Chapter IV: Boundaries, boundary layers, and inviscid limit

We consider in this chapter some examples of viscous flows in domains with boundaries, and we give an informal introduction to the difficult problem of the inviscid limit in presence of boundaries.

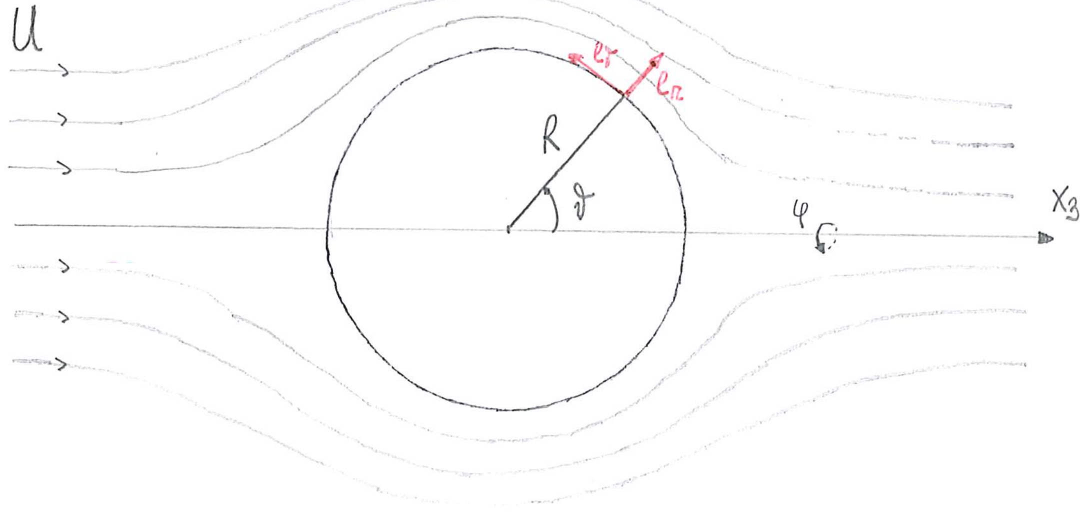
A) The Stokes flow past a sphere

For creeping flows ("Écoulements rampants") with $Re \ll 1$, the non-linear term $(v\cdot\nabla)v$ in the incompressible NS eq. can be neglected, and we are left with the Stokes flow: ($\rho > 0$ is a constant)

$$\rho\, \partial_t v = \mu \Delta v - \nabla p, \quad \operatorname{div} v = 0. \qquad \text{(Stokes)}$$

This linear equation is considered in a domain $\Omega \subset \mathbb{R}^d$ with no-slip boundary condition $v|_{\partial\Omega} = 0$. It is equivalent to $\partial_t v = -\nu A v$, where $A = -\mathbb{P}\Delta$ is the Stokes operator in Ω.

As an example, we consider the stationary Stokes flow in $\Omega = \{x \in \mathbb{R}^3 ;\ |x| > R\}$ with constant velocity $v = U e_3$ at infinity



Setting $q = p/\mu$, we want to solve the equation

$$0 = \Delta v - \nabla q \,, \quad \operatorname{div} v = 0 \qquad \text{in } \Omega$$

with boundary conditions:

$$v\big|_{\partial\Omega} = 0 \,, \qquad v(x) \xrightarrow[|x|\to\infty]{} U e_3 \,.$$

We use spherical coordinates $x = (r\sin\vartheta\cos\varphi, r\sin\vartheta\sin\varphi, r\cos\vartheta)$, see picture, and we decompose: $v = v_r e_r + v_\vartheta e_\vartheta + v_\varphi e_\varphi$.

From the symmetry of the domain and of the boundary conditions, we expect that $\partial_\varphi v = 0$, $\partial_\varphi q = 0$, $v_\varphi = 0$. (axisymmetric flow without swirl). In such a case the equations read:

$$\begin{cases} \Delta v_r - \dfrac{2v_r}{r^2} - \dfrac{2}{r^2\sin\vartheta}\partial_\vartheta(v_\vartheta\sin\vartheta) = \partial_r q \\[2mm] \Delta v_\vartheta + \dfrac{2}{r^2}\partial_\vartheta v_r - \dfrac{v_\vartheta}{r^2\sin^2\vartheta} = \dfrac{1}{r}\partial_\vartheta q \\[2mm] \dfrac{1}{r^2}\partial_r(r^2 v_r) + \dfrac{1}{r\sin\vartheta}\partial_\vartheta(v_\vartheta\sin\vartheta) = 0 \end{cases} \qquad (S)$$

where $\Delta f = \dfrac{1}{r^2}\partial_r(r^2\partial_r f) + \dfrac{1}{r^2\sin\vartheta}\partial_\vartheta(\sin\vartheta\,\partial_\vartheta f)$.

Since $e_3 = e_r\cos\vartheta - e_\vartheta\sin\vartheta$, the boundary conditions become:

$$\begin{cases} v_r = v_\vartheta = 0 \qquad \text{when } r = R \\[2mm] v_r \sim U\cos\vartheta \,, \quad v_\vartheta \sim -U\sin\vartheta \qquad \text{when } r \to +\infty. \end{cases} \qquad (C)$$

This suggests the following Ansatz:

$$v_r = a(r)\cos\vartheta \,, \quad v_\vartheta = b(r)\sin\vartheta \,, \quad q(r) = c(r)\cos\vartheta.$$

A direct calculation shows that (S) becomes

$$\begin{cases} a''(r) + \frac{2}{r} a'(r) - \frac{4}{r^2}(a(r)+b(r)) = c'(r) \\ b''(r) + \frac{2}{r} b'(r) - \frac{2}{r^2}(a(r)+b(r)) = -\frac{1}{r} c(r) \\ a'(r) + \frac{2}{r}(a(r)+b(r)) = 0 \end{cases}$$

Or equivalently:

$$\begin{cases} a''(r) + \frac{4}{r} a'(r) = c'(r) \\ b''(r) + \frac{2}{r} b'(r) + \frac{1}{r} a'(r) = -\frac{1}{r} c(r) \\ a'(r) + \frac{2}{r}(a(r)+b(r)) = 0 \end{cases} \qquad (S')$$

(S') can be written as a first-order homogeneous ODE system for the 4 quantities a, b, b', c $\Rightarrow$ (S') has 4 linearly independent solutions.

