Chapter V : vortices and filaments

A) Inviscid vortices in $\mathbb{R}^2$

We consider here the incompressible Euler equations in $\mathbb{R}^2$, in vorticity formulation:

$$\partial_t \omega(x,t) + u(x,t)\cdot\nabla\omega(x,t) = 0, \qquad x\in\mathbb{R}^2,\ t\in\mathbb{R} \tag{V}$$

where u is the velocity field defined by:

$$\partial_1 u_1 + \partial_2 u_2 = 0, \quad \partial_1 u_2 - \partial_2 u_1 = \omega.$$

We recall the explicit Biot-Savart formula:

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\,\omega(y,t)\,dy, \tag{BS}$$

where $x=(x_1,x_2) \Rightarrow x^\perp = (-x_2,x_1)$, $|x|^2 = x_1^2+x_2^2$.

We assume for simplicity that the initial data for (V) are smooth and localized, for instance

$$\omega_0 \in C_c^{2,\alpha}(\mathbb{R}^2), \quad \text{for some } \alpha>0.$$

Then (V), (BS) have a unique global, classical solution and it is known that $\omega(t)\in C_c^{2,\alpha}(\mathbb{R}^2)\ \forall t\in\mathbb{R}$.

Conserved quantities:

- Mass: $M = \int_{\mathbb{R}^2}\omega(x,t)\,dx$

$$\frac{dM}{dt} = \int_{\mathbb{R}^2}\partial_t\omega\,dx = -\int_{\mathbb{R}^2}\operatorname{div}(u\omega)\,dx = 0.$$

- First order moments: $P_j = \int_{\mathbb{R}^2} x_j\,\omega(x,t)\,dx$, $j=1,2$.

$$\frac{dP_j}{dt} = \int_{\mathbb{R}^2} x_j\,\partial_t\omega\,dx = -\int_{\mathbb{R}^2} x_j\operatorname{div}(u\omega)\,dx = \int_{\mathbb{R}^2} u_j\,\omega\,dx = 0$$

Indeed:

$$\int_{\mathbb{R}^2} u\,\omega\,dx = \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \underbrace{\frac{(x-y)^\perp}{|x-y|^2}\,\omega(y)\,\omega(x)}_{\text{odd with respect to } x \leftrightarrow y.}\,dy\,dx = 0$$

- <u>Symmetric second-order moment:</u> $I = \int_{\mathbb{R}^2} |x|^2 \omega\,dx$

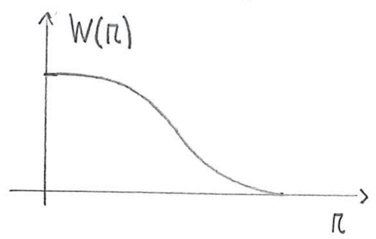
$$\frac{dI}{dt} = -\int_{\mathbb{R}^2} |x|^2 \,\mathrm{div}(u\omega)\,dx = 2\int_{\mathbb{R}^2} (x\cdot u)\,\omega\,dx$$

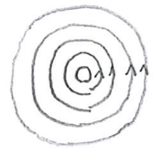
$$= \frac{1}{\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} x\cdot \frac{(x-y)^\perp}{|x-y|^2}\,\omega(y)\,\omega(x)\,dy\,dx$$

$$\underset{\text{symmetrization}}{=} \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \underbrace{(x-y)\cdot\frac{(x-y)^\perp}{|x-y|^2}}_{=0}\,\omega(y)\,\omega(x)\,dy\,dx = 0.$$

⚠ In general $E := \frac{1}{2}\int_{\mathbb{R}^2} |u(x,t)|^2 dx = +\infty$! (Infinite-energy solutions)

<u>Example:</u> (<u>steady vortex</u>) Assume that $\omega(x) = W(|x|)$, $x \in \mathbb{R}^2$.





