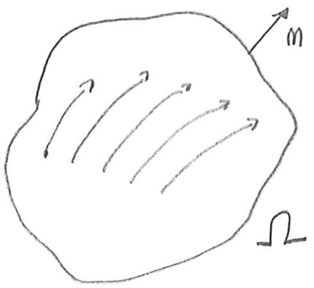
Chapter II : Perfect fluids

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a domain with smooth boundary $\partial\Omega$.
We consider the incompressible Euler equations in Ω:

$$\begin{cases} \partial_t v + (v\cdot\nabla)v = -\nabla p & t \in \mathbb{R},\, x \in \Omega \\ \operatorname{div} v = 0 & t \in \mathbb{R},\, x \in \Omega \\ v\cdot n = 0 & t \in \mathbb{R},\, x \in \partial\Omega \end{cases} \qquad (E)$$

where n denotes the outward unit normal on the boundary $\partial\Omega$.



The unknown functions are the Eulerian velocity $v(t,x)$ and the "pressure" $p(t,x)$. From a mathematical point of view, the first task is to study the

"initial-boundary value problem"

Cauchy problem: Given initial data $v_0 : \Omega \to \mathbb{R}^d$ satisfying $\operatorname{div} v_0 = 0$ in Ω and $v_0\cdot n = 0$ on $\partial\Omega$, prove that (E) has a unique solution $v : I \times \Omega \to \mathbb{R}^d$ such that $v(0) = v_0$. (I = time-interval, $0 \in \mathring{I}$)

To make the question precise, we have to specify appropriate function spaces for the initial data and the solution, and explain in which sense the evolution eq. (E) has to be understood. It turns out that the technical details depend (a little bit) on the domain Ω under consideration.

We shall give some details in the idealized situation where $\Omega = \mathbb{R}^d$, which is technically simpler. See below for some comments on the case where Ω has a boundary.

- Step 1: Elimination of the pressure

We apply the Leray-Hopf projection $\mathbb{P}$ onto divergence-free vector fields:

$$\partial_t v + \mathbb{P}(v\cdot\nabla)v = 0, \qquad t\in\mathbb{R}, x\in\mathbb{R}^d. \qquad (E1)$$

We recall that $\mathbb{P}$ is given by the explicit formula (in Fourier):

$$\widehat{(\mathbb{P}w)}(\xi) = \hat{w}(\xi) - \frac{\xi\cdot\hat{w}(\xi)}{|\xi|^2}\xi .$$

In particular $\mathbb{P}$ commutes with all differential operators (with constant coefficients).

Rem: Once the solution of is constructed, the pressure is uniquely determined (up to harmonic functions) by solving the linear elliptic equation:

$$-\Delta p = \mathrm{div}((v\cdot\nabla)v), \qquad x\in\mathbb{R}^d. \qquad (P_{eq})$$

- Step 2: Regularization of the equation

The "vector field" in (E1) is very irregular: it looses one derivative!

$\Longrightarrow$ no hope to apply directly a local existence result such as Cauchy-Lipschitz.

To avoid this problem, a possibility is to regularize the nonlinearity in (E1)

We give ourselves a function $\rho\in C_c^\infty(\mathbb{R}^d)$ such that

$$\rho(x) = \tilde{\rho}(|x|) \geq 0, \qquad \int_{\mathbb{R}^d}\rho(x)\,dx = 1.$$

Given any $\varepsilon > 0$ we define:

$$(J_\varepsilon v)(x) = \frac{1}{\varepsilon^d} \int_{\mathbb{R}^d} \rho\left(\frac{x-y}{\varepsilon}\right) v(y)\, dy. \qquad \text{"mollifier"}$$

If (for instance) $v \in L^2(\mathbb{R}^d)$, then $J_\varepsilon v \in H^m(\mathbb{R}^d)$ [the Sobolev space] for all $m \in \mathbb{N}$ and $J_\varepsilon v \xrightarrow[\varepsilon \to 0]{L^2} v$. The mollifier J_ε commutes with the Leray projection $\mathbb{P}$ and with any derivative ∂^α (easily seen in Fourier).

[Rem: In domains with boundaries, on cannot use a mollifier such as J_ε
$\Longrightarrow$ other ideas are needed to regularize Euler's equation.

We now consider the regularized equation:

$$\partial_t v + J_\varepsilon \mathbb{P}\left((J_\varepsilon v) \cdot \nabla\right) J_\varepsilon v = 0, \qquad t \in \mathbb{R},\ x \in \mathbb{R}^d. \qquad (E2)$$

$\uparrow$ to preserve the conservation of energy, see below.

We choose to study (E2) in L^2-based Sobolev spaces, because of the easy characterization in Fourier space:

$$H^s(\mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d)\,;\ \int_{\mathbb{R}^d} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi < \infty \right\}, \qquad s \geq 0$$

$$\|f\|_{H^s} \cong \left(\int_{\mathbb{R}^d} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2}.$$

Note that $\|J_\varepsilon f\|_{H^s} \leq \|f\|_{H^s}$ because $\widehat{(J_\varepsilon f)}(\xi) = \hat{\rho}(\varepsilon\xi)\hat{f}(\xi)$ with $\|\hat{\rho}\|_{L^\infty} = 1$.

For divergence-free vector fields, we define

$$V^s(\mathbb{R}^d) = \left\{ v \in H^s(\mathbb{R}^d)^d\,;\ \operatorname{div} v = 0 \right\}, \qquad s \geq 0$$

$$H^s_\sigma(\mathbb{R}^d) = \left\{ v \in L^2(\mathbb{R}^d)^d\,;\ \xi \cdot \hat{v}(\xi) = 0,\ \int_{\mathbb{R}^d} (1+|\xi|^2)^s |\hat{v}(\xi)|^2 d\xi < \infty \right\}$$

Note also that $\|\mathbb{P} v\|_{H^s} \leq \|v\|_{H^s} \quad \forall v \in H^s(\mathbb{R}^d)^d$.

<u>Rem:</u> In what follows we take $s = m \in \mathbb{N}$ for simplicity, but fractional Sobolev spaces can be considered as well.

It is rather easy to solve the regularized eq. (E2):

<u>Proposition 1</u> For any $\varepsilon > 0$ and any $V_0 \in V^0$, Eq. (E2) has a unique global solution $V^\varepsilon \in C^1(\mathbb{R}, V^0)$ such that $V^\varepsilon(0) = V_0$. Moreover:

i) $\|V^\varepsilon(t)\|_{L^2} = \|V_0\|_{L^2} \quad \forall t \in \mathbb{R}$ $\quad (*)$

ii) If moreover $V_0 \in V^m$ for some $m \in \mathbb{N}$, then $V^\varepsilon \in C^1(\mathbb{R}, V^m)$.

<u>Proof:</u> Fix $\varepsilon > 0$, and consider the bilinear map

$$\Phi : V^0 \times V^0 \longrightarrow V^0, \qquad (V, W) \longmapsto J_\varepsilon \mathbb{P}\big((J_\varepsilon V)\cdot\nabla\big)(J_\varepsilon W).$$

We have

$$\|\Phi(V,W)\|_{L^2} \leq \|[(J_\varepsilon V)\cdot\nabla]J_\varepsilon W\|_{L^2} \leq \|J_\varepsilon V\|_{L^4}\|\nabla J_\varepsilon W\|_{L^4}$$
$$\leq C\|J_\varepsilon V\|_{H^1}\|J_\varepsilon W\|_{H^2} \leq C_\varepsilon \|V\|_{L^2}\|W\|_{L^2}.$$

Here we used the fact that:

- $H^1(\mathbb{R}^d) \hookrightarrow L^4(\mathbb{R}^d)$ for $d \leq 4$
- $\|\partial J_\varepsilon f\|_{L^2} \leq \frac{C}{\varepsilon}\|f\|_{L^2} \implies \|J_\varepsilon V\|_{H^m} \leq C_{\varepsilon,m}\|V\|_{L^2}$.

Thus the map Φ is bilinear and continuous. By the Cauchy-Lipschitz theorem, the V^0-valued ODE

$$\left\| \; \partial_t V^\varepsilon(t) + \Phi(V^\varepsilon(t), V^\varepsilon(t)), \quad V^\varepsilon(0) = V_0 \right.$$

has a unique $\underset{\text{maximal}}{\uparrow}$ solution $V^\varepsilon \in C^1(I, V^0)$, I = open interval $\ni 0$.

On the existence interval $I \subset \mathbb{R}$ we can compute:

$$\frac{1}{2}\frac{d}{dt}\|V^\varepsilon(t)\|^2 = (V^\varepsilon(t), \partial_t V^\varepsilon(t))_{L^2}$$
$$= -\left(V^\varepsilon(t), J_\varepsilon \mathbb{P}\left((J_\varepsilon V^\varepsilon(t))\cdot\nabla\right) J_\varepsilon V^\varepsilon(t)\right)_{L^2}$$
$$= -\left(J_\varepsilon V^\varepsilon(t), (J_\varepsilon V^\varepsilon(t)\cdot\nabla)(J_\varepsilon V^\varepsilon(t))\right)_{L^2} = 0.$$

Indeed, if $V, W \in V^1$, we have

$$(V, (W\cdot\nabla)V)_{L^2} = \int_{\mathbb{R}^d} V_i (W_j\partial_j)V_i\, dx = \frac{1}{2}\int_{\mathbb{R}^d}(W_j\partial_j)|V|^2 dx$$
$$= -\frac{1}{2}\int_{\mathbb{R}^d}(\operatorname{div} W)|V|^2 dx = 0. \qquad (SS)$$

The nonlinearity in the Euler equation (and in its regularized version) is "skew-symmetric" in L^2.

Thus we see that $\|V^\varepsilon(t)\|_{L^2} = \|V_0\|_{L^2}$ $\forall t \in I$. In particular $\|V^\varepsilon(t)\|_{L^2}$ cannot blow up in finite time $\Rightarrow I = \mathbb{R}$.

Finally, if $V_0 \in V^m$, then

$$V^\varepsilon(t) = V_0 - \int_0^t \underset{\uparrow \text{ regularizing}}{J_\varepsilon} \underbrace{\mathbb{P}\left(J_\varepsilon V^\varepsilon(\tau)\cdot\nabla\right) J_\varepsilon V^\varepsilon(\tau)}_{\in C^0(\mathbb{R}, V^0)} d\tau \in C^1(\mathbb{R}, V^m). \quad \square$$

<u>Step 3:</u> <u>A priori estimates</u>

Our goal is to derive an estimate on the solution $V^\varepsilon(t)$ of (E1) which is <u>uniform in ε</u> for $\varepsilon \leq 1$ (because eventually we want to take the limit $\varepsilon \to 0+$). Estimate (*) is indep. of ε, but too weak to pass to the limit, whereas the estimates we have so far on $\|V^\varepsilon(t)\|_{H^m}$ (if $V_0 \in H^m$) behave badly as $\varepsilon \to 0$.

