Mathematical fluid mechanics

An introduction to the dynamics of incompressible viscous or perfect fluids

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Chapter I: Derivation of the fundamental equations.

A *fluid* (= liquid, gas, or plasma) is a continuous medium which is totally *deformable*: it does not oppose any resistance to a shear force, at least in the adiabatic regime. It consists of molecules or ions which are either *densely packed* (liquids) or *widely separated* (gases, plasmas).

There is currently no rigorous derivation of the eq. of motion for fluids, except in the case of dilute gases or plasmas. The considerations below are thus *phenomenological*.

Fluid particles:

In the spirit of thermodynamics, we do not try to describe the motion of all elementary particles in the fluid (molecules, atoms, or ions). Our interest is in *averaged quantities* such as the fluid *density* or *mean velocity*.

A fluid particle is an idealized portion of the fluid which is:

- large enough to contain a very large number of elementary particles ($\Rightarrow$ it makes sense to speak about averaged quantities)
- very small compared to the characteristic (macroscopic) scale of the flow

Ex: 1 $(\mu m)^3$ of water in a basin containing 10 L.

In what follows, our goal is to write an equation for the motion of a fluid particle, following the laws of classical mechanics.

A fluid particle has a well defined *mass* and *velocity*, and it undergoes forces, but unlike point particles in classical mechanics it has *nonzero extension* and it can get *deformed* under evolution.

<u>Remark:</u> Fluid particles are introduced to distinguish the average velocity with the chaotic motion of elementary particles. But since they are very small compared to the macroscopic scale of the flow, we invoke a <u>continuum limit</u> to make a continuous description of the fluid.

<u>Lagrange's description:</u>

We label the fluid particles, for instance by their initial position x_0.
We then define

$X(t, x_0)$ = position at time $t \geq 0$ of the fluid particle that was at position x_0 when $t = 0$.

By definition, we then have:

$u(t, x_0) = \frac{d}{dt} X(t, x_0)$ = speed at time $t \geq 0$ of the fluid particle labeled with x_0.

This point of view is complicated in practice (how to keep track of the particles' initial position?), but it is useful in particular situations (such as free-surface flows, or non-homogeneous flows with rough density).

<u>Euler's description:</u>

We do not give any permanent labels to the fluid particles, but we identify them at each time t by their current position x. We thus define

$v(t, x)$ = velocity at time t of the fluid particle whose position at that time is x

$\rho(t, x)$ = density of the fluid at position x at time t.

(and similarly for other quantities: pressure, momentum, energy...)

Rem: If the Eulerian velocity is known, one can reconstruct the Lagrangian flow map by solving the ODE:

$$\begin{cases} \frac{d}{dt} X(t, X_0) = V(t, X(t, X_0)) & t \geq 0 \\ X(0, X_0) = X_0 \end{cases}$$

Conversely, the Eulerian velocity can be formulated as

$$V(t,x) = \frac{dX}{dt}(t, X_0(t,x))$$

where $X_0(t,x)$ is the so-called "back to labels" map.

Material derivative ("dérivée particulaire")

Assume that $X(t)$ is the trajectory of a fluid particle, so that $X'(t) = V(t, X(t))$. If $f(t,x)$ is a (smooth) function, we observe that:

$$\frac{d}{dt} f(t, X(t)) = \frac{\partial f}{\partial t}(t, X(t)) + X'(t) \cdot \nabla f(t, X(t))$$

$$= \frac{\partial f}{\partial t}(t, X(t)) + V(t, X(t)) \cdot \nabla f(t, X(t)).$$

This leads to the definition:

$$\frac{Df}{Dt}(t,x) = \frac{\partial f}{\partial t}(t,x) + \underbrace{V(t,x)}_{\text{Eulerian velocity}} \cdot \nabla f(t,x).$$

This is the Lie derivative of f along the flow of V.

