following holds. If $\omega_0 = \Gamma\,\delta_{\mathsf{C}}$ where C is an oriented circle of radius r_0, the unique axisymmetric solution ω of the Cauchy problem (1.1), (1.2) established in [33] can be expressed for $t \in (0, T_{\mathrm{adv}}\,\mathrm{Re}^\sigma)$ as*

$$\omega(x,t) = \omega_{\mathrm{lin}}(x - a(t), t) + \omega_{\mathrm{cor}}(x,t)\,, \quad \text{with} \quad \|\omega_{\mathrm{cor}}(\cdot\,,t)\| \le K\,\Gamma\left(\frac{\sqrt{\nu t}}{r_0}\right)^{1-3\sigma}, \tag{1.7}$$

where $a(t)$ describes the translation of the ring along its symmetry axis according to the Kelvin-Saffman formula (1.5). Specifically, if $\mathsf{C} = \{(r_0\cos\theta, r_0\sin\theta, 0)\,;\, \theta \in [0,2\pi]\}$ is oriented positively, one has $a(t) = (0,0,a_3(t))$ where $a_3(t) = \int_0^t V(s)\,\mathrm{d}s$ and V is given by (1.5).

An extended version of our result, including a more precise approximate solution and a much stronger control of the correction term, is formulated as Theorem 2.6 below, after the necessary notation has been introduced in Section 2. In particular, the exponent $1 - 3\sigma$ in (1.7) can be improved to 1 if we take into account higher-order corrections to the Kelvin-Saffman formula.

In Theorem 1.1, the constants K and R_0 are large, whereas the exponent $\sigma > 0$ is taken small. We conjecture that an approximation result of the form (1.7) remains valid on longer time scales of order $T_{\mathrm{adv}}\mathrm{Re}^{\sigma'}$ with σ' close to 1, but we have no proof so far. In view of (1.4), the advection time T_{adv} can be interpreted as the time needed for a vortex ring of circulation Γ and small (but not infinitesimal) aspect ratio d/r_0 to travel over a distance comparable to its radius r_0. In contrast, the diffusion time $T_{\mathrm{dif}} = T_{\mathrm{adv}}\mathrm{Re}$ is the time at which the diffusion length $\sqrt{\nu t}$ becomes equal to the radius r_0, so that the vortex ring structure is essentially lost. The assumption that $\mathrm{Re} \gg 1$ means that the vortex ring can travel along its symmetry axis over a very long distance, compared to its radius r_0, before being destroyed by diffusion. In particular, on the time scale $T = T_{\mathrm{adv}}\,\mathrm{Re}^\sigma$ where Theorem 1.1 provides a rigorous control we find, using (1.5) and (1.6),

$$|a(T)| = \int_0^T V(t)\,\mathrm{d}t = \frac{r_0}{4\pi}\,\mathrm{Re}^\sigma\Big(\log\big(\mathrm{Re}^{\frac{1-\sigma}{2}}\big) + C'\Big)\,,$$

for some constant C'. Obviously the quantity in the right-hand side grows boundlessly as $\mathrm{Re} \to +\infty$, even in the limiting case where $\sigma = 0$ and $T = T_{\mathrm{adv}}$.

It is instructive to compare the situation for vortex rings with the case of a rectilinear filament, where the vorticity is initially concentrated on a straight line ℓ in $\mathbb{R}^3$. Let us denote this initial vorticity field by $\omega_0 = \Gamma\delta_\ell$. In that case the solution of the full vorticity equation is given by $\omega(\cdot\,,t) = \Gamma e^{\nu t\Delta}\delta_\ell$, because the nonlinear terms vanish identically due to symmetries when evaluated on the solution of the heat equation $\partial_t \omega = \nu \Delta \omega$. Although the evolution of the velocity and the vorticity fields does not look very dramatic, the fluid particles in the vicinity of ℓ do move at very large speeds when νt is small, and the inertial forces in the fluid are therefore significant. However, these forces are exactly balanced by the pressure gradient.

When the rectilinear filament is bent into a vortex ring (as already considered in Helmholtz's 1858 paper), the inertial forces are no longer in balance and the ring has to move. To achieve a relatively smooth motion, the bent vortex has to be "well-prepared" so that the inertial forces generated by the fast-moving fluid particles are still mostly canceled and do not generate fast oscillations. The initial condition $\omega_0 = \Gamma \delta_{\mathsf{C}}$ has the advantage of letting the equation to adjust the vorticity field into a well-prepared state without trying to achieve this "by hand". Quite remarkably, this adjustment is made in exactly such a way that the oscillations are avoided.[3] The largest inertial forces still cancel and the situation remains somewhat close to the rectilinear case with only two significant differences: (a) some motion of the ring along its axis of rotational symmetry is needed to balance the forces, but the speed of this motion is much lower than the speed of the fast particles in the fluid; (b) once the thickness of the ring becomes comparable to its radius, new effects (not discussed in this work) appear.

Theorem 2.6 can be compared with a result by Brunelli and Marchioro [9], where the authors consider general axisymmetric vorticities that are initially supported in a torus of major radius $r_0 > 0$ and minor radius $0 < \rho_0 \ll r_0$. Under certain technical assumptions, they show that the solution of the Navier-Stokes equations remains essentially concentrated in a thin torus which moves along the symmetry axis according to Kelvin's law. If the vortex strength Γ is kept fixed, the solution is under control on a time interval of length $T \log(r_0/\rho_0)^{-1}$, which therefore shrinks to zero as $\rho_0 \to 0$. Also, the authors assume that the viscosity satisfies $\nu T \le \rho_0^2$ (up to a logarithmic correction), so that the viscous effects can be treated perturbatively. In the same spirit, the case of several vortex rings with a common symmetry axis was recently considered in [11], see also [7, 12] for similar results in the inviscid case. Our Theorem 2.6 is restricted to specific initial data, which correspond to $\rho_0 = 0$, but it provides a more precise control of the solution on a much longer time scale, and the diffusive effects are not treated perturbatively.

1.1 Main ideas of the proof of Theorem 1.1

Our analysis starts with the construction of a precise approximation of the solution $\omega(x, t)$. This is achieved by writing the solution in suitable self-similar coordinates that capture well the singular behavior of the solution at $t = 0$ through explicit rescalings of a smooth "profile" η that can be thought of as a perturbation of a suitable Gaussian η_0. The perturbed profile η is expressed as an asymptotic series in the time-dependent parameter $\epsilon = \sqrt{\nu t}/\bar{r}$, with $\bar{r} = \bar{r}(t)$ denoting the instantaneous radius of the ring. To achieve a precision that is sufficient for our purposes, we need an expansion up to the fourth order: $\eta = \eta_0 + \epsilon\eta_1 + \epsilon^2\eta_2 + \epsilon^3\eta_3 + \epsilon^4\eta_4 + \eta_{\mathrm{cor}}$. The profiles η_j with $j \ge 1$ are obtained by inverting operators containing the small parameter $\delta = 1/\mathrm{Re} = \nu/\Gamma$, and in that sense we really deal with a two-parameter expansion. As far as we know, this is somewhat different from other expansions in the literature, such as [14, 55, 30]. A one-parameter formal expansion in ϵ would treat δ as $\sim \epsilon^2$, in view of the relation $\bar{r}^2\epsilon^2 = \delta\,\Gamma\, t$. Keeping both parameters makes it easier to cover the regimes when ϵ^2 and δ are not really comparable, as is the case for very small and very large times. For the sake of completeness, we mention that the vorticity profiles η_j for $j \ge 1$ can also depend on $\log \epsilon$. That phenomenon is well known, and the leading term in the speed of the ring is precisely related to choosing a moving coordinate system in which the terms with $\log \epsilon$ in η_1 are eliminated.

[3]In the related situation of interacting vortices in $\mathbb{R}^2$, this was already observed in [31].

The main difficulty in the proof of Theorem 1.1, however, is not in the computation of an approximate solution, but in showing that the true solution remains close to this approximation on a large time interval. This requires fairly strong stability properties for the linearization of the vorticity equation at the approximate solution, which is very singular in the low viscosity regime. When the initial condition corresponds to a finite number of parallel rectilinear vortices, a stability analysis was carried on in [31] by using suitable weighted L^2 spaces adapted to the specific features of the rectilinear vortices with Gaussian profiles. In the vortex ring case the nonlinearity of the equations starts affecting the formal expansions earlier and it is unclear whether the setup in [31] can be used to show that the vortex ring will not disintegrate on time-scales approaching zero as $\nu \to 0$. A recent important work [6] extends some of the 2d methods for proving stability to a relevant 3d situation, but the length of the time interval over which the solution is under control may approach 0 as ν tends to 0.

In physical flows and numerical experiments one observes a remarkable degree of stability of vortex rings as well as signs of instabilities with respect to non-axisymmetric perturbations, see for example [59, 52]. At a rigorous mathematical level the stability issues have not been well understood. In fact, when Γ/ν is not small, not only the stability, but even the uniqueness of the solutions of the Cauchy problem above with $\omega_0 = \Gamma\,\delta_{\mathsf{C}}$ (and also with $\omega_0 = \Gamma\,\delta_\ell$) is open in classes of solutions that do not share the symmetry of the initial data.

In the 1960s, V. I. Arnold suggested a variational method for proving stability of steady solutions to Euler's equation based on a geometric insight that can be summarized as follows, using the Hamiltonian setup of [49]:

(a) The incompressible Euler equation can be viewed as a Poisson system with a Hamiltonian function given by the usual kinetic energy.

(b) The steady states are critical points of the energy on the symplectic leaves. The latter coincide with the *coadjoint orbits*, called just *orbits* in what follows, of the group of the volume-preserving diffeomorphisms of the fluid domain acting by push-forward on the vorticity fields.

(c) When the critical point is a local maximum or a local minimum on an orbit, the corresponding steady state should be stable.

These ideas fit into a broader family of methods used for proving stability of solutions of Hamiltonian systems by invoking extremality properties of a conserved quantity under constraints given by other conserved quantities. For example, a circular planetary orbit in the three-dimensional Kepler problem is stable because it minimizes energy for a given angular momentum.[4] In the applications to vortex rings, it is natural to restrict the analysis to *axisymmetric flows with no swirl*, which means that the velocity field is invariant under rotations about a symmetry axis and under reflection across any plane containing that axis.

Arnold's method has found many applications to Euler flows in 2d (see, for example, [4]), and has also been invoked in the work of Benjamin [8] on inviscid vortex rings that is directly relevant for our purposes here. Although some arguments in [8] may not be fully rigorous, they provide important suggestions for investigating stability of inviscid vortex rings in the class of axisymmetric solutions. In a different direction, the conservation of energy, impulse, and vortex strength has been used to control the evolution of a general class of concentrated solutions of the Euler equations describing vortex rings, see for example [7, 12].

There is voluminous literature on the stationary vortex ring solutions of the Euler equation, starting with the explicit solution of Hill [42], see e.g. [1, 2, 5, 10, 15–18, 26–29, 53, 54, 57]. Many of these works rely in one way or another on variational aspects of the underlying PDEs that have connections to the work of Arnold and Benjamin, albeit in an indirect way. Roughly speaking, if

[4] It is well-known that this is no longer the case in dimension four and higher [39].

we compare Arnold's setup to the maximization of a function $f(x)$ under constraints $g_j(x) = c_j$, one can compare some of the variational approaches in the references above to searching for critical points of $f(x) - \lambda_1 g_1(x) - \cdots - \lambda_m g_m(x)$ when the Lagrange multipliers $\lambda_1, \ldots, \lambda_m$ are given. Readers interested in related links can find more details in [34].

In our asymptotic expansions of the solutions of (1.1), (1.2) inviscid vortex ring solutions can also be discerned. For each fixed time $t > 0$ the third-order expansion in our parameter $\epsilon = \sqrt{\nu t}/\bar{r}$ is a good approximation of an inviscid vortex ring, at least in the limiting case where our second parameter $\delta = \nu/\Gamma$ is taken equal to zero. A part of our stability analysis can be thus understood in terms of the stability properties of this ring, see Remark 2.3 and Section 3.8 for more details.

If one wishes to apply Arnold's ideas to the solutions of (1.1), (1.2), there appears to be a non-trivial obstacle: The viscous flows do not preserve the geometric structures that are at the basis of Arnold's considerations and the influence of the viscosity is too large to treat the viscous terms perturbatively. At first this may seem to be a serious problem: If the preservation of the orbits and the Hamiltonian nature of the equations are violated beyond the reach of the perturbative approach (such as [9, 11]), can the geometric structure relying on maximization of the energy on symplectic leaves be helpful? In our previous work [34] we showed, in a much simpler situation, that the answer to this question can be positive. It turns out that the quadratic forms coming up in Arnold's stability analysis, although originally envisaged as quadratic forms on the tangent spaces to the orbits, are often well-behaved on much larger subspaces. This point can still be conceptually explained by the geometry of the Euler equation. What we find more surprising is that Arnold's forms also have favorable behavior with respect to the dissipative term generated by the viscosity. We can show this by direct calculation, but we do not have a good conceptual explanation of this fortuitous circumstance. In the paper [34] we showed that the above ideas can be used to prove the stability of the rectilinear vortex solution (in self-similar variables) with respect to perturbations for which the vorticity field stays parallel to the original vortex line. This result has been established previously by a different method [36]. The new proof in [34] can be thought of as a proof of concept that the ideas of Arnold can be applied even in the presence of viscosity. The application to vortex rings presented here is more complicated, but we are not aware of any simpler approach in that case.

To conclude this section, we mention a recent important work by Dávila, Del Pino, Musso, and Wei [22], where the authors rigorously establish "leapfrogging" of inviscid vortex rings. The construction uses "gluing methods" that were previously developed in [21] to study the interaction of vortices in the plane. The approach shares similarities with ours, as it relies on the construction of accurate approximate solutions and their stability analysis. The stability part also uses an Arnold-type energy functional, although the connection to Arnold's geometric viewpoint is not explicit. In the inviscid case, the expansion parameter $\epsilon > 0$ does not need to change during the motion, and the solution is controlled on a time interval of size $T/|\log \epsilon|$. This is shorter than in Theorem 1.1, but our result is restricted to a single vortex ring, and uses viscous effects. One expects the viscosity to have a stabilizing role, but its interaction with the geometric structures of Arnold requires a careful analysis. One needs to show that the solutions will stay "coherent" for a sufficiently long time and the viscous effects will not be enhanced too much by the high velocities inside and near the ring.

Another description of the leapfrogging motion of vortex rings, in a different parameter regime, can be found in [13].

1.2 Comments on the local induction approximation for general filaments

The problem studied in this paper can be considered as a special case of the viscous version of the *local induction approximation conjecture.* In the setup considered here the conjecture could be formulated as follows: if we replace the circle C be a general smooth closed curve and consider the Cauchy problem (1.1), (1.2) with $\omega_0 = \Gamma\delta_{\mathsf{C}}$, the motion of the filament C should still be determined essentially by two effects: the diffusion, which transforms the filament into a vortex tube of thickness $d(t) \approx \sqrt{\nu t}$ at time t, and the advection by the self-induced velocity field. The latter is described by a geometric equation that represents an extension of Kelvin's formula to general smooth curves, and was derived by Da Rios [20] in 1906:

$$\mathbf{V} \approx \left(\frac{\Gamma}{4\pi r}\log\frac{8r}{d}\right)\mathbf{b}\,. \tag{1.8}$$

Here $\mathbf{V}$ is the vector representing the local velocity of the filament, $\mathbf{b}$ denotes the local binormal vector, r is the local radius of the curvature, and d denotes the local thickness of the filament. (All these quantities may be time- and position-dependent.) In the limit $\nu \to 0$ the approximation should be valid until the geometric evolution of the curve by the binormal flow leads to a self-intersection. For general initial curves C the time of the first self-intersection may be approaching zero as ν approaches zero. The first significant step towards this general case, a local-in time well-posedness result for a fixed $\nu > 0$, was obtained in [6]. Some formal computations related to the conjecture are presented in [14] and we also refer the reader to the important conditional result in [43]. Our result can be viewed as a proof of the viscous formulation of the conjecture in the special case where the curve C is a circle.

For a general smooth curve C and a sufficiently small Reynolds number Γ/ν, the Cauchy problem (1.1), (1.2) is globally well-posed as first shown in [38] by a perturbation analysis, see also [45] for a more general result in the same spirit. Accurate calculations in the recent noteworthy preprint [25] suggest that even in these perturbative regimes the motion by the local induction approximation can still be discerned, although its effect is small and the distance traveled by the ring due to the velocity field (1.8) seems to be quite shorter than its thickness.

The general case of the local induction approximation conjecture for the setup considered in this paper seems to be difficult. In fact, it is unclear whether the strongest version of the conjecture is valid even for small perturbations of the circle, as the perturbed filaments may perhaps become unstable to general 3d perturbations before possible self-intersections. For example, the instabilities studied in [59, 52] may be relevant.

2 Preliminaries and sketch of the proof

In this section we introduce the notation that is necessary to formulate our result in its stronger form, and we give a pretty detailed sketch of the overall proof. The construction of the approximate solution will be performed in Section 3, and the stability analysis in Section 4. Technical calculations are postponed to Appendix A and B.

2.1 Formulation of the problem in cylindrical coordinates

In a suitable Cartesian coordinate system, the circle of radius $r_0 > 0$ which represents the support of the initial vorticity (1.3) is given by $\mathsf{C} = \{(r_0\cos\theta, r_0\sin\theta, 0)\,;\, \theta \in [0, 2\pi]\}$. Due to the symmetries of the problem, it is natural to introduce the standard cylindrical coordinates (r, θ, z) defined by $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $x_3 = z$ and to restrict our attention to velocity and vorticity fields of the form

$$u(x,t) \;=\; u_r(r,z,t)e_r + u_z(r,z,t)e_z\,, \qquad \omega(x,t) \;=\; \omega_\theta(r,z,t)e_\theta\,, \tag{2.1}$$

where e_r, e_θ, e_z denote unit vectors in the radial, azimuthal, and vertical directions, respectively. In the usual terminology, we thus consider *axisymmetric flows with no swirl*, see [48]. Due to the incompressibility condition $\operatorname{div} u := r^{-1}\partial_r(ru_r) + \partial_z(u_z) = 0$, the velocity components u_r, u_z can be expressed in terms of the Stokes stream function ψ:

$$u_r \;=\; -\frac{1}{r}\,\partial_z\psi\,, \qquad u_z \;=\; \frac{1}{r}\,\partial_r\psi\,. \tag{2.2}$$

With this notation the vorticity formulation of the Navier-Stokes equation (1.1) becomes

$$\partial_t\omega_\theta + \Big\{\psi, \frac{\omega_\theta}{r}\Big\} \;=\; \nu\Big[(\partial_r^2 + \partial_z^2)\omega_\theta + \partial_r\frac{\omega_\theta}{r}\Big]\,, \tag{2.3}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket defined by $\{\psi,\phi\} = \partial_r\psi\,\partial_z\phi - \partial_z\psi\,\partial_r\phi$. Eq. (2.3) is to be solved in the domain $\Omega = \{(r,z)\in\mathbb{R}^2\,|\,r>0\}$. The smoothness of the fields in the original variables imposes the "boundary conditions" $\omega_\theta(r,z,t) = r\zeta(r,z,t)$ and $\psi(r,z,t) = r^2\Psi(r,z,t)$ near $r=0$, where ζ and Ψ can be extended to smooth functions on $\mathbb{R}^2\times\mathbb{R}_+$ that are even functions of r.

The Stokes stream function can be represented in terms of the vorticity $\omega_\theta = \partial_z u_r - \partial_r u_z$ by the Biot-Savart law

$$\psi(r,z) \;=\; \frac{1}{2\pi}\int_\Omega \sqrt{r\bar r}\,F\left(\frac{(r-\bar r)^2 + (z-\bar z)^2}{r\bar r}\right)\omega_\theta(\bar r,\bar z)\,\mathrm{d}\bar r\,\mathrm{d}\bar z\,, \tag{2.4}$$

where $F:(0,\infty)\to\mathbb{R}$ is defined by

$$F(s) \;=\; \int_0^{\pi/2}\frac{1-2\sin^2\psi}{\sqrt{\sin^2\psi + s/4}}\,\mathrm{d}\psi\,, \qquad s>0\,. \tag{2.5}$$

Formula (2.4) provides a solution to the equation

$$\operatorname{curl}\operatorname{curl}\left(\frac{\psi}{r}\,e_\theta\right) \;=\; \omega_\theta\,e_\theta \quad\text{or, equivalently,}\quad -\,\partial_r\Big(\frac{\partial_r\psi}{r}\Big) - \frac{\partial_z^2\psi}{r} \;=\; \omega_\theta\,, \tag{2.6}$$

which is familiar in magnetostatics, see for example [51, §701]. The same expression can also be found in the classical book [46, §161]. It is well-known (and not hard to check) that

$$F(s) \;=\; \begin{cases}\log\frac{8}{\sqrt{s}} - 2 + \mathcal{O}(s\log s) & \text{as } s\to 0\,,\\ \frac{\pi}{2s^{3/2}} + \mathcal{O}(s^{-5/2}) & \text{as } s\to\infty\,.\end{cases} \tag{2.7}$$

Since we wish to solve the Cauchy problem (1.1), (1.2) with initial data $\omega_0 = \Gamma\delta_{\mathsf{C}}$, we assume that the vorticity ω_θ in (2.1) satisfies the initial condition

$$\omega_\theta\Big|_{t=0} \;=\; \Gamma\delta_{(r_0,0)}\,, \tag{2.8}$$

where $\delta_{(r_0,z_0)}$ denotes the Dirac mass at the location $(r_0,z_0)\in\Omega$. Our starting point is the following global well-posedness result for the vorticity equation (2.3) with such initial data.

Theorem 2.1. **[33]** *For any $\Gamma > 0$, any $\nu > 0$, and any $(r_0, z_0) \in \Omega$, the axisymmetric vorticity equation* (2.3) *has a unique global mild solution $\omega_\theta \in C^0((0,\infty), L^1(\Omega) \cap L^\infty(\Omega))$ such that*

$$\sup_{t>0} \|\omega_\theta(t)\|_{L^1(\Omega)} < \infty\,, \qquad \text{and} \qquad \omega_\theta(t)\,\mathrm{d}r\,\mathrm{d}z \rightharpoonup \Gamma\,\delta_{(r_0,z_0)} \quad \text{as } t \to 0\,. \tag{2.9}$$

Moreover there exists a constant $C > 0$, depending only on the ratio Γ/ν, such that

$$\int_\Omega \Big|\omega_\theta(r,z,t) - \frac{\Gamma}{4\pi\nu t}\, e^{-\frac{(r-r_0)^2+(z-z_0)^2}{4\nu t}}\Big|\,\mathrm{d}r\,\mathrm{d}z \,\le\, C\,\Gamma\,\frac{\sqrt{\nu t}}{r_0}\,\log\Big(\frac{r_0}{\sqrt{\nu t}} + 1\Big)\,, \tag{2.10}$$

whenever $t \in (0, T_{\mathrm{dif}})$, where $T_{\mathrm{dif}} = r_0^2/\nu$.

Here and in what follows, it is understood that $L^1(\Omega) = L^1(\Omega, \mathrm{d}r\,\mathrm{d}z)$, and similarly for the other Lebesgue spaces. Theorem 2.1 establishes the existence of a four-dimensional family of vortex ring solutions to (2.3) parametrized by the circulation $\Gamma > 0$, the viscosity $\nu > 0$, the initial radius $r_0 > 0$, and the initial vertical position $z_0 \in \mathbb{R}$. Due to translation invariance in the vertical direction, we may assume without loss of generality that $z_0 = 0$, and we can also take $r_0 = 1$ by rescaling the space variables. Then a rescaling of time allows us to change the values of both ν and Γ, while keeping the ratio Γ/ν fixed. Therefore, up to symmetries, the viscous vortex ring solutions we consider here form a *one-parameter family* indexed by the circulation Reynolds number $\mathrm{Re} := \Gamma/\nu$.

The uniqueness of the vortex ring solution under the minimal assumptions (2.9) is discussed in some detail in [33], so we concentrate here on the short-time estimate (2.10), which is of limited use despite appearances. For a fixed value of the Reynolds number $\mathrm{Re} = \Gamma/\nu$, the right-hand side of (2.10) is small whenever $t \ll T_{\mathrm{dif}}$, which means that the solution of (2.3) with initial data (2.8) is well approximated by a Gaussian vortex ring of thickness proportional to $\sqrt{\nu t}$, located a the *initial position* $(r_0, z_0) \in \Omega$. However, since the constant C depends on the Reynolds number in a very bad way, estimate (2.10) gives no information on the solution at a fixed time $t > 0$ in the low viscosity regime $\nu \to 0$. This limitation is not surprising: due to the translational motion along the vertical axis predicted by the Kelvin-Saffman formula (1.5), the vortex ring at time $t > 0$ is actually located at a new position which is rather far from the initial one if ν is small.

Our goal in this paper is to replace (2.10) by an improved estimate of the form

$$\int_\Omega \Big|\omega_\theta(r,z,t) - \frac{\Gamma}{4\pi\nu t}\, e^{-\frac{(r-\bar r(t))^2+(z-\bar z(t))^2}{4\nu t}}\Big|\,\mathrm{d}r\,\mathrm{d}z \,\le\, K\,\Gamma\,\frac{\sqrt{\nu t}}{r_0}\,, \qquad t \in (0, T_{\mathrm{adv}}\,\mathrm{Re}^\sigma)\,, \tag{2.11}$$

where the constant K is now independent of the Reynolds number, if $\mathrm{Re} \gg 1$. Comparing with (2.10), we observe that (2.11) is valid up to the intermediate time $T_{\mathrm{adv}}\,\mathrm{Re}^\sigma$, for some $\sigma \in (0, \frac{1}{3})$, which is shorter than $T_{\mathrm{dif}} \equiv T_{\mathrm{adv}}\,\mathrm{Re}$. But the main difference is that (2.11) compares the solution $\omega_\theta(r,z,t)$ to a vortex ring located at a *time-dependent position* $(\bar r(t), \bar z(t))$, which has to be determined. As we shall see, we can take $\bar r(t), \bar z(t)$ to be continuous functions of time which are smooth for $t > 0$ and satisfy $\bar r(0) = r_0$, $\bar z(0) = z_0$. Moreover

$$\dot{\bar r}(t) \,=\, \mathcal{O}\Big(\frac{\nu}{r_0}\Big)\,, \qquad \dot{\bar z}(t) \,=\, \frac{\Gamma}{4\pi r_0}\Big(\log\frac{1}{\epsilon(t)} + \hat v\Big)\Big(1 + \mathcal{O}\big(\epsilon(t)^2 + \delta^2\big)\Big)\,, \tag{2.12}$$

where $\epsilon(t) = \sqrt{\nu t}/\bar r(t)$, $\hat v = \frac{3}{2}\log(2) + \frac{1}{2}(\gamma_E - 1)$, and $\delta = \nu/\Gamma$. The first relation in (2.12) implies that $\bar r(t) = r_0\big(1 + \mathcal{O}(\epsilon(t)^2)\big)$, which means that the change in the radius of the vortex ring over the time interval under consideration is much smaller than the diffusion length $\sqrt{\nu t}$. The second equality coincides with the Kelvin-Saffman formula (1.5), up to higher order corrections.

2.2 Self-similar variables

From now on, we fix the circulation $\Gamma > 0$ and the position $(r_0, 0) \in \Omega$ of the initial filament, and we consider the vortex ring solution given by Theorem 2.1, in the regime where the viscosity $\nu > 0$ is small. In view of the approximation formula (2.11), which is our objective, it is natural to make the following self-similar Ansatz for the axisymmetric vorticity and the associated Stokes stream function:

$$\begin{aligned}
\omega_\theta(r,z,t) \,&=\, \frac{\Gamma}{\nu t}\,\eta\Big(\frac{r-\bar r(t)}{\sqrt{\nu t}}\,,\, \frac{z-\bar z(t)}{\sqrt{\nu t}}\,,\, t\Big)\,,\\
\psi(r,z,t) \,&=\, \Gamma\,\bar r(t)\,\phi\Big(\frac{r-\bar r(t)}{\sqrt{\nu t}}\,,\, \frac{z-\bar z(t)}{\sqrt{\nu t}}\,,\, t\Big)\,,
\end{aligned} \tag{2.13}$$

where the time-dependent position $(\bar r(t), \bar z(t)) \in \Omega$ has to be determined. We introduce the important notation

$$\delta \,=\, \frac{\nu}{\Gamma}\,, \qquad \epsilon \,=\, \frac{\sqrt{\nu t}}{\bar r(t)}\,, \qquad R \,=\, \frac{r-\bar r(t)}{\sqrt{\nu t}}\,, \qquad Z \,=\, \frac{z-\bar z(t)}{\sqrt{\nu t}}\,. \tag{2.14}$$

The evolution equation for the rescaled vorticity $\eta(R,Z,t)$ is found to be

$$t\partial_t\eta + \frac{\Gamma}{\nu}\Big\{\phi\,,\,\frac{\eta}{1+\epsilon R}\Big\} - \sqrt{\frac{t}{\nu}}\Big(\dot{\bar r}\,\partial_R\eta + \dot{\bar z}\,\partial_Z\eta\Big) \,=\, \mathcal{L}\eta + \partial_R\Big(\frac{\epsilon\eta}{1+\epsilon R}\Big)\,, \tag{2.15}$$

where $\big\{\phi,\chi\big\} = \partial_R\phi\,\partial_Z\chi - \partial_Z\phi\,\partial_R\chi$ is the Poisson bracket in the new variables (R,Z), and $\mathcal{L}$ is the Fokker-Planck operator

$$\mathcal{L} \,=\, \partial_R^2 + \partial_Z^2 + \frac{1}{2}\big(R\partial_R + Z\partial_Z\big) + 1\,. \tag{2.16}$$

Eq. (2.15) is to be solved in the time-dependent domain

$$\Omega_\epsilon \,=\, \big\{(R,Z) \in \mathbb{R}^2 \,\big|\, 1+\epsilon R > 0\big\}\,, \tag{2.17}$$

with the Dirichlet boundary condition $\eta(-1/\epsilon, Z, t) = 0$ for all $(Z,t) \in \mathbb{R}\times\mathbb{R}_+$.

As in [33], it is useful to introduce the velocity field $U = (U_R, U_Z)$ defined by

$$U_R \,=\, -\frac{\partial_Z\phi}{1+\epsilon R}\,, \qquad U_Z \,=\, \frac{\partial_R\phi}{1+\epsilon R}\,, \tag{2.18}$$

in terms of which the nonlinearity in (2.15) reads $\big\{\phi\,,\,\frac{\eta}{1+\epsilon R}\big\} \,=\, \partial_R\big(U_R\,\eta\big) + \partial_Z(U_Z\,\eta)$. The stream function ϕ in (2.15) satisfies the elliptic equation

$$\eta \,=\, \partial_Z U_R - \partial_R U_Z \,\equiv\, -\partial_R\Big(\frac{\partial_R\phi}{1+\epsilon R}\Big) - \frac{\partial_Z^2\phi}{1+\epsilon R}\,, \qquad (R,Z) \in \Omega_\epsilon\,, \tag{2.19}$$

with boundary conditions $\phi(-1/\epsilon, Z, t) \,=\, \partial_R\phi(-1/\epsilon, Z, t) \,=\, 0$ for all $(Z,t) \in \mathbb{R}\times\mathbb{R}_+$. Using (2.4), we easily obtain the representation formula [33]

$$\phi(R,Z) \,=\, \frac{1}{2\pi}\int_{\Omega_\epsilon}\sqrt{(1{+}\epsilon R)(1{+}\epsilon R')}\,F\bigg(\epsilon^2\,\frac{(R{-}R')^2 + (Z{-}Z')^2}{(1{+}\epsilon R)(1{+}\epsilon R')}\bigg)\eta(R',Z')\,\mathrm{d}R'\,\mathrm{d}Z'\,, \tag{2.20}$$

where F is as in (2.5). In what follows we write $\phi = \mathrm{BS}^\epsilon[\eta]$ when (2.20) holds.

The quantities introduced in (2.14) are all dimensionless. The first one is the inverse Reynolds number $\delta > 0$, a fixed parameter that is assumed to be small. The second one is the time-dependent aspect ratio $\epsilon > 0$, which appears in the evolution equation (2.15), in the definition of the domain (2.17), and in the Biot-Savart formula (2.20). Finally, the variables R, Z are self-similar coordinates centered at the time-dependent location $(\bar{r}(t), \bar{z}(t))$ and normalized according to the size $\sqrt{\nu t}$ of the vortex core. Note that the rescaled functions η, ϕ defined in (2.13) are also dimensionless.

Remark 2.2. *Recalling that $\delta = \nu/\Gamma$ and $T_{\rm adv} = r_0^2/\Gamma$, we observe that*

$$\epsilon^2 \,=\, \frac{\nu t}{r_0^2}\,\frac{r_0^2}{\bar{r}(t)^2} \,=\, \frac{\delta t}{T_{\rm adv}}\,\frac{r_0^2}{\bar{r}(t)^2} \,\approx\, \frac{\delta t}{T_{\rm adv}}\,, \tag{2.21}$$

as long as the ratio $r_0/\bar{r}(t)$ remains close to unity, which will always be the case thanks to (2.12)*. It follows in particular that ϵ^2 is comparable to δ whenever t is comparable to $T_{\rm adv}$. Our goal is to control the solution of* (2.3) *when $t \le T_{\rm adv}\delta^{-\sigma}$ for some $\sigma \in (0, \frac{1}{3})$, and on that interval it follows from* (2.21) *that $\epsilon^2 \lesssim \delta^{1-\sigma}$.*

2.3 Approximate solution

The first important step in our analysis is the construction of an approximate solution of (2.15) with initial data

$$\eta_0(R,Z) \,=\, \frac{1}{4\pi}\,e^{-(R^2+Z^2)/4}\,, \qquad (R,Z) \in \Omega_0 = \mathbb{R}^2\,. \tag{2.22}$$

The associated stream function satisfies $-\Delta_0\phi_0 = \eta_0$, where $\Delta_0 = \partial_R^2 + \partial_Z^2$. As η_0, ϕ_0 are both radially symmetric, it is clear that $\{\phi_0, \eta_0\} = 0$, and the Gaussian profile (2.22) has the property that $\mathcal{L}\eta_0 = 0$. Since $\epsilon = 0$ when $t = 0$ in view of (2.14), we conclude that equation (2.15) is satisfied at initial time if η_0 is given by (2.22).

For $t > 0$, we construct our approximate solution as a power series in the time-dependent parameter $\epsilon = \sqrt{\nu t}/\bar{r}$, the coefficients of which depend on the small parameter δ. To this end, we multiply both sides of (2.15) by δ and rewrite the equation in the equivalent form

$$\delta\, t\partial_t\eta + \Big\{\phi\,, \frac{\eta}{1+\epsilon R}\Big\} - \frac{\epsilon\bar{r}}{\Gamma}\Big(\dot{\bar{r}}\,\partial_R\eta + \dot{\bar{z}}\,\partial_Z\eta\Big) \,=\, \delta\Big[\mathcal{L}\eta + \partial_R\Big(\frac{\epsilon\eta}{1+\epsilon R}\Big)\Big]\,. \tag{2.23}$$

This equation is defined on the time-dependent domain Ω_ϵ, but expanding the factors $(1+\epsilon R)^{-1}$ in powers of ϵ we get at each order a relation that can be solved in the whole plane $\Omega_0 = \mathbb{R}^2$. The corresponding approximation for the stream function ϕ is obtained in a self-consistent way by expanding the integrand in (2.20) in powers of ϵ, and integrating order by order over the whole plane $\mathbb{R}^2$. As is shown in Section 3, this results in an asymptotic expansion of the form

$$\eta_{\rm app}(R,Z,t) \,=\, \sum_{m=0}^{M} \epsilon^m\,\eta_m(R,Z,\beta_\epsilon)\,, \qquad \phi_{\rm app}(R,Z,t) \,=\, \sum_{m=0}^{M} \epsilon^m\,\phi_m(R,Z,\beta_\epsilon)\,, \tag{2.24}$$

where the dependence of the profiles η_m and ϕ_m on $\beta_\epsilon := \log(1/\epsilon)$ is polynomial. The profiles also depend on the small parameter δ, but to make the notation lighter this dependence is not indicated explicitly. The velocity of the vortex center is not known a priori, but can be approximated in a similar way as a power series in ϵ:

$$\dot{\bar{r}}(t) \,=\, \sum_{m=0}^{M-1} \epsilon^m\,\dot{\bar{r}}_m(\beta_\epsilon)\,, \qquad \dot{\bar{z}}_*(t) \,=\, \sum_{m=0}^{M-1} \epsilon^m\,\dot{\bar{z}}_m(\beta_\epsilon)\,, \tag{2.25}$$

where the quantities $\dot{\bar r}_m(\beta_\epsilon)$, $\dot{\bar z}_m(\beta_\epsilon)$ depend on δ and are polynomials in β_ϵ. As will be explained below, the quantity $\dot{\bar z}_*(t)$ in (2.25) is only an initial approximation of the vertical speed of the vortex ring; the final approximation $\dot{\bar z}(t)$ will be obtained from it by a small adjustment. It is perhaps worth emphasizing that, throughout the paper, the point $(\bar r(t), \bar z(t))$ is not necessarily the exact center of our vortex. Rather, it is its suitably chosen approximation.