We shall use the following <u>calculus inequalities</u>:

<u>Lemma 1</u>: Assume that $u, v \in H^m(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, for some $m \in \mathbb{N}$.

1) There exists $C > 0$ such that

$$\|uv\|_{H^m} \leq C\left(\|u\|_{L^\infty}\|v\|_{H^m} + \|u\|_{H^m}\|v\|_{L^\infty}\right).$$

2) If $\alpha \in \mathbb{N}^d$ is a multi-index of order $|\alpha| = m$, then

$$\|\partial^\alpha(uv) - u\,\partial^\alpha v\|_{L^2} \leq C\left(\|\nabla u\|_{L^\infty}\|v\|_{H^{m-1}} + \|u\|_{H^m}\|v\|_{L^\infty}\right).$$

Here it is assumed that $m \geq 1$ and $\nabla u \in L^\infty$.

<u>Rem:</u> Part 1) says that <u>$H^m \cap L^\infty$ is an algebra</u> for the usual product.
If $m > d/2$, we know that $H^m(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ (Sobolev embedding)
$\Longrightarrow$ $H^m(\mathbb{R}^d)$ is an algebra and $\|uv\|_{H^m} \leq C\|u\|_{H^m}\|v\|_{H^m}$.

<u>Proposition 2</u> (a priori estimates, uniform in ε)
Assume that $V_0 \in H^m_\sigma(\mathbb{R}^d)$ for some $m > \frac{d}{2} + 1$. Then there exists a constant $k > 0$ such that, for all $\varepsilon > 0$, the solution V^ε of (E2) satisfies

$$\|V^\varepsilon(t)\|_{H^m} \leq \frac{\|V_0\|_{H^m}}{1 - k|t|\,\|V_0\|_{H^m}}, \quad |t| < \frac{1}{k\|V_0\|_{H^m}}. \qquad (A\ell)$$

The constant $k > 0$ in $(A\ell)$ depends only on m, so $(A\ell)$ gives an estimate that is uniform in the regularization parameter $\varepsilon > 0$. This bounds V^ε on a finite time interval $(-T_*, T_*)$, where $T_* = \frac{1}{k\|V_0\|_{H^m}}$.

<u>Rem:</u> $V \in H^m(\mathbb{R}^d)$ for $m > \frac{d}{2} + 1 \Longrightarrow \nabla V \in L^\infty(\mathbb{R}^d)$

$\Longrightarrow$ the vector field V is <u>Lipschitz</u> continuous.

Proof of Proposition 2: We estimate $\|V^\varepsilon(t)\|_{H^m}$.

Given a multi-index $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq m$, we have:

$$\frac{1}{2}\frac{d}{dt}\|\partial^\alpha V^\varepsilon(t)\|_{L^2}^2 = \left(\partial^\alpha V^\varepsilon(t), \partial^\alpha \partial_t V^\varepsilon(t)\right)_{L^2}$$

$$= -\left(\partial^\alpha V^\varepsilon(t), \partial^\alpha J_\varepsilon \mathbb{P}\left((J_\varepsilon V^\varepsilon(t)\cdot\nabla) J_\varepsilon V^\varepsilon(t)\right)\right)_{L^2}$$

Notation: $W_\varepsilon = J_\varepsilon V^\varepsilon(t)$

$$= -\left(\partial^\alpha W_\varepsilon, \partial^\alpha[(W_\varepsilon\cdot\nabla)W_\varepsilon]\right)_{L^2}$$

$$= -\left(\partial^\alpha W_\varepsilon, \partial^\alpha[(W_\varepsilon\cdot\nabla)W_\varepsilon] - (W_\varepsilon\cdot\nabla)\partial^\alpha W_\varepsilon\right)_{L^2} \text{ by (SS).}$$

Using Lemma 1, we thus find:

$$\frac{1}{2}\frac{d}{dt}\|\partial^\alpha V^\varepsilon(t)\|_{L^2}^2 \leq \|\partial^\alpha W_\varepsilon\|_{L^2}\,\|\partial^\alpha[(W_\varepsilon\cdot\nabla)W_\varepsilon] - (W_\varepsilon\cdot\nabla)\partial^\alpha W_\varepsilon\|_{L^2}$$

$$\leq C\|\partial^\alpha W_\varepsilon\|_{L^2}\left(\|\nabla W_\varepsilon\|_{L^\infty}\|\nabla W_\varepsilon\|_{H^{m-1}} + \|W_\varepsilon\|_{H^m}\|\nabla W_\varepsilon\|_{L^\infty}\right)$$

$$\leq C\|V^\varepsilon(t)\|_{H^m}^2\,\|\nabla J_\varepsilon V^\varepsilon(t)\|_{L^\infty} \leq C\|V^\varepsilon(t)\|_{H^m}^3.$$

$\uparrow$ Sobolev embedding

If we sum over all multi-indices $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq m$, we arrive at:

$$\frac{1}{2}\frac{d}{dt}\|V^\varepsilon(t)\|_{H^m}^2 \leq k\|V^\varepsilon(t)\|_{H^m}^3, \text{ or equivalently}$$

$$\frac{d}{dt}\|V^\varepsilon(t)\|_{H^m} \leq k\|V^\varepsilon(t)\|_{H^m}^2.$$

Integrating this differential inequality over $[0,t]$ for $t>0$ small enough gives

$$\|V^\varepsilon(t)\|_{H^m} \leq \frac{\|V_0\|_{H^m}}{1-kt\|V_0\|_{H^m}}, \quad 0 \leq t < \frac{1}{k\|V_0\|_{H^m}}$$

and a similar argument gives

$$\|V^\varepsilon(t)\|_{H^m} \leq \frac{\|V_0\|_{H^m}}{1+kt\|V_0\|_{H^m}}, \quad -\frac{1}{k\|V_0\|_{H^m}} < t \leq 0. \quad \square$$

Step 4: Taking the limit $\varepsilon \to 0+$

We assume throughout that $m > \frac{d}{2}+1$ and we fix $T>0$ such that $kT\|V_0\|_{H^m} \leq 1/2$. From Proposition 2, we know that the approximate solutions $V^\varepsilon \in C^0([-T,T], V^m) \cap C^1([-T,T], V^{m-1})$ satisfy

$$\sup_{|t|\leq T} \|V^\varepsilon(t)\|_{H^m} \leq 2\|V_0\|_{H^m} \quad : \text{Directly from (AP)}$$

$$\sup_{|t|\leq T} \|\partial_t V^\varepsilon(t)\|_{H^{m-1}} \leq C\|V_0\|_{H^m}^2 \quad : \text{Use (E2) + } H^{m-1} \text{ is an algebra.}$$

Proposition 3 (contraction in L^2 norm)

If $kT\|V_0\|_{H^m} \leq 1/2$, there exists $C = C(T, \|V_0\|_{H^m})$ such that

$$\sup_{|t|\leq T} \|V^\varepsilon(t) - V^{\varepsilon'}(t)\|_{L^2} \leq C\max(\varepsilon,\varepsilon'), \qquad \forall \varepsilon,\varepsilon'>0.$$

In the proof we use the estimate

$$\|J_\varepsilon f - f\|_{L^2} \leq C\varepsilon\|\nabla f\|_{L^2} \qquad \forall f \in H^1(\mathbb{R}^d)\ \ \forall \varepsilon>0 \qquad (L2)$$

Indeed, we have in Fourier:

$$(\widehat{J_\varepsilon f - f})(\xi) = (\hat\rho(\varepsilon\xi)-1)\hat f(\xi) = \frac{\hat\rho(\varepsilon\xi)-1}{\varepsilon|\xi|}\,\varepsilon|\xi|\,\hat f(\xi)$$

with $1 = \hat\rho(0)$.

$$\Rightarrow \|J_\varepsilon f - f\|_{L^2} \leq \varepsilon\,\mathrm{Lip}(\hat\rho)\,\|\nabla f\|_{L^2}.$$

It also follows that

$$\|J_\varepsilon f - f\|_{H^{m-1}} \leq C\varepsilon\|f\|_{H^m}$$

$$\|J_\varepsilon f - J_{\varepsilon'} f\|_{L^2} \leq C\max(\varepsilon,\varepsilon')\|\nabla f\|_{L^2}.$$

Proof of Proposition 3 : given $\varepsilon, \varepsilon' > 0$ we have, for $|t| \leq T$:

$$\partial_t V^\varepsilon(t) = - J_\varepsilon \mathbb{P} (W_\varepsilon \cdot \nabla) W_\varepsilon , \qquad W_\varepsilon = J_\varepsilon V^\varepsilon(t) ,$$
$$\partial_t V^{\varepsilon'}(t) = - J_{\varepsilon'} \mathbb{P} (W_{\varepsilon'} \cdot \nabla) W_{\varepsilon'} , \qquad W_{\varepsilon'} = J_{\varepsilon'} V^{\varepsilon'}(t) .$$

Thus :

$$\frac{1}{2} \partial_t \| V^\varepsilon(t) - V^{\varepsilon'}(t) \|_{L^2}^2 = \left(V^\varepsilon(t) - V^{\varepsilon'}(t), - J_\varepsilon \mathbb{P} (W_\varepsilon \cdot \nabla) W_\varepsilon + J_{\varepsilon'} \mathbb{P} (W_{\varepsilon'} \cdot \nabla) W_{\varepsilon'} \right)_{L^2}$$
$$= A_1 + A_2 + A_3 .$$

- $A_1 = - \left(V^\varepsilon(t) - V^{\varepsilon'}(t), (J_\varepsilon - J_{\varepsilon'}) \mathbb{P} (W_\varepsilon \cdot \nabla) W_\varepsilon \right)_{L^2}$

$$|A_1| \leq \| V^\varepsilon(t) - V^{\varepsilon'}(t) \|_{L^2} \, C \max(\varepsilon, \varepsilon') \, \| (W_\varepsilon \cdot \nabla) W_\varepsilon \|_{H^1}$$
$$\leq C \| V^\varepsilon(t) - V^{\varepsilon'}(t) \|_{L^2} \max(\varepsilon, \varepsilon') \, \| V_0 \|_{H^m}^2 ,$$

because $\| (W_\varepsilon \cdot \nabla) W_\varepsilon \|_{H^1} \leq \| (W_\varepsilon \cdot \nabla) W_\varepsilon \|_{H^{m-1}} \leq \| W_\varepsilon \|_{H^m}^2 \leq \| V^\varepsilon(t) \|_{H^m}^2 \leq C \| V_0 \|_{H^m}^2 .$

- $A_2 = - \left(V^\varepsilon(t) - V^{\varepsilon'}(t), J_{\varepsilon'} \mathbb{P} \left((W_\varepsilon - W_{\varepsilon'}) \cdot \nabla W_\varepsilon \right) \right)_{L^2}$

