# Arnold's variational principle and its application to the stability of planar vortices

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**Abstract**

We consider variational principles related to V. I. Arnold's stability criteria for steady-state solutions of the two-dimensional incompressible Euler equation. Our goal is to investigate under which conditions the quadratic forms defined by the second variation of the associated functionals can be used in the stability analysis, both for the Euler evolution and for the Navier-Stokes equation at low viscosity. In particular, we revisit the classical example of Oseen's vortex, providing a new stability proof with stronger geometric flavor. Our analysis involves a fairly detailed functional-analytic study of the inviscid case, which may be of independent interest, and a careful investigation of the influence of the viscous term in the particular example of the Gaussian vortex.

## 1 Introduction

In this paper we investigate the applicability of V. I. Arnold's geometric methods to certain stability problems related to Navier-Stokes vortices at high Reynolds number. Our main goal is a "proof of concept" that such applications are possible, at least in simple cases, even though much of the geometric structure behind the inviscid stability analysis does not survive the addition of the viscosity term. In particular, we give a new proof of a known result concerning the stability of Oseen's vortex as a steady state of the Navier-Stokes equation in self-similar variables. We expect that the approach we advertise here will be useful to tackle stability problems involving solutions that are less symmetric and less explicit than the classical Oseen vortex. In such cases one may not have good alternative methods for proving stability in the presence of viscosity. Our investigation leads to a detailed study of the quadratic forms naturally arising in Arnold's approach. Some of their functional-analytic properties, which are established in the course of our analysis, may be of independent interest.

### 1.1 A finite-dimensional model

Following V. I. Arnold's seminal paper [2], we first illustrate the issues we want to address in a model situation where the "phase space" is finite-dimensional. We consider the ordinary differential equation

$$\dot{x} \,=\, b(x)\,, \qquad x \in \mathbb{R}^n\,, \tag{1.1}$$

where $b$ is a smooth vector field in $\mathbb{R}^n$. Let us assume that $f, g_1, \dots, g_m : \mathbb{R}^n \to \mathbb{R}$ are (sufficiently smooth) conserved quantities for the evolution (1.1), with $m < n$. This means

$$f'(x)b(x) \,=\, 0 \quad \text{and} \quad g_j'(x)b(x) \,=\, 0\;, \quad x \in \mathbb{R}^n\,, \quad j = 1, \dots, m\,, \tag{1.2}$$

where we adopt the standard notation $f'(x)$ for the linear form given by the first derivative of $f$ at $x$. The situation we have ultimately in mind is somewhat more specific: it corresponds to the case where the phase space $\mathbb{R}^n$ is equipped with a Poisson bracket $\{\cdot,\cdot\}$, where system (1.1) is of the form

$$\dot{x} = \{f, x\}, \tag{1.3}$$

and where $g_1, \ldots, g_m$ are Casimir functions. The Poisson structure is of course important in many respects, but for our arguments here it does not play a big role. We can therefore proceed in the general context of (1.1) and (1.2).

For any $c = (c_1, \ldots, c_m) \in \mathbb{R}^m$, let us denote $X_c = \{x \in \mathbb{R}^n\,;\, g_1(x) = c_1, \ldots, g_m(x) = c_m\}$. We assume that, for some $c \in \mathbb{R}^m$, the function $f$ attains a *non-degenerate local maximum* on $X_c$ at some point $\bar{x} \in X_c$, and that the derivatives $g_1'(\bar{x}), \ldots, g_m'(\bar{x})$ are linearly independent. The stationarity condition at $\bar{x}$ gives the linear relation

$$f'(\bar{x}) - \sum_{j=1}^{m} \lambda_j g_j'(\bar{x}) = 0, \tag{1.4}$$

for some Lagrange multipliers $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$. Moreover, the second order differential[1] of the function $f|_{X_c}$ (the restriction of $f$ to $X_c$) at $\bar{x}$ is given by the restriction to the tangent space $T_{\bar{x}}X_c$ of the quadratic form

$$\mathcal{Q} = f''(\bar{x}) - \sum_{j=1}^{m} \lambda_j g_j''(\bar{x}), \tag{1.5}$$

where we denote by $f''(\bar{x})$ the quadratic form given by the Hessian of $f$ at $\bar{x}$, and similarly for $g_1''(\bar{x}), \ldots, g_m''(\bar{x})$. Our non-degeneracy assumption means that the restriction of the form $\mathcal{Q}$ to $T_{\bar{x}}X_c$ is strictly negative definite. Now, let $B = b'(\bar{x})$ be the $n \times n$ matrix corresponding to the linearization of (1.1) at the point $\bar{x}$, which is a steady state by construction [2]. If we differentiate twice the relations (1.2) and use (1.4) together with $b(\bar{x}) = 0$, we see that the evolution defined by the linearized equation $\dot{\xi} = B\xi$ leaves the form $\mathcal{Q}$ invariant. In other words,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Q}(\xi, \xi) = \mathcal{Q}(B\xi, \xi) + \mathcal{Q}(\xi, B\xi) = 0, \qquad \forall\, \xi \in \mathbb{R}^n. \tag{1.6}$$

The above structure[2] gives various options for the stability analysis of the equilibrium $\bar{x}$ of (1.1), depending on the index of the quadratic form $\mathcal{Q}$ in (1.5). Our assumptions readily imply that $\bar{x}$ is stable in the sense of Lyapunov with respect to perturbations on the invariant submanifold $X_c$. Moreover, since a neighborhood of $\bar{x}$ in $\mathbb{R}^n$ is foliated by submanifolds of this form for nearby values of the parameter $c = (c_1, \ldots, c_m)$, one can show that $\bar{x}$ is in fact Lyapunov stable with respect to small *unconstrained* perturbations [2]. The perspective changes qualitatively if we add to the vector field $b$ in (1.1) a small "dissipative" term, with the effect that the quantities $f$ and $g_1, \ldots, g_m$ are no longer exactly conserved under the modified evolution. This is in the spirit of what we intend to do in the infinite-dimensional case, when we consider the Navier-Stokes equation as a perturbation of the Euler equation. Since the evolution no longer takes place on the manifolds $X_c$, the argument above leading to unconstrained Lyapunov stability is not applicable anymore. However, in good situations, stability can still be obtained if the quadratic form $\mathcal{Q}$ in (1.5) happens to be negative definite not just on $T_{\bar{x}}X_c$, but on larger

[1] We recall that the second order differential of a function on a manifold is intrinsically defined at the points where the first order differential vanishes.

[2] Pointed out in [2] in the form we use here, although in the finite-dimensional case these ideas go back to the founders of the analytical mechanics.

subspaces as well, for instance on the whole space $\mathbb{R}^n$. This is, roughly speaking, the idea we shall pursue in the infinite-dimensional case, to study the stability of vortex-like solutions of the Navier-Stokes equation.

To conclude with the (unmodified) evolution (1.1), we emphasize that the problem of determining the index of the form (1.5) is also very natural from the viewpoint of the usual constrained optimization theory. Clearly, the "Lagrange function"

$$\mathscr{L}(x) \;=\; f(x) - \sum_{j=1}^{m} \lambda_j g_j(x)\,, \qquad x \in \mathbb{R}^n\,, \tag{1.7}$$

when considered on the whole space $\mathbb{R}^n$, has a critical point at $\bar{x}$ (and a local maximum at $\bar{x}$ when restricted to $X_c$). The form $\mathscr{Q}$ will be strictly negative definite[3] in the whole space $\mathbb{R}^n$ if and only if $\mathscr{L}$ has a non-degenerate *unconstrained* maximum at $\bar{x}$. As is explained in Section 2.4, this is related to the concavity of the function

$$(c_1,\dots,c_m) \longmapsto M(c_1,\dots,c_m) \;:=\; \sup_{x\in X_c} f(x)\,. \tag{1.8}$$

## 1.2 Arnold's geometric view of the 2d incompressible Euler equation

V. I. Arnold [3, 4, 5] carried out the analogue of the above calculations in an infinite-dimensional setting to handle in particular the 2d incompressible Euler equation $\partial_t\omega + u\cdot\nabla\omega = 0$, where $u$ denotes the velocity of the fluid and $\omega = \operatorname{curl} u$ is the associated vorticity. In this case the evolution is generated by the Hamiltonian function, which represents the kinetic energy of the fluid, and the constraints are given by the Casimir functionals

$$\mathscr{C}_\Phi(\omega) \;=\; \int_\Omega \Phi(\omega(x))\,\mathrm{d}x\,, \tag{1.9}$$

where $\Omega \subset \mathbb{R}^2$ is the fluid domain and $\Phi$ is an "arbitrary" function on $\mathbb{R}$. The idea of maximizing or minimizing the energy on the set of vorticities satisfying suitable constraints has been widely used since then to study the stability of steady-state solutions of the 2d Euler equations and related fluid models, see [5, 8, 9] and the references therein.

Let us briefly recall the setup relevant for our goals here, making the similarities with the finite-dimensional case as transparent as possible. Our main objects will be the following:

(i) The *phase space* $\mathscr{P} = \{\omega\colon \mathbb{R}^2 \to (0,\infty)\,;\,\omega$ is smooth and decays "sufficiently fast" at $\infty\}$. This is our infinite-dimensional replacement for the manifold $\mathbb{R}^n$ in the finite-dimensional model. We restrict ourselves to positive vorticity distributions defined on $\Omega = \mathbb{R}^2$, because this is the appropriate framework to study the stability of radially symmetric vortices in the whole plane. Admittedly, the definition above is somewhat vague, but it serves only as a motivation and our results will be independent of the vague parts of the definitions. There is a natural Poisson structure on $\mathscr{P}$ that is relevant for the Euler equation, see Section A.5, but here we only need some of its Casimir functionals (to be specified now).

(ii) The *Casimir functionals*, which play the role of the constraints $g_j$ in the finite-dimensional example. These are linear combinations of elementary functionals of the form

$$h(a,\omega) \;=\; |\{\omega > a\}| \;=\; \int_{\mathbb{R}^2} \chi\big(\omega(x) - a\big)\,\mathrm{d}x\,, \qquad a > 0\,, \tag{1.10}$$

[3] Our use of the terms "positive definite" and "negative definite" allows for vanishing along some directions. When this is not the case, we speak of strictly positive definite or strictly negative definite forms.

where $\chi = \mathbf{1}_{(0,\infty)}$ is the indicator function of $(0,\infty)$. Here and in what follows, we denote by $|S|$ the Lebesgue measure of any (Borel) set $S \subset \mathbb{R}^2$. Due to our assumptions on the vorticities in $\mathscr{P}$, the functions $a \mapsto h(a,\omega)$ are finite and nonincreasing on $(0,\infty)$. In general, they do not have to be continuous in $a$ but they will have this property in the examples considered later. Similarly, the functionals $\omega \mapsto h(a,\omega)$ may in general not be differentiable in every direction, but they will be in our examples. It is useful to single out the quantity

$$M_0(\omega) \;=\; \int_{\mathbb{R}^2} \omega(x)\,\mathrm{d}x \;=\; \int_0^\infty h(a,\omega)\,\mathrm{d}a\,, \tag{1.11}$$

which will be referred to as the "mass" of the vorticity distribution $\omega \in \mathscr{P}$.

(iii) The *orbits* defined for any $\bar\omega \in \mathscr{P}$ by

$$\mathcal{O}_{\bar\omega} \;=\; \big\{\omega \in \mathscr{P}\,;\, h(a,\omega) = h(a,\bar\omega) \text{ for all } a \in (0,\infty)\big\}\,. \tag{1.12}$$

These subsets of the phase space are the analogues of the manifolds $X_c$ defined by the constraints, and can be considered as a measure-theoretical replacement for the symplectic leaves

$$\mathcal{O}_{\bar\omega}^{\mathrm{SDiff}} \;=\; \big\{\omega \in \mathscr{P}\,;\, \omega = \bar\omega \circ \phi \text{ for some } \phi \in \mathrm{SDiff}\big\} \;\subset\; \mathcal{O}_{\bar\omega}\,,$$

where SDiff denotes the group of area-preserving diffeomorphisms in $\mathbb{R}^2$. In contrast to $\mathcal{O}_{\bar\omega}^{\mathrm{SDiff}}$, the orbit $\mathcal{O}_{\bar\omega}$ does not carry any topological information about $\bar\omega$, since $\omega \in \mathcal{O}_{\bar\omega}$ as soon as $\omega$ is a measure-preserving rearrangement of $\bar\omega$.

(iv) The *Hamiltonian* (or energy functional) $E\colon \mathscr{P} \to \mathbb{R}$, given by

$$E(\omega) \;=\; -\frac{1}{2}\int_{\mathbb{R}^2} \psi(x)\omega(x)\,\mathrm{d}x \;=\; -\frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y|\,\omega(x)\,\omega(y)\,\mathrm{d}x\,\mathrm{d}y\,, \tag{1.13}$$

where $\psi = \Delta^{-1}\omega$ is the stream function defined by

$$\psi(x) \;=\; \frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y|\,\omega(y)\,\mathrm{d}y\,, \qquad x \in \mathbb{R}\,. \tag{1.14}$$

This is an analogue of the function $f$ in the finite-dimensional example. Note that the usual kinetic energy defined by $\frac{1}{2}\int_{\mathbb{R}^2}|u|^2\,\mathrm{d}x$, where $u = \nabla^\perp\psi$, is infinite for $\omega \in \mathscr{P}$. However, both definitions of the energy coincide when $\int_{\mathbb{R}^2}\omega\,\mathrm{d}x = 0$, which is the case for instance if $\omega$ is the difference of two vorticities in $\mathscr{P}$ with the same mass. It is also worth observing that the functional $E$ is not invariant under the scaling transformation $\omega(x) \mapsto \omega^{(\lambda)}(x) := \lambda^2\omega(\lambda x)$ when $M_0 = \int_{\mathbb{R}^2}\omega\,\mathrm{d}x \neq 0$. In fact, one can easily check that

$$E(\omega^{(\lambda)}) \;=\; E(\omega) + \frac{M_0^2}{4\pi}\log\lambda\,, \qquad \text{for all } \lambda > 0\,.$$

(v) The *conserved quantities* induced by Euclidean symmetries. These are the first order moments $M_1, M_2$ and the symmetric second order moment $I$ defined by

$$M_j(\omega) \;=\; \int_{\mathbb{R}^2} x_j\omega(x)\,\mathrm{d}x\,, \qquad j = 1,2\,, \qquad I(\omega) \;=\; \int_{\mathbb{R}^2} |x|^2\omega(x)\,\mathrm{d}x\,. \tag{1.15}$$

Note that $M_1, M_2$ are associated to the translational symmetry, via Noether's theorem, and $I$ to the rotational symmetry.

With these definitions, the Euler equation can be written in the form $\partial_t\omega = \{E(\omega),\omega\}$, where $\{\cdot,\cdot\}$ denotes the Poisson bracket on $\mathscr{P}$, see Section A.5. Any steady state $\bar\omega \in \mathscr{P}$ is a critical point of the Hamiltonian $E$ on the orbit $\mathcal{O}_{\bar\omega}$. Stability can be inferred when the restriction of the energy $E$ to $\mathcal{O}_{\bar\omega}$ has a strict local extremum at $\bar\omega$. In what follows, we focus on the maximizers of the energy, which correspond to radially symmetric vortices.

## 1.3 The constrained maximization of the energy in $\mathscr{P}$

Under our assumptions, it is easy to determine the maximizers of the Hamiltonian $E$ under the constraints given by the functions $h(a,\omega)$ for $a \in (0,\infty)$. Indeed, for any $\omega \in \mathscr{P}$, the orbit $\mathbb{O}_\omega$ contains a unique element $\omega^*$ that is radially symmetric and nonincreasing in the radial direction; this is the *symmetric decreasing rearrangement* of $\omega$ [20]. The Riesz's rearrangement inequality then shows that $E(\omega) \le E(\omega^*)$ for all $\omega \in \mathbb{O}_{\omega^*}$, with equality if and only if $\omega$ is a translate of $\omega^*$, see [10, Lemma 2]. Of course $\omega^*$ is a stationary solution of the Euler equation, which represents a radially symmetric vortex with nonincreasing vorticity profile. Our main focus here will be on the analogue of the quadratic form (1.5) for the steady state $\bar\omega = \omega^*$.

First, the analogue of the Lagrange function (1.7) is

$$E(\omega) - \int_0^\infty \Lambda(a) h(a,\omega)\,\mathrm{d}a \;=\; E(\omega) - \int_0^\infty \Lambda(a)\left(\int_{\mathbb{R}^2} \chi\big(\omega(x) - a\big)\,\mathrm{d}x\right)\mathrm{d}a\,,$$

where the quantities $\Lambda(a)$ for $a \in (0,\infty)$ can be thought of as the Lagrange multipliers. The role of the discrete index $j$ in (1.7) is now played by the continuous parameter $a > 0$. Defining[4]

$$\Phi(s) \;=\; -\int_0^\infty \Lambda(a)\chi(s-a)\,\mathrm{d}a \;=\; -\int_0^s \Lambda(a)\,\mathrm{d}a\,, \qquad s > 0\,, \tag{1.16}$$

we see that the Lagrange function can also be expressed as

$$F(\omega) \;=\; E(\omega) + \int_{\mathbb{R}^2} \Phi(\omega(x))\,\mathrm{d}x\,, \qquad \omega \in \mathscr{P}\,. \tag{1.17}$$

This quantity will be referred to later as the "free energy" of the vorticity distribution $\omega$, a terminology that will be discussed in Section 1.4 below.

Next, the analogue of the stationarity condition (1.4) at $\bar\omega = \omega^*$ is $F'(\bar\omega) = 0$, where the linear form $\eta \mapsto F'(\bar\omega)\eta$ is defined for all $\eta \in T_{\bar\omega}\mathscr{P}$ by

$$F'(\bar\omega)\eta \;=\; \int_{\mathbb{R}^2}\Big(-\bar\psi(x) + \Phi'(\bar\omega(x))\Big)\eta(x)\,\mathrm{d}x\,, \qquad \bar\psi(x) \;=\; \frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y|\,\bar\omega(y)\,\mathrm{d}y\,.$$

Stationarity is thus equivalent to the relation $\bar\psi(x) = \Phi'(\bar\omega(x))$ for all $x \in \mathbb{R}^2$. Finally the analogue of (1.5) is the quadratic form $\eta \mapsto F''(\bar\omega)[\eta,\eta]$, where

$$F''(\bar\omega)[\eta,\eta] \;=\; \int_{\mathbb{R}^2}\left(-\varphi\eta + \Phi''(\bar\omega)\eta^2\right)\mathrm{d}x\,, \qquad \varphi(x) \;=\; \int_{\mathbb{R}^2}\frac{1}{2\pi}\log|x-y|\,\eta(y)\,\mathrm{d}y\,.$$

Using the relation $\nabla\bar\psi(x) = \Phi''(\bar\omega(x))\nabla\bar\omega(x)$, the second variation can be rewritten in the form

$$F''(\bar\omega)[\eta,\eta] \;=\; \int_{\mathbb{R}^2}\Big(-\varphi\eta + \frac{\nabla\bar\psi}{\nabla\bar\omega}\,\eta^2\Big)\,\mathrm{d}x \;=\; 2E(\eta) + \int_{\mathbb{R}^2}\frac{\nabla\bar\psi}{\nabla\bar\omega}\,\eta^2\,\mathrm{d}x\,, \tag{1.18}$$

which is well-known from Arnold's work. Note that the ratio $\frac{\nabla\bar\psi}{\nabla\bar\omega}$ is meaningful only when the vector $\nabla\bar\omega(x)$ is nonzero and collinear with $\nabla\bar\psi(x)$ for almost all $x \in \mathbb{R}^2$. This condition is obviously satisfied for all radially symmetric vortices with strictly decreasing vorticity profile.

## 1.4 Overview of our results

We are now able to describe more precisely the results of this paper. We consider a general family of radially symmetric vortices $\bar\omega \in \mathscr{P}$ with vorticity profile satisfying Hypotheses 2.1 below. Typical examples are the "algebraic vortex" $\bar\omega(x) = (1+|x|^2)^{-\kappa}$, where $\kappa > 1$ is a parameter, and the Oseen vortex for which $\bar\omega(x) = e^{-|x|^2/4}$.

[4] The reason for the minus sign in (1.16) will become clear later.

### 1.4.1 Arnold's quadratic forms with and without constraints

In Section 2, we study in detail the quadratic form (1.18) associated with the second variation of the Lagrange function (1.17) at the steady state $\bar{\omega} \in \mathscr{P}$, paying some attention to the functional-analytic questions. First of all, while we know from the constrained maximization result that the restriction of that form to the tangent space $T_{\bar{\omega}}\mathcal{O}_{\bar{\omega}}$ is negative, it is not clear if this restriction is strictly negative definite, and if so in which function space. Our first main result is Theorem 2.5, where we show that, if two neutral directions corresponding to translational symmetry are disregarded, the restriction to $T_{\bar{\omega}}\mathcal{O}_{\bar{\omega}}$ of the quadratic form (1.18) is indeed strictly negative in an appropriate weighted $L^2$ space. The proof ultimately relies on a variant of the Krein-Rutman theorem.

We next investigate the index of the quadratic form (1.18) on a much larger subspace, corresponding to perturbations $\eta \in T_{\bar{\omega}}\mathscr{P}$ satisfying $\int_{\mathbb{R}^2} \eta(x)\,\mathrm{d}x = 0$. In other words, we relax all constraints given by the Casimir functions (1.10), except for the mass $M_0$ defined in (1.11), which is still supposed to be constant. A priori there is no reason why the form (1.18) should be negative definite in this larger sense, and indeed Theorem 2.8 shows that this is not always the case. More precisely, we show that negativity holds in the large sense if and only if the optimal constant in some weighted Hardy inequality (where the weight function depends on the vorticity profile $\bar{\omega}$) is smaller than 1. While that condition is not easy to check in general, we deduce from Corollary 2.11 that it is fulfilled at least for the Oseen vortex, as well as for the algebraic vortex $\bar{\omega}(x) = (1 + |x|^2)^{-\kappa}$ if $\kappa \geq 2$.

Although the mass constraint is rather natural, one may wonder if, for some vorticity profiles, the quadratic form (1.18) can be negative definite for all perturbations $\eta \in T_{\bar{\omega}}\mathscr{P}$; this question is briefly discussed in Section 2.3. Finally, in Section 2.4, we give a fairly explicit expression of the energy $E(\bar{\omega})$ in terms of the constraints $h(a, \bar{\omega})$ for all $a > 0$, see Proposition 2.16. One obtains in this way an infinite-dimensional analogue of the quantity $M(c_1, \ldots, c_n)$ defined in (1.8). Among other things, we justify our claim that the index of the quadratic form (1.5) is related to the concavity of the function (1.8) (which is a well known fact), and we discuss a similar link in the infinite-dimensional case.