The outcome of the analysis carried out in Section 3 below is that, if we want our expansions (2.24), (2.25) to hold uniformly with respect to the parameter δ in the limit where $\delta \to 0$, there is a *unique choice* of the profiles η_m, ϕ_m and of the velocities $\dot{\bar r}_m$, $\dot{\bar z}_m$ such that:

a) Both members of (2.23) agree up to order $\mathcal{O}(\epsilon^{M+1})$, modulo powers of β_ϵ;

b) The point $(\bar r(t), \bar z_*(t)) \in \Omega$ is the center of the vorticity distribution $\eta_{\mathrm{app}}(R, Z, t)$.

The integer M in (2.24), (2.25) determines the accuracy of our approximate solution. It turns out that $M = 4$ will be sufficient for our purposes. The velocities $\dot{\bar r}(t), \dot{\bar z}_*(t)$ given by (2.25) are found to satisfy estimate (2.12) with $\delta = 0$.

Remark 2.3. *If we set $\delta = \dot{\bar r} = 0$, equation* (2.23) *reduces to*

$$\Big\{\phi\,,\frac{\eta}{1+\epsilon R}\Big\} - \frac{\epsilon\bar r}{\Gamma}\,\dot{\bar z}\,\partial_Z\eta \;\equiv\; \Big\{\phi - \frac{\bar r\dot{\bar z}}{2\Gamma}\,(1+\epsilon R)^2\,,\frac{\eta}{1+\epsilon R}\Big\} \;=\; 0\,, \tag{2.26}$$

which is the relation satisfied by the vorticity η and the stream function ϕ of a stationary solution of the Euler equations in a frame moving with speed $\dot{\bar z}\,e_z$. These are precisely the vortex rings constructed, for instance, in [26,28,29,1,10]. In that situation the aspect ratio $\epsilon > 0$ is fixed and, as in (2.14)*, the dimensionless variables (R, Z) are defined so that $(r, z) = (\bar r, \bar z) + \epsilon\bar r\,(R, Z)$. An approximate solution of* (2.26) *can be constructed in the form of a power series in ϵ, as in* (2.24)*, where all profiles η_m, ϕ_m are even functions of the variable $Z \in \mathbb{R}$, since this is the case for the coefficients in* (2.26) *and for the initial approximation* (2.22)*. Returning to the approximate solution* (2.24)*, we deduce by uniqueness that η_{app}, ϕ_{app} are even functions of Z in the limit $\delta \to 0$, and that $\dot{\bar r} = \frac{\Gamma}{r_0}\,\mathcal{O}(\delta)$ as $\delta \to 0$.*

Remark 2.4. *In view of* (2.14) *and* (2.25)*, the function $\epsilon(t)$ is implicitly defined by the relation*

$$\frac{\sqrt{\nu t}}{\epsilon(t)} \;=\; \bar r(t) \;=\; r_0 + \sum_{m=0}^{M-1}\int_0^t \epsilon(s)^m\,\dot{\bar r}_m\big(\beta_{\epsilon(s)}\big)\,\mathrm{d}s\,, \tag{2.27}$$

which should hold when $0 < t \ll T_{\mathrm{dif}}$. As was mentioned in the previous remark, the radial velocities $\dot{\bar r}_m$ are small when $\delta \ll 1$, so that Eq. (2.27) *will be easy to solve, see Section 3.6.*

The asymptotic approximation $\eta_{\mathrm{app}}(R, Z, t)$ is defined on the whole plane and does not vanish on the boundary $\partial\Omega_\epsilon$. To obtain a valid approximate solution of (2.15), we fix $\sigma_0 \in (0, 1)$ and we truncate η_{app} outside a large ball of radius $\epsilon^{-\sigma_0}$ by setting

$$\eta_*(R, Z, t) \;=\; \chi_0\big(\epsilon^{\sigma_0}(R^2{+}Z^2)^{1/2}\big)\,\eta_{\mathrm{app}}(R, Z, t)\,, \qquad \phi_*(\cdot, t) \;=\; \mathrm{BS}^\epsilon[\eta_*(\cdot, t)]\,, \tag{2.28}$$

where $\chi_0 : \mathbb{R}_+ \to [0, 1]$ is a smooth function such that $\chi_0(r) = 1$ for $r \le 1$ and $\chi_0(r) = 0$ for $r \ge 2$. The remainder of that approximation is defined as

$$\mathrm{Rem}(R, Z, t) \;=\; \mathcal{L}\eta_* + \partial_R\Big(\frac{\epsilon\eta_*}{1{+}\epsilon R}\Big) - t\partial_t\eta_* - \frac{1}{\delta}\Big\{\phi_*\,,\frac{\eta_*}{1{+}\epsilon R}\Big\} + \frac{\epsilon\bar r}{\delta\Gamma}\Big(\dot{\bar r}\,\partial_R\eta_* + \dot{\bar z}_*\,\partial_Z\eta_*\Big)\,. \tag{2.29}$$

By construction this quantity depends on time only through the parameter $\epsilon = \sqrt{\nu t}/\bar r(t)$.

The accuracy of our approximate solution is quantified by the following result, which is established in Section 3.7 below:

Proposition 2.5. *Given any $\gamma_0 < 1$ and any $\gamma_5 < 5$, there exist a constant $C > 0$ such that the remainder* (2.29) *satisfies*

$$\sup_{(R,Z)\in\Omega_\epsilon} e^{\gamma_0(R^2+Z^2)/4}\,|\mathrm{Rem}(R,Z,t)| \,\le\, C\big(\epsilon\delta + \epsilon^{\gamma_5}\delta^{-1}\big)\,, \tag{2.30}$$

whenever the parameters ϵ, δ are small enough.

2.4 Stability estimates

In our previous work [33], the evolution equation (2.15) was carefully studied in the particular case where $\bar{r}(t) = r_0$ and $\bar{z}(t) = z_0$. This does not make any substantial difference for the initial value problem at fixed viscosity, and we can thus infer from the results of [33] that Eq. (2.15) has a unique solution $\eta(R,Z,t)$ with initial data η_0 given by (2.22). Our purpose is to show that, if the inverse Reynolds number $\delta = \nu/\Gamma$ is sufficiently small, the solution $\eta(R,Z,t)$ remains close to the approximation (2.28) on a long time interval of the form $(0, T_{\rm adv}\delta^{-\sigma})$, for some small $\sigma > 0$. We use the decomposition:

$$\eta(R,Z,t) \,=\, \eta_*(R,Z,t) + \delta\,\tilde{\eta}(R,Z,t)\,, \qquad \phi(R,Z,t) \,=\, \phi_*(R,Z,t) + \delta\,\tilde{\phi}(R,Z,t)\,, \tag{2.31}$$

where $\tilde{\phi} = \mathrm{BS}^\epsilon[\tilde{\eta}]$ in the sense of (2.20). Similarly we assume that the vertical speed of the vortex ring takes the form

$$\dot{\bar{z}}(t) \,=\, \dot{\bar{z}}_*(t) + \delta\,\dot{\tilde{z}}(t)\,, \tag{2.32}$$

where $\dot{\bar{z}}_*(t)$ is given by (2.25) and $\dot{\tilde{z}}(t)$ is a small correction which is chosen so that the perturbation $\tilde{\eta}$ has vanishing first order moment in the vertical direction, see Section 4.1. The equation satisfied by $\tilde{\eta}$ then reads

$$\begin{aligned} t\partial_t\tilde{\eta} + \frac{1}{\delta}\Big\{\phi_*\,,\frac{\tilde{\eta}}{1+\epsilon R}\Big\} + \frac{1}{\delta}\Big\{\tilde{\phi}\,,\frac{\eta_*}{1+\epsilon R}\Big\} + \Big\{\tilde{\phi}\,,\frac{\tilde{\eta}}{1+\epsilon R}\Big\} - \frac{\epsilon\bar{r}}{\delta\Gamma}\Big(\dot{\bar{r}}\,\partial_R\tilde{\eta} + \dot{\bar{z}}_*\,\partial_Z\tilde{\eta}\Big) \\ = \mathcal{L}\tilde{\eta} + \partial_R\Big(\frac{\epsilon\tilde{\eta}}{1+\epsilon R}\Big) + \frac{1}{\delta}\,\mathrm{Rem}(R,Z,t) \,+\, \frac{\epsilon\bar{r}}{\delta\Gamma}\,\dot{\tilde{z}}\,\partial_Z\eta\,. \end{aligned} \tag{2.33}$$

Since $\eta_*(R,Z,0) = \eta_0(R,Z)$, the nonlinear evolution equation (2.33) is to be solved with zero initial data. The solution is therefore driven by the source term $\delta^{-1}\mathrm{Rem}(R,Z,t)$, which is small in view of Proposition 2.5 and Remark 2.2 if the parameter σ is small enough. As long as $\tilde{\eta}$ stays small, the nonlinear term $\{\tilde{\phi}, (1+\epsilon R)^{-1}\tilde{\eta}\}$ is of course harmless. The most serious difficulty in controlling $\tilde{\eta}$ using (2.33) comes from the linear terms with a large prefactor $\delta^{-1} = \Gamma/\nu$. These terms could conceivably trigger violent instabilities that might lead to strong amplification of $\tilde{\eta}$ in a short time. Our goal is to show that this scenario does not occur, due to the special structure of the advection terms in (2.33). A similar strategy was applied in the previous work [31] devoted to the vanishing viscosity limit of interacting vortices in the plane, but the specific estimates used there do not seem to be easily adaptable to the present situation.

To control the time evolution of the solution of (2.33), we use the energy functional

$$E_\epsilon(t) \,=\, \frac{1}{2}\int_{\Omega_\epsilon} W_\epsilon\,\tilde{\eta}^2\,\mathrm{d}R\,\mathrm{d}Z \,-\, \frac{1}{2}\int_{\Omega_\epsilon}\tilde{\phi}\,\tilde{\eta}\,\mathrm{d}R\,\mathrm{d}Z\,, \tag{2.34}$$

where $W_\epsilon : \Omega_\epsilon \to (0,+\infty)$ is a weight function that will be described below. The first term in the right-hand side of (2.34) is a weighted L^2 integral of the vorticity $\tilde{\eta}$, similar to weighted enstrophies that were used for the same purposes in [36,31,33], for instance. The second term is just the kinetic energy associated with the vorticity perturbation $\tilde{\eta}$, as can be seen by invoking (2.18), (2.19) and integrating by parts:

$$\frac{1}{2}\int_{\Omega_\epsilon} \tilde{\phi}\,\tilde{\eta}\,\mathrm{d}R\,\mathrm{d}Z \;=\; \frac{1}{2}\int_{\Omega_\epsilon} \frac{|\partial_R\tilde{\phi}|^2 + |\partial_Z\tilde{\phi}|^2}{1+\epsilon R}\,\mathrm{d}R\,\mathrm{d}Z \;=\; \frac{1}{2}\int_{\Omega_\epsilon} \bigl(|\tilde{U}_R|^2 + |\tilde{U}_Z|^2\bigr)(1+\epsilon R)\,\mathrm{d}R\,\mathrm{d}Z\,.$$

To construct the weight W_ϵ in (2.34), we consider three different regions:

1) The *inner region* where $\rho := (R^2+Z^2)^{1/2} \lesssim \epsilon^{-\sigma_1}$, for some small $\sigma_1 > 0$. Here we choose

$$W_\epsilon \;=\; \frac{1}{1+\epsilon R}\,\Phi_\epsilon'\Bigl(\frac{\eta_*}{1+\epsilon R}\Bigr)\,, \tag{2.35}$$

where η_* is the approximate solution (2.28) and $\Phi_\epsilon : (0,+\infty) \to \mathbb{R}$ is a smooth function with the property that, in the region under consideration,

$$\phi_* - \frac{\bar{r}\dot{\bar{z}}_*}{2\Gamma}\,(1+\epsilon R)^2 \;=\; \Phi_\epsilon\Bigl(\frac{\eta_*}{1+\epsilon R}\Bigr) \;+\; \mathcal{O}(\epsilon\delta + \epsilon^{\gamma_3})\,, \tag{2.36}$$

for some $\gamma_3 < 3$ that can be arbitrarily close to 3. It is not difficult to understand intuitively why such a function should exist. Indeed, in the dimensionless variables (2.14), the left-hand side of (2.36) is nothing but the stream function of the approximate solution η_* in a frame moving with constant speed $\dot{\bar{z}}_*$ in the vertical direction, see Remark 2.3. If we drop the remainder term $\mathcal{O}(\epsilon\delta + \epsilon^{\gamma_3})$ and consider $\epsilon > 0$ as a fixed parameter, Eq. (2.36) expresses a functional relation between the potential vorticity $\zeta_* := (1+\epsilon R)^{-1}\eta_*$ and the stream function, which implies that η_* is a stationary solution of the Euler equation in the moving frame. This is not exactly true, of course, but the estimate on the remainder $\mathrm{Rem}(R,Z,t)$ in Proposition 2.5 ensures that the approximate solution η_* (for a fixed value of $\epsilon > 0$) is not far from a stationary solution of Euler, and in Section 3.8 we verify that this implies the existence of a function Φ_ϵ satisfying (2.36). Moreover, an easy calculation shows that

$$\frac{1}{1+\epsilon R}\,\Phi_\epsilon'\Bigl(\frac{\eta_*}{1+\epsilon R}\Bigr) \;=\; \frac{4}{\rho^2}\bigl(e^{\rho^2/4} - 1\bigr) + \mathcal{O}(\epsilon)\,, \qquad \rho \;:=\; \sqrt{R^2+Z^2} \;\le\; \epsilon^{-\sigma_1}\,.$$

2) The *intermediate region* where $\epsilon^{-\sigma_1} \lesssim \rho \le \epsilon^{-\sigma_2}$, for some $\sigma_2 > 1$. In this area we assume that the weight is approximately constant in space, with value $W_\epsilon \approx \exp(\epsilon^{-2\sigma_1}/4)$.

3) The *far field region* where $\rho \ge \epsilon^{-\sigma_2}$. Here we take $W_\epsilon \approx \exp(\rho^{2\gamma}/4)$, where $\gamma = \sigma_1/\sigma_2$.

The actual construction of the weight is more complicated, and ensures that W_ϵ is Lipschitz continuous at the boundaries of the three regions under consideration, see Section 4 below for details. For the moment, we just mention that our choice of the energy functional in the inner region is related to Arnold's variational characterization of the steady states of the Euler equation, as discussed in our previous work [34]. In fact, if we suppose that η_* is a stationary solution of the axisymmetric Euler equation in a moving frame (which not exactly true), then the functional (2.34) with the weight (2.35) corresponds, up to a constant factor, to the second variation of the kinetic energy on the isovortical surface, which is the set of (potential) vorticities $\zeta := (1+\epsilon R)^{-1}\eta$ that are measure-preserving rearrangements of ζ_* [3, 34]. Since the kinetic energy is conserved under the inviscid dynamics, the advection terms involving δ^{-1} in (2.33), which originate from the linearization of Euler's equation at the "steady state" ζ_*, do not contribute to the time evolution of the functional E_ϵ. In reality ζ_* is only an approximate steady state of Euler, and the cancellations alluded to above only occur up to correction terms of order $\mathcal{O}(\epsilon\delta + \epsilon^{\gamma_3})$, but this is sufficient to cancel the dangerous factors δ^{-1} in (2.33). On the

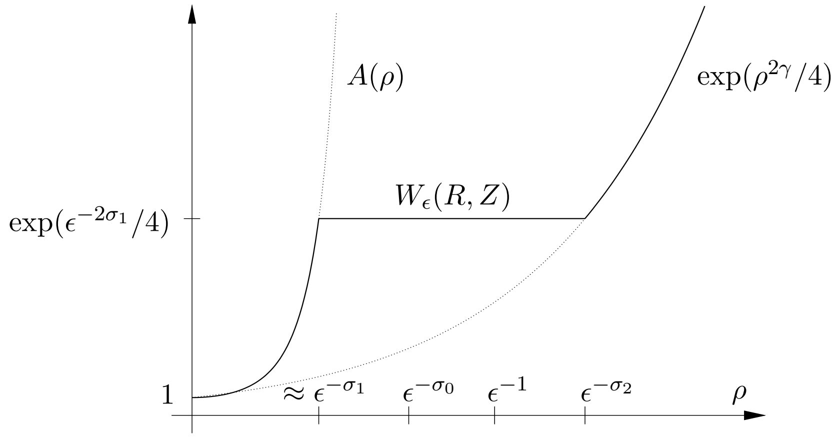

Figure 2: When $\epsilon > 0$ is small, the weight $W_\epsilon(R,Z)$ entering the energy functional (2.34) is close to a piecewise smooth radially symmetric function, which satisfies $W_\epsilon \approx A(\rho) := (4/\rho^2)\big(e^{\rho^2/4} - 1\big)$ in the inner region where $\rho := (R^2+Z^2)^{1/2} \lesssim \epsilon^{-\sigma_1}$. When W_ϵ reaches the threshold value $\exp(\epsilon^{-2\sigma_1}/4)$, the weight is taken approximately constant until $\rho = \epsilon^{-\sigma_2}$, and outside that region we set $W_\epsilon \approx \exp(\rho^{2\gamma}/4)$ with $\gamma = \sigma_1/\sigma_2$. The dashed lines reflect the fact that $\exp(\rho^{2\gamma}/4) \lesssim W_\epsilon \lesssim A(\rho)$ where the implicit constants do not depend on the parameter ϵ. The intermediate scales $\epsilon^{-\sigma_0}$, where the truncation (2.28) occurs, and ϵ^{-1}, which is the distance from the origin to the boundary $\partial\Omega_\epsilon$, are indicated for completeness.

other hand, away from the inner region, the last term in (2.34) is extremely small, so that our functional E_ϵ reduces to a weighted enstrophy. We assume that the weight W_ϵ is approximately constant in the intermediate region, so that the advection terms in (2.33) do not contribute to the evolution of E_ϵ, and in the far field region the dynamics is dominated by the diffusion operator $\mathcal{L}$ in (2.33) so that we can just take any radially symmetric weight with appropriate growth at infinity.

A technical difficulty inherent to our approach is the fact that the functional E_ϵ is not coercive, unless the perturbed vorticity $\tilde\eta$ satisfies some moment conditions. The problem comes from the inner region, where the last term in (2.34) plays an important role. The results established in [34, Section 2] indicate that E_ϵ is positive definite provided $\tilde\eta$ has zero mean and vanishing first order moments with respect to the space variables R, Z. In practice this means that, in addition to the information provided by the energy E_ϵ, we must control the integral and the first order moments of the perturbed vorticity $\tilde\eta$. It turns out that $\int \tilde\eta \,\mathrm{d}R\,\mathrm{d}Z$ is always extremely small, of the order of $\mathcal{O}(\exp(-c/\epsilon^2))$ for some $c > 0$. The radial moment $\int R\,\tilde\eta\,\mathrm{d}R\,\mathrm{d}Z$ may take larger values, but can be controlled using the conservation of the total impulse of the vortex ring. Finally, we choose the correction $\dot{\tilde z}(t)$ of the vertical speed (2.32) in such a way that $\int Z\,\tilde\eta\,\mathrm{d}R\,\mathrm{d}Z = 0$, see Section 4.1 below for further details. This correction thus plays the role of a "modulation parameter", see [58, 50] for a similar idea in the context of the stability analysis of solitary waves.

Disregarding these technical questions for the moment, we briefly indicate how the argument is concluded. If we differentiate E_ϵ with respect to time, and use the evolution equation (2.33) together with the estimate (2.30) on the source term, we obtain after lengthy calculations a differential inequality of the form

$$tE_\epsilon'(t) \le -c_1 E_\epsilon(t) + c_2\Big(\epsilon^2 + \frac{\epsilon^{2\gamma_3}}{\delta^2}\Big), \qquad t \in (0, T_{\mathrm{adv}}\delta^{-\sigma}), \tag{2.37}$$

for some positive constants c_1, c_2. Here we assume that $\epsilon^{2\gamma_3} \ll \delta^2$ so that the source term in (2.37) is small. Since $\epsilon^2 \lesssim \delta^{1-\sigma}$ by Remark 2.2, this is the case if $\sigma < 1 - 2/\gamma_3$, which is possible if $\sigma < 1/3$ and γ_3 is close enough to 3. Integrating (2.37) with initial condition $E_\epsilon(0) = 0$, we find

$$E_\epsilon(t) \,\le\, c_3\Big(\epsilon^2 + \frac{\epsilon^{2\gamma_3}}{\delta^2}\Big)\,, \qquad t \in (0, T_{\mathrm{adv}}\delta^{-\sigma})\,, \tag{2.38}$$

and using in addition the bounds on the moments of $\tilde\eta$ that are obtained by a different argument we arrive at an estimate of the form $\delta\|\tilde\eta(t)\|_{\mathcal{X}_\epsilon} \le c(\epsilon\delta + \epsilon^{\gamma_3})$, where $\mathcal{X}_\epsilon$ is the weighted L^2 space equipped with the norm

$$\|\tilde\eta\|_{\mathcal{X}_\epsilon} \,=\, \Big(\int_{\Omega_\epsilon} W_\epsilon(R,Z)\,|\tilde\eta(R,Z)|^2 \,\mathrm{d}R\,\mathrm{d}Z\Big)^{1/2}\,. \tag{2.39}$$

This space depends on time through the parameter $\epsilon > 0$, but we recall that the weight function satisfies a uniform lower bound of the form $W_\epsilon(R,Z) \gtrsim \exp(\rho^{2\gamma}/4)$, see Figure 2.

The main result of this paper can now be formulated as follows:

Theorem 2.6. *For any $\gamma_3 \in (2,3)$, there exist constants $K > 0$, $\delta_0 > 0$, and $\sigma \in (0, \frac{1}{3})$ such that, for all $\Gamma > 0$, all $r_0 > 0$, and all $\nu > 0$ satisfying $\delta := \nu/\Gamma \le \delta_0$, the following holds. There exist continuous functions $\bar r(t), \bar z(t)$ which are smooth for positive times and satisfy* (2.12) *with $\bar r(0) = r_0, \bar z(0) = 0$ such that the unique solution η of* (2.15) *with initial data* (2.22) *satisfies*

$$\|\eta(t) - \eta_*(t)\|_{\mathcal{X}_\epsilon} \,\le\, K\big(\epsilon\delta + \epsilon^{\gamma_3}\big)\,, \qquad t \in (0, T_{\mathrm{adv}}\delta^{-\sigma})\,, \tag{2.40}$$

where $\epsilon = \sqrt{\nu t}/\bar r(t)$ and η_* is the approximate solution defined by (2.24), (2.28).

We recall that estimate (2.12) for the radial velocity $\dot{\bar r}$ implies that $\bar r(t) = r_0\big(1 + \mathcal{O}(\epsilon^2)\big)$, meaning that the major radius of the vortex ring remains essentially constant on the time interval $(0, T_{\mathrm{adv}}\delta^{-\sigma})$. As for the vertical velocity, it is given by (2.32) where the approximate speed $\dot{\bar z}_*$ defined in (2.25) and the correction $\dot{\tilde z}$ satisfy

$$\dot{\bar z}_* \,=\, \frac{\Gamma}{4\pi r_0}\Big(\log\frac{1}{\epsilon} + \hat v\Big)\Big(1 + \mathcal{O}(\epsilon^2)\Big)\,, \qquad \dot{\tilde z} \,=\, \frac{\Gamma}{r_0}\,\mathcal{O}\Big(\Big(\epsilon + \frac{\epsilon^{\gamma_3}}{\delta}\Big)\log\frac{1}{\epsilon} + \delta\Big)\,. \tag{2.41}$$

This gives the announced formula (2.12) for the full velocity $\dot{\bar z} = \dot{\bar z}_* + \delta\dot{\tilde z}$.

It is not difficult to verify that Theorem 2.6 implies Theorem 1.1, see Section 4.9 for details. Here we just show how to derive estimate (2.11), which is essentially a reformulation of (1.7). By construction, our approximate solution satisfies $\|\eta_*(t) - \eta_0\|_{\mathcal{X}_\epsilon} = \mathcal{O}(\epsilon)$, where η_0 is the Gaussian function (2.22). Moreover, the lower bound $W_\epsilon(R,Z) \gtrsim \exp(\rho^{2\gamma}/4)$ implies that $\mathcal{X}_\epsilon \hookrightarrow L^1(\Omega_\epsilon)$ uniformly in ϵ. It thus follows from (2.40) that

$$\|\eta(t) - \eta_0\|_{L^1(\Omega_\epsilon)} \,\le\, C_1\Big(\|\eta(t) - \eta_*(t)\|_{\mathcal{X}_\epsilon} + \|\eta_*(t) - \eta_0\|_{\mathcal{X}_\epsilon}\Big) \,\le\, C_2\epsilon\,,$$

for any $t \in (0, T_{\mathrm{adv}}\delta^{-\sigma})$, and returning to the original variables we arrive at estimate (2.11).

Remark 2.7. *It follows from* (2.24) *and* (2.40) *that the solution of* (2.15) *satisfies*

$$\eta(R,Z,t) \,=\, \eta_0(R,Z) + \epsilon\eta_1(R,Z) + \epsilon^2\eta_2(R,Z,\beta_\epsilon) + \mathcal{O}\big(\delta\epsilon + \epsilon^{\gamma_3}\big)\,, \tag{2.42}$$

where the remainder term is estimated in the topology of $\mathcal{X}_\epsilon$ as $\epsilon \to 0$. Here η_0 is the Gaussian function (2.22)*, and the vorticity profiles η_1, η_2 are explicitly constructed in Section 3. Since $\delta \lesssim \epsilon^2$ except for very small times, see Remark 2.2, we see that* (2.42) *determines the shape of the vortex core up to third order in ϵ.*

3 Construction of the approximate solution

In this section we construct perturbatively an approximate solution of (2.23) such that the corresponding remainder satisfies (2.30). Approximations of vortex rings with varying degrees of accuracy were obtained by many authors, and typically rely on matched asymptotics expansions where the inner core of the vortex and the outer region are considered separately, see [44, 41, 23, 26, 27] in the inviscid case and [56, 14, 30] in the viscous case. Here we rather follow the direct approach introduced in [31] for interacting vortices in the plane, which does not rely on matched asymptotics techniques.

3.1 Expansion of the Biot-Savart formula

Our first task is to compute an accurate asymptotic expansion of the function $F(s)$ defined by (2.5) in the limit where $s \to 0$. This can be done by expressing F in terms of elliptic integrals, a procedure initiated in the early references [40, 51].

Lemma 3.1. *For $0 < s < 4$ we have the power series representation*

$$F(s) \,=\, \log\Bigl(\frac{8}{\sqrt{s}}\Bigr) \sum_{m=0}^{\infty} A_m s^m \,+\, \sum_{m=0}^{\infty} B_m s^m\,, \tag{3.1}$$

where A_m, B_m are real numbers. Moreover

$$A_0 = 1\,, \quad A_1 = \frac{3}{16}\,, \quad A_2 = -\frac{15}{1024}\,, \quad B_0 = -2\,, \quad B_1 = -\frac{1}{16}\,, \quad B_2 = \frac{31}{2048}\,. \tag{3.2}$$

Proof. If $s > 0$ and $k = 2/\sqrt{s+4} \in (0,1)$, it is straightforward to verify that

$$F(s) \,=\, \int_0^{\pi/2} \frac{1 - 2\sin^2\psi}{\sqrt{\sin^2\psi + s/4}}\,\mathrm{d}\psi \,=\, \frac{2-k^2}{k}\,K(k) - \frac{2}{k}\,E(k)\,, \tag{3.3}$$

where $K(k)$, $E(k)$ are the complete elliptic integrals with modulus k:

$$K(k) \,=\, \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2\theta}}\,\mathrm{d}\theta\,, \qquad E(k) \,=\, \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\,\mathrm{d}\theta\,.$$

We are interested in the limit where $s \to 0$, namely $k \to 1$. Introducing the complementary modulus $\kappa = \sqrt{1-k^2}$, we have the power series expansions (see [19])

$$\begin{aligned} K(k) \,&=\, \sum_{m=0}^{\infty} a_m^2\,\kappa^{2m}\Bigl(\log\frac{1}{\kappa} + 2b_m\Bigr)\,, \\ E(k) \,&=\, 1 + \sum_{m=0}^{\infty} \frac{2m+1}{2m+2}\,a_m^2\,\kappa^{2m+2}\Bigl(\log\frac{1}{\kappa} + b_m + b_{m+1}\Bigr)\,, \end{aligned} \tag{3.4}$$

where $a_0 = 1$, $b_0 = \log(2)$, and

$$a_m \,=\, \frac{1}{2}\cdot\frac{3}{4}\cdot\,\ldots\,\cdot\frac{2m{-}1}{2m}\,, \qquad b_m \,=\, \log(2) + \sum_{\ell=1}^{2m} \frac{(-1)^\ell}{\ell}\,, \qquad m \in \mathbb{N}^*\,.$$

Combining (3.3), (3.4), we obtain a representation of the form

$$F(s) \,=\, \frac{1+\kappa^2}{\sqrt{1-\kappa^2}}\,K(k) - \frac{2}{\sqrt{1-\kappa^2}}\,E(k) \,=\, \log\Bigl(\frac{4}{\kappa}\Bigr) \sum_{m=0}^{\infty} C_m \kappa^{2m} \,+\, \sum_{m=0}^{\infty} D_m \kappa^{2m}\,, \tag{3.5}$$

which converges for $0 < \kappa < 1$. Moreover, a direct calculation shows that

$$C_0 = 1\,, \quad C_1 = \frac{3}{4}\,, \quad C_2 = \frac{33}{64}\,, \quad D_0 = -2\,, \quad D_1 = -\frac{3}{4}\,, \quad D_2 = -\frac{81}{128}\,. \tag{3.6}$$

As $\kappa^2 = s/(s+4)$, the right-hand side of (3.5) can be written in the form (3.1), and using (3.6) we see that the first coefficients satisfy (3.2). $\square$

Remark 3.2. *Various asymptotic expansions of the stream function given by the Biot-Savart law (2.4) can be found in the literature [41, 23, 46, 56, 30], and are easily recovered using Lemma 3.1.*

We next consider the rescaled Biot-Savart formula (2.20), which can be written in the equivalent form

$$\phi(R,Z) \,=\, \frac{1}{2\pi}\int_{\Omega_\epsilon} K_\epsilon(R,Z;R',Z')\,\eta(R',Z')\,\mathrm{d}R'\,\mathrm{d}Z'\,, \tag{3.7}$$

where

$$K_\epsilon \,=\, \sqrt{(1{+}\epsilon R)(1{+}\epsilon R')}\;F\biggl(\frac{\epsilon^2 D^2}{(1{+}\epsilon R)(1{+}\epsilon R')}\biggr)\,, \qquad D^2 \,=\, (R{-}R')^2 + (Z{-}Z')^2\,. \tag{3.8}$$

To simplify the notations below, we define

$$\beta_\epsilon \,=\, \log\frac{1}{\epsilon}\,, \qquad L(R,Z;R',Z') \,=\, \log\Bigl(\frac{8}{D}\Bigr)\,. \tag{3.9}$$

Lemma 3.3. *For any (R,Z), $(R',Z') \in \mathbb{R}^2$ with $(R,Z) \neq (R',Z')$ and any sufficiently small $\epsilon > 0$, we have the expansion*

$$K_\epsilon \,=\, (\beta_\epsilon + L)\sum_{m=0}^{\infty}\epsilon^m P_m \,+\, \sum_{m=0}^{\infty}\epsilon^m Q_m\,, \tag{3.10}$$

where $P_m(R,Z;R',Z')$, $Q_m(R,Z;R',Z')$ are homogeneous polynomials of degree m in the three variables R, R', and $Z - Z'$. Moreover

$$\begin{aligned}
P_0 &= 1 & Q_0 &= -2\\
P_1 &= \tfrac{1}{2}(R+R') & Q_1 &= -\tfrac{1}{2}(R+R')\\
P_2 &= \tfrac{1}{16}(R-R')^2 + \tfrac{3}{16}(Z-Z')^2 \qquad & Q_2 &= \tfrac{1}{4}(R^2+R'^2) - \tfrac{1}{16}\,D^2\,.
\end{aligned} \tag{3.11}$$

Proof. If (R,Z), (R',Z') are as in the statement, we take $\epsilon > 0$ small enough so that

$$\max\bigl(|R|,|R'|\bigr) \,<\, \frac{1}{\epsilon}\,, \qquad \text{and} \qquad s \,:=\, \frac{\epsilon^2 D^2}{(1{+}\epsilon R)(1{+}\epsilon R')} \,<\, 4\,. \tag{3.12}$$

As $D \neq 0$ by assumption, we have $0 < s < 4$, so that we can apply expansion (3.1) to the quantity $F(s)$ in (3.8). In view of definitions (3.9) we have

$$\log\Bigl(\frac{8}{\sqrt{s}}\Bigr) \,=\, \beta_\epsilon + L + \frac{1}{2}\log(1+\epsilon R) + \frac{1}{2}\log(1+\epsilon R')\,. \tag{3.13}$$

We observe that the last two terms in (3.13), as well as the prefactor $\sqrt{(1{+}\epsilon R)(1{+}\epsilon R')}$ in (3.8) and each monomial s^m in the series (3.1), can be expanded into a power series in the three variables ϵR, $\epsilon R'$, and $\epsilon(Z-Z')$. Thus, combining (3.1) and (3.8), we obtain a representation of the form (3.10), where the first homogeneous polynomials P_m, Q_m are easily computed using the explicit values (3.2). $\square$

Remark 3.4. *In what follows, with a slight abuse of notation, we denote by L the integral operator on $\mathbb{R}^2$ given by the kernel* (3.9)*. For any continuous and rapidly decreasing function $\eta : \mathbb{R}^2 \to \mathbb{R}$, we thus have*

$$\big(L\eta\big)(R,Z) \,=\, \int_{\mathbb{R}^2} \log\biggl(\frac{8}{\sqrt{(R-R')^2+(Z-Z')^2}}\biggr)\,\eta(R',Z')\,\mathrm{d}R'\,\mathrm{d}Z'\,. \tag{3.14}$$

Similarly, we associate integral operators to the homogeneous polynomials P_m, Q_m in (3.10)*, and to the functions LP_m for all $m \in \mathbb{N}^*$.*

Definition 3.5. *Using the notation introduced in Remark 3.4, we define the linear operators*

$$\mathrm{BS}_0 \,=\, \frac{1}{2\pi}\,L\,, \quad \textit{and} \quad \mathrm{BS}_m \,=\, \frac{1}{2\pi}\Bigl(\beta_\epsilon P_m + LP_m + Q_m\Bigr)\,, \quad \textit{for all } m \in \mathbb{N}^*\,. \tag{3.15}$$

Note that, for $m \ge 1$, the linear operator BS_m depends on the parameter ϵ through the constant factor $\beta_\epsilon = \log(1/\epsilon)$, but for simplicity this mild dependence is not indicated explicitly. For convenience, we do not include the constant term $\beta_\epsilon P_0 + Q_0 \equiv \beta_\epsilon - 2$ in the definition of BS_0, because the stream function is only defined up to an additive constant. It is important to observe that, in (3.14) and in the corresponding definition of the integral operators P_m, Q_m, and LP_m, the integration is performed on the whole plane $\mathbb{R}^2$, rather than on the half-plane Ω_ϵ. This is justified because these operators will always be applied to functions that decay rapidly at infinity, so that the integration on $\mathbb{R}^2 \setminus \Omega_\epsilon$ gives a contribution of order $\mathcal{O}(\epsilon^\infty)$ as $\epsilon \to 0$, which can be neglected in our perturbative expansion. If $\eta : \mathbb{R}^2 \to \mathbb{R}$ is compactly supported, then according to Lemma 3.3 the following equality holds in any bounded region of $\mathbb{R}^2$:

$$\mathrm{BS}^\epsilon[\eta] \,=\, \frac{\beta_\epsilon - 2}{2\pi}\int_{\mathbb{R}^2} \eta(R',Z')\,\mathrm{d}R'\,\mathrm{d}Z' \,+\, \sum_{m=0}^{\infty} \epsilon^m\,\mathrm{BS}_m[\eta]\,, \tag{3.16}$$

provided $\epsilon > 0$ is sufficiently small. As was already mentioned, the first term in the right-hand side of (3.16) is a constant that can be omitted.

3.2 Function spaces and linear operators

We next introduce the function spaces in which we shall construct our approximate solution of (2.23). These spaces consist of functions that are defined on the whole space $\mathbb{R}^2$, and not just on the half-plane Ω_ϵ. Indeed, at each step of the approximation, the vorticity profile $\eta_m(R,Z,\beta_\epsilon)$ and the stream function $\phi_m(R,Z,\beta_\epsilon)$ in (2.24) are defined for all $(R,Z) \in \mathbb{R}^2$. To simplify the writing we often denote $X = (R,Z)$, and we use polar coordinates (ρ,ϑ) in $\mathbb{R}^2$ defined by the relations $R = \rho\cos\vartheta$, $Z = \rho\sin\vartheta$.