$|A_2| \leq \| V^\varepsilon(t) - V^{\varepsilon'}(t) \|_{L^2} \| W_\varepsilon - W_{\varepsilon'} \|_{L^2} \| \nabla W_\varepsilon \|_{L^\infty}$ and

i) $$\| W_\varepsilon - W_{\varepsilon'} \|_{L^2} \leq \| (J_\varepsilon - J_{\varepsilon'}) V^\varepsilon(t) \|_{L^2} + \| J_{\varepsilon'} (V^\varepsilon(t) - V^{\varepsilon'}(t)) \|_{L^2}$$
$$\leq C \max(\varepsilon, \varepsilon') \| V_0 \|_{H^m} + \| V^\varepsilon(t) - V^{\varepsilon'}(t) \|_{L^2}$$

ii) $$\| \nabla W_\varepsilon \|_{L^\infty} \leq \| W_\varepsilon \|_{H^m} \leq \| V^\varepsilon(t) \|_{H^m} \leq C \| V_0 \|_{H^m}$$

$$\Rightarrow |A_2| \leq C \| V^\varepsilon(t) - V^{\varepsilon'}(t) \|_{L^2} \left(\max(\varepsilon, \varepsilon') \| V_0 \|_{H^m}^2 + \| V^\varepsilon(t) - V^{\varepsilon'}(t) \|_{L^2} \| V_0 \|_{H^m} \right) .$$

- $A_3 = - \left(V^\varepsilon(t) - V^{\varepsilon'}(t), J_{\varepsilon'} \mathbb{P} \, W_{\varepsilon'} \cdot \nabla (W_\varepsilon - W_{\varepsilon'}) \right)_{L^2}$

$$= - \left(V^\varepsilon(t) - V^{\varepsilon'}(t), J_{\varepsilon'} \mathbb{P} \, W_{\varepsilon'} \cdot \nabla (J_\varepsilon - J_{\varepsilon'}) V^\varepsilon(t) \right)_{L^2} + 0 \quad \text{by (SS) !}$$

$$|A_3| \leq \|V^\varepsilon(t)-V^{\varepsilon'}(t)\|_{L^2} \|W_{\varepsilon'}\|_{L^\infty} \|\nabla(J_\varepsilon - J_{\varepsilon'})W_\varepsilon\|_{L^2}$$

$$\leq \|V^\varepsilon(t)-V^{\varepsilon'}(t)\|_{L^2} \|W_{\varepsilon'}\|_{H^{m-1}}\, C \max(\varepsilon,\varepsilon') \|W_\varepsilon\|_{H^2} \qquad (m \geq 2\,!)$$

$$\leq C \|V^\varepsilon(t)-V^{\varepsilon'}(t)\|_{L^2} \max(\varepsilon,\varepsilon') \|V_0\|^2_{H^m}.$$

Summarizing, we have shown:

$$\left| \frac{d}{dt} \|V^\varepsilon(t)-V^{\varepsilon'}(t)\|_{L^2} \right| \leq C_1 \max(\varepsilon,\varepsilon') \|V_0\|^2_{H^m} + C_2 \|V^\varepsilon(t)-V^{\varepsilon'}(t)\|_{L^2} \|V_0\|_{H^m}.$$

Since $V^\varepsilon(0) = V^{\varepsilon'}(0) = V_0$, we deduce by Gronwall's lemma:

$$\left[\ \|V^\varepsilon(t)-V^{\varepsilon'}(t)\|_{L^2} \leq C_1 \max(\varepsilon,\varepsilon') \|V_0\|^2_{H^m} \exp\Big(C_2 \underbrace{|t|\, \|V_0\|_{H^m}}_{\leq 1/2k} \Big), \quad |t| \leq T. \ \square \right.$$

Proposition 3 shows that $(V^\varepsilon)_{\varepsilon>0}$ is a <u>Cauchy sequence</u> in $C^0([-T,T], L^2(\mathbb{R}^d))$ as $\varepsilon \to 0+$. We also know that $(V^\varepsilon)_{\varepsilon>0}$ is <u>uniformly bounded</u> in $C^0([-T,T], H^m(\mathbb{R}^d))$. By a simple <u>interpolation argument</u>, we deduce that, for any $s \in [0,m)$, $(V^\varepsilon)_{\varepsilon>0}$ is a Cauchy sequence in $C^0([-T,T], H^s(\mathbb{R}^d))$.

<u>Lemma 2</u> (interpolation inequalities)
If $0 \leq s \leq m$, then $\forall f \in H^m(\mathbb{R}^d)$:

$$\| |\nabla|^s f \|_{L^2} \leq \|f\|_{L^2}^{1-s/m} \| |\nabla|^m f \|_{L^2}^{s/m}.$$

(This is just Hölder's inequality in Fourier space.)

Consequently, if $\|f_j\|_{H^m} \leq C$ and $\|f_j\|_{L^2} \xrightarrow[j\to+\infty]{} 0$, then

$$\|f_j\|_{H^s} \leq C\left(\|f_j\|_{L^2} + \|f_j\|_{L^2}^{1-s/m} \|f_j\|_{H^m}^{s/m} \right) \xrightarrow[j\to+\infty]{} 0 \quad \underline{\forall s < m}.$$

Given $V_0 \in V^m$ for some $m > \frac{d}{2}+1$, we thus denote

$$V(t) = \lim_{\varepsilon \to 0+} V^\varepsilon(t), \qquad t \in [-T,T]$$

where V^ε is the solution of (E2) with initial data V_0, and $kT\|V_0\|_{H^m} \leq 1/2$.

We know that

$$V \in C^0([-T,T], H^s(\mathbb{R}^d)), \qquad 0 \leq s < m.$$

Taking $\frac{d}{2}+1 < s < m$ and using $\sup_{|t|\leq T} \|V^\varepsilon(t) - V(t)\|_{H^s} \longrightarrow 0$, we easily deduce that

$$\|J_\varepsilon \mathbb{P}(J_\varepsilon V^\varepsilon(t)\cdot\nabla) J_\varepsilon V^\varepsilon(t) - \mathbb{P}(V(t)\cdot\nabla)V(t)\|_{H^{s-1}} \xrightarrow[\varepsilon \to 0+]{} 0,$$

uniformly for $t \in [-T,T]$. (Proceed as in the proof of Proposition 3)

Thus we can take the limit $\varepsilon \to 0+$ in the integral equation

$$V^\varepsilon(t) = V_0 - \int_0^t J_\varepsilon \mathbb{P}(J_\varepsilon V^\varepsilon(\tau)\cdot\nabla) J_\varepsilon V^\varepsilon(\tau)\, d\tau, \qquad |t| \leq T$$

and obtain

$$V(t) = V_0 - \int_0^t \mathbb{P}(V(\tau)\cdot\nabla)V(\tau)\, d\tau, \qquad |t| \leq T. \qquad \text{(IE)}$$

This shows that

$$V \in C^1([-T,T], H^{s-1}(\mathbb{R}^d)), \qquad 0 \leq s < m$$

and that

$$V'(t) + \mathbb{P}(V(t)\cdot\nabla)V(t) = 0, \qquad |t| \leq T.$$

Note that $V \in H^s \Longrightarrow V \in C^1(\mathbb{R}^d)$ since $s > \frac{d}{2}+1$ $\Longrightarrow$ we have constructed a classical solution of Euler which is C^1 in t and x.

Step 5 : Additional properties of the solution

Proposition 4: Given $V_0 \in H^m_\sigma(\mathbb{R}^d)$ $(m > \frac{d}{2}+1)$ and $T>0$ such that $kT\|V_0\|_{H^m} \leq 1/2$, the solution V constructed above satisfies

$$V \in C^0([-T,T], H^m(\mathbb{R}^d)) \cap C^1([-T,T], H^{m-1}(\mathbb{R}^d)).$$

Moreover V is the unique solution of (E1) in that class, with $V(0)=V_0$.

$\Rightarrow$ The Cauchy problem for (E1) is locally well-posed in $H^m_\sigma(\mathbb{R}^d)$, $m > \frac{d}{2}+1$.

Proof: i) Fix $t \in [-T,T]$. Since $\|V^\varepsilon(t)\|_{H^m} \leq 2\|V_0\|_{H^m}$, we can find a sequence $\varepsilon_n \to 0$ such that $V^{\varepsilon_n}(t) \xrightarrow[n\to+\infty]{H^m} W \in H^m_\sigma(\mathbb{R}^d)$ (weak convergence).

But we also know that $V^{\varepsilon_n}(t) \xrightarrow[n\to\infty]{H^s} V(t)$ $\forall s<m$ $\Rightarrow$ $W = V(t)$.

Thus $V^\varepsilon(t) \xrightarrow[\varepsilon\to 0]{H^m} V(t)$ $\Rightarrow$ $\|V(t)\|_{H^m} \leq \liminf_{\varepsilon\to 0} \|V^\varepsilon(t)\|_{H^m} \leq 2\|V_0\|_{H^m}$.

ii) Let $s<m$. For any $\varphi \in H^{-s}(\mathbb{R}^d)$, we have

$$\sup_{|t|\leq T} \left|(\varphi, V^\varepsilon(t))_{L^2} - (\varphi, V(t))_{L^2}\right| \leq \|\varphi\|_{H^{-s}} \sup_{|t|\leq T}\|V^\varepsilon(t)-V(t)\|_{H^s} \xrightarrow[\varepsilon\to 0]{} 0$$

Since $\|V^\varepsilon(t)\|_{H^m} \leq 2\|V_0\|_{H^m}$ and $H^{-s}(\mathbb{R}^d)$ is dense in $H^{-m}(\mathbb{R}^d)$, an $\varepsilon/3$ argument gives:

$$\forall \varphi \in H^{-m}(\mathbb{R}^d): \quad (\varphi, V^\varepsilon(t))_{L^2} \xrightarrow[\varepsilon\to 0]{\text{unif. in } t} (\varphi, V(t))_{L^2} \Rightarrow V \in C^0_w([-T,T], H^m(\mathbb{R}^d)).$$

iii) To prove that $V \in C^0([-T,T], H^m)$ it remains to show that $t \longmapsto \|V(t)\|_{H^m}$ is continuous. Let us check continuity at $t=0$ (for instance).

- $\|V_0\|_{H^m} \leq \liminf_{t\to 0} \|V(t)\|_{H^m}$ by weak continuity
- $\|V(t)\|_{H^m} \leq \dfrac{\|V_0\|_{H^m}}{1-k|t|\,\|V_0\|_{H^m}} \xrightarrow[t\to 0]{} \|V_0\|_{H^m}$

$\Big\} \Rightarrow$ OK.

The integral equation (IE) then shows that $V \in C^1([-T,T], H^{m-1})$.