The stream function $\Delta\varphi = \omega$ is of the form $\varphi(x) = \psi(|x|)$, where

$\psi''(r) + \frac{1}{r}\psi'(r) = W(r) \Rightarrow \psi'(r) = \frac{1}{r}\int_0^r s\,W(s)\,ds$. Thus

$u(x) = \nabla^\perp\varphi(x) = \frac{x^\perp}{|x|^2}\int_0^r s\,W(s)\,ds$. We deduce that:

- $x\cdot u(x) = 0 \Rightarrow u\cdot\nabla\omega = 0$: ω is a <u>stationary sol.</u> of (V)!
- $u(x) = \frac{x^\perp}{2\pi|x|^2}\int_{|y|\le|x|}\omega(y)\,dy \sim \frac{x^\perp}{|x|^2}\,\frac{M}{2\pi}$ as $|x| \to +\infty$

$\Rightarrow \|u\|_{L^2} = +\infty$ if $M \neq 0$.

Remark: One can check that the quantity

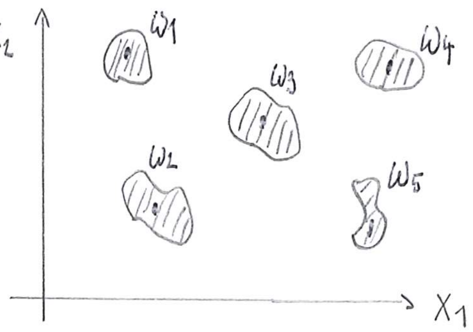
$$\tilde{E} = -\frac{1}{2}\int_{\mathbb{R}^2} \psi\,\omega\,dx = \frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\frac{1}{|x-y|}\,\omega(x)\omega(y)\,dx\,dy$$

is conserved, and that $\tilde{E} = E$ when $M = 0$. But $\tilde{E}$ has no definite sign.

N separated vortices We consider solutions of (V) of the form

$$\omega(x,t) = \sum_{j=1}^{N} \omega_j(x,t), \qquad u(x,t) = \sum_{j=1}^{N} u_j(x,t).$$

We assume that the supports of the "components" ω_j, which we call vortices, are concentrated near widely separated points in $\mathbb{R}^2$.



The evolution of the j^{th} vortex is given by

$$\begin{cases} \partial_t \omega_j + u\cdot\nabla\omega_j = 0, \qquad u = \sum_{k=1}^{N} u_k, \text{ where} \\ u_k(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\,\omega_k(y)\,dy. \end{cases}$$

Assuming that ω_j has a definite sign for each $j \in \{1,..,N\}$, we define:

$$\begin{cases} \Gamma_j = \int_{\mathbb{R}^2} \omega_j(x,t)\,dx \quad : \text{ "mass" or "circulation" of the } j^{th} \text{ vortex, } \Gamma_j \neq 0 \\ X_j(t) = \frac{1}{\Gamma_j}\int_{\mathbb{R}^2} x\,\omega_j(x,t)\,dx : \text{ "center of vorticity" of the } j^{th} \text{ vortex.} \end{cases}$$

The circulations Γ_j are conserved quantities.

Evolution of the centers:

$$X_j'(t) = \frac{1}{\Gamma_j}\int_{\mathbb{R}^2} x\,\partial_t \omega_j(x,t)\,dx = \frac{-1}{\Gamma_j}\int_{\mathbb{R}^2} x\,\mathrm{div}(u\omega_j)\,dx$$

$$= \frac{1}{\Gamma_j}\int_{\mathbb{R}^2} u(x,t)\,\omega_j(x,t)\,dx = \frac{1}{\Gamma_j}\sum_{k\neq j}\int_{\mathbb{R}^2} u_k(x,t)\,\omega_j(x,t)\,dx.$$

⚠ $\int_{\mathbb{R}^2} u_j(x,t)\,\omega_j(x,t)\,dx = 0$ by the calculation above.

Assume finally that the vortices are <u>sharply concentrated near their centers</u>, so that

$$u_k(x,t) \cong \frac{\Gamma_k}{2\pi}\frac{(x-X_k)^\perp}{|x-X_k|^2} \cong \frac{\Gamma_k}{2\pi}\frac{(X_j-X_k)^\perp}{|X_j-X_k|^2} \qquad \forall x \in \mathrm{supp}(\omega_j)$$

In this limit we obtain

$$X_j'(t) = \frac{1}{2\pi}\sum_{\substack{k=1\\k\neq j}}^{N} \Gamma_k \frac{(X_j(t)-X_k(t))^\perp}{|X_j(t)-X_k(t)|^2}, \qquad t\in\mathbb{R}. \qquad (PV)$$