Following [35,36] we introduce the weighted L^2 space

$$\mathcal{Y} \,=\, \Bigl\{\eta \in L^2(\mathbb{R}^2)\,\Big|\, \int_{\mathbb{R}^2} |\eta(X)|^2\, e^{|X|^2/4}\,\mathrm{d}X < \infty\Bigr\}\,, \tag{3.17}$$

equipped with the scalar product $(\eta_1,\eta_2)_{\mathcal{Y}} = \int_{\mathbb{R}^2} \eta_1(X)\eta_2(X)\,e^{|X|^2/4}\,\mathrm{d}X$ and the associated norm. We also introduce the differential operator $\mathcal{L} : D(\mathcal{L}) \to \mathcal{Y}$ corresponding to (2.16), namely

$$\mathcal{L}\eta \,=\, \Delta\eta + \frac{1}{2}\,X\cdot\nabla\eta + \eta\,, \qquad \eta \in D(\mathcal{L}) \,=\, \Bigl\{\eta \in \mathcal{Y}\,\Big|\, \Delta\eta \in \mathcal{Y}\,,\ X\cdot\nabla\eta \in \mathcal{Y}\Bigr\}\,, \tag{3.18}$$

as well as the integro-differential operator $\Lambda : D(\Lambda) \to \mathcal{Y}$ defined by

$$\Lambda\eta \,=\, \frac{1}{2\pi}\Big(\big\{L\eta_0\,,\eta\big\} + \big\{L\eta\,,\eta_0\big\}\Big)\,, \qquad \eta \in D(\Lambda) \,=\, \Big\{\eta \in \mathcal{Y}\,\Big|\,\big\{L\eta_0\,,\eta\big\} \in \mathcal{Y}\Big\}\,, \tag{3.19}$$

where η_0 is the Gaussian function (2.22) and L denotes the integral operator (3.14). Here and in what follows the Poisson bracket is understood with respect to the rescaled variables (R, Z), so that $\{\phi, \eta\} = \partial_R\phi\,\partial_Z\eta - \partial_Z\phi\,\partial_R\eta$. We recall the following well-known properties:

Proposition 3.6. [35, 36, 47]
1) The linear operator $\mathcal{L}$ is self-adjoint *in $\mathcal{Y}$, with purely discrete spectrum*

$$\sigma(\mathcal{L}) \,=\, \Big\{-\frac{n}{2}\,\Big|\, n = 0, 1, 2, \dots\Big\}\,.$$

The kernel of $\mathcal{L}$ is one-dimensional and spanned by the Gaussian function η_0. More generally, for any $n \in \mathbb{N}$, the eigenspace corresponding to the eigenvalue $\lambda_n = -n/2$ is spanned by the $n+1$ Hermite functions $\partial^\alpha\eta_0$ where $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ and $\alpha_1 + \alpha_2 = n$.
2) The linear operator Λ is skew-adjoint *in $\mathcal{Y}$, so that $\Lambda^* = -\Lambda$. Moreover,*

$$\mathrm{Ker}(\Lambda) \,=\, \mathcal{Y}_0 \oplus \big\{\beta_1\partial_R\eta_0 + \beta_2\partial_Z\eta_0 \,\big|\, \beta_1, \beta_2 \in \mathbb{R}\big\}\,, \tag{3.20}$$

where $\mathcal{Y}_0 \subset \mathcal{Y}$ is the subspace of all radially symmetric elements of $\mathcal{Y}$.

A crucial observation is that both operators $\mathcal{L}$, Λ are invariant under rotations about the origin in $\mathbb{R}^2$. It is therefore advantageous to decompose the space $\mathcal{Y}$ into a direct sum

$$\mathcal{Y} \,=\, \overset{\infty}{\underset{n=0}{\oplus}}\, \mathcal{Y}_n\,, \tag{3.21}$$

where $\mathcal{Y}_0 \subset \mathcal{Y}$ is as in Proposition 3.6 and, for all $n \ge 1$, the subspace $\mathcal{Y}_n \subset \mathcal{Y}$ consists of all functions $\eta \in \mathcal{Y}$ such that $\eta(\rho\cos\vartheta, \rho\sin\vartheta) = a_1(\rho)\cos(n\vartheta) + a_2(\rho)\sin(n\vartheta)$ for some $a_1, a_2 : \mathbb{R}_+ \to \mathbb{R}$. It is clear that $\mathcal{Y}_n \perp \mathcal{Y}_{n'}$ if $n \neq n'$. In particular, in view of (3.20), we have $\mathcal{Y}_n \in \mathrm{Ker}(\Lambda)^\perp$ for all $n \ge 2$. When $n = 1$, the functions $\partial_R\eta_0$, $\partial_Z\eta_0$ belong to $\mathcal{Y}_1 \cap \mathrm{Ker}(\Lambda)$, and we define

$$\mathcal{Y}_1' \,=\, \mathcal{Y}_1 \cap \mathrm{Ker}(\Lambda)^\perp \,=\, \Big\{\eta \in \mathcal{Y}_1 \,\Big|\, \int_{\mathbb{R}^2} \eta(R, Z) R \,\mathrm{d}R\,\mathrm{d}Z = \int_{\mathbb{R}^2} \eta(R, Z) Z \,\mathrm{d}R\,\mathrm{d}Z = 0\Big\}\,. \tag{3.22}$$

Since Λ is skew-adjoint, we know that $\mathrm{Ker}(\Lambda)^\perp = \overline{\mathrm{Ran}(\Lambda)}$, but the image of Λ is not dense in $\mathcal{Y}$ and, therefore, we cannot solve the equation $\Lambda\eta = f$ for any $f \in \mathrm{Ker}(\Lambda)^\perp$. As is shown in [37, 31], the problem disappears if one assumes in addition that f belongs to a smaller space such as

$$\mathcal{Z} \,=\, \Big\{\eta : \mathbb{R}^2 \to \mathbb{R} \,\Big|\, e^{|X|^2/4}\eta \in \mathcal{S}_*(\mathbb{R}^2)\Big\} \,\subset\, \mathcal{Y}\,, \tag{3.23}$$

where $\mathcal{S}_*(\mathbb{R}^2)$ denotes the space of all smooth functions which are slowly growing at infinity. More precisely, a smooth function $w : \mathbb{R}^2 \to \mathbb{R}$ belongs to $\mathcal{S}_*(\mathbb{R}^2)$ if, for any $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, there exist $C > 0$ and $N \in \mathbb{N}$ such that $|\partial^\alpha w(X)| \le C(1 + |X|)^N$ for all $X \in \mathbb{R}^2$.

Remark 3.7. *We do not need to specify the topology of the space $\mathcal{Z}$, but the following notation will be useful. If $f \in \mathcal{Z}$ depends on a small parameter $\epsilon > 0$, we say that $f = \mathcal{O}(\epsilon)$ in $\mathcal{Z}$ if, for any $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, there exist $C > 0$ and $N \in \mathbb{N}$ such that $|\partial^\alpha f(X)| \le C\epsilon(1+|X|)^N e^{-|X|^2/4}$ for all $X \in \mathbb{R}^2$.*

To formulate the main technical result of this section, we introduce the notation

$$\varphi(\rho) \;=\; \frac{1}{2\pi\rho^2}\bigl(1-e^{-\rho^2/4}\bigr)\,, \qquad h(\rho) \;=\; \frac{\rho^2/4}{e^{\rho^2/4}-1}\,, \qquad \rho>0\ . \tag{3.24}$$

The following statement is a slight extension of [31, Lemma 4]. For the reader's convenience, we give a short proof of it in Section A.1, emphasizing the case $n=1$ which was not treated in [31].

Proposition 3.8. *If $n\ge 2$ and $f\in\mathcal{Y}_n\cap\mathcal{Z}$, or if $n=1$ and $f\in\mathcal{Y}_1'\cap\mathcal{Z}$, there exists a unique $\eta\in\mathcal{Y}_n\cap\mathcal{Z}$ (respectively, a unique $\eta\in\mathcal{Y}_1'\cap\mathcal{Z}$) such that $\Lambda\eta=f$. In particular, if $f=a(\rho)\sin(n\vartheta)$, then $\eta=\omega(\rho)\cos(n\vartheta)$, where*

$$\omega(\rho) \;=\; h(\rho)\Omega(\rho)+\frac{a(\rho)}{n\varphi(\rho)}\,, \qquad \rho>0\ , \tag{3.25}$$

and where $\Omega:(0,\infty)\to\mathbb{R}$ is the unique solution of the differential equation

$$-\Omega''(\rho)-\frac{1}{\rho}\,\Omega'(\rho)+\Bigl(\frac{n^2}{\rho^2}-h(\rho)\Bigr)\Omega(\rho) \;=\; \frac{a(\rho)}{n\varphi(\rho)}\,, \qquad \rho>0\,, \tag{3.26}$$

such that $\Omega(\rho)=\mathcal{O}(\rho^n)$ as $\rho\to 0$ and $\Omega(\rho)=\mathcal{O}(\rho^{-n})$ as $\rho\to\infty$.

Remark 3.9. *As was observed in [31], if $f=a(\rho)\cos(n\vartheta)$, then $\eta=-\omega(\rho)\sin(n\vartheta)$, where ω is still given by* (3.25), (3.26). *The general case where $f=a_1(\rho)\cos(n\vartheta)+a_2(\rho)\sin(n\vartheta)$ follows by linearity.*

In the construction of an approximate solution of (2.23), we shall encounter linear equations of the form

$$\delta\bigl(\kappa-\mathcal{L}\bigr)\eta^\delta+\Lambda\eta^\delta \;=\; f\,, \tag{3.27}$$

where $\kappa>0$ is fixed and the parameter $\delta>0$ can be arbitrarily small. Proposition 3.6 implies that the linear operator $\delta(\kappa-\mathcal{L})+\Lambda$ is invertible in $\mathcal{Y}$, so that (3.27) has a unique solution η^δ for any $f\in\mathcal{Y}$. In general, the best estimate we can hope for is

$$\|\eta^\delta\|_{\mathcal{Y}} \;=\; \bigl\|\bigl(\delta(\kappa-\mathcal{L})+\Lambda\bigr)^{-1}f\bigr\|_{\mathcal{Y}} \;\le\; \frac{1}{\kappa\delta}\,\|f\|_{\mathcal{Y}}\,. \tag{3.28}$$

However, if f satisfies the assumptions of Proposition 3.8, the solution η^δ admits a regular expansion in powers of the small parameter δ. More precisely:

Proposition 3.10. *Assume that $n\ge 2$ and $f\in\mathcal{Y}_n\cap\mathcal{Z}$, or that $n=1$ and $f\in\mathcal{Y}_1'\cap\mathcal{Z}$. For any fixed $\kappa>0$ and any $\delta>0$, equation* (3.27) *has a unique solution $\eta^\delta\in\mathcal{Y}_n$ (respectively, $\eta^\delta\in\mathcal{Y}_1'$). Moreover, for each nonzero $N\in\mathbb{N}$, there exists a constant $C>0$, depending only on f and N, such that*

$$\Bigl\|\eta^\delta-\sum_{m=0}^{N-1}\delta^m\hat\eta_m\Bigr\|_{\mathcal{Y}} \;\le\; C\delta^N\,, \tag{3.29}$$

where the profiles $\hat\eta_m\in\mathcal{Y}_n\cap\mathcal{Z}$ (respectively, $\hat\eta_m\in\mathcal{Y}_1'\cap\mathcal{Z}$) are determined by the relations $\Lambda\hat\eta_0=f$ and $\Lambda\hat\eta_m=(\mathcal{L}-\kappa)\hat\eta_{m-1}$ for $m\ge 1$.

Proof. Assume first that $n\ge 2$. Since the space $\mathcal{Y}_n$ is invariant under the action of both operators $\mathcal{L}$ and Λ, it is clear that $\eta^\delta\in\mathcal{Y}_n$ if $f\in\mathcal{Y}_n$. If we suppose in addition that $f\in\mathcal{Z}$, Proposition 3.8 shows that there is a unique $\hat\eta_0\in\mathcal{Y}_n\cap\mathcal{Z}$ such that $\Lambda\hat\eta_0=f$. A direct calculation then shows that $(\mathcal{L}-\kappa)\hat\eta_0\in\mathcal{Y}_n\cap\mathcal{Z}$, so that we can define $\hat\eta_1\in\mathcal{Y}_n\cap\mathcal{Z}$ as the unique solution of $\Lambda\hat{\eta}_1 = (\mathcal{L}-\kappa)\hat{\eta}_0$. Repeating this procedure, we construct the profiles $\hat{\eta}_m$ for $m = 0,\dots,N$, and we define $\tilde{\eta} = \eta^\delta - \bigl(\hat{\eta}_0 + \delta\hat{\eta}_1 + \cdots + \delta^N\hat{\eta}_N\bigr)$, so that

$$\Bigl(\delta(\kappa-\mathcal{L})+\Lambda\Bigr)\tilde{\eta} \,=\, f - \Bigl(\delta\bigl(\kappa-\mathcal{L}\bigr)+\Lambda\Bigr)\sum_{m=0}^{N}\delta^m\hat{\eta}_m \,=\, \delta^{N+1}\bigl(\mathcal{L}-\kappa\bigr)\hat{\eta}_N\,. \tag{3.30}$$

Estimate (3.28) then gives the crude bound $\|\tilde{\eta}\|_{\mathcal{Y}} \le C\delta^N$, which nevertheless implies (3.29). The proof is identical if $n = 1$ and $f \in \mathcal{Y}_1' \cap \mathcal{Z}$. □

3.3 First order approximation

We now begin the construction of an approximate solution of (2.23) in the form (2.24), (2.25). We recall that, for an exact solution, the stream function is determined by the relation (2.20), which we write in the compact form $\phi = \mathrm{BS}^\epsilon[\eta]$. For our approximate solution, we expand the Biot-Savart operator as in (3.16), omitting the constant term in the right-hand side. We thus obtain the formal relation

$$\sum_{m=0}^{\infty}\epsilon^m\,\mathrm{BS}_m\biggl[\,\sum_{m=0}^{M}\epsilon^m\eta_m\biggr] \,=\, \sum_{m=0}^{M}\epsilon^m\phi_m + \mathcal{O}\bigl(\epsilon^{M+1}\bigr)\,,$$

which we assume to be satisfied order by order in ϵ, up to order M. This leads to the relations $\phi_0 = \mathrm{BS}_0[\eta_0]$, $\phi_1 = \mathrm{BS}_0[\eta_1] + \mathrm{BS}_1[\eta_0]$, and more generally

$$\phi_m \,=\, \mathrm{BS}_0[\eta_m] + \mathrm{BS}_1[\eta_{m-1}] + \cdots + \mathrm{BS}_{m-1}[\eta_1] + \mathrm{BS}_m[\eta_0]\,. \tag{3.31}$$

In particular, in view of (2.22) and (3.15), the leading order of our approximation is

$$\eta_0(R,Z) \,=\, \frac{1}{4\pi}\,e^{-(R^2+Z^2)/4}\,, \qquad \phi_0(R,Z) \,=\, \frac{1}{2\pi}\bigl(L\eta_0\bigr)(R,Z)\,, \tag{3.32}$$

where L is the integral operator (3.14). The stream function ϕ_0 has the expression

$$\phi_0(R,Z) \,=\, \phi_0(0) - \frac{1}{4\pi}\mathrm{Ein}\biggl(\frac{R^2{+}Z^2}{4}\biggr)\,, \qquad \text{where} \quad \mathrm{Ein}(x) \,=\, \int_0^x \frac{1-e^{-t}}{t}\,\mathrm{d}t\,, \tag{3.33}$$

so that ϕ_0 is radially symmetric and $\phi_0(R,Z) \sim -(2\pi)^{-1}\log\rho$ as $\rho := (R^2+Z^2)^{1/2} \to +\infty$. The value at the origin does not play a big role in our analysis, but can be computed too, see Section A.2 :

$$\phi_0(0) \,=\, \frac{\log(2)}{\pi} + \frac{\gamma_E}{4\pi}\,, \qquad \text{where } \gamma_E \text{ is Euler's constant.}$$

Before proceeding further, we estimate the time derivative of the quantity $\epsilon = \sqrt{\nu t}/\bar{r}(t)$ introduced in (2.14). In view of (2.25), we have

$$t\dot{\epsilon} \,=\, \frac{\epsilon}{2} - \frac{\epsilon t\dot{\bar{r}}}{\bar{r}} \,=\, \frac{\epsilon}{2} - \frac{\epsilon t}{\bar{r}}\sum_{m=0}^{M-1}\epsilon^m\,\dot{\bar{r}}_m\,. \tag{3.34}$$

At this stage the radial velocity profiles $\dot{\bar{r}}_m$ are not determined yet, but in view of Remark 2.3 we can anticipate the fact that $|\dot{\bar{r}}| = (\Gamma/r_0)\cdot\mathcal{O}(\delta)$ as $\delta \to 0$. Since $\delta t = (r_0^2/\Gamma)\cdot\mathcal{O}(\epsilon^2)$ by Remark 2.2, it follows that $\bar{r}(t) = r_0\bigl(1+\mathcal{O}(\epsilon^2)\bigr)$ and that $t\dot{\epsilon} = \epsilon/2 + \mathcal{O}(\epsilon^3)$ as $\epsilon \to 0$.

With that observation in mind, we substitute the expansions (2.24), (2.25) into the evolution equation (2.23), keeping only the terms that are exactly of order ϵ or $\epsilon\beta_\epsilon$. This gives the relation

$$\bigl\{\phi_1\,,\eta_0\bigr\}+\bigl\{\phi_0\,,\eta_1\bigr\}+\eta_0\partial_Z\phi_0-\frac{r_0}{\Gamma}\Bigl(\dot{\bar r}_0\,\partial_R\eta_0+\dot{\bar z}_0\,\partial_Z\eta_0\Bigr) \;=\; \delta\Bigl[\partial_R\eta_0+\bigl(\mathcal{L}-\tfrac{1}{2}\bigr)\eta_1-t\partial_t\eta_1\Bigr]\,. \tag{3.35}$$

To solve (3.35) we first impose the relation

$$\dot{\bar r}_0 \;=\; -\frac{\Gamma\delta}{r_0}\,, \tag{3.36}$$

which ensures that the terms involving $\partial_R\eta_0$ cancel exactly. We also assume that η_1 does not depend on β_ϵ, so that $\partial_t\eta_1=0$ (this property will be verified later). On the other hand, from (3.31) with $m=1$ we deduce that $\{\phi_1\,,\eta_0\}=\{\mathrm{BS}_0[\eta_1]\,,\eta_0\}+\{\mathrm{BS}_1[\eta_0]\,,\eta_0\}$, where BS_0, BS_1 are defined in (3.15). Using (3.32) and the definition (3.19) of the linear operator Λ, we thus find

$$\begin{aligned}\bigl\{\phi_1\,,\eta_0\bigr\}+\bigl\{\phi_0\,,\eta_1\bigr\} \;&=\; \frac{1}{2\pi}\Bigl(\bigl\{L\eta_1\,,\eta_0\bigr\}+\bigl\{L\eta_0\,,\eta_1\bigr\}\Bigr)+\bigl\{\mathrm{BS}_1[\eta_0]\,,\eta_0\bigr\}\\ &=\; \Lambda\eta_1+\frac{\beta_\epsilon-1}{2\pi}\bigl\{P_1\eta_0\,,\eta_0\bigr\}+\frac{1}{2\pi}\bigl\{LP_1\eta_0\,,\eta_0\bigr\}\,,\end{aligned}$$

where in the second line we used the definition (3.15) of BS_1 and the fact that $Q_1=-P_1$ in view of (3.11). Now, elementary calculations that are reproduced in Section A.2 show that

$$\bigl\{P_1\eta_0\,,\eta_0\bigr\} \;=\; \frac{1}{2}\,\partial_Z\eta_0\,, \quad\text{and}\quad \frac{1}{2\pi}\bigl\{LP_1\eta_0\,,\eta_0\bigr\} \;=\; \frac{1}{2}\,\partial_Z\bigl(\phi_0\eta_0\bigr)\,. \tag{3.37}$$

It follows that we can write (3.35) in the equivalent form

$$\Lambda\eta_1+\delta\bigl(\tfrac{1}{2}-\mathcal{L}\bigr)\eta_1 \;=\; \Bigl(\frac{r_0}{\Gamma}\,\dot{\bar z}_0-\frac{\beta_\epsilon-1}{4\pi}\Bigr)\partial_Z\eta_0-\frac{3}{2}\,(\partial_Z\phi_0)\eta_0-\frac{1}{2}\,\phi_0\partial_Z\eta_0\,. \tag{3.38}$$

Using the explicit expressions (3.32), (3.33) of the profiles η_0,ϕ_0, it is straightforward to verify that the right-hand side of (3.38), which we denote by $-\mathcal{R}_1$, belongs to $\mathcal{Y}_1\cap\mathcal{Z}$, where $\mathcal{Y}_1$, $\mathcal{Z}$ are the function spaces defined in (3.21), (3.23). Therefore, according to Proposition 3.10, the linear equation (3.38) has a unique solution $\eta_1\in\mathcal{Y}_1$ for any $\delta>0$, and that solution has a well-defined limit as $\delta\to 0$ if and only if $\mathcal{R}_1\in(\ker\Lambda)^\perp$, namely if $\mathcal{R}_1\in\mathcal{Y}_1'$. In view of (3.22), this gives the solvability condition $\int_{\mathbb{R}^2}\mathcal{R}_1 Z\,\mathrm{d}R\,\mathrm{d}Z=0$, which determines uniquely the value of the constant $\dot{\bar z}_0$ in (3.38). The calculations are reproduced in Section A.2, and yield the following expression of the vertical velocity to leading order:

$$\dot{\bar z}_0 \;=\; \frac{\Gamma}{4\pi r_0}\Bigl(\beta_\epsilon-1+2v\Bigr)\,, \qquad\text{where}\quad v \;=\; \frac{3}{4}\,\log(2)+\frac{1}{4}\,\gamma_E+\frac{1}{4}\,. \tag{3.39}$$

Here again $\gamma_E=0,5772\ldots$ denotes Euler's constant.

Remark 3.11. *The formula* (3.39)*, including the leading term* $\beta_\epsilon=\log(1/\epsilon)$ *and the correct value of the constant* $2v-1$*, was established by Saffman [55], see also Fukumoto & Moffatt [30].*

We assume henceforth that $\dot{\bar z}_0$ is given by (3.39), so that (3.38) reduces to

$$\Lambda\eta_1+\delta\bigl(\tfrac{1}{2}-\mathcal{L}\bigr)\eta_1 \;=\; \frac{v}{2\pi}\,\partial_Z\eta_0-\frac{3}{2}\,(\partial_Z\phi_0)\eta_0-\frac{1}{2}\,\phi_0\partial_Z\eta_0\,, \tag{3.40}$$

where the right-hand side $-\mathcal{R}_1$ now belongs to $\mathcal{Y}_1'\cap\mathcal{Z}$ and is independent of ϵ. Equation (3.40) is of the form (3.27), and can be solved using Proposition 3.10. For our purposes, it is sufficient to consider the *approximate* solution corresponding to the choice $N=2$ in (3.29), which reads

$$\eta_1(R,Z) \;=\; R\,\eta_{10}(\rho)+\delta Z\,\eta_{11}(\rho)\,, \qquad \rho=\sqrt{R^2+Z^2}\,, \tag{3.41}$$

where $\Lambda(R\,\eta_{10}) = -\mathcal{R}_1$ and $\Lambda(Z\,\eta_{11}) = (\mathcal{L} - \frac{1}{2})(R\,\eta_{10})$. Note that $\eta_1 \in \mathcal{Y}' \cap \mathcal{Z}$, which implies in particular that the functions η_{10}, η_{11} are smooth and have a Gaussian decay at infinity. The corresponding stream function $\phi_1 = \mathrm{BS}_0[\eta_1] + \mathrm{BS}_1[\eta_0]$ is computed in Section A.2 and takes the form

$$\phi_1(R, Z, \beta_\epsilon) \;=\; \frac{\beta_\epsilon - 1}{4\pi}\,R + \frac{R}{2}\,\phi_0 - \partial_R\phi_0 + R\,\phi_{10}(\rho) + \delta Z\,\phi_{11}(\rho)\,, \tag{3.42}$$

where $R\,\phi_{10} = \mathrm{BS}_0[R\,\eta_{10}]$ and $Z\,\phi_{11} = \mathrm{BS}_0[Z\,\eta_{11}]$. One can check that the functions ϕ_{10}, ϕ_{11} are smooth and decay at least like $1/\rho^2$ as $\rho \to +\infty$. Note that ϕ_1 involves the time-dependent term $\beta_\epsilon = \log(1/\epsilon)$, so that $\partial_t\phi_1 \neq 0$. With the choices (3.36), (3.39), (3.41), and (3.42), the relation (3.35) is not satisfied exactly, but the difference of both members is $\mathcal{O}(\delta^2)$ in the topology of $\mathcal{Z}$, which is all we need.

3.4 Second order approximation

We next compute the second order terms in the asymptotic expansion (2.24). As we shall see, it is consistent at this stage to take

$$\dot{\bar{r}}_1 \;=\; \dot{\bar{z}}_1 \;=\; 0\,, \tag{3.43}$$

so we make that assumption from now on. As before, we deduce from (3.34), (3.36), (3.43) that $\bar{r}(t) = r_0\big(1 + \mathcal{O}(\epsilon^2)\big)$ and $t\dot{\epsilon} = \epsilon/2 + \mathcal{O}(\epsilon^3)$ as $\epsilon \to 0$. Substituting (2.24), (2.25) into (2.23) and keeping only the terms involving ϵ^2 or $\epsilon^2\beta_\epsilon$, we obtain the relation

$$\begin{aligned} \big\{\phi_2\,,\eta_0\big\} + \big\{\phi_1\,,\eta_1 - R\eta_0\big\} + \big\{\phi_0\,,\eta_2 - R\eta_1 + R^2\eta_0\big\} - \frac{r_0}{\Gamma}\Big(\dot{\bar{r}}_0\,\partial_R\eta_1 + \dot{\bar{z}}_0\,\partial_Z\eta_1\Big)& \\ = \delta\Big[\big(\mathcal{L} - 1\big)\eta_2 + \partial_R(\eta_1 - R\eta_0) - t\partial_t\eta_2\Big]&\,. \end{aligned} \tag{3.44}$$

In view of (3.36), the terms involving $\partial_R\eta_1$ cancel exactly. Moreover, we know from (3.15), (3.31) that

$$\phi_2 \;=\; \frac{1}{2\pi}\Big(L\eta_2 + \big(\beta_\epsilon P_1 + LP_1 + Q_1\big)\eta_1 + \big(\beta_\epsilon P_2 + LP_2 + Q_2\big)\eta_0\Big)\,, \tag{3.45}$$

where the notations are introduced in Lemma 3.3. Recalling the definition (3.19) of the operator Λ, we can thus write (3.44) in the equivalent form

$$\Lambda\eta_2 + \delta\Big(t\partial_t\eta_2 + \big(1 - \mathcal{L}\big)\eta_2\Big) + \mathcal{R}_2 \;=\; 0\,, \tag{3.46}$$

where

$$\begin{aligned} \mathcal{R}_2 \;=&\; \frac{1}{2\pi}\big\{(\beta_\epsilon - 1)P_1\eta_1 + LP_1\eta_1\,,\eta_0\big\} + \frac{1}{2\pi}\big\{\beta_\epsilon P_2\eta_0 + LP_2\eta_0 + Q_2\eta_0\,,\eta_0\big\} \\ &+ \big\{\phi_1\,,\eta_1\big\} + (\partial_Z\phi_1)\eta_0 + (\partial_Z\phi_0)\eta_1 - R\Big(\big\{\phi_1\,,\eta_0\big\} + \big\{\phi_0\,,\eta_1\big\} + 2(\partial_Z\phi_0)\eta_0\Big) \\ &+ \delta\partial_R(R\eta_0) - \frac{r_0\dot{\bar{z}}_0}{\Gamma}\,\partial_Z\eta_1\,. \end{aligned} \tag{3.47}$$

We have the following result, whose proof is postponed to Section A.3 :

Lemma 3.12. *The function $\mathcal{R}_2$ defined in* (3.47) *belongs to $(\delta\mathcal{Y}_0 + \mathcal{Y}_2) \cap \mathcal{Z}$ and satisfies*

$$\mathcal{R}_2 \;=\; \frac{3\beta_\epsilon}{16\pi}\,RZ\eta_0 + RZ\chi_{20} + \delta\Big(\chi_{21} + (R^2 - Z^2)\chi_{22}\Big) + \delta^2 RZ\chi_{23}\,, \tag{3.48}$$

for some (time-independent) radially symmetric functions $\chi_{20}, \chi_{21}, \chi_{22}, \chi_{23} \in \mathcal{Y}_0 \cap \mathcal{Z}$.

In view of (3.48), we look for a solution of (3.46) in the form $\eta_2 = \beta_\epsilon \hat\eta_{20} + \hat\eta_{21} + \hat\eta_{22}$, where $\hat\eta_{20}, \hat\eta_{21} \in \mathcal{Y}_2$ and $\hat\eta_{22} \in \mathcal{Y}_0$ do not depend on β_ϵ. Inserting this ansatz into (3.46) and using the fact that $t\partial_t \beta_\epsilon = -1/2 + \mathcal{O}(\epsilon^2)$, we obtain the system

$$\begin{aligned}
&\Lambda\hat\eta_{20} + \delta\big(1-\mathcal{L}\big)\hat\eta_{20} + \frac{3}{16\pi}\,RZ\eta_0 \,=\, 0\,,\\
&\Lambda\hat\eta_{21} + \delta\big(1-\mathcal{L}\big)\hat\eta_{21} - \frac{\delta}{2}\,\hat\eta_{20} + \mathcal{P}_2\Big(\mathcal{R}_2 - \frac{3\beta_\epsilon}{16\pi}\,RZ\eta_0\Big) \,=\, 0\,,\\
&\delta\big(1-\mathcal{L}\big)\hat\eta_{22} + \mathcal{P}_0\mathcal{R}_2 \,=\, 0\,,
\end{aligned} \tag{3.49}$$

where $\mathcal{P}_n$ denotes the orthogonal projection in $\mathcal{Y}$ onto the subspace $\mathcal{Y}_n$. The first two equations in (3.49) have a unique solution by Proposition 3.10, and as in Section 3.3 we are satisfied with the approximate solution corresponding to (3.29) with $N = 2$. Since $\mathcal{P}_0\mathcal{R}_2 = \delta\chi_{21}$ by (3.48), the third equation reduces to $(1-\mathcal{L})\hat\eta_{22} + \chi_{21} = 0$, which also has a unique solution due to Proposition 3.6. We conclude that we can choose η_2 in the form

$$\eta_2(R,Z,\beta_\epsilon) \,=\, \beta_\epsilon\Big((R^2{-}Z^2)\eta_{20} + \delta RZ\eta_{21}\Big) + (R^2{-}Z^2)\eta_{22} + \delta RZ\eta_{23} + \eta_{24}\,, \tag{3.50}$$

where all functions η_{2j} belong to $\mathcal{Y}_0 \cap \mathcal{Z}$. Using (3.45) and the calculations at the beginning of Section A.3, we obtain a similar expression for the corresponding stream function

$$\phi_2(R,Z,\beta_\epsilon) \,=\, \beta_\epsilon\Big((R^2{-}Z^2)\phi_{20} + \delta RZ\phi_{21}\Big) + (R^2{-}Z^2)\phi_{22} + \delta RZ\phi_{23} + \beta_\epsilon\phi_{24} + \phi_{25}\,, \tag{3.51}$$

where the functions ϕ_{2j} are radially symmetric and belong to $\mathcal{S}_*(\mathbb{R}^2)$. With these choices, the left-hand side of (3.46) is of size $\mathcal{O}(\beta_\epsilon\delta^2 + \epsilon^2\delta)$ in the topology of $\mathcal{Z}$.

3.5 Third order approximation

The third order in the asymptotic expansion (2.24) can be computed in a similar way. According to (3.36), (3.43) and Remark 2.3, we have $\bar r(t) = r_0\big(1 - \epsilon^2 + \mathcal{O}(\epsilon^{4-})\big)$ as $\epsilon \to 0$, and using (3.34) we deduce that $t\dot\epsilon = \epsilon/2 + \epsilon^3 + \mathcal{O}(\epsilon^{5-})$. So, if we substitute (2.24), (2.25) into (2.23) and keep only the terms involving ϵ^3 or $\epsilon^3\beta_\epsilon$, we find

$$\begin{aligned}
&\big\{\phi_3\,,\eta_0\big\} + \big\{\phi_2\,,\eta_1 - R\eta_0\big\} + \big\{\phi_1\,,\eta_2 - R\eta_1 + R^2\eta_0\big\} + \big\{\phi_0\,,\eta_3 - R\eta_2 + R^2\eta_1 - R^3\eta_0\big\}\\
&\qquad - \frac{r_0}{\Gamma}\Big(\dot{\bar r}_0\,\partial_R\eta_2 + \big(\dot{\bar r}_2{-}\dot{\bar r}_0\big)\,\partial_R\eta_0 + \dot{\bar z}_0\,\partial_Z\eta_2 + \big(\dot{\bar z}_2{-}\dot{\bar z}_0\big)\,\partial_Z\eta_0\Big)\\
&\qquad = \delta\Big[\big(\mathcal{L} - \tfrac{3}{2}\big)\eta_3 + \partial_R(\eta_2 - R\eta_1 + R^2\eta_0) - t\partial_t\eta_3 - \eta_1\Big]\,.
\end{aligned} \tag{3.52}$$

On the other hand, using (3.31) with $m = 3$ and (3.15), we obtain

$$\phi_3 \,=\, \sum_{m=0}^{3} \mathrm{BS}_m[\eta_{3-m}] \,=\, \frac{1}{2\pi}\,L\eta_3 + \frac{1}{2\pi}\sum_{m=1}^{3}\Big((\beta_\epsilon + L)P_m + Q_m\Big)\eta_{3-m}\,, \tag{3.53}$$

where the polynomials P_m, Q_m are defined in (3.11) for $m \le 2$ and in (A.19) for $m = 3$. We can thus write (3.52) in the form

$$\Lambda\eta_3 + \delta\Big(t\partial_t\eta_3 + \big(\tfrac{3}{2} - \mathcal{L}\big)\eta_3\Big) + \mathcal{R}_3 \,=\, 0\,, \tag{3.54}$$

where

$$
\begin{aligned}
\mathcal{R}_3 \;=\; & \frac{1}{2\pi}\Big\{\sum_{m=1}^{3}\big((\beta_\epsilon + L)P_m + Q_m\big)\eta_{3-m}\,,\,\eta_0\Big\} + \big\{\phi_2\,,\eta_1 - R\eta_0\big\} \\
& + \big\{\phi_1\,,\eta_2 - R\eta_1 + R^2\eta_0\big\} - \big\{\phi_0\,,R\eta_2 - R^2\eta_1 + R^3\eta_0\big\} \\
& - \frac{r_0}{\Gamma}\Big(\big(\dot{\bar r}_2 - \dot{\bar r}_0\big)\partial_R\eta_0 + \big(\dot{\bar z}_2 - \dot{\bar z}_0\big)\partial_Z\eta_0 + \dot{\bar z}_0\partial_Z\eta_2\Big) + \delta\partial_R\big(R\eta_1 - R^2\eta_0\big) + \delta\eta_1\,.
\end{aligned}
\tag{3.55}
$$

Lemma 3.13. *The function* $\mathcal{R}_3$ *defined in* (3.55) *belongs to* $(\mathcal{Y}_1 + \mathcal{Y}_3) \cap \mathcal{Z}$ *and satisfies*

$$
\mathcal{R}_3 \;=\; \beta_\epsilon\Big(R^2 Z\chi_{30} + Z\chi_{31}\Big) + R^2 Z\chi_{32} + Z\chi_{33} + \mathcal{O}(\delta)\,,
\tag{3.56}
$$

for some (time-independent) radially symmetric functions $\chi_{30}, \chi_{31}, \chi_{32}, \chi_{33} \in \mathcal{Y}_0 \cap \mathcal{Z}$.

The proof of Lemma 3.13 is a direct calculation that is briefly outlined in Section A.4. In particular we verify there that the quantity $\mathcal{R}_3$ does not contain any factor β_ϵ^2, which is perhaps surprising since ϕ_1, ϕ_2, and η_2 all contain at least one term multiplied by β_ϵ. We do not need the expression of the $\mathcal{O}(\delta)$ terms in (3.56), but they can be computed too and are found to be of the form $\delta\beta_\epsilon\big(R^3\tilde\chi_{30} + R\tilde\chi_{31}\big) + \delta\big(R^3\tilde\chi_{32} + R\tilde\chi_{33}\big)$, where $\tilde\chi_{3j}$ are radially symmetric functions. Finally we mention that $\mathcal{R}_3$ also includes terms of the form (3.56) that are multiplied by δ^2.