Example: acceleration of a fluid particle

$$a(t,x) = \frac{DV}{Dt}(t,x) = \frac{\partial V}{\partial t}(t,x) + (V(t,x) \cdot \nabla) V(t,x).$$

In coordinates: $a_i = \frac{\partial v_i}{\partial t} + V_j \partial_j V_i$ (summation over j understood)

Integral quantity advected by the fluid

Let us denote by $X = \varphi_t(X_0) \equiv X(t, X_0)$ the Lagrangian flow map. For each $t \geq 0$, φ_t is a local diffeomorphism in $\mathbb{R}^3$. Given a smooth subset Ω_0 of the fluid domain, we denote $\Omega_t = \varphi_t(\Omega_0)$ (the image of Ω_0 under φ_t) for $t \geq 0$.

Lemma:

$$\frac{d}{dt} \int_{\Omega_t} f(t,x)\,dx = \int_{\Omega_t} \left(\frac{\partial f}{\partial t} + \operatorname{div}(f v) \right)(t,x)\,dx.$$

Proof: We perform the change of variables $x = \varphi_t(y)$ and obtain

determinant

$$\int_{\Omega_t} f(t,x)\,dx = \int_{\Omega_0} f(t, \varphi_t(y))\, |J\varphi_t(y)|\,dy$$

where $J\varphi_t$ is the Jacobian matrix of the map φ_t. By definition we have:

$$\frac{d}{dt} f(t, \varphi_t(y)) = \frac{Df}{Dt}(t, \varphi_t(y)) = \left(\frac{\partial f}{\partial t} + v \cdot \nabla f \right)(t, \varphi_t(y)). \qquad (1)$$

On the other hand, we claim that

$$\frac{d}{dt} |J\varphi_t(y)| = (\operatorname{div} v)(t, \varphi_t(y))\, |J\varphi_t(y)|. \qquad (2)$$

Let us prove (2) for $t=0$ (the general case follows). We have

because $\varphi_{t+s} = \varphi_t \circ \varphi_s$.

$$\varphi_t(y) = y + t\, v(0,y) + O(t^2), \text{ hence}$$

$$J\varphi_t(y) = \mathbb{1} + t \left(\frac{\partial v_i(0,y)}{\partial y_j} \right)_{ij} + O(t^2), \text{ hence}$$

$$|J\varphi_t(y)| = 1 + t \operatorname{Tr} \left(\frac{\partial v_i(0,y)}{\partial y_j} \right) + O(t^2)$$

$$= 1 + t \operatorname{div} v(0,y) + O(t^2) \implies (2) \text{ for } t=0.$$

We thus find:

$$\frac{d}{dt}\int_{\Omega_t} f(t,x)\,dx = \int_{\Omega_0}\left(\frac{Df}{Dt} + f\,\mathrm{div}\,v\right)(t,\varphi_t(y))\,|J\varphi_t(y)|\,dy$$

$$\overset{x=\varphi_t(y)}{=} \int_{\Omega_t}\left(\frac{Df}{Dt} + f\,\mathrm{div}\,v\right)(t,x)\,dx$$

$$= \int_{\Omega_t}\left(\frac{\partial f}{\partial t} + \mathrm{div}(fv)\right)(t,x)\,dx. \quad \square$$

Conservation of mass : Continuity equation.

Let $\rho(t,x)$ denote the density (of mass) in the fluid. By conservation of mass, for any domain Ω_t advected by the fluid, we have

$$0 = \frac{d}{dt}\int_{\Omega_t}\rho(t,x)\,dx = \int_{\Omega_t}\left(\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho v)\right)(t,x)\,dx.$$

Thus we must have:

(C) $$\boxed{\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho v) = 0}$$ everywhere inside the fluid.

Particular case: incompressible flows

Suppose that $\rho = $ const. everywhere in the fluid (incompressible fluid) or more generally that the flow is "isochorous" ($\frac{D\rho}{Dt} = 0$ everywhere). Then the Eulerian velocity is divergence-free:

$$\mathrm{div}(v) = \sum_{i=1}^{3}\frac{\partial v_i}{\partial x_i} = 0.$$

All real fluids are (at least slightly) compressible, but (especially in liquids) the incompressible hypothesis is often a good approximation, which allows to simplify the eq. of motion.