This a system of ODEs for the vortex positions (or centers), called the (Helmholtz-Kirchhoff) <u>point vortex system.</u>

<u>Rigorous justification</u> (Marchioro & Pulvirenti, 1993)

Assume that $X_1(t), \dots, X_N(t)$ is a solution of (PV) on $[0,T]$ such that

$$\min_{t\in[0,T]}\ \min_{i\neq j} |X_i(t)-X_j(t)| > 0 \qquad \text{(no collisions!)}$$

For $\varepsilon>0$ small enough, consider initial data $\omega_\varepsilon(x,0) = \sum_{j=1}^{N} \omega_{\varepsilon,j}(x,0)$ such that:

- $\omega_{\varepsilon,j}$ has a definite sign $\forall j\in\{1,\dots,N\}$
- $\mathrm{supp}(\omega_{\varepsilon,j}(\cdot,0)) \subset B(X_j(0),\varepsilon) \quad \forall j\in\{1,\dots,N\}$

- $\int_{\mathbb{R}^2} \omega_{\varepsilon,j}(x,0)\,dx = \Gamma_j \quad \forall j \in \{1,\dots,N\}$
- $\omega_{\varepsilon,j}(\cdot,0) \in C_c^{2,\alpha}, \quad |\omega_{\varepsilon,j}(x,0)| \leq C\varepsilon^{-\eta}, \quad 0<\eta<8/3.$

Theorem: Under these assumptions, for any $d>0$, there exists $\varepsilon_0 = \varepsilon_0(d,T)>0$ such that, if $0<\varepsilon<\varepsilon_0$, the solution of (V) with initial data above satisfies
$$\omega_\varepsilon(x,t) = \sum_{j=1}^{N} \omega_{\varepsilon,j}(x,t) \qquad x\in\mathbb{R}^2;\ t\in[0,T]$$
where
$$\mathrm{supp}\,(\omega_{\varepsilon,j}(\cdot,t)) \subset B(X_j(t),d) \qquad \forall j\in\{1,\dots,N\}\ \forall t\in[0,T].$$
In particular
$$\omega_\varepsilon(\cdot,t) \xrightarrow[\varepsilon\to 0]{} \sum_{j=1}^{N} \Gamma_j\,\delta(\cdot - X_j(t)).$$

The key step in the proof is the control of the size of the support of $\omega_{\varepsilon,j}$ $\forall j$. One uses the assumption that $\omega_{\varepsilon,j}\geq 0$ or $\omega_{\varepsilon,j}\leq 0$ + the evolution satisfied by the second order moment $\int_{\mathbb{R}^2} |x-X_j(t)|^2\omega_{\varepsilon,j}(x,t)\,dt$.

General properties of the point vortex system:

The system (PV) has 3 conserved quantities:

- Moment: $P = \sum_{j=1}^{N} \Gamma_j X_j(t).$
- Second order Moment: $I = \sum_{j=1}^{N} \Gamma_j |X_j(t)|^2.$
- Energy: $$E = \frac{1}{4\pi} \sum_{\substack{j,k=1\\ j\neq k}}^{N} \Gamma_j\Gamma_k \log\frac{1}{|X_j(t)-X_k(t)|}.$$

It is easy to verify that (PV) can be written in the form

$$\Gamma_j X_j'(t) = -\nabla^{\perp}_{X_j} E\left(X_1(t), \dots, X_N(t)\right) \qquad j = 1, \dots, N$$

$\Longrightarrow$ (PV) is a Hamiltonian system in $\mathbb{R}^{2N}$ with Hamiltonian function given by the energy E (up to unessential change of variables).

One can check that the conserved quantities P, I, E are independent $\Longrightarrow$ (PV) is <u>integrable in the sense of Liouville</u> if $N \leq 3$.

<u>The case $N = 2$</u> Denote $\Gamma = \Gamma_1 + \Gamma_2$.

Since $E = \frac{1}{2\pi}\Gamma_1\Gamma_2 \log \frac{1}{|X_1 - X_2|} = \text{const.}$, we see that $L := |X_1 - X_2|$ is a conserved quantity. Let $P = \Gamma_1 X_1 + \Gamma_2 X_2$.