Suppose now that $V, \tilde{V}$ are two solutions of (E1) in (Vclass), with the same initial data. We have

$$\frac{1}{2}\frac{d}{dt}\|V(t)-\tilde{V}(t)\|_{L^2}^2 = -\left(V(t)-\tilde{V}(t),\ (V(t)\cdot\nabla)V(t)-(\tilde{V}(t)\cdot\nabla)\tilde{V}(t)\right)_{L^2}$$

$$= -\left(V(t)-\tilde{V}(t),\ (V(t)-\tilde{V}(t))\cdot\nabla V(t)\right)_{L^2} + 0 \quad \text{by (SS)}$$

$$\leq \|V(t)-\tilde{V}(t)\|_{L^2}^2\ \|\nabla V(t)\|_{L^\infty};$$

hence by Gronwall: (e.g. for $t \geq 0$)

$$\|V(t)-\tilde{V}(t)\|_{L^2} \leq \underbrace{\|V(0)-\tilde{V}(0)\|_{L^2}}_{=0} \exp\left(\int_0^t \|\nabla V(\tau)\|_{L^\infty} d\tau\right) = 0. \quad (\text{Idem for } t \leq 0)$$

Thus $V = \tilde{V}$. □

<u>Remark:</u> For solutions $V, \tilde{V}$ with different initial data $V_0, \tilde{V}_0 \in H^m$, the last estimate + the interpolation argument shows that the solution $V(t)$ depends <u>continuously on the initial data</u> in the topology of $H^s(\mathbb{R}^d)$, for any $s < m$. Continuity in $H^m(\mathbb{R}^d)$ requires a different argument.

<u>Criteria for global existence</u>

Given $V_0 \in V^m$ for some $m > \frac{d}{2}+1$, we know that (E1) has a unique maximal solution (for positive times)

$$V \in C^0([0,T_*), H^m) \cap C^1([0,T_*), H^{m-1}), \quad V(0) = V_0,$$

for some $T_* \in (0, +\infty]$. Two possibilities could occur:

- $T_* = +\infty$: the solution V is <u>global</u> for positive times
- $T_* < \infty$: the solution V <u>blows up</u> as $t \to T_*$.

To explain what is meant by "blow up", we recall the estimate

$$\|V(t)\|_{H^m} \leq \|V_0\|_{H^m} \exp\left(C_m \int_0^t \|\nabla V(\tau)\|_{L^\infty} d\tau\right), \quad 0 \leq t < T_* \qquad (*)$$

See the proof of Proposition 2. If $T_* < \infty$, we necessarily have:

$$\int_0^{T_*} \|\nabla V(t)\|_{L^\infty} dt = +\infty \qquad \text{(blow up criterion)}$$

Otherwise, the solution would stay uniformly bounded on $[0, T_*)$, and since the local existence time depends only on $\|V_0\|_{H^m}$, the solution could be extended beyond T_* (i.e. T_* would not be maximal).

It is customary to formulate a blow-up criterion in terms of the vorticity

$$\omega = \operatorname{curl} V,$$

which is a physically meaningful quantity (local rotation of the fluid).
We have $\|\omega\|_{L^\infty} \leq C \|\nabla V\|_{L^\infty}$, but the converse inequality is not true in general. Nevertheless:

Proposition 5 (Beale-Kato-Majda)
With the notations above, if $T_* < \infty$, then

$$\int_0^{T_*} \|\omega(t)\|_{L^\infty} dt = +\infty.$$

The proof uses the following "elliptic estimate": (see below for a proof)

Lemma 3: If $V \in H^m_\sigma(\mathbb{R}^d)$ for some $m > \frac{d}{2} + 1$, then

$$\|\nabla V\|_{L^\infty} \leq C \|\omega\|_{L^\infty} \left(1 + \log_+ \|V\|_{H^m}\right) \qquad \text{(Ell)}$$

where $\log_+(x) = \max(0, \log x)$, $x > 0$.

Proof of Prop. 5: Combining (Ell) and (*), we obtain for any $t \in [0, T_*)$:

$$\|\nabla V(t)\|_{L^\infty} \leq C \|\omega(t)\|_{L^\infty} \left(1 + \log_+ \|V(t)\|_{H^m}\right)$$
$$\leq C \|\omega(t)\|_{L^\infty} \underbrace{\left(1 + \log_+ \|V_0\|_{H^m} + C_m \int_0^t \|\nabla V(\tau)\|_{L^\infty} d\tau\right)}_{=: \Phi(t)}.$$

$\Phi'(t) = C_m \|\nabla V(t)\|_{L^\infty} \leq C C_m \Phi(t) \|\omega(t)\|_{L^\infty}$, hence

$$C_m \int_0^t \|\nabla V(\tau)\|_{L^\infty} d\tau \leq \Phi(t) \leq \left(1 + \log_+ \|V_0\|_{H^m}\right) e^{\int_0^t C C_m \|\omega(\tau)\|_{L^\infty} d\tau}. \quad (**)$$

In particular, we must have $\int_0^{T_*} \|\omega(t)\|_{L^\infty} dt = +\infty$ because $\int_0^{T_*} \|\nabla V(t)\|_{L^\infty} dt = +\infty$. $\square$

Remark: The Euler equation can be written in the form

$$\partial_t V + \omega \wedge V = -\nabla\left(p + \tfrac{1}{2}|u|^2\right), \quad \mathrm{div}\, V = 0$$

because $(V \cdot \nabla) V = \frac{1}{2} \nabla |V|^2 - V \wedge \omega$. Taking the curl and using the identity $\mathrm{rot}(\omega \wedge V) = (V \cdot \nabla)\omega - (\omega \cdot \nabla) V$, we obtain the vorticity equation

$$\partial_t \omega + (V \cdot \nabla)\omega - (\omega \cdot \nabla) V = 0 \qquad (V3d)$$

In the 2D case, $V = (V_1, V_2, 0)$, $\omega = (0, 0, W)$ where $W = \partial_1 V_2 - \partial_2 V_1$, and the vorticity eq. becomes:

$$\partial_t W + u \cdot \nabla W = 0, \qquad u = (V_1, V_2). \qquad (V2d)$$

This means that the (scalar) vorticity W is advected by the 2D vector field $u = (V_1, V_2)$. Note that $\mathrm{div}\, u = \partial_1 V_1 + \partial_2 V_2 = 0$.

<u>Proposition 6:</u> If $d=2$ and $V_0 \in H^m_\sigma(\mathbb{R}^2)$ for some $m>2$, the solution V of (E1) satisfies

$$\|\omega(t)\|_{L^p} = \|\omega(0)\|_{L^p}, \qquad \forall p \in [1,+\infty]$$

for all t in the maximal interval of existence.

Combining this with the BKM criterion (Prop. 5), we obtain:

<u>Theorem:</u> Assume $d=2$. For any $V_0 \in H^m_\sigma(\mathbb{R}^2)$ with $m>2$, Eq. (E1) has a unique global solution $V \in C^0(\mathbb{R}, H^m) \cap C^1(\mathbb{R}, H^{m-1})$ such that $V(0)=V_0$. Moreover $\|\omega(t)\|_{L^p} = \|\omega(0)\|_{L^p}$ $\forall t \in \mathbb{R}$ $\forall p \in [1,+\infty]$.

(Global well-posedness for the 2D Euler equation)

<u>Remark:</u> From (*) and (**) we obtain the global estimate

$$\|V(t)\|_{H^m} \leq C(\|V_0\|_{H^m}) \exp(\exp(C|t|)), \quad \forall t \in \mathbb{R}.$$

Examples can be constructed which show that a double exponential growth is sharp!

<u>Proof of Prop. 6:</u>

<u>First method</u> (Euler proof): Let $p \in 2\mathbb{N}$. From (V_{2d}) we have:

$$\frac{d}{dt}\|W(t)\|^p_{L^p} = \frac{d}{dt}\int_{\mathbb{R}^2} |W(t,x)|^p dx = p\int_{\mathbb{R}^2} W^{p-1}\partial_t W \, dx$$

$$= -p\int_{\mathbb{R}^2} W^{p-1} u \cdot \nabla W \, dx = -\int_{\mathbb{R}^2} u \cdot \nabla(W^p)\, dx = \int_{\mathbb{R}^2} (\operatorname{div} u) W^p dx = 0$$

$$\Longrightarrow \quad \|W(t)\|_{L^p} = \|W_0\|_{L^p} \quad \forall t.$$

Taking the limit $p \to +\infty$, we obtain $\|W(t)\|_{L^\infty} = \|W_0\|_{L^\infty}$.

<u>Second method</u> (Lagrange proof):

We recall that the trajectory $X(t)$ of the fluid particle starting at $X_0 \in \mathbb{R}^2$ is determined by the ODE:

$$\| \quad X'(t) = u(t, X(t)), \quad X(0) = X_0.$$

N.B. $v \in H^m(\mathbb{R}^2)$ for $m > 2 \Rightarrow \nabla u \in H^{m-1}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2) \Rightarrow u$ is Lipschitz (and even C^{m-2}) with respect to $x \Rightarrow$ local existence and uniqueness. The trajectories are global in that case.

By construction

$$\frac{d}{dt} W(t, X(t)) = \partial_t W(t, X(t)) + u(t, X(t)) \cdot \nabla W(t, X(t)) = 0$$

$$\Longrightarrow W(t, X(t)) = W_0(X_0).$$

More generally, denoting $X(t) = \varphi_t(X_0)$ (φ_t = Lagrangian flow), we have:

$$\| \quad W(t, \varphi_t(X_0)) = W_0(X_0), \quad \text{or} \quad W(t,x) = W_0(\varphi_{-t}(x)).$$

But φ_t is a volume-preserving diffeomorphism, hence all L^p norms are preserved:

$$\| W(t,\cdot) \|_{L^p} = \| W_0 \circ \varphi_{-t} \|_{L^p} = \| W_0 \|_{L^p}. \quad \square$$

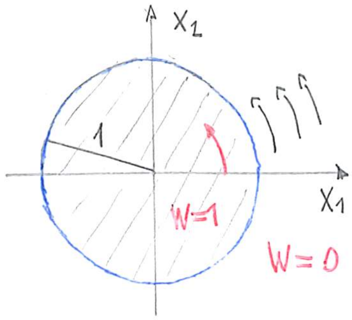
Remark: Other methods can be used to prove existence and uniqueness of classical ($\cong$ smooth) solutions to the Euler equations, also in bounded or exterior domains.

- A Galerkin method can be used to regularize the system in domains with boundaries, instead of the mollifier J_ε, see e.g. Constantin-Foias, Taylor.
- A fixed point argument can also be made on the Lagrangian flow map rather than on the velocity field (especially convenient in 2D), see Majda-Bertozzi, Marchioro-Pulvirenti.

Weak solutions of 2D Euler (Yudovich's theory)

It is important for applications to consider more singular solutions of Euler's equations than those constructed so far.