Deformation of a fluid particle

Let us consider the velocity of the fluid near a given point (which we assume to be $X=0$), at a given time (omitted in the notation). Using a Taylor expansion, we find:

$$V_i(x) = V_i(0) + \partial_j V_i(0) X_j + O(|x|^2) \qquad \text{(summation over } j \text{ understood)}$$

$$= V_i(0) + \underbrace{\tfrac{1}{2}(\partial_j V_i + \partial_i V_j)(0)}_{S_{ij}} X_j + \underbrace{\tfrac{1}{2}(\partial_j V_i - \partial_i V_j)(0)}_{a_{ij}} X_j + O(|x|^2)$$

- $V_i(0)$ indep. of $X \Rightarrow$ rigid translation
- S_{ij} = symmetric part of ∇V. In a suitable basis:

$$S_{ij} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad \text{with } \lambda_1+\lambda_2+\lambda_3 = \mathrm{Tr}(S) = \mathrm{div}\, V.$$

$\lambda_1, \lambda_2, \lambda_3$ are the local rates of strain in the 3 directions of the basis

S_{ij} is called the <u>strain rate tensor</u> ("tenseur des vitesses de déformation")

- a_{ij} = anti-symmetric part of ∇V

$$(a_{ij} X_j) = \frac{1}{2}\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \frac{1}{2}\,\omega \wedge X, \qquad \omega = \mathrm{curl}\, V.$$

"vorticity"

ω describes the local rotation of the fluid particle.

To this order, the deformation of the fluid particle is described by the rate of strain tensor $S = \frac{1}{2}(\nabla V + \nabla V^t)$. The motion is isochorous if $0 = \mathrm{Tr}(S) = \mathrm{div}(v)$.

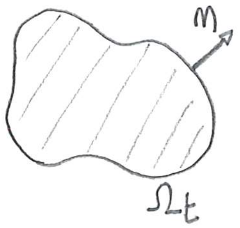
The momentum balance

Consider again a smooth subset Ω_t advected by the fluid.
By Newton's equation, we have:

$$\frac{d}{dt}\int_{\Omega_t} \rho(t,x)V(t,x)\,dx = \int_{\Omega_t} F(t,x)\,dx + \int_{\partial\Omega_t} \sigma(t,x)\cdot n\,dS$$

(variation of momentum) (volume force) (surface force)

- F is the sum of all external forces that act everywhere in the fluid, e.g. $F = -g e_z$ (gravitational force).
- The last term describes the forces exerted by the rest of the fluid on the "fluid particle" contained in Ω_t.



$$dF = \sigma\cdot n\,dS$$

n = outward unit normal on Ω_t
σ = <u>Cauchy stress tensor</u> of the fluid ("tenseur des contraintes")

From the Lemma p.4:

$$\frac{d}{dt}\int_{\Omega_t} \rho V_i\,dx = \int_{\Omega_t}\left(\frac{\partial}{\partial t}(\rho V_i) + \frac{\partial}{\partial x_j}(\rho V_i V_j)\right)dx, \qquad i = 1,2,3.$$

Integrating by parts (Stokes' theorem):

$$\int_{\partial\Omega_t} \sigma_{ij} n_j\,dS = \int_{\Omega_t} \partial_j(\sigma_{ij})\,dx.$$

Thus Newton's equation becomes:

$$\int_{\Omega_t} \left(\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) \right) dx = \int_{\Omega_t} \left(F_i + \partial_j \sigma_{ij} \right) dx .$$

Since the domain Ω_t was arbitrary, we arrive at the momentum equation

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) = F_i + \partial_j \sigma_{ij} .$$

Taking into account the continuity eq., this is equivalent to

$$\rho \left(\partial_t v_i + v_j \partial_j v_i \right) = F_i + \partial_j \sigma_{ij} ,$$

or returning to vector notations:

$$\boxed{\rho \left(\partial_t v + (v \cdot \nabla) v \right) = F + \operatorname{div} \sigma} \qquad (M)$$

It remains to specify the expression of σ.