Ansatz: $a(r) = A r^{\alpha}$, $b(r) = B r^{\alpha}$, $c(r) = C r^{\alpha-1}$.

$$\Longrightarrow \begin{pmatrix} \alpha(\alpha+3) & 0 & 1-\alpha \\ \alpha & \alpha(\alpha+1) & 1 \\ \alpha+2 & 2 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0.$$

The determinant of the matrix is $\alpha(\alpha+1)(\alpha+3)(\alpha-2)$. The 3 admissible values are $\alpha = 0, -1, -3$ ($\alpha = 2$ corresponds to solutions that grow at ∞). It follows that

$$\begin{pmatrix} a(r) \\ b(r) \\ c(r) \end{pmatrix} = \beta \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \frac{\gamma}{r} \begin{pmatrix} -2 \\ 1 \\ -2/r \end{pmatrix} + \frac{\delta}{r^3} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

where the coefficients β, γ, δ have to be determined by the boundary conditions.

As $r \to +\infty$ we have by (c): $a(r) \to U$, $b(r) \to -U$

$\Longrightarrow$ $\beta = U$; At $r = R$ we have $a(R) = b(R) = 0$, hence

$$\beta - \frac{2\gamma}{R} + \frac{2\delta}{R^3} = 0, \quad -\beta + \frac{\gamma}{R} + \frac{\delta}{R^3} = 0$$

$$\Longrightarrow \gamma = \frac{3R}{4} U, \quad \delta = \frac{R^3}{4} U.$$

Summarizing, we found a unique solution:

$$\begin{cases} V_r = U\cos\vartheta \left(1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\right) \\ V_\vartheta = U\sin\vartheta \left(-1 + \frac{3R}{4r} + \frac{R^3}{4r^3}\right) \\ p = -\frac{3RU}{2r^2}\mu\cos\vartheta + p_\infty \end{cases}$$

(Stokes, 1851)

The associated vorticity is $\omega = \operatorname{curl} V = \omega_\varphi e_\varphi$ where

$$\omega_\varphi = \frac{1}{r}\partial_r(rV_\vartheta) - \frac{1}{r}\partial_\vartheta V_r = -\frac{3R}{2r^2}U\sin\vartheta.$$

It is also interesting to compute the Cauchy stress tensor σ on the sphere $r = R$:

$$\sigma_{rr} = -p + 2\mu\,\partial_r V_r = -p = \frac{3U}{2R}\mu\cos\vartheta$$

$$\sigma_{r\vartheta} = \mu\left(\frac{1}{r}\partial_\vartheta V_r + \partial_r V_\vartheta - \frac{V_\vartheta}{r}\right) = \mu\,\partial_r V_\vartheta = \frac{-3U}{2R}\mu\sin\vartheta$$

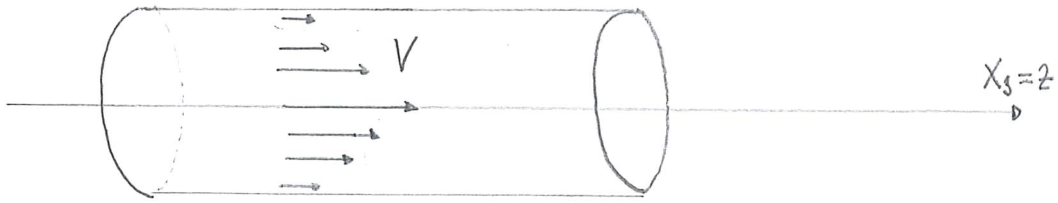
$$\Longrightarrow \sigma e_r = \frac{3U\mu}{2R}\left(\cos\vartheta\, e_r - \sin\vartheta\, e_\vartheta\right) = \frac{3U\mu}{2R} e_3.$$

Integrating over the sphere, we obtain the celebrated Stokes law:

$$F = \int_{|x|=R} (\sigma e_r)\, dS = 4\pi R^2 \frac{3U\mu}{2R} e_3 = 6\pi R U\mu\, e_3.$$

B) The cylindrical Poiseuille flow

We consider a steady flow in the infinite cylindrical pipe
$\Omega = \{ x \in \mathbb{R}^3 ;\ x_1^2 + x_2^2 < R^2,\ x_3 \in \mathbb{R} \}$.



Using cylindrical coordinates $x = (r\cos\vartheta, r\sin\vartheta, z)$, we assume that

- the only nonzero component of the velocity v is along $e_3 = e_z$;
- v does not depend on ϑ (axisymmetric flow without swirl).

Denoting $v = (0,0,u)$, the incompressibility condition gives $\partial_z u = 0$
$\Rightarrow$ $\underline{u = u(r)}$. It easily follows that $(v \cdot \nabla)v = 0$.

