Stability and Interaction of Vortices in Two-Dimensional Viscous Flows

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Abstract

The aim of these notes is to present in a comprehensive and relatively self-contained way some recent developments in the mathematical analysis of two-dimensional viscous flows. We consider the incompressible Navier-Stokes equations in the whole plane $\mathbb{R}^2$, and assume that the initial vorticity is a finite measure. This general setting includes vortex patches, vortex sheets, and point vortices. We first prove the existence of a unique global solution, for any value of the viscosity parameter, and we investigate its long-time behavior. We next consider the particular case where the initial flow is a finite collection of point vortices. In that situation, we show that the solution behaves, in the vanishing viscosity limit, as a superposition of Oseen vortices whose centers evolve according to the Helmholtz-Kirchhoff point vortex system. The proof requires a careful stability analysis of the Oseen vortices in the large Reynolds number regime, as well as a precise computation of the deformations of the vortex cores due to mutual interactions.

Contents

1 Introduction

Although real flows are always three-dimensional, it sometimes happens that the motion of a fluid is essentially planar in the sense that the fluid velocity in some distinguished spatial direction is negligible compared to the velocity in the orthogonal plane. This situation often occurs for fluids in thin layers, or for rapidly rotating fluids where the Coriolis force strongly penalizes displacements along the axis of rotation. Typical examples are geophysical flows, for which the geometry of the domain (the atmosphere or the ocean) and the effect of the Earth's rotation concur to make a two-dimensional approximation accurate and efficient [12].

From a mathematical point of view, planar flows are substantially easier to study than three-dimensional ones. For instance, it is known since the pioneering work of J. Leray [46, 47] that the two-dimensional incompressible Navier-Stokes equations are globally well-posed in the energy space, whereas global well-posedness is still an open problem in the three-dimensional case, no matter which function space is used [75]. The situation is essentially the same for the incompressible Euler equations, which describe the motion of inviscid fluids [52, 13]. However, having global solutions at hand does not mean that we fully understand the dynamics of the system. As a matter of fact, we are not able to establish on a rigorous basis the phenomenological laws of two-dimensional freely decaying turbulence, and the stability properties of 2D boundary layers in the high Reynolds number regime are not fully understood.

In these notes, we consider the idealized situation of an incompressible viscous fluid filling the whole plane $\mathbb{R}^2$ and evolving freely without exterior forcing. Following the approach of Helmholtz [36], we use the vorticity formulation of the problem, which is more appropriate to investigate the qualitative behavior of the solutions. Our goal is to understand the stability properties and the interactions of localized vortical structures at high Reynolds numbers. This question is important because carefully controlled experiments [17, 68], as well as numerical simulations of two-dimensional freely decaying turbulence [49, 5], suggest that vortex interactions, and especially vortex mergers, play a crucial role in the long-time dynamics of viscous planar flows, and are responsible in particular for the inverse energy cascade. Although nonperturbative phenomena such as vortex mergers may be very hard to describe mathematically [59, 45], we shall see that vortex interactions can be rigorously studied in the asymptotic regime where the distances between the vortex centers are much larger than the typical size of the vortex cores.

We begin our analysis of the two-dimensional viscous vorticity equation in Section 2. We first recall standard estimates for the two-dimensional Biot-Savart law, which expresses the velocity field in terms of the vorticity distribution, and we enumerate in Section 2.1 some general properties of the vorticity equation, including conservation laws, Lyapunov functions, and scaling invariance. In Section 2.2, we introduce the space of finite measures, which allows to consider nonsmooth flows such as vortex patches, vortex sheets, or point vortices. It is a remarkable fact that the vorticity equation is globally well-posed in such a large function space, for any value of the viscosity parameter, see [34, 25] and Theorem 2.8 below. Although a complete proof of that result is beyond the scope of these notes, we show in Section 2.3 that the classical approach of Fujita and Kato [24] applies to our problem and yields global existence and uniqueness of the solution provided that the atomic part of the initial vorticity distribution is sufficiently small. For larger initial data, existence of a global solution can be proved by an approximation scheme [16, 34, 43] but additional arguments are needed to establish uniqueness [25].

Section 3 collects a few results which describe the behavior of global solutions of the vorticity equation in $L^1(\mathbb{R}^2)$. In particular we prove convergence as $t \to \infty$ to a family of self-similar solutions called *Lamb-Oseen vortices*, see [31] and Theorem 3.1 below. To establish these results, we use accurate estimates on the fundamental solution of convection-diffusion equations, which were obtained by Osada [63] and are reproduced in Section 3.1. Another fundamental tool is a transformation into self-similar variables, which compactifies the trajectories of the system and allows to consider ω-limit sets, see Section 3.2. Using a pair of Lyapunov functions, one of which is only defined for positive solutions, we establish a "Liouville Theorem", which characterizes all complete trajectories of the vorticity equation in $L^1(\mathbb{R}^2)$ that are relatively compact in the self-similar variables. The conclusion is that all these trajectories necessarily coincide with Oseen vortices, see Proposition 3.4. In Section 3.3, we show that Liouville's theorem implies Theorem 3.1, and proves at the same time that the vorticity equation has a unique solution when the initial flow is a point vortex of arbitrary circulation. This is an important particular case of Theorem 2.8, which cannot be established by a standard application of Gronwall's lemma.

In Section 4 we investigate in some detail the stability properties of the Oseen vortices, which are steady states of the vorticity equation in the self-similar variables. We introduce in Section 4.1 an appropriate weighted space for the admissible perturbations, and we show that the linearized operator at Oseen's vortex has a remarkable structure in that space : it is the sum of a self-adjoint operator, which is essentially the harmonic oscillator, and a skew-symmetric relatively compact perturbation, which is multiplied by the circulation of the vortex. This structure almost immediately implies that Oseen vortices are stable equilibria of the rescaled vorticity equation, and that the size of the local basin of attraction is uniform in the circulation parameter. This is in sharp contrast with many classical examples in fluid mechanics, such as the Poiseuille flow or the Couette-Taylor flow, for which hydrodynamic instabilities are known to occur when the Reynolds number becomes large [23, 78]. In the case of Oseen vortices, we show in Section 4.2 that a rapid rotation (i.e., a large circulation number) has a *stabilizing effect* on the vortex : the size of the local basin of attraction increases, and the non-radially symmetric perturbations have a faster decay as $t \to \infty$. These empirically known facts can be rigorously established, although optimal spectral and pseudospectral estimates are not available yet.

In the final section of these notes, we consider the particular situation where the initial vorticity is a superposition of N Dirac masses (or point vortices). The corresponding solution of the two-dimensional vorticity equation is called the *viscous N-vortex solution*, and the goal of Section 5 is to investigate its behavior in the vanishing viscosity limit. Our main result, which is stated in Section 5.1, asserts that the viscous N-vortex solution is nicely approximated by a superposition of Oseen vortices whose centers evolve according to the Helmholtz-Kirchhoff point vortex dynamics [57, 62]. This approximation is accurate as long as the distance between the vortex centers remains much larger than the typical size of the vortex cores, which increases through diffusion. In Section 5.2, we decompose the viscous N-vortex solution into a sum of Gaussian vortex patches, and we introduce appropriate self-similar variables which allow us to formulate a stronger version of our result, taking into account the deformation of the vortices due to mutual interactions. The proof involves many technical issues which cannot be addressed here, but we sketch the main arguments in Section 5.3 and refer the interested reader to [28] for more details. In particular, we show in Section 5.3 how to systematically construct an asymptotic expansion of the viscous N-vortex solution, and we briefly indicate how the error terms can be controlled once a sufficiently accurate approximation is obtained.

The content of the present notes is strongly biased toward the scientific interests of the author, and does not provide a comprehensive survey of all important questions in two-dimensional fluid dynamics. We chose to focus on self-similar vortices, but other types of flows such as vortex patches or vortex sheets also lead to interesting and challenging problems, especially in the vanishing viscosity limit. Also, we should keep in mind that all real fluids are contained in domains with boundaries, so a comprehensive discussion of two-dimensional fluid mechanics should certainly include a description of the flow near the boundary, a question that is totally eluded here. On the other hand, we do not claim for much originality in these notes: all results collected here have already been published elsewhere, although they were never presented together in a unified way. In particular, most of the results of Sections 3 and 4 were obtained in collaboration with C.E. Wayne [30, 31], and the content of Section 5 is entirely borrowed from [28]. The material presented in Section 2 is rather standard, although our proof of Theorem 2.5 is perhaps not explicitly contained in the existing literature. The uniqueness part in Theorem 2.8, which is briefly discussed at the end of Section 3.3, was obtained in collaboration with I. Gallagher [25]. Finally, in Section 4.2, our approach to study the properties of the linearized operator at Oseen's vortex in the large circulation regime was developed in a collaboration with I. Gallagher and F. Nier [26].