As an aside, we mention here that the stability of radially symmetric vortices for the 2d Euler equations can also be studied using other conserved quantities, such as the second order symmetric moment $I$ defined in (1.15), see e.g. [21, Chapter 3].

### 1.4.2 The global maximizers of the free energy

Let $\bar{\psi}$ be the stream function associated with the radially symmetric vortex $\bar{\omega}$. We have seen that the analogue of the Lagrange function (1.7) is given by the "free energy" (1.17), where the function $\Phi$ is defined, up to an additive constant, by the relation $\bar{\psi}(x) = \Phi'(\bar{\omega}(x))$. The appellation "free energy" is partially justified by a (loose) analogy of formula (1.17) with the classical thermodynamical expression for the free energy

$$F = U - TS\,. \tag{1.19}$$

Here $U$ is the internal energy (of a suitable system), $T$ is the temperature, and $S$ is the entropy. In (1.17), the energy $E$ is analogous to $U$, the integral $\int_{\mathbb{R}^2} \Phi(\omega(x))\,\mathrm{d}x$ is analogous to $S$, and one can argue that it is reasonable to take $T = -1$. Of course, $T$ has nothing to do with the real temperature of the fluid, but should roughly be thought of as the statistical mechanics temperature of our system in the sense of Onsager [24]. We have not attempted to make this connexion rigorous, which would take us in a different direction.

In Section 3, we consider vortices $\bar{\omega}$ which are *global maximizers* of the free energy $F(\omega)$ for all $\omega \in \mathscr{P}$ satisfying $\int_{\mathbb{R}^2} \omega \,\mathrm{d}x = \int_{\mathbb{R}^2} \bar{\omega} \,\mathrm{d}x$. Such equilibria can be expected to have strong stability properties, and may be useful for other purposes too. Using a direct approach, in the sense of the calculus of variations, we prove the existence of global maximizers under fairly general assumptions on the function $\Phi$, see Theorem 3.4. However, we do not have any efficient method to determine if a given vortex $\bar{\omega}$ is a global maximizer or not. A necessary condition is of course that the quadratic form (1.18) be negative on perturbations $\eta$ with zero mean, see Theorem 2.8, but there is no reason to believe that this is sufficient. Numerical evidence indicates that the Oseen vortex is a global maximizer, and so are the algebraic vortices $\bar{\omega}(x) = (1 + |x|^2)^{-\kappa}$ for $\kappa \geq 2$. In the particular case $\kappa = 2$, maximality can be deduced from the logarithmic Hardy-Littlewood-Sobolev inequality

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log \frac{1}{|x-y|}\, \omega(x)\omega(y) \,\mathrm{d}x \,\mathrm{d}y \;\leq\; \frac{1}{2} \int_{\mathbb{R}^2} \omega(x) \log(\omega(x)) + \frac{1 + \log(\pi)}{2}\,, \tag{1.20}$$

which holds for all $\omega \in \mathscr{P}$ with $M_0(\omega) = 1$, see [10]. We mention that (1.20) is related to Onofri's sharp version of the Moser-Trudinger inequality [23].

### 1.4.3 The effect of viscosity — application to Oseen vortices

In Section 4, we consider the stability of the Gaussian vortex under the evolution defined by the Navier-Stokes equation $\partial_t \omega + u \cdot \nabla \omega = \nu \Delta \omega$, where $\nu > 0$ is the viscosity parameter. More precisely, we show that the quadratic form (1.18) can be used to give an alternative proof of the local stability results established in [17]. We believe that a proof relying on the second variation of the energy is of some interest, because the analogue of the form (1.18) can be defined for more complicated vortex structures as well, whereas the simpler approach in [17] may be more difficult to adapt.

The addition of the viscous term results in important new issues: the radial vortices are no longer steady states and the orbits (1.12) are no longer invariant under the evolution, so that much of the geometric picture underlying the Euler equation is destroyed. The first problem is settled by introducing self-similar variables and restricting ourselves to Oseen's vortex, which is a stationary solution of the Navier-Stokes equation in these new coordinates. Thanks to Theorem 2.8 and Corollary 2.11, the quadratic form (1.18) is positive definite for all perturbations with zero mean. This form is invariant under the evolution defined by the linearized Euler equation at the vortex, but not under the Navier-Stokes evolution due to the viscous term and the nonlinearity. The effect of viscosity is measured by a second quadratic form, which happens to have a favorable sign, see Theorem 4.2. We do not know if this is just a lucky coincidence, or if there are deeper reasons behind that. In any event, this nice structure allows us to recover the local stability result of [17], except for a slight difference in the choice of the function space, see Theorem 4.5. Again, we emphasize that the functional setting used in [17] relies in an essential way on the radial symmetry of Oseen's vortex, through the existence of conserved quantities such as the moment $I$ in (1.15), whereas our new approach can, at least in principle, be adapted to more general situations, where other methods do not work.

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## 2 The second variation of the energy

In this section we study the coercivity, on various subspaces, of the quadratic form (1.18) which represents the second variation of the free energy (1.17) at a radially symmetric vortex $\bar{\omega} \in \mathscr{P}$. We assume that $\bar{\omega}(x) = \omega_*(|x|)$ for all $x \in \mathbb{R}^2$, and that the vorticity profile $\omega_* : [0, +\infty) \to \mathbb{R}$ is a $C^2$ function with the following properties:

**Hypotheses 2.1.** *The vorticity profile $\omega_* \in C^2([0, +\infty))$ satisfies*
*1) $\omega_*(0) > 0$, $\omega_*'(0) = 0$, and $\omega_*''(0) < 0$;*
*2) $\omega_*'(r) < 0$ for all $r > 0$, and $\omega_*(r) \to 0$ as $r \to +\infty$;*
*3) there exists $C > 0$ and $\beta > 2$ such that $|\omega_*'(r)| \le C(1 + r)^{-\beta-1}$ for all $r > 0$.*

It follows in particular from 2), 3) that $\omega_*(r) = -\int_r^\infty \omega_*'(s)\,\mathrm{d}s$, so that

$$0 \,<\, \omega_*(r) \,\le\, \frac{C}{(1+r)^\beta} \quad \forall\, r > 0\,, \qquad \text{and} \quad 0 \,<\, \int_0^\infty r\omega_*(r)\,\mathrm{d}r \,<\, \infty\,. \tag{2.1}$$

Let $\bar{\psi}$ be the stream function associated with $\bar{\omega}$ as in (1.14). We have $\bar{\psi}(x) = \psi_*(|x|)$, where the stream profile $\psi_* : [0, +\infty) \to \mathbb{R}$ satisfies

$$\psi_*''(r) + \frac{1}{r}\,\psi_*'(r) \,=\, \omega_*(r)\,, \qquad \text{hence} \quad \psi_*'(r) \,=\, \frac{1}{r}\int_0^r s\omega_*(s)\,\mathrm{d}s\,, \quad \forall\, r > 0\,. \tag{2.2}$$

We introduce the weight function $A : [0, +\infty) \to \mathbb{R}$ defined by $A(0) = -\omega_*(0)/(2\omega_*''(0))$ and

$$A(r) \,=\, -\frac{\psi_*'(r)}{\omega_*'(r)} \,=\, -\frac{1}{r\omega_*'(r)}\int_0^r s\omega_*(s)\,\mathrm{d}s\,, \quad r > 0\,. \tag{2.3}$$

Hypotheses 2.1 ensure that $A \in C^0([0, +\infty)) \cap C^1((0, +\infty))$. Moreover, there exists a constant $C > 0$ such that $A(r) \ge C(1 + r)^\beta$ for all $r \ge 0$.

Let $\mathscr{A} : \mathbb{R}^2 \to (0, \infty)$ be the radially symmetric extension of $A$ to $\mathbb{R}^2$, namely $\mathscr{A}(x) = A(|x|)$ for all $x \in \mathbb{R}^2$. We introduce the weighted $L^2$ space $X$ defined by

$$X \,=\, \Bigl\{\omega \in L^2(\mathbb{R}^2)\ ;\ \|\omega\|_X^2 := \int_{\mathbb{R}^2} \mathscr{A}(x)|\omega(x)|^2\,\mathrm{d}x < \infty\Bigr\}\,, \tag{2.4}$$

so that $\omega \in X$ if and only if $\mathscr{A}^{1/2}\omega \in L^2(\mathbb{R}^2)$. Our assumptions ensure that $\mathscr{A}^{-1} \in L^1(\mathbb{R}^2)$, and using Hölder's inequality we easily deduce that $X \hookrightarrow L^1(\mathbb{R}^2)$. We also consider the closed subspaces $X_1 \subset X_0 \subset X$ defined by

$$\begin{aligned} X_0 \,&=\, \Bigl\{\omega \in X\ ;\ \int_{\mathbb{R}^2} \omega(x)\,\mathrm{d}x = 0\Bigr\}\,, \\ X_1 \,&=\, \Bigl\{\omega \in X_0\ ;\ \int_{\mathbb{R}^2} \frac{x_j}{|x|}\,\omega(x)\,\mathrm{d}x = 0 \ \text{ for }\ j = 1, 2\Bigr\}\,. \end{aligned} \tag{2.5}$$

We observe that, for any $\omega \in X$, the energy $E(\omega)$ introduced in (1.13) is well defined. This a consequence of the following classical estimate, whose proof is reproduced in Section A.1 for the reader's convenience.

**Proposition 2.2.** *Assume that $\omega \in L^1(\mathbb{R}^2)$ satisfies*

$$\int_{\mathbb{R}^2} |\omega(x)|\,\log(1 + |x|)\,\mathrm{d}x \,<\, \infty\,, \quad \textit{and} \quad \int_{\mathbb{R}^2} |\omega(x)|\,\log\bigl(1 + |\omega(x)|\bigr)\,\mathrm{d}x \,<\, \infty\,. \tag{2.6}$$

*Then the last member in* (1.13) *is well defined, and the energy* $E(\omega)$ *satisfies the bound*

$$|E(\omega)| \,\le\, C\|\omega\|_{L^1}\biggl(\int_{\mathbb{R}^2}|\omega(x)|\,\log(2+|x|)\,\mathrm{d}x + \int_{\mathbb{R}^2}|\omega(x)|\,\log_+\frac{|\omega(x)|}{\|\omega\|_{L^1}}\,\mathrm{d}x\biggr)\,, \tag{2.7}$$

*where* $\log_+(a) = \max\bigl(\log(a),0\bigr)$. *If moreover* $\int_{\mathbb{R}^2}\omega(x)\,\mathrm{d}x = 0$, *then* $E(\omega) = \frac{1}{2}\int_{\mathbb{R}^2}|u|^2\,\mathrm{d}x$ *where*

$$u(x) \,=\, \nabla^\perp\psi(x) \,=\, \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(x-y)^\perp}{|x-y|^2}\,\omega(y)\,\mathrm{d}y\,, \qquad x\in\mathbb{R}^2\,. \tag{2.8}$$

Since any $\omega \in X$ obviously satisfies (2.6), we can consider the quadratic form $J$ on $X$ defined by $J(\omega) = \frac{1}{2}\,\|\omega\|_X^2 - E(\omega)$, or explicitly

$$J(\omega) \,=\, \frac{1}{2}\int_{\mathbb{R}^2}\mathcal{A}(x)\omega(x)^2\,\mathrm{d}x \,+\, \frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log|x-y|\,\omega(x)\omega(y)\,\mathrm{d}x\,\mathrm{d}y\,, \quad \omega\in X\,. \tag{2.9}$$

In the particular case where $\omega \in X_0$, namely when $\omega$ has zero average over $\mathbb{R}^2$, Proposition 2.2 gives the alternative expression

$$J(\omega) \,=\, \frac{1}{2}\int_{\mathbb{R}^2}\Bigl(\mathcal{A}(x)\omega(x)^2 - |u(x)|^2\Bigr)\,\mathrm{d}x\,, \qquad \omega\in X_0\,, \tag{2.10}$$

where $u$ is the velocity field associated with $\omega$ via the Biot-Savart formula (2.8). In view of (1.18) and (2.3), we have $J = -\frac{1}{2}F''(\bar\omega)$, where $F''(\bar\omega)$ is the second variation of the free energy (1.17) at the equilibrium $\bar\omega$. It is clear that $X$ is the largest function space on which this second variation is well defined.

Our main goal in this section is to study the positivity and coercivity properties of the quadratic form $J$ on the spaces $X$, $X_0$, and $X_1$ defined in (2.4), (2.5). To formulate our results, it is useful to decompose $X = X_{\mathrm{rs}} \oplus X_{\mathrm{rs}}^\perp$, where

$$X_{\mathrm{rs}} \,=\, \bigl\{\omega\in X\,;\,\omega \text{ is radially symmetric}\bigr\}\,, \tag{2.11}$$

and $X_{\mathrm{rs}}^\perp$ is the orthogonal complement of $X_{\mathrm{rs}}$ in the Hilbert space $X$. Referring to the geometric picture of Section 1.2, we consider $X_{\mathrm{rs}}^\perp$ as the *tangent space to the orbit* $\mathcal{O}_{\bar\omega}$ at $\bar\omega$. This interpretation can be formally justified as follows: if $\bar\omega \in X$ is smooth, the tangent space $T_{\bar\omega}\mathcal{O}_{\bar\omega}$ is spanned by vorticities of the form $v\cdot\nabla\bar\omega$, where $v$ is a (smooth and localized) divergence-free vector field, and using polar coordinates as in Section 2.1 below one verifies that such vorticities are indeed orthogonal in $X$ to all radially symmetric functions. A contrario, since there is a one-to-one correspondence in $\mathcal{P}$ between orbits and symmetric decreasing rearrangements, it is clear that any radially symmetric perturbation of the equilibrium $\bar\omega$ is transverse to the orbit $\mathcal{O}_{\bar\omega}$.

It is easy to verify that $J(\omega_1+\omega_2) = J(\omega_1)+J(\omega_2)$ when $\omega_1\in X_{\mathrm{rs}}$ and $\omega_2\in X_{\mathrm{rs}}^\perp$, so that the restrictions of $J$ to $X_{\mathrm{rs}}$ and $X_{\mathrm{rs}}^\perp$ can be studied separately. We first consider the tangent space $X_{\mathrm{rs}}^\perp$ in Section 2.1, and postpone the study of radially symmetric perturbations (with zero or nonzero mass) to Sections 2.2 and 2.3.

**Remark 2.3.** *Differentiating the first equality in* (2.2), *we see that the function* $\phi = \psi_*'$ *satisfies*

$$\bigl(L_0\phi\bigr)(r) \,:=\, -\phi''(r) - \frac{1}{r}\,\phi'(r) + \frac{1}{r^2}\,\phi(r) \,=\, \frac{1}{A(r)}\,\phi(r)\,, \qquad r>0\,, \tag{2.12}$$

*where* $A(r) \ge C(1+r)^\beta$. *Since* $\phi > 0$, *Sturm-Liouville theory asserts that* $\mu = 1$ *is the lowest eigenvalue of the (generalized) eigenvalue problem* $L_0\phi = \mu A^{-1}\phi$ *on* $\mathbb{R}_+$, *with boundary conditions* $\phi(0) = \phi(+\infty) = 0$, *see [11, 18]. This observation will be used later.*

**Remark 2.4.** *Hypotheses 2.1 are sufficient for our results to hold, but can be relaxed in several ways. In particular, we can consider vortices that are not smooth at the origin, but the assumption that $\omega_*'(r) < 0$ for all $r > 0$ seems essential. This excludes vortices with compact support from our considerations, but as our motivation comes from applications to the Navier-Stokes equations, hypotheses 2.1 are good enough for our purposes here. Of course, extensions of the theory that would include compactly supported vortices might be relevant in other situations and can probably be constructed, although they may require additional work.*

## 2.1 Positivity of the quadratic form $J$ on $X_{\mathrm{rs}}^{\perp}$

**Theorem 2.5.** *Under Hypotheses 2.1, the quadratic form $J$ defined by* (2.10) *is nonnegative on the space $X_{\mathrm{rs}}^{\perp} \subset X_0$. Moreover, there exists a constant $\gamma > 0$ such that*

$$J(\omega) \,\geq\, \frac{\gamma}{2}\int_{\mathbb{R}^2} \mathcal{A}(x)\omega(x)^2 \,\mathrm{d}x\,, \qquad \textit{for all } \omega \in X_{\mathrm{rs}}^{\perp} \cap X_1\,. \tag{2.13}$$

**Proof.** We introduce polar coordinates $(r, \theta)$ in $\mathbb{R}^2$, and given any $\omega \in X_{\mathrm{rs}}^{\perp}$ we use the Fourier decomposition

$$\omega\bigl(r\cos(\theta), r\sin(\theta)\bigr) \,=\, \sum_{k\neq 0} \omega_k(r)\, e^{ik\theta}\,, \qquad r > 0\,, \quad \theta \in \mathbb{R}/(2\pi\mathbb{Z})\,, \tag{2.14}$$

where the sum runs over all nonzero integers $k \in \mathbb{Z} \setminus \{0\}$. By Parseval's relation we have

$$\begin{aligned} \int_{\mathbb{R}^2} \mathcal{A}(x)\omega(x)^2 \,\mathrm{d}x \,&=\, 2\pi \sum_{k\neq 0} \int_0^\infty A(r)\,|\omega_k(r)|^2\, r\,\mathrm{d}r\,, \\ \int_{\mathbb{R}^2} |u(x)|^2 \,\mathrm{d}x \,=\, \int_{\mathbb{R}^2} \bigl(-\Delta^{-1}\omega\bigr)(x)\,\omega(x)\,\mathrm{d}x \,&=\, 2\pi \sum_{k\neq 0} \int_0^\infty B_k[\omega_k](r)\,\overline{\omega_k}(r)\, r\,\mathrm{d}r\,, \end{aligned} \tag{2.15}$$

where $B_k$ is the integral operator on the half-line $\mathbb{R}_+$ defined by the formula

$$\bigl(B_k[f]\bigr)(r) \,=\, \frac{1}{2|k|}\int_0^\infty \min\Bigl(\frac{r}{s}, \frac{s}{r}\Bigr)^{|k|} f(s)\, s\,\mathrm{d}s\,, \qquad r > 0\,. \tag{2.16}$$

Note that $g = B_k[f]$ is the unique solution of the ODE

$$-g''(r) - \frac{1}{r}\,g'(r) + \frac{k^2}{r^2}\,g(r) \,=\, f(r)\,, \qquad r > 0\,, \tag{2.17}$$

which is regular at the origin and converges to zero at infinity.

In view of (2.15), the proof of Theorem 2.5 reduces to the study of the one-dimensional inequality

$$\int_0^\infty \bigl(B_k[f]\bigr)(r)\,\overline{f}(r)\, r\,\mathrm{d}r \,\leq\, C_k \int_0^\infty A(r)|f(r)|^2\, r\,\mathrm{d}r\,, \tag{2.18}$$

which depends on the angular Fourier parameter $k \in \mathbb{Z}\setminus\{0\}$. More precisely, the quadratic form $J$ is nonnegative on $X_{\mathrm{rs}}^{\perp}$ if and only if, for all $k \neq 0$, inequality (2.18) holds with some constant $C_k \leq 1$. In addition, we have the lower bound (2.13) on the subspace $X_{\mathrm{rs}}^{\perp} \cap X_1$ if and only if inequality (2.18) holds with a better constant $C_k \leq 1 - \gamma$ for all $k \neq 0$, assuming when $|k| = 1$ that $f$ satisfies the additional condition

$$\int_0^\infty f(r)\, r\,\mathrm{d}r \,=\, 0\,. \tag{2.19}$$

It remains to establish inequality (2.18) for all $k \in \mathbb{Z} \setminus \{0\}$. We obviously have the pointwise bound $|(B_k[f])(r)| \le (B_k[|f|])(r)$, so that we can restrict ourselves to nonnegative functions $f$. Moreover the operator $B_k$ preserves positivity, and an inspection of the formula (2.16) reveals that $0 \le B_k[f] \le |k|^{-1} B_1[f]$ if $f \ge 0$. As a consequence, to show that $J$ is nonnegative on $X_{\mathrm{rs}}^{\perp}$, it is sufficient to prove inequality (2.18) in the particular case where $|k| = 1$ and $f \ge 0$. Setting $h = A^{1/2} f$, we write that inequality in the equivalent form

$$\int_0^\infty \bigl(\widetilde{B}_1[h]\bigr)(r)\, h(r)\, r \,\mathrm{d}r \;\le\; C_1 \int_0^\infty h(r)^2\, r\,\mathrm{d}r\,, \tag{2.20}$$

where $\widetilde{B}_1[h] = A^{-1/2} B_1[A^{-1/2} h]$. The following assertions play a crucial role in our argument:

**Claim 1 :** The operator $\widetilde{B}_1$ is *selfadjoint and compact* in the (real) space $Y = L^2(\mathbb{R}_+, r\,\mathrm{d}r)$. Indeed, take $h \in Y$ with $\|h\|_Y \le 1$, and denote $f = A^{-1/2} h$, $g = B_1[f] = A^{1/2} \widetilde{B}_1[h]$. Applying (2.16) with $|k| = 1$, we see that

$$g(r) \;=\; \frac{1}{2r} \int_0^r A(s)^{-1/2} h(s)\, s^2 \,\mathrm{d}s \;+\; \frac{r}{2} \int_r^\infty A(s)^{-1/2} h(s)\,\mathrm{d}s\,, \qquad r > 0\,,$$

and using Hölder's inequality we deduce

$$|g(r)| \;\le\; \left\{ \frac{1}{2r} \Bigl( \int_0^r A(s)^{-1}\, s^3 \,\mathrm{d}s \Bigr)^{1/2} + \frac{r}{2} \Bigl( \int_r^\infty A(s)^{-1}\, s^{-1} \,\mathrm{d}s \Bigr)^{1/2} \right\} \|h\|_Y\,. \tag{2.21}$$

As $A(r) \ge C(1+r)^\beta$ with $\beta > 2$, the right-hand side of (2.21) is uniformly bounded, so that $\|g\|_{L^\infty} \le C$ for some universal constant $C$. It also follows from (2.21) that $g(r) \to 0$ as $r \to 0$ and $r \to +\infty$. On the other hand, since $g$ satisfies the ODE (2.17) with $k = 1$ and $f = A^{-1/2} h$, a standard energy estimate yields the bound

$$\int_0^\infty \Bigl( g'(r)^2 + \frac{g(r)^2}{r^2} \Bigr) r \,\mathrm{d}r \;=\; \int_0^\infty g(r) A(r)^{-1/2} h(r)\, r\,\mathrm{d}r \;\le\; \|g\|_{L^\infty} \|A^{-1/2}\|_Y \|h\|_Y \;\le\; C\,. \tag{2.22}$$

In view of (2.21) and (2.22), the Fréchet-Kolmogorov theorem [25, Thm XIII.66] implies that the function $\widetilde{B}_1[h] = A^{-1/2} g$ lies in a compact set of $Y$, so that the operator $\widetilde{B}_1$ is compact. To prove that $\widetilde{B}_1$ is selfadjoint, we take $h_1, h_2 \in Y$ and observe that

$$\int_0^\infty \bigl(\widetilde{B}_1[h_1]\bigr)(r)\, h_2(r)\, r\,\mathrm{d}r \;=\; \int_0^\infty \Bigl( g_1'(r) g_2'(r) + \frac{g_1(r) g_2(r)}{r^2} \Bigr) r\,\mathrm{d}r\,,$$

where $g_j = B_1[A^{-1/2} h_j]$ for $j = 1, 2$. This expression is clearly a symmetric function of $(h_1, h_2)$.