Example: (circular vortex patch in 2D)



$$w(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

$$u(x) = V(|x|)\, e_\theta$$

$$V(r) = \begin{cases} \frac{r}{2} & 0 \le r \le 1 & : \text{rigid rotation} \\ \frac{1}{2r} & r \ge 1 & : \text{irrotational flow} \end{cases}$$

Formally $u \cdot \nabla w = \frac{V(r)}{r} \partial_\theta w = 0 \Rightarrow u$ is a stationary solution of 2D Euler with discontinuous vorticity w.

To define weak solutions more precisely, we start from the vorticity eq. in $\mathbb{R}^2$:

$$\frac{\partial w}{\partial t}(t,x) + u(t,x) \cdot \nabla w(t,x) = 0 \qquad t \in \mathbb{R},\ x \in \mathbb{R}^2 \qquad (V2d)$$

where $\operatorname{div} u = 0$, $\partial_1 u_2 - \partial_2 u_1 = w$. If w is sufficiently regular and localized, we have the Biot-Savart formula:

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} w(y)\, dy =: (K * w)(y) \qquad (BS2)$$

Quick proof: Assume $w \in S(\mathbb{R}^2)$ and look for u in the form $u = \nabla^\perp \varphi = (-\partial_2\varphi, \partial_1\varphi)$.

$\Rightarrow \Delta \varphi = w$ hence

$$\varphi(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y|\, w(y)\, dy + \text{harmonic function.}$$

Taking $\nabla^\perp$ gives (BS2): unique solution s.th. $u(x) \xrightarrow[|x|\to\infty]{} 0$.

<u>Lemma 4:</u> (Estimates for the 2D Biot-Savart formula)

1) (<u>Hardy-Littlewood-Sobolev</u>). Let $1<p<2<q<\infty$, $\frac{1}{q}=\frac{1}{p}-\frac{1}{2}$.
Then
$$\|u\|_{L^q} \leq C_p \|W\|_{L^p}. \qquad \text{(HLS)}$$

2) (<u>Calderon - Zygmund</u>). For $1<p<\infty$:
$$\|\nabla u\|_{L^p} \leq C \frac{p^2}{p-1} \|W\|_{L^p}. \qquad \text{(CZ)}$$

3) Let $1\leq p<2<q\leq\infty$. Then
$$\|u\|_{L^\infty} \leq C_{pq} \|W\|_{L^p}^{\vartheta} \|W\|_{L^q}^{1-\vartheta}, \qquad \frac{1}{2}=\frac{\vartheta}{p}+\frac{1-\vartheta}{q}.$$
In particular $\|u\|_{L^\infty} \leq C(\|W\|_{L^1}+\|W\|_{L^\infty})$.

<u>Def:</u> Given $W_0\in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$, we say that $W\in L^\infty([0,T], L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2))$ is a <u>weak solution</u> of (V2d) if, for any $\varphi\in C_c^1([0,T]\times\mathbb{R}^2)$:
$$\int_{\mathbb{R}^2}\varphi(T,x)W(T,x)\,dx-\int_{\mathbb{R}^2}\varphi(0,x)W_0(x)\,dx=\int_{\mathbb{R}^2}\int_0^T\{\partial_t\varphi+u\cdot\nabla\varphi\}(t,x)W(t,x)\,dt\,dx \qquad \text{(WS)}$$
where $u=K*W\in L^\infty([0,T]\times\mathbb{R}^2)$.

By construction, any smooth solution given by Proposition 4 is a weak solution (provided W_0 is integrable). Conversely, it is not difficult to verify that, if a weak solution satisfies $W\in C^1([0,T]\times\mathbb{R}^2)$, then Eq. (V2d) is satisfied in the classical sense.

<u>Exercise:</u> Check that the circular vortex patch is a weak solution in the sense of (WS), for any $T>0$.

<u>Theorem</u> (Yudovich, 1963)

For any initial data $\omega_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, Eq. (V2d) has a unique global weak solution $\omega \in L^\infty([0,+\infty), L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$ in the sense of (WS).

$\Longrightarrow$ The Cauchy problem for the vorticity eq. (V2d) is <u>globally well-posed</u> in the space $X = L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, in the sense of weak solutions.

<u>Comments on the existence part:</u>

As before, for regularized initial data $\omega_0^\varepsilon = J_\varepsilon \omega_0$, there exists a unique global solution ω^ε of (V2d), with velocity field $u^\varepsilon = K * \omega^\varepsilon$. We have the following <u>a priori estimates</u>:

i) $\forall p \in [1,+\infty]$: Vorticity

$$\|\omega^\varepsilon(t)\|_{L^p} = \|\omega_0^\varepsilon\|_{L^p} \leq \|\omega_0\|_{L^p} \leq C \,|||\omega_0||| := \|\omega_0\|_{L^1} + \|\omega_0\|_{L^\infty}.$$

All L^p norms of the vorticity are conserved under evolution.

ii) By lemma 4, we have $\forall q \in (2,+\infty]$: Velocity

$$\|u^\varepsilon(t)\|_{L^q} \leq C_q \left(\|\omega^\varepsilon(t)\|_{L^1} + \|\omega^\varepsilon(t)\|_{L^\infty}\right) \leq C_q \,|||\omega_0|||.$$

⚠ In general $u^\varepsilon(t) \notin L^2(\mathbb{R}^2) \Longrightarrow$ infinite energy solutions!

We also have the uniform <u>log-Lipschitz estimate</u> (see Appendix):

$$|u^\varepsilon(t,x) - u^\varepsilon(t,x')| \leq C \,|||\omega_0|||\, |x-x'| \left(1 + \log_+ \frac{1}{|x-x'|}\right).$$

iii) Lagrangian flow

$$\frac{d}{dt}\varphi_t^\varepsilon(x) = u^\varepsilon(t, \varphi_t^\varepsilon(x)), \qquad \varphi_0^\varepsilon(x) = x.$$

Uniquely defined thanks to the log-Lipschitz estimate (Osgood lemma).

We have uniform Hölder estimates with <u>time-varying exponent</u>:

$$|\varphi_t^\varepsilon(x)-\varphi_t^\varepsilon(x')| \leq C\,|x-x'|^{\beta(t)} \qquad t\geq 0$$

$$|\varphi_{t_1}^\varepsilon(x)-\varphi_{t_2}^\varepsilon(x)| \leq C\,|t_1-t_2|^{\beta(t)} \qquad 0\leq t_1,t_2\leq t$$

where $\beta(t) = \exp(-C|||w_0|||\,t)$.

These estimates are enough to pass to the limit $\varepsilon\to 0$ and obtain a weak solution w of (V_2d), see Majda-Bertozzi for details.

<u>Sketch of the uniqueness part.</u>

Assume that $w_1,w_2\in L^\infty([0,T],L^1\cap L^2)$ are two weak solutions of (V_2d) with the same initial data $w_0\in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$. Assume also, for simplicity, that <u>supp w_0 is compact</u>. We take for granted that:

- $\int_{\mathbb{R}^2} w_j(t,x)\,dx = \int_{\mathbb{R}^2} w_0\,dx \qquad j=1,2,\quad 0\leq t\leq T$
- $\forall p\in[1,+\infty]:\ \|w_j(t)\|_{L^p}\leq C\,|||w_0||| \qquad j=1,2,\quad 0\leq t\leq T$

In particular, the velocities $u_j = K*w_j$ are uniformly bounded

$\Longrightarrow$ $w_j(t)$ are compactly supported $\forall t\in[0,T]$.

Finally a simple asymptotic expansion in (BS2) shows that

$$u_j(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2}\int_{\mathbb{R}^2} w_j(y)\,dy + O\Big(\frac{1}{|x|^2}\Big),\quad |x|\to+\infty$$ (here we use the compact support)

$$\Longrightarrow\ |u_1(x)-u_2(x)|\leq \frac{C}{1+|x|^2} \ \Longrightarrow\ u_1-u_2\in L^2(\mathbb{R}^2)\,!$$

The idea is to perform an energy estimate, controlling the quantity

$$E(t) = \|u_1(t)-u_2(t)\|^2_{L^2(\mathbb{R}^2)},\qquad E(0)=0.$$

The velocities u_1, u_2 satisfy (in the sense of distributions):

$$\frac{\partial u_i}{\partial t} + (u_i \cdot \nabla) u_i = -\nabla p_i, \quad i = 1, 2.$$

We denote $u = u_1 - u_2$. Then:

$$\frac{\partial u}{\partial t} + (u_1 \cdot \nabla) u + (u \cdot \nabla) u_2 = -\nabla(p_1 - p_2).$$

Testing both sides against u gives:

$$\frac{1}{2} E'(t) + \underbrace{(u, (u_1 \cdot \nabla) u)_{L^2}}_{=0} + (u, (u \cdot \nabla) u_2)_{L^2} + \underbrace{(u, \nabla(p_1 - p_2))_{L^2}}_{=0} = 0.$$

Thus, using Hölder's inequality, we find: $\quad (p \geq 2)$

$$E'(t) \leq 2 \int_{\mathbb{R}^2} |u(t,x)|^2 |\nabla u_2(t,x)| \, dx \qquad \left(\text{Hölder with } p, \tfrac{p}{p-1}\right)$$

$$\leq 2 \, \|\nabla u_2(t)\|_{L^p} \, \|u(t)\|^2_{L^{2p/p-1}} \qquad (\text{Interpolation})$$

$$\leq 2 \, \|\nabla u_2(t)\|_{L^p} \, \|u(t)\|_{L^\infty}^{2/p} \, \|u(t)\|_{L^2}^{2(1-1/p)}$$

But:

- $\|\nabla u_2(t)\|_{L^p} \overset{CZ}{\leq} C p \, \|W_2(t)\|_{L^p} \leq C p \, |||W_0|||$
- $\|u(t)\|_{L^\infty} \leq C \, \|W_1(t) - W_2(t)\|_{L^1 \cap L^\infty} \leq C \, |||W_0|||$

Finally, we have shown that, for any $p \in [2, +\infty)$:

$$E'(t) \leq M p \, E(t)^{1-1/p}, \quad 0 \leq t \leq T \qquad (\text{DI})$$

where M depends on $|||W_0|||$ (but not on p).

⚠ For a fixed $p < \infty$, the differential inequality (DI) does not imply $E = 0$!

Ex: Given $t_0 \geq 0$, the function $E(t) = \begin{cases} 0 & t \leq t_0 \\ (M(t-t_0))^p & t \geq t_0 \end{cases}$

solves the ODE $E'(t) = Mp\, E(t)^{1-1/p}$ with $E(0) = 0$.

But it is known that any solution of (DI) with $E(0)=0$ satisfies

$\| \; E(t) \leq (Mt)^p$ ("maximal solution").

So, if $0 \leq t \leq \frac{1}{2M}$ we have $E(t) \leq \left(\frac{1}{2}\right)^p \xrightarrow[p \to +\infty]{} 0$. Thus

We have shown that $E(t) = 0$, hence $W_1(t) = W_2(t)$, $\forall t \in [0, \frac{1}{2M}]$.