We thus have to solve the Stokes eq.

$$0 = \nu \Delta v - \nabla q, \quad \text{where } q = \frac{p}{\rho}.$$

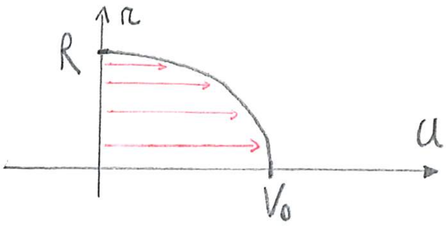
- first two components: $\partial_{x_1} q = \partial_{x_2} q = 0 \Rightarrow \underline{q = q(z)}$.
- third component: $\nu \left(u''(r) + \frac{1}{r} u'(r) \right) - q'(z) = 0$.

It follows that $q'(z) = -\alpha\nu$ and $u''(r) + \frac{1}{r} u'(r) + \alpha = 0$ for some constant $\alpha \in \mathbb{R}$. The unique solution satisfying $u(R) = 0$ (no-slip boundary condition) is:

$$u(r) = \frac{\alpha}{4}(R^2 - r^2) = V_0 \left(1 - \frac{r^2}{R^2}\right), \quad V_0 = \frac{\alpha R^2}{4}$$

$$q(z) = q_0 - \alpha\nu z \iff p(z) = p_0 - \alpha\mu z = p_0 - \frac{4V_0\mu}{R^2} z.$$

We can assume without of generality that $\alpha > 0$.
The velocity profile is <u>parabolic</u>, with a maximum V_0 on the axis:



The <u>pressure gradient</u> $\nabla p = -\frac{4V_0\mu}{R^2} e_z$ is constant, and drives the Poiseuille flow.

- <u>Velocity flux:</u>

$$\Phi = 2\pi \int_0^R u(r)\, r\, dr = 2\pi V_0 \int_0^R \left(1 - \frac{r^2}{R^2}\right) r\, dr$$
$$= 2\pi V_0 R^2 \int_0^1 (1-x^2)\, x\, dx = \frac{\pi R^2}{2} V_0 .$$

- <u>Drag force per unit length:</u> On the boundary we have

$$\sigma_{rr} = -p, \quad \sigma_{rz} = \mu u'(R) = -\frac{2}{R}\mu V_0$$
$$\Rightarrow \quad \sigma(-e_r) = p\, e_r + \frac{2}{R}\mu V_0 e_z$$
$$F_L = \int_0^L \int_0^{2\pi} \sigma(-e_r)\, R\, d\theta\, dz = 4\pi\mu V_0 L\, e_z .$$

- <u>Reynolds number:</u> $$Re := \frac{V_0 R}{\nu} = \frac{\rho V_0 R}{\mu} .$$

The cylindrical Poiseuille flow is an <u>exact solution</u> of NS which exists for all values of Re. It is believed to be stable for all Re, but experimentally it becomes unstable around $Re \simeq 5000$.

c) The problem of the inviscid limit

In a smooth domain $\Omega \subset \mathbb{R}^d$, we consider the NS eq.

$$\begin{cases} \partial_t v + (v\cdot\nabla)v = \nu\Delta v - \nabla p, \quad \operatorname{div} v = 0 \\ v(t,x) = 0, \quad x \in \partial\Omega \leftarrow \\ v(0,x) = v_0(x), \quad x\in\Omega \end{cases} \qquad (NS_\nu)$$

Taking formally the limit $\nu \to 0$, we obtain the Euler eq.

$$\begin{cases} \partial_t v + (v\cdot\nabla)v = -\nabla p, \quad \operatorname{div} v = 0 \\ v(t,x)\cdot n = 0, \quad x \in \partial\Omega \leftarrow \\ v(0,x) = v_0(x), \quad x\in\Omega \end{cases} \qquad (E)$$

⚠ Boundary conditions are different in (NS_ν) and (E)!

Suppose that v_0 is smooth enough, for instance $v_0 \in H^s(\Omega)$, $s > \frac{d}{2}+1$, $\operatorname{div} v_0 = 0$, $v_0\cdot n = 0$ on $\partial\Omega$. Then:

- There exists $T>0$ such that (E) has a unique solution v on $[0,T]$;
- Possibly restricting T, one can also show that (NS_ν) has a unique sol. v^ν on $[0,T]$ for any $\nu>0$.

Fundamental question: do we have $v^\nu(t,x) \xrightarrow[\nu\to 0]{} v(t,x)$? (IL)
If yes, in which sense?

Case 1: If $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$ (no boundary!), the answer is positive, and it is not difficult to show that (see Majda-Bertozzi):

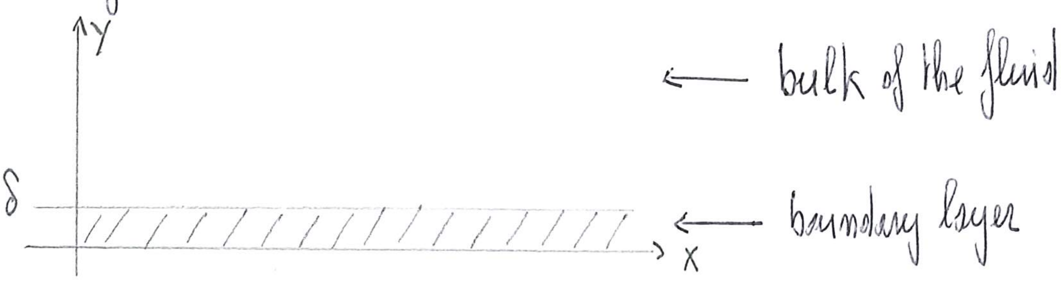
$$\sup_{0\le t\le T} \| v^\nu(t,\cdot) - v(t,\cdot)\|_{L^2} \le C\nu T \xrightarrow[\nu\to 0]{} 0.$$

Case 2: If $\partial\Omega \neq \emptyset$, it is clear that (IL) cannot hold in a strong topology, because $v^\nu(t,x) = 0 \; \forall x \in \partial\Omega \; \forall t > 0$, whereas in general $v(t,x) \neq 0$ for $x \in \partial\Omega$.