**Claim 2 :** The *spectral radius* of $\widetilde{B}_1$ is equal to 1, and $\lambda = 1$ is a *simple eigenvalue* of $\widetilde{B}_1$. To see that, we first observe that $\lambda = 1$ is an eigenvalue of $\widetilde{B}_1$ with a positive eigenfunction. Indeed, using (2.2), it is straightforward to verify that the function $g = \psi_*'$ satisfies the ODE (2.17) with $k = 1$ and $f = -\omega_*'$. This shows that $B_1[-\omega_*'] = \psi_*'$, hence defining $h = A^{-1/2} \psi_*' = -A^{1/2} \omega_*'$ we conclude that $\widetilde{B}_1[h] = h$. On the other hand, assume that $\lambda > 0$ is an eigenvalue of $\widetilde{B}_1$, with eigenfunction $h \in Y$. Defining $f = A^{-1/2} h$, we see that $B_1[f] = \lambda A f$, so that the function $g = B_1[f]$ satisfies the generalized eigenvalue problem

$$-g''(r) - \frac{1}{r}\, g'(r) + \frac{1}{r^2}\, g(r) \;=\; \mu\, \frac{g(r)}{A(r)}\,, \qquad r > 0\,, \tag{2.23}$$

with the boundary conditions $g(0) = g(+\infty) = 0$, where $\mu = 1/\lambda$, We already observed that $\mu = 1$ is the lowest eigenvalue of (2.23), see Remark 2.3. It follows that $\lambda = 1$ is the largest eigenvalue of the integral operator $\widetilde{B}_1$, whose spectral radius is therefore equal to 1. The argument above also shows that all positive eigenvalues of $\widetilde{B}_1$ are simple, because (2.23) is a second-order differential equation.

It is now a simple task to conclude the proof of Theorem 2.5. Claims 1 and 2 imply the validity of inequality (2.20) with $C_1 = 1$. We deduce that (2.18) holds for $|k| = 1$ with $C_k = 1$, and (since $B_k \le |k|^{-1}B_1$) for $|k| \ge 2$ with $C_k \le 1/|k|$. This shows that the quadratic form $J$ is nonnegative on $X_{\rm rs}^\perp$. On the other hand, if we assume that $\omega \in X_{\rm rs}^\perp \cap X_1$, the function $f = \omega_{\pm 1}$ satisfies condition (2.19), which means that $h = A^{1/2}f$ is orthogonal in $Y$ to the one-dimensional subspace $Y_0$ spanned by the positive function $\chi = A^{-1/2}$. It is clear that $Y_0^\perp$ does not contain any positive function, and in particular does not include the principal eigenfunction $h_0 = -A^{1/2}\omega_*'$ of the operator $\widetilde{B}_1$. So, applying Lemma 4.7 and Remark 4.8 below, we deduce that $\mathbf{1} - \widetilde{B}_1 > 0$ on $Y_0^\perp$, which means that inequality (2.20) holds on $Y_0^\perp$ with some constant $C_1' < 1$. Taking into account the other values of $k$, for which $C_k \le 1/|k| \le 1/2$, we conclude that estimate (2.13) holds with $\gamma = \min(1/2, 1{-}C_1')$. $\square$

**Remark 2.6.** *The Krein-Rutman theorem [12, Thm 19.2] asserts that the spectral radius of the compact and positivity-preserving operator $\widetilde{B}_1$ is an eigenvalue with positive eigenfunction. However, since the cone of positive functions has empty interior in $Y$, we cannot apply Theorem 19.3 in [12] to conclude that $\widetilde{B}_1$ has a* unique *eigenvalue with positive eigenfunction, which is thus equal to the spectral radius. For this reason, we prefer invoking Sturm-Liouville theory to prove that 1 is the largest eigenvalue of $\widetilde{B}_1$.*

**Remark 2.7.** *If $\beta > 4$ in Hypotheses 2.1, the conclusion of Theorem 2.5 remains valid, with the same proof, if the subspace $X_1$ is replaced by*

$$\mathcal{X}_1 \,=\, \Bigl\{\omega \in X_0\;;\; \int_{\mathbb{R}^2} x_j\,\omega(x)\,\mathrm{d}x = 0 \;\textit{ for }\; j = 1,2\Bigr\}\,. \tag{2.24}$$

*This possibility will be used in Section 4.*

## 2.2 Positivity of the quadratic form $J$ on $X_{\rm rs} \cap X_0$

The quadratic form $J$ is not necessarily positive when considered on the subspace $X_{\rm rs} \cap X_0$, which consists of radially symmetric functions with zero mean. This question is related to the optimal constant in the weighted Hardy inequality

$$\int_0^\infty f(r)^2\,\frac{\mathrm{d}r}{r} \,\le\, C_H \int_0^\infty A(r)f'(r)^2\,\frac{\mathrm{d}r}{r}\,, \tag{2.25}$$

where $f : [0,+\infty) \to \mathbb{R}$ is an absolutely continuous function with $f(0) = f(+\infty) = 0$. Weighted Hardy inequalities are extensively studied in the literature, see e.g. [22, Section 1.3.2]. In particular, it is known that (2.25) holds for *some* constant $C_H > 0$ if and only if the positive function $A$ satisfies

$$\limsup_{r\to 0}\Bigl(\log\frac{1}{r}\Bigr)\int_0^r \frac{s}{A(s)}\,\mathrm{d}s \,<\, \infty\,, \quad \text{and} \quad \limsup_{r\to+\infty}\,\log(r)\int_r^\infty \frac{s}{A(s)}\,\mathrm{d}s \,<\, \infty\,. \tag{2.26}$$

Both conditions in (2.26) are fulfilled in our case, since $A(r) \ge C(1+r)^\beta$ for some $\beta > 2$.

**Theorem 2.8.** *Under Hypotheses 2.1, the quadratic form $J$ defined by* (2.10) *is coercive on $X_{\rm rs} \cap X_0$ if and only if Hardy's inequality* (2.25) *holds for some $C_H < 1$. In that case we have*

$$J(\omega) \,\ge\, \frac{\gamma}{2}\int_{\mathbb{R}^2} \mathcal{A}(x)\omega(x)^2\,\mathrm{d}x\,, \qquad \textit{for all } \omega \in X_{\rm rs} \cap X_0\,, \tag{2.27}$$

*where $\gamma = 1 - C_H$.*

**Proof.** Given $\omega \in X_{\rm rs} \cap X_0$, we write $\omega(x) = \omega_0(|x|)$ and we consider the stream function $\psi_0$ defined (up to an irrelevant additive constant) by

$$\psi_0'(r) \,=\, \frac{1}{r}\int_0^r s\omega_0(s)\,\mathrm{d}s \,=\, -\frac{1}{r}\int_r^\infty s\omega_0(s)\,\mathrm{d}s\,, \qquad r > 0\,.$$

Defining $f(r) = r\psi_0'(r)$, we see that $f$ is absolutely continuous on $\mathbb{R}_+$ with $f(0) = f(+\infty) = 0$. Moreover we have $\omega_0(r) = f'(r)/r$ and $u_0(r) := \psi_0'(r) = f(r)/r$ by construction. Finally the assumption that $\omega_0 \in X_{\rm rs} \cap X_0$ ensures that $A^{1/2}\omega_0$ and $u_0$ belong to the space $Y = L^2(\mathbb{R}_+, r\,\mathrm{d}r)$. We thus have

$$J(\omega) \,=\, \pi\int_0^\infty \Bigl(A(r)\omega_0(r)^2 - u_0(r)^2\Bigr)\,r\,\mathrm{d}r \,=\, \pi\int_0^\infty \Bigl(A(r)f'(r)^2 - f(r)^2\Bigr)\frac{\mathrm{d}r}{r}\,, \tag{2.28}$$

and using (2.25) we conclude that (2.27) holds with $\gamma = 1 - C_H$. This proves that the quadratic form $J$ is coercive on $X_{\rm rs} \cap X_0$ if $C_H < 1$. Conversely, if (2.27) holds for some $\gamma > 0$, it follows from (2.28) that inequality (2.25) is valid with $C_H = 1 - \gamma$. $\square$

As is well known, the optimal constant in Hardy's inequality (2.25) is related to the lowest eigenvalue of a selfadjoint operator. A convenient way of seeing this is to apply the change of variables $r = e^x$, $h(x) = f(e^x)$, $B(x) = e^{-2x}A(e^x)$, which transforms (2.25) into the equivalent inequality

$$\int_{\mathbb{R}} h(x)^2\,\mathrm{d}x \,\le\, C_H\int_{\mathbb{R}} B(x)h'(x)^2\,\mathrm{d}x\,. \tag{2.29}$$

The integral in the right-hand side of (2.29) defines a closed quadratic form on the Hilbert space $H = L^2(\mathbb{R})$, with dense domain $D = \{h \in H\,;\, B^{1/2}h' \in H\}$. Let

$$\mathbb{B} : D(\mathbb{B}) \longrightarrow H\,, \qquad h \longmapsto -\partial_x(B(x)\partial_x h)$$

be the selfadjoint operator in $H$ associated with the quadratic form (2.29) by Friedrich's representation theorem [19]. Since $B(x) > 0$ for all $x \in \mathbb{R}$ we know that $\mathbb{B}$ is positive, and using the fact that $x^2B(x)^{-1} \to 0$ as $|x| \to \infty$ it is easy to verify that $\mathbb{B}$ has compact resolvent in $H$, hence purely discrete spectrum. The optimal constant in $C_H$ in (2.29) is precisely the inverse of the lowest eigenvalue of $\mathbb{B}$:

$$C_H \,=\, \max\bigl\{\lambda^{-1}\,;\, \lambda \in \mathrm{spec}(\mathbb{B})\bigr\}\,. \tag{2.30}$$

By Sturm-Louville's theory, if $\mu = C_H^{-1}$ is the lowest eigenvalue of $\mathbb{B}$, there exists a positive eigenfunction $h \in D(\mathbb{B})$ such that $\mathbb{B}h = \mu h$. Setting $h(x) = f(e^x)$, we see that $f$ is a positive solution of the ODE

$$-\partial_r\biggl(\frac{A(r)}{r}\,\partial_r f(r)\biggr) \,=\, \mu\,\frac{f(r)}{r}\,, \qquad r > 0\,, \tag{2.31}$$

satisfying the boundary conditions $f(0) = f(+\infty) = 0$. Moreover $\int_0^\infty A(r)f'(r)^2\,\mathrm{d}r/r < \infty$ by construction. It is not easy to guess from (2.31) whether $\mu$ is smaller or larger than 1, but under additional assumptions on the vortex profile it is possible to make another change of variables which puts (2.31) into a form that allows for a comparison with (2.12).

**Lemma 2.9.** *If the function $A$ in* (2.3) *satisfies*

$$A \,\in\, C^2([0,+\infty))\,, \qquad \textit{and} \qquad \sup_{r\ge 1}\biggl(\frac{A(r)}{r^2} + \frac{A'(r)^2}{r^2A(r)}\biggr)\int_r^\infty \frac{s}{A(s)}\,\mathrm{d}s \,<\, \infty\,, \tag{2.32}$$

*then the function $g : [0,+\infty) \to \mathbb{R}$ defined by $g(r) = A(r)^{1/2} f(r)/r$ is a solution of the ODE*

$$-g''(r) - \frac{1}{r}\, g'(r) + \frac{1}{r^2}\, g(r) + V(r) g(r) \;=\; \frac{\mu}{A(r)}\, g(r)\,, \tag{2.33}$$

*with boundary conditions $g(0) = g(+\infty) = 0$, where*

$$V(r) \;=\; \chi''(r) - \frac{1}{r}\,\chi'(r) + \chi'(r)^2\,, \qquad \textit{and} \qquad \chi(r) \;=\; \frac{1}{2}\,\log(A(r))\,. \tag{2.34}$$

**Proof.** Since $f$ satisfies (2.31), a direct calculation shows that $g(r) := A(r)^{1/2} f(r)/r$ is a solution of (2.33), where the potential $V$ is defined by (2.34). As for the boundary conditions, we recall that $\int_0^\infty A(r) f'(r)^2 \,\mathrm{d}r/r < \infty$, hence $\int_0^\infty |f'(r)| \,\mathrm{d}r < \infty$. As $f(r) = \int_0^r f'(s)\,\mathrm{d}s$, we have

$$\frac{|f(r)|}{r} \;\le\; \frac{1}{r}\Big(\int_0^r \frac{s}{A(s)}\,\mathrm{d}s\Big)^{1/2}\Big(\int_0^r A(s) f'(s)^2\,\frac{\mathrm{d}s}{s}\Big)^{1/2} \;\xrightarrow[r\to 0]{}\; 0\,,$$

which shows that $g(r) \to 0$ as $r \to 0$. Similarly, since $f(r) = -\int_r^\infty f'(s)\,\mathrm{d}s$, we have

$$|g(r)| \;\le\; \frac{A(r)^{1/2}}{r}\Big(\int_r^\infty \frac{s}{A(s)}\,\mathrm{d}s\Big)^{1/2}\Big(\int_r^\infty A(s) f'(s)^2\,\frac{\mathrm{d}s}{s}\Big)^{1/2} \;\xrightarrow[r\to +\infty]{}\; 0\,,$$

thanks to (2.32). This concludes the proof. □

**Remark 2.10.** *The same arguments show that $r^2 g'(r) \to 0$ as $r \to 0$ and $g'(r) \to 0$ as $r \to +\infty$, at least along appropriate sequences.*

Let $L$ be the differential operator defined by

$$L \;=\; L_0 + V \;=\; -\partial_r^2 - \frac{1}{r}\,\partial_r + \frac{1}{r^2} + V(r)\,, \tag{2.35}$$

where $L_0$ was introduced in (2.12). We know from (2.33) that $Lg = \mu A^{-1} g$, where $\mu = C_H^{-1}$ and $g$ is the positive function defined in Lemma 2.9. On the other hand, we observed in Remark 2.3 that $L_0\phi = A^{-1}\phi$, where $\phi = \psi_*'$ is also a positive function vanishing at the origin and at infinity. Using Sturm-Liouville's theory, we easily deduce the following useful criterion:

**Corollary 2.11.** *Under assumptions* (2.32)*, if the function $V$ defined by* (2.34) *does not change sign, the optimal constant in Hardy's inequality* (2.25) *satisfies $C_H \le 1$ if $V \ge 0$, and $C_H \ge 1$ if $V \le 0$; moreover $C_H = 1$ only if $V$ is identically zero.*

**Proof.** With the notations above, we have $L_0\phi - A^{-1}\phi = 0$ and

$$L_0 g - A^{-1} g \;=\; Lg - \big(A^{-1} + V\big) g \;=\; \mathcal{R}\,, \quad \text{where} \quad \mathcal{R} \;=\; (\mu - 1) A^{-1} g - V g\,. \tag{2.36}$$

Since $r\mathcal{R}\phi = r\big(\phi(L_0 g) - g(L_0\phi)\big) = \frac{\mathrm{d}}{\mathrm{d}r}\big(r(\phi' g - g'\phi)\big)$, we have for $r_1 > r_0 > 0$ the identity

$$\int_{r_0}^{r_1} \mathcal{R}(r)\phi(r) r \,\mathrm{d}r \;=\; r\Big(\phi'(r) g(r) - g'(r)\phi(r)\Big)\Big|_{r=r_0}^{r=r_1}\,. \tag{2.37}$$

Now, we let $r_0$ tend to 0 and $r_1$ to $+\infty$ along appropriate sequences, in such a way that the right-hand side of (2.37) converges to zero. This possible, because we know that $\phi(r) = \mathcal{O}(r)$ and $\phi'(r) = \mathcal{O}(1)$ as $r \to 0$, while $\phi(r) = \mathcal{O}(1/r)$ and $\phi'(r) = \mathcal{O}(1/r^2)$ as $r \to +\infty$; moreover the behavior of $g$ in these limits is given in Lemma 2.9 and Remark 2.10. We thus deduce from (2.37) that $\int_0^\infty \mathcal{R}\phi r \,\mathrm{d}r = 0$, which is impossible if the function $\mathcal{R}$ has a constant sign and is not identically zero. So, if $V$ does not change sign, we must have $\mu \ge 1$ if $V \ge 0$ and $\mu \le 1$ if $V \le 0$; moreover $\mu = 1$ is possible only if $V \equiv 0$. Since $\mu = C_H^{-1}$, this gives the desired conclusion. □

**Remark 2.12.** *As is easily verified, the optimal constant $C_H$ in Hardy's inequality* (2.25) *is unchanged if the function $A(r)$ is replaced by $\lambda^{-2}A(\lambda r)$ for some $\lambda > 0$. This corresponds to a rescaling of the vortex profile $\omega_*$.*

We now give two important examples where the sign of $C_H - 1$ can be determined.

**Example 1 : Algebraic vortex.** Given $\kappa > 1$, we define

$$\omega_*(r) \,=\, \frac{1}{(1+r^2)^\kappa}\,, \qquad \psi_*'(r) \,=\, \frac{1}{2(\kappa-1)r}\Bigl(1 - \frac{1}{(1+r^2)^{\kappa-1}}\Bigr)\,. \tag{2.38}$$

We have

$$A(r) \,=\, -\frac{\psi_*'(r)}{\omega_*'(r)} \,=\, \frac{1}{4\kappa(\kappa-1)r^2}\Bigl((1+r^2)^{\kappa+1} - (1+r^2)^2\Bigr)\,.$$

When $\kappa = 2$ (Kaufmann-Scully vortex), inequality (2.25) holds with optimal constant $C_H = 1$, and is saturated for $f(r) = r^2/(1+r^2)^2$. Indeed, it is easy to verify that $A(r) = (1+r^2)^2/8$ and $V(r) = 0$ in that particular case. Taking $g(r) = r/(1+r^2)$, a direct calculation shows that $Lg = A^{-1}g$, so that $C_H = 1$.

If $\kappa > 2$, we prove in Section A.2 that the potential $V$ is positive, so that $C_H < 1$ by Corollay 2.11. Finally, if $1 < \kappa < 2$, the potential $V$ is negative, implying that $C_H > 1$. Summarizing, for the family of algebraic vortices (2.38), the quadratic form $J$ is coercive on $X_{\mathrm{rs}} \cap X_0$ if and only if $\kappa > 2$.

**Example 2 : Gaussian vortex.** We next consider the Oseen vortex given by

$$\omega_*(r) \,=\, e^{-r^2/4}\,, \qquad \psi_*'(r) \,=\, \frac{2}{r}\Bigl(1 - e^{-r^2/4}\Bigr)\,, \qquad A(r) \,=\, \frac{4}{r^2}\Bigl(e^{r^2/4} - 1\Bigr)\,. \tag{2.39}$$

In that case too, the potential $V$ defined in (2.34) is positive, see Section A.2. By Corollary 2.11, we conclude that $C_H < 1$, so that the quadratic form $J$ is coercive on $X_{\mathrm{rs}} \cap X_0$. A numerical calculation gives the approximate value $C_H \approx 0.57$, so that $\gamma \approx 0.43$.

**Remark 2.13.** *In a finite-dimensional situation, one can use statements such as Theorems 2.5 and 2.8 for showing the nonlinear Lyapunov stability of the corresponding steady solution, at least if the smoothness class of the relevant objects is $C^2$. More precisely, if a flow $\dot{x} = b(x)$ on a finite-dimensional manifold preserves a $C^2$ function $f$ which attains a non-degenerate local maximum at $\bar{x}$, then the sets $\{f(x) > f(\bar{x}) - \varepsilon\}$ are invariant under the flow and for small $\varepsilon$ are well approximated by the small balls given by the quadratic form $-\frac{1}{2}f''(\bar{x})[x-\bar{x}, x-\bar{x}]$. A standard way to see this is to write $f(x) > f(\bar{x}) - \varepsilon$ as*

$$-\frac{1}{2}f''(\bar{x})[x-\bar{x},x-\bar{x}] - \int_0^1 (1-t)\bigl(f''((1-t)\bar{x}+tx) - f''(\bar{x})\bigr)[x-\bar{x},x-\bar{x}]\,\mathrm{d}t \,<\, \varepsilon\,.$$

*When $f''$ is continuous at $\bar{x}$ and $x$ is close to $\bar{x}$, the integral in this inequality is dominated by a small multiple of $-\frac{1}{2}f''(\bar{x})[x-\bar{x},x-\bar{x}]$ and the usual Lyapunov stability statements follow. In our situation here the set $\mathcal{O}_{\bar{\omega}}$ is not a $C^2$ submanifold and the free energy functional $\omega \mapsto E(\omega) + \int_{\mathbb{R}^2}\Phi(\omega(x))\,\mathrm{d}x$ is not of class $C^2$. It is not hard to see directly that the expression*

$$-\int_{\mathbb{R}^2}\int_0^1 (1-t)\Phi''\bigl((1-t)\bar{\omega}(x) + t\omega(x)\bigr)(\omega(x)-\bar{\omega}(x))^2\,\mathrm{d}t\,\mathrm{d}x$$

*cannot be dominated by $-\frac{1}{2}\int_{\mathbb{R}^2}\Phi''(\bar{\omega})(\omega(x)-\bar{\omega}(x))^2\,\mathrm{d}x$ in a suitable way. One may still use the invariance of the sets $\mathcal{U}_{\bar{\omega},\varepsilon} := \{\omega \in \mathcal{O}_{\bar{\omega}} \cap X_1,\, E(\omega) > E(\bar{\omega}) - \varepsilon\}$ under the Euler evolution,* *and possibly also the conservation of the second order moment $I(\omega)$ defined in* (1.15)*, to obtain Lyapunov-type stability statements. For results in this spirit when the domain occupied by the fluid is compact the reader can consult [8] and [5, Section II.4]. Our situation here is somewhat complicated by the non-compactness of our flow domain $\mathbb{R}^2$, but under our assumptions one still has $\cap_{\varepsilon>0}\mathcal{U}_{\bar\omega,\varepsilon} = \{\bar\omega\}$ (by using the uniqueness of the maximizers discussed in [10], for example). This could be turned into Lyapunov-type stability statements, although not quite of the same form as in the $C^2$ case. The important point is that there are estimates for the proximity of "almost maximizers" to the exact maximizers, an issue that also appears in other problems, such as the stability of the isoperimetric inequality [13], and of the Sobolev inequality [7].*

*In the present work our focus is on quadratic forms, due to their applicability to the viscous case. Of course, at the level of the linearized inviscid equation $\omega_t + \bar u\cdot\nabla\omega + u\cdot\nabla\bar\omega = 0$, the quadratic form $J$ does provide Lyapunov stability in the space $X_1$ if inequality* (2.25) *holds with $C_H < 1$. We note that the linearized analysis in other topologies can be more complicated, see for example [6].*

## 2.3 The quadratic form $J$ without mass constraint

In this short section we make a few remarks on the index of the quadratic form (2.9) when considered on the whole space $X$ defined by (2.4), and not only on the subspace $X_0$ given by (2.5). Our first observation is that, due to lack of scale invariance in this context, the form $J$ cannot be positive on $X$ if the underlying steady state $\bar\omega$ is sharply concentrated near the origin. To see this, we consider the rescaled vortex $\bar\omega_\lambda(x) = \lambda^2\bar\omega(\lambda x)$ and the associated weight function $\mathscr{A}_\lambda(x) = \lambda^{-2}\mathscr{A}(\lambda x)$, see Remark 2.12. We denote by $J_\lambda$ the quadratic form on $X$ corresponding to the steady state $\bar\omega_\lambda$, namely the form (2.9) where $\mathscr{A}$ is replaced by $\mathscr{A}_\lambda$. If $\omega \in X$ and $\omega_\lambda(x) = \lambda^2\omega(\lambda x)$, a simple calculation shows that

$$J_\lambda(\omega_\lambda) \,=\, J(\omega) - \frac{M_0^2}{4\pi}\,\log(\lambda)\,, \qquad \text{where} \quad M_0 \,=\, \int_{\mathbb{R}^2}\omega(x)\,\mathrm{d}x\,.$$

If $M_0 \neq 0$, it is clear that $J_\lambda(\omega_\lambda) < 0$ when $\lambda > 0$ is sufficiently large, so that the quadratic form $J_\lambda$ cannot be positive in this regime.