2 The Cauchy Problem for the 2D Vorticity Equation

We consider the two-dimensional incompressible Navier-Stokes equations:

$$\begin{cases} \partial_t u(x,t) + (u(x,t)\cdot\nabla)u(x,t) \;=\; \nu\Delta u(x,t) - \dfrac{1}{\rho}\nabla p(x,t)\;, \\ \operatorname{div} u(x,t) \;=\; 0\;, \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^2$ is the space variable and $t \geq 0$ is the time variable. The unknown functions are the velocity field $u(x,t) = (u_1(x,t), u_2(x,t)) \in \mathbb{R}^2$, which represents the speed of a fluid particle at point x and time t, and the pressure field $p(x,t) \in \mathbb{R}$. Eq. (2.1) contains two physical parameters, the kinematic viscosity $\nu > 0$ and the fluid density $\rho > 0$, which are both assumed to be constant.

Equivalently, the motion of a planar fluid can be described by the *vorticity* field:

$$\omega(x,t) \;=\; \partial_1 u_2(x,t) - \partial_2 u_1(x,t)\;,$$

which represents the angular rotation of the fluid particles. If we take the two-dimensional curl of the first equation in (2.1), we obtain the evolution equation

$$\partial_t\omega(x,t) + u(x,t)\cdot\nabla\omega(x,t) \;=\; \nu\Delta\omega(x,t)\;, \tag{2.2}$$

which is the starting point of our analysis. Note that $u\cdot\nabla\omega = \operatorname{div}(u\omega)$ because $\operatorname{div} u = 0$. In the case of a perfect fluid ($\nu = 0$), Eq. (2.2) simply means that the vorticity is advected by the velocity field $u(x,t)$ like a material particle. For real fluids ($\nu > 0$), the vorticity also diffuses at a rate given by the kinematic viscosity.

The vorticity equation (2.2) is definitely simpler than the original Navier-Stokes system (2.1), but it still contains the velocity field $u(x,t)$. To make (2.2) independent of (2.1), it is possible to express u in terms of ω by solving the elliptic system

$$\partial_1 u_1 + \partial_2 u_2 \,=\, 0 \;, \qquad \partial_1 u_2 - \partial_2 u_1 \,=\, \omega \;. \tag{2.3}$$

If ω decays sufficiently fast at infinity (see Lemma 2.1 below), the solution is given by the two-dimensional *Biot-Savart* formula:

$$u(x,t) \,=\, \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\,\omega(y,t)\,\mathrm{d}y \;, \tag{2.4}$$

where $x^\perp = (-x_2, x_1)$ and $|x|^2 = x_1^2 + x_2^2$ if $x = (x_1, x_2) \in \mathbb{R}^2$. The vorticity equation (2.2), supplemented with the Biot-Savart law (2.4), is now formally equivalent to the Navier-Stokes equations (2.1). Once a solution to (2.2), (2.4) is found, the pressure field can be recovered up to an additive constant by solving, for each $t > 0$, the Poisson equation

$$-\Delta p \,=\, \rho\,\mathrm{div}((u\cdot\nabla)u) \;, \qquad p(x,t) \to 0 \quad \text{as} \quad |x| \to \infty \;.$$

The following standard result shows that the Biot-Savart formula (2.4) is well defined if the vorticity ω lies in $L^p(\mathbb{R}^2)$ for some $p \in (1,2)$.

Lemma 2.1 *Assume that $\omega \in L^p(\mathbb{R}^2)$ for some $p \in (1,2)$. Then the velocity field u defined (for almost every $x \in \mathbb{R}^2$) by the Biot-Savart formula* (2.4) *satisfies:*
i) $u \in L^q(\mathbb{R}^2)$, where $q \in (2,\infty)$ is such that $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$. Moreover

$$\|u\|_{L^q(\mathbb{R}^2)} \,\le\, C\|\omega\|_{L^p(\mathbb{R}^2)} \;. \tag{2.5}$$

ii) $\nabla u \in L^p(\mathbb{R}^2)$, and (2.3) *holds in $L^p(\mathbb{R}^2)$. Moreover*

$$\|\nabla u\|_{L^p(\mathbb{R}^2)} \,\le\, C\|\omega\|_{L^p(\mathbb{R}^2)} \;. \tag{2.6}$$

Proof. According to (2.4) we have $u = K * \omega$, where

$$K(x) \,=\, \frac{1}{2\pi}\frac{x^\perp}{|x|^2} \;, \qquad x \in \mathbb{R}^2 \setminus \{0\} \;,$$

is the Biot-Savart kernel. Since K belongs to the weak L^2 space $L^{2,\infty}(\mathbb{R}^2)$, the Hardy-Littlewood-Sobolev (or weak Young) inequality [48, Section 4.3] shows that

$$\|u\|_{L^q(\mathbb{R}^2)} \,=\, \|K*\omega\|_{L^q(\mathbb{R}^2)} \,\le\, \|K\|_{L^{2,\infty}(\mathbb{R}^2)}\|\omega\|_{L^p(\mathbb{R}^2)} \;,$$

if $1 < p < 2 < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$. This proves (2.5). Moreover $\nabla u = \nabla K * \omega$, where ∇K is an integral kernel of Calderón-Zygmund type which is associated to a bounded Fourier multiplier. It follows [72, Section I.5] that K defines a bounded linear operator in $L^p(\mathbb{R}^2)$ for $p \in (1,2)$, as asserted in (2.6). □

Remark 2.2 *If $\omega \in L^1(\mathbb{R}^2)$, the velocity field $u = K*\omega$ belongs to $L^{2,\infty}(\mathbb{R}^2)$, but $u \notin L^2(\mathbb{R}^2)$ in general, see [34]. If $\omega \in L^2(\mathbb{R}^2)$, one can solve* (2.3) *directly by a simple calculation in Fourier space, which shows that u belongs to the homogeneous Sobolev space $\dot H^1(\mathbb{R}^2)$, but $u \notin L^\infty(\mathbb{R}^2)$*

in general. Finally, if $\omega \in L^p(\mathbb{R}^2)$ for some $p \in (2,\infty)$, the solution of (2.3) *is no longer given by the Biot-Savart law* (2.4)*, but by a modified formula of the form*

$$u(x) \;=\; \frac{1}{2\pi}\int_{\mathbb{R}^2}\left(\frac{(x-y)^\perp}{|x-y|^2} + \frac{y^\perp}{|y|^2}\right)\omega(y)\,\mathrm{d}y\ .$$

In that case, $u(x)$ may grow like $|x|^{1-\frac{2}{p}}$ as $|x| \to \infty$, and the solution of (2.3) *is only defined up to an additive constant.*

2.1 General Properties of the Vorticity Equation

In this section, we list a few important properties of the vorticity equation (2.2), including conservation laws, Lyapunov functions, and scaling invariance. The presentation is formal in the sense that we assume here that we are given a solution of (2.2) with the appropriate smoothness and integrability properties. However, all calculations can be justified once suitable existence theorems have been established [52].

2.1.1 Conservations Laws

Let $\omega(x,t)$ be a solution of (2.2) with initial data $\omega_0(x) = \omega(x,0)$. If $\omega_0 \in L^1(\mathbb{R}^2)$, then $x \mapsto \omega(x,t)$ is integrable for all $t \ge 0$ and the *total circulation* is conserved:

$$\int_{\mathbb{R}^2}\omega(x,t)\,\mathrm{d}x \;=\; \int_{\mathbb{R}^2}\omega_0(x)\,\mathrm{d}x\ . \tag{2.7}$$

If moreover $|x|\omega_0 \in L^1(\mathbb{R}^2)$, then the first order moments of $\omega(x,t)$ are also conserved:

$$\int_{\mathbb{R}^2} x_i\omega(x,t)\,\mathrm{d}x \;=\; \int_{\mathbb{R}^2} x_i\omega_0(x)\,\mathrm{d}x\ , \qquad i = 1,2\ . \tag{2.8}$$

Finally, if in addition $|x|^2\omega_0 \in L^1(\mathbb{R}^2)$, then the symmetric second order moment of $\omega(x,t)$ satisfies:

$$\int_{\mathbb{R}^2}|x|^2\omega(x,t)\,\mathrm{d}x \;=\; \int_{\mathbb{R}^2}|x|^2\omega_0(x)\,\mathrm{d}x + 4\nu t\int_{\mathbb{R}^2}\omega_0(x)\,\mathrm{d}x\ . \tag{2.9}$$

Thus the symmetric moment is conserved for perfect fluids ($\nu = 0$), and for viscous fluids ($\nu > 0$) if the total circulation vanishes. These elementary properties are easily established by direct calculations, using (2.2) and the Biot-Savart law $u = K * \omega$. In particular, to prove (2.8), one uses the fact that

$$\int_{\mathbb{R}^2} u_i(x)\omega(x)\,\mathrm{d}x \;=\; \int_{\mathbb{R}^2\times\mathbb{R}^2} K_i(x-y)\omega(y)\omega(x)\,\mathrm{d}x\,\mathrm{d}y \;=\; 0\ , \qquad i = 1,2\ ,$$

because the Biot-Savart kernel K is odd. Similarly, the proof of (2.9) relies on the identity

$$\int_{\mathbb{R}^2}(x\cdot u(x))\omega(x)\,\mathrm{d}x \;=\; \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}^2}\Big((x-y)\cdot K(x-y)\Big)\omega(y)\omega(x)\,\mathrm{d}x\,\mathrm{d}y \;=\; 0\ , \tag{2.10}$$

which holds because K is odd and $x \cdot K(x) = 0$ for all $x \in \mathbb{R}^2$.

