Chapter III : Viscous fluids without boundaries

In this chapter we study the initial value problem for the incompressible Navier-Stokes equations:

$$\partial_t v + (v\cdot\nabla)v = \nu\Delta v - \nabla p\,, \quad \operatorname{div} v = 0 \tag{NS}$$

in the whole space $\mathbb{R}^d$, $d = 2$ or 3. We recall that:

- $v(t,x) \in \mathbb{R}^d$ is the fluid velocity
- $p(t,x) \in \mathbb{R}$ is the pressure divided by the (constant) density
- $\nu > 0$ is the kinematic viscosity

Rescaling: Suppose that $L > 0$ is a characteristic length and $U > 0$ a characteristic speed of the flow under consideration. We introduce the dimensionless independent and dependent variables:

$$\tilde{t} = \frac{Ut}{L}\,, \quad \tilde{x} = \frac{x}{L}\,, \quad \tilde{v} = \frac{v}{U}\,, \quad \tilde{p} = \frac{p}{U^2}\,.$$

More precisely, we set:

$$v(t,x) = U\tilde{v}\Big(\frac{Ut}{L}, \frac{x}{L}\Big), \quad p(t,x) = U^2\tilde{p}\Big(\frac{Ut}{L}, \frac{x}{L}\Big).$$

Inserting this Ansatz into (NS) and dropping all tildes, we find:

$$\partial_t v + (v\cdot\nabla)v = \frac{1}{Re}\Delta v - \nabla p\,, \quad \operatorname{div} v = 0 \tag{NS'}$$

where Re is the dimensionless Reynolds number defined by:

$$\boxed{Re = \frac{UL}{\nu}\,.}$$

Re = (strength of the advection terms) / (strength of the viscous terms).

By a further time rescaling:

$$v(t,x) \longmapsto \lambda v(\lambda t, x), \quad p(t,x) \longmapsto \lambda^2 p(\lambda t, x), \quad \lambda = \frac{1}{Re}$$

one can always assume that $Re = 1$!

<u>Important remark</u>: For a fixed value of ν or Re, equations (NS), (NS') are invariant under the rescaling:

$$v(t,x) \longmapsto \lambda v(\lambda^2 t, \lambda x), \quad p(t,x) \longmapsto \lambda^2 p(\lambda^2 t, \lambda x), \quad \lambda > 0 \qquad (S)$$

This observation plays a crucial role in the analysis of the Cauchy problem.

We consider henceforth the Navier-Stokes eq. (NS) with $\nu = 1$. Existence of solutions can be proved in three steps.

<u>Step 1</u>: <u>Elimination of the pressure</u>

Applying the Leray-Hopf projection $\mathbb{P}$ we formally obtain:

$$\partial_t v + \mathbb{P}(v \cdot \nabla) v = \Delta v, \qquad v|_{t=0} = v_0. \qquad (NS'')$$

⚠ Since we are in the whole space $\mathbb{R}^d$ (no boundaries), the projection $\mathbb{P}$ <u>commutes with the Laplace operator</u>: $\mathbb{P}\Delta v = \Delta \mathbb{P} v = \Delta v$ if $\mathbb{P}v = v$.

We recall that $\mathbb{P}$ is a Fourier multiplier with matrix-valued symbol

$$\mathbb{P}_{jk} = \delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2} \qquad (\text{homogeneous of degree } 0).$$

In other words $\mathbb{P}_{jk} = \delta_{jk} + R_j R_k$, where R_j is the <u>Riesz transform</u>:

$$\widehat{R_j f}(\xi) = \frac{i\xi_j}{|\xi|} \hat{f}(\xi).$$

These operators can also be expressed as <u>singular integrals</u>:

$$(R_j f)(x) = C_d \lim_{\varepsilon \to 0+} \int_{\mathbb{R}^d \setminus B_\varepsilon(x)} \frac{(x-y)_j}{|x-y|^{d+1}} f(y)\, dy$$

where C_d depends only on the space dimension d.

It is clear that $\| \mathbb{P} v \|_{L^2} \leq \| v \|_{L^2}$ as $\mathbb{P}$ is a continuous projection in L^2.

By Calderón-Zygmund theory, the Riesz transforms are also bounded in $L^p(\mathbb{R}^d)$ for $1<p<\infty$, hence:

<u>Property 1:</u> If $1<p<\infty$ and $v \in L^p(\mathbb{R}^d)^d$, then

$$\| \mathbb{P} v \|_{L^p} \leq C_p \| v \|_{L^p}.$$

<u>Step 2:</u> <u>Integral equation</u>

By Duhamel's formula, the integral equation associated with (NS'') is

$$\begin{aligned} v(t) &= S(t) v_0 - \int_0^t S(t-s)\, \mathbb{P}\, (v(s)\cdot\nabla) v(s)\, ds \\ &= S(t) v_0 - \int_0^t S(t-s)\, \mathbb{P}\, \mathrm{div}\,(v(s)\otimes v(s))\, ds \end{aligned} \qquad (IE)$$

where $S(t) = \exp(t\Delta)$ is the <u>heat semi-group</u>. Explicitly:

$$(S(t)f)(x) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} f(y)\, dy \qquad (t>0)$$

$$= \frac{1}{(4\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|z|^2}{4}} f(x+\sqrt{t}\, z)\, dz. \qquad (t \geq 0)$$

The following properties are well known:

<u>Property 2</u> (L^p-L^q estimates)

If $1 \leq p \leq q \leq \infty$ and $\alpha \in \mathbb{N}^d$, there exists $C > 0$ such that

$$\| \partial^\alpha S(t) f \|_{L^q} \leq \frac{C}{t^{\frac{|\alpha|}{2} + \frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)}} \|f\|_{L^p}, \quad t > 0.$$

(Rem: $C = 1$ if $\alpha = 0$)

The integral term in (IE) involves the combination $S(t) \mathbb{P} \partial^\alpha$ with $|\alpha| = 1$. For such an operator we have:

<u>Property 3</u> (L^p-L^q estimates, continued)

If $1 \leq p \leq q \leq \infty$ and $\alpha \in \mathbb{N}^d$, $|\alpha| \geq 1$, there exists $C > 0$ such that

$$\| S(t) \mathbb{P} \partial^\alpha v \|_{L^q} \leq \frac{C}{t^{\frac{|\alpha|}{2} + \frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)}} \|v\|_{L^p}, \quad t > 0.$$

⚠ If $p = q = 1$ or $p = q = \infty$, property 3 <u>does not</u> follow from properties 1,2 because the Leray projection $\mathbb{P}$ is not bounded in L^1 or L^∞.

<u>Proof:</u> We use the relation

$$\left(\delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2} \right) e^{-|\xi|^2 t} = \delta_{jk} e^{-|\xi|^2 t} - \xi_j \xi_k \int_t^\infty e^{-|\xi|^2 \tau} \, d\tau,$$

which gives the useful identity:

$$(S(t) \mathbb{P} \partial^\alpha v)_j = \partial^\alpha S(t) v_j + \int_t^\infty \partial^\alpha \partial_j \partial_k S(\tau) v_k \, d\tau.$$

The first term is estimated by property 2. As for the integral term, if $|\alpha| \geq 1$:

$$\left\| \int_t^\infty \partial^\alpha \partial_j \partial_k S(\tau) v_k \, d\tau \right\|_{L^q} \leq C \int_t^\infty \frac{1}{\tau^{\frac{|\alpha|}{2} + 1 + \frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)}} \|v\|_{L^p} \, d\tau$$

(exponent is $\geq 3/2$)

$$= \frac{C}{t^{\frac{|\alpha|}{2} + \frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)}} \|v\|_{L^p}, \quad t > 0. \quad \square$$

Step 3: Fixed point argument, non-critical case

We assume here that $V_0 \in L^p(\mathbb{R}^d)$ for some $p \in (d, +\infty)$, $\operatorname{div} V_0 = 0$.