$\Rightarrow$ the inviscid limit is problematic due to the boundary conditions.

The approach of Prandtl (1904)

Assume for simplicity that $\Omega = \{(x,y) \in \mathbb{R}^2 ; y > 0\}$. We denote by (u,v) the velocity field and by p the pressure (divided by ρ). We consider the eq. (NS_ν) with small viscosity $\nu > 0$. We divide the domain Ω in two regions:



i) In the bulk of the fluid $(y \geq \delta > 0)$, we are far enough from the boundary so that the viscous term $\nu\Delta\vec{v}$ is negligible for small ν. We thus have (to leading order)

$$(u,v) \simeq (u^E, v^E),$$

where (u^E, v^E) is the solution of (E) in Ω. Note that the traces

$$U(t,x) := u^E(t,x,0) \quad \text{and} \quad P(t,x) := p^E(t,x,0)$$

do not vanish in general. They are linked by the relation

$$\| \; \partial_t U + U \partial_x U = -\partial_x P, \quad x \in \mathbb{R}, \; t \geq 0.$$

ii) In a boundary layer ($0<y<\delta$), the horizontal velocity varies rapidly to satisfy the no-slip boundary condition. Prandtl's Ansatz is:

$$u(t,x,y) = \tilde{u}(t,x,\tfrac{y}{\delta}), \quad v(t,x,y) = \delta\,\tilde{v}(t,x,\tfrac{y}{\delta}), \quad p(t,x,y) = \tilde{p}(t,x,\tfrac{y}{\delta}).$$

Denoting $z = y/\delta$, we obtain the new equations:

$$\begin{cases} \partial_t \tilde{u} + (\tilde{u}\partial_x + \tilde{v}\partial_z)\tilde{u} = \nu \partial_x^2 \tilde{u} + \dfrac{\nu}{\delta^2}\partial_z^2 \tilde{u} - \partial_x \tilde{p} \\ \delta\left(\partial_t \tilde{v} + (\tilde{u}\partial_x + \tilde{v}\partial_z)\tilde{v}\right) = \nu\delta\partial_x^2 \tilde{v} + \dfrac{\nu}{\delta}\partial_z^2 \tilde{v} - \dfrac{1}{\delta}\partial_z \tilde{p} \\ \partial_x \tilde{u} + \partial_z \tilde{v} = 0. \end{cases}$$

In view of the first equation, we set $\delta = \sqrt{\nu}$ (size of the boundary layer) so that $\nu/\delta^2 = 1$. Dropping the tildes and keeping only the leading order terms in δ, we obtain the Prandtl equations:

$$\left[\begin{array}{l} \partial_t u + (u\partial_x + v\partial_z)u = \partial_z^2 u - \partial_x p \\ \partial_z p = 0, \quad \partial_x u + \partial_z v = 0 \end{array}\right. \qquad (\mathrm{Pr})$$

This equation is considered in the domain $\Omega = \{(x,z) \in \mathbb{R}^2 ;\ z>0\}$.
The boundary conditions are:

- $u = v = 0$ when $z = 0$
- $u(t,x,z) \xrightarrow[z\to+\infty]{} U(t,x)$ (trace of the Euler solution)

Note that

- $p(t,x,z) = P(t,x) \quad \forall (x,z)\in\Omega$ (the pressure is constant in z)
- $v(t,x,z) = -\displaystyle\int_0^z \partial_x u(t,x,z')\,dz'$.

Summarizing, we are led to consider the <u>scalar equation</u>:

$$\begin{cases} \partial_t u + (u\partial_x + v\partial_z)u = \partial_z^2 u - \partial_x P \\ v = -\int_0^z \partial_x u \, dz' \end{cases} \qquad (P_{\mathbb{R}}')$$

with the boundary conditions $u = 0$ as $z = 0$, $u \to U$ as $z \to +\infty$.
Here U and P are given (trace of the Euler solution).

Eq. $(P_{\mathbb{R}}')$ is a <u>degenerate parabolic equation</u> (no diffusion on x!) with a very bad, nonlocal non-linearity. It is extremely difficult to solve in general, but special cases are well understood.

<u>The stationary case</u> (u, v, P, U are time-independent)

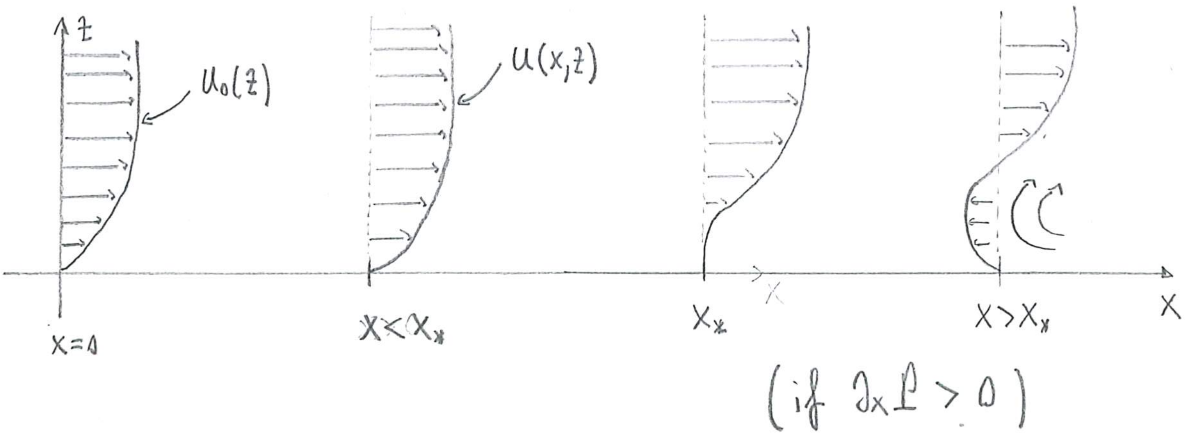
Here we typically want to solve the "parabolic equation"