Remark 2.3 *By Green's theorem, the total circulation satisfies*

$$\gamma := \int_{\mathbb{R}^2}\omega\,\mathrm{d}x \;=\; \lim_{R\to\infty}\oint_{|x|=R}(u_1(x)\,\mathrm{d}x_1 + u_2(x)\,\mathrm{d}x_2)\ ,$$

hence γ represents the circulation of the velocity field at infinity. This quantity is conserved due to the structure of Eq. (2.2). In contrast, the conservation of the first-order moments and the simple evolution law for the symmetric second order moment are related to translation and rotation invariance, and would not hold if the vorticity equation was considered in a nontrivial domain $\Omega \subset \mathbb{R}^2$.

2.1.2 Lyapunov Functions

If $\omega_0 \in L^p(\mathbb{R}^2)$ for some $p \in [1, \infty]$, the solution of (2.2) with initial data ω_0 lies in $L^p(\mathbb{R}^2)$ for all times and

$$\|\omega(\cdot, t)\|_{L^p(\mathbb{R}^2)} \le \|\omega_0\|_{L^p(\mathbb{R}^2)} , \quad \text{for all } t \ge 0 . \tag{2.11}$$

This again follows from the fact that the advection field u in (2.2) is divergence-free. If the vorticity equation is considered in a domain $\Omega \subset \mathbb{R}^2$, and if one assumes as usual that $u = 0$ on $\partial\Omega$ (no-slip boundary condition), then vorticity is created in the boundary layer near $\partial\Omega$ and (2.11) does not hold.

If $u_0 = K * \omega_0 \in L^2(\mathbb{R}^2)$, the solution $u(x, t)$ of the Navier-Stokes equation (2.1) with initial data u_0 satisfies $u(\cdot, t) \in L^2(\mathbb{R}^2)$ for all times, and the *kinetic energy*

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^2} |u(x, t)|^2 \,\mathrm{d}x ,$$

is a nonincreasing function of time:

$$E'(t) = -\nu \int_{\mathbb{R}^2} |\nabla u(x, t)|^2 \,\mathrm{d}x \le 0 .$$

The kinetic energy is the most famous Lyapunov function in fluid mechanics, and often the only one that is available. In particular, it plays a crucial role in the construction of global weak solutions of the three-dimensional Navier-Stokes equations [47]. Somewhat paradoxically, in unbounded two-dimensional domains, this quantity is not always useful because it is finite only if the total circulation vanishes. Indeed, we have the following elementary result:

Lemma 2.4 *If $u \in L^2(\mathbb{R}^2)^2$ satisfies $\omega = \partial_1 u_2 - \partial_2 u_1 \in L^1(\mathbb{R}^2)$, then $\int_{\mathbb{R}^2} \omega \,\mathrm{d}x = 0$.*

Proof. By assumption the Fourier transform $\hat{\omega}(k) = ik_1 \hat{u}_2(k) - ik_2 \hat{u}_1(k)$ is a continuous function of $k \in \mathbb{R}^2$, hence

$$\int_{\mathbb{R}^2} \omega \,\mathrm{d}x = \hat{\omega}(0) = \lim_{\epsilon \to 0} \frac{1}{\pi\epsilon^2} \int_{|k| \le \epsilon} \hat{\omega}(k) \,\mathrm{d}k .$$

But

$$\frac{1}{\pi\epsilon^2} \int_{|k| \le \epsilon} |\hat{\omega}(k)| \,\mathrm{d}k \le \frac{1}{\pi\epsilon^2} \int_{|k| \le \epsilon} |k| |\hat{u}(k)| \,\mathrm{d}k \le \frac{1}{\sqrt{2\pi}} \Big(\int_{|k| \le \epsilon} |\hat{u}(k)|^2 \,\mathrm{d}k \Big)^{1/2} ,$$

thus taking the limit $\epsilon \to 0$ we obtain $\int_{\mathbb{R}^2} \omega \,\mathrm{d}x = 0$. □

As a substitute for the kinetic energy, one can consider the *pseudo-energy*

$$\mathcal{E}_d(t) = \frac{1}{4\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{d}{|x - y|} \, \omega(x, t) \omega(y, t) \,\mathrm{d}x \,\mathrm{d}y ,$$

where $d > 0$ is an arbitrary length scale. This quantity is well defined under mild integrability assumptions on ω, and can be finite even if the total circulation is nonzero. Morever, $\mathcal{E}_d(t)$ is a Lyapunov function:

$$\mathcal{E}_d'(t) = -\nu \int_{\mathbb{R}^2} \omega(x, t)^2 \,\mathrm{d}x \le 0 ,$$

and it is straightforward to verify that $\mathcal{E}_d(t) = E(t)$ if the velocity field associated to $\omega(x, t)$ is square integrable. In this particular situation, $\mathcal{E}_d(t)$ does not depend on the parameter d.

2.1.3 Scaling Invariance

In the whole plane $\mathbb{R}^2$, the Navier-Stokes equations (2.1) and the vorticity equation (2.2) are invariant under the rescaling

$$u(x,t) \mapsto \lambda u(\lambda x, \lambda^2 t) \ , \qquad \omega(x,t) \mapsto \lambda^2 \omega(\lambda x, \lambda^2 t) \ , \tag{2.12}$$

for any $\lambda > 0$. To solve the Cauchy problem, it is rather natural to use space-time norms which are invariant under the transformation (2.12). The corresponding function spaces are called *critical spaces* (with respect to the scaling). A typical example is the *energy space*

$$u \in X = C_b^0(\mathbb{R}_+, L^2(\mathbb{R}^2)) \ , \qquad \|u\|_X = \sup_{t\geq 0} \|u(\cdot,t)\|_{L^2} \ , \tag{2.13}$$

where $C_b^0(\mathbb{R}_+, L^2(\mathbb{R}^2))$ is the space of all bounded and continuous maps from $\mathbb{R}_+ = [0,\infty)$ into $L^2(\mathbb{R}^2)$, and $u(\cdot,t)$ denotes the map $x \mapsto u(x,t)$. Alternatively, to include solutions with nonzero total circulation, one can assume that the vorticity is integrable and use the space

$$\omega \in Y = C_b^0(\mathbb{R}_+, L^1(\mathbb{R}^2)) \ , \qquad \|\omega\|_Y = \sup_{t\geq 0} \|\omega(\cdot,t)\|_{L^1} \ . \tag{2.14}$$

A well-known "metatheorem" in nonlinear PDE's asserts that, for a nonlinear partial differential equation with scaling invariance, the critical spaces are the largest ones (in terms of local regularity of the solutions) in which we can hope that the Cauchy problem is locally well-posed. In fact, there may be other obstructions to local well-posedness, but according to the above claim it is certainly not reasonable to try to solve the Cauchy problem in function spaces that are "strictly larger" than the critical ones.

As far as the Navier-Stokes equation is concerned, we know since the work of Leray [46] that Eq. (2.1) is globally well-posed in the energy space (2.13), and a similar result holds for the vorticity equation (2.2) in the critical space (2.14), see [3, 6]. In the rest of this section, we discuss the Cauchy problem for (2.2) in the space of finite measures, which is a natural extension of $L^1(\mathbb{R}^2)$. As we shall see in Section 5, this generalization is essential to study the interaction of vortices in the vanishing viscosity limit.