We have:

$$\begin{cases} X_1' = \dfrac{\Gamma_2}{2\pi}\dfrac{(X_1-X_2)^{\perp}}{|X_1-X_2|^2} = \dfrac{1}{2\pi L^2}\left(\Gamma_2 X_1^{\perp} + \Gamma_1 X_1^{\perp} - P^{\perp}\right) = \dfrac{\Gamma X_1^{\perp}}{2\pi L^2} - \dfrac{P^{\perp}}{2\pi L^2} \\[2ex] X_2' = \dfrac{\Gamma_1}{2\pi}\dfrac{(X_2-X_1)^{\perp}}{|X_2-X_1|^2} = \dfrac{1}{2\pi L^2}\left(\Gamma_1 X_2^{\perp} + \Gamma_2 X_2^{\perp} - P^{\perp}\right) = \dfrac{\Gamma X_2^{\perp}}{2\pi L^2} - \dfrac{P^{\perp}}{2\pi L^2}. \end{cases}$$

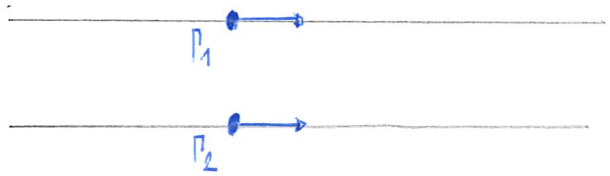
- If $\Gamma = \Gamma_1 + \Gamma_2 \neq 0$, then by a translation we can assume that $P = 0$.

$\Longrightarrow$ $X_j' = \Omega X_j^{\perp}$, $\quad \Omega = \dfrac{\Gamma}{2\pi L^2}$.

Both vortices rotate with the same angular speed around the origin.

• If $\Gamma = \Gamma_1 + \Gamma_2 = 0$, then $X_1' = X_2' = -\dfrac{\ell^\perp}{2\pi L^2}$

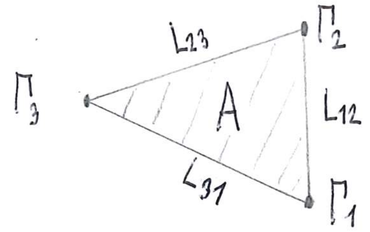
$\Rightarrow$ Both vortices move with the same speed on parallel lines.



$\Gamma_1 > 0 > \Gamma_2$, $\Gamma_1 + \Gamma_2 = 0$.

The case $N = 3$

The trajectories here are more complicated, and system (PV) is not well-posed for all initial data!



The area A of the triangle with vertices X_1, X_2, X_3 is given by

$$\| \; A = \frac{1}{2}(X_1 - X_2)^\perp \cdot (X_2 - X_3), \quad \text{if } X_1, X_2, X_3 \text{ are located counterclockwise.}$$

We denote $L_{12} = |X_1 - X_2|$, $L_{23} = |X_2 - X_3|$, $L_{31} = |X_3 - X_1|$.

Since

$$\begin{cases} X_1' = \dfrac{\Gamma_2}{2\pi} \dfrac{(X_1 - X_2)^\perp}{L_{12}^2} + \dfrac{\Gamma_3}{2\pi} \dfrac{(X_1 - X_3)^\perp}{L_{31}^2} \\[2ex] X_2' = \dfrac{\Gamma_3}{2\pi} \dfrac{(X_2 - X_3)^\perp}{L_{23}^2} + \dfrac{\Gamma_1}{2\pi} \dfrac{(X_2 - X_1)^\perp}{L_{12}} \end{cases}$$

we have:

$$\frac{d}{dt} L_{12}^2 = 2(X_1 - X_2) \cdot (X_1' - X_2')$$

$$= \frac{\Gamma_3}{\pi} \left\{ \frac{(X_1 - X_2) \cdot (X_1 - X_3)^\perp}{L_{31}^2} - \frac{(X_1 - X_2) \cdot (X_2 - X_3)^\perp}{L_{23}^2} \right\}$$

$$= \frac{2A\Gamma_3}{\pi} \left(\frac{1}{L_{23}^2} - \frac{1}{L_{31}^2} \right).$$

$\|$ Idem for L_{23}, L_{31} by cyclic permutations. (*)

Interesting information can be deduced from (*). For instance, if the vortices are located at the vertices of an equilateral triangle ($L_{12} = L_{23} = L_{31}$), the same property holds for all time. If $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 \neq 0$, the triangle rotates with constant angular speed around the vorticity center $\bar{x} = P/\Gamma$. If $\Gamma = 0$, the motion is a rigid translation.