Proposition 1: Assume $d < p < \infty$. For any $R > 0$, there exists $T = T(R) > 0$ such that, for all $V_0 \in L^p(\mathbb{R}^d)$ with $\operatorname{div} V_0 = 0$ and $\|V_0\|_{L^p} \leq R$, eq. (IE) has a unique solution

$$V \in C^0([0,T], L^p(\mathbb{R}^d)).$$

Proof: Given $V_0 \in L^p_\sigma(\mathbb{R}^d)$ and $T > 0$ (to be specified later), we want to solve (IE) by a fixed point argument in the Banach space:

$$X = C^0([0,T], L^p(\mathbb{R}^d)), \qquad \|V\|_X = \sup_{t \in [0,T]} \|V(t)\|_{L^p}.$$

For $V \in X$ we define

$$(FV)(t) = S(t)V_0 - \int_0^t S(t-s)\,\mathbb{P}\operatorname{div}(V(s) \otimes V(s))\,ds, \qquad t \in [0,T].$$

i) We have the estimate:

note that $p > 2$

$$\|(FV)(t)\|_{L^p} \leq \|V_0\|_{L^p} + C\int_0^t \frac{1}{(t-s)^{\frac{1}{2} + \frac{d}{2}\left(\frac{2}{p} - \frac{1}{p}\right)}} \|V(s) \otimes V(s)\|_{p/2}\,ds$$

$$\leq \|V_0\|_{L^p} + C\int_0^t \frac{1}{(t-s)^{1/2 + d/2p}} \|V(s)\|_{L^p}^2\,ds \qquad \left(\tfrac{1}{2} + \tfrac{d}{2p} < 1\right)$$

$$\leq \|V_0\|_{L^p} + C t^{\vartheta} \|u\|_X^2 \quad \text{where } \vartheta = \tfrac{1}{2}\left(1 - \tfrac{d}{p}\right) > 0. \quad \|$$

This shows that $FV \in L^\infty([0,T], L^p)$ with

$$\|FV\|_X \leq \|V_0\|_{L^p} + C_1 T^{\vartheta} \|V\|_X^2. \tag{A}$$

ii) We next verify that $t \longmapsto (FV)(t)$ is continuous in $L^p(\mathbb{R}^d)$.
For instance, let us check the continuity <u>to the right</u> at $t \in [0,T)$.
For $0 < \varepsilon < T-t$, we have : $\quad$ ($t \longmapsto S(t)V_0$ is continuous !)

$$\int_0^{t+\varepsilon} S(t+\varepsilon-s)\,\mathbb{P}\,\mathrm{div}(v(s)\otimes v(s))\,ds - \int_0^t S(t-s)\,\mathbb{P}\,\mathrm{div}(v(s)\otimes v(s))\,ds$$
$$= \int_t^{t+\varepsilon} S(t+\varepsilon-s)\,\mathbb{P}\,\mathrm{div}(v(s)\otimes v(s))\,ds + \int_0^t \big(S(t+\varepsilon-s)-S(t-s)\big)\,\mathbb{P}\,\mathrm{div}(v(s)\otimes v(s))\,ds$$
$$= \qquad I \qquad + \qquad J.$$

$$\|I\|_{L^p} \leq \int_t^{t+\varepsilon} \frac{C}{(t+\varepsilon-s)^{1-\vartheta}}\,\|V(s)\|_{L^p}^2\,ds \leq C\,\|V\|_X^2\,\varepsilon^{\vartheta} \xrightarrow[\varepsilon\to 0]{} 0.$$

$\|J\|_{L^p} \xrightarrow[\varepsilon\to 0]{} 0$ by dominated convergence, since:

- $\|(S(t+\varepsilon-s)-S(t-s))\,\mathbb{P}\,\mathrm{div}(u(s)\otimes u(s))\|_{L^p} \leq \dfrac{C}{(t-s)^{1-\vartheta}}\|V\|_X^2$: integrable over $s \in [0,t]$
- $\|(S(t+\varepsilon-s)-S(t-s))\,\mathbb{P}\,\mathrm{div}(u(s)\otimes u(s))\|_{L^p} \xrightarrow[\varepsilon\to 0]{} 0 \quad \forall s \in [0,t)$ (C_0-semigroup).

Continuity to the left at any $t \in (0,T]$ is proved similarly $\Rightarrow FV \in X$.

iii) If $V_1, V_2 \in X$, then

$$(FV_1)(t) - (FV_2)(t) = -\int_0^t S(t-s)\,\mathbb{P}\,\mathrm{div}\big((V_1-V_2)(s)\otimes V_1(s)\big)\,ds$$
$$\qquad\qquad - \int_0^t S(t-s)\,\mathbb{P}\,\mathrm{div}\big(V_2(s)\otimes(V_1-V_2)(s)\big)\,ds$$

hence :

$$\|FV_1 - FV_2\|_X \leq C_1 T^{\vartheta}\big(\|V_1\|_X + \|V_2\|_X\big)\|V_1-V_2\|_X. \tag{B}$$

Summarizing, $F : X \to X$ is well defined and satisfies the Lipschitz estimate (B), as well as the bound (A).

We now choose $R>0$ such that $\|v_0\|_{L^p} \leq R$ and $T>0$ small enough so that

$$4 C_1 R T^{\vartheta} < 1. \qquad (*)$$

If $B = \{ v \in X \,;\, \|v\|_X \leq 2R \}$, it follows from (A), (B) that

- $F : B \longrightarrow B$
- $\exists\, k<1$ such that $\|Fv_1 - Fv_2\|_{L^p} \leq k \|v_1 - v_2\|_{L^p} \quad \forall v_1, v_2 \in B.$

By the contraction mapping principle, F has a unique fixed point v in B. By construction, v is a solution of (IE).

Finally, assume that $\tilde{v} \in X$ is another solution of (IE). We do not know a priori if $\tilde{v} \in B$! We thus define:

$$\begin{cases} \tilde{T} = \sup \{ t \in [0,T] \,;\, \|\tilde{v}(t)\|_{L^p} \leq 2R \} \\ \tilde{X} = C^0([0,\tilde{T}], L^p(\mathbb{R}^d)) \text{ equipped with the sup norm.} \end{cases}$$

For $t \in [0,\tilde{T}]$ we have

$$v(t) - \tilde{v}(t) = - \int_0^t S(t-s)\, \mathbb{P} \,\mathrm{div}\left(v(s) \otimes v(s) - \tilde{v}(s) \otimes \tilde{v}(s) \right) ds$$

hence proceeding as above:

$$\|v - \tilde{v}\|_{\tilde{X}} \leq \underbrace{C_1 \tilde{T}^{\vartheta} (4R)}_{<1} \|v - \tilde{v}\|_{\tilde{X}} \implies v(t) = \tilde{v}(t) \;\; \forall t \in [0,\tilde{T}].$$

If $\tilde{T} < T$, we have $\|\tilde{v}(\tilde{T})\|_{L^p} = \underbrace{\|v(\tilde{T})\|_{L^p}}_{\leq R + 4C_1 R \tilde{T}^{\vartheta}} < 2R$, which contradicts the definition of $\tilde{T}$. Thus we must have $\tilde{T} = T$, and $v = \tilde{v}$ in X. $\square$

Remark (Continuous dependence on the data):

Proposition 1 provides a uniform local existence time for all initial data v_0 in the ball $B_0(r) = \{ v_0 \in L^p ; \|v_0\|_{L^p} \leq r \}$. If $v, \tilde{v} \in X$ are two solutions with initial data $v_0, \tilde{v}_0 \in B_0(r)$, then

$$v(t) - \tilde{v}(t) = S(t)(v_0 - \tilde{v}_0) - \int_0^t S(t-s)\mathbb{P}\,\mathrm{div}\big(v(s)\otimes v(s) - \tilde{v}(s)\otimes\tilde{v}(s)\big)\,ds\,, \quad 0 \leq t \leq T.$$

By the same estimates as above, we deduce:

$$\|v-\tilde{v}\|_X \leq \|v_0 - \tilde{v}_0\|_{L^p} + C_1(4r)T^{\vartheta}\|v-\tilde{v}\|_X\,, \quad \text{hence}$$

$$\|v-\tilde{v}\|_X \leq \frac{\|v_0-\tilde{v}_0\|_{L^p}}{1-4C_1 r T^{\vartheta}} \quad : \quad \underline{v \text{ is a Lipschitz function of } v_0}.$$

Solutions given by Proposition 1 can be extended to maximal solutions, defined on $[0, T^*(v_0))$. Given $v_0 \in L^p(\mathbb{R}^d)$, two cases can occur:

a) $T^*(v_0) = +\infty$, i.e. the maximal solution is global in time, or:

b) $T^*(v_0) < \infty$ and $\|v(t)\|_{L^p} \xrightarrow[t \to T^*]{} +\infty$: blow up in finite time.

We even have $\int_0^{T^*} \|v(s)\|_{L^p}\,ds = +\infty$.

Unfortunately, at least if $d=3$, we do not know how to control the L^p norm of a solution for $p>d$. The only a priori estimate at hand is the dissipation of energy ($\Rightarrow$ a priori estimate on the L^2 norm).

In dimension $d=2$, the L^p norm of v ($p>2$) can be controlled using the $L^{\frac{2p}{2+p}}$ norm of the vorticity $\omega = \mathrm{curl}\, v$, see below.

The critical case $p = d$ (Kato, 1984).

The argument of Proposition 1 fails when $p = d$, because the map F is not continuous on $C^0([0,T], L^d(\mathbb{R}^d))$. Note that, when $p = d$, the norm is invariant under the scaling of the equation:

$$V_\lambda(t,x) = \lambda V(\lambda^2 t, \lambda x) \Rightarrow \sup_{t\in[0,T]} \|V_\lambda(t)\|_{L^d} = \sup_{t\in[0,\lambda^2 T]} \|V(t)\|_{L^d}.$$

Proposition 2: For any $V_0 \in L^d(\mathbb{R}^d)$ with $\operatorname{div} V_0 = 0$, there exists $T = T(V_0) > 0$ such that eq. (IE) has a unique solution

$$V \in C^0([0,T], L^d(\mathbb{R}^d)) \cap C^0((0,T], L^\infty(\mathbb{R}^d)) \quad \text{such that}$$

$$\sup_{t\in[0,T]} t^{1/2} \|V(t)\|_{L^\infty(\mathbb{R}^d)} < \infty.$$

Moreover there exists $\varepsilon_0 > 0$ such that $T(V_0) = +\infty$ if $\|V_0\|_{L^d} \le \varepsilon_0$.