2.2 The Space of Finite Measures

Let $\mathcal{M}(\mathbb{R}^2)$ denote the space of all real-valued Radon measures on $\mathbb{R}^2$, equipped with the total variation norm:

$$\|\mu\|_{\text{tv}} = \sup\Big\{ \int_{\mathbb{R}^2} \phi \, \mathrm{d}\mu \ \Big| \ \phi \in C_0(\mathbb{R}^2) \ , \ \|\phi\|_{L^\infty} \leq 1 \Big\} \ . \tag{2.15}$$

Here $C_0(\mathbb{R}^2)$ is the space of all continuous functions $\phi : \mathbb{R}^2 \to \mathbb{R}$ such that $\phi(x) \to 0$ as $|x| \to \infty$. It is well known that $\mathcal{M}(\mathbb{R}^2)$ equipped with the total variation norm is a Banach space, which contains $L^1(\mathbb{R}^2)$ as a closed subspace (if we identify any absolutely continuous measure with its density function). In particular, if $\omega \in L^1(\mathbb{R}^2)$, then $\|\omega\|_{\text{tv}} = \|\omega\|_{L^1}$. The total variation norm is scale invariant in the sense that $\|\mu_\lambda\|_{\text{tv}} = \|\mu\|_{\text{tv}}$ for all $\lambda > 0$, where μ_λ is the rescaled measure defined by

$$\int_{\mathbb{R}^2} \phi \, \mathrm{d}\mu_\lambda = \int_{\mathbb{R}^2} \phi(\lambda^{-1} \cdot) \, \mathrm{d}\mu \ , \qquad \text{for all } \phi \in C_0(\mathbb{R}^2) \ .$$

As is clear from the definition (2.15), the space $\mathcal{M}(\mathbb{R}^2)$ is the tolopogical dual of $C_0(\mathbb{R}^2)$. By the Banach-Alaoglu theorem, it follows that the unit ball in $\mathcal{M}(\mathbb{R}^2)$ is a (sequentially) compact set for the weak convergence defined by:

$$\mu_n \xrightarrow[n\to\infty]{} \mu \qquad \text{if} \qquad \int_{\mathbb{R}^2} \phi \, \mathrm{d}\mu_n \xrightarrow[n\to\infty]{} \int_{\mathbb{R}^2} \phi \, \mathrm{d}\mu \quad \text{for all } \phi \in C_0(\mathbb{R}^2) \ .$$

Any $\mu \in \mathcal{M}(\mathbb{R}^2)$ can be decomposed in a unique way as $\mu = \mu_{ac} + \mu_s$, where $\mu_{ac} \in \mathcal{M}(\mathbb{R}^2)$ is absolutely continuous with respect to Lebesgue's measure on $\mathbb{R}^2$, and μ_s is singular with respect to Lebesgue's measure [67]. By the Radon-Nikodym theorem, there exists a unique $\omega \in L^1(\mathbb{R}^2)$ such that $\mathrm{d}\mu_{ac} = \omega\,\mathrm{d}x$. Furthermore, the singular part μ_s can be decomposed as $\mu_s = \mu_{sc} + \mu_{pp}$, where μ_{pp} is the restriction of μ to the atomic set $\Sigma = \{x \in \mathbb{R}^2 \,|\, \mu(\{x\}) \neq 0\}$. Since μ is a finite measure, this set is at most countable, so that

$$\mu_{pp} \,=\, \sum_{i=1}^{\infty} \gamma_i \delta_{x_i}~, \tag{2.16}$$

for some $\gamma_i \in \mathbb{R}$ and some $x_i \in \mathbb{R}^2$. By construction, the singularly continuous part μ_{sc} has no atoms, yet is concentrated on a set of zero Lebesgue measure.

To summarize, we have $\mu = \mu_{ac} + \mu_{sc} + \mu_{pp}$, with $\mu_{ac} \perp \mu_{sc} \perp \mu_{pp}$. This notation means that the absolutely continuous, singularly continuous, and pure point parts of μ are concentrated on pairwise disjoint sets. It follows in particular that

$$\|\mu\|_{\mathrm{tv}} \,=\, \|\mu_{ac}\|_{\mathrm{tv}} + \|\mu_{sc}\|_{\mathrm{tv}} + \|\mu_{pp}\|_{\mathrm{tv}} \,=\, \|\omega\|_{L^1} + \|\mu_{sc}\|_{\mathrm{tv}} + \sum_{i=1}^{\infty} |\gamma_i|~.$$

If we now suppose that the measure $\mu \in \mathcal{M}(\mathbb{R}^2)$ is the vorticity of a two-dimensional flow, there is a nice correspondence between the abstract decomposition result presented above and the following standard catalogue of nonsmooth flows:

- *Vortex Patches.* A vortex patch can be defined as a flow for which the vorticity μ is a piecewise smooth function. For instance, μ can be the characteristic function of a smooth bounded domain $\Omega \subset \mathbb{R}^2$. In that case $\mu = \mu_{ac} \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, and the velocity field $u = K * \mu$ is Lipschitz continuous if the boundary $\partial\Omega$ is sufficiently smooth.
- *Vortex Sheets.* In contrast, vortex sheets are characterized by discontinuities of the velocity field. As a typical example, we mention the case where the vorticity μ is the Euclidean measure supported by a smooth closed curve $\gamma \subset \mathbb{R}^2$. In that case $\mu = \mu_{sc}$, and the tangential component of the velocity field $u = K * \mu$ is discontinuous on the curve γ.
- *Point Vortices.* This is the case where the vorticity μ is a collection of Dirac masses. Here $\mu = \mu_{pp}$, and the velocity field $u = K * \mu$ is unbounded near the center of each vortex.

2.3 The Cauchy Problem in $\mathcal{M}(\mathbb{R}^2)$

The aim of this section is to study the Cauchy problem for the two-dimensional vorticity equation (2.2) in the space of finite measures. We start with a preliminary result that is relatively easy to prove.

Theorem 2.5 [16, 34, 43]
There exists a universal constant $C_0 > 0$ such that, if the initial vorticity $\mu \in \mathcal{M}(\mathbb{R}^2)$ satisfies $\|\mu_{pp}\|_{\mathrm{tv}} \le C_0\,\nu$, then the vorticity equation (2.2) has a unique global solution

$$\omega \in C^0((0,\infty), L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$$

such that $\|\omega(\cdot,t)\|_{L^1} \le \|\mu\|_{\mathrm{tv}}$ for all $t > 0$, and $\omega(\cdot,t) \rightharpoonup \mu$ as $t \to 0$.

Existence of global solutions to (2.2) with initial data in $\mathcal{M}(\mathbb{R}^2)$ was first proved by G.-H. Cottet [16], by Y. Giga, T. Miyakawa and H. Osada [34], and by T. Kato [43]. In these works, the strategy is : i) to approximate the initial measure by a smooth vorticity distribution; ii) to construct a global solution, with good a priori estimates, by applying classical existence results for smooth initial data; iii) to obtain a solution of the original problem by extracting an appropriate subsequence as the regularization parameter converges to zero. This yields the existence part of Theorem 2.5 without any smallness condition on the initial data. The restriction $\|\mu_{pp}\|_{\rm tv} \le C_0\,\nu$ arises when one tries to prove uniqueness of the solution by a standard application of Gronwall's lemma. As we shall see later, this smallness condition is purely technical and can be completely relaxed, see Theorem 2.8.

In the rest of this section, we present a more direct proof of Theorem 2.5, which is nevertheless inspired by [34]. Without loss of generality, we assume from now on that the kinematic viscosity is equal to 1. Given initial data $\mu \in \mathcal{M}(\mathbb{R}^2)$, we consider the integral equation associated with (2.2) :

$$\omega(t) \,=\, e^{t\Delta}\mu - \int_0^t \mathrm{div}\Big(e^{(t-s)\Delta}\, u(s)\omega(s)\Big)\,\mathrm{d}s\;, \qquad t > 0\;, \tag{2.17}$$

where $\omega(t) = \omega(\cdot,t)$, $u(t) = u(\cdot,t)$, and $e^{t\Delta}$ denotes the heat semigroup defined by

$$(e^{t\Delta}\mu)(x) \,=\, \frac{1}{4\pi t}\int_{\mathbb{R}^2} e^{-|x-y|^2/(4t)}\,\mathrm{d}\mu_y\;, \qquad t > 0\;, \quad x \in \mathbb{R}^2\;. \tag{2.18}$$

Our goal is to solve the integral equation (2.17) by a fixed point argument in an appropriate function space. This is a classical method which, in the context of the Navier-Stokes equations, goes back to the work of Kato and Fujita [24]. Solutions of the integral equation (2.17) are often called *mild* solutions of the vorticity equation (2.2). It is rather straightforward to show that, if $\omega \in C^0((0,T), L^p(\mathbb{R}^2))$ is a mild solution of (2.2) with $p \ge 4/3$, then $\omega(x,t)$ is smooth and satisfies (2.2) in the classical sense. Thus there is no loss of generality in considering (2.17) instead of (2.2).