As can be seen from the last line of (3.55), there is a unique way to choose the quantities $\dot{\bar r}_2$ and $\dot{\bar z}_2$ so that $\mathcal{R}_3 \in \mathcal{Y}_1' + \mathcal{Y}_3$, where $\mathcal{Y}_1'$ is the subspace defined in (3.22). In view of (3.56), (3.36), (3.39), the velocities obtained in this way satisfy

$$
\frac{r_0}{\Gamma}\,\dot{\bar r}_2 \;=\; \big(c_1\beta_\epsilon + c_2\big)\delta\,, \qquad \frac{r_0}{\Gamma}\,\dot{\bar z}_2 \;=\; c_3\beta_\epsilon + c_4 + \mathcal{O}(\delta^2)\,,
\tag{3.57}
$$

for some constants c_1, c_2, c_3, c_4. Now, decomposing $\mathcal{R}_3 = \beta_\epsilon\mathcal{R}_{31} + \mathcal{R}_{32}$ where $\mathcal{R}_{31}, \mathcal{R}_{32}$ are independent of β_ϵ, we look for a solution of (3.54) in the form $\eta_3 = \beta_\epsilon\hat\eta_{31} + \hat\eta_{32}$ where

$$
\Lambda\hat\eta_{31} + \delta\big(\tfrac{3}{2} - \mathcal{L}\big)\hat\eta_{31} + \mathcal{R}_{31} \;=\; 0\,, \qquad \Lambda\hat\eta_{32} + \delta\big(\tfrac{3}{2} - \mathcal{L}\big)\hat\eta_{32} - \frac{\delta}{2}\,\hat\eta_{31} + \mathcal{R}_{32} \;=\; 0\,.
\tag{3.58}
$$

Since $\mathcal{R}_{31}, \mathcal{R}_{32} \in \mathcal{Y}_1' + \mathcal{Y}_3$, both equations in (3.58) can be solved using Proposition 3.10. However, at this stage, it is sufficient to use the crude approximation corresponding to $N = 1$ in (3.29). This means that we can determine our profiles by solving the equations $\Lambda\hat\eta_{3j} + \mathcal{R}_{3j} = 0$ for $j = 1, 2$ using Proposition 3.8. We thus obtain an approximate solution of (3.54) of the form

$$
\eta_3(R, Z, \beta_\epsilon) \;=\; \beta_\epsilon\Big(R^3\eta_{30} + R\eta_{31}\Big) + R^3\eta_{32} + R\eta_{33}\,,
\tag{3.59}
$$

where all functions η_{3j} belong to $\mathcal{Y}_0 \cap \mathcal{Z}$. Using (3.53) we deduce the corresponding expression of the stream function

$$
\phi_3(R, Z, \beta_\epsilon) \;=\; \beta_\epsilon\Big(R^3\phi_{30} + R\phi_{31}\Big) + R^3\phi_{32} + R\phi_{33}\,,
\tag{3.60}
$$

where the functions ϕ_{3j} are radially symmetric and belong to $\mathcal{S}_*(\mathbb{R}^2)$. Note that (3.60) does not contain any factor β_ϵ^2. With the choices (3.59), (3.60), the left-hand side of (3.54) is of size $\mathcal{O}(\beta_\epsilon\delta)$ in the topology of $\mathcal{Z}$.

3.6 Fourth order approximation

Finally we compute the fourth order approximation, which is the final step in our construction. No modification of the vortex speed is needed at this stage, so we can take

$$\dot{\bar{r}}_3 \,=\, \dot{\bar{z}}_3 \,=\, 0\,. \tag{3.61}$$

The full expansion of the vortex speed is therefore

$$\dot{\bar{r}}(t) \,=\, \dot{\bar{r}}_0 + \epsilon^2\dot{\bar{r}}_2(\beta_\epsilon)\,, \qquad \dot{\bar{z}}_*(t) \,=\, \dot{\bar{z}}_0(\beta_\epsilon) + \epsilon^2\dot{\bar{z}}_2(\beta_\epsilon)\,, \tag{3.62}$$

where $\dot{\bar{r}}_0, \dot{\bar{z}}_0$ are defined in (3.36), (3.39) and $\dot{\bar{r}}_2, \dot{\bar{z}}_2$ satisfy (3.57). As is easily verified, this implies that $\bar{r}(t) = r_0\bigl(1-\epsilon^2+\mathcal{O}(\epsilon^4\beta_\epsilon)\bigr)$ and $t\dot\epsilon = \epsilon/2 + \epsilon^3 + \mathcal{O}(\epsilon^5\beta_\epsilon)$ as $\epsilon \to 0$.

We look for an approximate solution of (2.23) of the form

$$\eta_{\rm app}(R,Z,t) \,=\, \sum_{m=0}^4 \epsilon^m \eta_m(R,Z,\beta_\epsilon)\,, \qquad \phi_{\rm app}(R,Z,t) \,=\, \sum_{m=0}^4 \epsilon^m \phi_m(R,Z,\beta_\epsilon)\,, \tag{3.63}$$

where the profiles η_m, ϕ_m for $m \le 3$ have been constructed in the previous steps, and η_0, η_1, ϕ_0 are actually independent of β_ϵ. In analogy with (3.53), we have

$$\phi_4 \,=\, \frac{1}{2\pi}\,L\eta_4 + \frac{1}{2\pi}\sum_{m=1}^4 \Bigl(\bigl(\beta_\epsilon + L\bigr)P_m + Q_m\Bigr)\eta_{4-m}\,, \tag{3.64}$$

where the polynomials P_m, Q_m are defined in (3.11) for $m \le 2$, in (A.19) for $m = 3$, and in (A.20) for $m = 4$. Replacing (3.62), (3.63), (3.64) into (2.23) and proceeding as in the previous sections, we obtain the following equation for the profile η_4:

$$\Lambda\eta_4 + \delta\Bigl(t\partial_t\eta_4 + \bigl(2-\mathcal{L}\bigr)\eta_4\Bigr) + \mathcal{R}_4 \,=\, 0\,, \tag{3.65}$$

where

$$\begin{aligned}
\mathcal{R}_4 \,=\, &\frac{1}{2\pi}\Bigl\{\sum_{m=1}^4 \bigl((\beta_\epsilon + L)P_m + Q_m\bigr)\eta_{4-m}\,,\,\eta_0\Bigr\} + \bigl\{\phi_3\,,\eta_1 - R\eta_0\bigr\} + \bigl\{\phi_2\,,\eta_2 - R\eta_1 + R^2\eta_0\bigr\}\\
&+ \bigl\{\phi_1\,,\eta_3 - R\eta_2 + R^2\eta_1 - R^3\eta_0\bigr\} - \bigl\{\phi_0\,,R\eta_3 - R^2\eta_2 + R^3\eta_1 - R^4\eta_0\bigr\}\\
&- \frac{r_0}{\Gamma}\Bigl(\bigl(\dot{\bar{r}}_2 - \dot{\bar{r}}_0\bigr)\partial_R\eta_1 + \bigl(\dot{\bar{z}}_2 - \dot{\bar{z}}_0\bigr)\partial_Z\eta_1 + \dot{\bar{z}}_0\partial_Z\eta_3\Bigr) + \delta\partial_R\bigl(R\eta_2 - R^2\eta_1 + R^3\eta_0\bigr) + 2\delta\eta_2\,.
\end{aligned} \tag{3.66}$$

Lemma 3.14. *The function $\mathcal{R}_4$ defined in* (3.66) *belongs to $(\delta\mathcal{Y}_0 + \mathcal{Y}_2 + \mathcal{Y}_4) \cap \mathcal{Z}$ and satisfies*

$$\mathcal{R}_4 \,=\, \sum_{k=0}^2 \beta_\epsilon^k\Bigl(R^3Z\chi_{4k} + RZ\chi_{5k}\Bigr) + \mathcal{O}(\delta)\,, \tag{3.67}$$

for some (time-independent) radially symmetric functions $\chi_{4k}, \chi_{5k} \in \mathcal{Y}_0 \cap \mathcal{Z}$.

The proof of Lemma 3.14 is the same as that of Lemma 3.13, and can therefore be omitted. The only important observation is that the projection of $\mathcal{R}_4$ onto the subspace $\mathcal{Y}_0$ is of size $\mathcal{O}(\delta)$. This can be seen as a consequence of Remark 2.3: when $\delta = \dot{\bar{r}} = 0$, all profiles η_m, ϕ_m are even functions of Z, so that the quantities $\mathcal{R}_m$ are odd in Z.

We now project Eq. (3.65) on the subspace $\mathcal{Y}_m$ for $m = 0, 2, 4$, and compute an (approximate) solution $\mathcal{P}_m\eta_4$ invoking either Proposition 3.8 (for $m = 2, 4$) or Proposition 3.6 (for $m = 0$). In the latter case, we use the observation that $\mathcal{P}_0\mathcal{R}_4 = \mathcal{O}(\delta)$ to show that $\mathcal{P}_0\eta_4$ is regular in the limit $\delta \to 0$. Altogether, we obtain an approximate solution of (3.65) in the form

$$\eta_4(R,Z,\beta_\epsilon) \;=\; \sum_{k=0}^{2} \beta_\epsilon^k \Big(R^2Z^2\eta_{4k} + \big(R^2-Z^2\big)\eta_{5k} + \eta_{6k}\Big)\,, \tag{3.68}$$

where the functions $\eta_{jk} \in \mathcal{Y}_0 \cap \mathcal{Z}$ are radially symmetric and time-independent. Using (3.64) we deduce a similar expression for the stream function

$$\phi_4(R,Z,\beta_\epsilon) \;=\; \sum_{k=0}^{2} \beta_\epsilon^k \Big(R^2Z^2\phi_{4k} + \big(R^2-Z^2\big)\phi_{5k} + \phi_{6k}\Big)\,, \tag{3.69}$$

and with these choices the left-hand side of (3.65) is of size $\mathcal{O}(\beta_\epsilon^2\delta)$ in the topology of $\mathcal{Z}$.

Since we have now completed the construction of our approximate solution, we explain precisely how to define the vortex radius $\bar r(t)$ and the time-dependent aspect ratio $\epsilon(t) = \sqrt{\nu t}/\bar r(t)$. In view of (3.36), (3.57), and (3.62), the function $\bar r(t)$ satisfies the differential equation

$$\dot{\bar r}(t) \;=\; -\frac{\Gamma\delta}{r_0}\Big(1 - \epsilon(t)^2\big(c_1\beta_{\epsilon(t)} + c_2\big)\Big) \;=\; -\frac{\Gamma\delta}{r_0}\bigg(1 - \frac{\nu t}{\bar r(t)^2}\Big(c_1 \log\frac{\bar r(t)}{\sqrt{\nu t}} + c_2\Big)\bigg)\,, \tag{3.70}$$

with initial condition $\bar r(0) = r_0$. The right-hand side of (3.70) is a smooth function of $\bar r > 0$, uniformly in $t \in (0, T_{\rm dif})$, and also a $C^{0,\alpha}$ function of time for any $\alpha < 1$. Applying the Cauchy-Lipschitz theorem, we obtain a unique local solution of (3.70), which can be extended as long as $\bar r(t) > 0$. Now, if we define $\epsilon(t) = \sqrt{\nu t}/\bar r(t)$, it follows that $\bar r(t) = r_0\big(1 - \epsilon(t)^2 + \mathcal{O}(\epsilon^4\beta_\epsilon)\big)$, and it is easy to see that the solution of (3.70) is well-defined and has the required properties on the time intervals relevant for our considerations.

Remark 3.15. *It is useful to notice that the approximate solution $\eta_{\rm app}$ given by* (3.63) *satisfies, for all $t > 0$,*

$$\int_{\mathbb{R}^2} \eta_{\rm app}(R,Z,t)\,{\rm d}R\,{\rm d}Z \;=\; 1\,, \tag{3.71}$$

$$\int_{\mathbb{R}^2} R\,\eta_{\rm app}(R,Z,t)\,{\rm d}R\,{\rm d}Z \;=\; \int_{\mathbb{R}^2} Z\,\eta_{\rm app}(R,Z,t)\,{\rm d}R\,{\rm d}Z \;=\; 0\,. \tag{3.72}$$

Indeed, at each step $m \ge 1$, the vorticity profile η_m is constructed by solving equations of the form $\Lambda\eta_m + \big(\frac{m}{2} - \mathcal{L}\big)\eta_m + \mathcal{R}_m = 0$, where the source term $\mathcal{R}_m$ has vanishing integral (by definition) and zero first order moments (due to the choice of the speeds $\dot{\bar r}_{m-1}$, $\dot{\bar z}_{m-1}$). These properties are inherited by the profile η_m, due to Proposition 3.6, and in view of (3.32) *this leads to* (3.71), (3.72).

3.7 Estimate of the remainder

This section is devoted to the proof of Proposition 2.5. Our task is to estimate the remainder (2.29), where η_*, ϕ_* are defined in (2.28), and for this we need bounds on the derivatives of the stream function in terms of the vorticity. If $\phi = {\rm BS}^\epsilon[\eta]$, where the Biot-Savart operator is defined in (2.20), we have the formulas

$$\begin{aligned}
\partial_Z\phi(R,Z) \;&=\; -\frac{1}{2\pi}\int_{\Omega_\epsilon} \sqrt{(1{+}\epsilon R)(1{+}\epsilon R')}\;\tilde F(s)\,\frac{(Z{-}Z')\,\eta(R',Z')}{(R{-}R')^2+(Z{-}Z')^2}\,{\rm d}R'\,{\rm d}Z'\,,\\
\partial_R\phi(R,Z) \;&=\; -\frac{1}{2\pi}\int_{\Omega_\epsilon} \sqrt{(1{+}\epsilon R)(1{+}\epsilon R')}\;\tilde F(s)\,\frac{(R{-}R')\,\eta(R',Z')}{(R{-}R')^2+(Z{-}Z')^2}\,{\rm d}R'\,{\rm d}Z'\\
&\quad\; + \frac{\epsilon}{4\pi}\int_{\Omega_\epsilon} \frac{\sqrt{1{+}\epsilon R'}}{\sqrt{1+\epsilon R}}\,\big(F(s)+\tilde F(s)\big)\,\eta(R',Z')\,{\rm d}R'\,{\rm d}Z'\,,
\end{aligned} \tag{3.73}$$

where $\tilde{F}(s) = -2sF'(s)$, see [33, Section 4.2]. Here, as in (3.12), we use the shorthand notation

$$s \,=\, \frac{\epsilon^2 D^2}{(1+\epsilon R)(1+\epsilon R')} \,\equiv\, \epsilon^2\, \frac{(R-R')^2 + (Z-Z')^2}{(1+\epsilon R)(1+\epsilon R')}\,. \tag{3.74}$$

In view of (2.7), we have $\tilde{F}(s) \to 1$ as $s \to 0$ and $\tilde{F}(s) = \mathcal{O}(s^{-3/2})$ as $s \to +\infty$.

Throughout the proof, we fix $t > 0$ and we assume that the parameters $\epsilon = \sqrt{\nu t}/\bar{r}(t)$ and $\delta = \nu/\Gamma$ are small enough. By construction the vorticity $\eta_*(R,Z,t)$ defined by (2.28) vanishes identically when $\rho := (R^2{+}Z^2)^{1/2} \ge 2\epsilon^{-\sigma_0}$, so we can assume henceforth that $\rho \le 2\epsilon^{-\sigma_0}$. In that region, we have for any $\gamma \in (0,1)$ the a priori bounds

$$\sum_{|\alpha|\le 2} |\partial^\alpha \eta_*(R,Z,t)| \,\le\, C\, e^{-\gamma\rho^2/4}\,, \qquad \sum_{|\alpha|=1} |\partial^\alpha \phi_*(R,Z,t)| \,\le\, C\,, \tag{3.75}$$

where $\alpha = (\alpha_1,\alpha_2) \in \mathbb{N}^2$ and $\partial^\alpha = \partial_R^{\alpha_1}\partial_Z^{\alpha_2}$. Indeed, the first estimate in (3.75) holds because η_* is obtained by truncating the asymptotic approximation $\eta_{\rm app}(R,Z,t)$ which belongs to the space $\mathcal{Z}$ defined in (3.23). The second estimate can then be obtained using the expressions (3.73) with $\phi = \phi_*$ and $\eta = \eta_*$. To see this, we first observe that $1 + \epsilon R \approx 1$ and $1 + \epsilon R' \approx 1$ in (3.73), because both quantities ρ and $\rho' := (R'^2{+}Z'^2)^{1/2}$ are smaller than $2\epsilon^{-\sigma_0} \ll \epsilon^{-1}$. If we use the estimates $|\tilde{F}(s)| \le C$ in the first two lines of (3.73) and $|F(s) + \tilde{F}(s)| \le Cs^{-1/2}$ in the third line, we thus obtain

$$|\partial_R\phi_*(R,Z,t)| + |\partial_Z\phi_*(R,Z,t)| \,\le\, C\int_{\mathbb{R}^2} \frac{|\eta_*(R',Z',t)|}{\sqrt{(R-R')^2 + (Z-Z')^2}}\,{\rm d}R'\,{\rm d}Z' \,\le\, C\,,$$

which concludes the proof of (3.75). Finally, since

$$t\partial_t\eta_*(R,Z,t) \,=\, \chi_0\big(\epsilon^{\sigma_0}\rho\big)\, t\partial_t\eta_{\rm app}(R,Z,t) + \sigma_0\,\epsilon^{\sigma_0}\rho\chi_0'\big(\epsilon^{\sigma_0}\rho\big)\,\eta_{\rm app}(R,Z,t)\, t\partial_t\log(\epsilon)\,,$$

it follows from the expressions given in Sections 3.3–3.6 that $t\partial_t\eta_*$ satisfies the same bound as η_* in (3.75). Summarizing, in view of (3.75), the remainder ${\rm Rem}(R,Z,t)$ satisfies

$$e^{\gamma_0\rho^2/4}\,|{\rm Rem}(R,Z,t)| \,\le\, C\,\delta^{-1}(1+\rho)\,e^{-(\gamma-\gamma_0)\rho^2/4}\,, \qquad \text{when } \rho \le 2\epsilon^{-\sigma_0}\,, \tag{3.76}$$

for any $\gamma_0 \in (0,1)$. If we assume that $\gamma \in (\gamma_0,1)$, we conclude that the right-hand side of (3.76) is $\mathcal{O}(\delta^{-1}\epsilon^\infty)$ if $\rho \ge \epsilon^{-\sigma_0}$. So from now on we may concentrate on the inner region $\rho \le \epsilon^{-\sigma_0}$, where $\eta_* = \eta_{\rm app}$ is given by (3.63).

In that region we decompose the stream function as $\phi_* = {\rm BS}^\epsilon[\chi_0\,\eta_{\rm app}] = \phi_*^0 - \phi_*^1 + \phi_*^2$, where

$$\phi_*^0 \,=\, \sum_{m=0}^4 \epsilon^m\,{\rm BS}_m[\eta_{\rm app}]\,, \quad \phi_*^1 \,=\, \sum_{m=0}^4 \epsilon^m\,{\rm BS}_m[(1{-}\chi_0)\,\eta_{\rm app}]\,, \quad \phi_*^2 \,=\, \sum_{m=5}^\infty \epsilon^m\,{\rm BS}_m[\chi_0\,\eta_{\rm app}]\,.$$

Here χ_0 is a shorthand notation for $\chi_0(\epsilon^{\sigma_0}\rho)$. The convergence of the series defining ϕ_*^2 is easily justified using Lemmas 3.1 and 3.3, if we observe that both inequalities in (3.12) are satisfied since $\rho,\rho' \ll \epsilon^{-1}$. The principal term ${\rm BS}_5[\chi_0\,\eta_{\rm app}]$ can be estimated using the explicit representation (3.15), where P_5, Q_5 are homogeneous polynomials of degree 5, and this leads to a bound of the form

$$|\partial_R\phi_*^2(R,Z,t)| + |\partial_Z\phi_*^2(R,Z,t)| \,\le\, C\epsilon^5\beta_\epsilon\,(1+\rho)^5\,, \qquad \rho \le \epsilon^{-\sigma_0}\,,$$

where $\beta_\epsilon = \log(1/\epsilon)$. Moreover we have $|\partial_R\phi_*^1| + |\partial_Z\phi_*^1| = \mathcal{O}(\epsilon^\infty)$ because $(1-\chi_0)\eta_{\rm app} = \mathcal{O}(\epsilon^\infty)$. Finally, in view of (3.31) and (3.63), we have the identity

$$\phi_*^0 \,=\, \phi_{\rm app} \,+\, \sum_{m=5}^{8} \epsilon^m \sum_{k=m-4}^{4} \mathrm{BS}_k[\eta_{m-k}]\,.$$

from which we easily deduce

$$|\partial_R\bigl(\phi_*^0 - \phi_{\rm app}\bigr)| + |\partial_Z\bigl(\phi_*^0 - \phi_{\rm app}\bigr)| \,\le\, C\epsilon^5\beta_\epsilon^3\,(1+\rho)^5\,.$$

Collecting the estimates above, it is straightforward to verify that the remainder (2.29) satisfies

$$\bigl|\mathrm{Rem}(R,Z,t) - \widehat{\mathrm{Rem}}(R,Z,t)\bigr| \,\le\, C\delta^{-1}\epsilon^5\beta_\epsilon^3\,(1+\rho)^5\,e^{-\gamma\rho^2/4}\,, \qquad \rho \le \epsilon^{-\sigma_0}\,, \tag{3.77}$$

where $\widehat{\mathrm{Rem}}(R,Z,t)$ is the quantity defined for all $(R,Z) \in \mathbb{R}^2$ by the formula

$$\mathcal{L}\eta_{\rm app} + \epsilon\partial_R\bigl(S_4\eta_{\rm app}\bigr) - t\partial_t\eta_{\rm app} - \frac{1}{\delta}\,\bigl\{\phi_{\rm app}\,,S_4\eta_{\rm app}\bigr\} + \frac{\epsilon\bar r}{\delta\Gamma}\Bigl(\dot{\bar r}\,\partial_R\eta_{\rm app} + \dot{\bar z}_*\,\partial_Z\eta_{\rm app}\Bigr)\,, \tag{3.78}$$

with $S_4 = 1 - \epsilon R + (\epsilon R)^2 - (\epsilon R)^3 + (\epsilon R)^4$.

Now the crucial observation is that the asymptotic approximation (2.24), (2.25) was constructed precisely so as to make the quantity (3.78) small in the topology of $\mathcal{Z}$. More precisely, the results of Sections 3.3–3.6 can be rephrased as follows:

$$\delta\,\widehat{\mathrm{Rem}}(R,Z,t) \,=\, \mathcal{O}_{\mathcal{Z}}\Bigl(\epsilon\delta^2 + \epsilon^2\beta_\epsilon\delta^2 + \epsilon^3\beta_\epsilon\delta + \epsilon^4\beta_\epsilon^2\delta + \epsilon^5\beta_\epsilon^3\Bigr)\,. \tag{3.79}$$

Inside the parenthesis in the right-hand side, the first four terms represent what remains from the quantities $\epsilon^m\bigl(\Lambda\eta_m + \delta\bigl[t\partial_t + \frac{m}{2} - \mathcal{L}\bigr]\eta_m + \mathcal{R}_m\bigr)$ for $m = 1,2,3,4$ after the profiles η_m have been determined, and the last one corresponds to those terms in (3.78) which are of order $\mathcal{O}(\epsilon^5)$ or higher and were therefore not considered in the construction of $\eta_{\rm app}$. Combining (3.77), (3.79) and using Young's inequality, we obtain

$$\sup_{\rho\le\epsilon^{-\sigma_0}} e^{\gamma_0\rho^2/4}\,|\mathrm{Rem}(R,Z,t)| \,\le\, \frac{C}{\delta}\Bigl(\epsilon\delta^2 + \epsilon^3\beta_\epsilon\delta + \epsilon^5\beta_\epsilon^3\Bigr) \,\le\, C\bigl(\epsilon\delta + \epsilon^{\gamma_5}\delta^{-1}\bigr)\,,$$

for any $\gamma_5 < 5$. This concludes the proof of (2.30). □

3.8 The Eulerian approximation

As was already observed in Remark 2.3, if we set $\delta = \dot{\bar r} = 0$ in the expansion (2.24), we obtain an approximate solution $\eta_{\rm app}^E, \phi_{\rm app}^E, \dot{\bar z}_E$ of equation (2.26), which is nothing but the stationary Euler equation in a frame moving with (constant) velocity $\dot{\bar z}_E\,e_z$. As is well known [3], steady states of the Euler system are often characterized by a global functional relation between the vorticity and the stream function. In our case, in view of (2.26), we expect finding a function $\Phi_\epsilon : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\phi_{\rm app}^E(R,Z) - \frac{r_0\dot{\bar z}_E}{2\Gamma}\,(1+\epsilon R)^2 \,=\, \Phi_\epsilon\biggl(\frac{\eta_{\rm app}^E(R,Z)}{1+\epsilon R}\biggr) + \mathcal{O}\bigl(\epsilon^{M+1-}\bigr)\,, \tag{3.80}$$

for all $(R,Z) \in \mathbb{R}^2$ such that $\rho := \sqrt{R^2+Z^2} \ll \epsilon^{-1}$.

In this section, we first verify that a relation of the form (3.80) holds to second order, namely with $M = 2$. Using the expressions (3.41), (3.42), (3.50), (3.51) with $\delta = 0$ and simplifying somehow the notation, we can write our approximate solution in the form

$$\begin{aligned}
\eta^E_{\rm app}(R,Z) \;&=\; \eta_0 + \epsilon R\eta_1 + \epsilon^2(R^2{-}Z^2)\eta_2 + \epsilon^2\eta_3\,, \\
\phi^E_{\rm app}(R,Z) \;&=\; \phi_0 + \epsilon R\phi_1 + \epsilon^2(R^2{-}Z^2)\phi_2 + \epsilon^2\phi_3\,,
\end{aligned} \tag{3.81}$$

where η_0, ϕ_0 are given by (3.32), and the profiles $\eta_1, \eta_2, \eta_3 \in \mathcal{Z}$ and $\phi_1, \phi_2, \phi_3 \in \mathcal{S}_*(\mathbb{R}^2)$ are all radially symmetric. Note that η_m, ϕ_m may include factors of $\beta_\epsilon = \log(1/\epsilon)$ when $m \ge 1$, but this dependence is not explicitly indicated. We also expand the unknown function Φ_ϵ in (3.80) in powers of ϵ:

$$\Phi_\epsilon(s) \;=\; \Phi_0(s) + \epsilon\Phi_1(s) + \epsilon^2\Phi_2(s)\,. \tag{3.82}$$

Finally, to simplify the writing, we denote

$$\omega \;=\; \frac{1}{4\pi}\bigl(\beta_\epsilon - 1 + 2v\bigr) \;=\; \frac{r_0\dot{\bar z}_E}{\Gamma} + \mathcal{O}(\epsilon^2\beta_\epsilon)\,, \tag{3.83}$$

where the last equality follows from (3.39), (3.43), (3.57).

If we consider equality (3.80) to leading order in ϵ, thus neglecting all terms that are $\mathcal{O}(\epsilon)$ or $\mathcal{O}(\epsilon\beta_\epsilon)$, we obtain the relation $\phi_0 - \omega/2 = \Phi_0(\eta_0)$, which determines the principal term Φ_0 in the expansion (3.82). In view of (3.32), (3.33) we thus have

$$\Phi_0(s) \;=\; \phi_0(0) - \frac{\omega}{2} - \frac{1}{4\pi}\,\mathrm{Ein}\Bigl(\log\frac{1}{4\pi s}\Bigr)\,, \qquad s > 0\,. \tag{3.84}$$

The constant in (3.84) has no relevance, but it is important to note that $\Phi_0(s) \sim -\frac{1}{4\pi}\log\log\frac{1}{s}$ as $s \to 0$. For later use we define

$$A(\rho) \;=\; \Phi_0'\bigl(\eta_0(\rho)\bigr) \;=\; \frac{\partial_R\phi_0}{\partial_R\eta_0} \;=\; \frac{\partial_Z\phi_0}{\partial_Z\eta_0} \;=\; \frac{4}{\rho^2}\Bigl(e^{\rho^2/4} - 1\Bigr)\,, \qquad \rho > 0\,. \tag{3.85}$$

Incidentally we observe that $A(\rho) = 1/h(\rho)$ where h is defined in (3.24).

To the next order in ϵ, we deduce from (3.80) the relation

$$(\phi_1 - \omega)R \;=\; \Phi_0'(\eta_0)(\eta_1 - \eta_0)R + \Phi_1(\eta_0)\,, \tag{3.86}$$

which can be satisfied only if $\Phi_1 = 0$, because $\Phi_1(\eta_0)$ is the only radially symmetric term in (3.86). Dividing by R, we obtain the equality $\phi_1 - \omega = A(\eta_1 - \eta_0)$, which happens to be satisfied in view of our definitions of the profiles η_1, ϕ_1. This fact can be verified by following carefully the calculations in Section 3.3.

Finally we exploit (3.80) to order ϵ^2, keeping in mind that $\Phi_1 = 0$. In this calculation, we neglect the $\mathcal{O}(\epsilon^2\beta_\epsilon)$ correction in (3.83), because this term would only add an irrelevant constant to the function Φ_2. We thus obtain the relation

$$\begin{aligned}
(R^2{-}Z^2)\phi_2 + \phi_3 - \frac{\omega}{2}\,R^2 \;&=\; \Phi_0'(\eta_0)\Bigl((R^2{-}Z^2)\eta_2 + \eta_3 + (\eta_0 - \eta_1)R^2\Bigr) \\
&\quad + \frac{1}{2}\Phi_0''(\eta_0)(\eta_0 - \eta_1)^2R^2 + \Phi_2(\eta_0)\,,
\end{aligned}$$

where it is useful to substitute $R^2 = \frac{1}{2}(R^2{+}Z^2) + \frac{1}{2}(R^2{-}Z^2)$. The terms containing $R^2{-}Z^2$ cancel exactly due to the identity

$$\phi_2 - \frac{1}{2}\Psi - A\eta_2 \;=\; 0\,, \qquad \text{where} \quad \Psi \;=\; \frac{\omega}{2} + \Phi_0'(\eta_0)(\eta_0 - \eta_1) + \frac{1}{2}\Phi_0''(\eta_0)(\eta_0 - \eta_1)^2\,,$$

which is satisfied by definition of the profiles ϕ_2, η_2, as can be verified by following the calculations in Section 3.4. We are thus left with a relation involving only radially symmetric terms

$$\phi_3 - \frac{1}{2}(R^2{+}Z^2)\Psi - A\eta_3 \;=\; \Phi_2(\eta_0)\,, \tag{3.87}$$

which provides the definition of the second order correction Φ_2 in (3.82). Summarizing, if Φ_ϵ is defined by (3.82) with $\Phi_1 = 0$, Φ_0 given by (3.84) and Φ_2 by (3.87), we have shown that (3.80) holds with $M = 2$.

We now come back to the approximate solution η_*, ϕ_* of (2.23) constructed in Sections 3.3–3.6, and we show that it also satisfies a relation of the form (3.80), in a sufficiently small region near the origin. To formulate that result, we denote

$$\Theta(R,Z,t) \;=\; \phi_*(R,Z,t) - \frac{\bar r \dot{\bar z}_*}{2\Gamma}\,(1+\epsilon R)^2 - \Phi_\epsilon\biggl(\frac{\eta_*(R,Z,t)}{1+\epsilon R}\biggr)\,, \qquad (R,Z) \in \Omega_\epsilon\,. \tag{3.88}$$

Proposition 3.16. *There exist $\sigma_1 \in (0,\sigma_0)$ and $N \in \mathbb{N}$ such that, for any $\gamma_3 < 3$, the quantity Θ defined by* (3.88) *satisfies, for some $C > 0$,*

$$|\partial_R\Theta(R,Z,t)| + |\partial_Z\Theta(R,Z,t)| \;\le\; C(\epsilon\delta + \epsilon^{\gamma_3})(1+\rho)^N\,, \qquad \rho \le \epsilon^{-\sigma_1}\,, \tag{3.89}$$

whenever ϵ and δ are small enough.

Proof. The idea is to compare Θ with the second order Eulerian approximation

$$\Theta^E_{\rm app}(R,Z,t) \;=\; \phi^E_{\rm app}(R,Z,t) - \frac{r_0\dot{\bar z}_E}{2\Gamma}\,(1+\epsilon R)^2 - \Phi_\epsilon\biggl(\frac{\eta^E_{\rm app}(R,Z,t)}{1+\epsilon R}\biggr)\,, \tag{3.90}$$

which is of size $\mathcal{O}(\epsilon^{3-})$ in view of (3.80). Here we consider both quantities $\eta^E_{\rm app}, \phi^E_{\rm app}$ as time-dependent, because we deal with the viscous case where $\epsilon = \sqrt{\nu t}/\bar r(t)$. We already estimated the difference $\phi_* - \phi_{\rm app}$ in the proof of Proposition 2.5, and by construction we know that $\phi_{\rm app} = \phi^E_{\rm app} + \mathcal{O}(\epsilon\delta + \epsilon^3\beta_\epsilon)$. These arguments lead to the bound

$$\bigl|\partial_R\bigl(\phi_* - \phi^E_{\rm app}\bigr)\bigr| + \bigl|\partial_Z\bigl(\phi_* - \phi^E_{\rm app}\bigr)\bigr| \;\le\; C\bigl(\epsilon\delta + \epsilon^3\beta_\epsilon\bigr)\,(1+\rho)^5\,, \qquad \rho \le \epsilon^{-\sigma_0}\,. \tag{3.91}$$

On the other hand, we know that that $\bar r(t) = r_0(1 + \mathcal{O}(\epsilon^2))$, and in view of (3.43), (3.57) the difference between the vertical speed $\dot{\bar z}_*$ and its second order Eulerian approximation $\dot{\bar z}_E$ is of size $(\Gamma/r_0)\cdot\mathcal{O}(\epsilon^2\beta_\epsilon)$. We thus find

$$\biggl|\frac{\bar r\dot{\bar z}_*}{2\Gamma} - \frac{r_0\dot{\bar z}_E}{2\Gamma}\biggr|\,\bigl|\partial_R(1+\epsilon R)^2\bigr| \;\le\; C\epsilon^3\beta_\epsilon\,, \qquad \rho \le \epsilon^{-\sigma_0}\,. \tag{3.92}$$

Finally η_* is just a truncation of $\eta_{\rm app}$ and by definition $\eta_{\rm app} - \eta^E_{\rm app} = \mathcal{O}(\epsilon\delta + \epsilon^3\beta_\epsilon)$ in the topology of $\mathcal{Z}$. This gives the following bound

$$\sum_{|\alpha|\le 1}\bigl|\partial^\alpha\bigl(\eta_* - \eta^E_{\rm app}\bigr)(R,Z,t)\bigr| \;\le\; C\bigl(\epsilon\delta + \epsilon^3\beta_\epsilon\bigr)\,(1+\rho)^N e^{-\rho^2/4}\,, \qquad \rho \le \epsilon^{-\sigma_0}\,, \tag{3.93}$$

for some $N \in \mathbb{N}$.

At this point we observe that $\eta_* - \eta_0 = \mathcal{O}(\epsilon)$ in the topology of $\mathcal{Z}$ when $\rho \le \epsilon^{-\sigma_0}$. In particular, there exists $N \in \mathbb{N}$ such that $|\eta_* - \eta_0| \le C\epsilon(1+\rho)^N\eta_0$ in that region, and one can verify that $N = 3$ is in fact sufficient. If we choose $\sigma_1 > 0$ small enough so that $N\sigma_1 < 1$, it follows that

$$\frac{1}{2}\,\eta_0(\rho) \;\le\; \frac{\eta_*(R,Z,t)}{1+\epsilon R} \;\le\; 2\,\eta_0(\rho)\,, \qquad \rho \le \epsilon^{-\sigma_1}\,, \tag{3.94}$$

whenever $\epsilon > 0$ is small enough. The same estimate holds for the Eulerian approximation $\eta^E_{\rm app}$.

To conclude the proof of Proposition 3.16, we need bounds on the derivatives of the function Φ_ϵ defined in (3.82). We begin with the leading order term Φ_0 which is given by the explicit formula (3.84). We have

$$\Phi_0'\Big(\frac{s}{4\pi}\Big) \,=\, \frac{1-s}{s\log(1/s)}\,, \qquad \frac{1}{4\pi}\,\Phi_0''\Big(\frac{s}{4\pi}\Big) \,=\, -\frac{s-1+\log(1/s)}{s^2\big(\log(1/s)\big)^2}\,, \qquad s>0\,.$$

Thanks to (3.94) we only need to evaluate these expressions when the argument $s/(4\pi)$ takes its values in the interval $\big[\frac12\eta_0(\rho), 2\eta_0(\rho)\big]$. In view of Lemma 3.17 below, there exists $C > 1$ such that, for all $\lambda \in [1/2, 2]$ and all $\rho > 0$,

$$\frac{A(\rho)}{C} \,\le\, \Phi_0'\big(\lambda\eta_0(\rho)\big) \,\le\, CA(\rho)\,, \qquad \big|\Phi_0''\big(\lambda\eta_0(\rho)\big)\big| \,\le\, CB(\rho)\,, \tag{3.95}$$

where $A(\rho)$ is defined in (3.85) and

$$B(\rho) \,=\, -\Phi_0''(\eta_0(\rho)) \,=\, \frac{16\pi}{\rho^4}\,\Big((\rho^2-4)e^{\rho^2/2} + 4e^{\rho^2/4}\Big)\,, \qquad \rho > 0\,. \tag{3.96}$$

The second order contribution Φ_2 is not known explicitly, but from the definition (3.87), where the left-hand side belongs to $\mathcal{S}_*(\mathbb{R}^2)$, we deduce that there exist $C > 0$ and $N \in \mathbb{N}$ such that

$$\big|\Phi_2'\big(\lambda\eta_0(\rho)\big)\big| \,\le\, CA(\rho)(1+\rho)^N\,, \qquad \big|\Phi_2''\big(\lambda\eta_0(\rho)\big)\big| \,\le\, CB(\rho)(1+\rho)^N\,, \tag{3.97}$$

for all $\rho > 0$ and all $\lambda \in [1/2, 2]$.