The Cauchy stress tensor for Newtonian fluids

At equilibrium ($v = 0$), isotropy indicates that

$$\sigma = -p \mathbb{1} , \qquad p = \text{scalar function} = \text{pressure in the fluid.}$$

When $v \neq 0$, we assume (Newton's hypothesis) that σ is an affine function of the rate of strain tensor $S = \frac{1}{2}(\nabla v + \nabla v^t)$.

Rem: . In viscoelastic fluids, σ also depends on the strain tensor!

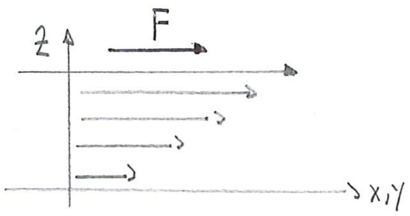
. In many fluids, the stress depends on the strain rate in a nonlinear way $\Rightarrow$ non-Newtonian behavior.

Ex: honey, maiz flour + water, foam, snow, granular media, ...

Under Newton's hypothesis, general considerations (homogeneity + isotropy) leads to the general form:

$$\sigma_{ij} = \underbrace{-p\,\delta_{ij}}_{\text{pressure}} + \underbrace{\mu\left(\partial_i v_j + \partial_j v_i - \tfrac{2}{3}(\partial_k v_k)\,\delta_{ij}\right)}_{\text{shear (traceless)}} + \underbrace{\xi\,(\partial_k v_k)\,\delta_{ij}}_{\text{dilation}}$$

μ = <u>shear viscosity</u> (as in the Couette experiment)



$$\frac{F}{S} = \mu\,\frac{\partial v}{\partial z}$$

ξ = <u>second viscosity</u> (linked to compression, difficult to measure)

In real fluids μ, ξ may depend on the <u>density</u> ρ and the <u>temperature</u>.

For simplicity we assume here that μ, ξ are <u>constants</u>. Then

$$\operatorname{div}\sigma = -\nabla p + \mu\,\Delta v + (\mu/3 + \xi)\,\nabla \operatorname{div} v.$$

Combining (C) and (M), we thus obtain the <u>(compressible) Navier-Stokes equations</u>:

$$\boxed{\begin{aligned} &\partial_t \rho + \operatorname{div}(\rho v) = 0 \\ &\rho\,(\partial_t v + (v\cdot\nabla)v) = -\nabla p + \mu\,\Delta v + (\xi + \mu/3)\,\nabla \operatorname{div} v + F \end{aligned}} \qquad \text{(NSC)}$$

This evolution eq. for ρ, v is complemented with an <u>equation of state</u> of the form (e.g.)

$$p = C\rho^{\gamma} \qquad \text{(barotropic gas)}, \quad \gamma \geq 1 \text{ parameter.}$$

Fluids with constant density

In liquids like water, the density is approximatively constant in most laboratory experiments. If $\rho = \text{const.}$, the continuity eq. reduces to the incompressibility condition $\operatorname{div} v = 0$. The incompressible Navier-Stokes eq. thus take the form:

$$\rho(\partial_t v + (v\cdot\nabla)v) = -\nabla p + \mu \Delta v + F, \quad \operatorname{div} v = 0.$$

Dividing the first eq. by ρ, we obtain: Navier, ~1822

$$\partial_t v + (v\cdot\nabla)v = -\nabla\tilde{p} + \nu\Delta v + \tilde{F}, \quad \operatorname{div} v = 0 \qquad \text{(NSI)}$$

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity. Also $\tilde{p} = \frac{p}{\rho}$, $\tilde{F} = \frac{F}{\rho}$.

For the water at 20 °C, we have $\nu \approx 10^{-6}\, m^2/s$, so $\nu \ll 1$.

In regions where the velocity gradients are too large, one can thus assume that the fluid is perfect ($\nu = 0$). We thus obtain the incompressible Euler eq.