2.3.1 Heat Kernel Estimates

The following lemma plays a crucial role in the proof of Theorem 2.5 :

Lemma 2.6 *Let $\mu \in \mathcal{M}(\mathbb{R}^2)$.*
a) For $1 \le p \le \infty$ and $t > 0$, we have

$$\|e^{t\Delta}\mu\|_{L^p} \,\le\, \frac{1}{(4\pi t)^{1-\frac{1}{p}}}\,\|\mu\|_{\rm tv}\;, \qquad \|\nabla e^{t\Delta}\mu\|_{L^p} \,\le\, \frac{C}{t^{\frac{3}{2}-\frac{1}{p}}}\,\|\mu\|_{\rm tv}\;. \tag{2.19}$$

b) For $1 < p \le \infty$, we have

$$L_p(\mu) \,:=\, \limsup_{t\to 0}\,(4\pi t)^{1-\frac{1}{p}}\|e^{t\Delta}\mu\|_{L^p} \,\le\, \|\mu_{pp}\|_{\rm tv}\;. \tag{2.20}$$

Proof. Estimates (2.19) are easily obtained from (2.18) if $p = 1$ or $p = \infty$, and the general case follows by interpolation. Thus we concentrate on estimate (2.20), which we establish using a simplified version of the argument given by Giga, Miyakawa, and Osada [34]. First, if we decompose $\mu = \mu_{ac} + \mu_{sc} + \mu_{pp}$, we have

$$L_p(\mu) \,\le\, L_p(\mu_{ac}) + L_p(\mu_{sc}) + L_p(\mu_{pp}) \,\le\, L_p(\mu_{ac}) + L_p(\mu_{sc}) + \|\mu_{pp}\|_{\rm tv}\;.$$

Thus, to prove (2.20), it is sufficient to show that $L_p(\mu) = 0$ for all $\mu \in \mathcal{M}(\mathbb{R}^2)$ such that $\mu_{pp} = 0$. Furthermore, since

$$L_p(\mu) \le L_1(\mu)^{1/p} L_\infty(\mu)^{1-1/p} \le \|\mu\|_{\mathrm{tv}}^{1/p} L_\infty(\mu)^{1-1/p} ,$$

it is sufficient to consider the case where $p = \infty$.

Assume now that $\mu \in \mathcal{M}(\mathbb{R}^2)$ satisfies $\mu_{pp} = 0$, and fix $\epsilon > 0$. We claim that there exists $\delta > 0$ such that

$$\sup_{x \in \mathbb{R}^2} |\mu|(B(x,\delta)) \le \epsilon , \quad \text{where} \quad B(x,\delta) = \{y \in \mathbb{R}^2 \,|\, |y-x| \le \delta\} . \tag{2.21}$$

Here and in what follows, we denote by $|\mu| \in \mathcal{M}(\mathbb{R}^2)$ the total variation of the measure $\mu \in \mathcal{M}(\mathbb{R}^2)$. If μ^+, μ^- are the positive and negative variations of μ, then $\mu = \mu^+ - \mu^-$ (Jordan's decomposition) and $|\mu| = \mu^+ + \mu^-$ [67]. To prove (2.21), assume on the contrary that there exist $\delta_n > 0$ and $x_n \in \mathbb{R}^2$ such that $\delta_n \to 0$ as $n \to \infty$ and $|\mu|(B(x_n,\delta_n)) > \epsilon$ for all $n \in \mathbb{N}$. Since $|\mu|$ is a finite measure, the sequence (x_n) is necessarily bounded, thus after extracting a subsequence we can assume that $x_n \to \tilde{x}$ as $n \to \infty$, for some $\tilde{x} \in \mathbb{R}^2$. As a consequence, for any $\delta > 0$, we have $B(\tilde{x},\delta) \supset B(x_n,\delta_n)$ for all sufficiently large n, so that $|\mu|(B(\tilde{x},\delta)) \ge |\mu|(B(x_n,\delta_n)) > \epsilon$. Taking the limit $\delta \to 0$, we obtain $|\mu(\{\tilde{x}\})| \ge \epsilon$, which contradicts the assumption that $\mu_{pp} = 0$. This proves (2.21).

Now, for any $t > 0$, we can take $\bar{x}(t) \in \mathbb{R}^2$ such that $|(e^{t\Delta}\mu)(\bar{x}(t))| = \|e^{t\Delta}\mu\|_{L^\infty}$. Thus, using (2.18) and (2.21), we obtain

$$4\pi t \, \|e^{t\Delta}\mu\|_{L^\infty} \le \int_{B(\bar{x}(t),\delta)} e^{-\frac{|\bar{x}(t)-y|^2}{4t}} \,\mathrm{d}|\mu|_y + \int_{B(\bar{x}(t),\delta)^c} e^{-\frac{|\bar{x}(t)-y|^2}{4t}} \,\mathrm{d}|\mu|_y .$$

The first term on the right-hand side is bounded by ϵ for all $t > 0$, and the second one vanishes as $t \to 0$. Thus

$$L_\infty(\mu) = \limsup_{t \to 0} \, (4\pi t) \, \|e^{t\Delta}\mu\|_{L^\infty} \le \epsilon .$$

Since $\epsilon > 0$ was arbitrary, we conclude that $L_\infty(\mu) = 0$ whenever $\mu_{pp} = 0$, which is the desired result. ☐

Remark 2.7 *Estimate* (2.20) *was strengthened by Kato [43] in the following way:*

$$\lim_{t \to 0} \, (4\pi t)^{1-\frac{1}{p}} \|e^{t\Delta}\mu\|_{L^p} = p^{-1/p} \|\{\gamma_i\}_{i=1}^\infty\|_{\ell^p} \le \sum_{i=1}^\infty |\gamma_i| , \tag{2.22}$$

where γ_i, x_i are as in (2.16). *In both* (2.20), (2.22), *the assumption that $p > 1$ is crucial: if $\mu \in \mathcal{M}(\mathbb{R}^2)$ is a positive measure, then $\|e^{t\Delta}\mu\|_{L^1} = \|\mu\|_{\mathrm{tv}}$ for all $t > 0$, hence $L_1(\mu) = \|\mu\|_{\mathrm{tv}}$.*

2.3.2 Fixed Point Argument

To prove the existence of solutions to the integral equation (2.17), we fix $T > 0$ and introduce the function space

$$X_T = \Big\{\omega \in C^0((0,T], L^{4/3}(\mathbb{R}^2)) \,\Big|\, \|\omega\|_{X_T} < \infty\Big\} , \tag{2.23}$$

equipped with the norm

$$\|\omega\|_{X_T} = \sup_{0 < t \le T} t^{1/4} \|\omega(t)\|_{L^{4/3}} .$$

Given $\mu \in \mathcal{M}(\mathbb{R}^2)$, we denote $\omega_0(t) = e^{t\Delta}\mu$ for all $t > 0$. In view of Lemma 2.6, we have $\omega_0 \in X_T$ and there exist positive constants C_1, C_2 (independent of T) such that

a) $\|\omega_0\|_{X_T} \le C_1 \|\mu\|_{\rm tv}$ (for any $T > 0$);

b) $\|\omega_0\|_{X_T} \le C_2 \|\mu_{pp}\|_{\rm tv} + \epsilon$ (if $T > 0$ is sufficiently small, depending on μ,ϵ).

Here $\epsilon > 0$ is an arbitrarily small positive constant.