**Remark 2.14.** *The negative direction arising by such a rescaling is related to a particular choice of the unit of length implicitly involved in the kernel $\frac{1}{2\pi}\log|x|$. In writing $\log|x|$, we imply that $x$ is dimensionless. In case $x$ is measured in some units of length, we should write the kernel as $\frac{1}{2\pi}\log\frac{|x|}{r_0}$, where $r_0$ is a reference length. The choice of $r_0$ does not affect the behavior of the system, and in the stability analysis based on $J$ it can be compensated for by adding to the quadratic form $J$ a suitable multiple of the quantity $\bigl(\int_{\mathbb{R}^2}\omega(x,t)\,\mathrm{d}x\bigr)^2$, which is preserved by the evolution. Hence, as one can expect, the stability analysis is independent of the choice of the reference length $r_0$, or, equivalently, of the scaling parameter $\lambda$ above.*

We next argue that, for any vortex $\bar\omega$ satisfying Hypotheses 2.1, the index of the quadratic form is well defined in the sense that $J$ has (at most) a finite number of negative directions. In view of Theorem 2.5, it is sufficient to evaluate $J$ on radially symmetric functions $\omega \in X_{\mathrm{rs}}$. The following expression will be useful:

**Lemma 2.15.** *For any $\omega \in X_{\mathrm{rs}}$, we have*

$$J(\omega) \,=\, \pi\int_0^\infty A(r)\omega(r)^2 r\,\mathrm{d}r \,+\, \pi\int_0^\infty\!\!\int_0^\infty \log\bigl(\max(r,s)\bigr)\,r\omega(r)s\omega(s)\,\mathrm{d}r\,\mathrm{d}s\,. \tag{2.40}$$

**Proof.** Here and below, with a slight abuse of notation, we consider any $\omega \in X_{\rm rs}$ as a function of the one-dimensional variable $r = |x|$. For such vorticities, the first integral in (2.9) obviously gives the first term in (2.40), so it remains to establish the following expression of the energy:

$$E(\omega) = -\pi \int_0^\infty\int_0^\infty \log\bigl(\max(r,s)\bigr)\, r\omega(r)\, s\omega(s)\,\mathrm{d}r\,\mathrm{d}s\,, \qquad \omega \in X_{\rm rs}\,. \tag{2.41}$$

To this end, we introduce polar coordinates $x = r\,e^{i\theta}$, $y = s\,e^{i\zeta}$ to compute the right-hand side of (1.13), and we use the identity

$$\int_0^{2\pi}\int_0^{2\pi} \log|re^{i\theta} - se^{i\zeta}|\,\mathrm{d}\theta\,\mathrm{d}\zeta \;=\; 2\pi\int_0^{2\pi}\log|re^{i\theta} - s|\,\mathrm{d}\theta \;=\; 4\pi^2\log\bigl(\max(r,s)\bigr)\,. \tag{2.42}$$

The formula (2.42) is well known and can be derived in many ways. For example, assuming that $r$ is a fixed positive number, we interpret the last integral as a function of $s \in \mathbb{C}$. This expression obviously depends only on $|s|$, is continuous everywhere, and is analytic both inside and outside of the circle $|s| = r$. Inside the circle it has to be constant and outside the circle it coincides with the potential of a point particle of mass $2\pi$ located at the origin, which is $2\pi\log|s|$. This gives (2.42), and (2.41) follows. □

Applying the change of variables $w(r) = \omega(r)A(r)^{1/2}$, so that $w \in Y = L^2(\mathbb{R}_+, r\,\mathrm{d}r)$ when $\omega \in X_{\rm rs}$, the formula (2.40) becomes

$$\frac{1}{\pi}\,J(\omega) \;=\; \int_0^\infty w(r)^2 r\,\mathrm{d}r \;-\; \int_0^\infty\int_0^\infty \mathscr{k}(r,s)w(r)w(s)rs\,\mathrm{d}r\,\mathrm{d}s\,, \tag{2.43}$$

where $\mathscr{k}(r,s) = -\log\bigl(\max(r,s)\bigr)\,A(r)^{-1/2}\,A(s)^{-1/2}$. Under Hypotheses 2.1, we have the lower bound $A(r) \ge C(1+r)^\beta$ for some $\beta > 2$, which implies that

$$\int_0^\infty\int_0^\infty \mathscr{k}(r,s)^2\,rs\,\mathrm{d}r\,\mathrm{d}s \;<\; \infty\,.$$

This means that the right-hand side of (2.43) is the quadratic form in $Y$ associated with a selfadjoint operator of the form $\mathbf{1} - \mathscr{K}$, where $\mathbf{1}$ is the identity and $\mathscr{K}$ is a Hilbert-Schmidt perturbation. By compactness, this operator has (at most) a finite number of negative eigenvalues, which means that the index of the quadratic form $J$ on $X$ is well defined.

The eigenvalues of $\mathscr{K}$ can also be thought of as eigenvalues of the quadratic form (2.41) with respect to the reference form $\omega \mapsto \pi\int_0^\infty A(r)\omega(r)^2 r\,\mathrm{d}r$. As is easily verified, if $\lambda$ is such an eigenvalue, the corresponding eigenfunction $\omega$ satisfies

$$-\psi(r) \;=\; \lambda A(r)\omega(r)\,, \qquad \text{where} \quad \psi(r) \;=\; \int_0^\infty \log\bigl(\max(r,s)\bigr)\,s\omega(s)\,\mathrm{d}s\,. \tag{2.44}$$

Since $\omega(r) = \psi''(r) + \frac{1}{r}\,\psi'(r)$, the first relation in (2.44) is an ordinary differential equation for the stream function $\psi : \mathbb{R}_+ \to \mathbb{R}$, to be solved with the boundary conditions

$$\psi'(0) \;=\; 0\,, \qquad \text{and} \qquad \lim_{r\to+\infty}\Bigl(\psi(r)\log(2r) - \psi(2r)\log(r)\Bigr) \;=\; 0\,,$$

which can be deduced from the expression of $\psi$ in (2.44). For the Lamb-Oseen vortex (2.39) a numerical computation gives the largest eigenvalue $\lambda \approx 0.7127$, thus suggesting that the form $J$ is strictly positive definite on the whole space $X_{\rm rs}$ in that case. In contrast, the largest eigenvalue for the algebraic vortices (2.38) seems to exceed the threshold value 1, indicating that for those vortices the form $J$ is not positive definite without additional constraints on $\omega$.

## 2.4 The maximal energy as a function of the constraints

In Section 1.1 we considered the classical problem of maximizing a function $f : \mathbb{R}^n \to \mathbb{R}$ under a family of constraints of the form $g_1 = c_1, \dots, g_m = c_m$, where $g_1, \dots, g_m : \mathbb{R}^n \to \mathbb{R}$. Given $c = (c_1, \dots, c_m) \in \mathbb{R}^m$, we recall the notation $X_c = \{x \in \mathbb{R}^n\,;\, g_1(x) = c_1, \dots, g_m(x) = c_m\}$. Assuming that $f$ reaches a non-degenerate maximum on $X_c$ at some point $\bar{x} \in X_c$ where the first-order derivatives $g_1'(\bar{x}), \dots, g_m'(\bar{x})$ are linearly independent, we introduced the quadratic form $\mathcal{Q}$ defined by (1.5), which is the second order differential of the Lagrange function (1.7) at $\bar{x}$. In the present section, we are interested in the index of the form $\mathcal{Q}$ on larger subspaces than $T_{\bar{x}}X_c$. As was already mentioned, this question is closely related to concavity properties of the function $M$ defined by (1.8) or, almost equivalently, to convexity properties of the set $S = \{(g_1(x), \dots, g_m(x), f(x))\,;\, x \in \mathbb{R}^n\} \subset \mathbb{R}^{m+1}$ near its "upper boundary".

The situation becomes particularly transparent if we use adapted coordinates which, as it turns out, have a fairly complete analogy in 2d Euler case. Let us assume that we can introduce new coordinates $(c_1, \dots, c_m, y_1, \dots, y_{n-m})$ in $\mathbb{R}^n$ such that, as before, $c_1, \dots, c_m$ are the values of the constraints $g_1, \dots, g_m$, and the additional coordinates $y_1, \dots, y_{n-m}$ are chosen so that the points having coordinates $(c_1, \dots, c_m, 0, \dots, 0)$ are those where $f$ attains its maximum on $X_c$.[5] Denoting $M(c_1, \dots, c_m) = f(c_1, \dots, c_m, 0, \dots, 0)$ as in (1.8), one verifies that

$$\frac{\partial M}{\partial c_j}(c_1, \dots, c_m) \;=\; \lambda_j\,, \qquad j = 1, \dots, m\,, \tag{2.45}$$

where $\lambda_1, \dots, \lambda_m$ are the Lagrange multipliers introduced in (1.4). Moreover the extremality condition on $X_c$ implies that

$$\frac{\partial f}{\partial y_k}(c_1, \dots, c_m, 0, \dots, 0) \;=\; 0\,, \qquad k = 1, \dots, n-m\,.$$

We infer that

$$D^2 f(c_1, \dots, c_m, 0, \dots, 0) \;=\; \begin{pmatrix} \left(\frac{\partial^2 f}{\partial c_i \partial c_j}\right)_{i,j=1}^{m} & 0 \\ 0 & \left(\frac{\partial^2 f}{\partial y_k \partial y_\ell}\right)_{k,\ell=1}^{n-m} \end{pmatrix}\,, \tag{2.46}$$

where all derivatives are evaluated at the point $(c_1, \dots, c_m, 0, \dots, 0)$. The first submatrix in the right-hand side of (2.46) is precisely the Hessian of $M$, and the second submatrix is always negative definite, due to our assumption that $f$ reaches a maximum at $(y_1, \dots, y_{n-m}) = (0, \dots, 0)$ for any fixed value of $c_1, \dots, c_m$. So we conclude that the quadratic form $\mathcal{Q}$ defined in (1.5) is negative definite at $\bar{x}$ if and only if the Hessian of $M$ is negative definite at $(c_1, \dots, c_m)$, where $c_j = g_j(\bar{x})$ for $j = 1, \dots, m$.

Another interesting object is the function

$$\begin{aligned} N(\lambda_1, \dots, \lambda_m) \;&=\; \sup_{x \in \mathbb{R}^n} \Big(f(x) - \lambda_1 g_1(x) - \ldots - \lambda_m g_m(x)\Big) \\ &=\; \sup_{c \in \mathbb{R}^m} \Big(M(c_1, \dots, c_m) - \lambda_1 c_1 - \ldots - \lambda_m c_m\Big)\,, \end{aligned} \tag{2.47}$$

which is the *Legendre transform* of $M$. Under appropriate assumptions, the main one being the concavity of $M$, this quantity is well defined and the relation (2.45) can be inverted (at least

[5] In a non-degenerate situation, the local existence of such a coordinate system is clear by standard arguments, but globally the situation can, of course, be more complicated.

locally) via the formula

$$c_j \;=\; -\frac{\partial N}{\partial \lambda_j}(\lambda_1,\dots,\lambda_m)\,, \qquad j=1,\dots,m\,. \tag{2.48}$$

We now return to the infinite-dimensional framework of the 2d Euler equation, with the manifold $\mathbb{R}^n$ replaced by the phase space $\mathscr{P}$ introduced in Section 1.2, the function $f$ replaced by the energy $E$ in (1.13), the constraints $g_j$ replaced by the Casimir functionals $h(a,\omega)$ in (1.10), and the submanifolds $X_c$ replaced by the orbits $\mathbb{O}_\omega$ in (1.12). In that case we have

$$\max_{\omega\in\mathbb{O}_{\bar\omega}} E(\omega) \;=\; E(\bar\omega^*)\,, \tag{2.49}$$

where, as before, $\bar\omega^*$ denotes the symmetric decreasing rearrangement of an element $\bar\omega\in\mathscr{P}$. As $\mathbb{O}_{\bar\omega}$ is characterized in terms of the functionals $h(a,\omega)$ defined in (1.10), the energy of the maximizer $\bar\omega^*$ in $\mathbb{O}_{\bar\omega}$ can also be expressed in terms of the constraint function $a\to h(a,\bar\omega)$. It turns out that the representation formula is quite explicit.

**Proposition 2.16.** *Given $\bar\omega\in\mathscr{P}$, we define $\hbar(a)=\pi^{-1}h(a,\bar\omega)=\pi^{-1}|\{\bar\omega>a\}|$ for any $a>0$. Then*

$$\mathscr{E}(\hbar) \;:=\; \max_{\substack{\omega\in\mathscr{P}\\ h(\cdot,\omega)=\pi\hbar}} E(\omega) \;=\; \frac{\pi}{8}\int_0^m\!\!\int_0^m L(\hbar(a),\hbar(b))\,\mathrm{d}a\,\mathrm{d}b \;+\; \frac{1}{8\pi}M_0^2\,, \tag{2.50}$$

*where $m=\max\bar\omega$, $M_0=\int_{\mathbb{R}^2}\bar\omega\,\mathrm{d}x=\pi\int_0^m\hbar(a)\,\mathrm{d}a$, and*

$$L(R,S) \;=\; -RS\log\max(R,S)-\frac{1}{2}\min(R,S)^2\,. \tag{2.51}$$

**Proof.** Replacing $\bar\omega$ with $\bar\omega^*$ (an operation that does not affect the function $\hbar$), we can assume that $\bar\omega$ is radially symmetric and nonincreasing in the radial direction. In view of of (2.49), we then have $\mathscr{E}(\hbar)=E(\bar\omega)$, and if we consider $\bar\omega$ as a function of the radius $r=|x|$ we observe that $\hbar(a)=(\bar\omega^{-1}(a))^2$ wherever $\bar\omega$ is strictly decreasing. To compute $E(\bar\omega)$, we start from the expression (2.41), and we introduce the functions

$$k(r,s) \;=\; -rs\log\max(r,s)\,, \qquad K(R,S) \;=\; L(R,S)+RS\,.$$

Clearly $K(R,0)=0$, $K(0,S)=0$ for $R,S>0$, and one can verify by direct calculation that $K(R,S)$ is twice continuously differentiable on $(0,\infty)\times(0,\infty)$ with

$$\frac{\partial^2 K}{\partial R\partial S}(R,S) \;=\; -\log\max(R,S)\,, \qquad R,S>0\,.$$

So the function $(r,s)\mapsto K(r^2,s^2)$ is twice continuously differentiable on $[0,\infty)\times[0,\infty)$ and

$$\frac{1}{8}\,\frac{\partial^2}{\partial r\partial s}K(r^2,s^2) \;=\; k(r,s)\,.$$

Integrating by parts in (2.41) and recalling that $m=\max\bar\omega$, we can thus write

$$\begin{aligned}
E(\bar\omega) \;&=\; \frac{\pi}{8}\int_0^\infty\!\!\int_0^\infty \frac{\partial^2}{\partial r\partial s}K(r^2,s^2)\,\bar\omega(r)\,\bar\omega(s)\,\mathrm{d}r\,\mathrm{d}s \;=\; \frac{\pi}{8}\int_0^\infty\!\!\int_0^\infty K(r^2,s^2)\,\mathrm{d}\bar\omega(r)\,\mathrm{d}\bar\omega(s)\\
&=\; \frac{\pi}{8}\int_0^m\!\!\int_0^m K\big((\bar\omega^{-1}(a))^2,(\bar\omega^{-1}(b))^2\big)\,\mathrm{d}a\,\mathrm{d}b \;=\; \frac{\pi}{8}\int_0^m\!\!\int_0^m K(\hbar(a),\hbar(b))\,\mathrm{d}a\,\mathrm{d}b\\
&=\; \frac{\pi}{8}\int_0^m\!\!\int_0^m L(\hbar(a),\hbar(b))\,\mathrm{d}a\,\mathrm{d}b+\frac{1}{8\pi}M_0^2\,,
\end{aligned} \tag{2.52}$$

where we have formally used the substitutions $\bar\omega(r)=a$, $\bar\omega(s)=b$. This is straighforward when $\bar\omega$ is strictly decreasing, and the general case where $\bar\omega$ is nonincreasing can be treated by integrating only over the intervals where $\bar\omega$ is strictly decreasing. $\square$

We now make a more precise comparison with the finite-dimensional situation above. Let us assume that $\bar\omega \in \mathscr{P}$ is radially symmetric with $\partial_r\bar\omega(r) < 0$ for all $r > 0$ and $\partial_r^2\bar\omega(0) < 0$. To eliminate the translational symmetries, we work with the manifold

$$\tilde{\mathscr{P}} = \big\{\omega \in \mathscr{P}\,;\, M_0(\omega) = M_0(\bar\omega)\,,\, M_j(\omega) = 0\,,\, j = 1,2\big\}\,, \tag{2.53}$$

where $M_0, M_j$ are as in (1.11), (1.15). If $\eta \in \mathscr{X}_1$ (see (2.24)) is smooth and compactly supported with suffiently small $C^2$ norm, then $\bar\omega + \eta \in \tilde{\mathscr{P}}$. Denoting by $\eta_{\rm rs}$ the projection of $\eta$ onto the subspace $X_{\rm rs}$ defined in (2.11), we can take the quantities $h(a, \bar\omega + \eta_{\rm rs})$ and $\eta_{\rm rs}^\perp := \eta - \eta_{\rm rs}$ as the (approximate) analogues of the coordinates $c_j$ and $y_k$, respectively. The analogy is not perfect, due to the stronger-than-ideal assumptions on $\eta$, but it is sufficient for concluding that when $\bar\omega = \bar\omega^*$, the negative-definiteness of Arnold's form (1.18) on the tangent space $T_{\bar\omega}\tilde{\mathscr{P}}$ is strongly related to the concavity of the energy $E$ in the variable[6] $\hbar$ at the function $\bar\hbar(a) = \pi^{-1}h(a,\bar\omega)$. In some sense the expression (2.50) is "trying to be concave", although not quite achieving this: the function $L(R, S)$ is separately concave, but not concave. The second variation on the space $X_0$ is given by the quadratic form which takes a function $\xi(a)$ with $\int_0^m \xi(a)\,\mathrm{d}a = 0$ to

$$\frac{\pi}{8}\int_0^m\!\!\int_0^m \big(D_1^2L(\hbar(a),\hbar(b))\xi(a)^2 + 2D_1D_2L(\hbar(a),\hbar(b))\xi(a)\xi(b) + D_2^2L(\hbar(a),\hbar(b))\xi(b)^2\big)\,\mathrm{d}a\,\mathrm{d}b\,.$$

Due to the separate concavity of $L$ the first term and the third term are negative, but the second one can lead to the form being indefinite. In view of our previous considerations, the negativity of the form is equivalent to the validity of the Hardy inequality (2.25) with $C_H \le 1$, and it is not hard to verify directly that this is indeed the case. As an analogue of (2.45), we also note that the variational derivative of $\mathscr{E}$ with respect to $\hbar$ is

$$\frac{1}{\pi}\,\frac{\delta\mathscr{E}}{\delta\hbar}(a) \,=\, \Lambda(a) \,=\, -\Phi'(a)\,. \tag{2.54}$$

We will not go into the details as we will not work with this expression. The reader can also derive the analogue of (2.48) (under appropriate assumptions).

## 3 Global maximization of the free energy

In the previous section we observed that some radially symmetric vortices $\bar\omega$, including the Gaussian vortex (2.39) and the algebraic vortex (2.38) with $\kappa > 2$, are non-degenerate local maxima of the associated free energy functional (1.17) once restricted to the manifold $\tilde{\mathscr{P}}$ defined in (2.53). This was established by showing that the second order differential $F''(\bar\omega)$ is strictly negative definite on the tangent space $T_{\bar\omega}\tilde{\mathscr{P}}$. We now follow a different approach, which relies on the direct method in the calculus of variations: under appropriate assumptions on the function $\Phi$ in (1.17), we show that the free energy $F(\omega)$ has a global maximum on the set of all vorticity distributions with a fixed mass $M$. By construction, if $\bar\omega$ is any maximizer obtained in this way, the conclusion of Theorem 2.8 applies with $\gamma \ge 0$, so that Hardy's inequality (2.25) holds with $C_H \le 1$. Note also that, according to the discussion in Section 2.4, prescribing $\Phi$ amounts to fixing the "Lagrange multipliers" in our constrained maximization problem.

We start with a preliminary result, which is probably well known. For the reader's convenience, the proof is reproduced in Section A.1.