Euler, ~1755

$$\partial_t v + (v\cdot\nabla)v = -\nabla\tilde{p} + \tilde{F}, \quad \operatorname{div} v = 0. \qquad \text{(EI)}$$

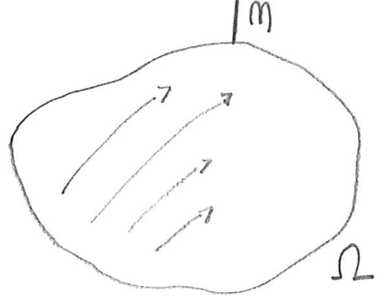
Remark: In (NSI) or (EI), the only dynamical variable is the Eulerian velocity $v(t,x)$. The pressure is no longer given by an equation of state, but solves the elliptic equation

$$\operatorname{div}((v\cdot\nabla)v) = -\Delta\tilde{p} + \operatorname{div}\tilde{F}$$

which is obtained by taking the divergence of the momentum equation. Boundary conditions have to be specified, see below.

Boundary conditions

Suppose that we consider a fluid filling a domain $\Omega \subset \mathbb{R}^3$ with impermeable boundaries (no inflow or outflow on $\partial\Omega$).



The impermeability condition is $\quad V(x,t)\cdot n(x) = 0 \quad \forall x \in \partial\Omega.$

- for a perfect fluid ($\xi = \mu = 0$): this is the only boundary condition
- for a viscous fluid, this condition is not sufficient. The standard boundary condition is

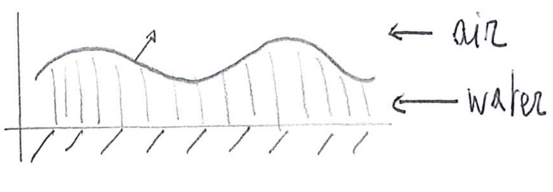
$$V(x,t) = 0 \quad \forall x \in \partial\Omega$$

"Dirichlet" or "no-slip" boundary condition. Largely valid for liquids in normal conditions.

Rem: Due to the impermeability condition, there is no need for bound. cond. on the density ρ (the characteristics are tangent to the boundary).

Other boundary conditions are sometimes considered:

- Navier friction condition: the tangential stress on the boundary is proportional to the tangential velocity.
- Free boundary conditions: at a free surface, or at an interface



(not considered here)

Energy balance

Consider the Navier-Stokes equations without forcing:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0 \\ \rho(\partial_t v + (v\cdot\nabla)v) = \operatorname{div}\sigma \end{cases} \qquad \begin{cases} \sigma_{ij} = -p\delta_{ij} + \mu C_{ij} + \xi(\operatorname{div} v)\delta_{ij} \\ C_{ij} = (\partial_i v_j + \partial_j v_i - \frac{2}{3}(\operatorname{div} v)\delta_{ij}) \end{cases}$$

The kinetic energy density inside the fluid is $\frac{1}{2}\rho|v|^2$. One finds:

$$\partial_t\left(\tfrac{1}{2}\rho|v|^2\right) + \partial_i\left(\tfrac{1}{2}\rho|v|^2 v_i\right) = \partial_j(v_i\sigma_{ij}) + D \qquad (EB)$$

where $D = -(\partial_j v_i)\sigma_{ij} = p\operatorname{div} v - \mu(\partial_j v_i)C_{ij} - \xi(\operatorname{div} v)^2$.

But:

$$(\partial_j v_i)C_{ij} \overset{C_{ij}\text{ symmetric}}{=} \frac{1}{2}(\partial_i v_j + \partial_j v_i)C_{ij} \overset{C_{ij}\text{ traceless}}{=} \frac{1}{2}(\partial_i v_j + \partial_j v_i - \tfrac{2}{3}(\operatorname{div} v)\delta_{ij})C_{ij}$$
$$= \frac{1}{2}C_{ij}C_{ij},$$

hence

$$D = p\operatorname{div} v - \frac{\mu}{2}|C|^2 - \xi(\operatorname{div} v)^2.$$

- $p\operatorname{div} v$: work of the pressure
- $-\frac{\mu}{2}|C|^2$: dissipation due to shear viscosity
- $-\xi(\operatorname{div} v)^2$: dissipation due to volume viscosity

In general $D \neq 0$ (for instance, $D<0$ for a nontrivial incompressible viscous flow). But the total energy in the fluid should be conserved $\Rightarrow$ we need to introduce the <u>internal energy</u> of the fluid, related to the temperature (we won't do that in this course).