On the other hand, given $\omega \in X_T$, we define $F\omega : (0, T] \to L^{4/3}(\mathbb{R}^2)$ by

$$(F\omega)(t) \;=\; \int_0^t \mathrm{div}\Big(e^{(t-s)\Delta}\, u(s)\omega(s)\Big)\,\mathrm{d}s\ , \quad 0 < t \le T\ , \tag{2.24}$$

where $u(s) = K * \omega(s)$ is the velocity field obtained via the Biot-Savart law (2.4). Using the second estimate in (2.19), Hölder's inequality, and estimate (2.5), we find

$$\begin{aligned} t^{1/4}\|(F\omega)(t)\|_{L^{4/3}} \;&\le\; t^{1/4}\int_0^t \frac{C}{(t-s)^{\frac{3}{4}}}\,\|u(s)\omega(s)\|_{L^1}\,\mathrm{d}s \\ &\le\; t^{1/4}\int_0^t \frac{C}{(t-s)^{\frac{3}{4}}}\,\|u(s)\|_{L^4}\|\omega(s)\|_{L^{4/3}}\,\mathrm{d}s \\ &\le\; t^{1/4}\int_0^t \frac{C}{(t-s)^{\frac{3}{4}}}\,\|\omega(s)\|_{L^{4/3}}^2\,\mathrm{d}s \\ &\le\; C\|\omega\|_{X_T}^2\, t^{1/4}\int_0^t \frac{C}{(t-s)^{\frac{3}{4}}s^{\frac{1}{2}}}\,\mathrm{d}s \;\le\; C\|\omega\|_{X_T}^2\ , \end{aligned} \tag{2.25}$$

for all $t \in (0, T]$. This shows that $\|F\omega\|_{X_T} \le C_3\|\omega\|_{X_T}^2$ for some $C_3 > 0$ (independent of T), and it is not difficult to verify that $(F\omega)(t)$ depends continuously on t in the topology of $L^{4/3}(\mathbb{R}^2)$, so that $F\omega \in X_T$. Similarly, one can show that

$$\|F\omega - F\tilde{\omega}\|_{X_T} \;\le\; C_3(\|\omega\|_{X_T} + \|\tilde{\omega}\|_{X_T})\|\omega - \tilde{\omega}\|_{X_T}\ , \tag{2.26}$$

for all $\omega, \tilde{\omega} \in X_T$.

Now, fix $R > 0$ such that $2C_3R < 1$, and consider the closed ball

$$B \;=\; \{\omega \in X_T \,|\, \|\omega\|_{X_T} \le R\} \;\subset\; X_T\ .$$

As a consequence of the estimates above, if $\|\omega_0\|_{X_T} \le R/2$, the map $\omega \mapsto \omega_0 - F\omega$ is a strict contraction in B, hence has a unique fixed point in B. By construction, this fixed point ω is a solution of the integral equation (2.17) on $(0, T]$. The condition $\|\omega_0\|_{X_T} \le R/2$ is fulfilled if either

1) $2C_1\|\mu\|_{\rm tv} \le R$ and $T > 0$ is arbitrary, or

2) $2C_2\|\mu_{pp}\|_{\rm tv} < R$ and $T > 0$ is small enough, depending on μ.

In other words, the proof above shows that equation (2.17) is *globally well-posed* for small initial data $\mu \in \mathcal{M}(\mathbb{R}^2)$, and *locally well-posed* for large initial data with small atomic part μ_{pp}. However, if $2C_2\|\mu_{pp}\|_{\rm tv} \ge R$, it is not possible to meet the condition $\|\omega_0\|_{X_T} \le R/2$ by an appropriate choice of T, so the argument breaks down and does not give any information on the existence of solutions to (2.17).

2.3.3 End of the Proof of Theorem 2.5

The solution $\omega \in X_T$ of (2.17) constructed by the fixed point argument automatically satisfies $\omega \in C^0((0,T], L^1(\mathbb{R}^2))$. Indeed, arguing as in (2.25), we find

$$\|\omega(t) - \omega_0(t)\|_{L^1} \le \int_0^t \frac{C}{(t-s)^{\frac{1}{2}}} \|\omega(s)\|_{L^{4/3}}^2 \,\mathrm{d}s \le C\|\omega\|_{X_T}^2 , \qquad (2.27)$$

for $t \in (0,T]$, where $\omega_0(t) = e^{t\Delta}\mu$. Since $\|\omega_0(t)\|_{L^1} \le \|\mu\|_{\mathrm{tv}}$ by (2.19), we conclude that $\omega(t) \in L^1(\mathbb{R}^2)$ for all $t \in (0,T]$, and the continuity with respect to time is again easy to verify. In fact, one can even show that

$$\lim_{t\to 0} \|\omega(t) - \omega_0(t)\|_{L^1} = 0 . \qquad (2.28)$$

Since $\omega_0(t) \rightharpoonup \mu$ as $t \to 0$, this implies that $\omega(t) \rightharpoonup \mu$ as $t \to 0$. Moreover, as $\|\omega(t)\|_{L^1}$ is a nonincreasing function of t, we deduce from (2.28) that $\|\omega(t)\|_{L^1} \le \|\mu\|_{\mathrm{tv}}$ for all $t \in (0,T]$.

To prove (2.28), we denote

$$\delta := \limsup_{t\to 0} t^{1/4}\|\omega(t) - \omega_0(t)\|_{L^{4/3}} \equiv \limsup_{T\to 0} \|\omega - \omega_0\|_{X_T} .$$

Since $\omega - \omega_0 = (F\omega_0 - F\omega) - F\omega_0$ and $\|\omega\|_{X_T} + \|\omega_0\|_{X_T} \le 2R$, it follows from (2.26) that $\delta \le (2C_3R)\delta + \ell_{4/3}(\mu)$, where

$$\ell_p(\mu) = \limsup_{t\to 0} t^{1-\frac{1}{p}} \|(F\omega_0)(t)\|_{L^p} , \qquad 1 \le p \le \infty .$$

Now, a direct calculation, which is postponed to Lemma 2.9 below, shows that $\ell_p(\mu) = 0$ for any $\mu \in \mathcal{M}(\mathbb{R}^2)$ and any $p \in [1,\infty]$. This implies that $\delta = 0$ because $2C_3R < 1$. Using again the identity $\omega - \omega_0 = (F\omega_0 - F\omega) - F\omega_0$ and arguing as in (2.27), we conclude that

$$\limsup_{t\to 0} \|\omega(t) - \omega_0(t)\|_{L^1} \le CR\delta + \ell_1(\mu) = 0 .$$

To finish the proof of Theorem 2.5, it remains to show that all solutions of (2.2) are global in time. We first observe that, if $\mu \in \mathcal{M}(\mathbb{R}^2) \cap L^{4/3}(\mathbb{R}^2)$, then $\|e^{t\Delta}\mu\|_{L^{4/3}} \le \|\mu\|_{L^{4/3}}$ for all $t > 0$, hence $\|\omega_0\|_{X_T} \le T^{1/4}\|\mu\|_{L^{4/3}}$. Since the local existence time $T > 0$ is determined by the condition $\|\omega_0\|_{X_T} \le R/2$, we see that *an upper bound* on the $L^{4/3}$-norm of the initial data provides a *lower bound* on the local existence time T. Assume now that $\mu \in \mathcal{M}(\mathbb{R}^2)$ satisfies $2C_2\|\mu_{pp}\|_{\mathrm{tv}} < R$, and let $\omega \in X_T$ be the local solution of (2.2) constructed by the fixed point argument. If $T_1 > 0$ satisfies $T_1^{1/4}\|\omega(T)\|_{L^{4/3}} \le R/2$, the same argument provides a solution $\tilde{\omega}$ of (2.2) on the time interval $[T, T+T_1]$ with $\tilde{\omega}(T) = \omega(T)$, and arguing as in [3] one can verify that $\tilde{\omega} \in C^0([T,T+T_1], L^1(\mathbb{R}^2)\cap L^{4/3}(\mathbb{R}^2))$. Thus, if we glue together ω and $\tilde{\omega}$, we obtain a local solution $\omega \in C^0((0,T+T_1], L^1(\mathbb{R}^2)\cap L^{4/3}(\mathbb{R}^2))$ of (2.2). Iterating this procedure, and using the fact that $\|\omega(t)\|_{L^{4/3}}$ is a nonincreasing function of time, see (2.11), we can construct a (unique) local solution on the time interval $(0, T + kT_1]$ for any $k \in \mathbb{N}$. This means that the solution of (2.2) is global if $2C_2\|\mu_{pp}\|_{\mathrm{tv}} < R$.

The proof of Theorem 2.5 is now complete except for two minor points. First, once we know that the solution ω of (2.2) lies in $L^{4/3}(\mathbb{R}^2)$ for positive times, a standard bootstrap argument shows that $\omega(t) \in L^\infty(\mathbb{R}^2)$ for $t > 0$, as asserted in Theorem 2.5. More importantly, while the fixed point argument only shows that the solution is unique in a ball of the space X_T, one can in fact prove uniqueness in the whole space $C_b^0((0,T], L^1(\mathbb{R}^2)) \cap C^0((0,T], L^\infty(\mathbb{R}^2))$, for any

$T > 0$. Establishing this improved uniqueness property requires additional arguments which will be presented in Section 3.1 below.