[6] It is perhaps worth recalling that $E$ is convex in $\omega$ on the subspace given by $\int_{\mathbb{R}^2}\omega\,\mathrm{d}x = 0$. However, in some regions it may be concave in $\hbar$, at least on the subspace given by $\int_0^\infty \hbar(a)\,\mathrm{d}a = 0$.

**Proposition 3.1.** *Assume that $f \in L^1(\mathbb{R}^n)$ is nonnegative and that $M := \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x > 0$. Then*

$$M + \int_{\mathbb{R}^n} \bigl(\log_- |x|\bigr)\, f(x)\,\mathrm{d}x \;\lesssim\; M + \int_{\mathbb{R}^n} \Bigl(\log_+ \frac{f(x)}{M}\Bigr)\, f(x)\,\mathrm{d}x\,, \tag{3.1}$$

$$M + \int_{\mathbb{R}^n} \bigl(\log_+ |x|\bigr)\, f(x)\,\mathrm{d}x \;\gtrsim\; M + \int_{\mathbb{R}^n} \Bigl(\log_- \frac{f(x)}{M}\Bigr)\, f(x)\,\mathrm{d}x\,, \tag{3.2}$$

*where the implicit constants only depend on the space dimension $n$. Moreover, if $f$ is radially symmetric and nonincreasing in the radial direction, then the reverse inequalities also hold.*

We next specify the function space in which we shall solve our maximization problem.

**Definition 3.2.** *Given any $M > 0$, we denote by $X_M$ the set of all $\omega \in L^1(\mathbb{R}^2)$ such that $\omega(x) \ge 0$ for almost all $x \in \mathbb{R}^2$ and*

$$\int_{\mathbb{R}^2} \omega(x)\,\mathrm{d}x \;=\; M\,, \quad \int_{\mathbb{R}^2} \omega(x)\log\bigl(1+|x|\bigr)\,\mathrm{d}x \;<\; \infty\,, \quad \int_{\mathbb{R}^2} \omega(x)\log\bigl(1+\omega(x)\bigr)\,\mathrm{d}x \;<\; \infty\,. \tag{3.3}$$

For later use we observe that, if $\omega \in X_M$ and if $\omega^*$ denotes the symmetric nonincreasing rearrangement of $\omega$, then $\int_{\mathbb{R}^2} \omega^*(x)\,\mathrm{d}x = \int_{\mathbb{R}^2} \omega(x)\,\mathrm{d}x = M$ and

$$\int_{\mathbb{R}^2} \omega^*(x)\log\bigl(1+|x|\bigr)\,\mathrm{d}x \;\le\; \int_{\mathbb{R}^2} \omega(x)\log\bigl(1+|x|\bigr)\,\mathrm{d}x \;<\; \infty\,,$$

$$\int_{\mathbb{R}^2} \omega^*(x)\log\bigl(1+\omega^*(x)\bigr)\,\mathrm{d}x \;=\; \int_{\mathbb{R}^2} \omega(x)\log\bigl(1+\omega(x)\bigr)\,\mathrm{d}x \;<\; \infty\,.$$

This shows that the set $X_M \subset L^1(\mathbb{R}^2)$ is invariant under the action of the symmetric nonincreasing rearrangement.

For $\omega \in X_M$, we consider the free energy defined by $F(\omega) = E(\omega) + S(\omega)$, where

$$E(\omega) \;=\; \frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\frac{1}{|x-y|}\,\omega(x)\omega(y)\,\mathrm{d}x\,\mathrm{d}y\,, \qquad S(\omega) \;=\; \int_{\mathbb{R}^2} \Phi(\omega(x))\,\mathrm{d}x\,.$$

We have shown in Proposition 2.2 that the energy $E(\omega)$ is finite for any $\omega \in X_M$. Unlike in Section 2, the function $\Phi$ in the entropy term is not related here to any radially symmetric vortex, but is an arbitrary function satisfying the following properties:

**Hypotheses 3.3.** *The function $\Phi : [0,+\infty) \to \mathbb{R}$ is continuous with $\Phi(0) = 0$. Moreover, there exist constants $C_1 \in \mathbb{R}$, $C_2 < M/(8\pi)$, and $C_3 > M/(8\pi)$ such that*

$$\begin{aligned} \Phi(\omega) \;&\le\; C_1\omega + C_2\,\omega\log\frac{M}{\omega} \quad \textit{when } \omega \le M\,,\\ \Phi(\omega) \;&\le\; C_1\omega - C_3\,\omega\log\frac{\omega}{M} \quad \textit{when } \omega \ge M\,. \end{aligned} \tag{3.4}$$

Under Hypotheses 3.3, the positive part of $\Phi$ satisfies $\Phi_+(\omega) \le C\omega(1+|\log(\omega/M)|)$ for some constant $C > 0$, and this implies in particular that the entropy $S(\omega)$ is well defined in $\mathbb{R}\cup\{-\infty\}$ for any $\omega \in X_M$. We are now in a position to state the main result of this section.

**Theorem 3.4.** *Fix any $M > 0$. Under Hypotheses 3.3, there exists $\bar\omega \in X_M$ such that*

$$F(\bar\omega) \;=\; E(\bar\omega) + S(\bar\omega) \;=\; \sup_{\omega\in X_M}\bigl(E(\omega) + S(\omega)\bigr)\,.$$

*Moreover $\bar\omega$ can be chosen to be radially symmetric and nonincreasing in the radial direction.*

The proof of Theorem 3.4 is divided into two parts. The first one consists in showing that the free energy $F$ is bounded from above on $X_M$, and that there exists a maximizing sequence which is convergent in $L^1(\mathbb{R}^2)$. We formulate this in a separate statement:

**Proposition 3.5.** *Under Hypotheses 3.3, the free energy $F = E + S$ is bounded from above on the space $X_M$:*

$$F_M \,:=\, \sup_{\omega \in X_M} \bigl(E(\omega) + S(\omega)\bigr) \,<\, \infty\,.$$

*Moreover, there exists a maximizing sequence $(\omega_j)_{j\in\mathbb{N}}$ in $X_M$ which converges in $L^1(\mathbb{R}^2)$ to some limiting profile $\bar\omega = \bar\omega^* \in X_M$ as $j \to +\infty$, and we have $S(\bar\omega) > -\infty$.*

**Proof.** Our starting point is the logarithmic Hardy-Littlewood-Sobolev inequality

$$E(\omega) + \frac{M}{8\pi}\int_{\mathbb{R}^2} \omega \log\frac{M}{\omega}\,\mathrm{d}x \,\le\, \frac{M^2}{8\pi}\bigl(1 + \log\pi\bigr)\,, \tag{3.5}$$

which holds for all $\omega \in X_M$, see [10]. In view of (3.4), we deduce from (3.5) that

$$\begin{aligned} E(\omega) + S(\omega) + \Bigl(\frac{M}{8\pi} - C_2\Bigr)\int_{\omega<M} \omega\log\frac{M}{\omega}\,\mathrm{d}x + \Bigl(C_3 - \frac{M}{8\pi}\Bigr)\int_{\omega>M}\omega\log\frac{\omega}{M}\,\mathrm{d}x \\ \le\, E(\omega) + C_1 M + \frac{M}{8\pi}\int_{\mathbb{R}^2}\omega\log\frac{M}{\omega}\,\mathrm{d}x \,\le\, C_1 M + \frac{M^2}{8\pi}\bigl(1+\log\pi\bigr)\,. \end{aligned} \tag{3.6}$$

Since $C_2 < M/(8\pi)$ and $C_3 > M/(8\pi)$, this proves that $F_M \le C_1 M + M^2(1+\log\pi)/(8\pi)$.

Now, let $(\omega_j)_{j\in\mathbb{N}}$ be a sequence in $X_M$ such that $E(\omega_j) + S(\omega_j) \to F_M$ as $j \to +\infty$. If we denote by $(\omega_j)^* \in X_M$ the symmetric nonincreasing rearrangement of $\omega_j$, we know that $E((\omega_j)^*) \ge E(\omega_j)$ and $S((\omega_j)^*) = S(\omega_j)$ for all $j \in \mathbb{N}$, so that $\bigl((\omega_j)^*\bigr)_{j\in\mathbb{N}}$ is a fortiori a maximizing sequence. So we assume henceforth that $\omega_j = (\omega_j)^*$, i.e. $\omega_j$ is radially symmetric and nonincreasing in the radial direction. In that case, there exists a constant $C_0 > 0$ such that

$$\int_{\mathbb{R}^2}\omega_j(x)\Bigl|\log\frac{\omega_j(x)}{M}\Bigr|\,\mathrm{d}x \,\le\, C_0\,, \qquad \text{and} \qquad \int_{\mathbb{R}^2}\omega_j(x)\bigl|\log|x|\bigr|\,\mathrm{d}x \,\le\, C_0\,, \tag{3.7}$$

for all $j \in \mathbb{N}$. Indeed, the first inequality in (3.7) follows directly from (3.6), and the second one is a consequence of the first inequality and of Proposition 3.1, since $\omega_j = (\omega_j)^*$.

It remains to verify that one can extract from $(\omega_j)_{j\in\mathbb{N}}$ a convergent subsequence in $L^1(\mathbb{R}^2)$. We recall that $\omega_j(x)$ is a nonincreasing function of the radial variable $|x|$, which satisfies the uniform pointwise estimate $0 \le \omega_j(x) \le M/(\pi|x|^2)$, see (A.3) below. By Helly's selection theorem [26], there exists a subsequence, still denoted by $(\omega_j)_{j\in\mathbb{N}}$, which converges pointwise to some limit $\bar\omega : \mathbb{R}^2 \to \mathbb{R}_+$ as $j \to +\infty$. It is clear that $\bar\omega$ is radially symmetric and nonincreasing, so that $\bar\omega = \bar\omega^*$, and Fatou's lemma implies that $\int_{\mathbb{R}^2}\bar\omega(x)\,\mathrm{d}x \le M$. Using in addition (3.7), we obtain similarly

$$\int_{\mathbb{R}^2}\bar\omega(x)\Bigl|\log\frac{\bar\omega(x)}{M}\Bigr|\,\mathrm{d}x \,\le\, C_0\,, \qquad \text{and} \qquad \int_{\mathbb{R}^2}\bar\omega(x)\bigl|\log|x|\bigr|\,\mathrm{d}x \,\le\, C_0\,. \tag{3.8}$$

To prove the convergence in $L^1(\mathbb{R}^2)$ we decompose, for any $\epsilon \in (0,1)$,

$$\int_{\mathbb{R}^2}|\omega_j(x) - \bar\omega(x)|\,\mathrm{d}x \,=\, \int_{A_\epsilon}|\omega_j(x) - \bar\omega(x)|\,\mathrm{d}x + \int_{\mathbb{R}^2\setminus A_\epsilon}|\omega_j(x) - \bar\omega(x)|\,\mathrm{d}x\,, \tag{3.9}$$

where $A_\epsilon = \{x \in \mathbb{R}^2\,;\, \epsilon \le |x| \le \epsilon^{-1}\}$. The integral over $A_\epsilon$ converges to zero as $j \to +\infty$ by the dominated convergence theorem, and in view of (3.7), (3.8) the integral over $\mathbb{R}^2 \setminus A_\epsilon$ is bounded by $2C_0/|\log\epsilon|$ uniformly in $j$. It thus follows from (3.9) that

$$\limsup_{j\to+\infty} \int_{\mathbb{R}^2} |\omega_j(x) - \bar\omega(x)|\,\mathrm{d}x \,\le\, \frac{2C_0}{|\log\epsilon|} \,\xrightarrow[\epsilon\to 0]{}\, 0\,,$$

which shows that $\omega_j \to \bar\omega$ in $L^1(\mathbb{R}^2)$. In particular $\int_{\mathbb{R}^2}\bar\omega(x)\,\mathrm{d}x = M$, so that $\bar\omega \in X_M$.

Finally, if we decompose $\Phi = \Phi_+ - \Phi_-$, where $\Phi_+, \Phi_-$ denote the positive and negative parts of $\Phi$, we have the lower bound

$$S(\bar\omega) \,\ge\, -\int_{\mathbb{R}^2} \Phi_-(\bar\omega(x))\,\mathrm{d}x \,\ge\, -\liminf_{j\to+\infty}\int_{\mathbb{R}^2}\Phi_-(\omega_j(x))\,\mathrm{d}x\,, \tag{3.10}$$

where the second inequality is again obtained by Fatou's lemma. But we have the identity

$$\int_{\mathbb{R}^2}\Phi_-(\omega_j(x))\,\mathrm{d}x \,=\, \int_{\mathbb{R}^2}\Phi_+(\omega_j(x))\,\mathrm{d}x - S(\omega_j) \,=\, \int_{\mathbb{R}^2}\Phi_+(\omega_j(x))\,\mathrm{d}x + E(\omega_j) - F(\omega_j)\,,$$

where the first two terms in the right-hand side are bounded uniformly in $j$ by (3.7), in view of Hypotheses 3.3 and Proposition 2.2, whereas $F(\omega_j)$ is bounded from below since $(\omega_j)$ is a maximizing sequence for $F$. We conclude that the right-hand side of (3.10) is finite, so that $S(\bar\omega) > -\infty$. □

To conclude the proof of Theorem 3.4, it remains to show that the free energy is upper semicontinuous along the maximizing sequence constructed in Proposition 3.5, namely

$$E(\bar\omega) + S(\bar\omega) \,\ge\, \limsup_{j\to+\infty}\Big(E(\omega_j) + S(\omega_j)\Big) \,=\, F_M\,. \tag{3.11}$$

This will imply that $E(\bar\omega) + S(\bar\omega) = F_M$, which is the desired result.

**Proof of Theorem 3.4.** Let $(\omega_j)_{j\in\mathbb{N}}$ be the maximizing sequence defined in Proposition 3.5, and $\bar\omega \in X_M$ be the limiting profile. Given any sufficiently large $R > 0$, we decompose

$$\begin{aligned}
\omega_j(x) \,&=\, \omega_j(x)\,\mathbf{1}_{\{|x|\le R\}} + \omega_j(x)\,\mathbf{1}_{\{|x|>R\}} \,=:\, \omega_{jR}^1(x) + \omega_{jR}^2(x)\,,\\
\bar\omega(x) \,&=\, \bar\omega(x)\,\mathbf{1}_{\{|x|\le R\}} + \bar\omega(x)\,\mathbf{1}_{\{|x|>R\}} \,=:\, \bar\omega_R^1(x) + \bar\omega_R^2(x)\,,
\end{aligned}$$

for all $x \in \mathbb{R}^2$. We thus have

$$\begin{aligned}
E(\omega_j) + S(\omega_j) \,&=\, E(\omega_{jR}^1) + S(\omega_{jR}^1) + 2E(\omega_{jR}^1,\omega_{jR}^2) + E(\omega_{jR}^2) + S(\omega_{jR}^2)\,,\\
E(\bar\omega) + S(\bar\omega) \,&=\, E(\bar\omega_R^1) + S(\bar\omega_R^1) + 2E(\bar\omega_R^1,\bar\omega_R^2) + E(\bar\omega_R^2) + S(\bar\omega_R^2)\,,
\end{aligned}$$

where $E(\omega_1,\omega_2)$ is the bilinear form associated with the energy functional:

$$E(\omega_1,\omega_2) \,=\, -\frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log|x-y|\,\omega_1(x)\,\omega_2(y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

The upper-semicontinuity property (3.11) can be deduced from the following assertions:

$$\limsup_{j\to+\infty}\Big(E(\omega_{jR}^1) + S(\omega_{jR}^1)\Big) \,\le\, E(\bar\omega_R^1) + S(\bar\omega_R^1)\,, \tag{3.12}$$

$$\sup_{j\in\mathbb{N}}\Big(2E(\omega_{jR}^1,\omega_{jR}^2) + E(\omega_{jR}^2) + S(\omega_{jR}^2)\Big) \,\le\, \delta_1(R) \,\xrightarrow[R\to+\infty]{}\, 0\,, \tag{3.13}$$

$$2E(\bar\omega_R^1,\bar\omega_R^2) + E(\bar\omega_R^2) + S(\bar\omega_R^2) \,=\, \delta_2(R) \,\xrightarrow[R\to+\infty]{}\, 0\,. \tag{3.14}$$

Indeed, assuming that (3.12)–(3.14) hold, we obtain

$$\limsup_{j\to+\infty}\Big(E(\omega_j)+S(\omega_j)\Big) - \Big(E(\bar\omega)+S(\bar\omega)\Big) \,\le\, \delta_1(R)-\delta_2(R) \xrightarrow[R\to+\infty]{} 0\,.$$

It remains to verify the assertions (3.12)–(3.14) above. We recall that the functions $\omega_j, \bar\omega$ are radially symmetric and nonincreasing in the radial direction. With a slight abuse of notation, we write $\omega_j(r)$ instead of $\omega_j(x)$ when $r=|x|$, and similarly for $\bar\omega$. Accordingly, using (2.41), we obtain the following expressions for the energy of $\omega_j$ and $\bar\omega$:

$$E(\omega_j) = -\int_0^\infty M_j(r)\log(r)\,r\omega_j(r)\,\mathrm{d}r\,, \qquad E(\bar\omega) = -\int_0^\infty \overline{M}(r)\log(r)\,r\bar\omega(r)\,\mathrm{d}r\,, \tag{3.15}$$

where

$$M_j(r) \,=\, 2\pi\int_0^r s\omega_j(s)\,\mathrm{d}s\,, \qquad \overline{M}(r) \,=\, 2\pi\int_0^r s\bar\omega(s)\,\mathrm{d}s\,, \qquad r>0\,. \tag{3.16}$$

Since $\omega_j \to \bar\omega$ in $L^1(\mathbb{R}^2)$, we see that $M_j(r)\to\overline{M}(r)$ uniformly in $r$ as $j\to+\infty$. Moreover, since $\omega_j\in X_M$ satisfies (3.7), the quantity $M_j(r)$ converges to $M$ as $r\to+\infty$ uniformly in $j$. In particular, we can choose $R\ge 1$ large enough so that $M_j(r)\ge M/2$ for all $j\in\mathbb{N}$ when $r\ge R$.

To prove (3.12), we first decompose

$$E(\omega^1_{jR})-E(\bar\omega^1_R) \,=\, -\int_0^R \big(M_j(r)-\overline{M}(r)\big)\log(r)\,r\omega_j(r)\,\mathrm{d}r - \int_0^R \overline{M}(r)\log(r)\,r\big(\omega_j(r)-\bar\omega(r)\big)\,\mathrm{d}r\,,$$

and we deduce that

$$\begin{aligned}\big|E(\omega^1_{jR})-E(\bar\omega^1_R)\big| \,\le\, &\sup_{0\le r\le R}\Big(|M_j(r)-\overline{M}(r)|\Big)\int_0^R|\log(r)|\,r\omega_j(r)\,\mathrm{d}r\\ &+\sup_{0\le r\le R}\Big(|\log(r)|\,\overline{M}(r)\Big)\int_0^R r\big|\omega_j(r)-\bar\omega(r)\big|\,\mathrm{d}r \xrightarrow[j\to+\infty]{} 0\,.\end{aligned} \tag{3.17}$$

Here we used the convergence of $\omega_j$ to $\bar\omega$ in $L^1(\mathbb{R}^2)$, the a priori estimates (3.7), and the fact that $\log(r)\overline{M}(r)$ is bounded as $r\to 0$, as a consequence of (3.8). On the other hand, since the function $-\Phi$ is continuous and bounded from below, and since we integrate on the bounded domain $\{x\in\mathbb{R}^2\,;\,|x|\le R\}$, we can apply Fatou's lemma to obtain

$$-S(\bar\omega^1_R) \,=\, \int_{|x|\le R} -\Phi(\bar\omega(x))\,\mathrm{d}x \,\le\, \liminf_{j\to+\infty}\int_{|x|\le R}-\Phi(\omega_j(x))\,\mathrm{d}x \,=\, -\limsup_{j\to+\infty} S(\omega^1_{jR})\,. \tag{3.18}$$

Combining (3.17) and (3.18), we obtain (3.12).

We next prove (3.13). Recalling that $R\ge 1$, we first observe that

$$E(\omega^2_{jR}) = -\int_R^\infty M_j(r)\log(r)\,r\omega_j(r)\,\mathrm{d}r \,\le\, 0\,,$$

which means that the contribution of $E(\omega^2_{jR})$ can be disregarded since we only need an upper bound. The other terms in (3.13) have the following expressions

$$2E(\omega^1_{jR},\omega^2_{jR}) = -M_j(R)\int_R^\infty \log(r)\,r\omega_j(r)\,\mathrm{d}r\,, \qquad S(\omega^2_{jR}) \,=\, 2\pi\int_R^\infty \Phi(\omega_j(r))\,r\,\mathrm{d}r\,.$$

Since $\omega_j$ is decreasing, we have $\omega_j(r)\le M_j(r)/(\pi r^2)\le M$ for $r\ge R$. So, using Hypotheses 3.3, we deduce that $\Phi(\omega_j)\le C_1\omega_j + C_2\omega_j\log(M/\omega_j)$, where $C_1\in\mathbb{R}$ and $C_2<M/(8\pi)$. It follows that

$$2E(\omega^1_{jR},\omega^2_{jR}) + S(\omega^2_{jR}) \,\le\, 2\pi C_1\int_R^\infty \omega_j(r)r\,\mathrm{d}r + \int_R^\infty \Delta_j(r)\omega_j(r)r\,\mathrm{d}r\,, \tag{3.19}$$

where

$$\Delta_j(r) \;=\; 2\pi C_2 \log \frac{M}{\omega_j(r)} - M_j(R)\log(r)\,.$$

In view of (3.7), the first term in the right-hand side of (3.19) converges to zero uniformly in $j$ as $R \to +\infty$, and can therefore be absorbed in the quantity $\delta_1(R)$. To treat the second term, we fix a positive number $\alpha > 2$ such that $4\pi C_2 \alpha \le M$, and we introduce the mutually disjoints sets

$$I(\alpha,R) \;=\; \bigl\{r \ge R\,;\, \omega_j(r) \;\ge\; Mr^{-\alpha}\bigr\}\,, \qquad I(\alpha,R)^c \;=\; \bigl\{r \ge R\,;\, \omega_j(r) \;<\; M\,r^{-\alpha}\bigr\}\,. \tag{3.20}$$

As $M_j(R) \ge M/2$, it follows from (3.20) that $\Delta_j(r) \le 0$ when $r \in I(\alpha,R)$, so the last integral in (3.19) can be restricted to the complement $I(\alpha,R)^c$. But on that set we have the upper bound $\omega_j(r) < Mr^{-\alpha}$, where $\alpha > 2$, and we easily deduce that $\int_{I(\alpha,R)^c} \Delta_j(r)\omega_j(r) r\,\mathrm{d}r$ converges to zero as $R \to +\infty$, uniformly in $j$. Altogether we obtain (3.13).