The <u>isentropic</u> system (NSC) decouples from the eq. for the internal energy <u>provided</u> we assume that the coefficients μ, ξ <u>do not depend on the temperature</u>.

Linearization at a uniform steady state

To have a glimpse of the dynamics of the full system (NSC), consider the linearization at a uniform steady state $(\rho, v) = (\rho_0, 0)$, where $\rho_0 > 0$ is a constant. We denote

$$\rho = \rho_0(1+r), \quad p = P(\rho_0 + \rho_0 r) \approx P(\rho_0) + P'(\rho_0)\rho_0 r.$$

Neglecting all non-linear terms in (r, v), we obtain from (NSC) with $F=0$:

$$\begin{cases} \rho_0 \partial_t r + \rho_0 \operatorname{div} v = 0 \\ \rho_0 \partial_t v = -\rho_0 P'(\rho_0)\nabla r + \mu \Delta v + (\xi + \mu/3)\nabla \operatorname{div} v \end{cases}$$

We denote

$$\underline{c^2 = P'(\rho_0) > 0}, \quad \nu = \frac{\mu}{\rho_0}, \quad \eta = \frac{\xi + \mu/3}{\rho_0}.$$

We thus obtain:

$$\left| \begin{aligned} &\partial_t r + \operatorname{div} v = 0 \\ &\partial_t v + c^2 \nabla r = \nu \Delta v + \eta \nabla \operatorname{div} v. \end{aligned} \right. \qquad \text{(Lin1)}$$

To elucidate the nature of this system, we apply Hodge's decomposition:

$$\| \; v = w + \nabla q, \quad \text{where } \operatorname{div} w = 0.$$

In Fourier space where $\nabla = i\xi$, we have the explicit formulas:

$$\hat{w}(\xi) = \hat{v}(\xi) - \frac{\xi \cdot \hat{v}(\xi)}{|\xi|^2}\xi, \quad \hat{q}(\xi) = -i\frac{\xi \cdot \hat{v}(\xi)}{|\xi|^2}.$$

$$\Rightarrow \hat{v}(\xi) = \hat{w}(\xi) + i\xi\,\hat{q}(\xi).$$

Rem: $W = \mathbb{P} V$, $\mathbb{P}$ = Leray-Hopf projection onto divergence-free vector fields.

System (Lin 1) decomposes into two parts:

a) Solenoidal part (divergence free component of V):

$$\boxed{\partial_t W = \nu \Delta W}$$

Linear heat equation! Diffusion of vorticity

b) Irrotational part

$$\boxed{\begin{aligned} \partial_t \pi + \Delta q &= 0 \\ \partial_t q + c^2 \pi &= (\nu+\eta)\Delta q \end{aligned}}$$

Damped wave equation!

$$\Big\Vert \quad \partial_t^2 q - c^2 \Delta q = (\nu+\eta)\partial_t \Delta q .$$

- inviscid case $\nu+\eta = 0$: wave equation, sound waves.

  $C = \sqrt{P'(\rho_0)}$ = speed of sound!

- viscous case $\nu+\eta > 0$:

  $$\partial_t^2 \hat{q}(\xi,t) + (\nu+\eta)|\xi|^2 \partial_t \hat{q}(\xi,t) + c^2|\xi|^2 \hat{q}(\xi,t) = 0$$

  $\Longrightarrow$ damped oscillations for each non zero Fourier mode.

Thus the compressible NS system (NSC) is a mixture of nonlinear parabolic and nonlinear (damped) wave equations. Considering only the incompressible part is a reasonable application when the Mach number

$$M = \frac{|V|}{c} \ll 1.$$