Theorem 2.5 shows that the vorticity equation (2.2) is globally well-posed if the atomic part of the initial data satisfies $\|\mu_{pp}\|_{\rm tv} \le C_0\nu$ for some $C_0 > 0$. As was already mentioned, this smallness condition can be completely relaxed, and the optimal result is:

Theorem 2.8 [34, 25]
For any $\mu \in \mathcal{M}(\mathbb{R}^2)$, the vorticity equation (2.2) has a unique global solution

$$\omega \in C^0((0,\infty), L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$$

such that $\|\omega(\cdot,t)\|_{L^1} \le \|\mu\|_{\rm tv}$ for all $t > 0$, and $\omega(\cdot,t) \rightharpoonup \mu$ as $t \to 0$.

The existence part in Theorem 2.8 can be established using a relatively standard approximation procedure, see [34], but proving uniqueness without any restriction on the initial data requires a careful treatment of the large Dirac masses in the initial measure. We refer to Section 3 below for a sketch of the uniqueness proof, and to [25] for full details.

2.3.4 Short Time Self-Interaction

To conclude this section, we state and prove an auxiliary result which was used in the proof of Theorem 2.5.

Lemma 2.9 *Let $\mu \in \mathcal{M}(\mathbb{R}^2)$. For any $t > 0$, let $\omega_0(t) = e^{t\Delta}\mu$ and let $u_0(t) = K * \omega_0(t)$ be the corresponding velocity field, given by the Biot-Savart law (2.4). Then*

$$\ell_p(\mu) \,:=\, \limsup_{t\to 0}\, t^{1-\frac{1}{p}}\|(F\omega_0)(t)\|_{L^p} \,=\, 0~, \qquad 1 \le p \le \infty~, \tag{2.29}$$

where

$$(F\omega_0)(t) \,=\, \int_0^t e^{(t-s)\Delta}\, u_0(s)\cdot\nabla\omega_0(s)\,{\rm d}s~, \qquad t > 0~.$$

Proof. We shall prove (2.29) for $p \in [1,2)$. The other values of p can be treated in a similar way. Arguing as in (2.25), we find

$$t^{1-\frac{1}{p}}\|(F\omega_0)(t)\|_{L^p} \,\le\, t^{1-\frac{1}{p}}\int_0^t \frac{C}{(t-s)^{\frac{3}{2}-\frac{1}{p}}}\,\|\omega_0(s)\|_{L^{4/3}}^2\,{\rm d}s~, \quad t > 0~.$$

It follows that $\ell_p(\mu) \le C_p(L_{4/3}(\mu))^2$, where C_p is a constant depending only on p and $L_p(\mu)$ is defined in (2.20). This already shows that $\ell_p(\mu) = 0$ if $\mu_{pp} = 0$. More generally, the argument above implies that

$$|\ell_p(\mu_1+\mu_2) - \ell_p(\mu_1) - \ell_p(\mu_2)| \,\le\, 2C_pL_{4/3}(\mu_1)L_{4/3}(\mu_2)~, \tag{2.30}$$

for any $\mu_1, \mu_2 \in \mathcal{M}(\mathbb{R}^2)$. Taking $\mu_1 = \mu - \mu_{pp}$ and $\mu_2 = \mu_{pp}$, we conclude that $\ell_p(\mu) = \ell_p(\mu_{pp})$. This means that the non-atomic part of the measure μ does not contribute to the limit $\ell_p(\mu)$. It is thus sufficient to prove (2.29) in the case where μ is purely atomic, and in view of (2.30) we can even suppose that μ is a finite linear combination of Dirac masses: $\mu = \sum_{i=1}^N \alpha_i\delta_{x_i}$. In that case,

$$\omega_0(x,t) \,=\, \sum_{i=1}^N \frac{\alpha_i}{t}\,G\Big(\frac{x-x_i}{\sqrt{t}}\Big)~, \qquad u_0(x,t) \,=\, \sum_{i=1}^N \frac{\alpha_i}{\sqrt{t}}\,v^G\Big(\frac{x-x_i}{\sqrt{t}}\Big)~,$$

where G and v^G are defined in (3.2) below. Let

$$H(x,t) \;=\; u_0(x,t)\cdot\nabla\omega_0(x,t) \;=\; \sum_{i\neq j}\frac{\alpha_i\alpha_j}{t^2}\,v^G\Big(\frac{x-x_i}{\sqrt{t}}\Big)\cdot\nabla G\Big(\frac{x-x_j}{\sqrt{t}}\Big)\;.$$

Here the sum runs over indices $i\neq j$ because the contributions of the self-interaction terms $i=j$ are identically zero. Using this observation, it is rather straightforward to verify that

$$\limsup_{t\to 0}\, t^{\frac{3}{2}-\frac{1}{p}}\|H(t)\|_{L^p} \;<\; \infty\;, \qquad 1\le p\le\infty\;.$$

As $p<\infty$, it follows that

$$t^{1-\frac{1}{p}}\|(F\omega_0)(t)\|_{L^p} \;\le\; t^{1-\frac{1}{p}}\int_0^t \frac{C}{(t-s)^{1-\frac{1}{p}}}\,\|H(s)\|_{L^1}\mathrm{d}s \;\xrightarrow[t\to 0]{}\; 0\;.$$

This concludes the proof. □

3 Self-Similar Variables and Long-Time Behavior

Explicit examples of two-dimensional viscous flows are easily constructed if one assumes radial symmetry. Indeed, if the vorticity $\omega(x,t)$ is radially symmetric, the velocity field $u=K*\omega$ given by the Biot-Savart law (2.4) is *azimuthal*, in the sense that $x\cdot u(x,t)\equiv 0$. As a consequence, the nonlinearity $u\cdot\nabla\omega$ in (2.2) vanishes identically, and (2.2) therefore reduces to the linear heat equation $\partial_t\omega=\nu\Delta\omega$ (which preserves radial symmetry).

As an example, consider the particular case where the initial vorticity is a point vortex of circulation $\gamma\in\mathbb{R}$ located at the origin: $\mu=\gamma\delta_0$. The unique solution of (2.2) given by Theorem 2.8 is the *Lamb-Oseen vortex*:

$$\omega(x,t) \;=\; \frac{\gamma}{\nu t}\,G\Big(\frac{x}{\sqrt{\nu t}}\Big)\;, \qquad u(x,t) \;=\; \frac{\gamma}{\sqrt{\nu t}}\,v^G\Big(\frac{x}{\sqrt{\nu t}}\Big)\;, \tag{3.1}$$

where the vorticity and velocity profiles are given by

$$G(\xi) \;=\; \frac{1}{4\pi}\,e^{-|\xi|^2/4}\;, \qquad v^G(\xi) \;=\; \frac{1}{2\pi}\,\frac{\xi^\perp}{|\xi|^2}\Big(1-e^{-|\xi|^2/4}\Big)\;, \qquad \xi\in\mathbb{R}^2\;. \tag{3.2}$$

Note that $\int_{\mathbb{R}^2}G(\xi)\,\mathrm{d}\xi \;=\; 1$, so that $\gamma \;=\; \int_{\mathbb{R}^2}\omega(x,t)\,\mathrm{d}x$ is the total circulation of the vortex. Oseen's vortex is thus a *self-similar solution* of (2.2) with Gaussian vorticity profile G. The velocity profile v^G is azimuthal, vanishes at the origin, and satisfies $|v^G(\xi)|\;\sim\;(2\pi|\xi|)^{-1}$ as $|\xi|\;\to\;\infty$. In particular, $v^G\;\notin\;L^2(\mathbb{R}^2)$, in agreement with Lemma 2.4. Oseen vortices are therefore infinite energy solutions of the two-dimensional Navier-Stokes equation.

Oseen's vortex plays an important role in the dynamics of the vorticity equation (2.2). As was already observed, it can be considered as the "fundamental solution" of (2.2); i.e., the solution with a single Dirac mass as initial data. In addition, the following result shows that it describes the long-time asymptotics of all integrable solutions of (2.2):

Theorem 3.1 [33, 31]
For all initial data $\mu\in\mathcal{M}(\mathbb{R}^2)$, the solution $\omega(x,t)$ of (2.2) given by Theorem 2.8 satisfies

$$\lim_{t\to\infty}\Big\|\omega(x,t)-\frac{\gamma}{\nu t}G\Big(\frac{x}{\sqrt{\nu t}}\Big)\Big\|_{L^1} \;=\; 0\;, \qquad \textit{where}\quad \gamma \;=\; \int_{\mathbb{R}^2}\mathrm{d}\mu\;. \tag{3.3}$$

It follows immediately from (3.3) that Oseen vortices are the only self-similar solutions of the Navier-Stokes equation in $\mathbb{R}^2$ for which the vorticity profile is integrable. As an aside, we mention that, following the approach of Cannone and Planchon [9], one can construct an infinite-dimensional family of small self-similar solutions of (2.1) for which $u \in L^{2,\infty}(\mathbb{R}^2)$ but $\omega \notin L^1(\mathbb{R}^2)$. Assuming, as in Theorem 3.1, that the initial measure is finite allows to eliminate all these "artificial" solutions, which are produced by fat tails in the initial data and not by the intrinsinc dynamics of Eq. (2.2). Also, Theorem 3.1 strongly suggests that Oseen vortices are *stable* solutions of (2.2) for all values of the circulation γ. This important question will be discussed in Section 4 below.