$$\begin{cases} u\partial_x u + v\partial_z u = \partial_z^2 u - \partial_x P \\ \partial_x u + \partial_z v = 0 \end{cases} \qquad (SP)$$

in the quadrant $\Omega_+ = \{(x,z) \in \mathbb{R}^2, x \geq 0, z \geq 0\}$ with

- <u>boundary conditions</u>: $u = v = 0$ for $z = 0$, $u \to U$ as $z \to +\infty$
- <u>initial data</u>: $u(0,z) = u_0(z)$ $\forall z \geq 0$.

We consider $x \geq 0$ as the "evolutionary variable" in this picture.



Assume that $u(x,z) > 0 \ \forall x \geq 0 \ \forall z > 0$. We introduce the stream function ψ defined by

$$\partial_z \psi = u, \quad \partial_x \psi = -v, \qquad \psi|_{z=0} = 0$$

and we define a new function W by

$$u(x,z) = W(x, \psi(x,z))^{1/2} \qquad \text{(von Mise transform).}$$

N.B. This is well-defined as long as $\partial_z \psi = u > 0$.

We have:

- $2uu_x = \partial_x u^2 = \partial_x W + \partial_\psi W \cdot \partial_x \psi = \partial_x W - v\,\partial_\psi W$
- $2uu_z = \partial_z u^2 = \partial_\psi W \, \partial_z \psi = u\,\partial_\psi W \implies 2\partial_z u = \partial_\psi W$
- $2\partial_z^2 u = \partial_\psi^2 W \, \partial_z \psi = u\,\partial_\psi^2 W$

hence (SL) becomes

$$\partial_x W = \sqrt{W}\,\partial_\psi^2 W - 2\partial_x P. \qquad (*)$$

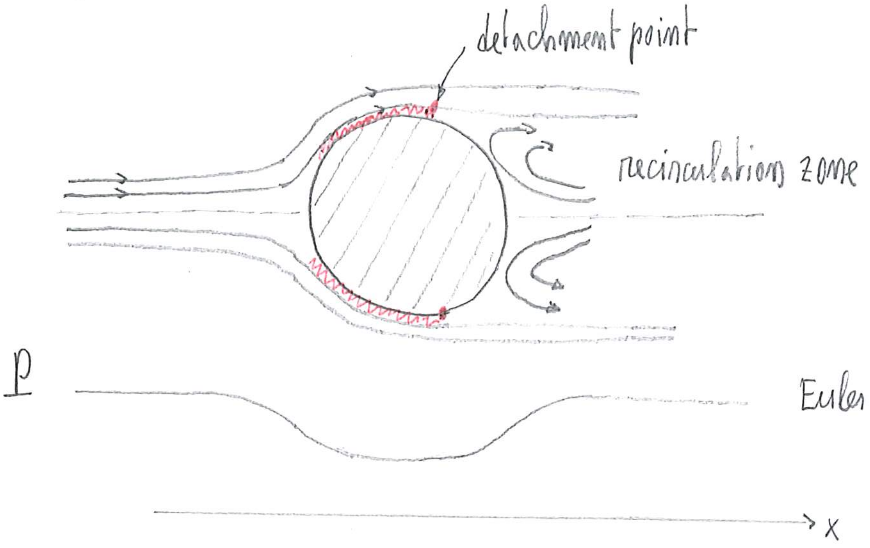
This is a degenerate parabolic equation, to be solved in the domain $\psi \in [0,+\infty)$ with boundary conditions $W|_{\psi=0} = 0$, $W|_{\psi=+\infty} = U^2$ and initial data $W|_{x=0} = u_0^2$.

⚠ We need the solution to stay positive, which is OK if $\partial_x P \leq 0$.
Under that assumption, Oleinik proved in 1962 that $(*)$ is globally well-posed for $x \geq 0$.

In case of adverse pressure gradient ($\partial_x P < 0$), the solution of $(*)$ may become singular at a finite "time" x_*:

$$u(x,z) \sim (x_* - x)^{1/2}\, U_0\left(\frac{z}{(x_*-x)^{1/4}}\right), \qquad x \to x_*.$$

The point x_* is often interpreted as the point where the boundary layer "detaches" from the boundary.



Stationary Prandtl is a good model only before boundary layer detachment

$$p + \frac{1}{2} U^2 = \text{const.}$$

The time-dependent case

Here we assume that u is a monotone function of z, for instance

$$\omega = \partial_z u > 0 \qquad (\omega \text{ is the vorticity in the boundary layer})$$

The equation satisfied by ω is:

$$\partial_t \omega + u \partial_x \omega + v \partial_z \omega = \partial_z^2 \omega$$

Note that $\partial_z \omega = \partial_x p$ at $z=0$, and $\partial_z \omega \longrightarrow 0$ as $z \to +\infty$.

We now introduce the new function W defined by

$$\omega(t,x,z) = W(t,x,u(t,x,z)) \qquad (\text{Crocco transform})$$

A direct calculation shows that W satisfies the degenerate parabolic PDE

$$\partial_t W + u \partial_x W = W^2 \partial_u^2 W + (\partial_x p) \partial_u W \qquad (**)$$

in the domain $\Omega_t = \{(x,u);\ x \in \mathbb{R},\ 0 < u < U(t,x)\}$ with boundary conditions

$$\partial_u W\big|_{u=0} = \partial_x p, \quad \partial_u W\big|_{u=U} = 0.$$

N.B. The domain Ω_t depends on time, but this can be cured by introducing the variable $\tilde{u} = u/U \in (0,1)$. Eq. (**) then becomes slightly more complicated.