Let us consider another instructive case. Assume that

- $\Gamma := \Gamma_1 + \Gamma_2 + \Gamma_3 \neq 0$
- $\dfrac{1}{\Gamma_1} + \dfrac{1}{\Gamma_2} + \dfrac{1}{\Gamma_3} = 0$ ($\Rightarrow$ the vortices do not have all the same sign!)
- $P = \sum_{j=1}^{3} \Gamma_j x_j = 0$ ($=$ the center of vorticity is at the origin)
- $I = \sum_{j=1}^{3} \Gamma_j |x_j|^2 = 0$

We observe that

$$
\begin{aligned}
F &:= \Gamma_1\Gamma_2 L_{12}^2 + \Gamma_2\Gamma_3 L_{23}^2 + \Gamma_3\Gamma_1 L_{31}^2 \\
&= (\Gamma_1+\Gamma_2+\Gamma_3)(\Gamma_1|x_1|^2+\Gamma_2|x_2|^2+\Gamma_3|x_3|^2) - |\Gamma_1x_1+\Gamma_2x_2+\Gamma_3x_3|^2 \\
&= \Gamma I - |P|^2 = 0.
\end{aligned}
$$

We now compute:

$$
\begin{aligned}
\frac{d}{dt}\frac{L_{12}^2}{L_{23}^2} &= \frac{1}{L_{23}^4}\left\{ L_{23}^2 \frac{d}{dt} L_{12}^2 - L_{12}^2 \frac{d}{dt} L_{23}^2 \right\} \\
&= \frac{1}{L_{23}^4}\frac{2A}{\pi}\left\{ \Gamma_3 L_{23}^2\left(\frac{1}{L_{23}^2} - \frac{1}{L_{31}^2}\right) - \Gamma_1 L_{12}^2\left(\frac{1}{L_{31}^2} - \frac{1}{L_{12}^2}\right)\right\} \\
&= \frac{2A}{\pi}\frac{1}{L_{23}^4 L_{31}^2}\left\{ (\Gamma_1+\Gamma_3) L_{31}^2 - \Gamma_1 L_{12}^2 - \Gamma_3 L_{23}^2 \right\} \\
&= -\frac{2A}{\pi\Gamma_2}\frac{1}{L_{23}^4 L_{31}^2}\left\{ \underbrace{\Gamma_1\Gamma_2 L_{12}^2 + \Gamma_2\Gamma_3 L_{23}^2 + \Gamma_3\Gamma_1 L_{31}^2}_{F} \right\} = 0
\end{aligned}
$$

Where we have used the relation $\Gamma_2(\Gamma_1+\Gamma_3) = -\Gamma_1\Gamma_3$.

Thus the ratios $\frac{L_{12}}{L_{23}}$ and (similarly) $\frac{L_{23}}{L_{31}}$ stay constant.

Since the center of vorticity is fixed (at the origin), the motion is a rotation + a dilation. By (*), we have

$$\frac{d}{dt} L_{12}^2 = \text{const.} \text{ (because } A \text{ varies as } L_{23}^2 \text{ or } L_{31}^2\text{)}$$

and similarly $\frac{d}{dt} L_{23}^2 = \text{const.}$, $\frac{d}{dt} L_{31}^2 = \text{const.}$ Thus:

$$\frac{dA}{dt} = \alpha = \text{const}, \quad \text{hence } A(t) = A_0 + \alpha t.$$

From (*) we deduce that $\alpha = 0$ iff $L_{12} = L_{23} = L_{31}$ (equilateral triangle).

If $\alpha < 0$, the triangle shrinks in a self-similar way and collapses at the origin when $t = \frac{A_0}{|\alpha|}$.

If $\alpha > 0$, the triangle expands in a self-similar way.

For $N \geqslant 4$, the dynamics of (PV) is chaotic in general. It can be shown that (PV) is globally well-posed for almost all initial data (w.r.t. Lebesgue's measure), see Marchioro & Pulvirenti.