Repeating the argument a finite number of times, we conclude that

$W_1(t) = W_2(t)$ $\forall t \in [0,T]$. $\square$

Remarks:

- If $\Omega_0 \subset \mathbb{R}^2$ is a bounded domain with smooth boundary, the unique weak solution of 2D Euler with initial vorticity $W_0 = \mathbb{1}_{\Omega_0}$ is of the form

$$W(t,\cdot) = \mathbb{1}_{\Omega(t)}, \quad t \in \mathbb{R}$$

where $\Omega(t) \subset \mathbb{R}^2$ is again a bounded domain with smooth boundary (Chemin 1993).

$\Longrightarrow$ smooth vortex patches remain smooth (but the boundary may become wild).

- It is possible to prove existence of solutions to 2D Euler under much weaker assumptions, e.g.

$$\begin{cases} W_0 \in \mathcal{M}(\mathbb{R}^2), \; u_0 = K * W_0 \in L^2(\mathbb{R}^2) \\ + \text{ the singular part of the measure } W_0 \text{ has a definite sign} \end{cases} \qquad \text{(Delort, 1991)}$$

This class contains vortex sheets (but no uniqueness is known).

Kelvin's circulation theorem

Assume that $V(t,x)$ is a smooth solution of Euler's equation in a three-dimensional domain $\Omega \subset \mathbb{R}^3$:

$$\partial_t V + (V\cdot\nabla)V = -\nabla p, \quad \operatorname{div} V = 0.$$

(Conservative forces such as gravity may be included too.) Let $C(t)$ be a closed curve in Ω advected by the flow (= made of fluid particles). Then

$$\frac{d}{dt}\oint_{C(t)} V(t,\cdot)\cdot dl = 0. \qquad \text{(KCT)}$$

The circulation of the velocity along any curve advected by the flow is constant.

Proof: We prove (KCT) at $t=0$. Let φ_t denote the Lagrangian flow:

$$\frac{d}{dt}\varphi_t(x_0) = V(t,\varphi_t(x_0)), \qquad \varphi_0(x_0) = x_0.$$

If $\gamma : [0,2\pi] \to \Omega$ is a paramétrisation of C_0, then $\varphi_t\circ\gamma$ is a paramétrisation of $C(t)$ for $t \neq 0$. We thus have:

$$\oint_{C(t)} V(t,\cdot)\cdot dl = \int_0^{2\pi} V(t,\varphi_t(\gamma(s)))\cdot\frac{d}{ds}\varphi_t(\gamma(s))\,ds, \quad \text{hence}$$

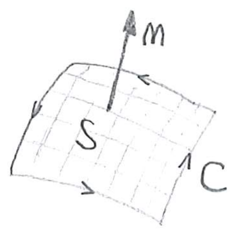
$$\frac{d}{dt}\oint_{C(t)} V\cdot dl = \int_0^{2\pi}\Big\{\frac{\partial V}{\partial t} + (V\cdot\nabla)V\Big\}(t,\varphi_t(\gamma(s)))\cdot\frac{d}{ds}\varphi_t(\gamma(s))\,ds$$

$$+ \int_0^{2\pi} V(t,\varphi_t(\gamma(s)))\cdot\frac{d}{ds}\underbrace{\frac{d}{dt}\varphi_t(\gamma(s))}_{=V(t,\varphi_t(\gamma(s)))}\,ds \quad \Longrightarrow \text{this integral vanishes!}$$

$$= \oint_{C(t)}\Big\{\frac{\partial V}{\partial t} + (V\cdot\nabla)V\Big\}\cdot dl = -\oint_{C(t)} \nabla p\cdot dl = 0. \quad \square$$

Remark: By Stokes' theorem:

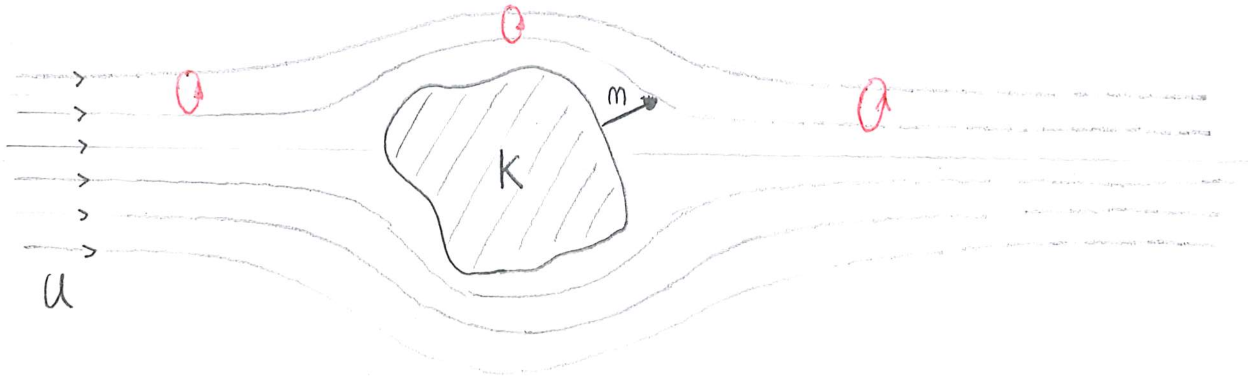
$$\oint_C v \cdot dl = \int_S (\operatorname{curl} v) \cdot n \, dS$$



where S is a piece of surface with boundary C.
In particular, for irrotational flows $\omega = \operatorname{curl} v = 0$, the circulation of v on any contractile curve is zero. Conversely, if $\oint_C v \cdot dl = 0$ for any closed curve C, then $\operatorname{curl} v = 0$.

The potential flow past an obstacle

Consider a stationary flow around a smooth, bounded obstacle $K \subset \mathbb{R}^3$;



In $\Omega = \mathbb{R}^3 \setminus K$, the velocity v satisfies:

$$\begin{cases} (v \cdot \nabla) v = -\nabla p, \quad \operatorname{div} v = 0 & \text{in } \Omega \\ v \cdot n = 0 & \text{on } \partial\Omega \\ v(x) \xrightarrow[|x| \to \infty]{} u \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & \end{cases} \qquad \text{(SE)}$$

Remark: This is equivalent (by Galilean invariance) to a solid of shape K moving with constant speed $-u \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ in a fluid that is at rest at infinity. The latter point of view may be more physical.

How to construct a solution of (SE)?

1°) We must have $\underline{\omega = \text{curl}\, V = 0}$ everywhere in Ω.
Indeed, by Kelvin's theorem, the circulation of V along any closed curve C can be computed by "pushing the curve at $x_1 = -\infty$", where the circulation is zero because $V = u\binom{1}{0}$.

2°) Assuming that $\Omega = \mathbb{R}^3 \setminus K$ is simply connected, it follows that V is the gradient of a scalar function. $\Rightarrow$ <u>potential flow.</u>

We choose to write

$$\| \quad V(x) = u e_1 + \nabla\varphi(x), \qquad p(x) = p_0 - \frac{1}{2}|V(x)|^2$$

where

$$\begin{cases} \Delta\varphi(x) = 0 & x \in \Omega \\ (\nabla\varphi(x) + u e_1)\cdot n = 0 & x \in \partial\Omega \\ \varphi(x) \xrightarrow[|x|\to\infty]{} 0 & \end{cases} \qquad \text{(SE')}$$

We are thus lead to a classical problem in <u>potential theory</u>: solve the Laplace equation in the exterior domain $\Omega = \mathbb{R}^3 \setminus K$ with (inhomogeneous) Neumann condition on $\partial\Omega$.

$\Rightarrow$ (SE') has a unique solution, which satisfies:

$$\| \quad \varphi(x) = O\left(\frac{1}{|x|^2}\right), \quad |\nabla\varphi(x)| = O\left(\frac{1}{|x|^3}\right), \quad |x| \to \infty.$$

⚠ In the potential flow, the velocity V is <u>nonzero on $\partial\Omega$</u>: the fluid flows along the boundary without friction!
$\neq$ viscous fluids!

The d'Alembert theorem

Let F be the force exerted by the fluid on the obstacle K:

$$F = \int_{\partial\Omega} \sigma n \, ds = -\int_{\partial\Omega} p n \, ds = 0 \;!!$$

Indeed, let us compute the k^{th} component F_k:

$$F_k = -\int_{\partial\Omega} p n_k \, ds = -\int_{\partial\Omega} (p e_k)\cdot n \, ds = \int_{\Omega} \mathrm{div}(p e_k)\, dx = \int_{\Omega} \partial_k p \, dx.$$

by Gauss' theorem (no contribution at infinity because p converges suff. fast to a constant as $|x| \to \infty$).

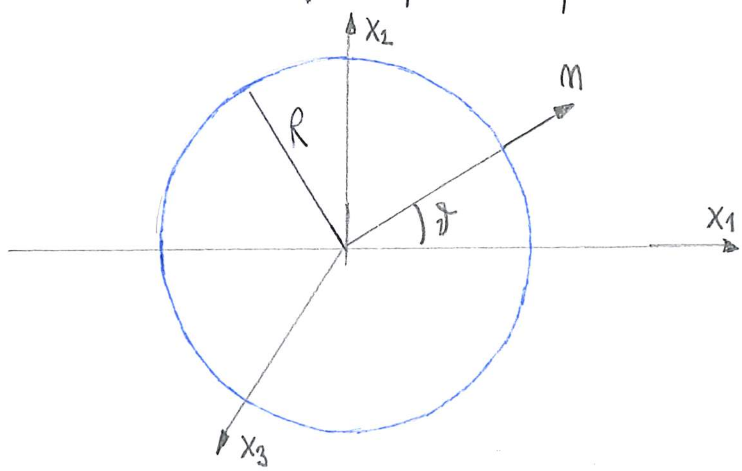
But $\partial_k p = -(V\cdot\nabla)V_k = -V_j \partial_j V_k = -\partial_j (V_j V_k)$, hence

$$F_k = -\int_{\Omega} \partial_j (V_j V_k)\, dx \underset{\text{Gauss}}{=} \int_{\partial\Omega} V_k \underbrace{(V\cdot n)}_{=0 \text{ on } \partial\Omega} ds = 0.$$

So there is no drag and no lift for a potential flow around an obstacle!

$\Rightarrow$ necessity to consider viscous fluids to account for observed phenomena.

Example: Potential flow past a sphere



$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} r\cos\vartheta \\ r\sin\vartheta\cos\alpha \\ r\sin\vartheta\sin\alpha \end{pmatrix}$$

In spherical coordinates (around x_1), the problem (SE') becomes:

$$\begin{cases} \partial_r^2 \varphi + \frac{2}{r}\partial_r \varphi + \frac{1}{r^2 \sin\vartheta}\partial_\vartheta(\sin\vartheta\,\partial_\vartheta \varphi) + \frac{1}{r^2\sin^2\vartheta}\partial_\alpha^2\varphi = 0, & r > R \\ \partial_r \varphi + u\cos\vartheta = 0 & r = R \\ \varphi \longrightarrow 0 \text{ as } r \to +\infty. \end{cases}$$

Ansatz: $\varphi(r,\vartheta,\alpha) = \psi(r)\cos\vartheta$.