Now, if $\partial^\alpha = \partial_R$ or ∂_Z, we decompose

$$\begin{aligned}\partial^\alpha\Phi_\epsilon\Big(\frac{\eta_*}{1{+}\epsilon R}\Big) - \partial^\alpha\Phi_\epsilon\Big(\frac{\eta^E_{\rm app}}{1{+}\epsilon R}\Big) \,&=\, \Phi_\epsilon'\Big(\frac{\eta_*}{1{+}\epsilon R}\Big)\Big(\partial^\alpha\Big(\frac{\eta_*}{1{+}\epsilon R}\Big) - \partial^\alpha\Big(\frac{\eta^E_{\rm app}}{1{+}\epsilon R}\Big)\Big) \\ &\quad + \Big(\Phi_\epsilon'\Big(\frac{\eta_*}{1{+}\epsilon R}\Big) - \Phi_\epsilon'\Big(\frac{\eta^E_{\rm app}}{1{+}\epsilon R}\Big)\Big)\,\partial^\alpha\Big(\frac{\eta^E_{\rm app}}{1{+}\epsilon R}\Big)\,,\end{aligned}$$

and we estimate the right-hand side using (3.93), (3.95), and (3.97). Taking into account the preliminary bounds (3.91), (3.92), we arrive at an estimate of the form

$$\sum_{|\alpha|=1}\big|\partial^\alpha\big(\Theta(R,Z,t) - \Theta^E_{\rm app}(R,Z,t)\big)\big| \,\le\, C(\epsilon\delta + \epsilon^3\beta_\epsilon)(1+\rho)^N\,, \qquad \rho \le \epsilon^{-\sigma_1}\,.$$

As was already mentioned, the approximation $\Theta^E_{\rm app}(R,Z,t)$ is $\mathcal{O}(\epsilon^{3-})$ in the topology of $\mathcal{S}_*(\mathbb{R}^2)$, so altogether we arrive at (3.89). $\square$

In the argument above we used the following elementary result, whose proof can be omitted.

Lemma 3.17. *Let $f, g : (0,+\infty) \to (0,+\infty)$ be defined by*

$$f(s) \,=\, \frac{1-s}{s\log(1/s)}\,, \qquad g(s) \,=\, \frac{s-1+\log(1/s)}{s^2\big(\log(1/s)\big)^2} \,=\, -f'(s)\,, \qquad s>0\,.$$

Then given any $\Lambda > 1$ there exists $C > 1$ such that, for any $\lambda \in [\Lambda^{-1}, \Lambda]$ and any $s > 0$,

$$\frac{1}{C} \,\le\, \frac{f(\lambda s)}{f(s)} \,\le\, C\,, \qquad \frac{1}{C} \,\le\, \frac{g(\lambda s)}{g(s)} \,\le\, C\,.$$

4 Energy estimates and stability proof

In the previous section we constructed an approximate solution η_* of the rescaled vorticity equation (2.15) which corresponds, in the original variables, to a sharply concentrated vortex ring of radius $\bar r(t)$ located at the approximate vertical position $\bar z_*(t)$. Our goal is now to control the difference between this approximation and the actual solution η of (2.15) with initial data η_0, which is located at the modified vertical position $\bar z(t) = \bar z_*(t) + \delta\tilde z(t)$ given by (2.32). This will conclude the proof of our main results, Theorems 1.1 and 2.6.

Our starting point is the evolution equation (2.33) for the perturbation $\tilde\eta$ defined in (2.31), which can be written in the form

$$\begin{aligned}
t\partial_t\tilde\eta + \frac{1}{\delta}\{\phi_*\,,\tilde\zeta\} + \frac{1}{\delta}\{\tilde\phi\,,\zeta_*\} + \{\tilde\phi\,,\tilde\zeta\} - \frac{\epsilon\bar r}{\delta\Gamma}\Big(\dot{\bar r}\,\partial_R\tilde\eta + \dot{\bar z}_*\,\partial_Z\tilde\eta\Big) \\
= \mathcal{L}\tilde\eta + \epsilon\partial_R\tilde\zeta + \frac{1}{\delta}\,\mathrm{Rem}(R,Z,t) + \frac{\epsilon\bar r\dot{\tilde z}}{\delta\Gamma}\Big(\partial_Z\eta_* + \delta\partial_Z\tilde\eta\Big)\,,
\end{aligned} \tag{4.1}$$

where to simplify the notation we use the letter ζ to denote the potential vorticity:

$$\tilde\zeta(R,Z,t) \,=\, \frac{\tilde\eta(R,Z,t)}{1+\epsilon R}\,, \qquad \zeta_*(R,Z,t) \,=\, \frac{\eta_*(R,Z,t)}{1+\epsilon R}\,. \tag{4.2}$$

From our previous work [33] we know that Eq. (4.1) has a unique solution $\tilde\eta$, in an appropriate weighted L^2 space, with zero initial data. Our goal is to control the evolution of that solution on a large time interval, uniformly with respect to the viscosity in the limit $\nu \to 0$. This is not an easy task, because several terms in (4.1) are multiplied by the Reynolds number $\delta^{-1} = \Gamma/\nu$, which becomes arbitrarily large in the regime we consider. As was explained in the introduction, we shall use energy estimates to control the solution of (4.1), but a few preliminary steps are necessary before starting the actual calculations.

4.1 Control of the lowest order moments

To implement our strategy based on energy estimates, we need a precise information on the lowest order moments of the solution of (4.1). We first define, for all $t > 0$,

$$\mu_0(t) \,=\, \int_{\Omega_\epsilon}\tilde\eta(R,Z,t)\,\mathrm{d}X\,, \qquad \mu_1(t) \,=\, \int_{\Omega_\epsilon}\big(R+\epsilon R^2/2\big)\,\tilde\eta(R,Z,t)\,\mathrm{d}X\,, \tag{4.3}$$

where $\mathrm{d}X = \mathrm{d}R\,\mathrm{d}Z$ denotes the Lebesgue measure in $\mathbb{R}^2$.

Lemma 4.1. *The moments defined in* (4.3) *satisfy* $\mu_0(t) = \mathcal{O}(\epsilon^\infty\delta^{-1})$ *and* $\mu_1(t) = \mathcal{O}(\epsilon + \epsilon^{\gamma_5}\delta^{-2})$ *for any* $\gamma_5 < 1$*, whenever* ϵ *and* δ *are small enough.*

Proof. The conclusion can be obtained by direct calculations, but we find it more illuminating to use the conserved quantities of the original equation (2.3). The first one is the total circulation

$$M(t) \,=\, \int_\Omega \omega_\theta(r,z,t)\,\mathrm{d}r\,\mathrm{d}z \,=\, \Gamma\int_{\Omega_\epsilon}\big(\eta_* + \delta\tilde\eta\big)(R,Z,t)\,\mathrm{d}X \,=\, \Gamma\int_{\Omega_\epsilon}\eta_*\,\mathrm{d}X + \Gamma\delta\mu_0(t)\,, \tag{4.4}$$

which satisfies $M(0) = \Gamma$ and is almost constant in time. In fact it is proved in [33, Section 4.4] that $0 \le 1 - M(t)/\Gamma \le C\exp(-c/\epsilon^2)$ for some positive constants C and c. Moreover, since the approximate solution η_{app} lies in the space $\mathcal{Z}$ defined by (3.23), it follows from (2.28) and (3.71) that $\int_{\Omega_\epsilon}\eta_*\,\mathrm{d}X = 1 + \mathcal{O}(\exp(-c/\epsilon^{2\sigma_0}))$. Therefore $\mu_0(t) = \mathcal{O}(\exp(-c/\epsilon^{2\sigma_0})\,\delta^{-1})$ by (4.4).

We next consider the total impulse in the vertical direction

$$I \,=\, \int_{\Omega} r^2 \omega_\theta(r,z,t)\,\mathrm{d}r\,\mathrm{d}z \,=\, \Gamma\bar r(t)^2 \int_{\Omega_\epsilon} (1+\epsilon R)^2\bigl(\eta_* + \delta\tilde\eta\bigr)(R,Z,t)\,\mathrm{d}X\,, \tag{4.5}$$

which is known to be exactly conserved [48, 32], so that $I = \Gamma r_0^2$ for all times. Equality (4.5) can be rephrased as $I/\Gamma = I_*(t) + \delta\bar r(t)^2\mu(t)$, where

$$I_*(t) \,=\, \bar r(t)^2 \int_{\Omega_\epsilon} (1+\epsilon R)^2 \eta_*(R,Z,t)\,\mathrm{d}X\,, \qquad \mu(t) \,=\, \mu_0(t) + 2\epsilon\mu_1(t)\,. \tag{4.6}$$

It is not difficult to show that

$$tI_*'(t) \,=\, -\bar r(t)^2 \int_{\Omega_\epsilon} (1+\epsilon R)^2\,\mathrm{Rem}(R,Z,t)\,\mathrm{d}X\,. \tag{4.7}$$

The easiest way to establish (4.7) is to observe that the impulse $I_*(t)$ would be conserved if η_* was an exact solution of (2.15), so that the remainder $\mathrm{Rem}(R,Z,t)$ defined in (2.29) is the only term that contributes to the evolution of $I_*(t)$. However equality (4.7) can also be verified by a direct calculation. In any case, since $\mathrm{Rem}(R,Z,t)$ satisfies estimate (2.30) and $\int_{\Omega_\epsilon} \mathrm{Rem}(R,Z,t)\,\mathrm{d}x = \mathcal{O}(\epsilon^\infty)$, we deduce from (4.7) that $|tI_*'(t)| \le Cr_0^2\bigl(\epsilon^2\delta + \epsilon^{\gamma_5+1}\delta^{-1}\bigr)$, hence

$$|I_*(t) - r_0^2| \,\le\, \int_0^t |I_*'(s)|\,\mathrm{d}s \,\le\, Cr_0^2 \int_0^t \frac{\epsilon(s)^2\delta + \epsilon(s)^{\gamma_5+1}\delta^{-1}}{s}\,\mathrm{d}s \,\le\, Cr_0^2\bigl(\epsilon^2\delta + \epsilon^{\gamma_5+1}\delta^{-1}\bigr)\,.$$

As $r_0^2 - I_*(t) = \delta\bar r(t)^2\mu(t)$, we conclude that $\mu(t) = \mathcal{O}\bigl(\epsilon^2 + \epsilon^{\gamma_5+1}\delta^{-2}\bigr)$, which gives the desired estimate for $\mu_1(t)$. □

It is not clear if the strategy above can be applied to control the first order moment of the perturbation $\tilde\eta$ with respect to the vertical variable Z. In particular, we are not aware of any (approximately) conserved quantity that we could use for this purpose. Instead we choose the modulation parameter $\tilde z(t)$ in (2.32) so that the vertical moment vanishes identically:

$$\mu_2(t) \,:=\, \int_{\Omega_\epsilon} Z\,\tilde\eta(R,Z,t)\,\mathrm{d}X \,=\, 0\,. \tag{4.8}$$

Differentiating (4.8) with respect to time and using (4.1), we obtain the relation

$$\dot{\tilde z}(t) \int_{\Omega_\epsilon} Z\bigl(\partial_Z\eta_* + \delta\partial_Z\tilde\eta\bigr)\,\mathrm{d}X \,=\, \frac{\delta\Gamma}{\epsilon\bar r} \int_{\Omega_\epsilon} Z\,\mathcal{R}(R,Z,t)\,\mathrm{d}X\,, \tag{4.9}$$

where

$$\begin{aligned} \mathcal{R} = &\frac{1}{\delta}\bigl\{\phi_*\,,\tilde\zeta\bigr\} + \frac{1}{\delta}\bigl\{\tilde\phi\,,\zeta_*\bigr\} + \bigl\{\tilde\phi\,,\tilde\zeta\bigr\} - \frac{\epsilon\bar r}{\delta\Gamma}\Bigl(\dot{\bar r}\,\partial_R\tilde\eta + \dot{\bar z}_*\,\partial_Z\tilde\eta\Bigr) \\ &- \mathcal{L}\tilde\eta - \epsilon\partial_R\tilde\zeta - \frac{1}{\delta}\,\mathrm{Rem}(R,Z,t)\,. \end{aligned} \tag{4.10}$$

In view of Lemma 4.1 the integral in the left-hand side of (4.9) is equal to $-1 + \mathcal{O}(\epsilon^\infty)$, and is therefore bounded away from zero if ϵ is small enough. The integral in the right-hand side is a priori of size $\mathcal{O}(\delta^{-1})$, but we observe that $\mathcal{R} = \delta^{-1}\Lambda\tilde\eta + \mathcal{O}(\epsilon\delta^{-1})$, where Λ is the linear operator defined in (3.19). Using the properties established in Proposition 3.6, we see that the leading term gives no contribution:

$$\frac{1}{4\pi}\int_{\mathbb{R}^2} Z\,\Lambda\tilde\eta\,\mathrm{d}X \,=\, \bigl(Z\eta_0\,,\Lambda\tilde\eta\bigr)_{\mathcal{Y}} \,=\, -\bigl(\Lambda(Z\eta_0)\,,\tilde\eta\bigr)_{\mathcal{Y}} \,=\, 0\,,$$

since $Z\eta_0 = -2\partial_Z\eta_0$ is in the kernel of Λ. These considerations, which will be made rigorous in Section 4.8 below, show that the modulation speed $\dot{\tilde z}$ is uniquely determined by (4.9), and suggest that $\dot{\tilde z}(t) = \mathcal{O}(\|\tilde\eta\|_{\mathcal{X}_\epsilon})$ as long as $\|\tilde\eta\|_{\mathcal{X}_\epsilon}$ remains of size $\mathcal{O}(1)$. In particular $\delta\tilde z(t)$ is indeed a small correction to the vertical position of the vortex ring.

4.2 Definition and properties of the weight function

We now provide the precise definition of the weight function $W_\epsilon : \Omega_\epsilon \to (0, +\infty)$ which appears in the energy functional (2.34). We give ourselves three positive numbers $\sigma_1, \sigma_2, \gamma$ such that

$$0 \,<\, \sigma_1 \,<\, \sigma_0 \,<\, 1 \,<\, \sigma_2\,, \qquad \gamma \,=\, \sigma_1/\sigma_2\,, \tag{4.11}$$

where $\sigma_0 \in (0,1)$ is the cut-off exponent already introduced in (2.28). As we shall see $\sigma_2 > 1$ can be chosen arbitrarily, but $\sigma_1 > 0$ has to be taken sufficiently small. In particular σ_1 should be small enough so that Proposition 3.16 holds.

As in (4.2), if $\epsilon > 0$ and $\delta > 0$ are sufficiently small, we denote $\zeta_* = \eta_*/(1+\epsilon R)$, where η_* is the approximate solution of (2.15) given by (2.28). We recall that ζ_* and $\phi_* := \mathrm{BS}^\epsilon[\eta_*]$ satisfy the relation (2.36), where $\Phi_\epsilon : \mathbb{R}_+ \to \mathbb{R}$ is the function constructed in Section 3.8. We decompose the domain $\Omega_\epsilon = \bigl\{(R,Z)\,;\, 1+\epsilon R > 0\bigr\}$ into a disjoint union $\Omega_\epsilon' \cup \Omega_\epsilon'' \cup \Omega_\epsilon'''$, where

$$\begin{aligned}
\Omega_\epsilon' \,&=\, \Bigl\{(R,Z) \in \Omega_\epsilon\,;\, \Phi_\epsilon'(\zeta_*(R,Z)) < \exp\bigl(\epsilon^{-2\sigma_1}/4\bigr)\Bigr\}\,,\\
\Omega_\epsilon'' \,&=\, \Bigl\{(R,Z) \in \Omega_\epsilon \setminus \Omega_\epsilon'\,;\, \rho \le \epsilon^{-\sigma_2}\Bigr\}\,,\\
\Omega_\epsilon''' \,&=\, \Bigl\{(R,Z) \in \Omega_\epsilon\,;\, \rho > \epsilon^{-\sigma_2}\Bigr\}\,.
\end{aligned} \tag{4.12}$$

Here and in what follows, if $(R,Z) \in \mathbb{R}^2$, we denote $\rho = (R^2+Z^2)^{1/2}$. The domains $\Omega_\epsilon', \Omega_\epsilon''$ also depend (mildly) on δ, but for simplicity this dependence is not indicated explicitly.

Lemma 4.2. *If $\epsilon > 0$ is small enough, the inner region Ω_ϵ' defined in* (4.12) *is diffeomorphic to a open disk, and there exists $\kappa > 0$ such that*

$$\bigl\{(R,Z)\,;\, \rho \le \epsilon^{-\sigma_1}\bigr\} \subset \Omega_\epsilon' \,\subset\, \Bigl\{(R,Z)\,;\, \rho^2 \le \epsilon^{-2\sigma_1} + \kappa \log\frac{1}{\epsilon}\Bigr\}\,. \tag{4.13}$$

Proof. The main properties of the function Φ_ϵ are established in the proof of Proposition 3.16. In particular, using estimates (3.94), (3.95), (3.97), it is easy to verify that

$$\frac{1}{2}\, A(\rho) \,\le\, \Phi_\epsilon'\bigl(\zeta_*(R,Z)\bigr) \,\le\, 2A(\rho)\,, \qquad \text{when } \rho \le 2\epsilon^{-\sigma_1}\,. \tag{4.14}$$

Here $A(\rho) = (4/\rho^2)\bigl(e^{\rho^2/4} - 1\bigr)$, see (3.85). Since $2A(\epsilon^{-\sigma_1}) < \exp(\epsilon^{-2\sigma_1}/4)$ as soon as $\epsilon^{-\sigma_1} \ge 3$, we deduce that $(R,Z) \in \Omega_\epsilon'$ if $\rho \le \epsilon^{-\sigma_1}$. Similarly, using the lower bound in (4.14), it is easy to verify that the inner region Ω_ϵ' is contained in the disk $\rho^2 \le \epsilon^{-2\sigma_1} + \kappa \log\frac{1}{\epsilon}$ if $\kappa > 4\sigma_1$ and $\epsilon > 0$ is small enough. Finally Ω_ϵ' is diffeomorphic to a disk because $\Phi_\epsilon'(\zeta_*)$ is C^2-close to a strictly increasing radially symmetric function when $\epsilon > 0$ is small, see (3.82). $\square$

We next choose a smooth cut-off function $\chi_1 : \mathbb{R} \to [\frac{1}{2}, 3]$ such that

$$\chi_1(x) \,=\, \frac{1}{1+x} \quad \text{for } |x| \le \frac{1}{2}\,, \qquad \chi_1'(x) \,=\, 0 \quad \text{for } |x| \ge \frac{3}{4}\,. \tag{4.15}$$

The weight $W_\epsilon : \Omega_\epsilon \to (0, +\infty)$ is defined by

$$W_\epsilon(R,Z) \,=\, \chi_1(\epsilon R) \times \begin{cases} \Phi_\epsilon'\bigl(\zeta_*(R,Z)\bigr) & \text{in } \Omega_\epsilon'\,,\\ \exp\bigl(\epsilon^{-2\sigma_1}/4\bigr) & \text{in } \Omega_\epsilon''\,,\\ \exp\bigl(\rho^{2\gamma}/4\bigr) & \text{in } \Omega_\epsilon'''\,,\end{cases} \tag{4.16}$$

where $\gamma = \sigma_1/\sigma_2 < 1$ and $\Omega'_\epsilon, \Omega''_\epsilon, \Omega'''_\epsilon$ are the regions defined in (4.12). In other words, we assume that $W_\epsilon = \Phi'_\epsilon(\zeta_*)/(1+\epsilon R)$ as long as the numerator remains smaller than the threshold value $\exp(\epsilon^{-2\sigma_1}/4)$. Outside this inner region, the weight is radially symmetric except for the geometric factor $\chi_1(\epsilon R)$, and the radial profile remains constant as long as $\rho \le \epsilon^{-\sigma_2}$ before increasing again like $\exp(\rho^{2\gamma}/4)$ when $\rho > \epsilon^{-\sigma_2}$. By construction the function W_ϵ is locally Lipschitz continuous in Ω_ϵ, and smooth in the interior of all three regions (4.12). The (mild) dependence of W_ϵ upon the parameter $\delta > 0$ is not indicated explicitly. A schematic representation of the graph of W_ϵ is given in Figure 2.

Further properties of the weight W_ϵ are collected in the following lemma.

Lemma 4.3. *There exist positive constants C_1, C_2 such that, if ϵ, δ, and σ_1 are small enough, the weight W_ϵ satisfies the uniform bounds*

$$C_1 \exp\bigl(\rho^{2\gamma}/4\bigr) \;\le\; W_\epsilon(R,Z) \;\le\; C_2 A(\rho)\,, \qquad (R,Z) \in \Omega_\epsilon\,, \tag{4.17}$$

where $\rho = (R^2+Z^2)^{1/2}$ and $A(\rho)$ is defined in (3.85). Moreover, given any $\gamma_1 < 1$ there exists $C_3 > 0$ such that the following estimates hold in the inner region

$$\bigl|W_\epsilon(R,Z) - A(\rho)\bigr| \,+\, \bigl|\nabla W_\epsilon(R,Z) - \nabla A(\rho)\bigr| \;\le\; C_3\, \epsilon^{\gamma_1} A(\rho)\,, \qquad (R,Z) \in \Omega'_\epsilon\,. \tag{4.18}$$

Proof. Since $\frac{1}{2} \le \chi_1(\epsilon R) \le 3$ and $\exp(\rho^{2\gamma}/4) \le CA(\rho)$, we deduce from (4.14) that the bounds (4.17) hold in the inner region Ω'_ϵ, as well as in the far field region Ω'''_ϵ. In the intermediate region Ω''_ϵ we know that $\rho \le \epsilon^{-\sigma_2}$, so that $\exp(\rho^{2\gamma}/4) \le \exp(\epsilon^{-2\sigma_1}/4)$ since $\gamma = \sigma_1/\sigma_2$, and this gives the lower bound in (4.17). If $\rho \ge 2\epsilon^{-\sigma_1}$, it is clear that $\exp(\epsilon^{-2\sigma_1}/4) \le A(\rho)$, which is the desired upper bound. Finally if $(R,Z) \in \Omega''_\epsilon$ and $\rho \le 2\epsilon^{-\sigma_1}$, we deduce from (4.14) that $\exp(\epsilon^{-2\sigma_1}/4) \le \Phi'_\epsilon\bigl(\zeta_*(R,Z)\bigr) \le 2A(\rho)$, which concludes the proof of the upper bound in (4.17).

To prove (4.18), we start from the expression (4.16) of the weight W_ϵ in the inner region Ω'_ϵ. We know from (3.85) that $A(\rho) = \Phi'_0(\eta_0)$, where η_0 is defined in (3.32). We thus find

$$|W_\epsilon(R,Z) - A(\rho)\bigr| \;\le\; \bigl|\chi_1(\epsilon R) - 1\bigr|\Phi'_\epsilon(\zeta_*) + \bigl|\Phi'_\epsilon(\zeta_*) - \Phi'_\epsilon(\eta_0)\bigr| + \bigl|\Phi'_\epsilon(\eta_0) - \Phi'_0(\eta_0)\bigr|\,. \tag{4.19}$$

Since $\chi_1(\epsilon R) = (1+\epsilon R)^{-1}$ when $(R,Z) \in \Omega'_\epsilon$, the first term in the right-hand of (4.19) is smaller than $C\epsilon|R|\,\Phi'_\epsilon(\zeta_*) \le C\epsilon^{1-\sigma_1} A(\rho)$. For the second term, we use the bounds (3.94), (3.95), and (3.97) to obtain

$$\bigl|\Phi'_\epsilon(\zeta_*) - \Phi'_\epsilon(\eta_0)\bigr| \;\le\; \sup_{\frac{1}{2}\le\lambda\le 2} \bigl|\Phi''_\epsilon(\lambda\eta_0)\bigr|\,|\zeta_* - \eta_0| \;\le\; CB(\rho)(1+\rho)^N \epsilon\eta_0 \;\le\; C\epsilon^{\gamma_1} A(\rho)\,,$$

where in the last inequality we assumed that $\sigma_1 > 0$ is small enough so that $N\sigma_1 \le 1-\gamma_1$. The last term in (4.19) is bounded by $\epsilon^2|\Phi'_2(\eta_0)| \le C\epsilon^{\gamma_1} A(\rho)$ in view of (3.97). Altogether we arrive at the estimate $|W_\epsilon(R,Z) - A(\rho)\bigr| \le C\epsilon^{\gamma_1} A(\rho)$. The corresponding inequality for the first order derivatives can be obtained in a similar way, and we omit the details $\square$

4.3 Coercivity of the energy functional

For $\epsilon \ge 0$ small enough, we introduce the weighted L^2 space $\mathcal{X}_\epsilon = \bigl\{\eta \in L^2(\Omega_\epsilon)\,;\, \|\eta\|_{\mathcal{X}_\epsilon} < \infty\bigr\}$ defined by the norm (2.39), namely

$$\|\eta\|^2_{\mathcal{X}_\epsilon} \,=\, \int_{\Omega_\epsilon} W_\epsilon(R,Z)\,|\eta(R,Z)|^2 \,\mathrm{d}R\,\mathrm{d}Z\,. \tag{4.20}$$

In the limiting case $\epsilon = 0$, it is understood that $\Omega_0 = \mathbb{R}^2$ and $W_0(R,Z) = A(\rho)$, in agreement with (4.18). Assuming that $\epsilon > 0$, we consider the energy functional (2.34), namely

$$E_\epsilon[\eta] \,=\, \frac{1}{2}\,\|\eta\|_{\mathcal{X}_\epsilon}^2 - E_\epsilon^{\mathrm{kin}}[\eta]\,, \qquad \eta \in \mathcal{X}_\epsilon\,, \tag{4.21}$$

where $E_\epsilon^{\mathrm{kin}}$ is the kinetic energy defined by

$$E_\epsilon^{\mathrm{kin}}[\eta] \,=\, \frac{1}{2}\int_{\Omega_\epsilon} \phi\,\eta\,\mathrm{d}R\,\mathrm{d}Z \,=\, \frac{1}{2}\int_{\Omega_\epsilon} \frac{|\nabla\phi|^2}{1+\epsilon R}\,\mathrm{d}R\,\mathrm{d}Z\,, \qquad \phi \,=\, \mathrm{BS}^\epsilon[\eta]\,. \tag{4.22}$$

Since we are interested in the regime where ϵ is small, it is important to observe that $E_\epsilon^{\mathrm{kin}}[\eta]$ becomes *singular* in the limit $\epsilon \to 0$, if the vorticity η has nonzero mean. This divergence is related to the well-known fact that any (nontrivial) nonnegative vorticity distribution in $\mathbb{R}^2$ has infinite kinetic energy. The regular part of $E_\epsilon^{\mathrm{kin}}[\eta]$ is given, to leading order, by the two-dimensional energy

$$E_0^{\mathrm{kin}}[\eta] \,=\, \frac{1}{4\pi}\int_{\mathbb{R}^2} (L\eta)\eta\,\mathrm{d}X \,=\, \frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\Bigl(\frac{8}{D}\Bigr)\,\eta(R,Z)\eta(R',Z')\,\mathrm{d}X\,\mathrm{d}X'\,, \tag{4.23}$$

where L is the integral operator (3.14) and $D^2 = (R{-}R')^2 + (Z{-}Z')^2$. More precisely, we have the following statement, whose proof is postponed to Section B.1.

Lemma 4.4. *If $\epsilon > 0$ is small and $\eta \in \mathcal{X}_\epsilon$ satisfies $\mathrm{supp}(\eta) \subset B_\epsilon := \{(R,Z) \in \Omega_\epsilon\,;\, \rho \le \epsilon^{-\sigma_1}\}$, we have the expansion*

$$E_\epsilon^{\mathrm{kin}}[\eta] \,=\, \frac{\beta_\epsilon - 2}{4\pi}\,\mu_0^2 + E_0^{\mathrm{kin}}[\eta] + \mathcal{O}\bigl(\epsilon\beta_\epsilon\|\eta\|_{\mathcal{X}_\epsilon}^2\bigr)\,, \qquad \textit{as } \epsilon \to 0\,, \tag{4.24}$$

where $\beta_\epsilon = \log(1/\epsilon)$ and $\mu_0 = \int_{\Omega_\epsilon} \eta\,\mathrm{d}R\,\mathrm{d}Z$.

We now consider the (formal) limit of the functional $E_\epsilon[\eta]$ as $\epsilon \to 0$, assuming that η has zero mean to avoid the logarithmic divergence in the right-hand side of (4.24). In view of (4.18) and Lemma 4.4, we obtain the limiting functional

$$E_0[\eta] \,=\, \frac{1}{2}\int_{\mathbb{R}^2} A(\rho)\,\eta(R,Z)^2\,\mathrm{d}R\,\mathrm{d}Z - E_0^{\mathrm{kin}}[\eta] \,=\, \frac{1}{2}\,\|\eta\|_{\mathcal{X}_0}^2 - E_0^{\mathrm{kin}}[\eta]\,, \tag{4.25}$$

which is studied in detail in our previous work [34]. In particular, we have the following property:

Proposition 4.5. *There exists constants $C_4 > 2$ and $C_5 > 0$ such that, for all $\eta \in \mathcal{X}_0$,*

$$\|\eta\|_{\mathcal{X}_0}^2 \,\le\, C_4 E_0[\eta] + C_5\bigl(\mu_0^2 + \mu_1^2 + \mu_2^2\bigr)\,, \tag{4.26}$$

where $\mu_0 = \int_{\mathbb{R}^2} \eta\,\mathrm{d}X$, $\mu_1 = \int_{\mathbb{R}^2} R\eta\,\mathrm{d}X$, $\mu_2 = \int_{\mathbb{R}^2} Z\eta\,\mathrm{d}X$.

Proof. The results of [34, Section 2] show that (4.26) holds when $\mu_0 = \mu_1 = \mu_2 = 0$, and the general case is easily deduced by the following argument. Given $\eta \in \mathcal{X}_0$ we define

$$\hat\eta \,=\, \eta - \mu_0\eta_0 + \mu_1\partial_R\eta_0 + \mu_2\partial_Z\eta_0\,, \qquad \hat\phi \,=\, \phi - \mu_0\phi_0 + \mu_1\partial_R\phi_0 + \mu_2\partial_Z\phi_0\,,$$

where $\phi = (2\pi)^{-1}L\eta$ and η_0, ϕ_0 are as in (3.32). By construction the integral and the first order moments of the new function $\hat\eta \in \mathcal{X}_0$ vanish, so that we can apply the results of [34] which give the bound $\|\hat\eta\|_{\mathcal{X}_0}^2 \,\le\, C_4 E_0[\hat\eta]$. On the other hand, expanding the quadratic expressions $\|\hat\eta\|_{\mathcal{X}_0}^2$ and $E_0[\hat\eta]$ and using Hölder's inequality, it is straightforward to verify that

$$\|\hat\eta\|_{\mathcal{X}_0}^2 \,\ge\, \frac{1}{2}\,\|\eta\|_{\mathcal{X}_0}^2 - C\bigl(\mu_0^2 + \mu_1^2 + \mu_2^2\bigr)\,, \quad E_0[\hat\eta] \,\le\, E_0[\eta] + \frac{1}{4C_4}\,\|\eta\|_{\mathcal{X}_0}^2 + C\bigl(\mu_0^2 + \mu_1^2 + \mu_2^2\bigr)\,,$$

for some $C > 0$. If we combine these estimates, we arrive at the bound (4.26) with a deteriorated constant C_4. $\square$

Using Proposition 4.5, we now establish a similar coercivity property for the functional E_ϵ when $\epsilon > 0$ is small. The proof of the following proposition is again postponed to Section B.1.

Proposition 4.6. *If the weight W_ϵ satisfies (4.17) and (4.18), there exist constants $C_6 > 0$ and $C_7 > 0$ such that, for all sufficiently small $\epsilon > 0$ and all $\eta \in \mathcal{X}_\epsilon$, we have the estimate*

$$\|\eta\|_{\mathcal{X}_\epsilon}^2 \,\le\, C_6 E_\epsilon[\eta] + C_7\big(\beta_\epsilon \mu_0^2 + \mu_1^2 + \mu_2^2\big)\,, \tag{4.27}$$

where $\beta_\epsilon = \log(1/\epsilon)$ and $\mu_0 = \int_{\Omega_\epsilon} \eta \,\mathrm{d}X$, $\mu_1 = \int_{\Omega_\epsilon} R\eta \,\mathrm{d}X$, $\mu_2 = \int_{\Omega_\epsilon} Z\eta \,\mathrm{d}X$.

In what follows we use the bound (4.27) to estimate the vorticity perturbation $\tilde\eta$ introduced in (2.31). The corresponding moments μ_0, μ_1 are under control thanks to Lemma 4.1, and $\mu_2 = 0$ according to (4.8). So it remains to bound the energy functional $E_\epsilon[\tilde\eta]$, which is the purpose of the remaining sections.

4.4 Time evolution of the energy

Let $\tilde\eta$ be the solution of (4.1) with zero initial data. Assuming that $\delta > 0$ and $\sigma > 0$ are sufficiently small, we consider for $t \in (0, T_{\mathrm{adv}}\delta^{-\sigma})$ the energy function

$$E_\epsilon(t) \,=\, \frac{1}{2}\int_{\Omega_\epsilon} W_\epsilon(R,Z)\,\tilde\eta(R,Z,t)^2 \,\mathrm{d}X \,-\, \frac{1}{2}\int_{\Omega_\epsilon} \tilde\phi(R,Z,t)\,\tilde\eta(R,Z,t)\,\mathrm{d}X\,, \tag{4.28}$$

where $\epsilon = \sqrt{\nu t}/\bar r(t)$ and W_ϵ is the weight function defined by (4.16). The first term in the right-hand side of (4.28) is equal to $\frac{1}{2}\|\tilde\eta\|_{\mathcal{X}_\epsilon}^2$, and the second one is the kinetic energy $E_\epsilon^{\mathrm{kin}}[\tilde\eta]$, which satisfies (4.22) and involves the stream function $\tilde\phi = \mathrm{BS}^\epsilon[\tilde\eta]$ defined by the Biot-Savart formula (2.20). Differentiating (4.28) with respect to time and using the relations (3.34), (4.22) together with the evolution equation (4.1), we obtain by a direct calculation

$$\begin{aligned} t\partial_t E_\epsilon \,&=\, \int_{\Omega_\epsilon}\Big(W_\epsilon\tilde\eta\, t\partial_t\tilde\eta + \frac{1}{2}t(\partial_t W_\epsilon)\tilde\eta^2\Big)\,\mathrm{d}X \,-\, \int_{\Omega_\epsilon}\Big(\tilde\phi\, t\partial_t\tilde\eta + \frac{t\dot\epsilon}{2}\frac{R|\nabla\tilde\phi|^2}{(1+\epsilon R)^2}\Big)\,\mathrm{d}X \\ &=\, I_1 + I_2 + I_3 + I_4 + I_5 + I_6\,, \end{aligned}$$

where the quantities $I_1, \dots, I_6$ collect the following terms.