Remark: In the original variables, the smallness condition ensuring global existence is

$$\underbrace{\frac{\|V_0\|_{L^d(\mathbb{R}^d)}}{\nu}}_{\text{Reynolds number!}} \le \varepsilon_0$$

$\uparrow$ universal constant, dep. on d.

We shall use the following property:

Property 4: If $1 \le p < q \le \infty$ and $f \in L^p(\mathbb{R}^d)$, then

$$\lim_{t\to 0} t^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|S(t)f\|_{L^q(\mathbb{R}^d)} = 0.$$

Proof: If $f \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$, then $\|S(t)f\|_{L^q} \le \|f\|_{L^q}$ and the results follows. In the general case, we use a density argument.

Let $f \in L^p(\mathbb{R}^d)$. Given $\varepsilon > 0$, $\exists\, g \in L^p \cap L^q$ such that $\|f-g\|_{L^p} \leq \varepsilon$.
For any $t > 0$, we have:

$$\|S(t)f\|_{L^q} \leq \|S(t)(f-g)\|_{L^q} + \|S(t)g\|_{L^q}$$
$$\leq \frac{C\varepsilon}{t^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}} + \|S(t)g\|_{L^q},$$

hence $\limsup_{t\to 0^+} \|S(t)f\|_{L^q}\, t^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \leq C\varepsilon + 0 = C\varepsilon$. Since $\varepsilon > 0$ was arbitrary, the result follows. $\square$

Proof of Prop. 2: Fix $V_0 \in L^d(\mathbb{R}^d)$, $\operatorname{div} V_0 = 0$. Given $T > 0$ to be defined later, we introduce the space

$$X = \left\{ V \in C^0([0,T], L^d(\mathbb{R}^d)) \cap C^0((0,T], L^\infty(\mathbb{R}^d))\ ;\ \|V\|_{X_2} < \infty \right\}$$

where

$$\|V\|_{X_1} = \sup_{t\in[0,T]} \|V(t)\|_{L^d}, \quad \|V\|_{X_2} = \sup_{t\in(0,T]} t^{1/2}\|V(t)\|_{L^\infty}.$$

X is a Banach space equipped with the norm $\|V\|_X = \max(\|V\|_{X_1}, \|V\|_{X_2})$.

We take $R_1 > 0$, $R_2 > 0$ such that

$$R_1 \geq \|V_0\|_{L^d}, \quad R_2 \geq \sup_{t\in(0,T]} t^{1/2}\|S(t)V_0\|_{L^\infty}. \qquad (*)$$

As before, given $V \in X$, we define:

$$(FV)(t) = S(t)V_0 - \int_0^t S(t-s)\mathbb{P}\operatorname{div}(V(s)\otimes V(s))\,ds, \quad 0 \leq t \leq T.$$

We shall show that F maps X into X, and is a strict contraction when restricted to a suitable domain.

i) Estimate in X_1:

not $L^{d/2}$ anymore!

$$\|(FV)(t)\|_{L^d} \leq \|V_0\|_{L^d} + \int_0^t \frac{C}{(t-s)^{1/2}} \|V(s)\otimes V(s)\|_{L^d}\, ds$$

$$\leq R_1 + \int_0^t \frac{C}{(t-s)^{1/2}} \|V(s)\|_{L^\infty} \|V(s)\|_{L^d}\, ds$$

$$\leq R_1 + \int_0^t \frac{C}{(t-s)^{1/2}} \frac{\|V\|_{X_2}}{s^{1/2}} \|V\|_{X_1}\, ds$$

$$\leq R_1 + C\|V\|_{X_1}\|V\|_{X_2}, \qquad \left(\int_0^t \frac{1}{(t-s)^{1/2}s^{1/2}}\, ds = \pi\right)$$

$$\Rightarrow \|FV\|_{X_1} \leq R_1 + C_1\|V\|_{X_1}\|V\|_{X_2}.$$

One also verifies that $t \longmapsto (FV)(t)$ is continuous from $[0,T]$ to $L^d(\mathbb{R}^d)$.

ii) Estimate in X_2:

$$t^{1/2}\|(FV)(t)\|_{L^\infty} \leq t^{1/2}\|S(t)V_0\|_{L^\infty} + t^{1/2}\int_0^t \frac{C}{(t-s)^{3/4}} \|V(s)\otimes V(s)\|_{L^{2d}}\, ds$$

$$\leq R_2 + t^{1/2}\int_0^t \frac{C}{(t-s)^{3/4}} \|V(s)\|_{L^\infty}^{3/2} \|V(s)\|_{L^d}^{1/2}\, ds$$

$$\leq R_2 + t^{1/2}\int_0^t \frac{C}{(t-s)^{3/4}} \frac{\|V\|_{X_2}^{3/2}}{s^{3/4}} \|V\|_{X_1}^{1/2}\, ds$$

$$\leq R_2 + C\|V\|_{X_1}^{1/2}\|V\|_{X_2}^{3/2}, \qquad \left(\int_0^t \frac{t^{1/2}}{(t-s)^{3/4}s^{3/4}}\, ds = \sqrt{2}\,\pi\right)$$

$$\Rightarrow \|FV\|_{X_2} \leq R_2 + C_2\|V\|_{X_1}^{1/2}\|V\|_{X_2}^{3/2}.$$

Again $t \longmapsto \|(FV)(t)\|_{L^\infty}$ is continuous from $(0,T]$ to $L^\infty(\mathbb{R}^d)$.

iii) Lipschitz estimates: (similar calculations)

$$\|FV - F\tilde V\|_{X_1} \leq C_1\left(\|V\|_{X_2} + \|\tilde V\|_{X_2}\right)\|V-\tilde V\|_{X_1}$$

$$\|FV - F\tilde V\|_{X_2} \leq C_2\left(\|V\|_{X_1} + \|\tilde V\|_{X_1}\right)^{1/2}\left(\|V\|_{X_2} + \|\tilde V\|_{X_2}\right)^{1/2}\|V-\tilde V\|_{X_2}.$$

iv) Invariant domain:

$$B = \{ v \in X ; \ \|v\|_{X_1} \leq 2R_1, \ \|v\|_{X_2} \leq 2R_2 \}$$

Then $F(B) \subset B$ and F is a strict contraction in B if

$$4C_1 R_2 < 1, \quad \text{and} \quad 4C_2 (R_1 R_2)^{1/2} < 1. \qquad (**)$$

So if $(**)$ is satisfied, F has a unique fixed point in B, which is a solution of (IE).

v) Checking inequalities $(**)$:

Here there are two possibilities:

a) (Small data) If $4C_0 \|v_0\|_{L^d} < 1$ where $C_0 = \max(C_1, C_2)$, we take $R_1 = R_2 = \|v_0\|_{L^d}$ and $T > 0$ arbitrary!

$\Longrightarrow$ global well-posedness for small data.

b) (Large data) We take

- $R_1 = \|v_0\|_{L^d}$, then
- $R_2 > 0$ small enough so that $(**)$ hold, then
- $T > 0$ small enough so that $\sup_{t \in [0,T]} t^{1/2} \|S(t) v_0\|_{L^\infty} \leq R_2$, see property 4.

$\Longrightarrow$ local well-posedness for arbitrary data.

vi) Uniqueness in X

As in Proposition 1, one verifies that uniqueness holds in the whole space X, and not only in the ball $B \subset X$. $\square$

Remark: Except for small data, the existence time in proposition 2 depends on the detailed properties of the initial data $V_0 \in L^d(\mathbb{R}^d)$, and not only on an upper bound of $\|V_0\|_{L^d}$.

$\Rightarrow$ it is not sufficient to control $\|V(t)\|_{L^d}$ to prove global existence!