Again, for monotone initial data corresponding to $W(0,x,u) > 0$, $+ \partial_x P \leq 0$, one can prove that (**) is globally well-posed (Oleinik, 1967).

Without monotonicity assumption, system (P_ν') is strongly ill-posed in Sobolev spaces, due to the "loss of derivative in x" in the non-linear term $V \partial_z u$. It can be shown to be locally well-posed in spaces with infinite regularity in x (analyticity or Gevrey regularity).

Moreover, the Prandtl Ansatz can be proved to be a good approximation of the NS equation for small ν only if the data are analytic (Sammartino & Caflish).

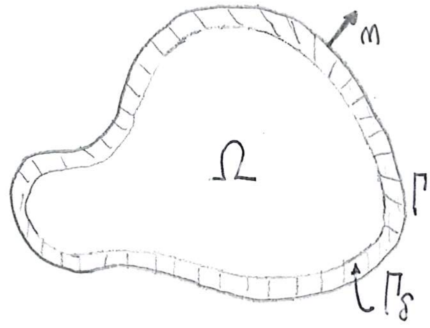
D) Kato's convergence criterion

In 1984 T. Kato gave a simple and intriguing necessary and sufficient condition for the vanishing viscosity limit in the energy space. We present here Kato's result in the 2D case (without external force), for simplicity.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\Gamma = \partial\Omega$.
For any $\delta > 0$, we denote

$$\Gamma_\delta = \{ x \in \Omega \, ; \, \mathrm{dist}(x, \Gamma) < \delta \}$$

Γ_δ is a strip of width δ near the boundary, inside Ω.



Given a smooth function $\bar{u}_0$ with $\operatorname{div}\bar{u}_0 = 0$, $\bar{u}_0 \cdot n = 0$ on Γ, we denote by $\bar{u}$ the solution of Euler's equations

$$\begin{cases} \partial_t \bar{u} + (u\cdot\nabla)\bar{u} = -\nabla \bar{p}, \ \operatorname{div}\bar{u} = 0 & x\in\Omega,\ t\in[0,T] \\ \bar{u}\cdot n = 0 & x\in\Gamma,\ t\in[0,T] \\ \bar{u}(0,x) = \bar{u}_0(x) & x\in\Omega \end{cases} \qquad (E)$$

This solution can be constructed by adapting the results of chapter II to the case of a bounded domain. The existence time $T>0$ is arbitrary here, since the solution is global.

Also given $\nu>0$ and initial data $u_0^\nu \in H = \{v\in L^2;\ \operatorname{div} v = 0,\ v\cdot n = 0 \text{ on } \partial\Omega\}$, let $u^\nu \in C^0([0,T],H)\cap L^2((0,T),V)$ be the unique solution of the Navier-Stokes equations

$$\begin{cases} \partial_t u^\nu + (u^\nu\cdot\nabla)u^\nu = \nu\Delta u^\nu - \nabla p^\nu, \ \operatorname{div} u^\nu = 0 & x\in\Omega,\ t\in(0,T] \\ u^\nu = 0 & x\in\Gamma,\ t\in(0,T] \\ u^\nu(0,x) = u_0^\nu(x) & x\in\Omega \end{cases} \qquad (NS_\nu)$$

We recall that:

$$\|\bar{u}(t)\|_{L^2} = \|\bar{u}_0\|_{L^2} \quad \forall t\in[0,T] \qquad \text{(conservation of energy for Euler)}$$

$$\frac{1}{2}\|u^\nu(t)\|_{L^2}^2 + \nu\int_0^t \|\nabla u^\nu(s)\|_{L^2}^2\, ds = \frac{1}{2}\|u_0^\nu\|_{L^2}^2 \qquad \text{(energy balance for NS)}$$

We also take for granted the fact that u^ν is a weak solution of NS: for all test function $\varphi \in C^1([0,T]\times\bar\Omega)^2$ with $\operatorname{div}\varphi = 0$ and $\varphi = 0$ on Γ, we have $\forall t \in [0,T]$:

$$\begin{aligned}(u^\nu(t),\varphi(t))_{L^2} - (u_0^\nu,\varphi(0))_{L^2} = \int_0^t \Big\{ &(u^\nu(s)\otimes u^\nu(s), \nabla\varphi(s))_{L^2} \\ &- \nu(\nabla u^\nu(s),\nabla\varphi(s))_{L^2} + (u^\nu(s),\partial_t\varphi(s))_{L^2}\Big\}\,ds.\end{aligned} \qquad \text{(WS)}$$

Here $(u\otimes u, \nabla\varphi)_{L^2} = (u_j u_k, \partial_j \varphi_k)_{L^2}$.

Theorem (Kato, 1984) Assume that $\|u_0^\nu - \bar u_0\|_{L^2} \xrightarrow[\nu\to 0]{} 0$. The following assertions are then equivalent:

i) $\sup_{t\in[0,T]} \|u^\nu(t)-\bar u(t)\|_{L^2} \xrightarrow[\nu\to 0]{} 0$ (vanishing viscosity limit in energy space)

ii) $u^\nu(T) \xrightarrow[\nu\to 0]{} \bar u(T)$ (weak convergence in L^2)

iii) $\nu \int_0^T \|\nabla u^\nu(t)\|^2_{L^2(\Omega)}\,dt \xrightarrow[\nu\to 0]{} 0$

iv) $\nu \int_0^T \|\nabla u^\nu(t)\|^2_{L^2(\Gamma_{c\nu})}\,dt \xrightarrow[\nu\to 0]{} 0$ $(c>0$ arbitrary$)$

Remarks: This result does not say if the Euler solution can be obtained from NS by taking the vanishing viscosity limit, but it provides necessary and sufficient conditions for that (none of which is easy to check!)