1. *Local advection terms :*

$$\begin{aligned} I_1 \,&=\, -\frac{1}{\delta}\int_{\Omega_\epsilon} W_\epsilon\tilde\eta\{\phi_*\,,\,\tilde\zeta\}\,\mathrm{d}X + \frac{\epsilon\bar r\dot{\bar z}_*}{\delta\Gamma}\int_{\Omega_\epsilon} W_\epsilon\tilde\eta\,\partial_Z\tilde\eta\,\mathrm{d}X \\ &=\, -\frac{1}{\delta}\int_{\Omega_\epsilon} W_\epsilon\tilde\eta\Big\{\phi_* - \frac{\bar r\dot{\bar z}_*}{2\Gamma}(1+\epsilon R)^2\,,\,\tilde\zeta\Big\}\,\mathrm{d}X \\ &=\, -\frac{1}{2\delta}\int_{\Omega_\epsilon}\Big\{W_\epsilon(1+\epsilon R)\,,\,\phi_* - \frac{\bar r\dot{\bar z}_*}{2\Gamma}(1+\epsilon R)^2\Big\}\tilde\zeta^2\,\mathrm{d}X\,. \end{aligned} \tag{4.29}$$

2. *Nonlocal advection terms :*

$$\begin{aligned} I_2 \,&=\, \frac{1}{\delta}\int_{\Omega_\epsilon}\tilde\phi\{\phi_*\,,\,\tilde\zeta\}\,\mathrm{d}X - \frac{\epsilon\bar r\dot{\bar z}_*}{\delta\Gamma}\int_{\Omega_\epsilon}\tilde\phi\,\partial_Z\tilde\eta\,\mathrm{d}X - \frac{1}{\delta}\int_{\Omega_\epsilon}\big(W_\epsilon\tilde\eta - \tilde\phi\big)\{\tilde\phi\,,\,\zeta_*\}\,\mathrm{d}X \\ &=\, \frac{1}{\delta}\int_{\Omega_\epsilon}\tilde\phi\Big\{\phi_* - \frac{\bar r\dot{\bar z}_*}{2\Gamma}(1+\epsilon R)^2\,,\,\tilde\zeta\Big\}\,\mathrm{d}X - \frac{1}{\delta}\int_{\Omega_\epsilon} W_\epsilon\tilde\eta\{\tilde\phi\,,\,\zeta_*\}\,\mathrm{d}X \\ &=\, \frac{1}{\delta}\int_{\Omega_\epsilon}\Big\{\tilde\phi\,,\,\phi_* - \frac{\bar r\dot{\bar z}_*}{2\Gamma}(1+\epsilon R)^2\Big\}\tilde\zeta\,\mathrm{d}X - \frac{1}{\delta}\int_{\Omega_\epsilon} W_\epsilon(1+\epsilon R)\{\tilde\phi\,,\,\zeta_*\}\tilde\zeta\,\mathrm{d}X\,. \end{aligned} \tag{4.30}$$

3. *Nonlinear terms :*

$$I_3 \;=\; -\int_{\Omega_\epsilon} (W_\epsilon\tilde\eta - \tilde\phi)\{\tilde\phi\,,\tilde\zeta\}\,\mathrm{d}X \;=\; -\int_{\Omega_\epsilon}\{W_\epsilon\tilde\eta\,,\tilde\phi\}\tilde\zeta\,\mathrm{d}X\,. \tag{4.31}$$

4. *Diffusive terms :*

$$I_4 \;=\; \int_{\Omega_\epsilon}\big(W_\epsilon\tilde\eta - \tilde\phi\big)\Big(\mathcal{L}\tilde\eta + \epsilon\partial_R\tilde\zeta\Big)\,\mathrm{d}X\,.$$

Integrating by parts as explained in Section B.2, we obtain the equivalent expression

$$\begin{aligned} I_4 \;=\; &-\int_{\Omega_\epsilon} W_\epsilon|\nabla\tilde\eta|^2\,\mathrm{d}X - \int_{\Omega_\epsilon}(\nabla W_\epsilon\cdot\nabla\tilde\eta)\tilde\eta\,\mathrm{d}X - \int_{\Omega_\epsilon} V_\epsilon\tilde\eta^2\,\mathrm{d}X \\ &- \frac{\epsilon}{2}\int_{\Omega_\epsilon}\partial_R\big(W_\epsilon(1+\epsilon R)\big)\tilde\zeta^2\,\mathrm{d}X + \frac{\epsilon}{4}\int_{\Omega_\epsilon}\frac{R|\nabla\tilde\phi|^2}{(1+\epsilon R)^2}\,\mathrm{d}X\,, \end{aligned} \tag{4.32}$$

where

$$V_\epsilon \;=\; \frac{1}{4}(R\partial_R + Z\partial_Z)W_\epsilon - \frac{1}{2}W_\epsilon - (1+\epsilon R)\,. \tag{4.33}$$

5. *Remainder term :*

$$I_5 \;=\; \frac{1}{\delta}\int_{\Omega_\epsilon}\big(W_\epsilon\tilde\eta - \tilde\phi\big)\,\mathrm{Rem}(R,Z,t)\,\mathrm{d}X\,. \tag{4.34}$$

6. *Additional terms :*

$$\begin{aligned} I_6 \;=\; &\frac{1}{2}\int_{\Omega_\epsilon} t(\partial_t W_\epsilon)\tilde\eta^2\,\mathrm{d}X + \frac{\epsilon\bar r\dot{\bar r}}{\delta\Gamma}\int_{\Omega_\epsilon}\big(W_\epsilon\tilde\eta - \tilde\phi\big)\,\partial_R\tilde\eta\,\mathrm{d}X \\ &- \frac{t\dot\epsilon}{2}\int_{\Omega_\epsilon}\frac{R|\nabla\tilde\phi|^2}{(1+\epsilon R)^2}\,\mathrm{d}X + \frac{\epsilon\bar r\dot{\bar z}}{\delta\Gamma}\int_{\Omega_\epsilon}\big(W_\epsilon\tilde\eta - \tilde\phi\big)\,\big(\partial_Z\eta_* + \delta\partial_Z\tilde\eta\big)\,\mathrm{d}X\,. \end{aligned} \tag{4.35}$$

For the purposes of our analysis, it is useful to reorganize some terms appearing in the quantities I_4 and I_6. First, using (2.19) and integrating by parts, it is easy to verify that

$$-\int_{\Omega_\epsilon}\tilde\phi\,\partial_R\tilde\eta\,\mathrm{d}X \;=\; \int_{\Omega_\epsilon}\tilde\eta\,\partial_R\tilde\phi\,\mathrm{d}X \;=\; \frac{\epsilon}{2}\int_{\Omega_\epsilon}\frac{|\nabla\tilde\phi|^2}{(1+\epsilon R)^2}\,\mathrm{d}X\,. \tag{4.36}$$

So, if we collect all terms involving $|\nabla\tilde\phi|^2$ in (4.32), (4.35), and (4.36), we obtain the quantity

$$\Big(\frac{\epsilon}{4} - \frac{t\dot\epsilon}{2}\Big)\int_{\Omega_\epsilon}\frac{R|\nabla\tilde\phi|^2}{(1+\epsilon R)^2}\,\mathrm{d}X + \frac{\epsilon^2\bar r\dot{\bar r}}{2\delta\Gamma}\int_{\Omega_\epsilon}\frac{|\nabla\tilde\phi|^2}{(1+\epsilon R)^2}\,\mathrm{d}X \;=\; \frac{t\dot{\bar r}}{2\bar r}\int_{\Omega_\epsilon}\frac{|\nabla\tilde\phi|^2}{1+\epsilon R}\,\mathrm{d}X\,,$$

where we used the expression (3.34) of $t\dot\epsilon$. Next, we prefer including the term involving $t\partial_t W_\epsilon$ in I_4 rather than I_6, because it will be combined with the diffusive terms in I_4 to obtain negative quantities that will allow us to control the evolution of the energy. Summarizing, if we define

$$\begin{aligned} \hat I_4 \;=\; &-\int_{\Omega_\epsilon} W_\epsilon|\nabla\tilde\eta|^2\,\mathrm{d}X - \int_{\Omega_\epsilon}(\nabla W_\epsilon\cdot\nabla\tilde\eta)\tilde\eta\,\mathrm{d}X - \int_{\Omega_\epsilon} V_\epsilon\tilde\eta^2\,\mathrm{d}X \\ &- \frac{\epsilon}{2}\int_{\Omega_\epsilon}\partial_R\big(W_\epsilon(1+\epsilon R)\big)\tilde\zeta^2\,\mathrm{d}X + \frac{1}{2}\int_{\Omega_\epsilon} t(\partial_t W_\epsilon)\tilde\eta^2\,\mathrm{d}X\,, \end{aligned} \tag{4.37}$$

and

$$\hat I_6 \;=\; \frac{\epsilon\bar r\dot{\bar r}}{\delta\Gamma}\int_{\Omega_\epsilon} W_\epsilon\tilde\eta\partial_R\tilde\eta\,\mathrm{d}X + \frac{t\dot{\bar r}}{\bar r}E_\epsilon^{\mathrm{kin}}[\tilde\eta] + \frac{\epsilon\bar r\dot{\bar z}}{\delta\Gamma}\int_{\Omega_\epsilon}\big(W_\epsilon\tilde\eta - \tilde\phi\big)\,\big(\partial_Z\eta_* + \delta\partial_Z\tilde\eta\big)\,\mathrm{d}X\,, \tag{4.38}$$

we obtain the identity $t\partial_t E_\epsilon = I_1 + I_2 + I_3 + \hat I_4 + I_5 + \hat I_6$, which we exploit in Sections 4.6–4.9.

4.5 Bounds on the stream function

In this section we collect a few estimates on the stream function $\phi = \mathrm{BS}^\epsilon[\eta]$, where BS^ϵ is the ϵ-dependent Biot-Savart operator (2.20). We are especially interested in bounds on the velocity field $U = (U_R, U_Z)$ defined by (2.18).

Lemma 4.7. *There exists a constant $C > 0$ such that, for all $\epsilon \in (0,1)$,*

$$\Big|\frac{\partial_R \phi}{1+\epsilon R}\Big| + \Big|\frac{\partial_Z \phi}{1+\epsilon R}\Big| \le \int_{\Omega_\epsilon} \frac{C}{\sqrt{(R-R')^2 + (Z-Z')^2}}\, |\eta(R', Z')| \,\mathrm{d}X' \,. \tag{4.39}$$

In particular, for any $q > 2$, we have $\|U\|_{L^q} \le C_q \|\eta\|_{\mathcal{X}_\epsilon}$ where U is the velocity field (2.18)*.*

Proof. Estimate (4.39) is established in the proof of [33, Lemma 4.1], which in turn relies on [32, Proposition 2.3]. Using the Hardy-Littlewood-Sobolev inequality, we deduce from (4.39) that $\|U\|_{L^q} \le C_q \|\eta\|_{L^p}$ if $q > 2$ and $p \in (1,2)$ satisfy the relation $1/p = 1/q + 1/2$. Finally, the lower bound on W_ϵ in (4.17) implies that $\|\eta\|_{L^p} \le C\|\eta\|_{\mathcal{X}_\epsilon}$ for any $p \in [1,2]$. □

The particular case where $\eta = \eta_*$ is the approximate solution (2.28) plays an important role.

Lemma 4.8. *The following estimates hold for the stream function $\phi_* = \mathrm{BS}^\epsilon[\eta_*]$:*

$$\Big|\frac{\partial_R \phi_*}{1+\epsilon R}\Big| + \Big|\frac{\partial_Z \phi_*}{1+\epsilon R}\Big| \le \frac{C}{1+\rho+\epsilon^2\rho^3}\,, \qquad \Big|\frac{\partial_Z \phi_*}{(1+\epsilon R)^2}\Big| \le \frac{C}{1+\rho+\epsilon^3\rho^4}\,, \tag{4.40}$$

where $\rho = (R^2 + Z^2)^{1/2}$.

Proof. In the region where $\rho \le 1/(2\epsilon)$, we can use estimate (4.39) with $\eta = \eta_*$. Since η_* satisfies the Gaussian bound (3.75), we easily deduce that $|U| \le C(1+\rho)^{-1}$, which gives estimate (4.40) in that case. We now concentrate on the region $\rho \ge 1/(2\epsilon)$, where a more careful analysis is needed. We start from the formulas (3.73) with $\eta = \eta_*$, and we first estimate the vertical derivative $\partial_Z \phi_*$. Since $|\tilde F(s)| \le C s^{-3/2}$ for all $s > 0$, we see that

$$\Big|\frac{\partial_Z \phi_*}{(1+\epsilon R)^2}\Big| \le \frac{C}{\epsilon^3} \int_{\Omega_\epsilon} \frac{(1+\epsilon R')^2\, |\eta_*(R', Z')|}{\big((R-R')^2 + (Z-Z')^2\big)^2} \,\mathrm{d}R' \,\mathrm{d}Z' \,. \tag{4.41}$$

Note that the integral is, in fact, taken over the support of η_*, which is included in the ball $\rho' := (R'^2 + Z'^2)^{1/2} \le 2\epsilon^{-\sigma_0}$ where $\sigma_0 < 1$. In particular we can disregard the factor $(1+\epsilon R')^2$ in the numerator, and the denominator is always larger that $\rho^4/2$ if ϵ is sufficiently small. So the right-hand side of (4.41) is bounded by $C\epsilon^{-3}\rho^{-4}$ when $\rho \ge 1/(2\epsilon)$, which concludes the proof of the second inequality in (4.40). Since $1 + \epsilon R \le 1 + \epsilon\rho$, the estimate on $\partial_Z \phi_*/(1+\epsilon R)$ in (4.40) follows immediately.

To conclude the proof of the first inequality in (4.40), we must estimate the quantity $\partial_R \phi_*$ which contains an additional term given by the last line in (3.73). In the region where $\rho \ge 1/(2\epsilon)$, using the fact that $|F(s)| + |\tilde F(s)| \le C s^{-3/2}$, we see that the contribution of that term to the vertical speed $U_Z = \partial_R \phi_*/(1+\epsilon R)$ is bounded by

$$\frac{C}{\epsilon^2} \int_{\Omega_\epsilon} \frac{(1+\epsilon R')^2\, |\eta_*(R', Z')|}{\big((R-R')^2 + (Z-Z')^2\big)^{3/2}} \,\mathrm{d}R' \,\mathrm{d}Z' \le \frac{C}{\epsilon^2 \rho^3}\,.$$

The proof of (4.40) is thus complete. □

4.6 Control of the advection terms

In what follows we always assume that $\delta > 0$ is sufficiently small and that $\epsilon^2 \lesssim \delta^{1-\sigma}$ for some small $\sigma > 0$, see Remark 2.2. As in Lemma 4.3, we also suppose that the exponent $\sigma_1 > 0$ is small enough. We first estimate the advection terms I_1, I_2 defined in (4.29), (4.30). These terms are potentially dangerous because they include a factor $1/\delta$ which is very large in the vanishing viscosity limit, but the energy functional (2.34) was designed precisely so that these contributions can be controlled.

Lemma 4.9. *There exist $\gamma_1 > 0$ and $C > 0$ such that*

$$|I_1| \le C\epsilon^{\gamma_1} \|\tilde{\eta}\|_{\mathcal{X}_\epsilon}^2 + \frac{C\epsilon^2}{\delta} \int_{\Omega''_\epsilon} W_\epsilon \tilde{\eta}^2 \,\mathrm{d}X\,. \tag{4.42}$$

Proof. To exploit the properties of the weight W_ϵ, we decompose the integral (4.29) defining I_1 in three pieces, which correspond to the subdomains (4.12). If $(R, Z) \in \Omega'_\epsilon$, we know from (4.16), (3.88) that

$$W_\epsilon = \frac{\Phi'_\epsilon(\zeta_*)}{1 + \epsilon R}\,, \qquad \phi_* - \frac{\bar{r}\dot{\bar{z}}_*}{2\Gamma}(1 + \epsilon R)^2 = \Phi_\epsilon(\zeta_*) + \Theta\,, \tag{4.43}$$

where Θ is a remainder term that is studied in Proposition 3.16. It follows that

$$\Big\{W_\epsilon(1 + \epsilon R)\,,\, \phi_* - \frac{\bar{r}\dot{\bar{z}}_*}{2\Gamma}(1 + \epsilon R)^2\Big\} = \big\{\Phi'_\epsilon(\zeta_*)\,,\, \Phi_\epsilon(\zeta_*) + \Theta\big\} = \big\{\Phi'_\epsilon(\zeta_*)\,,\, \Theta\big\}\,,$$

where the right-hand side can be controlled using the bounds (3.89) on Θ and the estimates (4.14), (4.18) on the weight W_ϵ in Ω'_ϵ. This gives, for some integer N and any $\gamma_3 \in (2, 3)$,

$$\big|\big\{\Phi'_\epsilon(\zeta_*)\,,\, \Theta\big\}\big| \le C\big(\epsilon\delta + \epsilon^{\gamma_3}\big)(1 + \rho)^N W_\epsilon \le C\big(\epsilon\delta + \epsilon^{\gamma_3}\big)\,\epsilon^{-N\sigma_1} W_\epsilon\,, \tag{4.44}$$

where we used the fact that $1 + \rho \le 2\epsilon^{-\sigma_1}$ when $(R, Z) \in \Omega'_\epsilon$. Since $\tilde{\zeta} \approx \tilde{\eta}$ in Ω'_ϵ and since $\delta^{-1} \lesssim \epsilon^{-2/(1-\sigma)}$ in the parameter regime we consider, it follows from (4.44) that

$$\frac{1}{\delta}\int_{\Omega'_\epsilon} \big|\big\{\Phi'_\epsilon(\zeta_*)\,,\, \Theta\big\}\big|\,\tilde{\zeta}^2 \,\mathrm{d}X \le C\Big(\epsilon + \frac{\epsilon^{\gamma_3}}{\delta}\Big)\,\epsilon^{-N\sigma_1} \int_{\Omega'_\epsilon} W_\epsilon \tilde{\eta}^2 \,\mathrm{d}X \le C\epsilon^{\gamma_1} \|\tilde{\eta}\|_{\mathcal{X}_\epsilon}^2\,, \tag{4.45}$$

where γ_1 is taken so that $0 < \gamma_1 < \gamma_3 - 2/(1-\sigma) - N\sigma_1$. As $\gamma_3 < 3$ is arbitrary, such a choice is always possible if we assume that $\sigma > 0$ and $\sigma_1 > 0$ are small enough.

We next consider the intermediate region Ω''_ϵ in which $W_\epsilon(1 + \epsilon R) = \chi_2(\epsilon R)\exp\big(\epsilon^{-2\sigma_1}/4\big)$, where $\chi_2(x) = (1 + x)\chi_1(x)$. In that region, we thus have

$$J_\epsilon := \Big\{W_\epsilon(1 + \epsilon R)\,,\, \phi_* - \frac{\bar{r}\dot{\bar{z}}_*}{2\Gamma}(1 + \epsilon R)^2\Big\} = \epsilon\chi'_2(\epsilon R)\exp\big(\epsilon^{-2\sigma_1}/4\big)\partial_Z\phi_*\,.$$

Since $\chi_2(x) = 1$ when $|x| \le \frac{1}{2}$, the quantity J_ϵ vanishes when $\rho := (R^2+Z^2)^{1/2} \le 1/(2\epsilon)$. In the region where $1/(2\epsilon) \le \rho \le \epsilon^{-\sigma_2}$, we know from (4.40) that $|\partial_Z\phi_*/(1+\epsilon R)^2| \le C\epsilon^{-3}\rho^{-4} \le C\epsilon$, and that $W_\epsilon \approx \exp\big(\epsilon^{-2\sigma_1}/4\big)$. Since χ'_2 is a bounded function, we deduce

$$\frac{1}{\delta}\int_{\Omega''_\epsilon} |J_\epsilon|\,\tilde{\zeta}^2 \,\mathrm{d}X = \frac{1}{\delta}\int_{\Omega''_\epsilon} \frac{|J_\epsilon|\,\tilde{\eta}^2}{(1+\epsilon R)^2} \,\mathrm{d}X \le \frac{C\epsilon^2}{\delta}\int_{\Omega''_\epsilon} W_\epsilon \tilde{\eta}^2 \,\mathrm{d}X\,. \tag{4.46}$$

Finally, in Ω'''_ϵ we have $W_\epsilon(1 + \epsilon R) = \chi_2(\epsilon R)\hat{W}_\epsilon$ where $\hat{W}_\epsilon = \exp(\rho^{2\gamma}/4)$, so that

$$J_\epsilon = \epsilon\chi'_2(\epsilon R)\hat{W}_\epsilon\,\partial_Z\phi_* + \frac{\epsilon\bar{r}\dot{\bar{z}}_*}{\Gamma}\chi_1(\epsilon R)(1 + \epsilon R)^2\partial_Z\hat{W}_\epsilon + \chi_2(\epsilon R)\big\{\hat{W}_\epsilon\,,\, \phi_*\big\}\,.$$

The first term in the right-hand side is estimated as above, with the difference that we now have the improved bound $|\partial_Z \phi_*/(1{+}\epsilon R)^2| \le C\epsilon^{-3}\rho^{-4} \le C\epsilon^{4\sigma_2-3}$. For the second one we observe that

$$\big|\partial_R \hat{W}_\epsilon\big| + \big|\partial_Z \hat{W}_\epsilon\big| \;\le\; \gamma\,\rho^{2\gamma-1}\,\hat{W}_\epsilon \;\le\; \gamma\,\epsilon^{\sigma_2-2\sigma_1}\,\hat{W}_\epsilon\,, \quad \text{since } \rho \ge \epsilon^{-\sigma_2}\,, \tag{4.47}$$

and the last term is estimated using (4.47) and the first bound in (4.40). Altogether we find

$$\frac{1}{\delta}\int_{\Omega_\epsilon'''} |J_\epsilon|\,\tilde{\zeta}^2\,\mathrm{d}X \;\le\; \frac{C}{\delta}\int_{\Omega_\epsilon'''}\Big(\frac{1}{\epsilon^2\rho^4} + \frac{\bar{r}|\dot{\bar{z}}_*|}{\Gamma}\frac{\epsilon}{\rho^{1-2\gamma}} + \frac{1}{\epsilon^2\rho^{4-2\gamma}}\Big)W_\epsilon\tilde{\eta}^2\,\mathrm{d}X \;\le\; C\epsilon^{\gamma_1}\,\|\tilde{\eta}\|_{\mathcal{X}_\epsilon}^2\,, \tag{4.48}$$

provided $0 < \gamma_1 < \sigma_2 + 1 - 2\sigma_1 - 2/(1{-}\sigma)$. Since $\sigma_2 > 1$, such a choice is again possible if $\sigma > 0$ and $\sigma_1 > 0$ are small enough. Combining (4.45), (4.46), (4.48), we arrive at (4.42). □

Lemma 4.10. *There exist $\gamma_1 > 0$ and $C > 0$ such that*

$$|I_2| \;\le\; C\epsilon^{\gamma_1}\,\|\tilde{\eta}\|_{\mathcal{X}_\epsilon}^2\,. \tag{4.49}$$

Proof. In Ω_ϵ' we have $W_\epsilon(1{+}\epsilon R) = \Phi_\epsilon'(\zeta_*)$ by (4.43), hence $W_\epsilon(1{+}\epsilon R)\big\{\tilde{\phi}\,,\,\zeta_*\big\} = \big\{\tilde{\phi}\,,\,\Phi_\epsilon(\zeta_*)\big\}$. Using the second relation in (4.43), we deduce that

$$\Big\{\tilde{\phi}\,,\,\phi_* - \frac{\bar{r}\dot{\bar{z}}_*}{2\Gamma}\,(1+\epsilon R)^2\Big\} - W_\epsilon(1+\epsilon R)\big\{\tilde{\phi}\,,\,\zeta_*\big\} \;=\; \big\{\tilde{\phi}\,,\,\Theta\big\}\,. \tag{4.50}$$

The first-order derivatives of Θ are estimated in Proposition 3.16. Proceeding as in the previous lemma, we thus obtain

$$\frac{1}{\delta}\int_{\Omega_\epsilon'}\big|\{\tilde{\phi}\,,\,\Theta\}\big|\,|\tilde{\zeta}|\,\mathrm{d}X \;\le\; C\Big(\epsilon + \frac{\epsilon^{\gamma_3}}{\delta}\Big)\,\epsilon^{-N\sigma_1}\int_{\Omega_\epsilon'}\frac{|\nabla\tilde{\phi}|}{1{+}\epsilon R}\,|\tilde{\eta}|\,\mathrm{d}X \;\le\; C\epsilon^{\gamma_1}\,\|\tilde{\eta}\|_{\mathcal{X}_\epsilon}^2\,, \tag{4.51}$$

where $0 < \gamma_1 < \gamma_3 - 2/(1{-}\sigma) - N\sigma_1$. In the last step, we used Hölder's inequality with exponents 3 and 3/2, and we invoked Lemma 4.7 to control the L^3 norm of $\nabla\tilde{\phi}/(1+\epsilon R)$.

In $\mathcal{D}_\epsilon := \Omega_\epsilon \setminus \Omega_\epsilon'$, we consider both terms in the left-hand side of (4.50) separately. The contribution of the first one to I_2 is estimated by

$$\frac{1}{\delta}\int_{\mathcal{D}_\epsilon}\frac{|\nabla\tilde{\phi}|\,|\nabla\phi_*|}{1+\epsilon R}|\,\tilde{\eta}|\,\mathrm{d}X + \frac{\epsilon\bar{r}|\dot{\bar{z}}_*|}{\delta\Gamma}\int_{\mathcal{D}_\epsilon}|\partial_Z\tilde{\phi}|\,|\tilde{\eta}|\,\mathrm{d}X \;=\; \mathcal{O}\big(\epsilon^\infty\|\tilde{\eta}\|_{\mathcal{X}_\epsilon}^2\big)\,, \tag{4.52}$$

because $|\nabla\phi_*| \le C$ by (4.40), $\|\nabla\tilde{\phi}/(1{+}\epsilon R)\|_{L^3} \le C\|\tilde{\eta}\|_{\mathcal{X}_\epsilon}$ by Lemma 4.7, and

$$\|\tilde{\eta}\|_{L^{3/2}(\mathcal{D}_\epsilon)} \;\le\; \Big(\int_{\mathcal{D}_\epsilon} W_\epsilon\tilde{\eta}^2\,\mathrm{d}X\Big)^{1/2}\Big(\int_{\mathcal{D}_\epsilon} W_\epsilon^{-3}\,\mathrm{d}X\Big)^{1/6} \;=\; \mathcal{O}\big(\epsilon^\infty\|\tilde{\eta}\|_{\mathcal{X}_\epsilon}\big)\,.$$

The second term in the left-hand side of (4.50) is nonzero only if $\rho \le 2\epsilon^{-\sigma_0}$, in view of (2.28). In that region, we know that $W_\epsilon|\nabla\zeta_*| \le C(1+\rho)^N$ for some integer N, because W_ϵ satisfies the upper bound in (4.17) and η_* belongs to the space $\mathcal{Z}$ defined in (3.23). The contribution of that term to I_2 can therefore be estimated in the same way as above:

$$\frac{1}{\delta}\int_{\mathcal{D}_\epsilon} W_\epsilon\,|\{\tilde{\phi},\zeta_*\}|\,|\tilde{\zeta}|\,\mathrm{d}X \;\le\; \frac{C}{\delta}\int_{\mathcal{D}_\epsilon}\frac{|\nabla\tilde{\phi}|\,|\tilde{\eta}|}{1+\epsilon R}\,(1+\rho)^N\,\mathrm{d}X \;=\; \mathcal{O}\big(\epsilon^\infty\|\tilde{\eta}\|_{\mathcal{X}_\epsilon}^2\big)\,. \tag{4.53}$$

Combining (4.51), (4.52), (4.53), we obtain (4.49). □

4.7 Control of the diffusive terms

Our next task is to estimate the diffusive terms collected in (4.37). To formulate the result, we introduce the continuous function $\rho_\gamma : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\rho_\gamma(R,Z,\epsilon) \,=\, \begin{cases} \rho & \text{if} \quad \rho \le \epsilon^{-\sigma_1}\,, \\ \epsilon^{-\sigma_1} & \text{if} \quad \epsilon^{-\sigma_1} < \rho < \epsilon^{-\sigma_2}\,, \\ \rho^\gamma & \text{if} \quad \rho \ge \epsilon^{-\sigma_2}\,, \end{cases} \tag{4.54}$$

where as usual $\rho = (R^2+Z^2)^{1/2}$. Our goal in this section is:

Proposition 4.11. *There exist $\kappa > 0$ and $C > 0$ such that*

$$\hat I_4 \,\le\, -\kappa \int_{\Omega_\epsilon} W_\epsilon\Bigl(|\nabla\tilde\eta|^2 + \rho_\gamma^2\tilde\eta^2 + \tilde\eta^2\Bigr)\,\mathrm{d}X + C\bigl(\mu_0^2+\mu_1^2+\mu_2^2\bigr)\,, \tag{4.55}$$

where μ_0, μ_1, μ_2 are defined in (4.3), (4.8).

The proof of Proposition 4.11 requires several steps. We first control the term in $\hat I_4$ that involves the time derivative of the weight function W_ϵ.

Lemma 4.12. *There exist $C > 0$ and $\gamma_1 > 0$ such that*

$$\int_{\Omega_\epsilon} t(\partial_t W_\epsilon)\tilde\eta^2\,\mathrm{d}X \,\le\, -\frac{\sigma_1}{5}\int_{\Omega''_\epsilon} W_\epsilon\rho_\gamma^2\tilde\eta^2\,\mathrm{d}X + C\int_{\Omega'''_\epsilon} W_\epsilon\tilde\eta^2\,\mathrm{d}X + C\epsilon^{\gamma_1}\|\tilde\eta\|_{\mathcal{X}_\epsilon}^2\,. \tag{4.56}$$

Proof. Following (4.16) we decompose $W_\epsilon(R,Z) = \chi_1(\epsilon R)\,\hat W_\epsilon(R,Z)$, so that

$$t\partial_t W_\epsilon \,=\, \chi_1(\epsilon R)\,t\partial_t\hat W_\epsilon(R,Z) + t\dot\epsilon R\chi_1'(\epsilon R)\,\hat W_\epsilon(R,Z)\,. \tag{4.57}$$

We first estimate the right-hand side in the region Ω'_ϵ defined by (4.12), where $\hat W_\epsilon = \Phi'_\epsilon(\zeta_*)$. As $\Phi_\epsilon = \Phi_0 + \epsilon^2\Phi_2$ according to (3.82), we have $t\partial_t\hat W_\epsilon = \Phi''_\epsilon(\zeta_*)\,t\partial_t\zeta_* + 2t\epsilon\dot\epsilon\,\Phi'_2(\zeta_*)$ in that region. We recall that $2t\dot\epsilon = \epsilon(1+\mathcal{O}(\epsilon^2))$ by (3.34), and that the functions Φ_0, Φ_2 satisfy the estimates (3.95), (3.97). It follows immediately that $|t\epsilon\dot\epsilon\,\Phi'_2(\zeta_*)| \le C\epsilon^{2-N\sigma_1}\hat W_\epsilon \le C\epsilon W_\epsilon$. Moreover, since $\zeta_* = \eta_*/(1+\epsilon R)$ with $\eta_* = \eta_{\mathrm{app}}$ in Ω'_ϵ, we also have $|\Phi''_\epsilon(\zeta_*)t\partial_t\zeta_*| \le C\epsilon(1+\rho)^N\hat W_\epsilon \le C\epsilon^{\gamma_1}W_\epsilon$, provided $0 < \gamma_1 < 1 - N\sigma_1$. Finally, the last term in (4.57) is bounded by $C\epsilon\rho W_\epsilon \le C\epsilon^{1-\sigma_1}W_\epsilon$. Altogether we have shown that $|t\partial_t W_\epsilon| \le C\epsilon^{\gamma_1}W_\epsilon$ in Ω'_ϵ.

In the intermediate region Ω''_ϵ we have $\hat W_\epsilon = \exp\bigl(\epsilon^{-2\sigma_1}/4\bigr)$ and $\rho_\gamma = \epsilon^{-\sigma_1}$, so that

$$t\partial_t\hat W_\epsilon \,=\, -\frac{\sigma_1}{2}\,\exp\bigl(\epsilon^{-2\sigma_1}/4\bigr)\,\frac{t\dot\epsilon}{\epsilon^{2\sigma_1+1}} \,=\, -\frac{\sigma_1}{2}\,\hat W_\epsilon\,\rho_\gamma^2\,\frac{t\dot\epsilon}{\epsilon} \,\approx\, -\frac{\sigma_1}{4}\,\hat W_\epsilon\,\rho_\gamma^2\,.$$

Since $|t\dot\epsilon R\chi_1'(\epsilon R)| \le |\epsilon R\chi_1'(\epsilon R)| \le C$, it follows that $t\partial_t W_\epsilon \le -(\sigma_1/5)W_\epsilon\rho_\gamma^2$ in Ω''_ϵ. Finally, in the exterior region Ω'''_ϵ, the function $\hat W_\epsilon = \exp(\rho^{2\gamma}/4)$ does not depend on time, and we deduce from (4.57) that $|t\partial_t W_\epsilon| \le CW_\epsilon$. Collecting all these estimates, we arrive at (4.56). $\square$

We next consider the term involving $\tilde\zeta$ in (4.37).

Lemma 4.13. *There exist $C > 0$ and $\gamma_1 > 0$ such that*

$$-\frac{\epsilon}{2}\int_{\Omega_\epsilon} \partial_R\bigl(W_\epsilon(1+\epsilon R)\bigr)\tilde\zeta^2\,\mathrm{d}X \,\le\, -\frac{\epsilon^2}{4}\int_{\Omega_\epsilon} W_\epsilon\tilde\zeta^2\,\mathrm{d}X + C\epsilon^{\gamma_1}\|\tilde\eta\|_{\mathcal{X}_\epsilon}^2\,. \tag{4.58}$$

Proof. If $\mathcal{D}_\epsilon$ denotes any of the three regions defined in (4.12), we have

$$-\frac{\epsilon}{2}\int_{\mathcal{D}_\epsilon}\partial_R\big(W_\epsilon(1+\epsilon R)\big)\tilde{\zeta}^2\,\mathrm{d}X \;=\; -\frac{\epsilon^2}{2}\int_{\mathcal{D}_\epsilon}W_\epsilon\tilde{\zeta}^2\,\mathrm{d}X-\frac{\epsilon}{2}\int_{\mathcal{D}_\epsilon}\big(\partial_R W_\epsilon\big)\tilde{\zeta}\tilde{\eta}\,\mathrm{d}X \tag{4.59}$$

$$\le\; -\frac{\epsilon^2}{4}\int_{\mathcal{D}_\epsilon}W_\epsilon\tilde{\zeta}^2\,\mathrm{d}X+\frac{1}{4}\int_{\mathcal{D}_\epsilon}\frac{(\partial_R W_\epsilon)^2}{W_\epsilon}\,\tilde{\eta}^2\,\mathrm{d}X\,, \tag{4.60}$$

where in the second line we used Young's inequality. In the inner region Ω'_ϵ we observe that $\tilde{\zeta}\approx\tilde{\eta}$, because $|\epsilon R|\le 2\epsilon^{1-\sigma_1}\ll 1$. Moreover we have $\epsilon|\partial_R W_\epsilon|\le C\epsilon^{\gamma_1}W_\epsilon$ for some $\gamma_1>0$, so taking $\mathcal{D}_\epsilon=\Omega'_\epsilon$ and using (4.59) we obtain the analogue of (4.58) in that region. Outside Ω'_ϵ, we cannot directly compare $\tilde{\zeta}$ and $\tilde{\eta}$, so we prefer using inequality (4.60). In the intermediate region Ω''_ϵ, we have $|\partial_R W_\epsilon|\le C\epsilon W_\epsilon$ by (4.16), and (4.58) easily follows. Finally, in the exterior region Ω'''_ϵ, we observe that

$$\partial_R W_\epsilon \;=\; \left(\frac{\epsilon\chi_1'(\epsilon R)}{\chi_1(\epsilon R)}+\frac{\gamma R}{2}\,\rho^{2\gamma-2}\right)W_\epsilon\,.$$

Taking σ_1 small enough so that $\gamma\equiv\sigma_1/\sigma_2<1/2$, and using the fact that $\rho\ge\epsilon^{-\sigma_2}$ in Ω'''_ϵ, we deduce that $|\partial_R W_\epsilon|\le C\epsilon^{\gamma_1}W_\epsilon$ for some $\gamma_1>0$, and this leads to (4.58). The proof is thus complete. ☐

To conclude the proof of Proposition 4.11, we consider the quadratic form given by the first line of (4.37), namely

$$Q_\epsilon[\eta] \;=\; \int_{\Omega_\epsilon}W_\epsilon|\nabla\eta|^2\,\mathrm{d}X+\int_{\Omega_\epsilon}(\nabla W_\epsilon\cdot\nabla\eta)\eta\,\mathrm{d}X+\int_{\Omega_\epsilon}V_\epsilon\eta^2\,\mathrm{d}X\,, \tag{4.61}$$

where V_ϵ is defined in (4.33). Taking formally the limit $\epsilon\to 0$ in (4.61), we obtain using (4.18)

$$Q_0[\eta] \;=\; \int_{\mathbb{R}^2}A|\nabla\eta|^2\,\mathrm{d}X+\int_{\mathbb{R}^2}(\nabla A\cdot\nabla\eta)\eta\,\mathrm{d}X+\int_{\mathbb{R}^2}V\eta^2\,\mathrm{d}X\,, \tag{4.62}$$

where A is defined by (3.85) and $V=\frac{1}{4}(R\partial_R+Z\partial_Z)A-\frac{1}{2}A-1$. The limiting quadratic form (4.62) is carefully studied in our previous work [34], and we have the following result:

Proposition 4.14. *There exists constants $C_8>2$ and $C_9>0$ such that, for all $\eta\in\mathcal{X}_0$ with $\rho\eta\in\mathcal{X}_0$ and $\nabla\eta\in\mathcal{X}_0^2$, we have*

$$\|\nabla\eta\|_{\mathcal{X}_0}^2+\|\rho\eta\|_{\mathcal{X}_0}^2+\|\eta\|_{\mathcal{X}_0}^2 \;\le\; C_8Q_0[\eta]+C_9\big(\mu_0^2+\mu_1^2+\mu_2^2\big)\,, \tag{4.63}$$

where $\mu_0=\int_{\mathbb{R}^2}\eta\,\mathrm{d}X$, $\mu_1=\int_{\mathbb{R}^2}R\eta\,\mathrm{d}X$, $\mu_2=\int_{\mathbb{R}^2}Z\eta\,\mathrm{d}X$.

Proof. In [34, Theorem 4.2] we prove that there exists $\delta_0>0$ such that $Q_0[\eta]\ge\delta_0\|\eta\|_{\mathcal{X}_0}^2$ for any $\eta\in\mathcal{X}_0$ such that $\mu_0=\mu_1=\mu_2=0$. On the other hand, if we apply Young's inequality to the middle term in the right-hand side of (4.62), we obtain the lower bound

$$Q_0[\eta] \;\ge\; \frac{1}{4}\int_{\mathbb{R}^2}A|\nabla\eta|^2\,\mathrm{d}X+\int_{\mathbb{R}^2}\Big(V-\frac{|\nabla A|^2}{3A}\Big)\eta^2\,\mathrm{d}X \;\ge\; \frac{1}{4}\|\nabla\eta\|_{\mathcal{X}_0}^2+\frac{1}{24}\|\rho\eta\|_{\mathcal{X}_0}^2-C\|\eta\|_{\mathcal{X}_0}^2\,,$$

because a direct calculation reveals that $V/A-|\nabla A|^2/(3A^2)\ge\rho^2/(24)-C$ for some constant $C>0$. Taking a convex combination of both estimates, we see that there exists $C_8>0$ such that

$$\|\nabla\eta\|_{\mathcal{X}_0}^2+\|\rho\eta\|_{\mathcal{X}_0}^2+\|\eta\|_{\mathcal{X}_0}^2 \;\le\; C_8Q_0[\eta]\,, \tag{4.64}$$

whenever $\eta\in\mathcal{X}_0$ satisfies $\mu_0=\mu_1=\mu_2=0$. It remains to deduce (4.63) from (4.64), which is easily done using exactly the same arguments as in the proof of Proposition 4.5. ☐

The analogue of Proposition 4.14 for the full quadratic form (4.61) is the following statement, whose proof is postponed to Section B.3.