- $\psi''(r) + \frac{2}{r}\psi'(r) - \frac{2}{r^2}\psi(r) = 0 \quad \Rightarrow \quad \psi(r) = C_1 r + \frac{C_2}{r^2}$.
- $\psi(r) \xrightarrow[r\to+\infty]{} 0 \quad \Rightarrow \quad C_1 = 0$
- $\psi'(R) + u = \frac{-2C_2}{R^3} + u = 0 \quad \Rightarrow \quad C_2 = \frac{uR^3}{2}$.

Finally $\varphi = \frac{uR^3}{2r^2}\cos\vartheta = \frac{uR^3 x_1}{2|x|^3}$.

Differentiating with respect to x_1, x_2, x_3, we obtain the expressions:

$$\begin{cases} v_1(x) = u + \frac{1}{2}uR^3\left(\frac{1}{|x|^3} - \frac{3x_1^2}{|x|^5}\right) \\ v_2(x) = -\frac{3}{2}uR^3\,\frac{x_1x_2}{|x|^5} \\ v_3(x) = -\frac{3}{2}uR^3\,\frac{x_1x_3}{|x|^5} \end{cases}$$

The velocity <u>vanishes</u> only for $x_2 = x_3 = 0$, $x_1 = \pm R$ (stagnation points)

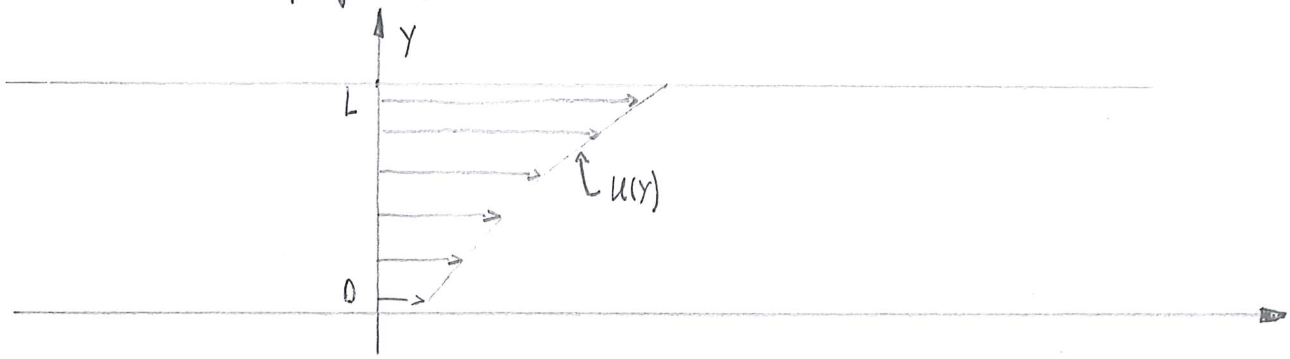
It is <u>maximal</u> when $x_1 = 0$, $|x| = R$:

$$v = \frac{3u}{2}e_1.$$

The pressure is everywhere positive if $p_0 > \frac{9}{8}u^2$.

Stability of shear flows

We consider a perfect fluid in the two-dimensional strip $\Omega = \mathbb{R} \times [0,L]$:



A shear flow (or parallel flow) is a stationary solution of Euler of the form

$$V = \begin{pmatrix} u(y) \\ 0 \end{pmatrix}, \qquad p = 0$$

where $u : [0,L] \longrightarrow \mathbb{R}$ is the shear velocity profile, which can be arbitrary. To investigate the stability of this flow, we consider perturbations of the form:

$$V = \begin{pmatrix} u(y) \\ 0 \end{pmatrix} + \begin{pmatrix} \tilde{V}_1 \\ \tilde{V}_2 \end{pmatrix}, \qquad \partial_x \tilde{V}_1 + \partial_y \tilde{V}_2 = 0.$$

The evolution equation for $\tilde{V}$ is:

nonlinear term

$$\partial_t \tilde{V} + u(y)\partial_x \tilde{V} + u'(y)\tilde{V}_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (\tilde{V}\cdot\nabla)\tilde{V} = -\nabla\tilde{p}, \quad \operatorname{div} \tilde{V} = 0.$$

In what follows, we drop the tildes for simplicity, and we consider the linearized equations where the nonlinearity is neglected:

$$\begin{cases} \partial_t V_1 + u(y)\partial_x V_1 + u'(y)V_2 = -\partial_1 p \\ \partial_t V_2 + u(y)\partial_x V_2 = -\partial_2 p \end{cases} \qquad \partial_1 V_1 + \partial_2 V_2 = 0 \qquad \text{(LE)}$$

Boundary conditions: $V_2(x,0) = V_2(x,L) = 0.$

Equations (LE) are invariant under translations in the x-direction.
Taking Fourier transform w.r.to $x \in \mathbb{R}$, so that $\partial_x \leftarrow ik$, we find:

$$\begin{cases} \partial_t V_1 + ik\,U(y) V_1 + U'(y) V_2 = -ikp \\ \partial_t V_2 + ik\,U(y) V_2 = -\partial_2 p \end{cases} \qquad ikV_1 + \partial_2 V_2 = 0. \qquad (LE')$$

Now $k \in \mathbb{R}$ is fixed, and V_1, V_2, p depend only on $y \in [0, L]$.
The pressure is obtained by solving the ODE:

$$\begin{cases} (k^2 - \partial_2^2) p = 2ik\,U'(y) V_2 & y \in [0,L] \\ \partial_2 p = 0 & y = 0 \text{ or } L \end{cases} \qquad (*)$$

System (LE') can be put in the abstract form:

$$\partial_t V + L_k V = 0, \qquad L_k = A_k + B_k$$

where A_k, B_k are the linear operators defined by

$$A_k V = ik\,U(y) V - U'(y) \begin{pmatrix} V_2 \\ 0 \end{pmatrix}, \qquad B_k V = \begin{pmatrix} 2U'(y) V_2 + ikp \\ \partial_2 p \end{pmatrix}.$$

We introduce the function space

$$X_k = \left\{ V \in L^2([0,L])^2 \,;\; ikV_1 + \partial_2 V_2 = 0,\; V_2(0) = V_2(L) = 0 \right\}.$$

<u>Proposition:</u>

- The operator A_k is the generator of a <u>group of bounded linear operators</u> in X_k, and $\sigma(A_k) \subset i\mathbb{R}$.
- The operator B_k is <u>compact</u> in X_k.

<u>Corollary:</u> $L_k = A_k + B_k$ generates a group of bounded linear operators.

<u>Proof:</u> • The equation $\partial_t V + A_k V = 0$ can be solved explicitly:

$$\begin{pmatrix} V_1(t,y) \\ V_2(t,y) \end{pmatrix} = e^{-iktU(y)} \begin{pmatrix} V_1(0,y) + tU'(y)V_2(0,y) \\ V_2(0,y) \end{pmatrix}.$$

Moreover, if $\lambda \in \mathbb{C} \setminus i\mathbb{R}$, the resolvent equation $(\lambda + A_k)V = f$ has a unique solution:

$$V_2 = \frac{f_2}{\lambda + ikU(y)}, \quad V_1 = \frac{f_1}{\lambda + ikU(y)} + \frac{U'(y) f_2}{(\lambda + ikU(y))^2}.$$

This proves that $\sigma(A_k) \subset i\mathbb{R}$.

• From $(*)$ we obtain by a standard energy estimate

$$k^2 \|p\|_{L^2}^2 + \|\partial_2 p\|_{L^2}^2 \leq 2|k| \|U'\|_{L^\infty} \|V\|_{L^2} \|p\|_{L^2}$$

$$\Longrightarrow |k| \|p\|_{L^2} + \|\partial_2 p\|_{L^2} \leq C \|V\|_{L^2} \Longrightarrow B_k \text{ is bounded in } X_k.$$

Moreover $\|\partial_2^2 p\|_{L^2} \leq k^2 \|p\|_{L^2} + C|k| \|V\|_{L^2}$, and $\|\partial_2 V_2\| \leq C|k| \|V\|_{L^2}$.

By Rellich-Kondrachov, we deduce that B_k is compact in X_k. □

By a theorem of Weyl, the spectrum of the operator L_k outside the imaginary axis consists of <u>discrete eigenvalues with finite multiplicity</u> (which can accumulate only on the axis). If such eigenvalues exist, the shear flow $\begin{pmatrix} u(y) \\ 0 \end{pmatrix}$ is exponentially unstable at the linear level: there exist solutions of (LE') which grow exponentially as $t \to \pm\infty$.

<u>Theorem</u> (Rayleigh, 1880) If U'' does not change sign in $[0,L]$, then $\sigma(L_k) \subset i\mathbb{R}$ $\forall k \in \mathbb{R}$.

(Rayleigh's inflexion point criterion)

Proof: Assume that $(\lambda + L_k)V = 0$ for some $\lambda \in \mathbb{C} \setminus i\mathbb{R}$ and some nonzero $V \in X_k$. Denoting $\gamma(y) = \lambda + ikU(y) \neq 0$, we have:

$$\begin{matrix} -\partial_2 \\ ik \end{matrix} \quad \begin{cases} \gamma(y) V_1 + U'(y) V_2 = -ikp \\ \gamma(y) V_2 = -\partial_2 p \end{cases}, \qquad ikV_1 + \partial_2 V_2 = 0.$$

Eliminating the pressure, we find:

$$0 = \gamma(y)(-\partial_2 V_1 + ikV_2) - U'(y)\underbrace{(ikV_1 + \partial_2 V_2)}_{=0} - U''(y) V_2$$

$$\Longrightarrow \quad \partial_2 V_1 - ikV_2 + \frac{U''(y)}{\gamma(y)} V_2 = 0.$$

Multiplying by ik and using the divergence-free condition, we arrive at the Rayleigh equation:

$$-\partial_2^2 V_2 + k^2 V_2 + \frac{ikU''(y)}{\gamma(y)} V_2 = 0, \qquad V_2(0) = V_2(L) = 0. \qquad \text{(RE)}$$

By assumption, this equation has a nontrivial solution V_2. Multiplying by $\overline{V_2}$, integrating over $[0,L]$, we find:

$$\int_0^L \left(|\partial_2 V_2|^2 + k^2 |V_2|^2\right) dy + ik \int_0^L \frac{U''(y)}{\gamma(y)} |V_2|^2 dy = 0.$$

Taking the imaginary part, we deduce:

$$k \, \mathrm{Re}(\lambda) \int_0^L \frac{U''(y)}{|\gamma(y)|^2} |V_2|^2 dy = 0.$$