The energy balance

Proposition 3: Assume that $V_0 \in L^d(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, $\operatorname{div} V_0 = 0$. If $T = T(V_0)$, the solution V of (IE) given by Proposition 2 satisfies in addition:

$$V \in C^0([0,T], L^2(\mathbb{R}^d)) \cap L^2((0,T), H^1(\mathbb{R}^d)),$$

and the following equality holds:

$$\frac{1}{2}\|V(t)\|_{L^2}^2 + \int_0^t \|\nabla V(s)\|_{L^2}^2\, ds = \frac{1}{2}\|V_0\|_{L^2}^2, \quad t \in [0,T]. \qquad (EB)$$

Sketch of proof: We first verify that $V \in C^0([0,T], L^2(\mathbb{R}^3))$ if $d = 3$ (nothing to prove if $d = 2$). We have

$$V(t) = S(t)V_0 - \int_0^t S(t-s)\mathbb{P}\operatorname{div}(V(s)\otimes V(s))\, ds, \qquad 0 \le t \le T$$

hence:

$$\|V(t)\|_{L^2} \le \|S(t)V_0\|_{L^2} + \int_0^t \frac{C}{(t-s)^{1/2+\frac{3}{2}\left(\frac{2}{3}-\frac{1}{2}\right)}} \|V(s)\otimes V(s)\|_{L^{3/2}}^2\, ds$$

$$\le \|V_0\|_{L^2} + \int_0^t \frac{C}{(t-s)^{3/4}} \|V(s)\|_{L^3}^2\, ds$$

$$\le \|V_0\|_{L^2} + C\,T^{1/4}\,\|V\|_{X_1}^2 < \infty.$$

Continuity in time can be verified in the usual way $\Rightarrow$ $V \in C^0([0,T], L^2)$.

Next, using the smoothing properties of the heat semigroup $S(t)$, it is not difficult to verify that the solution $v(t)$ is smooth for $t>0$. In particular

$$v \in C^1((0,T), L^2(\mathbb{R}^d)) \cap C^0((0,T), H^2(\mathbb{R}^d)),$$

and the NS equation is satisfied in $L^2(\mathbb{R}^d)$:

$$\frac{d}{dt} v(t) + \mathbb{P}(v(t)\cdot\nabla)v(t) = \Delta v(t), \qquad 0<t<T.$$

N.B. $\|\mathbb{P}(v\cdot\nabla)v\|_{L^2} \leq \|(v\cdot\nabla)v\|_{L^2} \leq \|v\|_{L^6}\|\nabla v\|_{L^3} \leq \|v\|_{H^1}\|v\|_{H^2} < \infty$.

We can thus compute:

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{L^2}^2 = (v(t), v'(t))_{L^2} = (v(t), -\mathbb{P}(v(t)\cdot\nabla)v(t))_{L^2} + (v(t), \Delta v(t))_{L^2}$$

$$= -\underbrace{(v(t),(v(t)\cdot\nabla)v(t))_{L^2}}_{=0} - (\nabla v(t), \nabla v(t))_{L^2} = -\|\nabla v(t)\|_{L^2}^2$$

$$\Rightarrow \quad \frac{1}{2}\|v(t)\|_{L^2}^2 + \int_{t_0}^t \|\nabla v(s)\|_{L^2}^2\,ds = \frac{1}{2}\|v(t_0)\|_{L^2}^2, \quad 0<t_0<t<T.$$

Letting $t_0 \to 0$, we obtain the energy balance (EB), which shows that $v \in L^2((0,T), H^1(\mathbb{R}^d))$. $\square$

When $d=3$, the energy balance (EB) is not sufficient to prove that the solutions are global! We have $H^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ for $p\in[2,6]$, thus $\|v\|_{L^3} \leq C\|v\|_{H^1}$, but (EB) does not prevent $\|v(t)\|_{H^1}$ to blow-up at some time $t = T_*$: (EB) gives an L^2 control in time of $\|\nabla v(t)\|_{L^2}$, not an L^∞ control!

To get a better control on v, one may try <u>higher-order energy estimates</u> (in Sobolev spaces H^s for $s>0$, or in L^p for $p>2$). The problem is then that the nonlinear term $\mathbb{P}(u\cdot\nabla)u$ gives a nonzero contribution!

<u>Example:</u> (H^1-energy estimate)

$$\frac{1}{2}\frac{d}{dt}\|\nabla v(t)\|_{L^2}^2 = (\nabla v(t), \nabla v'(t))_{L^2} = (-\Delta v(t), v'(t))_{L^2}$$

$$= (-\Delta v(t), \Delta v(t) - \mathbb{P}(v(t)\cdot\nabla)v(t))_{L^2}$$

$$= -\|\Delta v(t)\|_{L^2}^2 + (\Delta v(t), (v(t)\cdot\nabla)v(t))_{L^2}.$$

Using Sobolev's embeddings, the trilinear term can be estimated as follows:

$$(\Delta v, (v\cdot\nabla)v)_{L^2} \leq \|\Delta v\|_{L^2}\|(v\cdot\nabla)v\|_{L^2} \leq \|\Delta v\|_{L^2}\|v\|_{L^6}\|\nabla v\|_{L^3}$$

$$\leq C\|\Delta v\|_{L^2}\|\nabla v\|_{L^2}\left(\|\nabla v\|_{L^2}^{1/2}\cdot\|\nabla v\|_{L^6}^{1/2}\right)$$

$$\leq C\|\Delta v\|_{L^2}\|\nabla v\|_{L^2}^{3/2}\|\Delta v\|_{L^2}^{1/2} = C\|\Delta v\|_{L^2}^{3/2}\|\nabla v\|_{L^2}^{3/2}.$$

Thus

$$\frac{d}{dt}\|\nabla v(t)\|_{L^2}^2 \leq -2\|\Delta v(t)\|_{L^2}^2 + C\|\Delta v(t)\|_{L^2}^{3/2}\|\nabla v(t)\|_{L^2}^{3/2} \qquad (p=4/3,\ q=4)$$

$$\overset{\text{Young}}{\leq} -\|\Delta v(t)\|_{L^2}^2 + C\|\nabla v(t)\|_{L^2}^6. \qquad \text{(HEB)}$$

<u>Problem:</u> (HEB) <u>does not prevent</u> $\|\nabla v(t)\|_{L^2}$ of blowing-up in finite time, even if we exploit the fact that $\int_0^{T_*}\|\nabla v(t)\|_{L^2}^2\,dt < \infty$ by (EB).

Indeed:

- the negative term $-\|\Delta v(t)\|_{L^2}^2$ has a different scaling and cannot help;
- $\left.\begin{array}{l} z'(t) = \frac{1}{2}z(t)^3 \\ z(0) = z_0 \neq 0 \end{array}\right\} \Rightarrow z(t) = \dfrac{z_0}{\sqrt{1-t z_0^2}}$! $\quad T_* = 1/z_0^2, \ \int_0^{T_*} z(t)\,dt < \infty$!

$\quad (z(t) \approx \|\nabla v(t)\|_{L^2}^2)$

The two-dimensional case

When $d=2$ the situation is much more favorable. We know from (EB) that $\|v(t)\|_{L^2} \leq \|v_0\|_{L^2}$ $\forall t \in [0, T^*(v_0))$, but this information alone is not sufficient to prove that $T^*(v_0) = +\infty$, because T^* does not depend only on $\|v_0\|_{L^2}$. The integral term in (EB) does not help either.

However, further a priori estimates can be obtained using the vorticity equation.

We recall that $v = (v_1, v_2, 0)$, $\omega = \operatorname{curl} v = (0, 0, w)$, and

$$\partial_t w + u \cdot \nabla w = \Delta w\,, \qquad u = (v_1, v_2). \tag{VE}$$

For $t > 0$, w and u are smooth, and we have in particular

$$w \in C^1((0,T), L^2(\mathbb{R}^2)) \cap C^0((0,T), H^2(\mathbb{R}^2)).$$

Thus we can compute

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2}^2 &= \big(w(t), \Delta w(t) - u(t)\cdot\nabla w(t)\big)_{L^2} \\ &= -\|\nabla w(t)\|_{L^2}^2 - \big(w(t), u(t)\cdot\nabla w(t)\big)_{L^2} \\ &= -\|\nabla w(t)\|_{L^2}^2 \leq 0\end{aligned}$$

because $\int_{\mathbb{R}^2} w\, u\cdot\nabla w\, dx = \frac{1}{2}\int_{\mathbb{R}^2} u\cdot\nabla(w^2)\, dx = -\frac{1}{2}\int_{\mathbb{R}^2} (\operatorname{div} u)\, w^2\, dx = 0.$

Thus the L^2 norm of any solution of (VE) is nonincreasing with time.

A similar calculation shows that $\|w(t)\|_{L^p} \leq \|w(t_0)\|_{L^p}$ $(t \geq t_0)$ for all $p \in [2, +\infty]$.

Since $\|w\|_{L^2} = \|\nabla v\|_{L^2}$ (easy to verify in Fourier), we conclude that

$$\|\nabla v(t)\|_{L^2} \leq \|\nabla v(t_0)\|_{L^2}\,, \qquad 0 < t_0 \leq t < T_*(v_0). \tag{H}$$

This leads to:

<u>Proposition 4</u>: Assume that $d=2$. For any $v_0 \in L^2(\mathbb{R}^2)$ with $\operatorname{div} v_0 = 0$, the integral equation (IE) has a unique global solution

$$v \in C_b^0([0,+\infty), L^2(\mathbb{R}^2)) \cap C^0((0,+\infty), L^\infty(\mathbb{R}^2))$$

such that $\sup_{t>0} t^{1/2} \|v(t)\|_{L^\infty(\mathbb{R}^2)} < \infty$. Moreover $v \in C^0((0,+\infty), H^1(\mathbb{R}^2))$ and the energy balance (EB) holds for all times.