Proposition 4.15. *There exists constants $C_{10} > 2$ and $C_{11} > 0$ such that, for all sufficiently small $\epsilon > 0$ and all $\eta \in \mathcal{X}_\epsilon$ with $\rho_\gamma \eta \in \mathcal{X}_\epsilon$ and $\nabla\eta \in \mathcal{X}_\epsilon^2$, we have*

$$\|\nabla\eta\|_{\mathcal{X}_\epsilon}^2 + \|\eta\|_{\mathcal{X}_\epsilon}^2 + \int_{\Omega'_\epsilon \cup \Omega'''_\epsilon} W_\epsilon \rho_\gamma^2 \eta^2 \,\mathrm{d}X \;\le\; C_{10} Q_\epsilon[\eta] + C_{11}\Big(\mu^2 + \int_{\Omega''_\epsilon} W_\epsilon \eta^2 \,\mathrm{d}X\Big)\,, \tag{4.65}$$

where $\mu^2 = \mu_0^2 + \mu_1^2 + \mu_2^2$ and $\mu_0 = \int_{\Omega_\epsilon} \eta \,\mathrm{d}X$, $\mu_1 = \int_{\Omega_\epsilon} R\eta \,\mathrm{d}X$, $\mu_2 = \int_{\Omega_\epsilon} Z\eta \,\mathrm{d}X$.

End of the proof of Proposition 4.11. In view of (4.37) and (4.61) we have

$$\hat{I}_4 \;=\; -Q_\epsilon[\tilde\eta] - \frac{\epsilon}{2}\int_{\Omega_\epsilon} \partial_R\big(W_\epsilon(1+\epsilon R)\big)\tilde\zeta^2 \,\mathrm{d}X + \frac{1}{2}\int_{\Omega_\epsilon} t(\partial_t W_\epsilon)\tilde\eta^2 \,\mathrm{d}X\,.$$

The three terms in the right-hand side are estimated using (4.65), (4.58), and (4.56), respectively. Taking $\epsilon > 0$ sufficiently small and recalling that $\rho_\gamma \ge \epsilon^{-\sigma_1} \gg 1$ outside the inner region Ω'_ϵ, we arrive at (4.55). The slight discrepancy between the definitions of μ_1 in (4.3) and in Proposition 4.15 is completely harmless. □

4.8 Control of the remaining terms

In this section, we estimate the remaining terms I_3, I_5, and $\hat{I}_6$ defined in (4.31), (4.34), and (4.38), respectively.

Control of I_3. We deduce from (4.31) that

$$|I_3| \;\le\; \int_{\Omega_\epsilon} \frac{|\nabla\tilde\phi|}{1+\epsilon R}\,|\tilde\eta|\big|\nabla(W_\epsilon\tilde\eta)\big| \,\mathrm{d}X \;\le\; \int_{\Omega_\epsilon} \frac{|\nabla\tilde\phi|}{1+\epsilon R}\,|\tilde\eta|\Big(|\tilde\eta||\nabla W_\epsilon| + W_\epsilon|\nabla\tilde\eta|\Big) \,\mathrm{d}X\,. \tag{4.66}$$

To estimate the right-hand side, we use (4.39) and [35, Lemma 2.1] to obtain the uniform bound

$$\Big\|\frac{|\nabla\tilde\phi|}{1+\epsilon R}\Big\|_{L^\infty} \;\le\; C\|\tilde\eta\|_{L^{4/3}}^{1/2}\|\tilde\eta\|_{L^4}^{1/2} \;\le\; C\|\tilde\eta\|_{\mathcal{X}_\epsilon}^{1/2}\big(\|\tilde\eta\|_{\mathcal{X}_\epsilon}^{1/2} + \|\nabla\tilde\eta\|_{\mathcal{X}_\epsilon}^{1/2}\big)\,.$$

On the other hand it is easy to verify that $|\nabla W_\epsilon| \le C(1+\rho_\gamma)W_\epsilon$ where ρ_γ is defined in (4.54). It follows that

$$|I_3| \;\le\; C\|\tilde\eta\|_{\mathcal{X}_\epsilon}^{3/2}\big(\|\tilde\eta\|_{\mathcal{X}_\epsilon}^{1/2} + \|\nabla\tilde\eta\|_{\mathcal{X}_\epsilon}^{1/2}\big)\big(\|\tilde\eta\|_{\mathcal{X}_\epsilon} + \|\rho_\gamma\tilde\eta\|_{\mathcal{X}_\epsilon} + \|\nabla\tilde\eta\|_{\mathcal{X}_\epsilon}\big) \;\le\; C\|\tilde\eta\|_{\mathcal{X}_\epsilon} D_\epsilon[\tilde\eta]\,, \tag{4.67}$$

where for convenience we denote

$$D_\epsilon[\tilde\eta] \;=\; \|\nabla\tilde\eta\|_{\mathcal{X}_\epsilon}^2 + \|\rho_\gamma\tilde\eta\|_{\mathcal{X}_\epsilon}^2 + \|\tilde\eta\|_{\mathcal{X}_\epsilon}^2\,. \tag{4.68}$$

Control of I_5. Proposition 2.5 asserts that the remainder $\mathrm{Rem}(R,Z,t)$ satisfies the pointwise estimate (2.30), which implies in particular that $\mathrm{Rem} \in \mathcal{X}_\epsilon$. In view of (4.34), we thus find

$$|I_5| \;\le\; \frac{1}{\delta}\,\|\mathrm{Rem}\|_{\mathcal{X}_\epsilon}\Big(\|\tilde\eta\|_{\mathcal{X}_\epsilon} + \|W_\epsilon^{-1}\tilde\phi\|_{\mathcal{X}_\epsilon}\Big) \;\le\; C\Big(\epsilon + \frac{\epsilon^{\gamma_5}}{\delta^2}\Big)\Big(\|\tilde\eta\|_{\mathcal{X}_\epsilon} + \|W_\epsilon^{-1}\tilde\phi\|_{\mathcal{X}_\epsilon}\Big)\,.$$

It remains to estimate the norm of $W_\epsilon^{-1}\tilde\phi$ in the space $\mathcal{X}_\epsilon$. This can be done by decomposing the Biot-Savart kernel as in the proof of Lemma 4.4, see in particular Eq. (B.2) below. Neglecting contributions of order $\mathcal{O}(\epsilon^\infty)$, we can restrict the integrals to the region where $R^2 + Z^2 \le \epsilon^{-2\sigma_1}$

and $R'^2 + Z'^2 \le \epsilon^{-2\sigma_1}$. Invoking (B.3) and recalling that $\mu_0(t) = \mathcal{O}(\epsilon^\infty)$ by Lemma 4.1, we find that $\|W_\epsilon^{-1}\tilde\phi\|_{\mathcal{X}_\epsilon} = \|W_\epsilon^{-1/2}\tilde\phi\|_{L^2(\Omega_\epsilon)} \le C\|\tilde\eta\|_{\mathcal{X}_\epsilon}$. We conclude that

$$|I_5| \,\le\, C\Big(\epsilon + \frac{\epsilon^{\gamma_5}}{\delta^2}\Big)\,\|\tilde\eta\|_{\mathcal{X}_\epsilon}\,. \tag{4.69}$$

Control of $\hat I_6$. The first two terms in (4.38) are easily estimated, because $\dot{\bar r} = \mathcal{O}(\delta)$ by (3.70). Proceeding as in Lemma 4.4 to control the kinetic energy, and recalling that $\mu_0(t) = \mathcal{O}(\epsilon^\infty)$, we find

$$\mathcal{I}_0 \,:=\, \biggl|\frac{\epsilon\bar r\dot{\bar r}}{\delta\Gamma}\int_{\Omega_\epsilon} W_\epsilon\tilde\eta\partial_R\tilde\eta\,\mathrm{d}X + \frac{t\dot{\bar r}}{\bar r}\,E_\epsilon^{\mathrm{kin}}[\tilde\eta]\biggr| \,\le\, C\epsilon\|\tilde\eta\|_{\mathcal{X}_\epsilon}\|\nabla\tilde\eta\|_{\mathcal{X}_\epsilon} + C\epsilon^2\|\tilde\eta\|_{\mathcal{X}_\epsilon}^2\,.$$

So it remains to estimate the last term in (4.38), which involves the correction $\dot{\tilde z}(t)$ to the vertical speed introduced in (4.9). Using (2.19) and integrating by parts we first observe that

$$\begin{aligned}\mathcal{I}_1 \,:=\, \int_{\Omega_\epsilon}\bigl(W_\epsilon\tilde\eta - \tilde\phi\bigr)\partial_Z\eta_*\,\mathrm{d}X \,&=\, \int_{\Omega_\epsilon}\bigl(W_\epsilon\partial_Z\eta_* - \partial_Z\phi_*\bigr)\tilde\eta\,\mathrm{d}X \\ &=\, -\int_{\Omega'_\epsilon}(\partial_Z\Theta)\tilde\eta\,\mathrm{d}X + \int_{\Omega''_\epsilon\cup\Omega'''_\epsilon}\bigl(W_\epsilon\partial_Z\eta_* - \partial_Z\phi_*\bigr)\tilde\eta\,\mathrm{d}X\,,\end{aligned} \tag{4.70}$$

where Θ is defined in (3.88). In the second line, we used the expression (4.16) of W_ϵ in the inner region Ω'_ϵ to obtain the identity $W_\epsilon\partial_Z\eta_* - \partial_Z\phi_* \,=\, \Phi'_\epsilon(\zeta_*)\partial_Z\zeta_* - \partial_Z\phi_* \,=\, -\partial_Z\Theta$. The last integral in (4.70) is of order $\mathcal{O}(\epsilon^\infty\|\tilde\eta\|_{\mathcal{X}_\epsilon})$, and the integral over Ω'_ϵ can be controlled using Proposition 3.16. We thus obtain $|\mathcal{I}_1| \le C(\epsilon\delta + \epsilon^{\gamma_3})\|\tilde\eta\|_{\mathcal{X}_\epsilon}$. Moreover, we obviously have

$$\mathcal{I}_2 \,:=\, \biggl|\int_{\Omega_\epsilon}\bigl(W_\epsilon\tilde\eta - \tilde\phi\bigr)\partial_Z\tilde\eta\,\mathrm{d}X\biggr| \,\le\, C\|\tilde\eta\|_{\mathcal{X}_\epsilon}\|\nabla\tilde\eta\|_{\mathcal{X}_\epsilon}\,.$$

Finally, to control the velocity $\dot{\tilde z}(t)$, we need the following lemma:

Lemma 4.16. *Let $J(t) = \int_{\Omega_\epsilon} Z\mathcal{R}(R,Z,t)\,\mathrm{d}X$ where $\mathcal{R}$ is defined in* (4.10)*. Then there exists a constant $C > 0$ such that*

$$|J| \,\le\, \frac{C\epsilon\beta_\epsilon}{\delta}\Bigl(\|\tilde\eta\|_{\mathcal{X}_\epsilon} + \delta\|\tilde\eta\|_{\mathcal{X}_\epsilon}^2\Bigr) + C\Big(\epsilon + \frac{\epsilon^{\gamma_5}}{\delta^2}\Big)\,. \tag{4.71}$$

Proof. We consider separately the various terms in the right-hand side of (4.10). Integrating by parts, we find

$$J_1 \,:=\, \frac{1}{\delta}\int_{\Omega_\epsilon} Z\Bigl(\bigl\{\phi_*\,,\tilde\zeta\bigr\} + \bigl\{\tilde\phi\,,\zeta_*\bigr\}\Bigr)\,\mathrm{d}X \,=\, -\frac{1}{\delta}\int_{\Omega_\epsilon}\biggl(\frac{\tilde\eta\partial_R\phi_*}{1{+}\epsilon R} + \frac{\eta_*\partial_R\tilde\phi}{1{+}\epsilon R}\biggr)\,\mathrm{d}X\,.$$

In the right-hand side, we can restrict the integration to the region where $\rho \le \epsilon^{-\sigma_1}$, because the integral on the complement is of order $\mathcal{O}\bigl(\epsilon^\infty\|\tilde\eta\|_{\mathcal{X}_\epsilon}\bigr)$. Thus, expanding the Biot-Savart formula as in Section 3.1, we obtain

$$-\delta J_1 \,=\, \frac{1}{2\pi}\int_{\mathbb{R}^2}\Bigl(\tilde\eta\partial_R(L\eta_*) + \eta_*\partial_R(L\tilde\eta)\Bigr)\,\mathrm{d}X + \mathcal{O}\bigl(\epsilon\beta_\epsilon\|\tilde\eta\|_{\mathcal{X}_\epsilon}\bigr)\,, \tag{4.72}$$

where L is the convolution operator (3.14). Since L is symmetric in $L^2(\mathbb{R}^2)$ and commutes with ∂_R, the integral in (4.72) vanishes and we conclude that $|J_1| \lesssim \delta^{-1}\epsilon\beta_\epsilon\|\tilde\eta\|_{\mathcal{X}_\epsilon}$.

Similarly, we have

$$J_2 \,:=\, \int_{\Omega_\epsilon} Z\{\tilde\phi\,,\tilde\zeta\}\,\mathrm{d}X \,=\, \int_{\Omega_\epsilon}\{Z\,,\tilde\phi\}\tilde\zeta\,\mathrm{d}X \,=\, -\int_{\Omega_\epsilon}\frac{\tilde\eta\partial_R\tilde\phi}{1{+}\epsilon R}\,\mathrm{d}X\,.$$

Here again, up to a negligible error, we can assume that $\tilde\eta$ is supported in the ball $\rho \le \epsilon^{-\sigma_1}$. Proceeding as before, we thus find

$$J_2 \,=\, -\frac{1}{2\pi}\int_{\mathbb{R}^2} \tilde\eta\partial_R(L\tilde\eta)\,\mathrm{d}X + \mathcal{O}\big(\epsilon\beta_\epsilon\|\tilde\eta\|^2_{\mathcal{X}_\epsilon}\big) \,=\, \mathcal{O}\big(\epsilon\beta_\epsilon\|\tilde\eta\|^2_{\mathcal{X}_\epsilon}\big)\,. \tag{4.73}$$

The remaining terms in (4.10) are easier to treat. In view of (4.8) we have

$$\int_{\Omega_\epsilon} Z\Big(\mathcal{L}\tilde\eta + \epsilon\partial_R\tilde\zeta\Big)\,\mathrm{d}x \,=\, 0\,, \qquad \text{and} \qquad \int_{\Omega_\epsilon} Z\Big(\dot{\bar r}\,\partial_R\tilde\eta + \dot{\tilde z}_*\,\partial_Z\tilde\eta\Big)\,\mathrm{d}X \,=\, -\dot{\tilde z}_*\mu_0\,,$$

where $\mu_0(t) = \mathcal{O}(\epsilon^\infty)$ by Lemma 4.1. Finally, using estimate (2.30), we obtain

$$\frac{1}{\delta}\int_{\Omega_\epsilon} |Z|\,|\mathrm{Rem}(R,Z,t)|\,\mathrm{d}X \,\le\, C\Big(\epsilon + \frac{\epsilon^{\gamma_5}}{\delta^2}\Big)\,. \tag{4.74}$$

Combining (4.72), (4.73), and (4.74), we arrive at (4.71). ☐

Corollary 4.17. *There exists a constant $C > 0$ such that the velocity $\dot{\tilde z}$ defined by* (4.9) *satisfies*

$$\frac{\bar r|\dot{\tilde z}|}{\Gamma} \,\le\, C\beta_\epsilon\Big(\|\tilde\eta\|_{\mathcal{X}_\epsilon} + \delta\|\tilde\eta\|^2_{\mathcal{X}_\epsilon}\Big) + C\Big(\delta + \frac{\epsilon^{\gamma_5-1}}{\delta}\Big)\,. \tag{4.75}$$

We now conclude the estimate of the term $\hat I_6$. To simplify the writing, we assume that $\|\tilde\eta\|_{\mathcal{X}_\epsilon} \le 1$ and we use the shorthand notation (4.68). Also, since $\epsilon^2 \lesssim \delta^{1-\sigma}$ we observe that

$$\epsilon + \frac{\epsilon^{\gamma_5}}{\delta^2} \,\lesssim\, \mathfrak{R}_\epsilon(t)\,, \qquad \text{where} \quad \mathfrak{R}_\epsilon(t) \,:=\, \epsilon + \frac{\epsilon^{\gamma_3}}{\delta}\,. \tag{4.76}$$

Here $\gamma_3 = \gamma_5 - 2/(1-\sigma) < 3$, so that γ_3 can be chosen arbitrary close to $\gamma_5 - 2$ if $\sigma > 0$ is small enough. In view of (4.9) and (4.38) we have $|\hat I_6| \le \mathcal{I}_0 + |J|\big(|\mathcal{I}_1| + \delta\mathcal{I}_2\big)$, so that

$$\begin{aligned} |\hat I_6| \,&\le\, C\epsilon\|\tilde\eta\|_{\mathcal{X}_\epsilon}D_\epsilon^{1/2} + C\Big(\frac{\epsilon\beta_\epsilon}{\delta}\,\|\tilde\eta\|_{\mathcal{X}_\epsilon} + \mathfrak{R}_\epsilon\Big)\Big(\delta\mathfrak{R}_\epsilon\|\tilde\eta\|_{\mathcal{X}_\epsilon} + \delta\|\tilde\eta\|_{\mathcal{X}_\epsilon}D_\epsilon^{1/2}\Big) \\ &\le\, C\|\tilde\eta\|_{\mathcal{X}_\epsilon}\big(D_\epsilon^{1/2} + \mathfrak{R}_\epsilon\big)\big(\epsilon\beta_\epsilon + \delta\mathfrak{R}_\epsilon\big) \,\le\, C\epsilon\beta_\epsilon\|\tilde\eta\|_{\mathcal{X}_\epsilon}\big(D_\epsilon^{1/2} + \mathfrak{R}_\epsilon\big)\,. \end{aligned} \tag{4.77}$$

4.9 Conclusion of the proof

We are now in position to conclude the proof of Theorem 2.6, hence also of Theorem 1.1. Let $\tilde\eta$ be the unique solution of (2.33) with zero initial data. The associated energy (2.34) satisfies the evolution equation

$$t\partial_t E_\epsilon(t) \,=\, I_1 + I_2 + I_3 + \hat I_4 + I_5 + \hat I_6\,, \tag{4.78}$$

where the various terms in the right-hand side are defined in Section 4.4 and estimated in Sections 4.6–4.8. Using (4.42), (4.49), (4.67), (4.55), (4.69), and (4.77), we find that, as long as $t \le T_{\mathrm{adv}}\delta^{-\sigma}$ and $\|\tilde\eta\|_{\mathcal{X}_\epsilon} \le 1$, there exist positive constants C, C_*, κ such that

$$t\partial_t E_\epsilon(t) \,\le\, -\kappa D_\epsilon + C_*\|\tilde\eta\|_{\mathcal{X}_\epsilon}D_\epsilon + C\|\tilde\eta\|_{\mathcal{X}_\epsilon}\big(\mathfrak{R}_\epsilon + \epsilon\beta_\epsilon D_\epsilon^{1/2}\big) + \frac{C\epsilon^2}{\delta}\int_{\Omega''_\epsilon} W_\epsilon\tilde\eta^2\,\mathrm{d}X + C\mu^2\,,$$

where D_ϵ is defined in (4.68), $\mathfrak{R}_\epsilon$ in (4.76), and $\mu^2 := \mu_0^2 + \mu_1^2 + \mu_2^2 \le C\,\mathfrak{R}_\epsilon^2$ by Lemma 4.1. Since $\rho_\gamma \ge \epsilon^{-\sigma_1}$ in the region Ω''_ϵ, the integral term can be estimated as follows

$$\frac{\epsilon^2}{\delta}\int_{\Omega''_\epsilon} W_\epsilon\tilde\eta^2\,\mathrm{d}X \,\le\, \frac{\epsilon^{2+2\sigma_1}}{\delta}\int_{\Omega''_\epsilon} W_\epsilon\rho_\gamma^2\tilde\eta^2\,\mathrm{d}X \,\lesssim\, \epsilon^{\gamma_*}D_\epsilon\,,$$

where $\gamma_* = 2+2\sigma_1-2/(1-\sigma) > 0$ if $\sigma > 0$ is small enough. So, if we assume that $C_*\|\tilde\eta\|_{\mathcal{X}_\epsilon} \le \kappa/4$ and that ϵ is sufficiently small, we obtain by Young's inequality

$$t\partial_t E_\epsilon(t) \,\le\, -\frac{\kappa}{2}\,D_\epsilon + C\,\mathfrak{R}_\epsilon\|\tilde\eta\|_{\mathcal{X}_\epsilon} + C\mu^2 \,\le\, -\frac{\kappa}{4}\,D_\epsilon + C\,\mathfrak{R}_\epsilon^2\,.$$

Integrating that differential inequality over the time interval $(0,t)$ and recalling that $E_\epsilon(0) = 0$, we arrive at

$$E_\epsilon(t) + \frac{\kappa}{4}\int_0^t \frac{D_\epsilon(s)}{s}\,\mathrm{d}s \,\le\, C\int_0^t \frac{\mathfrak{R}_\epsilon(s)^2}{s}\,\mathrm{d}s \,\le\, C\,\mathfrak{R}_\epsilon(t)^2\,.$$

Finally, in view of (4.27), (4.8), and Lemma 4.1, we infer that

$$\|\tilde\eta(t)\|_{\mathcal{X}_\epsilon}^2 \,\le\, C_6 E_\epsilon(t) + C_7\big(\beta_\epsilon\mu_0(t)^2 + \mu_1(t)^2\big) \,\le\, C\,\mathfrak{R}_\epsilon(t)^2\,. \tag{4.79}$$

Inequality (4.79) holds as long as $\|\tilde\eta(t)\|_{\mathcal{X}_\epsilon} \le \min\big(1,\kappa/(4C_*)\big)$ and $t < T_{\mathrm{adv}}\delta^{-\sigma}$. But on that time interval we know that $\mathfrak{R}_\epsilon \lesssim \epsilon^{\gamma_3-2/(1-\sigma)} \ll 1$, so (4.79) is actually valid for all $t \in (0, T_{\mathrm{adv}}\delta^{-\sigma})$ if $\epsilon > 0$ is small enough. Returning to the solution of (2.15) with initial data (2.22), we obtain in view of (2.31), (4.79)

$$\|\eta(t) - \eta_*(t)\|_{\mathcal{X}_\epsilon} \,=\, \delta\|\tilde\eta(t)\|_{\mathcal{X}_\epsilon} \,\le\, C\delta\,\mathfrak{R}_\epsilon(t) \,=\, C\big(\epsilon\delta + \epsilon^{\gamma_3}\big)\,, \qquad t \in (0, T_{\mathrm{adv}}\delta^{-\sigma})\,,$$

which gives (2.40). This concludes the proof of Theorem 2.6. □

Remark 4.18. *The correction $\tilde z(t)$ to the vertical position of the vortex is small, and produces negligible effects in our calculations. Indeed, it follows from* (4.75) *and* (4.79) *that*

$$\frac{\bar r|\dot{\tilde z}(t)|}{\Gamma} \,\lesssim\, \big(\beta_\epsilon\,\mathfrak{R}_\epsilon + \delta\big)\,, \qquad \textit{hence} \qquad \delta|\tilde z(t)| \,\lesssim\, \epsilon^2\bar r(t)\big(\delta + \beta_\epsilon\,\mathfrak{R}_\epsilon\big)\,. \tag{4.80}$$

This gives in particular (2.41)*.*

Proof of Theorem 1.1. Let $\omega_{\mathrm{lin}}(r,z,t)$ be the solution of the (axisymmetric) heat equation in Ω with initial data $\Gamma\,\delta_{(r_0,z_0)}$. Using the same self-similar variables as in the proof of Theorem 2.6, we define the rescaled vorticity η_{lin} by the relation

$$\omega_{\mathrm{lin}}\big(r, z - a_3(t), t\big) \,=\, \frac{\Gamma}{\nu t}\,\eta_{\mathrm{lin}}\Big(\frac{r-\bar r(t)}{\sqrt{\nu t}}\,,\, \frac{z - \bar z_*(t) - \delta\tilde z(t)}{\sqrt{\nu t}}\,,\, t\Big)\,, \tag{4.81}$$

where $a_3(t) = \int_0^t V(s)\,\mathrm{d}s$ and V is given by (1.5). A direct calculation then shows that η_{lin} satisfies the linear equation

$$t\partial_t\eta_{\mathrm{lin}} - \frac{\epsilon\bar r}{\delta\Gamma}\Big(\dot{\bar r}\,\partial_R\eta_{\mathrm{lin}} + \dot s\,\partial_Z\eta_{\mathrm{lin}}\Big) \,=\, \mathcal{L}\eta_{\mathrm{lin}} + \partial_R\Big(\frac{\epsilon\eta_{\mathrm{lin}}}{1+\epsilon R}\Big)\,, \tag{4.82}$$

with initial data η_0, where the shift $s(t) = \bar z_*(t) - a_3(t) + \delta\tilde z(t)$ measures the difference between the vertical position of the vortex as computed in Theorem 2.6 and the approximation given by the Kelvin-Saffman formula (1.5) without correction terms. Since $\dot a_3 = \dot{\bar z}_0$, it follows from (3.62) that $\dot s = \epsilon^2\dot{\bar z}_2 + \delta\dot{\tilde z}$. Using (3.57) and (4.80), we thus obtain

$$\frac{\epsilon\bar r|\dot s(t)|}{\delta\Gamma} \,\le\, C\Big(\frac{\beta_\epsilon\epsilon^3}{\delta} + \beta_\epsilon\epsilon^2 + \epsilon\delta\Big) \,\le\, C\epsilon^{1-3\sigma}\,, \tag{4.83}$$

because $\epsilon^2 \lesssim \delta^{1-\sigma}$ so that $\beta_\epsilon\epsilon^3\delta^{-1} \le \epsilon^{1-3\sigma}$ if $0 < \sigma < 1/3$ and $\epsilon > 0$ is small enough.

The solution of (4.82) with initial data η_0 can be estimated as in [33, Section 4.4], with substantial simplifications. We use the approximate solution $\hat\eta_0(R,Z,t) := \chi_0(4\epsilon\rho)\eta_0(R,Z)$, where χ_0 is the cut-off function in (2.28). Decomposing $\eta_{\rm lin} = \hat\eta_0 + \hat\eta$, we see that the correction $\hat\eta$ satisfies

$$t\partial_t\hat\eta - \frac{\epsilon\bar r}{\delta\Gamma}\Bigl(\dot{\bar r}\,\partial_R\hat\eta + \dot s\,\partial_Z\hat\eta\Bigr) \,=\, \mathcal{L}\hat\eta + \partial_R\Bigl(\frac{\epsilon\hat\eta}{1+\epsilon R}\Bigr) + \mathcal{R}_0\,, \tag{4.84}$$

where

$$\mathcal{R}_0 \,=\, \mathcal{L}\hat\eta_0 + \partial_R\Bigl(\frac{\epsilon\hat\eta_0}{1+\epsilon R}\Bigr) + \frac{\epsilon\bar r}{\delta\Gamma}\Bigl(\dot{\bar r}\,\partial_R\hat\eta_0 + \dot s\,\partial_Z\hat\eta_0\Bigr) - t\partial_t\hat\eta_0\,.$$

To control the solution of (4.84), we introduce the space $\hat{\mathcal{X}}_\epsilon$ defined by the norm

$$\|\hat\eta\|_{\hat{\mathcal{X}}_\epsilon}^2 \,=\, \int_{\Omega_\epsilon} e^{(R^2+Z^2)/4}\,\hat\eta(R,Z)^2\,{\rm d}R\,{\rm d}Z\,.$$

In view of (4.83) we have $\|\mathcal{R}_0\|_{\hat{\mathcal{X}}_\epsilon} \le C\epsilon^{1-3\sigma}$, and using energy estimates as in [33] we deduce that the solution of (4.84) with zero initial data satisfies $\|\hat\eta\|_{\hat{\mathcal{X}}_\epsilon} \le C\epsilon^{1-3\sigma}$ for $t \in (0, T_{\rm adv}\delta^{-\sigma})$. Since $\hat{\mathcal{X}}_\epsilon \hookrightarrow \mathcal{X}_\epsilon$ by (4.17), (4.20), we conclude that $\|\eta_{\rm lin} - \eta_0\|_{\mathcal{X}_\epsilon} = \mathcal{O}\bigl(\epsilon^{1-3\sigma}\bigr)$ as $\epsilon \to 0$.

Now the solution of (2.3) with initial data $\Gamma\,\delta_{(r_0,z_0)}$ satisfies, instead of (2.13),

$$\omega_\theta(r,z,t) \,=\, \frac{\Gamma}{\nu t}\,\eta\Bigl(\frac{r-\bar r(t)}{\sqrt{\nu t}}\,,\,\frac{z-\bar z_*(t)-\delta\tilde z(t)}{\sqrt{\nu t}}\,,\,t\Bigr)\,, \tag{4.85}$$

so combining (4.81), (4.85) we obtain

$$\begin{aligned}\frac{1}{\Gamma}\int_\Omega\Bigl|\omega_\theta\bigl(r,z,t\bigr) - \omega_{\rm lin}\bigl(r,z-a_3(t),t\bigr)\Bigr|\,{\rm d}r\,{\rm d}z \,&=\, \|\eta(t)-\eta_{\rm lin}(t)\|_{L^1(\Omega_\epsilon)}\\ &\le\, C\|\eta(t)-\eta_{\rm lin}(t)\|_{\mathcal{X}_\epsilon} \,\le\, C\epsilon^{1-3\sigma}\,,\end{aligned} \tag{4.86}$$

because $\|\eta(t)-\eta_0\|_{\mathcal{X}_\epsilon} \le C\epsilon$ and $\|\eta_0-\eta_{\rm lin}\|_{\mathcal{X}_\epsilon} \le C\epsilon^{1-3\sigma}$. Using the notations of (1.7), inequality (4.86) exactly means that $\|\omega_{\rm cor}(\,\cdot\,,t)\| \le C\Gamma\epsilon^{1-3\sigma}$. This concludes the proof of Theorem 1.1. ☐

A Appendix to Section 3

A.1 Inverting the operator Λ

Following [31], we give here a short proof of Proposition 3.8. Assume that $n \ge 2$ and $f \in \mathcal{Y}_n \cap \mathcal{Z}$, or that $n = 1$ and $f \in \mathcal{Y}_1' \cap \mathcal{Z}$. In both cases, we have $f \in {\rm Ker}(\Lambda)^\perp$. We want to show that there exists a unique $\eta \in \mathcal{Y}_n \cap \mathcal{Z}$ (respectively, $\eta \in \mathcal{Y}_1' \cap \mathcal{Z}$ if $n = 1$) such that $\Lambda\eta = f$.

To make things concrete, we suppose without loss of generality that $f = a(\rho)\sin(n\vartheta)$, for some function $a : \mathbb{R}_+ \to \mathbb{R}$. Our hypotheses imply that a is smooth, that $a(\rho) = \mathcal{O}(\rho^n)$ as $\rho \to 0$, and that $e^{\rho^2/4}a(\rho)$ grows at most polynomially as $\rho \to \infty$. We look for a solution of the form $\eta = \omega(\rho)\cos(n\vartheta)$, where $\omega : \mathbb{R}_+ \to \mathbb{R}$ has to be determined. By (3.19), we have

$$\Lambda\eta \,=\, \bigl\{\phi_0\,,\eta\bigr\} + \bigl\{\Psi\,,\eta_0\bigr\}\,, \qquad \text{where} \quad \phi_0 \,=\, \frac{1}{2\pi}\,L\eta_0\,, \quad \Psi \,=\, \frac{1}{2\pi}\,L\eta\,. \tag{A.1}$$

The function ϕ_0 is radially symmetric and satisfies $\partial_\rho\phi_0 = -\rho\varphi(\rho)$, see (3.24) and (A.12) below. It follows that

$$\bigl\{\phi_0\,,\eta\bigr\} \,=\, \partial_\rho\phi_0\,\frac{1}{\rho}\,\partial_\vartheta\eta \,=\, n\varphi(\rho)\omega(\rho)\sin(n\vartheta)\,. \tag{A.2}$$

On the other hand, as $-\Delta\Psi = \eta$, we have $\Psi = \Omega(\rho)\cos(n\vartheta)$, where Ω is the unique regular solution of the differential equation

$$-\Omega''(\rho) - \frac{1}{\rho}\,\Omega'(\rho) + \frac{n^2}{\rho^2}\,\Omega(\rho) \,=\, \omega(\rho)\;, \qquad \rho > 0\;. \tag{A.3}$$

Since η_0 is radially symmetric and $\partial_\rho \eta_0 = -(\rho/2)\eta_0 = -\rho\varphi(\rho)h(\rho)$, see (3.24), we deduce

$$\{\Psi\,,\eta_0\} \;=\; -\partial_\rho\eta_0\,\frac{1}{\rho}\,\partial_\vartheta\Psi \;=\; -n\varphi(\rho)h(\rho)\Omega(\rho)\sin(n\vartheta)\,. \tag{A.4}$$

In view of (A.1), (A.2), (A.4), the equation $\Lambda\eta = f$ is equivalent to the relation (3.25), and using in addition (A.3) we obtain the differential equation (3.26) for the stream function Ω.

The main step in the proof is to show that (3.26) has a unique solution that is regular at the origin and decays to zero at infinity. Here we distinguish two cases according to the value of the angular Fourier mode n.

1. If $n \ge 2$, the homogeneous equation (3.26) with $a \equiv 0$ has two linearly independent solutions ψ_+, ψ_- which satisfy

$$\psi_-(\rho) \,\sim\, \begin{cases} \rho^n & \text{as } \rho \to 0\,, \\ \kappa\rho^n & \text{as } \rho \to \infty\,, \end{cases} \qquad\qquad \psi_+(\rho) \,\sim\, \begin{cases} \kappa\rho^{-n} & \text{as } \rho \to 0\,, \\ \rho^{-n} & \text{as } \rho \to \infty\,, \end{cases} \tag{A.5}$$

for some $\kappa > 0$, see [31]. Here we use the crucial observation that $(n^2/\rho^2) - h(\rho) > 0$ when $n \ge 2$, so that the differential operator in the left-hand side of (3.26) satisfies the Maximum Principle. We deduce the following representation formula for the solution of the inhomogeneous equation :

$$\Omega(\rho) \;=\; \psi_+(\rho)\int_0^\rho \frac{r}{w_0}\,\psi_-(r)\frac{a(r)}{n\varphi(r)}\,\mathrm{d}r + \psi_-(\rho)\int_\rho^\infty \frac{r}{w_0}\,\psi_+(r)\frac{a(r)}{n\varphi(r)}\,\mathrm{d}r\,, \tag{A.6}$$

where $w_0 = 2n\kappa$. It is then straightforward to verify that $\Omega(\rho) = \mathcal{O}(\rho^n)$ as $\rho \to 0$ and $\Omega(\rho) = \mathcal{O}(\rho^{-n})$ as $\rho \to \infty$. Moreover, if ω is defined by (3.25), the function $\eta = \omega(\rho)\cos(n\vartheta)$ lies in $\mathcal{Y}_n \cap \mathcal{Z}$ and satisfies $\Lambda\eta = f$ by construction. The details can be found in [31, Lemma 4].