It remains to establish (3.14), which is an easy task. Indeed $\bar\omega$ is a fixed function which satisfies the estimates (3.8), so that $2E(\bar\omega_R^1, \bar\omega_R^2) + E(\bar\omega_R^2) \to 0$ as $R \to +\infty$. In addition, we proved in Proposition 3.5 that the integral defining $S(\bar\omega)$ is absolutely convergent, and this implies that $S(\bar\omega_R^2) \to 0$ as $R \to +\infty$. We thus obtain (3.14), and the proof of Theorem 3.4 is complete. ☐

**Example 3.6.** *We consider the family of algebraic vortices with parameter $\kappa > 1$:*

$$\omega(r) \;=\; \frac{1}{(1+r^2)^\kappa}\,, \qquad M \;=\; 2\pi \int_0^\infty r\omega(r)\,\mathrm{d}r \;=\; \frac{\pi}{\kappa - 1}\,.$$

*The associated stream function $\psi$ satisfies $\psi(r) = \psi(0) + \int_0^r \psi'(s)\,\mathrm{d}s$ where*

$$\psi(0) \;=\; \int_0^\infty \log(r)\,\frac{r}{(1+r^2)^\kappa}\,\mathrm{d}r\,, \qquad \psi'(r) \;=\; \frac{1}{2(\kappa{-}1)r}\Bigl(1 - \frac{1}{(1+r^2)^{\kappa-1}}\Bigr)\,.$$

*We have $\Phi(\omega) = \int_0^\omega \phi(s)\,\mathrm{d}s$ where $\phi(\omega(r)) = \psi(r)$. Explicitly, for a few values of $\kappa$, we find*

$$\begin{array}{lll}
\kappa = \frac{3}{2}\;: & \psi(r) = \log\bigl(1 + \sqrt{1+r^2}\bigr) & \phi(\omega) = \log\Bigl(1 + \dfrac{1}{\omega^{1/3}}\Bigr) \\[2mm]
\kappa = 2\;: & \psi(r) = \dfrac{1}{4}\log\bigl(1+r^2\bigr) & \phi(\omega) = \dfrac{1}{8}\log\dfrac{1}{\omega} \\[2mm]
\kappa = 3\;: & \psi(r) = \dfrac{1}{8}\Bigl(\log\bigl(1+r^2\bigr) - \dfrac{1}{1+r^2}\Bigr) & \phi(\omega) = \dfrac{1}{24}\log\dfrac{1}{\omega} - \dfrac{\omega^{1/3}}{8}
\end{array}$$

*In all cases, we observe that*

$$\phi(\omega) \;=\; \Phi'(\omega) \;\sim\; \frac{1}{4\kappa(\kappa{-}1)}\,\log\frac{1}{\omega} \;=\; \frac{M}{4\pi\kappa}\,\log\frac{1}{\omega}\,, \quad \textit{as} \quad \omega \to 0\,.$$

*It follows that Hypotheses 3.3 are satisfied if and only if $\kappa > 2$.*

**Example 3.7.** *We next consider the Gaussian vortex $\omega(r) = e^{-r^2/4}$, where $M = 4\pi$. In that case we have $\psi(0) = \int_0^{+\infty} \log(r)\,e^{-r^2/4}\,\mathrm{d}r = 2\log(2) - \gamma_E$, so that the stream function satisfies*

$$\psi(r) \;=\; \psi(0) + \int_0^r \frac{2}{s}\Bigl(1 - e^{-s^2/4}\Bigr)\,\mathrm{d}s \;=\; 2\log(2) - \gamma_E + \mathrm{E_{in}}(r^2/4)\,,$$

*where*

$$\mathrm{E_{in}}(z) \;=\; \int_0^z \frac{1-e^{-t}}{t}\,\mathrm{d}t \;=\; \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k}\,\frac{z^k}{k!}\,, \qquad z \in \mathbb{C}\,.$$

*We conclude that*

$$\phi(\omega) \;=\; \Phi'(\omega) \;=\; 2\log(2) - \gamma_E + \mathrm{E_{in}}\Big(\log\frac{1}{\omega}\Big)\,.$$

*In particular* $\phi(\omega) \sim \log\log\frac{1}{\omega}$ *as* $\omega \to 0$*, and Hypotheses 3.3 are satisfied in that case.*

We do not have much information on the maximizer $\bar\omega$ whose existence is established in Theorem 3.4. We expect that, if $\Phi$ is as in Example 3.7, the maximizer is indeed the Gaussian vortex (2.39), but except from numerical evidence we have no proof so far. Similarly, we believe that the algebraic vortices (2.38) with $\kappa \ge 2$ are global maximizers, but this is known only in the particular case $\kappa = 2$, where maximality follows from the logarithmic HLS inequality (3.5).

The examples above also suggest that the decay rate of the maximizer $\bar\omega(x)$ as $|x| \to \infty$ strongly depends on the behavior of the function $\Phi(s)$ near $s = 0$. Extending the techniques in the proof of Theorem 3.4, one should be able to prove that, if $\Phi$ is differentiable to the right at the origin, the corresponding maximizer $\bar\omega$ is compactly supported. It is also worth mentioning that the entropy function $\Phi$ associated with any radially symmetric decreasing vortex $\bar\omega$ through the relation $\bar\psi(x) = \Phi'(\bar\omega(x))$ is necessarily concave on the range of $\bar\omega$, whereas no concavity assumption is included in Hypotheses 3.3. This suggests that the maximizer $\bar\omega$ corresponding to a non-concave function $\Phi$ should be discontinuous, so that its range does not include the intervals where $\Phi$ does not coincide with its concave hull.

# 4 Stability of viscous vortices

In this final section, we give a new proof of the nonlinear stability of the Oseen vortices, which are self-similar solutions of the Navier-Stokes equations in $\mathbb{R}^2$. Our approach relies on the functional-analytic tools developed in Section 2, in connexion with Arnold's variational principle, although we now consider a dissipative equation for which the Casimir functions (1.9) are no longer conserved quantities. Let $w = w(y,\tau) \in \mathbb{R}$ denote the vorticity of the fluid at point $y \in \mathbb{R}^2$ and time $\tau > 0$, and let $\phi = \phi(y,\tau) \in \mathbb{R}$ be the associated stream function. The vorticity formulation of the Navier-Stokes equations is

$$\partial_\tau w(y,\tau) + \big\{\phi, w\big\}(y,\tau) \;=\; \nu\Delta(y,\tau)\,, \qquad \Delta\phi(y,\tau) = w(y,\tau)\,, \tag{4.1}$$

where $\{\phi, w\} = \nabla^\perp\phi\cdot\nabla w$ is the Poisson bracket, $\nu > 0$ is the viscosity parameter, and the Laplace operator $\Delta$ acts on the space variable $y \in \mathbb{R}^2$. As in [16, 17], we introduce self-similar variables $x = y/\sqrt{\nu\tau}$ and $t = \log(\tau/T)$, where $T > 0$ is an arbitrary time scale. More precisely, we look for solutions of (4.1) in the form

$$w(y,\tau) \;=\; \frac{1}{\tau}\,\omega\Big(\frac{y}{\sqrt{\nu\tau}}\,,\,\log\frac{\tau}{T}\Big)\,, \qquad \phi(y,\tau) \;=\; \nu\,\psi\Big(\frac{y}{\sqrt{\nu\tau}}\,,\,\log\frac{\tau}{T}\Big)\,. \tag{4.2}$$

The evolution equation for the rescaled vorticity $\omega$ is

$$\partial_t\omega(x,t) + \big\{\psi,\omega\big\}(x,t) \;=\; \mathcal{L}\omega(x,t)\,, \qquad \Delta\psi(x,t) \;=\; \omega(x,t)\,, \tag{4.3}$$

where $\{\psi,\omega\} = \nabla^\perp\psi\cdot\nabla\omega$ and $\mathcal{L}$ is the Fokker-Planck operator

$$\mathcal{L} \;=\; \Delta + \frac{1}{2}\,x\cdot\nabla + 1\,. \tag{4.4}$$

Let $\bar\omega$ be the vortex with Gaussian profile (2.39), namely

$$\bar\omega(x) \;=\; \frac{1}{4\pi}\,e^{-|x|^2/4}\,, \qquad \bar u(x) \;=\; \nabla^\perp\bar\psi(x) \;=\; \frac{1}{2\pi}\,\frac{x^\perp}{|x|^2}\Big(1 - e^{-|x|^2/4}\Big)\,. \tag{4.5}$$

It is easy to verify that $\mathcal{L}\bar{\omega} = 0$ and $\{\bar{\psi}, \bar{\omega}\} = 0$. This implies that $\omega = \alpha\bar{\omega}$ is a stationary solution of (4.3) for any $\alpha \in \mathbb{R}$. This family of equilibria is known to be stable with respect to perturbations in various weighted $L^2$ spaces, see [17, 14]. We present here a new stability proof, which may be easier to adapt to more general situations.

## 4.1 Nonlinear stability of Oseen vortices

Given any $\alpha \in \mathbb{R}$, we consider solutions of (4.3) of the form $\omega = \alpha\bar{\omega} + \tilde{\omega}$, $\psi = \alpha\bar{\psi} + \tilde{\psi}$. The perturbation $\tilde{\omega}$ satisfies the modified equation

$$\partial_t \tilde{\omega} + \alpha\{\bar{\psi}, \tilde{\omega}\} + \alpha\{\tilde{\psi}, \bar{\omega}\} + \{\tilde{\psi}, \tilde{\omega}\} \;=\; \mathcal{L}\tilde{\omega}\,, \tag{4.6}$$

where it is understood that the stream function $\tilde{\psi}$ is expressed in terms of $\tilde{\omega}$ via the formula (1.14), so that $\Delta\tilde{\psi} = \tilde{\omega}$. We assume henceforth that the perturbation $\tilde{\omega}$ satisfies the moment conditions

$$\int_{\mathbb{R}^2} \tilde{\omega}\,\mathrm{d}x \;=\; 0\,, \qquad \text{and} \qquad \int_{\mathbb{R}^2} x_j \tilde{\omega}\,\mathrm{d}x \;=\; 0 \quad \text{for } j = 1, 2\,, \tag{4.7}$$

which are preserved under the evolution defined by (4.6). As is shown at the end of Ref. [17], this hypothesis does not restrict the generality, in the sense that stability with respect to general perturbations (with no moment conditions) can then deduced by a simple argument. As for the existence of solutions to (4.6), we have the following standard result:

**Lemma 4.1.** *The Cauchy problem for equation* (4.6) *is globally well-posed in the weighted $L^2$ space $X$ defined by* (2.4)*, where $\mathcal{A}(x) = 4|x|^{-2}(e^{|x|^2/4} - 1)$, and the subspace $\mathcal{X}_1 \subset X$ defined by* (2.24) *is invariant under the evolution.*

**Proof.** It is known that the vorticity equation (4.3) or (4.6) is globally well-posed in various weighted $L^2$ spaces, see e.g. [16, 14, 15]. The nearly gaussian weight $\mathcal{A}$ is not explicitly considered in those references, but the arguments therein can be easily modified to cover that case too. If $\mathcal{A}^{1/2}\tilde{\omega} \in L^2(\mathbb{R}^2)$, then all moments of $\tilde{\omega}$ are well defined, and a direct calculation shows that the conditions (4.7) are preserved under the evolution, so that (4.6) is globally well-posed in the subspace $\mathcal{X}_1$. ☐

Let $\tilde{\omega}_0 \in \mathcal{X}_1$, and let $\tilde{\omega} \in C^0([0, +\infty), \mathcal{X}_1)$ be the solution of (4.6) with initial data $\tilde{\omega}_0$. By parabolic regularization, we have $\tilde{\omega}(\cdot, t) \in Z_1 := Z \cap \mathcal{X}_1$ for all $t > 0$, where $Z$ is the weighted Sobolev space

$$Z \;=\; \Big\{\omega \in H^1(\mathbb{R}^2)\,;\, \mathcal{A}^{1/2}\omega \in L^2(\mathbb{R}^2)\,,\; \mathcal{A}^{1/2}\nabla\omega \in L^2(\mathbb{R}^2)\Big\}\,. \tag{4.8}$$

For later use, we introduce the following quadratic form on $Z$:

$$Q(\omega) \;=\; \int_{\mathbb{R}^2} \Big(\mathcal{A}(x)|\nabla\omega(x)|^2 - \mathcal{B}(x)\omega(x)^2\Big)\,\mathrm{d}x\,, \qquad \omega \in Z\,, \tag{4.9}$$

where

$$\mathcal{B} \;=\; 1 + \frac{1}{2}\Big(\Delta\mathcal{A} - \frac{x}{2}\cdot\nabla\mathcal{A} + \mathcal{A}\Big) \;=\; 1 + \mathcal{A} - \frac{x\cdot\nabla\mathcal{A}}{|x|^2}\,. \tag{4.10}$$

We shall verify in Section A.3 that $\mathcal{A}/2 \le \mathcal{B} \le 2\mathcal{A}$, so that the form $Q$ is well defined.

The following coercivity result plays a crucial role in our argument.

**Theorem 4.2.** *The quadratic form $Q$ defined by* (4.9) *is coercive on the subspace $Z_1 = Z \cap \mathcal{X}_1$ : there exists a constant $\delta > 0$ such that*

$$Q(\omega) \;\ge\; \delta \int_{\mathbb{R}^2} \mathcal{A}(x)\omega(x)^2 \,\mathrm{d}x\,, \qquad \textit{for all } \omega \in Z_1\,. \tag{4.11}$$

The proof of Theorem 4.2 requires a careful analysis, which is postponed to Section 4.2 below. In particular, we shall see that the quadratic form $Q$ is not positive on the whole space $Z$, because it takes negative values on a one-dimensional subspace made of radially symmetric functions. If we restrict ourselves to functions with zero mean, the form $Q$ is nonnegative but vanishes on a two-dimensional subspace due to translation invariance. Therefore, all moment conditions (4.7) are necessary to establish the coercivity of $Q$.

Returning to the solution $\tilde\omega \in C^0([0,+\infty),\mathcal{X}_1)$ of (4.6), we define for all $t > 0$ the quantities

$$\begin{aligned}
\tilde J(t) \;&=\; \frac{1}{2}\int_{\mathbb{R}^2}\Bigl(\mathcal{A}(x)\tilde\omega(x,t)^2 + \tilde\psi(x,t)\tilde\omega(x,t)\Bigr)\,\mathrm{d}x \;=\; J(\tilde\omega(t))\,,\\
\tilde Q(t) \;&=\; \int_{\mathbb{R}^2}\Bigl(\mathcal{A}(x)|\nabla\tilde\omega(x,t)|^2 - \mathcal{B}(x)\tilde\omega(x,t)^2\Bigr)\,\mathrm{d}x \;=\; Q(\tilde\omega(t))\,,\\
\tilde N(t) \;&=\; \frac{1}{2}\int_{\mathbb{R}^2}\bigl\{\mathcal{A}(x),\tilde\psi(x,t)\bigr\}\tilde\omega(x,t)^2\,\mathrm{d}x \;=:\; N(\tilde\omega(t))\,.
\end{aligned} \tag{4.12}$$

The key observation is:

**Proposition 4.3.** *If $\tilde\omega \in C^0([0,+\infty),\mathcal{X}_1)$ is a solution of* (4.6)*, the quantities defined in* (4.12) *satisfy*

$$\tilde J'(t) \;=\; -\tilde Q(t) - \tilde N(t)\,, \qquad \textit{for all } t > 0\,. \tag{4.13}$$

**Proof.** Using the evolution equation (4.6), we find

$$\begin{aligned}
\tilde J'(t) \;&=\; \int_{\mathbb{R}^2}\Bigl(\mathcal{A}(x)\tilde\omega(x,t) + \tilde\psi(x,t)\Bigr)\partial_t\tilde\omega(x,t)\,\mathrm{d}x\\
&=\; \int_{\mathbb{R}^2}\Bigl(\mathcal{A}\tilde\omega + \tilde\psi\Bigr)\Bigl(\mathcal{L}\tilde\omega - \alpha\{\bar\psi,\tilde\omega\} - \alpha\{\tilde\psi,\bar\omega\} - \{\tilde\psi,\tilde\omega\}\Bigr)(x,t)\,\mathrm{d}x\,.
\end{aligned} \tag{4.14}$$

We first consider the terms involving the diffusion operator $\mathcal{L}$ in (4.14). We observe that

$$\int_{\mathbb{R}^2}\tilde\psi(x,t)\,\mathcal{L}\tilde\omega(x,t)\,\mathrm{d}x \;=\; \int_{\mathbb{R}^2}\tilde\omega(x,t)^2\,\mathrm{d}x\,, \tag{4.15}$$

because $\mathcal{L}\tilde\omega = \Delta\tilde\omega + \frac{1}{2}\operatorname{div}(x\tilde\omega)$ and

$$\begin{aligned}
\int_{\mathbb{R}^2}\tilde\psi\,\Delta\tilde\omega\,\mathrm{d}x \;&=\; \int_{\mathbb{R}^2}\bigl(\Delta\tilde\psi\bigr)\,\tilde\omega\,\mathrm{d}x \;=\; \int_{\mathbb{R}^2}\tilde\omega^2\,\mathrm{d}x\,,\\
\int_{\mathbb{R}^2}\tilde\psi\,\operatorname{div}(x\tilde\omega)\,\mathrm{d}x \;&=\; -\int_{\mathbb{R}^2}\bigl(\Delta\tilde\psi\bigr)\bigl(x\cdot\nabla\tilde\psi\bigr)\,\mathrm{d}x \;=\; \frac{1}{2}\int_{\mathbb{R}^2}\operatorname{div}\bigl(x|\nabla\tilde\psi|^2\bigr)\,\mathrm{d}x \;=\; 0\,.
\end{aligned}$$

On the other hand, integrating by parts we obtain by direct calculation

$$\int_{\mathbb{R}^2}\mathcal{A}(x)\tilde\omega(x,t)\,\mathcal{L}\tilde\omega(x,t)\,\mathrm{d}x \;=\; -Q(\tilde\omega(t)) - \int_{\mathbb{R}^2}\tilde\omega(x,t)^2\,\mathrm{d}x\,. \tag{4.16}$$

We next compute the advection terms in (4.14), which are proportional to $\alpha$. We claim that

$$I(\tilde\omega) \;:=\; \int_{\mathbb{R}^2}\Bigl(\mathcal{A}\tilde\omega + \tilde\psi\Bigr)\Bigl(\{\bar\psi,\tilde\omega\} + \{\tilde\psi,\bar\omega\}\Bigr)\,\mathrm{d}x \;=\; 0\,. \tag{4.17}$$

This identity is not surprising, as it means that the quadratic form $J$ is invariant under the evolution defined by the linearized Euler equation at $\bar{\omega}$, see (1.6) for an analogue in the finite-dimensional case. It can also be verified by direct calculations:

$$\begin{aligned}
&\int_{\mathbb{R}^2} \mathcal{A}\tilde{\omega}\{\bar{\psi},\tilde{\omega}\}\,\mathrm{d}x \,=\, \frac{1}{2}\int_{\mathbb{R}^2} \mathcal{A}\{\bar{\psi},\tilde{\omega}^2\}\,\mathrm{d}x \,=\, \frac{1}{2}\int_{\mathbb{R}^2}\{\mathcal{A},\bar{\psi}\}\tilde{\omega}^2\,\mathrm{d}x \,=\, 0\,,\\
&\int_{\mathbb{R}^2} \tilde{\psi}\{\tilde{\psi},\bar{\omega}\}\,\mathrm{d}x \,=\, \int_{\mathbb{R}^2}\{\tilde{\psi},\tilde{\psi}\}\bar{\omega}\,\mathrm{d}x \,=\, 0\,,\\
&\int_{\mathbb{R}^2}\Big(\mathcal{A}\tilde{\omega}\{\tilde{\psi},\bar{\omega}\} + \tilde{\psi}\{\bar{\psi},\tilde{\omega}\}\Big)\,\mathrm{d}x \,=\, \int_{\mathbb{R}^2}\tilde{\omega}\Big(\mathcal{A}\{\tilde{\psi},\bar{\omega}\} + \{\tilde{\psi},\bar{\psi}\}\Big)\,\mathrm{d}x \,=\, 0\,.
\end{aligned}$$

Here we used the fact that $\{\mathcal{A},\bar{\psi}\} = 0$, because $\mathcal{A}$ and $\bar{\psi}$ are radially symmetric. Moreover,

$$\mathcal{A}\{\tilde{\psi},\bar{\omega}\} + \{\tilde{\psi},\bar{\psi}\} \,=\, (\nabla\tilde{\psi})^{\perp}\cdot\big(\mathcal{A}\nabla\bar{\omega} + \nabla\bar{\psi}\big) \,=\, 0\,,$$

by the very definition of $\mathcal{A}$, see (2.3). This proves (4.17).

Finally, integrating by parts the last term in (4.14), we find

$$N(\tilde{\omega}) \,:=\, \int_{\mathbb{R}^2}\Big(\mathcal{A}\tilde{\omega} + \tilde{\psi}\Big)\{\tilde{\psi},\tilde{\omega}\}\,\mathrm{d}x \,=\, \int_{\mathbb{R}^2}\mathcal{A}\tilde{\omega}\{\tilde{\psi},\tilde{\omega}\}\,\mathrm{d}x \,=\, \frac{1}{2}\int_{\mathbb{R}^2}\{\mathcal{A},\tilde{\psi}\}\tilde{\omega}^2\,\mathrm{d}x\,. \qquad (4.18)$$

Combining (4.14)–(4.18), we obtain the desired result. ☐

To control the nonlinear term $N(\tilde{\omega})$, we use the following estimate.