The first postulate of Kolmogorov's theory of turbulence is

$$\limsup_{\nu\to 0} \nu \|\nabla u^\nu\|^2_{L^2(\Omega)} = \varepsilon > 0$$

(the energy dissipation rate does not vanish as $\nu \to 0$ $\Rightarrow$ energy cascade)

This is obviously related to iii).

iv) Shows that it is enough to measure the energy dissipation in a "boundary layer" of size $C\nu$, for some $C>0$. This is much smaller than the size of Prandtl's boundary layer, which is $O(\nu^{1/2})$!!

Proof:

i) $\Rightarrow$ ii) obvious

ii) $\Rightarrow$ iii) If $u^\nu(T) \rightharpoonup \bar{u}(T)$, then $\|\bar{u}(T)\|_{L^2} \leq \liminf_{\nu \to 0} \|u^\nu(T)\|_{L^2}$, hence:

$$\limsup_{\nu \to 0} \nu \int_0^T \|\nabla u^\nu(s)\|_{L^2}^2 \, ds = \limsup_{\nu \to 0} \left(\frac{1}{2}\|u_0^\nu\|_{L^2}^2 - \frac{1}{2}\|u^\nu(T)\|_{L^2}^2 \right)$$

$$= \frac{1}{2}\|\bar{u}_0\|_{L^2}^2 - \frac{1}{2}\liminf_{\nu \to 0} \|u^\nu(T)\|_{L^2}^2 \leq \frac{1}{2}\|\bar{u}_0\|_{L^2}^2 - \frac{1}{2}\|\bar{u}(T)\|_{L^2}^2 = 0.$$

iii) $\Rightarrow$ iv) obvious

So the only nontrivial step is iv) $\Rightarrow$ i).

a) Construction of a boundary layer correction

For each $t \in [0,T]$, we denote by $\varphi = \varphi(t)$ the stream function associated with $\bar{u}(t)$, namely:

$$\bar{u}(t) = \nabla^\perp \varphi(t) = \begin{pmatrix} -\partial_2 \varphi(t) \\ \partial_1 \varphi(t) \end{pmatrix}, \qquad \varphi(t)\big|_{\partial\Omega} = 0.$$

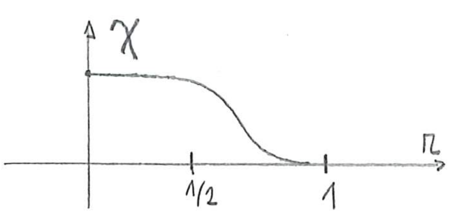
(Below we drop the time-dependence for notational simplicity.)

The stream function can be constructed by solving the elliptic problem:

$$\Delta \varphi = \omega = \partial_1 u_2 - \partial_2 u_1 \text{ in } \Omega, \quad \varphi = 0 \text{ on } \partial\Omega.$$

Let $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ be a smooth cut-off function such that

$$\chi(r) = \begin{cases} 1 & \text{for } 0 \leq r \leq 1/2 \\ 0 & \text{for } r \geq 1 \end{cases}$$



We now define $V \in C^1([0,T]\times\bar{\Omega})$ by

$$V(t,x) = \nabla^{\perp}\big(\varphi(t,x)\,\chi(\rho(x)/\delta)\big)$$

where $\rho(x) = \text{dist}(x,\partial\Omega)$, and $\delta = c\nu$ for some $c>0$.
By construction V is smooth (if $\delta>0$ is small enough) and satisfies

- $\text{div}\, V = 0$
- $V = \bar{u}$ whenever $\text{dist}(x,\Gamma) \leq \delta/2$.

Moreover, the following bounds are easy to check:

$$\begin{cases} \|V\|_{L^\infty} \leq C, \quad \|\nabla V\|_{L^\infty} \leq C\delta^{-1}, \quad \|\rho\nabla V\|_{L^\infty} \leq C, \quad \|\rho^2\nabla V\|_{L^\infty} \leq C\delta \\ \|V\|_{L^2} \leq C\delta^{1/2}, \quad \|\nabla V\|_{L^2} \leq C\delta^{-1/2}, \quad \|\rho\nabla V\|_{L^2} \leq C\delta^{1/2} \\ \|\partial_t V\|_{L^2} \leq C\delta^{1/2}. \end{cases} \qquad (B)$$

b) Convergence proof: Given $t\in[0,T]$, we observe that

$$\begin{aligned} \|u^\nu(t)-\bar{u}(t)\|_{L^2}^2 &= \|u^\nu(t)\|_{L^2}^2 + \|\bar{u}(t)\|_{L^2}^2 - 2\big(u^\nu(t),\bar{u}(t)\big)_{L^2} \\ &\leq \|u_0^\nu\|_{L^2}^2 + \|\bar{u}_0\|_{L^2}^2 - 2\big(u^\nu(t),\bar{u}(t)\big)_{L^2} \\ &\leq 2\|\bar{u}_0\|_{L^2}^2 - 2\big(u^\nu(t),\bar{u}(t)-V(t)\big)_{L^2} + \varepsilon_\nu, \end{aligned} \qquad (1)$$

where $\varepsilon_\nu \xrightarrow[\nu\to 0]{} 0$. Here we used the facts that

- $\|u_0^\nu\|_{L^2} \xrightarrow[\nu\to 0]{} \|\bar{u}_0\|_{L^2}$ by assumption
- $|(u^\nu(t),V(t))_{L^2}| \underset{\uparrow (B)}{\leq} \|u_0^\nu\|_{L^2}\|V\|_{L^2} \leq C\delta^{1/2} \xrightarrow[\nu\to 0]{} 0$.