2. The situation is quite different when $n = 1$, because the lower order term $1/\rho^2 - h(\rho)$ in (3.26) is no longer positive. In that case, it happens that the homogeneous equation (3.26) with $a \equiv 0$ has a solution $\psi(\rho) = \rho\varphi(\rho)$ which satisfies $\psi(\rho) \sim \rho/(8\pi)$ as $\rho \to 0$ and $\psi(\rho) \sim 1/(2\pi\rho)$ as $\rho \to \infty$. In other words, the linear operator in the left-hand side of (3.26) has a one-dimensional kernel, and for that reason we have to impose the solvability condition

$$f \,\in\, \mathcal{Y}_1' \,\subset\, \mathrm{Ker}(\Lambda)^\perp\,, \quad \text{or equivalently} \quad \int_0^\infty a(\rho)\rho^2\,\mathrm{d}\rho \;=\; 0\,. \tag{A.7}$$

To solve (3.26) for $n = 1$, we look for a solution of the form $\Omega(\rho) = b(\rho)\psi(\rho)$, which leads to a first-order differential equation for $b(\rho)$. In view of (A.7), we thus find

$$b'(\rho) \;=\; -\frac{1}{\rho\psi(\rho)^2}\int_0^\rho a(r)r^2\,\mathrm{d}r \;=\; \frac{1}{\rho\psi(\rho)^2}\int_\rho^\infty a(r)r^2\,\mathrm{d}r\,. \tag{A.8}$$

Integrating (A.8) gives the representation formula

$$b(\rho) \;=\; b_0 - \int_0^\rho a(r)r^2\Big(\mathcal{F}(\rho) - \mathcal{F}(r)\Big)\,\mathrm{d}r\,, \qquad \text{for some } b_0 \in \mathbb{R}\,,$$

where

$$\mathcal{F}(\rho) \;=\; 8\pi^2\left(\log(e^{\rho^2/4} - 1) - \frac{1}{e^{\rho^2/4} - 1}\right)\,, \qquad \mathcal{F}'(\rho) \;=\; \frac{1}{\rho\psi(\rho)^2}\,.$$

We now substitute $\Omega(\rho) = b(\rho)\psi(\rho)$ into (3.25) with $n = 1$, and we choose the constant b_0 so that $\int_0^\infty \omega(\rho)\rho^2\,\mathrm{d}\rho = 0$. This is always possible in a unique way, since

$$\int_0^\infty h(\rho)\psi(\rho)\rho^2\,\mathrm{d}\rho \;=\; \int_0^\infty h(\rho)\varphi(\rho)\rho^3\,\mathrm{d}\rho \;=\; \frac{1}{8\pi}\int_0^\infty e^{-\rho^2/4}\rho^3\,\mathrm{d}\rho \;=\; \frac{1}{\pi} \;\neq\; 0\,.$$

To conclude the proof, it remains to verify that the function $\eta = \omega(\rho)\cos(\vartheta)$ constructed above belongs to $\mathcal{Y}_1' \cap \mathcal{Z}$ and satisfies $\Lambda\eta = f$. These are straightforward calculations, which can be omitted. $\square$

A.2 First order calculations

We first establish the relations (3.37). As $\eta_0 \in \mathcal{Y}_0$ has unit mass we find, using (3.11),

$$\big(P_1\eta_0\big)(R,Z) \,=\, \int_{\mathbb{R}^2} \frac{R+R'}{2}\,\eta_0(R',Z')\,\mathrm{d}R'\,\mathrm{d}Z' \,=\, \frac{R}{2}\,, \tag{A.9}$$

hence $\{P_1\eta_0\,,\eta_0\} = \frac{1}{2}\,\partial_Z\eta_0$. On the other hand, since $\partial_R\eta_0 = -(R/2)\eta_0$ and L is a convolution operator, which therefore commutes with derivatives, we have

$$\big(LP_1\eta_0\big)(R,Z) \,=\, \frac{R}{2}\,(L\eta_0)(R,Z) + L\Big(\frac{R}{2}\eta_0\Big)(R,Z) \,=\, \frac{R}{2}\,(L\eta_0)(R,Z) - \partial_R\big(L\eta_0\big)(R,Z)\,.$$

Recalling that $L\eta_0 = 2\pi\phi_0$, and that $\{\phi_0,\eta_0\} = 0$ because both ϕ_0, η_0 are radially symmetric, we thus obtain

$$\begin{aligned}\frac{1}{2\pi}\big\{LP_1\eta_0\,,\eta_0\big\} \,&=\, \Big\{\frac{R}{2}\,\phi_0 - \partial_R\phi_0\,,\eta_0\Big\} \,=\, \frac{1}{2}\,\phi_0\,\partial_Z\eta_0 + \big\{\phi_0\,,\partial_R\eta_0\big\}\\ &=\, \frac{1}{2}\,\phi_0\,\partial_Z\eta_0 - \Big\{\phi_0\,,\frac{R}{2}\,\eta_0\Big\} \,=\, \frac{1}{2}\,\phi_0\,\partial_Z\eta_0 + \frac{1}{2}\,(\partial_Z\phi_0)\eta_0\,,\end{aligned}$$

which concludes the proof of (3.37).

We next prove formula (3.39) for the vertical velocity. Assuming that $\dot z_0$ is given by (3.39) for some $v \in \mathbb{R}$, we see that the right-hand side of (3.38) belongs to $\mathcal{Y}_1' = \mathcal{Y} \cap \mathrm{Ker}(\Lambda)^\perp$ if and only if

$$\int_{\mathbb{R}^2}\Big(\frac{v}{2\pi}\,\partial_Z\eta_0 - \frac{3}{2}(\partial_Z\phi_0)\eta_0 - \frac{1}{2}\phi_0\partial_Z\eta_0\Big)Z\;\mathrm{d}R\,\mathrm{d}Z \,=\, 0\,. \tag{A.10}$$

Since $\partial_Z\eta_0 = -(Z/2)\eta_0$ and $\int_{\mathbb{R}^2} Z^2\eta_0\,\mathrm{d}R\,\mathrm{d}Z = 2$, it is straightforward to verify that (A.10) is equivalent to

$$v \,=\, \pi\int_{\mathbb{R}^2}\phi_0\eta_0\big(3 - Z^2\big)\,\mathrm{d}R\,\mathrm{d}Z \,=\, \frac{\pi}{2}\int_{\mathbb{R}^2}\phi_0\eta_0\big(6 - |X|^2\big)\,\mathrm{d}X\,, \tag{A.11}$$

where $X = (R,Z)$ and $|X|^2 = R^2 + Z^2$.

To evaluate the right-hand side of (A.11), we temporarily denote $\psi_0 = 2\pi\phi_0 = L\eta_0$, namely

$$\psi_0(X) \,=\, \frac{1}{4\pi}\int_{\mathbb{R}^2}\log\Big(\frac{8}{|X-Y|}\Big)\,e^{-|Y|^2/4}\,\mathrm{d}Y\,, \qquad X \in \mathbb{R}^2\,.$$

This function satisfies $-\Delta\psi_0 = 2\pi\eta_0 = \frac{1}{2}\,e^{-|X|^2/4}$, so that

$$\psi_0(X) \,=\, \psi_0(0) \,-\, \int_0^{|X|}\frac{1 - e^{-\rho^2/4}}{\rho}\,\mathrm{d}\rho \,=:\, \tilde\psi_0(|X|)\,, \qquad X \in \mathbb{R}^2\,, \tag{A.12}$$

where

$$\psi_0(0) \,=\, \log(8) - \frac{1}{4\pi}\int_{\mathbb{R}^2}\log(|Y|)\,e^{-|Y|^2/4}\,\mathrm{d}Y \,=\, 2\log(2) + \frac{\gamma_E}{2}\,. \tag{A.13}$$

Using (A.12), (A.13) and integrating by parts, we easily find

$$\int_{\mathbb{R}^2}\psi_0\eta_0\,\mathrm{d}X \,=\, \frac{1}{2}\int_0^\infty\tilde\psi_0(\rho)e^{-\rho^2/4}\rho\,\mathrm{d}\rho \,=\, \psi_0(0) + \int_0^\infty\tilde\psi_0'(\rho)e^{-\rho^2/4}\,\mathrm{d}\rho \,=\, \frac{3}{2}\log(2) + \frac{\gamma_E}{2}\,,$$

and similarly

$$\int_{\mathbb{R}^2}\psi_0\eta_0|X|^2\,\mathrm{d}X \,=\, 4\psi_0(0) + \int_0^\infty\tilde\psi_0'(\rho)e^{-\rho^2/4}(\rho^2+4)\,\mathrm{d}\rho \,=\, 6\log(2) + 2\gamma_E - 1\,.$$

Returning to (A.11), we conclude that

$$v \,=\, \frac{1}{4}\int_{\mathbb{R}^2}\psi_0\eta_0\big(6 - |X|^2\big)\,\mathrm{d}X \,=\, \frac{3}{4}\log(2) + \frac{1}{4}\gamma_E + \frac{1}{4}\,. \tag{A.14}$$

A.3 Second order calculations

Our goal here is to prove Lemma 3.12. To establish (3.48), we consider separately the various terms in (3.47). As $\eta_1 \in \mathcal{Y}_1$ has zero mean, we find as in (A.9) that $P_1\eta_1$ is a constant, which can be disregarded. Moreover $LP_1\eta_1 = \frac{R}{2}L\eta_1 + L\bigl(\frac{R}{2}\eta_1\bigr)$, hence using the expression (3.41) of η_1 we find that

$$LP_1\eta_1 \,=\, (R^2 - Z^2)\chi_1(\rho) + \delta RZ\chi_2(\rho) + \chi_3(\rho)\,,$$

where $\chi_1, \chi_2, \ldots$ are functions of the radial variable $\rho = (R^2 + Z^2)^{1/2}$. As η_0 itself is radially symmetric, we deduce that

$$\bigl\{(\beta_\epsilon - 1)P_1\eta_1 + LP_1\eta_1\,,\eta_0\bigr\} \,=\, RZ\chi_4(\rho) + \delta(R^2 - Z^2)\chi_5(\rho)\,. \tag{A.15}$$

Next, using the expression (3.11) of P_2, we see that

$$\bigl(P_2\eta_0\bigr)(R,Z) \,=\, \frac{1}{16}\int_{\mathbb{R}^2}\Bigl((R-R')^2 + 3(Z-Z')^2\Bigr)\,\eta_0(R',Z')\,\mathrm{d}R'\,\mathrm{d}Z' \,=\, \frac{R^2}{16} + \frac{3Z^2}{16} + \frac{1}{2}\,,$$

and a similar calculation gives $Q_2\eta_0 = \frac{3R^2}{16} - \frac{Z^2}{16} + \frac{1}{4}$. Moreover,

$$\bigl(LP_2\eta_0\bigr)(R,Z) \,=\, \frac{1}{16}\int_{\mathbb{R}^2}\log\Bigl(\frac{8}{D}\Bigr)\Bigl(2D^2 + (Z-Z')^2 - (R-R')^2\Bigr)\,\eta_0(R',Z')\,\mathrm{d}R'\,\mathrm{d}Z'\,,$$

where $D^2 = (R-R')^2 + (Z-Z')^2$. Using the fact that η_0 given by (3.32) is radially symmetric, we easily obtain

$$\frac{1}{2\pi}\bigl(LP_2\eta_0\bigr)(R,Z) \,=\, \chi_6(\rho) + (R^2 - Z^2)\chi_7(\rho)\,.$$

Altogether, we arrive at

$$\frac{1}{2\pi}\bigl\{\beta_\epsilon P_2\eta_0 + LP_2\eta_0 + Q_2\eta_0\,,\eta_0\bigr\} \,=\, \frac{\beta_\epsilon}{16\pi}\,RZ\eta_0 + RZ\chi_8(\rho)\,. \tag{A.16}$$

The remaining terms in (3.47) are easier to treat. In view of (3.39), (3.41), (3.42), we have

$$\begin{aligned}
\bigl\{\phi_1\,,\eta_1\bigr\} - \frac{r_0\dot{\bar{z}}_0}{\Gamma}\,\partial_Z\eta_1 \,&=\, \Bigl\{\phi_1 - \frac{\beta_\epsilon - 1}{4\pi}\,R\,,\eta_1\Bigr\} - \frac{v}{2\pi}\,\partial_Z\eta_1 \\
&=\, \Bigl\{\frac{R}{2}\,\phi_0 - \partial_R\phi_0 + R\,\phi_{10}(\rho) + \delta Z\,\phi_{11}(\rho)\,,R\,\eta_{10}(\rho) + \delta Z\,\eta_{11}(\rho)\Bigr\} - \frac{v}{2\pi}\,\partial_Z\eta_1 \\
&=\, RZ\,\chi_9(\rho) + \delta\Bigl(\chi_{10}(\rho) + (R^2 - Z^2)\chi_{11}(\rho)\Bigr) + \delta^2 RZ\,\chi_{12}(\rho)\,.
\end{aligned} \tag{A.17}$$

It is also easy to verify that the terms $(\partial_Z\phi_1)\eta_0 + (\partial_Z\phi_0)\eta_1 - 2R(\partial_Z\phi_0)\eta_0 + \delta\partial_R(R\eta_0)$ are exactly of the same form. Finally, using again (3.41), (3.42), we obtain

$$R\Bigl(\bigl\{\phi_1\,,\eta_0\bigr\} + \bigl\{\phi_0\,,\eta_1\bigr\}\Bigr) \,=\, R\Bigl(\frac{\beta_\epsilon - 1}{4\pi}\,\partial_Z\eta_0 + Z\chi_{13}(\rho) + \delta R\chi_{14}(\rho)\Bigr)\,. \tag{A.18}$$

If we now combine (A.15), (A.16), (A.17), (A.18), we arrive at (3.48). ☐

A.4 Higher order order calculations

The calculations carried out in Sections 3.5 and 3.6 do not require new ideas, but a more compact notation is often helpful. To prove Lemma 3.13 and similar statements, it is important to understand how the decomposition (3.21) of the function space $\mathcal{Y}$ behaves under the Poisson bracket. If we use polar coordinates $R = \rho\cos\vartheta$, $Z = \rho\sin\vartheta$, we recall that $\mathcal{Y}_n$ is the subspace of $\mathcal{Y}$ spanned by functions of the form $a(\rho)\cos(n\vartheta)$ and $b(\rho)\sin(n\vartheta)$. Since

$$\big\{f\,,g\big\} \,=\, \partial_R f\partial_Z g - \partial_Z f\partial_R g \,=\, \frac{1}{\rho}\Big(\partial_\rho f\partial_\vartheta g - \partial_\vartheta f\partial_\rho g\Big)\,,$$

we easily obtain the following result:

Lemma A.1. *If $a, b : \mathbb{R}_+ \to \mathbb{R}$ are smooth functions and $n, m \in \mathbb{N}$, then*

$$\begin{aligned}
\big\{a(\rho)\cos(n\vartheta)\,,\,b(\rho)\cos(m\vartheta)\big\} \,&=\, c_{11}(\rho)\sin((n{-}m)\vartheta) + c_{12}(\rho)\sin((n{+}m)\vartheta)\,,\\
\big\{a(\rho)\sin(n\vartheta)\,,\,b(\rho)\sin(m\vartheta)\big\} \,&=\, c_{21}(\rho)\sin((n{-}m)\vartheta) + c_{22}(\rho)\sin((n{+}m)\vartheta)\,,\\
\big\{a(\rho)\sin(n\vartheta)\,,\,b(\rho)\cos(m\vartheta)\big\} \,&=\, c_{31}(\rho)\cos((n{-}m)\vartheta) + c_{32}(\rho)\cos((n{+}m)\vartheta)\,,
\end{aligned}$$

where $c_{ij} : \mathbb{R}_+ \to \mathbb{R}$ are smooth functions. In particular $\{\mathcal{Y}_n, \mathcal{Y}_m\} \subset \mathcal{Y}_{n-m} + \mathcal{Y}_{n+m}$ if $m \le n$.

It is also necessary to compute the homogeneous polynomials P_j, Q_j in (3.10) for higher values of j than in Lemma 3.3. This is a cumbersome calculation that can be done for instance using computer algebra. For $j = 3$ we find

$$\begin{aligned}
P_3 \,&=\, -\frac{1}{32}(R+R')\Big((R-R')^2 + 3(Z-Z')^2\Big)\,,\\
Q_3 \,&=\, -\frac{1}{48}(R+R')\Big((R+R')^2 - 6(Z-Z')^2\Big)\,,
\end{aligned} \tag{A.19}$$

and the calculation for $j = 4$ yields the more complicated expressions

$$\begin{aligned}
P_4 \,=&\, -\frac{15}{1024}\,(Z{-}Z')^4 + \frac{21}{512}\,(R{-}R')^2(Z{-}Z')^2 + \frac{3}{16}\,RR'\,(Z{-}Z')^2\\
&+ \frac{17}{1024}\,(R^2{-}R'^2)^2 - \frac{1}{256}\,RR'\,(R{-}R')^2\,,\\
Q_4 \,=&\, \frac{31}{2048}\,(Z{-}Z')^4 - \frac{89}{1024}\,(R{+}R')^2(Z{-}Z')^2 + \frac{1}{256}\,RR'\,(Z{-}Z')^2\\
&- \frac{19}{6144}\,(R^2{-}R'^2)^2 + \frac{35}{1536}\,RR'(R{+}R')^2 - \frac{1}{128}\,R^2R'^2\,.
\end{aligned} \tag{A.20}$$

The proof of Lemma 3.13 is similar to that of Lemma 3.12, and the details can be omitted. We use the expressions (3.41), (3.50) of the vorticities η_1, η_2, the formulas (3.42), (3.51) for the stream functions ϕ_1, ϕ_2, and the definition (3.15) of the Biot-Savart operators, which involve the polynomials (3.11) and (A.19). Using Lemma A.1, it is straightforward to verify that the quantity defined in (3.55) satisfies $\mathcal{R}_3 \in \mathcal{Y}_1 + \mathcal{Y}_3$ and takes the form

$$\mathcal{R}_3 \,=\, \chi_1(\rho)\sin(\vartheta) + \chi_2(\rho)\sin(3\vartheta) + \delta\Big(\chi_3(\rho)\cos(\vartheta) + \chi_4(\rho)\cos(3\vartheta)\Big) + \mathcal{O}(\delta^2)\,,$$

where $\chi_1, \chi_2, \chi_3, \chi_4$ are radially symmetric functions which may depend linearly on β_ϵ. To arrive at (3.56), it remains to verify that $\mathcal{R}_3$ does not contain any term involving β_ϵ^2. Indeed, according to (3.11), (3.50), we have

$$\frac{\beta_\epsilon}{2\pi}\,P_1\eta_2 \,=\, \frac{\beta_\epsilon}{4\pi}\int_{\mathbb{R}^2}(R+R')\,\eta_2(R',Z')\,\mathrm{d}R'\,\mathrm{d}Z' \,=\, \frac{\beta_\epsilon R}{4\pi}\int_{\mathbb{R}^2}\eta_{24}(R',Z')\,\mathrm{d}R'\,\mathrm{d}Z'\,,$$

so that the first term in (3.55) does not contain β_ϵ^2. The only other terms that we have to check are

$$\big\{\phi_1\,,\eta_2\big\} - \frac{r_0}{\Gamma}\,\dot{z}_0\partial_Z\eta_2 \,=\, \Big\{\phi_1 - \frac{\beta_\epsilon - 1 + 2v}{4\pi}\,R\,,\eta_2\Big\}\,,$$

but using the expressions (3.42), (3.50) we immediately see that the right-hand side does not contain any factor β_ϵ^2. Altogether we arrive at (3.56). $\square$

B Appendix to Section 4

B.1 Properties of the energy functional

Proof of Lemma 4.4. We use the first expression of $E_\epsilon^{\mathrm{kin}}[\eta]$ in (4.22) and the representation formula (2.20) for the stream function ϕ. Since $\mathrm{supp}(\eta) \subset B_\epsilon$ by assumption, we have

$$E_\epsilon^{\mathrm{kin}}[\eta] \;=\; \frac{1}{4\pi}\int_{B_\epsilon}\int_{B_\epsilon} K_\epsilon(R,Z;R',Z')\,\eta(R,Z)\,\eta(R',Z')\,\mathrm{d}X\,\mathrm{d}X'\,, \tag{B.1}$$

where the integral kernel K_ϵ is defined in (3.8). As $R^2+Z^2 \le \epsilon^{-2\sigma_1}$ and $R'^2+Z'^2 \le \epsilon^{-2\sigma_1}$, the argument of F in (3.8) is not larger than $C\epsilon^{2-2\sigma_1}$ for some $C>0$. Using the asymptotic expansion of $F(s)$ as $s\to 0$ and proceeding as in Section 3.1, we easily obtain the decomposition

$$K_\epsilon(R,Z;R',Z') \;=\; \beta_\epsilon - 2 + \log\frac{8}{D} + \tilde K_\epsilon(R,Z;R',Z')\,, \tag{B.2}$$

where $\beta_\epsilon = \log(1/\epsilon)$ and $D^2 = (R-R')^2 + (Z-Z')^2$. The remainder $\tilde K_\epsilon$ satisfies the estimate

$$|\tilde K_\epsilon(R,Z;R',Z')| \;\le\; C\epsilon\big(|R|+|R'|\big)\Big(\beta_\epsilon + 1 + \log\frac{8}{D}\Big) + \mathcal{O}\big(\beta_\epsilon\epsilon^{2-2\sigma_1}\big)\,. \tag{B.3}$$

If we insert the decomposition (B.2) into (B.1), the contributions of $\beta_\epsilon - 2$ and $\log(8/D)$ give exactly the first two terms in the right-hand side of (4.24), in view of (4.23). Moreover, taking into account estimate (B.3) where $\epsilon^{2-2\sigma_1}\le\epsilon$, we see that the contributions of $\tilde K_\epsilon$ to the kinetic energy (B.1) are of order $\mathcal{O}\big(\epsilon\beta_\epsilon\|\eta\|_{\mathcal{X}_\epsilon}^2\big)$, as stated in (4.24). $\square$

Proof of Proposition 4.6. Given $\eta\in\mathcal{X}_\epsilon$, we decompose $\eta = \eta_1+\eta_2$ where $\eta_1 = \eta\mathbf{1}_{B_\epsilon}$ and $\mathbf{1}_{B_\epsilon}$ is the indicator function of the ball $B_\epsilon = \{(R,Z)\in\Omega_\epsilon\,;\, R^2+Z^2\le\epsilon^{-2\sigma_1}\}$. We thus have

$$E_\epsilon[\eta] \;=\; \frac12\int_{\Omega_\epsilon} W_\epsilon\,\eta_1^2\,\mathrm{d}X + \frac12\int_{\Omega_\epsilon} W_\epsilon\,\eta_2^2\,\mathrm{d}X - \frac12\int_{\Omega_\epsilon}\big(\phi_1+\phi_2\big)\big(\eta_1+\eta_2\big)\,\mathrm{d}X\,, \tag{B.4}$$

where $\phi_j = \mathrm{BS}^\epsilon[\eta_j]$ for $j=1,2$. We claim that

$$\frac12\int_{\Omega_\epsilon}\big(\phi_1+\phi_2\big)\big(\eta_1+\eta_2\big)\,\mathrm{d}X \;=\; E_\epsilon^{\mathrm{kin}}[\eta_1] + \mathcal{O}\big(\epsilon^\infty\|\eta\|_{\mathcal{X}_\epsilon}^2\big)\,, \tag{B.5}$$

so that

$$E_\epsilon[\eta] \;=\; E_\epsilon[\eta_1] + \frac12\|\eta_2\|_{\mathcal{X}_\epsilon}^2 + \mathcal{O}\big(\epsilon^\infty\|\eta\|_{\mathcal{X}_\epsilon}^2\big)\,. \tag{B.6}$$

To prove (B.5), we recall that $\phi_j(R,Z) = \frac{1}{2\pi}\int_{\Omega_\epsilon} K_\epsilon(R,Z;R',Z')\eta_j(R',Z')\,\mathrm{d}X'$, where the kernel K_ϵ is given by (3.8). Using the crude estimate $|F(s)|\le C\big(|\log s|+1\big)$, we easily obtain

$$\big|K_\epsilon(R,Z;R',Z')\big| \;\le\; C\big(1{+}\epsilon|R|\big)^a\big(1{+}\epsilon|R'|\big)^a\big(\beta_\epsilon + \big|\log D\big| + 1\big)\,, \tag{B.7}$$

for some $a>1/2$. It follows in particular that

$$|\phi(R,Z)| \;\le\; C\big(\beta_\epsilon+1\big)(1+\rho)^b\|\eta\|_{\mathcal{X}_\epsilon}\,, \qquad \rho \;=\; \sqrt{R^2+Z^2}\,,$$

for some $b>1/2$, and using Hölder's inequality we deduce

$$\int_{\Omega_\epsilon}|\phi(R,Z)|\,|\eta_2(R,Z)|\,\mathrm{d}X \;\le\; C\big(\beta_\epsilon+1\big)\|\eta\|_{\mathcal{X}_\epsilon}^2\bigg(\int_{B_\epsilon^c}(1+\rho)^{2b}\,W_\epsilon(R,Z)^{-1}\,\mathrm{d}X\bigg)^{1/2}\,,$$

where the last integral is $\mathcal{O}(\epsilon^\infty)$ in view of (4.17). In a similar way we have

$$|\phi_2(R,Z)| \,\le\, C\big(\beta_\epsilon+1\big)(1+\rho)^b\bigg(\int_{B_\epsilon^c}(1+\rho')^{2b}|\eta(R',Z')|^2\,\mathrm{d}X'\bigg)^{1/2} \,=\, \mathcal{O}\big(\epsilon^\infty\|\eta\|_{\mathcal{X}_\epsilon}\big)(1+\rho)^b\,,$$

so that $\int_{\Omega_\epsilon}\phi_2\eta_1\,\mathrm{d}x = \mathcal{O}\big(\epsilon^\infty\|\eta\|^2_{\mathcal{X}_\epsilon}\big)$. Altogether we arrive at (B.5).

Now, since η_1 is supported in the ball B_ϵ, it follows from (4.18) and Lemma 4.4 that

$$\|\eta_1\|^2_{\mathcal{X}_\epsilon} \,=\, \|\eta_1\|^2_{\mathcal{X}_0} + \mathcal{O}\big(\epsilon^{\gamma_1}\|\eta\|^2_{\mathcal{X}_\epsilon}\big)\,, \qquad E_\epsilon^{\mathrm{kin}}[\eta_1] \,=\, \frac{\beta_\epsilon-2}{4\pi}\,\tilde\mu_0^2 + E_0^{\mathrm{kin}}[\eta_1] + \mathcal{O}\big(\epsilon\beta_\epsilon\|\eta\|^2_{\mathcal{X}_\epsilon}\big)\,. \quad \text{(B.8)}$$

Moreover we know from Proposition 4.5 that

$$\|\eta_1\|^2_{\mathcal{X}_0} \,\le\, C_4E_0[\eta_1] + C_5\big(\tilde\mu_0^2+\tilde\mu_1^2+\tilde\mu_2^2\big)\,, \qquad \text{(B.9)}$$

where $\tilde\mu_0,\tilde\mu_1,\tilde\mu_2$ are the moments of η_1, which satisfy $\tilde\mu_j = \mu_j + \mathcal{O}\big(\epsilon^\infty\|\eta\|_{\mathcal{X}_\epsilon}\big)$. Combining both estimates in (B.8) we obtain

$$E_0[\eta_1] \,=\, \frac12\|\eta_1\|^2_{\mathcal{X}_0} - E_0^{\mathrm{kin}}[\eta_1] \,\le\, \frac12\|\eta_1\|^2_{\mathcal{X}_\epsilon} - E_\epsilon^{\mathrm{kin}}[\eta_1] + \frac{\beta_\epsilon-2}{4\pi}\,\tilde\mu_0^2 + \mathcal{O}\big(\epsilon^{\gamma_1}\|\eta\|^2_{\mathcal{X}_\epsilon}\big)\,,$$

namely $E_0[\eta_1] \le E_\epsilon[\eta_1] + \frac{\beta_\epsilon-2}{4\pi}\tilde\mu_0^2 + \mathcal{O}\big(\epsilon^{\gamma_1}\|\eta\|^2_{\mathcal{X}_\epsilon}\big)$. Using in addition (B.9) we deduce

$$\|\eta_1\|^2_{\mathcal{X}_\epsilon} \,\le\, \|\eta_1\|^2_{\mathcal{X}_0} + \mathcal{O}\big(\epsilon^{\gamma_1}\|\eta\|^2_{\mathcal{X}_\epsilon}\big) \,\le\, C_4E_\epsilon[\eta_1] + C\big(\beta_\epsilon\tilde\mu_0^2+\tilde\mu_1^2+\tilde\mu_2^2\big) + \mathcal{O}\big(\epsilon^{\gamma_1}\|\eta\|^2_{\mathcal{X}_\epsilon}\big)\,.$$

Finally, invoking (B.6) and recalling that $C_4>2$, we find

$$\|\eta\|^2_{\mathcal{X}_\epsilon} \,\le\, \|\eta_1\|^2_{\mathcal{X}_\epsilon} + \frac{C_4}{2}\|\eta_2\|^2_{\mathcal{X}_\epsilon} \,\le\, C_4E_\epsilon[\eta] + C\big(\beta_\epsilon\tilde\mu_0^2+\tilde\mu_1^2+\tilde\mu_2^2\big) + \mathcal{O}\big(\epsilon^{\gamma_1}\|\eta\|^2_{\mathcal{X}_\epsilon}\big)\,,$$

and estimate (4.27) follows, since $\tilde\mu_j = \mu_j + \mathcal{O}\big(\epsilon^\infty\|\eta\|_{\mathcal{X}_\epsilon}\big)$ for $j=0,1,2$. □

B.2 Diffusive terms in the energy functional

We justify here the expression (4.32) of the quantity I_4. Integrating by parts as in [34], we find

$$\int_{\Omega_\epsilon} W_\epsilon\tilde\eta\,\mathcal{L}\tilde\eta\,\mathrm{d}X \,=\, -\int_{\Omega_\epsilon} W_\epsilon|\nabla\tilde\eta|^2\,\mathrm{d}X - \int_{\Omega_\epsilon}(\nabla W_\epsilon\cdot\nabla\tilde\eta)\tilde\eta\,\mathrm{d}X - \int_{\Omega_\epsilon}\tilde V_\epsilon\tilde\eta^2\,\mathrm{d}X\,,$$

where $\tilde V_\epsilon = \frac14(R\partial_R+Z\partial_Z)W_\epsilon - \frac12 W_\epsilon$. Similarly,

$$\epsilon\int_{\Omega_\epsilon} W_\epsilon\tilde\eta\,\partial_R\tilde\zeta\,\mathrm{d}X \,=\, \epsilon\int_{\Omega_\epsilon} W_\epsilon(1+\epsilon R)\tilde\zeta\,\partial_R\tilde\zeta\,\mathrm{d}X \,=\, -\frac{\epsilon}{2}\int_{\Omega_\epsilon}\partial_R\big(W_\epsilon(1+\epsilon R)\big)\tilde\zeta^2\,\mathrm{d}X\,.$$

On the other hand, integrating by parts and using the relation (2.19) between $\tilde\phi$ and $\tilde\eta$, we obtain

$$\begin{aligned}\int_{\Omega_\epsilon}\tilde\phi\Big(\mathcal{L}\tilde\eta+\epsilon\partial_R\tilde\zeta\Big)\,\mathrm{d}X \,&=\, \int_{\Omega_\epsilon}\tilde\eta\Big(\Delta\tilde\phi - \frac{\epsilon\partial_R\tilde\phi}{1+\epsilon R}\Big)\,\mathrm{d}X - \frac12\int_{\Omega_\epsilon}\tilde\eta\big(R\partial_R+Z\partial_Z\big)\tilde\phi\,\mathrm{d}X\\ &=\, -\int_{\Omega_\epsilon}\tilde\eta^2(1+\epsilon R)\,\mathrm{d}X - \frac12\int_{\Omega_\epsilon}\tilde\eta\big(R\partial_R+Z\partial_Z\big)\tilde\phi\,\mathrm{d}X\,.\end{aligned}$$

It remains to treat the last term in the right-hand side. Here again, we use the relation (2.19) and integrate by parts to obtain

$$\frac12\int_{\Omega_\epsilon}\tilde\eta\big(R\partial_R+Z\partial_Z\big)\tilde\phi\,\mathrm{d}X \,=\, \frac{\epsilon}{4}\int_{\Omega_\epsilon}\frac{R|\nabla\tilde\phi|^2}{(1+\epsilon R)^2}\,\mathrm{d}X\,.$$

Altogether we arrive at (4.32), with $V_\epsilon = \tilde V_\epsilon - (1+\epsilon R)$.

B.3 Coercivity of the diffusive quadratic form

This section is devoted to the proof of Proposition 4.15. Given $\epsilon > 0$ sufficiently small, we take a smooth partition of unity of the form $1 = \chi_3^2 + \chi_4^2$, where χ_3, χ_4 are radially symmetric and $\chi_3 = 1$ when $\rho \le \frac{1}{2}\epsilon^{-\sigma_1}$, $\chi_3 = 0$ when $\rho \ge \epsilon^{-\sigma_1}$. We can also assume that $|\nabla\chi_3| + |\nabla\chi_4| \le C\epsilon^{\sigma_1}$. Given η as in the statement of Proposition 4.15, we define $\eta_3 = \chi_3\eta$, $\eta_4 = \chi_4\eta$. We thus have the decompositions $\eta^2 = \eta_3^2 + \eta_4^2$, $\eta\nabla\eta = \eta_3\nabla\eta_3 + \eta_4\nabla\eta_4$, and

$$|\nabla\eta|^2 \,=\, |\nabla\eta_3|^2 + |\nabla\eta_4|^2 - \big(|\nabla\chi_3|^2 + |\nabla\chi_4|^2\big)\eta^2\,. \tag{B.10}$$

As a consequence, the quadratic form $Q_\epsilon[\eta]$ can be decomposed as

$$Q_\epsilon[\eta] \,=\, Q_\epsilon[\eta_3] + Q_\epsilon[\eta_4] - \int_{\Omega_\epsilon} W_\epsilon\big(|\nabla\chi_3|^2 + |\nabla\chi_4|^2\big)\eta^2 \,\mathrm{d}X\,. \tag{B.11}$$

The last term in (B.11) is bounded by $C\epsilon^{2\sigma_1}\|\eta\|_{\mathcal{X}_\epsilon}^2$ and is thus negligible when $\epsilon \ll 1$. So our main task is to estimate from below the terms $Q_\epsilon[\eta_3]$ and $Q_\epsilon[\eta_4]$.

We first consider the function η_3 which is supported in the region where $\rho \le \epsilon^{-\sigma_1}$. We recall that the weight W_ϵ in (4.16) satisfies the estimates (4.18), which read

$$|\nabla W_\epsilon(R,Z) - \nabla A(\rho)| + |W_\epsilon(R,Z) - A(\rho)| \,\le\, C\epsilon^{\gamma_1}A(\rho)\,, \qquad \text{when } \rho \le \epsilon^{-\sigma_1}\,, \tag{B.12}$$

where $\gamma_1 > 0$. We easily deduce that

$$Q_\epsilon[\eta_3] \,\ge\, Q_0[\eta_3] - C\epsilon^{\gamma_1}\big(\|\nabla\eta_3\|_{\mathcal{X}_0}^2 + \|\rho\eta_3\|_{\mathcal{X}_0}^2 + \|\eta_3\|_{\mathcal{X}_0}^2\big)\,, \tag{B.13}$$

where Q_0 is the limiting quadratic form (4.62). On the other hand, we know from Proposition 4.14 that

$$C_8 Q_0[\eta_3] \,\ge\, \|\nabla\eta_3\|_{\mathcal{X}_0}^2 + \|\rho\eta_3\|_{\mathcal{X}_0}^2 + \|\eta_3\|_{\mathcal{X}_0}^2 - C_9\big(\tilde\mu_0^2 + \tilde\mu_1^2 + \tilde\mu_2^2\big)\,, \tag{B.14}$$

where $\tilde\mu_0, \tilde\mu_1, \tilde\mu_2$ are the moments of η_3, which satisfy $\tilde\mu_j = \mu_j + \mathcal{O}\big(\epsilon^\infty\|\eta\|_{\mathcal{X}_\epsilon}\big)$. Combining (B.13), (B.14) and using (B.12) once again, we arrive at

$$\|\nabla\eta_3\|_{\mathcal{X}_\epsilon}^2 + \|\rho\eta_3\|_{\mathcal{X}_\epsilon}^2 + \|\eta_3\|_{\mathcal{X}_\epsilon}^2 \,\le\, 2C_8 Q_\epsilon[\eta_3] + C\big(\tilde\mu_0^2 + \tilde\mu_1^2 + \tilde\mu_2^2\big)\,. \tag{B.15}$$

We next consider the function η_4, which is nonzero only if $\rho \ge \frac{1}{2}\epsilon^{-\sigma_1}$. Our starting point is the lower bound

$$Q_\epsilon[\eta_4] \,\ge\, \frac{1}{4}\int_{\Omega_\epsilon} W_\epsilon|\nabla\eta_4|^2 \,\mathrm{d}X + \int_{\Omega_\epsilon}\Big(V_\epsilon - \frac{|\nabla W_\epsilon|^2}{3W_\epsilon}\Big)\eta_4^2 \,\mathrm{d}X\,,$$

which is obtained from (4.61) by applying Young's inequality to the middle term in the right-hand side. Using the expression (4.16) of the weight function, as well as the estimates (B.12) in the inner region Ω_ϵ', it is not difficult to verify that

$$\frac{V_\epsilon}{W_\epsilon} - \frac{|\nabla W_\epsilon|^2}{3W_\epsilon^2} \,\ge\, \begin{cases} C\rho^2 - \tilde C & \text{in } \Omega_\epsilon'\,,\\ -\tilde C & \text{in } \Omega_\epsilon''\,,\\ C\rho^{2\gamma} & \text{in } \Omega_\epsilon'''\,,\end{cases}$$

for some positive constants $C, \tilde C$. It follows that

$$Q_\epsilon[\eta_4] \,\ge\, \frac{1}{4}\|\nabla\eta_4\|_{\mathcal{X}_\epsilon}^2 + C\int_{\Omega_\epsilon'\cup\Omega_\epsilon'''} W_\epsilon\,\rho_\gamma^2\eta_4^2 \,\mathrm{d}X - \tilde C\int_{\Omega_\epsilon''} W_\epsilon\,\eta_4^2 \,\mathrm{d}X\,. \tag{B.16}$$

If we now combine (B.15) and (B.16), we obtain

$$\begin{aligned}\|\nabla\eta_3\|_{\mathcal{X}_\epsilon}^2 + \|\nabla\eta_4\|_{\mathcal{X}_\epsilon}^2 + \|\eta\|_{\mathcal{X}_\epsilon}^2 + \int_{\Omega'_\epsilon\cup\Omega'''_\epsilon} W_\epsilon\,\rho_\gamma^2\eta^2\,\mathrm{d}X \\ \le C_{10}\big(Q_\epsilon[\eta_3]+Q_\epsilon[\eta_4]\big) + C_{11}\Big(\tilde\mu^2 + \int_{\Omega''_\epsilon} W_\epsilon\eta^2\,\mathrm{d}X\Big)\,,\end{aligned} \tag{B.17}$$

for some positive constants C_{10}, C_{11}, where $\tilde\mu^2 = \tilde\mu_0^2 + \tilde\mu_1^2 + \tilde\mu_2^2$. Finally, using again (B.10) as well as (B.11), and recalling that $\tilde\mu_j = \mu_j + \mathcal{O}\big(\epsilon^\infty\|\eta\|_{\mathcal{X}_\epsilon}\big)$, we deduce (4.65) from (B.17). ☐

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