**Lemma 4.4.** *There exists a constant $C_0 > 0$ such that, for all $\tilde{\omega} \in Z$, the nonlinear term* (4.18) *satisfies*

$$|N(\tilde{\omega})| \,\le\, C_0\,\|\mathcal{A}^{1/2}\tilde{\omega}\|_{L^2}^2\Big(\|\mathcal{A}^{1/2}\tilde{\omega}\|_{L^2} + \|\mathcal{A}^{1/2}\nabla\tilde{\omega}\|_{L^2}\Big)\,. \qquad (4.19)$$

**Proof.** We have $|\{\mathcal{A},\tilde{\psi}\}| \le C|\nabla\mathcal{A}|\,|\nabla\tilde{\psi}| \le C|x|\mathcal{A}|\nabla\tilde{\psi}|$, hence

$$|N(\tilde{\omega})| \,\le\, C\int_{\mathbb{R}^2}|x|\,|\nabla\tilde{\psi}|\,\mathcal{A}\,\tilde{\omega}^2\,\mathrm{d}x \,\le\, C\,\||x||\nabla\tilde{\psi}|\|_{L^\infty}\,\|\mathcal{A}^{1/2}\tilde{\omega}\|_{L^2}^2\,.$$

On the other hand, using Proposition B.1 in [16], Hölder's inequality and Sobolev's embedding theorem, we find

$$\||x||\nabla\tilde{\psi}|\|_{L^\infty} \,\le\, C\big(\|\langle x\rangle\tilde{\omega}\|_{L^{3/2}} + \|\langle x\rangle\tilde{\omega}\|_{L^3}\big) \,\le\, C\big(\|\mathcal{A}^{1/2}\tilde{\omega}\|_{L^2} + \|\mathcal{A}^{1/2}\nabla\tilde{\omega}\|_{L^2}\big)\,,$$

where $\langle x\rangle = (1+|x|^2)^{1/2}$. Combining these estimates we arrive at (4.19). ☐

We are now able to state our final result:

**Theorem 4.5.** *There exist positive constants $C_1$, $\epsilon_0$, and $\mu$ such that, for any $\alpha \in \mathbb{R}$ and any $\tilde{\omega}_0 \in \mathcal{X}_1$ satisfying $\|\tilde{\omega}_0\|_X \le \epsilon_0$, the solution of* (4.6) *with initial data $\tilde{\omega}_0$ satisfies*

$$\|\tilde{\omega}(t)\|_X^2 \,\le\, C_1\,\|\tilde{\omega}_0\|_X^2\,e^{-\mu t}\,, \qquad \textit{for all } t \ge 0\,. \qquad (4.20)$$

**Proof.** If $\tilde{\omega} \in C^0([0,+\infty),\mathcal{X}_1)$ is the solution of (4.6) with initial data $\tilde{\omega}_0$, we define

$$m_0(t) \,=\, \|\tilde{\omega}(t)\|_X^2 \,=\, \|\mathcal{A}^{1/2}\tilde{\omega}(t)\|_{L^2}^2 \quad (t \ge 0)\,, \qquad m_1(t) \,=\, \|\mathcal{A}^{1/2}\nabla\tilde{\omega}(t)\|_{L^2}^2 \quad (t > 0)\,.$$

For the Gaussian vortex, we proved in Section 2 that Hardy's inequality (2.25) holds for some $C_H < 1$. Thus, by Theorems 2.5 and 2.8, there exists a constant $\gamma \in (0,1)$ such that

$$\frac{\gamma}{2}\, m_0(t) \,\le\, \tilde{J}(t) \,\le\, \frac{1}{2}\, m_0(t)\,, \qquad t \ge 0\,. \tag{4.21}$$

On the other hand, by Theorem 4.2, there exists $\delta > 0$ such that

$$\tilde{Q}(t) \,\ge\, \delta\, m_0(t)\,, \quad \text{and} \quad \tilde{Q}(t) \,\ge\, m_1(t) - 2m_0(t)\,, \qquad t > 0\,, \tag{4.22}$$

where the second inequality follows from the definition (4.9) and the inequality $\mathcal{B} \le 2\mathcal{A}$. Taking a convex combination of both estimates in (4.22), we deduce

$$\tilde{Q}(t) \,\ge\, \mu\big(m_0(t) + m_1(t)\big)\,, \qquad t > 0\,, \tag{4.23}$$

where $\mu = \delta/(3+\delta)$. Finally, it follows from Lemma 4.4 and Young's inequality that

$$|\tilde{N}(t)| \,\le\, C_0 m_0(t)\big(m_0(t)^{1/2} + m_1(t)^{1/2}\big) \,\le\, \frac{\mu}{4}\big(m_0(t) + m_1(t)\big) + \frac{2C_0^2}{\mu}\, m_0(t)^2\,. \tag{4.24}$$

Now, as long as $m_0(t) \le \epsilon^2 := \mu^2/(8C_0^2)$, we have by (4.13), (4.21), (4.23), (4.24)

$$\tilde{J}'(t) \,=\, -\tilde{Q}(t) - \tilde{N}(t) \,\le\, -\frac{\mu}{2}\big(m_1(t) + m_0(t)\big) \,\le\, -\mu \tilde{J}(t)\,,$$

which implies

$$\gamma m_0(t) \,\le\, 2\tilde{J}(t) \,\le\, 2\tilde{J}(0)\, e^{-\mu t} \,\le\, m_0(0)\, e^{-\mu t}\,.$$

As a consequence, if we assume that $\|\tilde{\omega}_0\|_X^2 = m_0(0) \le \epsilon_0^2 := \gamma\epsilon^2$, we have $m_0(t) \le \epsilon^2$ for all $t \ge 0$ and estimate (4.20) holds with $C_1 = \gamma^{-1}$. □

We briefly indicate here the meaning of our result for the Navier-Stokes equations in the original, unscaled variables. If $\omega = \alpha\bar{\omega} + \tilde{\omega}$ where $\tilde{\omega} \in C^0([0,+\infty), \mathcal{X}_1)$ is as in Theorem 4.5, the vorticity $w$ defined by (4.2) satisfies, in particular, the estimate

$$\int_{\mathbb{R}^2} \Big| w(y,\tau) - \frac{\alpha}{4\pi\tau}\, e^{-|y|^2/(4\tau)} \Big| \,\mathrm{d}y \,=\, \mathcal{O}\big(\tau^{-\mu/2}\big)\,, \qquad \text{as } \tau \to +\infty\,,$$

which means that $w(\cdot,\tau)$ converges to a self-similar solution with Gaussian profile as $\tau \to +\infty$. As is shown in [14, Theorem 1.2], that property holds in fact for all solutions of the vorticity equation (4.1) in $L^1(\mathbb{R}^2)$, although it is not possible to specify any decay rate in the general case. Note that the evolution defined by (4.1) in $L^1(\mathbb{R}^2)$ preserves the total mass, so that we necessarily have $\int_{\mathbb{R}^2} w(y,\tau)\,\mathrm{d}y = \alpha$ for all $\tau > 0$.

**Remark 4.6.** *Except for a slight difference in the definition of the function space $X$, Theorem 4.5 coincides with the well known stability result established in [14, Proposition 4.5]. The approach originally developed by C.E. Wayne and the first author relies on conserved quantities related to symmetries of the problem, such as the second order moment $I(\omega)$ in (1.15). In many respects, it is simpler than ours, and it provides an estimate of the form (4.20) with explicit constants $C_1$ and $\mu$. Note also that, in the limit of large circulation numbers $|\alpha| \to \infty$, the enhanced dissipation effect due to fast rotation can be used to improve both the decay rate of the perturbations and the size of the basin of attraction of the vortex, see [15].*

### 4.2 Coercivity of the diffusive quadratic form

This section is entirely devoted to the proof of Theorem 4.2, which is a key ingredient in Theorem 4.5. We first observe that the functions $\mathcal{A}(x), \mathcal{B}(x)$ in (4.9) are both radially symmetric, with radial profiles $A(r), B(r)$ given by the explicit expressions

$$A(r) \,=\, \frac{e^s - 1}{s}\,, \qquad B(r) \,=\, \frac{1}{2s^2}\Big(e^s(1+s) - 1 - 2s\Big) + 1\,, \qquad s \,=\, \frac{r^2}{4}\,. \tag{4.25}$$

On can also verify that $B/A$ is a decreasing function of $r$ satisfying $1/2 \le B(r)/A(r) \le 7/4$ for all $r > 0$, see Section A.3.

We next follow a similar approach as in Section 2. If $\omega \in Z$ is decomposed in Fourier series like in (2.14), we have

$$Q(\omega) \,=\, 2\pi \sum_{k\in\mathbb{Z}} \int_0^\infty \Big\{A(r)\Big(|\omega_k'(r)|^2 + \frac{k^2}{r^2}\,|\omega_k(r)|^2\Big) - B(r)|\omega_k(r)|^2\Big\} r\,\mathrm{d}r\,, \tag{4.26}$$

and we observe that $\omega \in Z_1$ if and only if

$$\int_0^\infty \omega_0(r)\, r\,\mathrm{d}r \,=\, 0\,, \qquad \text{and} \qquad \int_0^\infty \omega_{\pm 1}(r)\, r^2\,\mathrm{d}r \,=\, 0\,.$$

Introducing the new variables $w_k = A^{1/2}\omega_k \equiv e^{\chi}\omega_k$, where $\chi = \frac{1}{2}\log(A)$, we obtain after straightforward calculations

$$Q(\omega) \,=\, 2\pi \sum_{k\in\mathbb{Z}} \int_0^\infty \Big\{|w_k'(r)|^2 + \frac{k^2}{r^2}\,|w_k(r)|^2 + W(r)|w_k(r)|^2\Big\} r\,\mathrm{d}r\,, \tag{4.27}$$

where the potential $W$ is defined by

$$W(r) \,=\, \chi''(r) + \frac{1}{r}\,\chi'(r) + \chi'(r)^2 - \frac{B(r)}{A(r)} \,=\, \frac{r}{2}\,\chi'(r) - \chi'(r)^2 - \frac{1}{2} - e^{-2\chi(r)}\,. \tag{4.28}$$

The coercivity estimate (4.11) is thus equivalent to the inequality

$$\int_0^\infty \Big\{|w_k'(r)|^2 + \frac{k^2}{r^2}\,|w_k(r)|^2 + W(r)|w_k(r)|^2\Big\} r\,\mathrm{d}r \,\ge\, \delta \int_0^\infty |w_k(r)|^2\, r\,\mathrm{d}r\,, \tag{4.29}$$

which should hold for all $k \in \mathbb{Z}$ under the conditions

$$\int_0^\infty w_0(r)\, e^{-\chi(r)}\, r\,\mathrm{d}r \,=\, 0\,, \qquad \text{and} \qquad \int_0^\infty w_{\pm 1}(r)\, e^{-\chi(r)}\, r^2\,\mathrm{d}r \,=\, 0\,. \tag{4.30}$$

For any $k \in \mathbb{Z}$, we denote by $L_k$ the selfadjoint operator in $Y = L^2(\mathbb{R}_+, r\,\mathrm{d}r)$ defined by

$$L_k g \,=\, -\frac{1}{r}\partial_r\big(r\partial_r g\big) + \frac{k^2}{r^2}\, g + W g\,. \tag{4.31}$$

The domain of $L_k$ is exactly the same as for the harmonic oscillator in $\mathbb{R}^2$, because the potential $W$ defined by (4.28) satisfies

$$W(r) \,>\, \frac{r^2}{16} - \frac{3}{2} \quad \text{for all } r > 0\,, \quad \text{and} \quad W(r) \,\sim\, \begin{cases} -3/2 & \text{as } r \to 0\,, \\ r^2/16 & \text{as } r \to \infty\,, \end{cases} \tag{4.32}$$

see Section A.3. Our goal is to prove the lower bound $L_k \ge \delta$ in the entire space $Y$ when $|k| \ge 2$, and in the subspaces given by conditions (4.30) when $k = 0$ or $k = \pm 1$. We consider three cases separately.

**Case 1:** When $|k| \ge 2$, the desired inequality is simply obtained by comparing $L_k$ with the usual harmonic operator. Indeed, we know from (4.31), (4.32) that

$$L_k \;>\; -\partial_r^2 - \frac{1}{r}\,\partial_r + \frac{k^2}{r^2} + \frac{r^2}{16} - \frac{3}{2} \;\ge\; \frac{|k|}{2} - 1\,, \tag{4.33}$$

where inequalities are between selfadjoint operators on $Y$. Thus $L_k \ge 1/2$ when $|k| \ge 3$, and there exists $\delta > 0$ such that $L_k \ge \delta$ when $|k| = 2$, because the inequality in (4.32) is strict.

**Case 2:** When $|k| = 1$, the lower bound (4.33) is of no use, but it is easy to verify that $L_k \ge 0$ in that case. Indeed, we claim that $L_k g_1 = 0$ where $g_1(r) = e^{\chi(r)}\, r\, e^{-r^2/4}$. Since $g_1$ is a positive function vanishing at the origin and at infinity, this means that 0 is the lowest eigenvalue of $L_k$ in $Y$ when $k = \pm 1$. To prove the above claim, we first observe that, for any (smooth) function $f$ on $\mathbb{R}_+$, we have the identity

$$\widetilde{L}_k f \;:=\; e^{\chi} L_k(e^{\chi} f) \;=\; -\frac{1}{r}\partial_r\bigl(rA\partial_r f\bigr) + \frac{k^2}{r^2}\,Af - Bf\,, \tag{4.34}$$

because this is the property we used to go from (4.26) to (4.27). On the other hand, in view of (2.2) and (2.3), we have the identity

$$-\frac{1}{r}\partial_r\bigl(rA\partial_r\omega_*\bigr) \;=\; \omega_*\,, \tag{4.35}$$

which holds in fact for any vorticity profile $\omega_*$, if $A$ is defined by (2.3). In the case of the Lamb-Oseen vortex, if we differentiate the equality (4.35) with respect to $r$, we find that the function $f = -2\omega_*' = r\,e^{-r^2/4}$ satisfies the relation

$$-\frac{1}{r}\partial_r\bigl(rA\partial_r f\bigr) + \frac{1}{r^2}\,Af - \Bigl(A'' + \frac{2}{r}\,A' - \frac{r}{2}\,A'\Bigr)f \;=\; f\,. \tag{4.36}$$

But $A'' + 2A'/r - rA'/2 = B - 1$ by (4.10), so combining (4.34) and (4.36) we conclude that $\widetilde{L}_k f = 0$ if $|k| = 1$, which is the desired result.

To get coercivity, we now restrict ourselves to the subspace $Y_1 \subset Y$ of all functions $g$ satisfying $\langle g, h_1\rangle = 0$ where $h_1(r) = r\,e^{-\chi(r)}$, see the second relation in (4.30). It is important to observe that $h_1$ is not proportional to $g_1$, so that $Y_1$ is *not* the orthogonal complement in $Y$ of the eigenspace spanned by $g_1$. However, we have $\langle g_1, h_1\rangle = 8 \neq 0$, which means that the closed hyperplane $Y_1$ does not contain the eigenfunction $g_1$. In view of Remark 4.8 below, we conclude that there exists some $\delta > 0$ such that $L_k \ge \delta$ on $Y_1$ when $|k| = 1$.

**Case 3:** Finally, we consider the radially symmetric case where $k = 0$. The difficulty here is that the operator $L_0$ is not positive on the entire space $Y$. A numerical calculation indicates that $L_0$ has one negative eigenvalue $\mu_0 \approx -0.722$, and that the next eigenvalue $\mu_1 \approx 0.615$ is positive. So it is essential to use the first relation in (4.30), and to restrict our analysis to the subspace $Y_0$ of all $g \in Y$ such that $\langle g, h_0\rangle = 0$, where $h_0(r) = e^{-\chi(r)}$. Our strategy is to apply Lemma 4.7 below with $a = -\mu_0$, $b = \mu_1$, $\psi = h_0/\|h_0\|$, and $\phi = g_0/\|g_0\|$, where $g_0$ denotes a positive function in the kernel of $L_0 - \mu_0$. Estimate (4.41) can be used to prove coercivity of $L_0$ on $Y_0$ if we have good lower bounds on the eigenvalues $\mu_0, \mu_1$ and on the scalar product $|\langle\phi,\psi\rangle|$, which measures the angle between the linear spaces spanned by $g_0$ and $h_0$.

We first estimate the lowest eigenvalue $\mu_0$. We know from the previous step that $L_1 g_1 = 0$. Defining $g = c g_1/r = c e^{\chi} e^{-r^2/4}$, where $c = (2\log(2))^{-1/2}$ is a normalization factor chosen so that $\|g\| = 1$, we deduce that $L_0 g = (2/r)\partial_r g$. This gives the relation

$$\Bigl(L_0 + \frac{3}{4}\Bigr) g \,=\, R\,, \qquad \text{where} \quad R \,=\, \frac{2}{r}\,\partial_r g + \frac{3}{4}\,g \,=\, \Bigl(\frac{3}{4} - \frac{B-1}{A}\Bigr) g\,, \tag{4.37}$$

where we used the identity $(B-1)/A = 1 - A'/(rA) = 1 - 2\chi'/r$, see (4.10). In Section A.3 below, we show that $B - 1 < 3A/4$, so that $R > 0$. This means that the operator $L_0 + \frac{3}{4}$ admits a positive supersolution, and using Sturm-Liouville's theory we conclude that $L_0 + \frac{3}{4} > 0$, so that $\mu_0 > -3/4$. Actually the function $g$ is a remarkably accurate quasimode, in the sense that the remainder $R$ in (4.37) is small. The norm of $R$ in $Y = L^2(\mathbb{R}_+, r\,\mathrm{d}r)$ can be computed explicitly, see Section A.4. The result is

$$\int_0^\infty R(r)^2 r\,\mathrm{d}r \,=\, \frac{1}{16\log(2)}\Bigl(3 - \log(2) - 2\log(\pi)\Bigr)\,, \tag{4.38}$$

so that $\epsilon := \|R\|_Y \approx 0.0396$. Since $L_0$ is a selfadjoint operator, we deduce that $L_0$ has an eigenvalue in the interval $[-3/4, -3/4+\epsilon]$. Anticipating the fact (established below) that $L_0$ has a unique negative eigenvalue, we conclude that $\mu_0 \in [-3/4, -3/4+\epsilon]$.

We next estimate the second eigenvalue $\mu_1$ of $L_0$. It is convenient here to observe that, if $g = e^{\chi} f$, the relation $L_0 g = \mu g$ is equivalent to the generalized eigenvalue problem $\widetilde{L}_0 f = \mu A f$, where $\widetilde{L}_k$ is defined in (4.34). The second eigenvalue of that problem is characterized by the inf-sup formula

$$\mu_1 \,=\, \inf_{f\in\mathcal{F}} \sup_{r>0} \bigl(\mathcal{R}[f]\bigr)(r) \,=\, \sup_{f\in\mathcal{F}} \inf_{r>0} \bigl(\mathcal{R}[f]\bigr)(r)\,, \quad \text{where} \quad \mathcal{R}[f] \,=\, \frac{\widetilde{L}_0 f}{Af}\,. \tag{4.39}$$

Here $\mathcal{F}$ denotes the class of all (smooth) functions $f : [0,+\infty) \to \mathbb{R}$ such that $f(0) = 1$, $f(r) \to 0$ as $r \to +\infty$, and $f$ has exactly one zero in the interval $(0,+\infty)$. Our first trial function is $f(r) = e^{-s}(1-\alpha s)$, where $s = r^2/4$ and $\alpha = \log(2)^{-1}$. The value of $\alpha$ is chosen so that the Rayleigh quotient has no singularity:

$$\mathcal{R}[f] \,=\, \frac{e^{-s}(1 + (2{-}\alpha)s + 2\alpha s^2) - (1 + (1{-}\alpha)s + \alpha s^2)}{2s(1-e^{-s})(1-\alpha s)}\,, \qquad s = \frac{r^2}{4}\,.$$

It happens that $\mathcal{R}[f]$ is a decreasing function on $\mathbb{R}_+$, with $\mathcal{R}[f](0) = -3/4+\alpha$ and $\mathcal{R}[f](+\infty) = 1/2$. In view of (4.39), this implies that $1/2 < \mu_1 < -3/4 + \alpha \approx 0.69$. A better approximation is obtained using the improved try

$$f(r) \,=\, e^{-s}(1-\alpha s)(1+\beta s)\,, \qquad \text{where} \quad \beta \,=\, \frac{\alpha(1 - 2\,e^{-1/\alpha})}{2\alpha - 1 + 2\,e^{-1/\alpha}(1-\alpha)}\,.$$

If $1/2 < \alpha < \log(2)^{-1}$, then $\beta > 0$ and the Rayleigh quotient has no singularity in the interval $(0,+\infty)$. Taking for instance $\alpha = 1.4$ gives the excellent lower bound $\mu_1 \ge 0.6$.

Finally, we use the quasimode $g$ in (4.37) and a standard perturbation argument to estimate the true eigenfunction corresponding to the lowest eigenvalue $\mu_0$. We first look for a non-normalized eigenfunction of the form $g_0 = g - f$, where $f \perp g_0$. We have

$$0 \,=\, (L_0 - \mu_0) g_0 \,=\, (L_0 - \mu_0) g - (L_0 - \mu_0) f \,=\, R - \bigl(\mu_0 + \tfrac{3}{4}\bigr) g - (L_0 - \mu_0) f\,,$$

so that $f = (L_0 - \mu_0)^{-1}\bigl(R - (\mu_0 + \frac{3}{4}) g\bigr)$, where $(L_0 - \mu_0)^{-1}$ denotes the partial inverse of $L_0 - \mu_0$ on its range. The norm of that inverse is bounded by $1/d$, where $d = \mu_1 - \mu_0$ is the spectral gap. As $\|R\| = \epsilon$ and $|\mu_0 + \frac{3}{4}| \le \epsilon$, we conclude that $\|f\| \le 2\epsilon/d$. The normalized eigenfunction is

$$\phi \;=\; \frac{g_0}{\|g_0\|} \;=\; \frac{g-f}{\sqrt{1-\|f\|^2}}\,.$$

Let $\psi = \hat{c}h_0 = \hat{c}\,e^{-\chi}$, where $\hat{c} = \sqrt{3}/\pi$ is a normalization factor chosen so that $\|\psi\| = 1$. A direct calculation shows that

$$\langle \psi, g\rangle \;=\; c\hat{c}\int_0^\infty e^{-r^2/4}\,r\,\mathrm{d}r \;=\; 2c\hat{c} \;=\; \frac{1}{\pi}\sqrt{\frac{6}{\log(2)}} \approx 0.9365\,,$$

hence

$$\langle \psi, \phi\rangle \;=\; \frac{\langle \psi, g\rangle - \langle \psi, f\rangle}{\sqrt{1-\|f\|^2}} \;\ge\; 2c\hat{c} - \frac{2\epsilon}{d}\,. \tag{4.40}$$

We use Lemma 4.7 below with $a = -\mu_0 \le 3/4$, $d = a+b = \mu_1 - \mu_0 \ge 1.2$, and $\epsilon = \|R\| \le 0.04$. In view of (4.40), estimate (4.41) shows that there exists some $\delta > 0$ such that $\langle Lf, f\rangle \ge \delta\|f\|^2$ for all $f \in Y_0 = h_0^\perp$. This concludes the proof of Theorem 4.5. $\square$

Finally, we state an elementary lemma that was used twice in the above proof.