Some particular configurations have been studied, e.g. when the vortices form a regular N-gon with equal circulations.

B) Viscous vortices in $\mathbb{R}^2$

We next consider the Navier-Stokes equations in $\mathbb{R}^2$, in vorticity formulation:

$$\partial_t \omega(x,t) + u(x,t)\cdot\nabla\omega(x,t) = \Delta\omega(x,t) \qquad x\in\mathbb{R}^2, t>0 \qquad \text{(NS)}$$

where u is again given by (BS). The corresponding integral equation reads:

$$\omega(t) = S(t)\omega_0 - \int_0^t S(t-s)\,\mathrm{div}\,(u(s)\,\omega(s))\,ds, \qquad \text{(IE)}.$$

where $S(t) = \exp(t\Delta)$ is the heat semigroup in $\mathbb{R}^2$.

(IE) can be solved by a fixed point argument in the space equipped with the norm:

$$\|\omega\|_X = \sup_{0\le t\le T}\|\omega(t)\|_{L^1} + \sup_{0<t\le T} t^{1/4}\|\omega(t)\|_{L^{4/3}}$$

assuming that the initial data ω_0 belong to $L^1(\mathbb{R}^2)$. The solution can then be extended to all times $t\ge 0$ using the fact that

$$\|\omega(t)\|_{L^p} \le \|\omega(t_0)\|_{L^p} \qquad \forall t\ge t_0 \quad \forall p\in[1,+\infty].$$

In this way one obtains:

Proposition 1: For any initial data $\omega_0\in L^1(\mathbb{R}^2)$, (IE) has a unique global solution

$$\omega\in C^0([0,+\infty), L^1(\mathbb{R}^2)) \cap C^0((0,+\infty), L^\infty(\mathbb{R}^2))$$

such that $\omega(0)=\omega_0$. Moreover:

- $\int_{\mathbb{R}^2}\omega(x,t)\,dx = \int_{\mathbb{R}^2}\omega_0(x)\,dx \quad \forall t\ge 0$
- $\sup_{t\ge 0} t^{1-1/p}\|\omega(t)\|_{L^p(\mathbb{R}^2)} \le C_p\|\omega_0\|_{L^1(\mathbb{R}^2)}, \quad 1\le p\le\infty.$

<u>Oseen vortices:</u> If ω is radially symmetric, $u\cdot\nabla\omega = 0$

$\Rightarrow$ the vorticity evolves according to the <u>heat equation</u> $\partial_t\omega = \Delta\omega$.

The fundamental solution is self-similar:

$$\omega(x,t) = \frac{\Gamma}{t}\, G\!\left(\frac{x}{\sqrt{t}}\right), \qquad G(y) = \frac{1}{4\pi}\, e^{-|y|^2/4}.$$

$$u(x,t) = \frac{\Gamma}{\sqrt{t}}\, v^G\!\left(\frac{x}{\sqrt{t}}\right), \qquad v^G(y) = \frac{1}{2\pi}\,\frac{y^\perp}{|y|^2}\left(1 - e^{-|y|^2/4}\right)$$

Here $\Gamma \in \mathbb{R}$ is a free parameter. It turns out that these particular solutions describe the long-time behavior of all solutions of (NS) such that $\omega \in L^1(\mathbb{R}^2)$:

<u>Proposition 2:</u> For any initial data $\omega_0 \in L^1(\mathbb{R}^2)$, the solution of (NS) with initial data ω_0 given by Proposition 1 satisfies

$$\lim_{t\to+\infty} \int_{\mathbb{R}^2} \left| \omega(x,t) - \frac{\Gamma}{t}\, G\!\left(\frac{x}{\sqrt{t}}\right)\right| dx = 0,$$

where

$$\Gamma = \int_{\mathbb{R}^2} \omega_0(x)\, dx.$$

<u>Comments:</u>

- Oseen vortices are the only self-similar solutions of (NS) such that $\omega \in L^1(\mathbb{R}^2)$.
- The result is linked with the stability of the Oseen vortices, for all values of the circulation parameter Γ.
- Compare with the heat equation!

  Long-time asymptotics $\neq$ transient behavior.