<u>Proof</u>: Let $T_*(v_0) \in (0,+\infty]$ be the maximal existence time of the solution given by Proposition 2. For $0 < t_0 \leq t < T_*$ one has $\|v(t)\|_{L^2} \leq \|v_0\|_{L^2}$ by Proposition 3 and $\|\nabla v(t)\|_{L^2} \leq \|\nabla v(t_0)\|_{L^2}$ by (M). By Sobolev embedding, $\|v(t)\|_{L^p}$ is thus bounded from above for $t \geq t_0$, for any $p \in (2,+\infty)$. This implies that $T_*(v_0) = +\infty$, see the blow-up criterion on p.58 or the assumption on the local existence time T on p.62.

We know from (EB) that $\|v(t)\|_{L^2} \leq \|v_0\|_{L^2}$ $\forall t \geq 0 \Rightarrow v \in C_b^0([0,+\infty), L^2(\mathbb{R}^2))$. Moreover, using (EB) and (M) we find:

$$\|\nabla v(t)\|_{L^2}^2 \overset{(M)}{\leq} \frac{1}{t}\int_t^{2t} \|\nabla v(s)\|_{L^2}^2 \, ds \leq \frac{\|v_0\|_{L^2}^2}{2t}, \quad t > 0.$$

By Sobolev embedding, for $2 < p < \infty$:

$$\|v(t)\|_{L^p} \leq C_p \|v(t)\|_{L^2}^{2/p} \|\nabla v(t)\|_{L^2}^{1-2/p} \leq \frac{C_p \|v_0\|_{L^2}}{t^{\frac{1}{2}-\frac{1}{p}}}.$$

To obtain the same result in the limiting case $p = +\infty$, we use (IE):

$$v(t) = S(t)v_0 - \int_0^t S(t-s)\,\mathbb{P}\operatorname{div}(v(s) \otimes v(s))\, ds, \quad t > 0.$$

We know that $\|S(t)v_0\|_{L^\infty} \leq \dfrac{C\|v_0\|_{L^2}}{t^{1/2}}$. As for the integral term:

$$\Big\|\int_0^t S(t-s)\mathbb{P}\,\mathrm{div}(v(s)\otimes v(s))\,ds\Big\|_{L^\infty} \leq C\int_0^t \frac{1}{(t-s)^{\frac{1}{2}+\frac{2}{p}}}\|v(s)\otimes v(s)\|^2_{L^{p/2}}\,ds \qquad (4<p<\infty)$$

$$\leq C\int_0^t \frac{1}{(t-s)^{1/2+2/p}}\|v(s)\|^2_{L^p}\,ds$$

$$\leq C\int_0^t \frac{1}{(t-s)^{1/2+2/p}}\,\frac{C_p^2\|v_0\|^2_{L^2}}{s^{1-2/p}}\,ds = \frac{C}{t^{1/2}}\|v_0\|^2_{L^2}.$$

This proves that $\sup\limits_{t>0} t^{1/2}\|v(t)\|_{L^\infty} \leq C\left(\|v_0\|_{L^2}+\|v_0\|^2_{L^2}\right) < \infty$. $\square$

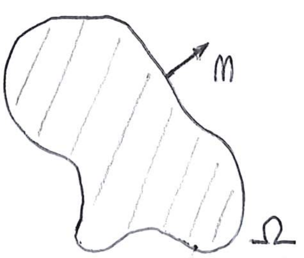
Remark: Under the assumptions of Prop. 4, one can further show that

$$\lim_{t\to+\infty}\|v(t)\|_{L^2} = 0, \qquad \lim_{t\to+\infty} t^{1/2}\|v(t)\|_{L^\infty} = 0$$

(Kato, Masuda, 1984).

The case of a bounded domain

In the rest of this chapter, we indicate how the arguments above have to be modified if one considers a fluid moving in a bounded domain $\Omega\subset\mathbb{R}^d$ with smooth boundary $\partial\Omega$.



n = outward unit normal on Ω.
σ = d-1 Lebesgue measure on $\partial\Omega$
$\partial\Omega$ = compact hypersurface of $\mathbb{R}^d$

We recall the Gauss formula: $\forall f\in C^1(\bar\Omega)^d$

$$\int_\Omega \mathrm{div} f\,dx = \int_{\partial\Omega} f\cdot n\,d\sigma$$

We recall the following well known facts (cf. Temam, Lions-Magenes):

a) <u>Trace operator</u> There exists a (unique) bounded linear operator

$$\gamma_0 : H^1(\Omega) \longrightarrow L^2(\partial\Omega) \quad \text{such that} \quad \gamma_0(u) = u|_{\partial\Omega} \quad \forall u \in C^1(\bar{\Omega}).$$

One has:

$$H_0^1(\Omega) := \operatorname{Ker}(\gamma_0) = \text{closure of } C_c^\infty(\Omega) \text{ in } H^1(\Omega)$$

$$H^{1/2}(\partial\Omega) := \operatorname{Im}(\gamma_0).$$ (One can show that this notation is justified)

b) <u>Lifting operator</u> There exists a (non-unique) bounded linear operator

$$\ell : H^{1/2}(\partial\Omega) \longrightarrow H^1(\Omega) \quad \text{such that} \quad \gamma_0 \circ \ell = \mathbb{1}_{H^{1/2}(\partial\Omega)}.$$

We also define $H^{-1/2}(\partial\Omega)$ as the <u>dual space</u> of $H^{1/2}(\partial\Omega)$.

<u>Def:</u> $E(\Omega) = \{ v \in L^2(\Omega)^d \,;\, \operatorname{div} v \in L^2(\Omega) \}$

div in the sense of distributions in Ω

$E(\Omega)$ is a Hilbert space equipped with the scalar product:

$$(u,v)_E = \int_\Omega u\, v \, dx = \int_\Omega (\operatorname{div} u)(\operatorname{div} v)\, dx.$$

<u>Lemma:</u> There exists a (unique) bounded linear operator ("normal trace")

$$\gamma : E(\Omega) \longrightarrow H^{-1/2}(\partial\Omega) \quad \text{such that} \quad \gamma(u) = u \cdot n|_{\partial\Omega} \quad \forall u \in C^1(\bar{\Omega}).$$

⚠ If $u \in L^2(\Omega)$, one cannot define the trace $u|_{\partial\Omega}$, because $\partial\Omega$ has zero Lebesgue measure in $\mathbb{R}^d$ (one needs $u \in H^s(\Omega)$ for $s > 1/2$).
But $u \cdot n|_{\partial\Omega}$ makes sense in $H^{-1/2}(\partial\Omega)$ $\forall u \in E(\Omega)$!

Proof: If $v \in C^1(\bar{\Omega})^d$ and $w \in C^1(\bar{\Omega})$, we have by Gauss:

$$\int_{\partial\Omega} w(v\cdot n)\,d\sigma = \int_\Omega \operatorname{div}(wv)\,dx = \int_\Omega (w \operatorname{div} v + v\cdot\nabla w)\,dx.$$

If $v \in E(\Omega)$, the right-hand side makes sense $\forall w \in H^1(\Omega)$, and thus $\forall w = \ell(u)$ with $u \in H^{1/2}(\partial\Omega)$. We can thus <u>define</u> $\gamma(v) = v\cdot n$ by

$$\int_{\partial\Omega} u\,\gamma(v)\,d\sigma = \int_\Omega \left(\ell(u) \operatorname{div} v + v\cdot\nabla\ell(u)\right)dx, \quad \forall u \in H^{1/2}(\partial\Omega).$$

By construction $\gamma(v) \in H^{-1/2}(\partial\Omega)$ and $\gamma(v) = v\cdot n$ if $v \in C^1(\bar{\Omega})$. □

Remark: For all $v \in E(\Omega)$, $w \in H^1(\Omega)$, we thus have the Gauss formula

$$\int_\Omega \gamma_0(w)\,\gamma(v)\,d\sigma = \int_\Omega (w \operatorname{div} v + v\cdot\nabla w)\,dx. \qquad \text{(Gauss)}$$

We are now able the closure of the smooth divergence-free vector fields in $L^2(\Omega)$ and $H^1(\Omega)$.

Def: $\mathcal{D}_\sigma(\Omega) = \{ v \in C_c^\infty(\Omega)^d ;\ \operatorname{div} v = 0 \}$

H = closure of $\mathcal{D}_\sigma(\Omega)$ in $L^2(\Omega)$

V = closure of $\mathcal{D}_\sigma(\Omega)$ in $H^1(\Omega)$

Proposition 1:

$$H = \{ v \in L^2(\Omega)^d ;\ \operatorname{div} v = 0,\ \gamma(v) = 0 \}$$

$$V = \{ v \in H_0^1(\Omega)^d ;\ \operatorname{div} v = 0 \}$$

Partial proof: If v_n is a sequence in $\mathcal{D}_\sigma(\Omega)$ such that $v_n \xrightarrow[n\to\infty]{L^2} v \in L^2(\Omega)^d$, then $0 = \operatorname{div} v_n \xrightarrow[n\to\infty]{\mathcal{D}'} \operatorname{div} v \Rightarrow \operatorname{div} v = 0$ in the sense of distributions.