- if $U'' \equiv 0$ or $k = 0$, we had a contradiction from (RE).
- otherwise, if U'' does not change sign, the integral is nonzero, hence we get a contradiction since $\mathrm{Re}(\lambda) \neq 0$. $\square$

Appendix: Proof of the lemmas

Lemma 1 (calculus inequalities)

The proof uses Leibniz's formula and Gagliardo-Nirenberg interpolation inequalities. For simplicity we consider the 1d case (the proof is similar in higher dimensions). We admit the following result:

If $u \in L^\infty(\mathbb{R}) \cap H^m(\mathbb{R})$, then

$$\| \partial^k u \|_{L^{2m/k}} \leq C \| u \|_{L^\infty}^{1-k/m} \| \partial^m u \|_{L^2}^{k/m}, \qquad 0 \leq k \leq m. \qquad \text{(GN)}$$

1) $\underline{m=1}$: $\|(uv)'\|_{L^2} \leq \|u'v\|_{L^2} + \|uv'\|_{L^2} \leq \|u\|_{L^\infty}\|v'\|_{L^2} + \|u'\|_{L^2}\|v\|_{L^\infty}$ ✓

$\underline{m=2}$: $\|(uv)''\|_{L^2} \leq \|uv''\|_{L^2} + 2\|u'v'\|_{L^2} + \|u''v\|_{L^2}$

But: $\|u'v'\|_{L^2} \leq \|u'\|_{L^4}\|v'\|_{L^4}$ and by GN:

$$\|u'\|_{L^4} \leq C \|u\|_{L^\infty}^{1/2} \|u''\|_{L^2}^{1/2}, \quad \|v'\|_{L^4} \leq C \|v\|_{L^\infty}^{1/2} \|v''\|_{L^2}^{1/2}$$

$$\Rightarrow \|(uv)''\|_{L^2} \leq \|u\|_{L^\infty}\|v''\|_{L^2} + \|u''\|_{L^2}\|v\|_{L^\infty} + C\left(\|u\|_{L^\infty}\|v''\|_{L^2}\right)^{1/2}\left(\|u''\|_{L^2}\|v\|_{L^\infty}\right)^{1/2}$$

$$\leq C\left(\|u\|_{L^\infty}\|v''\|_{L^2} + \|u''\|_{L^2}\|v\|_{L^\infty}\right).$$

$\underline{m=3}$: $\|(uv)'''\|_{L^2} \leq \|uv'''\|_{L^2} + 3\|u'v''\|_{L^2} + 3\|u''v'\|_{L^2} + \|u'''v\|_{L^2}$

But:
$$\|u'v''\|_{L^2} \leq \|u'\|_{L^6}\|v''\|_{L^3}$$
$$\leq C \|u\|_{L^\infty}^{2/3} \|u'''\|_{L^2}^{1/3} \|v\|_{L^\infty}^{1/3} \|v'''\|_{L^2}^{2/3}$$
$$\leq C\left(\|u\|_{L^\infty}\|v'''\|_{L^2} + \|u'''\|_{L^2}\|v\|_{L^\infty}\right) \text{ etc...}$$

2) $\partial^m(uv) - u\partial^m v$ contains the same terms as $\partial^{m-1}(u'v)$, hence

$$\|\partial^m(uv) - u\partial^m v\|_{L^2} \leq C\left(\|u'\|_{L^\infty}\|v\|_{H^{m-1}} + \|u\|_{H^m}\|v\|_{L^\infty}\right). \quad \square$$

Lemma 3 To reconstruct v from $\omega = \operatorname{curl} v$, we have to solve the linear elliptic system

$$\operatorname{div} v = 0, \quad \operatorname{curl} v = \omega \quad \text{in } \mathbb{R}^3.$$

We have $\operatorname{curl}\omega = \operatorname{curl}(\operatorname{curl} v) = \nabla \operatorname{div} v - \Delta v = -\Delta v$, hence formally

$$v(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\operatorname{curl}\omega(y)}{|x-y|}\,dy, \quad x \in \mathbb{R}^3.$$

Integrating by parts, we arrive at the Biot-Savart formula:

$$v(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(x-y)}{|x-y|^3} \wedge \omega(y)\,dy, \quad x \in \mathbb{R}^3. \qquad (BS3)$$

In the two-dimensional case, we obtain similarly:

$$u(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\, w(y)\,dy, \qquad x \in \mathbb{R}^2 \qquad (BS2)$$

The Biot-Savart kernel in $\mathbb{R}^d$ is homogeneous of degree $1-d$. If we differentiate formally (BS3) w.r.to x, we obtain a representation formula involving a singular integral kernel:

$$\nabla v(x) = \int_{\mathbb{R}^3} K(x,y)\,\omega(y)\,dy, \qquad x \in \mathbb{R}^3$$

$\uparrow$ homogeneous of degree -3!

The integral has to be understood in the sense of Cauchy's principal value. Calderon-Zygmund's theory provides L^p bounds of the form:

$$\|\nabla v\|_{L^p} \leq C\,\|\omega\|_{L^p}, \qquad 1 < p < \infty \qquad \text{(obvious for } p=2 \text{ in Fourier)}$$

Unfortunately, the case $p = +\infty$ is excluded $\Rightarrow$ we need a different argument.

We give here an elementary proof of (Ell) when $d=2$ (the case $d=3$ is similar). We assume that $u \in H^m_\sigma(\mathbb{R}^2)$ for some $m>2$, so that $w = \partial_1 u_2 - \partial_2 u_1 \in H^{m-1}(\mathbb{R}^2) \hookrightarrow C^\gamma(\mathbb{R}^2)$ for some $\gamma > 0$.

Fix $R > \varepsilon > 0$, and take $x, x' \in \mathbb{R}^2$ such that $|x-x'| \leq \varepsilon/2$. We write

$$u(x) - u(x') = \frac{1}{2\pi}\int_{\mathbb{R}^2} \left\{ \frac{(x-y)^\perp}{|x-y|^2} - \frac{(x'-y)^\perp}{|x'-y|^2} \right\} w(y)\,dy$$

$$= \frac{1}{2\pi}\int_{|y-x|\leq\varepsilon} \{\dots\} w(y)\,dy + \frac{1}{2\pi}\int_{\varepsilon<|y-x|<R} \{\dots\} w(y)\,dy + \frac{1}{2\pi}\int_{|y-x|\geq R} \{\dots\} w(y)\,dy$$

$$= \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

N.B. $\left|\frac{x}{|x|^2} - \frac{y}{|y|^2}\right|^2 = \frac{|x-y|^2}{|x|^2|y|^2}$!

We use the elementary estimate

$$\left| \frac{(x-y)^\perp}{|x-y|^2} - \frac{(x'-y)^\perp}{|x'-y|^2} \right| \leq \frac{|x-x'|}{|x-y||x'-y|} \qquad y \neq x,\ y \neq x'.$$

In regions II and III, we have $|y-x'| \geq |y-x| - |x-x'| \geq \frac{1}{2}|y-x|$, hence:

$$|\mathrm{II}| \leq C \int_{\varepsilon<|y-x|<R} \frac{|x-x'|}{|x-y|^2} |w(y)|\,dy \leq C|x-x'|\,\|w\|_{L^\infty} \log\frac{R}{\varepsilon},$$

$$|\mathrm{III}| \leq C \int_{|y-x|\geq R} \frac{|x-x'|}{|x-y|^2} |w(y)|\,dy \leq C|x-x'|\,\|w\|_{L^1} \frac{1}{R}.$$

We decompose $\mathrm{I} = \mathrm{I}_a + \mathrm{I}_b$ where

$$\mathrm{I}_a = \frac{1}{2\pi}\int_{|y-x|\leq\varepsilon} \left\{ \frac{(x-y)^\perp}{|x-y|^2} - \frac{(x'-y)^\perp}{|x'-y|^2} \right\} (w(y) - w(x))\,dy$$

$$|\mathrm{I}_a| \leq C \int_{|y-x|\leq\varepsilon} \frac{|x-x'|}{|x-y||x'-y|} \|w\|_{C^\gamma} |x-y|^\gamma\,dy \qquad \text{(max. reached when } x'=x\text{)}$$

$$\leq C \int_{|y-x|\leq\varepsilon} \frac{|x-x'|}{|x-y|^2} \|w\|_{C^\gamma} |x-y|^\gamma\,dy \leq C|x-x'|\,\|w\|_{C^\gamma}\varepsilon^\gamma.$$

$$I_b = \frac{w(x)}{2\pi} \int_{|y-x|\leq \varepsilon} \left\{ \frac{(x-y)^\perp}{|x-y|^2} - \frac{(x'-y)^\perp}{|x'-y|^2} \right\} dy = \frac{w(x)}{2} (x-x')^\perp$$

direct calculation!

$$|I_b| \leq \frac{1}{2} \|w\|_{L^\infty} |x-x'| .$$

Summarizing, we have shown that, when $|x-x'| \leq \varepsilon/2$:

$$|u(x)-u(x')| \leq C|x-x'| \left\{ \|w\|_{L^\infty}\left(1+\log\frac{R}{\varepsilon}\right) + \|w\|_{C^\gamma}\varepsilon^\gamma + \frac{1}{R}\|w\|_{L^2} \right\}.$$

In particular, we have (by Rademacher's thm)

$$\|\nabla u\|_{L^\infty} \leq C\left\{ \|w\|_{L^\infty}\left(1+\log\frac{R}{\varepsilon}\right) + \|w\|_{C^\gamma}\varepsilon^\gamma + \frac{1}{R}\|w\|_{L^2} \right\}$$

whenever $0<\varepsilon<R$. Finally, we choose

$$R = \frac{\|w\|_{L^2}}{\|w\|_{L^\infty}} , \quad \varepsilon = \min\left(\frac{R}{2} , \left(\frac{\|w\|_{L^\infty}}{\|w\|_{C^\gamma}}\right)^{1/\gamma} \right).$$

Since $\|w\|_{L^\infty} \leq C\|u\|_{H^m}$ and $\|w\|_{C^\gamma} \leq C\|u\|_{H^m}$, we thus obtain

$$\|\nabla u\|_{L^\infty} \leq C\|w\|_{L^\infty}\left(1+\log_+\|u\|_{H^m}\right). \qquad \square$$

Remark: If one only have $w \in L^\infty(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$, the velocity field given by (BS2) is not Lipschitz, but log-Lipschitz. Modifying the estimate of I above, we have:

$$|u(x)-u(x')| \leq C|x-x'| \left\{ \|w\|_{L^\infty}\left(1+\log\frac{R}{\varepsilon} + \log\frac{\varepsilon}{|x-x'|}\right) + \frac{1}{R}\|w\|_{L^2} \right\}$$

whenever $|x-x'| \leq \varepsilon/2$ and $0<\varepsilon\leq R$.