From now on, to simplify the notation, we drop the time dependence and write $\|\cdot\|$ instead of $\|\cdot\|_{L^2}$, etc.

To estimate the term $2(u^\nu, \bar{u}-v)$ in (1), we use (WS) with test function $\varphi = \bar{u} - v$ (note that $\operatorname{div}\varphi = 0$ and $\varphi = 0$ on $\partial\Omega$, hence the necessity of subtracting v). This gives

$$-2(u^\nu, \bar{u}-v) + 2(u_0^\nu, \bar{u}_0 - v_0) = \int_0^t \Big\{ -2(u^\nu \otimes u^\nu, \nabla(\bar{u}-v)) + 2\nu(\nabla u^\nu, \nabla(\bar{u}-v)) - 2(u^\nu, \partial_t(\bar{u}-v)) \Big\} ds .$$

But $(u_0^\nu, \bar{u}_0 - v_0) \xrightarrow[\nu \to 0]{} \|\bar{u}_0\|^2$, hence combining with (1) we obtain

$$\|u^\nu - \bar{u}\|^2 \leqslant \int_0^t \Big\{ -2(u^\nu \otimes u^\nu, \nabla(\bar{u}-v)) + 2\nu(\nabla u^\nu, \nabla(\bar{u}-v)) - 2(u^\nu, \partial_t(\bar{u}-v)) \Big\} ds + \varepsilon_\nu . \quad (2)$$

It remains to control the various terms in the integrand. We observe that:

- $-2(u^\nu, \partial_t \bar{u} - \partial_t v) = 2(u^\nu, (\bar{u}\cdot\nabla)\bar{u}) + \varepsilon_\nu$ (using Euler's equation)
- $-2(u^\nu \otimes u^\nu, \nabla \bar{u}) + 2(u^\nu, (\bar{u}\cdot\nabla)\bar{u}) = -2((u^\nu - \bar{u})\otimes(u^\nu - \bar{u}), \nabla\bar{u})$ (identity)

hence

$$\|u^\nu - \bar{u}\|^2 \leqslant 2\int_0^t \Big\{ (u^\nu \otimes u^\nu, \nabla v) - ((u^\nu - \bar{u})\otimes(u^\nu - \bar{u}), \nabla\bar{u}) + \nu(\nabla u^\nu, \nabla(\bar{u}-v)) \Big\} ds + \varepsilon_\nu .$$

But $|((u^\nu - \bar{u})\otimes(u^\nu - \bar{u}), \nabla\bar{u})| \leqslant \|\nabla\bar{u}\|_{L^\infty} \|u^\nu - \bar{u}\|^2 = K\|u^\nu - \bar{u}\|^2$, so that

$$\|u^\nu(t) - \bar{u}(t)\|^2 \leqslant \varepsilon_\nu + \int_0^t \big(K\|u^\nu(s) - \bar{u}(s)\|^2 + R(s) \big) ds, \text{ where}$$

$$R = (u^\nu \otimes u^\nu, \nabla v) + \nu(\nabla u^\nu, \nabla(\bar{u}-v)) . \quad (3)$$

This gives the desired result by Gronwall's lemma if we can show that

$$\int_0^T |R(t)|\, dt \xrightarrow[\nu \to 0]{} 0 . \quad (4)$$

To check that, we observe that

- $$|(u^\nu \otimes u^\nu, \nabla v)| \leq \|\tfrac{1}{\rho} u^\nu\|^2_{L^2_{\Gamma_\delta}} \|\rho^2 \nabla v\|_{L^\infty} \overset{\text{(B) + Hardy}}{\leq} C \|\nabla u^\nu\|^2_{L^2(\Gamma_\delta)} \delta$$

  where we used Hardy's inequality $\|\frac{1}{\rho} u^\nu\|_{L^2(\Gamma_\delta)} \leq C \|\nabla u^\nu\|_{L^2(\Gamma_\delta)}$.

  Since $\delta = c\nu$, we have

  $$\delta \int_0^T \|\nabla u^\nu(s)\|^2_{L^2(\Gamma_\delta)} ds \xrightarrow[\nu \to 0]{} 0 \text{ by assumption iv).}$$

- $$\nu |(\nabla u^\nu, \nabla(\bar{u} - v))| \leq \nu \|\nabla u^\nu\| \|\nabla \bar{u}\| + \nu \|\nabla u^\nu\|_{L^2(\Gamma_\delta)} \|\nabla v\|_{L^2(\Gamma_\delta)}$$

  $$\leq C\nu \|\nabla u^\nu\|_{L^2(\Omega)} + \frac{C\nu}{\delta^{1/2}} \|\nabla u^\nu\|_{L^2(\Gamma_\delta)}.$$

  But

  $$\nu \int_0^T \|\nabla u^\nu\|_{L^2(\Omega)} dt \leq \nu T^{1/2} \left(\int_0^T \|\nabla u^\nu\|^2_{L^2(\Omega)} dt\right)^{1/2} \leq (\nu T)^{1/2} \|u_0^\nu\|_{L^2} \xrightarrow[\nu \to 0]{} 0$$

  $$\frac{\nu}{\delta^{1/2}} \int_0^T \|\nabla u^\nu\|_{L^2(\Gamma_\delta)} dt \leq \frac{T^{1/2}}{c^{1/2}} \left(\nu \int_0^T \|\nabla u^\nu\|^2_{L^2(\Gamma_\delta)} dt\right)^{1/2} \xrightarrow[\nu \to 0]{} 0 \text{ by assumption iv).}$$

This shows (4), and (3) gives the desired result. $\square$