Thus $v \in E(\Omega)$ and $\|v - v_n\|_E \xrightarrow[n\to\infty]{} 0$. By the lemma,

$$0 = \gamma(v_n) \xrightarrow[n\to\infty]{H^{-1/2}(\partial\Omega)} \gamma(v) \implies \gamma(v) = 0. \text{ Therefore:}$$

$$H \subset H_1 := \{ v \in L^2(\Omega) ;\ \operatorname{div} v = 0,\ \gamma(v) = 0 \}.$$

To prove the converse inclusion, take $v \in H_1$ and define

$$\tilde{v}(x) = \begin{cases} v(x) & x \in \Omega \\ 0 & x \notin \Omega \end{cases} \qquad \text{(extension by zero outside } \Omega\text{)}$$

The crucial observation is that No singular term at the boundary!

$$(\operatorname{div} \tilde{v})(x) = \begin{cases} \operatorname{div} v(x) & x \in \Omega \\ 0 & x \notin \Omega \end{cases} \quad : \quad \operatorname{div} \tilde{v} = \widetilde{(\operatorname{div} v)}\ !$$

Indeed, $\forall w \in C_c^\infty(\mathbb{R}^d)$:

$$\langle \operatorname{div} \tilde{v}, w \rangle = - \langle \tilde{v}, \nabla w \rangle = - \int_\Omega v \cdot \nabla w \, dx \overset{\text{Gauss}}{=} \int_\Omega w \operatorname{div} v \, dx$$

$$= \int_{\mathbb{R}^d} w \, (\operatorname{div} v) \mathbb{1}_\Omega \, dx.$$

Thus $\tilde{v} \in E(\mathbb{R}^d)$. By a small translation inside the domain + convolution with a smooth mollifier, it is easy to approximate $\tilde{v}$ by smooth, divergence-free vector field with support in $\Omega \implies H_1 \subset H$.

The claim concerning V is more standard: we already know that $C_c^\infty(\Omega)$ is dense in $H^1(\Omega)$, and it is not difficult to approximate divergence-free vector fields by divergence-free vector fields. □

<u>Proposition 2</u>: $L^2(\Omega)^d = H \oplus H^\perp$ where

$$H^\perp = \{ v \in L^2(\Omega)^d ;\ v = \nabla p \text{ for some } p \in H^1(\Omega) \}.$$

We recall the following inequalities: (Ω bounded)

Poincaré: $\|p\|_{L^2(\Omega)} \leq C \|\nabla p\|_{L^2(\Omega)} \quad \forall p \in H^1_0(\Omega)$

Wirtinger: $\|p\|_{L^2(\Omega)} \leq C \|\nabla p\|_{L^2(\Omega)} \quad \forall p \in H^1(\Omega), \int_\Omega p\,dx = 0.$

Proof of prop 2: Let $M = \{ v \in L^2(\Omega)^d ; \ v = \nabla p \text{ for some } p \in H^1(\Omega) \}$
(N.B. we can always assume $\int_\Omega p\,dx = 0$)

i) M is a closed subset of $L^2(\Omega)^d$. If $v_n = \nabla p_n \xrightarrow[n\to\infty]{L^2} v$, then $\|p_n - p_m\|_{L^2} \leq C \|\nabla p_n - \nabla p_m\|_{L^2} \xrightarrow[n,m\to\infty]{} 0$ by Wirtinger $\Rightarrow (p_n)$ is a Cauchy sequence in $H^1(\Omega)$, so that $p_n \xrightarrow[n\to\infty]{H^1} p \in H^1(\Omega)$. Clearly $v = \nabla p$, thus $v \in M$.

ii) $M \perp H$: If $p \in H^1(\Omega)$ and $v \in H$, we have by Gauss:

$$\int_\Omega \nabla p \cdot v\,dx = \int_{\partial\Omega} \underbrace{\gamma_0(p)\gamma(v)}_{=0}\,d\sigma - \int_\Omega p \underbrace{\operatorname{div} v}_{=0}\,dx = 0.$$

iii) $L^2(\Omega)^d = H \oplus M$.

Assume that $w \in L^2(\Omega)^d$ satisfies $w \perp H$ and $w \perp M$: we want to show that $w = 0$.

- $w \perp M \Rightarrow$ in particular $\int_\Omega w \cdot \nabla p\,dx = 0 \ \forall p \in C_c^\infty(\Omega)$
  $\Rightarrow \operatorname{div} w = 0$ in the sense of distributions in Ω.
- $w \perp H \Rightarrow$ in particular $\int_\Omega w \cdot \operatorname{curl} \varphi = 0 \ \forall \varphi \in C_c^\infty(\Omega)^d$
  $\Rightarrow \operatorname{curl} w = 0$ in the sense of distributions in Ω.

But $\operatorname{div} w = 0$, $\operatorname{curl} w = 0 \Rightarrow w \in C^\infty(\Omega)$ and $w = \nabla p$, $\Delta p = 0$.
As moreover $w \in L^2(\Omega)^d$ we have $p \in H^1(\Omega)$, hence $w \in M$. But we assumed that $w \perp M$ hence $w = 0$. $\square$

Leray-Hopf projection and Stokes operator

Def: We denote by $\mathbb{P}$ the orthogonal projection in $L^2(\Omega)^d$ onto the closed subspace $H = \{ v \in L^2(\Omega)^d ;\ \operatorname{div} v = 0,\ \gamma(v) = 0 \}$.
We call $\mathbb{P}$ the Leray projection in Ω.

By proposition 2, any $v \in L^2(\Omega)^d$ can be decomposed as

$$v = \underbrace{\mathbb{P} v}_{\text{divergence-free}} + \underbrace{\operatorname{grad} p}_{\text{curl-free}}, \quad \text{for some } p \in H^1(\Omega)/\mathbb{R}$$

(Hodge!)

Moreover $\|v\|_{L^2}^2 = \|\mathbb{P} v\|_{L^2}^2 + \|\nabla p\|_{L^2}^2$.

Recall that $V = \{ v \in H_0^1(\Omega)^d ;\ \operatorname{div} v = 0 \}$. We equip V with the scalar product:

$$(u,v)_V = (\nabla u, \nabla v)_{L^2} = \int_\Omega \nabla u \cdot \nabla v \, dx .$$

Note that $\|u\|_V^2 = \|\nabla u\|_{L^2}^2$ is equivalent to $\|u\|_{H^1}^2$ by Poincaré's inequality.

We know that:

- $V \hookrightarrow H$: V is continuously embedded in H. Indeed $V \subset H$ and $\|u\|_{L^2} \leq C \|\nabla u\|_{L^2} = C \|u\|_V$ by Poincaré, $\forall u \in V$.
- V is dense in H (obvious)
- V is compactly embedded in H : the closed unit ball in V is a compact subset of H (Rellich-Kondrachov).

We now use the representation theorem in spectral theory to associate a self-adjoint operator to the pair of Hilbert spaces V, H.

Proposition 3: There exists a unique self-adjoint operator A in H such that $D(A) \subset V$ and

$$(Au, v)_{L^2} = (\nabla u, \nabla v)_{L^2} \quad \forall u \in D(A)\ \forall v \in V. \qquad (*)$$

Proof (standard):

i) We define

$$D(A) = \{ u \in V;\ \exists C > 0 \text{ s.th. } |(\nabla u, \nabla v)_{L^2}| \leq C \|v\|_{L^2}\ \forall v \in V \}.$$

[According to $(*)$, this is the maximal definition of $D(A)$.]

If $u \in D(A)$, we denote $f(v) = (\nabla u, \nabla v)_{L^2}$ $\forall v \in V$. Then f is a linear form on V which is continuous for the topology of H: $|f(v)| \leq C\|v\|_{L^2}$ $\forall v \in V$. Since V is dense in H, f extends to a unique continuous linear form f on H. By Riesz representation theorem, $\exists$ a unique element of H (denoted Au) such that $(Au, v) = f(v)$ $\forall v \in H$. This gives in particular $(*)$.

We have thus defined a linear operator $A : D(A) \longrightarrow H$, satisfying $(*)$.

ii) A is onto.