**Lemma 4.7.** *Let $X$ be a Hilbert space and $L : D(L) \to X$ be a selfadjoint operator in $X$. We assume that there exist $\phi \in D(L)$ with $\|\phi\| = 1$ and $a, b \in \mathbb{R}$ with $a + b \ge 0$ such that*
*i) $L\phi = -a\phi$, and*
*ii) $\langle Lg, g\rangle \ge b\|g\|^2$ for all $g \in D(L)$ with $g \perp \phi$.*
*Then, for any $\psi \in X$ with $\|\psi\| = 1$, we have the lower bound*

$$\langle Lf, f\rangle \ge \Big((a+b)|\langle \phi, \psi\rangle|^2 - a\Big)\|f\|^2\,, \quad \textit{for all } f \in D(L) \textit{ with } f \perp \psi\,. \tag{4.41}$$

**Proof.** Given $f \in D(L)$, we decompose $f = \langle f, \phi\rangle\phi + g$, so that $g \perp \phi$. Since $L\phi = -a\phi$, we find

$$\langle Lf, f\rangle \;=\; \langle Lg, g\rangle - a|\langle f, \phi\rangle|^2 \;\ge\; b\|g\|^2 - a|\langle f, \phi\rangle|^2 \;=\; b\|f\|^2 - (a+b)|\langle f, \phi\rangle|^2\,,$$

where the inequality follows from ii). We now assume that $f \perp \psi$ and decompose $\phi = \langle \phi, \psi\rangle\psi + h$. By Cauchy-Schwarz, we have

$$|\langle f, \phi\rangle|^2 \;=\; |\langle f, h\rangle|^2 \;\le\; \|f\|^2\,\|h\|^2 \;=\; \|f\|^2\big(1 - |\langle \phi, \psi\rangle|^2\big)\,,$$

and combining both inequalities we arrive at (4.41). $\square$

**Remark 4.8.** *In the particular case where $a = 0$ and $b > 0$, the kernel of $L$ is one-dimensional, and inequality (4.41) implies that the quadratic form of $L$ is strictly positive on any closed hyperplane that does not contain the eigenfunction $\phi$.*

# A Appendix

## A.1 Integral inequalities involving logarithmic weights

**Proof of Proposition 3.1.** Let $B_1 = \{x \in \mathbb{R}^n\,;\, |x| < 1\}$ and $D_M = \big\{x \in \mathbb{R}^n\,;\, f(x) < M\big\}$. To prove (3.1), we must verify that

$$\int_{B_1}\Big(\log\frac{1}{|x|}\Big)\,f(x)\,\mathrm{d}x \;\lesssim\; M \;+\; \int_{\mathbb{R}^n\setminus D_M}\Big(\log\frac{f(x)}{M}\Big)\,f(x)\,\mathrm{d}x\,. \tag{A.1}$$

Let $\Omega_1 = \big\{x\in B_1\,;\, f(x)\le M|x|^{-n/2}\big\}$ and $\Omega_2 = \big\{x\in B_1\,;\, f(x)> M|x|^{-n/2}\big\}\subset \mathbb{R}^n\setminus D_M$. We have $B_1 = \Omega_1\cup\Omega_2$ and

$$
\begin{aligned}
&\int_{\Omega_1}\Big(\log\frac{1}{|x|}\Big) f(x)\,\mathrm{d}x \,\le\, M\int_{B_1}\frac{1}{|x|^{n/2}}\,\log\frac{1}{|x|}\,\mathrm{d}x \,=\, CM\,,\\
&\int_{\Omega_2}\Big(\log\frac{1}{|x|}\Big) f(x)\,\mathrm{d}x \,\le\, \frac{2}{n}\int_{\Omega_2}\Big(\log\frac{f(x)}{M}\Big) f(x)\,\mathrm{d}x \,\le\, \frac{2}{n}\int_{\mathbb{R}^n\setminus D_M}\Big(\log\frac{f(x)}{M}\Big) f(x)\,\mathrm{d}x\,,
\end{aligned}
$$

hence (A.1) follows by adding both inequalities. We next consider (3.2), which reads

$$
\int_{D_M}\Big(\log\frac{M}{f(x)}\Big) f(x)\,\mathrm{d}x \,\lesssim\, M \,+\, \int_{\mathbb{R}^n\setminus B_1}\big(\log|x|\big) f(x)\,\mathrm{d}x\,. \tag{A.2}
$$

Let $e=\exp(1)$ and

$$
\Omega_3 \,=\, \Big\{x\in D_M\,;\, f(x)\le \frac{M}{e(1{+}|x|)^{2n}}\Big\}\,,\qquad \Omega_4 \,=\, \Big\{x\in D_M\,;\, f(x)> \frac{M}{e(1{+}|x|)^{2n}}\Big\}\,.
$$

Since $t\mapsto t\log(1/t)$ is increasing on $[0,e^{-1}]$ and $s\mapsto \log(s)$ is increasing on $[1,+\infty)$, we have

$$
\begin{aligned}
&\int_{\Omega_3}\Big(\log\frac{M}{f(x)}\Big) f(x)\,\mathrm{d}x \,\le\, M\int_{\mathbb{R}^n}\frac{1}{e(1{+}|x|)^{2n}}\log\big(e(1{+}|x|)^{2n}\big)\,\mathrm{d}x \,=\, CM\,,\\
&\int_{\Omega_4}\Big(\log\frac{M}{f(x)}\Big) f(x)\,\mathrm{d}x \,\le\, \int_{\Omega_4}\log\big(e(1{+}|x|)^{2n}\big) f(x)\,\mathrm{d}x \,\le\, CM + 2n\int_{\mathbb{R}^n\setminus B_1}\big(\log|x|\big) f(x)\,\mathrm{d}x\,,
\end{aligned}
$$

and (A.2) follows in the same way.

From now on, we assume that $f$ is radially symmetric and nonincreasing in the radial direction. In particular, we have for all $x\neq 0$:

$$
f(x) \,\le\, \frac{1}{\alpha_n|x|^n}\int_{|y|\le|x|} f(y)\,\mathrm{d}y \,\le\, \frac{M}{\alpha_n|x|^n}\,,\qquad \text{where}\quad \alpha_n \,=\, \frac{\pi^{n/2}}{\Gamma(1+\frac{n}{2})}\,. \tag{A.3}
$$

Since $t\mapsto \log_+(t)$ is increasing, we deduce that

$$
\int_{\mathbb{R}^n\setminus D_M}\Big(\log\frac{f(x)}{M}\Big) f(x)\,\mathrm{d}x \,\le\, \int_{\mathbb{R}^n}\Big(\log_+\frac{1}{\alpha_n|x|^n}\Big) f(x)\,\mathrm{d}x \,\le\, CM + n\int_{B_1}\Big(\log\frac{1}{|x|}\Big) f(x)\,\mathrm{d}x\,,
$$

which is the converse of (3.1). Note that, when $n\le 12$, the first term $CM$ in the right-hand side can be dropped, because $\alpha_n>1$. In a similar way, we find

$$
\int_{\mathbb{R}^n\setminus B_1}\big(\log|x|\big) f(x)\,\mathrm{d}x \,\le\, \frac{1}{n}\int_{\mathbb{R}^n}\Big(\log_+\frac{M}{\alpha_n f(x)}\Big) f(x)\,\mathrm{d}x \,\le\, CM + \int_{D_M}\Big(\log\frac{M}{f(x)}\Big) f(x)\,\mathrm{d}x\,,
$$

which is the converse of (3.2). Again, the first term $CM$ in the right-hand side can be dropped when $n\le 12$. □

**Proof of Proposition 2.2.** Throughout the proof we assume that $M := \|\omega\|_{L^1} > 0$. We decompose $E(\omega) = E_1(\omega) + E_2(\omega)$ where

$$
E_i(\omega) \,=\, \frac{1}{4\pi}\int_{\Omega_i}\log\frac{1}{|x-y|}\,\omega(x)\omega(y)\,\mathrm{d}x\,\mathrm{d}y\,,\qquad i=1,2\,,
$$

and $\Omega_1 = \{(x,y)\in\mathbb{R}^2\times\mathbb{R}^2\,;\,|x-y|<1\}$, $\Omega_2 = \{(x,y)\in\mathbb{R}^2\times\mathbb{R}^2\,;\,|x-y|\ge 1\}$. We have to verify that the integrals defining the quantities $E_1, E_2$ are convergent under assumptions (2.6).

First of all, using inequality (A.1) above with $n = 2$, we obtain for all $x \in \mathbb{R}^2$:

$$\int_{|y-x|<1} \log \frac{1}{|x-y|}\,|\omega(y)|\,\mathrm{d}y \;\le\; C\int_{\mathbb{R}^2}\Bigl(1+\log_+\frac{|\omega(y)|}{M}\Bigr)|\omega(y)|\,\mathrm{d}y\,.$$

If we multiply both sides by $|\omega(x)|$ and integrate over $x \in \mathbb{R}^2$, we thus find

$$|E_1(\omega)| \;\le\; CM\int_{\mathbb{R}^2}\Bigl(1+\log_+\frac{|\omega(y)|}{M}\Bigr)|\omega(y)|\,\mathrm{d}y\,. \tag{A.4}$$

On the other hand, we have $\log|x-y| \le \log(|x|+|y|) \le \log(1+|x|)+\log(1+|y|)$ when $|x-y| \ge 1$. This gives the bound

$$\begin{aligned}|E_2(\omega)| \;&\le\; \frac{1}{4\pi}\int_{\Omega_2}|\omega(x)|\,|\omega(y)|\,\Bigl(\log(1+|x|)+\log(1+|y|)\Bigr)\,\mathrm{d}x\,\mathrm{d}y\\ &\le\; \frac{M}{2\pi}\int_{\mathbb{R}^2}|\omega(y)|\log(1+|y|)\,\mathrm{d}y\,.\end{aligned} \tag{A.5}$$

Combining (A.4) and (A.5), we arrive at (2.7).

Finally, we assume that $\omega \in C^2_c(\mathbb{R}^2)$ and $\int_{\mathbb{R}^2}\omega(x)\,\mathrm{d}x = 0$. The associated stream function $\psi \in C^2(\mathbb{R})$ defined by (1.14) satisfies $|\psi(x)| = \mathcal{O}(|x|^{-1})$ and $|u(x)| = |\nabla\psi(x)| = \mathcal{O}(|x|^{-2})$ as $|x| \to \infty$, so that $u \in L^2(\mathbb{R}^2)$. This allows us to integrate by parts in the first expression (1.13) of the energy, using the relation $\Delta\psi = \omega$, to obtain the elegant formula $E(\omega) = \frac{1}{2}\int_{\mathbb{R}^2}|u|^2\,\mathrm{d}x$. By a density argument, the conclusion remains valid for all integrable vorticities with zero average satisfying a assumptions (2.6). $\square$

## A.2 Positivity of the potential $V$ in some examples

For the algebraic vortex (2.38) with $\kappa = 1+\nu > 1$, the potential $V$ defined in (2.34) has the following expression

$$V(r) \;=\; \frac{1}{r^2(1+r^2)^2}\Bigl(3-2(\nu-1)r^2+(\nu^2-1)r^4-2S-S^2\Bigr)\,,\quad \text{where}\quad S \;=\; \frac{\nu r^2}{(1+r^2)^\nu-1}\,.$$

If $\nu = 1$, then $S = 1$ hence $V \equiv 0$. Otherwise:

- If $\nu > 1$, we have $(1+r^2)^\nu > 1+\nu r^2$ for all $r > 0$, so that $S < 1$. We deduce

$$V(r) \;>\; \frac{\nu-1}{(1+r^2)^2}\Bigl(-2+(\nu+1)r^2\Bigr)\,, \tag{A.6}$$

so that $V(r) > 0$ if $r^2 \ge 2/(\nu+1)$. In the region where $r^2 \le 2/(\nu+1)$, we use the improved estimate

$$S \;=\; \frac{\nu r^2}{(1+r^2)^\nu-1} \;<\; 1-\frac{\nu-1}{2}\,r^2+\frac{\nu^2-1}{12}\,r^4\,,\qquad r > 0\,, \tag{A.7}$$

which can be established by a direct calculation. This gives the lower bound

$$V(r) \;>\; \frac{(\nu-1)r^2}{12(1+r^2)^2}\Bigl(5\nu+11+(\nu^2-1)r^2-\frac{(\nu-1)(\nu+1)^2}{12}\,r^4\Bigr)\,, \tag{A.8}$$

which implies that $V(r) > 0$ if $(\nu+1)r^2 \le 2$.

- If $0 < \nu < 1$, the calculations are entirely similar, except that all inequalities in (A.6)–(A.8) are reversed. This shows that $V(r) < 0$ in that case.

For the Gaussian vortex (2.39), a direct calculation shows that

$$V(r) \;=\; \frac{3}{4s} - \frac{1}{2} + \frac{s}{4} - \frac{1/2}{e^s-1} - \frac{s/4}{(e^s-1)^2}\,, \qquad \text{where} \quad s \;=\; \frac{r^2}{4}\,.$$

Using the lower bound $e^s - 1 \ge s(1+s/2+s^2/6)$, we obtain

$$\begin{aligned} V(r) &\ge \frac{1}{4s}\,\frac{1}{(1{+}s/2{+}s^2/6)^2}\Bigl((3-2s+s^2)(1+s/2+s^2/6)^2 - 2(1+s/2+s^2/6)-1\Bigr) \\ &= \frac{1}{4}\,\frac{s}{(6{+}3s{+}s^2)^2}\Bigl(15+12s+12s^2+4s^3+s^4\Bigr) \;>\; 0\,. \end{aligned}$$

## A.3 Properties of the Gaussian vortex

Given the expressions of $A, B$ in (4.25), we first verify that the ratio $B/A$ is a decreasing function of $r$. We have

$$\frac{B(r)-1}{A(r)} \;=\; \frac{1}{2}\Bigl(1+h(r^2/4)\Bigr)\,, \qquad \text{where} \quad h(s) \;=\; \frac{1}{s} - \frac{1}{e^s-1}\,. \tag{A.9}$$

Since

$$h'(s) \;=\; -\frac{(e^s-1)^2 - s^2e^s}{s^2(e^s-1)^2} \;=\; -4\,e^s\,\frac{\sinh(s/2)^2-(s/2)^2}{s^2(e^s-1)^2} \;<\; 0\,, \qquad s>0\,,$$

we see that $h$ is strictly decreasing on $(0,+\infty)$ with $h(0) = 1/2$ and $h(+\infty) = 0$. We conclude that $(B-1)/A$, hence also $B/A$, is a decreasing function of $r$, and that $1/2 \le B/A \le 7/4$.

We next prove the lower bound (4.32) on the potential $W$. Since $\chi = \log(A)/2$ with $A$ as in (4.25), a direct calculation shows that the potential $W$ defined by (4.28) has the following expression

$$W(r) \;=\; \frac{s}{4} - \frac{1}{2} - \frac{1}{4s} - \frac{s-1/2}{e^s-1} - \frac{s/4}{(e^s-1)^2}\,, \qquad \text{where} \quad s \;=\; \frac{r^2}{4}\,.$$

Inequality (4.32) is thus equivalent to the positivity of the function $G$ defined by

$$G(s) \;=\; 1 - \frac{1}{4s} - \frac{s-1/2}{e^s-1} - \frac{s/4}{(e^s-1)^2}\,, \qquad s>0\,. \tag{A.10}$$

If $s \ge 1/2$, we use the lower bound $e^s-1 \ge s(1+s/2)$ and obtain

$$G(s) \;\ge\; \frac{s}{4(2+s)^2}\Bigl(7+4s\Bigr) \;>\; 0\,.$$

If $0 < s < 1/2$, the third term in the right-hand side of (A.10) has the opposite sign. To estimate the denominator, we use the upper bound $e^s - 1 \le s(1+s/2)(1+s^2/5)$, which holds for $s \le 1/2$. This gives

$$G(s) \;\ge\; \frac{s}{4(2+s)^2(5+s^2)}\Bigl(27+32s+15s^2+4s^3\Bigr) \;>\; 0\,.$$

## A.4 Computing the norm of the quasimode (4.37)

In this section we compute the $L^2$ norm of the function $R$ defined by (4.37). We recall that $g = cA^{1/2}e^{-r^2/4}$, where $c = (2\log(2))^{-1/2}$, and using (A.9) we observe that

$$\frac{3}{4} - \frac{B-1}{A} \;=\; \frac{1}{4}\Bigl(1-2h(r^2/4)\Bigr)\,, \qquad \text{where} \quad h(s) \;=\; \frac{1}{s} - \frac{1}{e^s-1}\,.$$

It follows that

$$\|R\|_Y^2 \;=\; \frac{1}{16}\int_0^\infty \Big(1-2h(r^2/4)\Big)^2 g(r)^2 r\,\mathrm{d}r \;=\; \frac{1}{16\log(2)}\int_0^\infty \Big(1-2h(s)\Big)^2\,\frac{1}{s}\,\Big(e^{-s}-e^{-2s}\Big)\,\mathrm{d}s\,.$$

Expanding $(1-2h(s))^2 = 1-4h(s)+4h(s)^2$, we decompose

$$\|R\|_Y^2 \;\equiv\; \int_0^\infty R(r)^2 r\,\mathrm{d}r \;=\; \frac{I_1-4I_2+4I_3}{16\log(2)}\,, \tag{A.11}$$

where the integrals $I_1$, $I_2$, $I_3$ are defined and computed below.

• Evaluation of $I_1$:

$$I_1 \;=\; \int_0^\infty \frac{1}{s}\,\Big(e^{-s}-e^{-2s}\Big)\,\mathrm{d}s \;=\; \log(2)\,.$$

• Evaluation of $I_2$:

$$\begin{aligned}
I_2 \;&=\; \int_0^\infty \frac{h(s)}{s}\,\Big(e^{-s}-e^{-2s}\Big)\,\mathrm{d}s \\
&=\; \int_0^\infty \Big(\frac{1}{s}-\frac{1}{e^s-1}\Big)\Big\{\int_0^\infty e^{-st}\,\mathrm{d}t\Big\}\Big(e^{-s}-e^{-2s}\Big)\,\mathrm{d}s \\
&=\; \int_0^\infty \Big\{\int_0^\infty \Big(\frac{1}{s}-\frac{1}{e^s-1}\Big)\Big(e^{-s(1+t)}-e^{-s(2+t)}\Big)\,\mathrm{d}s\Big\}\,\mathrm{d}t \\
&=\; \int_0^\infty \Big(\log\frac{2+t}{1+t}-\frac{1}{2+t}\Big)\,\mathrm{d}t \;=\; (1+t)\log\frac{2+t}{1+t}\,\bigg|_{t=0}^{t=+\infty} \;=\; 1-\log(2)\,.
\end{aligned}$$

• Evaluation of $I_3$:

$$\begin{aligned}
I_3 \;&=\; \int_0^\infty \frac{h(s)^2}{s}\,\Big(e^{-s}-e^{-2s}\Big)\,\mathrm{d}s \\
&=\; \int_0^\infty \Big(\frac{1}{s}-\frac{1}{e^s-1}\Big)^2\Big\{\int_0^\infty ts\,e^{-st}\,\mathrm{d}t\Big\}\Big(e^{-s}-e^{-2s}\Big)\,\mathrm{d}s \\
&=\; \int_0^\infty \Big\{\int_0^\infty \Big(\frac{e^{-s(1+t)}-e^{-s(2+t)}}{s}-2\,e^{-s(2+t)}+\frac{s\,e^{-s(2+t)}}{e^s-1}\Big)\,\mathrm{d}s\Big\}\,t\,\mathrm{d}t \\
&=\; \int_0^\infty \Big(\log\frac{2+t}{1+t}-\frac{2}{2+t}+\psi^{(1)}(3+t)\Big)\,t\,\mathrm{d}t\,,
\end{aligned}$$

where $\psi^{(1)}$ denotes the trigamma function [1, Section 6.4]:

$$\psi^{(1)}(z) \;=\; \int_0^\infty \frac{s\,e^{-sz}}{1-e^{-s}}\,\mathrm{d}s \;=\; \frac{\mathrm{d}^2}{\mathrm{d}z^2}\,\log\Gamma(z)\,, \qquad \mathrm{Re}(z)>0\,.$$

It follows that

$$\begin{aligned}
I_3 \;&=\; \frac{t^2+4}{2}\,\log(2+t)-\frac{t^2-1}{2}\,\log(1+t)-\frac{3t}{2}+t\big(\log\Gamma\big)'(3+t)-\big(\log\Gamma\big)(3+t)\,\bigg|_{t=0}^{t=+\infty} \\
&=\; \frac{1}{4}\Big(7-6\log(2)-2\log(\pi)\Big)\,.
\end{aligned}$$

Here we used Stirling's formula to compute an asymptotic expansion of $\big(\log\Gamma\big)(3+t)$ and its derivative as $t\to+\infty$. Inserting the values of $I_1$, $I_2$, $I_3$ into (A.11), we arrive at (4.38).

### A.5 The Poisson structure on $\mathscr{P}$

For two functions $\phi, \psi$ on $\mathbb{R}^2$ we use the familiar notation $\{\phi, \psi\} = \partial_1\phi\partial_2\psi - \partial_2\phi\partial_1\psi$. Now, if $\mathcal{F}$ and $\mathcal{G}$ are two functionals of $\mathscr{P}$, the standard way of defining their Poisson bracket is

$$\boldsymbol{\{}\mathcal{F}, \mathcal{G}\boldsymbol{\}}(\omega) = -\int_{\mathbb{R}^2} \omega \left\{ \frac{\delta \mathcal{F}}{\delta \omega}, \frac{\delta \mathcal{G}}{\delta \omega} \right\} \mathrm{d}x\,, \tag{A.12}$$

where $\frac{\delta \mathcal{F}}{\delta \omega}$ is the usual "variational derivative" of $\mathcal{F}$, namely, the function on $\mathbb{R}^2$ defined by the relation

$$\Big(\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{F}(\omega + \varepsilon\eta)\Big)\Big|_{\varepsilon=0} \,=\, \int_{\mathbb{R}^2} \frac{\delta \mathcal{F}}{\delta \omega}(x)\eta(x)\,\mathrm{d}x\,,$$

for all (smooth and compactly supported) increments $\eta$. In particular, the variational derivative of the energy functional (1.13) is $\frac{\delta E}{\delta \omega} = -\psi$, where $\psi$ is the stream function (1.14). As an application, if $\omega$ evolves according to the Euler equation $\partial_t\omega + \{\psi, \omega\} = 0$, we have for any (smooth) functional $\mathcal{F}$ :

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(\omega) \,=\, -\int_{\mathbb{R}^2} \frac{\delta \mathcal{F}}{\delta \omega}\{\psi, \omega\}\,\mathrm{d}x \,=\, \int_{\mathbb{R}^2} \left\{ \frac{\delta \mathcal{F}}{\delta \omega}, \frac{\delta E}{\delta \omega} \right\} \omega\,\mathrm{d}x \,=\, \boldsymbol{\{}E, \mathcal{F}\boldsymbol{\}}(\omega)\,.$$

This is precisely the integrated form of the canonical equation $\partial_t\omega = \boldsymbol{\{}E, \omega\boldsymbol{\}}$.

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