Take $w \in H$ and define $f : V \to \mathbb{R}$, $f(v) = (w, v)_{L^2}$. The form f is linear and continuous on V, so by Riesz theorem (in V) there exist a unique $u \in V$ such that $f(v) = (\nabla u, \nabla v)_{L^2}$ $\forall v \in V$. By definition $u \in D(A)$ and $Au = w$.

iii) A is one-to-one, and A^{-1} is bounded

If $u \in D(A)$, then using $(*)$ with $v = u$ we find

$$\|\nabla u\|_{L^2}^2 = (Au, u)_{L^2} \leq \|Au\|_{L^2} \|u\|_{L^2} \leq \|Au\|_{L^2} C_p \|\nabla u\|_{L^2}$$

$$\Longrightarrow \|Au\|_{L^2} \geq \frac{1}{C_p} \|\nabla u\|_{L^2} \geq \frac{1}{C_p^2} \|u\|_{L^2} \qquad (C_p = \text{Poincaré's constant})$$

Thus A is one-to-one and $|||A^{-1}||| \leq C_P^2 \Rightarrow A^{-1} \in \mathcal{L}(H)$.

iv) A is densely defined

If $D(A)$ is not dense in H, there exists $w \in H$, $w \neq 0$ such that $(u, w)_{L^2} = 0 \ \forall u \in D(A)$. By Riesz, we can find $v \in V$, $v \neq 0$ such that $(u, w)_{L^2} = (\nabla u, \nabla v)_{L^2} \ \forall u \in V$. Thus

$$0 = (u, w)_{L^2} = (\nabla u, \nabla v)_{L^2} = (Au, v)_{L^2} \quad \forall u \in D(A)$$

$\Rightarrow v = 0$ by ii) which gives a contradiction.

v) A is self-adjoint in H

It follows easily from $(*)$ that $(Au, v)_{L^2} = (u, Av)_{L^2} \ \forall u, v \in D(A)$

$\Rightarrow A$ is symmetric : $A \subset A^*$. But:

$$\begin{aligned} v \in D(A^*) &\Leftrightarrow \exists C > 0 \text{ s.th. } |(Au, v)| \leq C \|u\|_{L^2} \ \forall u \in D(A) \\ &\Leftrightarrow \exists C > 0 \text{ s.th. } |(\nabla u, \nabla v)| \leq C \|u\|_{L^2} \ \forall u \in V \\ &\Leftrightarrow v \in D(A). \end{aligned}$$

Thus $A = A^*$. $\square$

Rem: The proof shows that $\|\nabla A^{-1} v\|_{L^2} \leq C_P \|v\|_{L^2} \ \forall v \in H$, hence $\|A^{-1} v\|_V \leq C_P \|v\|_H$. Since V is compactly embedded into H, this means that A^{-1} is a compact operator in H.

Proposition 4: $D(A) = H^2(\Omega)^d \cap V$ and $Au = -P\Delta u \ \forall u \in D(A)$.

We say that A is the Stokes operator in Ω.

⚠ Even if $u \in H^2(\Omega)^d \cap V$, we do not have $\Delta u \in H$ in general!

It is clear that $\Delta u \in L^2$ and $\operatorname{div}(\Delta u) = \Delta \operatorname{div} u = 0$, but there is no reason that $\gamma(\Delta u) = 0$! The problem comes from the boundary.

Partial proof of prop 4: Suppose that $u \in H^2(\Omega)^d \cap V$.

For any $v \in V$ we have by Gauss:

$$(\nabla u, \nabla v)_{L^2} = \int_\Omega \nabla u \cdot \nabla v \, dx = \int_{\partial\Omega} \overset{=0}{\gamma_0(v)} \gamma(\nabla u) \, d\sigma - \int_\Omega v \Delta u \, dx$$

$\Rightarrow |(\nabla u, \nabla v)_{L^2}| \leq \|\Delta u\|_{L^2} \|v\|_{L^2}$. Thus $u \in D(A)$ and

$$(Au, v)_{L^2} = -\int_\Omega v \Delta u \, dx = -(v, P\Delta u)_{L^2} \quad \forall v \in V$$

This implies that $Au = -P\Delta u$.

It remains to verify that $D(A) \subset H^2(\Omega)^d \cap V$. Fix $f \in H$ and define $u = A^{-1} f \in D(A)$. Then $Au = f$, or in PDE terms u solves the linear Stokes equation:

$$\begin{cases} -\Delta u + \nabla p = f & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad \text{(Stokes)}$$

We know that the solution of this problem satisfies $u \in V$ and $\|\nabla u\|^2 \leq \|f\|_{L^2} \|u\|_{L^2} \Rightarrow \|\nabla u\|_{L^2} \leq C_p \|f\|_{L^2}$. If the domain Ω is of class C^2, it was shown by Cattabriga (1961) that the solution u of (Stokes) satisfies $\|D^2 u\|_{L^2} \leq C \|f\|_{L^2}$, so that $u \in H^2(\Omega)^d \cap V$.

$\Rightarrow$ for a smooth domain one has $D(A) = H^2(\Omega)^d \cap V$. □

Remark: Since A is a positive, self-adjoint operator with compact resolvent (compact inverse), the spectrum of A is a countable family of discrete eigenvalues with finite multiplicity. Repeating the eigenvalues according to multiplicity, we thus have

$$\sigma(A) = \{\lambda_j \,;\, j = 1, 2, 3, \dots\}, \quad 0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j \xrightarrow[j \to +\infty]{} \infty$$

Moreover, there exist normalized eigenfunctions $w_j \in D(A)$ such that

- $A w_j = \lambda_j w_j \quad \forall j \in \mathbb{N}$
- $(w_j)_{j=1}^{\infty}$ is an orthonormal basis of H.

Thus any $v \in H$ can be decomposed as

$$v = \sum_{j=1}^{\infty} v_j w_j, \qquad v_j = (v, w_j)_{L^2} \quad \text{(convergence in } H\text{)}.$$

Moreover $v \in D(A)$ iff $\sum_{j=1}^{\infty} \lambda_j^2 |v_j|^2 < \infty$, in which case

$$Av = \sum_{j=1}^{\infty} \lambda_j v_j w_j.$$

This representation formula allows us to define easily:

- <u>fractional powers of A:</u> If $s > 0$:

$$D(A^s) = \Big\{ v \in H ; \sum_{j=1}^{\infty} \lambda_j^{2s} |v_j|^2 < \infty \Big\}$$

$$A^s v = \sum_{j=1}^{\infty} \lambda_j^s v_j w_j, \quad v \in D(A^s)$$

Ok for $s \le 0$ with $D(A^s) = H$

- <u>Semi-group generated by $-A$:</u>

$$e^{-tA} v = \sum_{j=1}^{\infty} e^{-t\lambda_j} v_j w_j, \quad t \geq 0, \ v \in H.$$

<u>Remark:</u> Since $\|\nabla w_j\|^2 = (A w_j, w_j)_{L^2} = \lambda_j \quad \forall j \geq 1$, we have <u>$D(A^{1/2}) = V$</u> and

$$\|\nabla v\|_{L^2}^2 = \sum_{j=1}^{\infty} \lambda_j |v_j|^2 \quad \forall v \in V.$$

One can also show that $\|v\|_{H^2} \leq C \|Av\|_{L^2} \quad \forall v \in D(A)$.

Remark: Since $\lambda_j \geq \lambda_1 > 0 \ \forall j \in \mathbb{N}$, it follows from (SG) that

$$\| e^{-tA} v \|_{L^2} \leq e^{-\lambda_1 t} \| v \|_{L^2} \quad \forall v \in H \ \forall t \geq 0.$$

The decay rate $\lambda_1 = \lambda_1(\Omega) > 0$ tends to zero when the domain Ω increases (we know that $\lambda_1 \geq 1/C_p^2$, C_p = Poincaré constant in Ω).

Moreover, for any $t > 0$, e^{-tA} maps H into $D(A^\alpha)$ $\forall \alpha \geq 0$ and we have

$$\| A^\alpha e^{-tA} v \|_{L^2} \leq C_{\alpha,\varepsilon} \frac{1}{t^\alpha} e^{-(1-\varepsilon)\lambda_1 t} \| v \|_{L^2} \quad \forall t > 0, \forall \alpha \geq 0, \forall \varepsilon > 0.$$

Indeed $\lambda_j^\alpha e^{-t\lambda_j} = \frac{1}{t^\alpha} (\lambda_j t)^\alpha e^{-\lambda_j t} \leq \frac{e^{-(1-\varepsilon)\lambda_j t}}{t^\alpha} \underbrace{\sup_{j \in \mathbb{N}^*} (\lambda_j t)^\alpha e^{-\varepsilon \lambda_j t}}_{\leq C_{\alpha,\varepsilon}}$.

The Navier-Stokes equation in Ω

Applying the Leray projection $\mathbb{P}$ to (NS) we obtain the abstract evolution problem in H:

$$\partial_t v + \mathbb{P}(v \cdot \nabla) v = -Av, \qquad v|_{t=0} = v_0.$$

The associated integral equation takes the form:

$$v(t) = e^{-At} v_0 - \int_0^t e^{-A(t-s)} \mathbb{P}(v(s) \cdot \nabla) v(s) \, ds, \quad t \geq 0. \qquad \text{(IE)}$$

This formulation incorporates the initial data and the (no-slip) boundary conditions, because $D(A) = H^2(\Omega)^d \cap V$ and $V = H^1_{0,\sigma}(\Omega)^d$.

By the remark above, $e^{-At} v_0 \in D(A)$ $\forall v_0 \in H$ $\forall t > 0$, hence the solution of (IE) will belong to V for any $t > 0$ (maybe not at initial time).

As in the case of $\mathbb{R}^d$, (IE) can be solved by a fixed point argument. We give some details in the three-dimensional case $d = 3$.

In the 3D case, it is convenient to write (IE) in the equivalent form:

$$V(t) = e^{-At} V_0 - \int_0^t A^{1/4} e^{-A(t-s)} N(V(s))\,ds, \qquad \text{(IE')}$$

where

$$N(V) = A^{-1/4} \mathbb{P}(V\cdot\nabla)V.$$

To bound the nonlinear term N, we use:

<u>Lemma:</u> $\exists\, C_1 > 0$ such that

$$\|N(V)\|_{L^2} \leq C_1 \|A^{1/2}V\|_{L^2}^2 \qquad \forall V \in D(A^{1/2}) = V$$

$$\|N(V) - N(\tilde V)\|_{L^2} \leq C_1 \left(\|A^{1/2}V\|_{L^2} + \|A^{1/2}\tilde V\|_{L^2}\right) \|A^{1/2}V - A^{1/2}\tilde V\|_{L^2} \quad \forall V, \tilde V \in V$$

<u>Partial proof:</u>

- We take for granted the fact that $D(A^{1/4}) \hookrightarrow H^{1/2}(\Omega)$, so that

$$\|V\|_{L^3} \underset{\substack{\uparrow \\ \text{Sobolev embedding}}}{\leq} C\|V\|_{H^{1/2}} \leq C\|A^{1/4}V\|_{L^2} \quad \forall V \in D(A^{1/4}).$$

By duality we deduce that

$$\|A^{-1/4}u\|_{L^2} \leq C\|u\|_{L^{3/2}} \quad \forall u \in H \qquad (*)$$

[Indeed, for any $V \in D(A^{1/4})$:

$$|(A^{-1/4}u, A^{1/4}V)_{L^2}| = |(u,V)_{L^2}| \overset{\text{Hölder}}{\leq} \|u\|_{L^{3/2}}\|V\|_{L^3} \leq C\|u\|_{L^{3/2}}\|A^{1/4}V\|_{L^2}$$

and $A^{1/4} : D(A^{1/4}) \to H$ is onto.]

- We also admit that the Leray projection $\mathbb{P}$ extends to a bounded operator on $L^p(\Omega)^d$ for $1 < p < \infty$ (Calderón-Zygmund theory)

This requires some (not-so-easy) arguments from harmonic analysis.
We need the result for $p = 3/2$.

Then, for all $v \in V = D(A^{1/2})$, we have:

$$\|N(v)\|_{L^2} \overset{(*)}{\leqslant} C\,\|\mathbb{P}(v\cdot\nabla)v\|_{L^{3/2}} \leqslant C\,\|(v\cdot\nabla)v\|_{L^{3/2}}$$

$\mathbb{P}$ is bounded

$$\leqslant C\,\|v\|_{L^6}\|\nabla v\|_{L^2} \leqslant C\,\|v\|^2_{H^1} \leqslant C_1\|A^{1/2}v\|^2_{L^2}.$$

Hölder   Sobolev

The Lipschitz estimate is proved similarly. $\square$

We now fix $v_0 \in D(A^{1/4})$ and we solve (IE') by a fixed point argument in the space

$$\begin{cases} X = \left\{ v \in C^0((0,T],V)\,;\ \sup\limits_{0<t\leq T} t^{1/4}\|A^{1/2}v(t)\|_{L^2} < \infty,\ \lim\limits_{t\to 0} t^{1/4}\|A^{1/2}v(t)\|_{L^2} = 0 \right\} \\ \|v\|_X = \sup\limits_{0<t\leq T} t^{1/4}\|A^{1/2}v(t)\|_{L^2} \end{cases}$$

where $T>0$ is fixed later. We denote $R = \sup\limits_{0<t\leq T} t^{1/4}\|A^{1/2}e^{-At}v_0\|_{L^2} < \infty$.
We observe that $R \to 0$ as $T \to 0$. $R \leqslant C\|A^{1/4}v_0\|_{L^2}$!

The nonlinear term in (IE') is estimated as follows:

$$t^{1/4}\left\|A^{1/2}\int_0^t A^{1/4}e^{-A(t-s)}N(v(s))\,ds\right\|_{L^2} \leqslant t^{1/4}\int_0^t \|A^{3/4}e^{-A(t-s)}N(v(s))\|_{L^2}\,ds$$

$$\leqslant t^{3/4}\int_0^t \frac{C}{(t-s)^{3/4}}\|N(v(s))\|_{L^2}\,ds \leqslant t^{1/4}\int_0^t \frac{C}{(t-s)^{3/4}}\|A^{1/2}v(s)\|^2_{L^2}\,ds$$

$$\leqslant t^{1/4}\int_0^t \frac{C}{(t-s)^{3/4}\,s^{1/2}}\,ds\,\|v\|^2_X = C_1\|v\|^2_X$$

The Lipschitz estimate is obtained similarly. Kato's argument gives a strict contraction in the ball $B = \{v\in X;\ \|v\|_X \leqslant 2R\}$ if $4C_1R < 1$.

- Small data: $R \leqslant C\|A^{1/4}v_0\|_{L^2} < 1/4C_1 \Rightarrow$ global existence.
- large data: take $T>0$ small enough so that $R < 1/4C_1$.

The 2D case : global well-posedness

In the 2D case, a modification of the argument above allows us to prove local well-posedness in H. We indicate here how the solution can be extended to all $t \in \mathbb{R}_+$ using energy estimates. The NS eq. reads

$$\Big\| \; \partial_t v + \mathbb{P}(v\cdot\nabla)v = -Av, \quad v|_{t=0} = v_0 \in H.$$

L² energy estimate: $\frac{1}{2}\partial_t \|v\|_{L^2}^2 = (v, -Av)_{L^2} = -\|A^{1/2}v\|_{L^2}^2$

$$\Rightarrow \quad \frac{1}{2}\|v(t)\|_{L^2}^2 + \int_0^t \|A^{1/2}v(s)\|_{L^2}^2\,ds = \frac{1}{2}\|v_0\|_{L^2}^2 . \qquad (1)$$

H¹ energy estimate: We know from (1) that $v(t) \in D(A^{1/2}) = V$ for "many" times $t > 0$. On a time interval where $v(t) \in V$ we can compute:

$$\frac{1}{2}\partial_t \|A^{1/2}v(t)\|_{L^2}^2 = (A^{1/2}v(t), A^{1/2}v'(t))_{L^2} = (Av(t), v'(t))_{L^2}$$
$$= -\|Av(t)\|_{L^2}^2 - (Av(t), (v(t)\cdot\nabla)v(t))_{L^2} .$$

We use here the "Ladyzhenskaia inequality":

$$\|v\|_{L^4}^2 \leq C\|v\|_{L^2}\|A^{1/2}v\|_{L^2} \simeq C\|v\|_{L^2}\|\nabla v\|_{L^2} \qquad \forall v \in V.$$

Thus:

$$\|(v\cdot\nabla)v\|_{L^2} \leq \|v\|_{L^4}\|\nabla v\|_{L^4} \leq C\|v\|_{L^2}^{1/2}\|A^{1/2}v\|_{L^2}\|Av\|_{L^2}^{1/2},$$

hence

$$|(Av, (v\cdot\nabla)v)_{L^2}| \leq C\|v\|_{L^2}^{1/2}\|A^{1/2}v\|_{L^2}\|Av\|_{L^2}^{3/2} \underset{\text{Young } 4,4/3}{\leq} \frac{1}{2}\|Av\|_{L^2}^2 + C\|v\|_{L^2}^2\|A^{1/2}v\|_{L^2}^4 .$$

Summarizing:

$$\partial_t \|A^{1/2}v(t)\|_{L^2}^2 \leq -\|Av(t)\|_{L^2}^2 + C\|v(t)\|_{L^2}^2\|A^{1/2}v(t)\|_{L^2}^4 . \qquad (2)$$

Integrating (2) over $[t_0, t]$ and using (1) we find (Gronwall)

$$\|A^{1/2}V(t)\|_{L^2}^2 + \int_{t_0}^t \|AV(s)\|_{L^2}^2\,ds \leq \|A^{1/2}V(t_0)\|_{L^2}^2 \exp\left(\int_{t_0}^t \|V(s)\|_{L^2}^2 \|A^{1/2}V(s)\|_{L^2}^2\,ds\right)$$

$$\leq \|A^{1/2}V(t_0)\|_{L^2}^2 \exp\left(\frac{1}{2}\|V_0\|_{L^2}^4\right).$$

This shows that the H^1 norm of $V(t)$ is under control for all $t \geq t_0$, as soon as $V(t_0) \in V$. Thus the local solution can be extended to a global one (since the local existence time is uniform on bounded sets in V).