It is worth mentioning that the convergence result (3.3) is not constructive, and does not provide any estimate of the time needed for the solution of (2.2) to approach Oseen's vortex. Under strong localization assumptions, such estimates were obtained in [29], but these results are certainly not optimal. We also refer to [8] for an interesting attempt toward a more precise description of the intermediate asymptotics, using ideas from statistical mechanics.

In the rest of this section, we give a sketch of the proof of Theorem 3.1 and of the uniqueness part of Theorem 2.8. We first recall classical estimates for solutions of convection-diffusion equations, which are due to Osada [63]. We then introduce self-similar variables, which allow to compactify the solutions of (2.2) in $L^1(\mathbb{R}^2)$, and to transform the self-similar Oseen vortices into equilibria. Finally, we use Lyapunov functions to determine the ω-limit sets of solutions of (2.2), and we establish a "Liouville theorem" which implies both the convergence result (3.3) and the uniqueness of the solution of (2.2) with a single Dirac mass as initial vorticity.

3.1 Estimates for Convection-Diffusion Equations

Given $\nu > 0$, we consider a linear convection-diffusion equation of the form

$$\partial_t \omega(x,t) + U(x,t) \cdot \nabla \omega(x,t) \;=\; \nu \Delta \omega(x,t) \;, \tag{3.4}$$

where $x \in \mathbb{R}^2$, $t \in (0,T)$, and $U : \mathbb{R}^2 \times (0,T) \to \mathbb{R}^2$ is a given divergence-free vector field. We assume that the curl $\Omega = \partial_1 U_2 - \partial_2 U_1$ of the advection field U satisfies $\Omega \in C^0((0,T), L^1(\mathbb{R}^2))$ and $\|\Omega(\cdot,t)\|_{L^1(\mathbb{R}^2)} \le K_0 \nu$ for $t \in (0,T)$, where K_0 is a positive constant. Then, in view of [63], any solution of (3.4) can be represented as

$$\omega(x,t) \;=\; \int_{\mathbb{R}^2} \Gamma^\nu_U(x,t;y,s)\omega(y,s)\,\mathrm{d}y \;, \quad x \in \mathbb{R}^2 \;, \quad 0 < s < t < T \;, \tag{3.5}$$

where Γ^ν_U is the *fundamental solution* of the convection-diffusion equation (3.4). The following properties of Γ^ν_U will be useful:

- There exist $\beta > 0$ and $K_1 > 0$ (depending only on K_0) such that

$$0 \;<\; \Gamma^\nu_U(x,t;y,s) \;\le\; \frac{K_1}{\nu(t-s)}\,\exp\Bigl(-\beta\frac{|x-y|^2}{4\nu(t-s)}\Bigr) \;, \tag{3.6}$$

for $x, y \in \mathbb{R}^2$ and $0 < s < t < T$, see [63]. In fact, a more precise result due to Carlen and Loss [10] shows that, if $(\nu t)^{1/2}\|U(\cdot,t)\|_{L^\infty} \le K_0 \nu$ for all $t \in (0,T)$, then (3.6) holds for any $\beta \in (0,1)$, with a constant K_1 depending on K_0 and β. It is also possible to establish a Gaussian lower bound on Γ^ν_U, see [63].

- There exists $\gamma \in (0,1)$ (depending only on K_0) and, for any $\delta > 0$, there exists $K_2 > 0$

(depending only on K_0 and δ) such that

$$|\Gamma_U^\nu(x,t;y,s) - \Gamma_U^\nu(x',t';y',s')| \\ \le K_2\Big(|x-x'|^\gamma + |\nu(t-t')|^{\gamma/2} + |y-y'|^\gamma + |\nu(s-s')|^{\gamma/2}\Big) \;, \qquad (3.7)$$

whenever $\nu(t-s) \ge \delta$ and $\nu(t'-s') \ge \delta$, see [63].

• For $0 < s < t < T$ and $x, y \in \mathbb{R}^2$,

$$\int_{\mathbb{R}^2} \Gamma_U^\nu(x,t;y,s)\,\mathrm{d}x \;=\; 1 \;, \qquad \int_{\mathbb{R}^2} \Gamma_U^\nu(x,t;y,s)\,\mathrm{d}y \;=\; 1 \;. \qquad (3.8)$$

For $0 < s < r < t < T$ and $x, y \in \mathbb{R}^2$,

$$\Gamma_U^\nu(x,t;y,s) \;=\; \int_{\mathbb{R}^2} \Gamma_U^\nu(x,t;z,r)\Gamma_U^\nu(z,r;y,s)\,\mathrm{d}z \;. \qquad (3.9)$$

Remark 3.2 *If $x, y \in \mathbb{R}^2$ and $t > 0$, it follows from* (3.7) *that the function $s \mapsto \Gamma_U^\nu(x,t;y,s)$ can be continuously extended up to $s = 0$, and that this extension (still denoted by Γ_U^ν) satisfies properties* (3.6) *to* (3.9) *with $s = 0$.*

As an application of these results, we give a new formulation of the uniqueness claims in Theorems 2.5 and 2.8. Assume that $\omega \in C^0((0,T), L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$ is a solution of (2.2) which is uniformly bounded in $L^1(\mathbb{R}^2)$: there exists $K_0 > 0$ such that $\|\omega(t)\|_{L^1} \le K_0\nu$ for $t \in (0,T)$. Let $\phi \in C_0^\infty(\mathbb{R}^2)$ be a test function. Using (2.2) and integrating by parts, we easily find

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} \omega(x,t)\phi(x)\,\mathrm{d}x \;=\; \int_{\mathbb{R}^2} \omega(x,t)u(x,t)\cdot\nabla\phi(x)\,\mathrm{d}x + \nu\int_{\mathbb{R}^2} \omega(x,t)\Delta\phi(x)\,\mathrm{d}x \;.$$

If we express u in terms of ω using the Biot-Savart law (2.4), we can rewrite the first integral on the right-hand side as

$$\frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \omega(x,t)\omega(y,t)\frac{(x-y)^\perp}{|x-y|^2}\cdot\Big(\nabla\phi(x) - \nabla\phi(y)\Big)\,\mathrm{d}x\,\mathrm{d}y \;,$$

see e.g. [20]. Since ω is uniformly bounded in $L^1(\mathbb{R}^2)$, we conclude that

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} \omega(x,t)\phi(x)\,\mathrm{d}x\Big| \;\le\; C\nu^2 \sum_{|\alpha|=2} \|\partial^\alpha\phi\|_{L^\infty} \;, \qquad t \in (0,T) \;,$$

for some constant $C > 0$ depending only on K_0. This implies that $\omega(t)$ has a limit in $D'(\mathbb{R}^2)$ as $t \to 0$, which we denote by μ. As $\|\omega(t)\|_{L^1} \le K_0\nu$, it follows that $\mu \in \mathcal{M}(\mathbb{R}^2)$ with $\|\mu\|_{\mathrm{tv}} \le K_0\nu$, and that $\omega(t) \rightharpoonup \mu$ as $t \to 0$.

On the other hand, since ω solves (3.4) with $U(x,t) = u(x,t)$, and since the curl of the advection field is uniformly bounded in $L^1(\mathbb{R}^2)$, we have the representation (3.5), where the fundamental solution $\Gamma_u^\nu(x,t;y,s)$ satisfies (3.6) to (3.9). In particular, using Remark 3.2, we have for all $x \in \mathbb{R}^2$ and all $t \in (0,T)$,

$$\omega(x,t) \;=\; \int_{\mathbb{R}^2} \Gamma_u^\nu(x,t;y,0)\omega(y,s)\,\mathrm{d}y \\ + \int_{\mathbb{R}^2} (\Gamma_u^\nu(x,t;y,s) - \Gamma_u^\nu(x,t;y,0))\omega(y,s)\,\mathrm{d}y \;, \quad 0 < s < t \;.$$

In view of (3.7), the second integral on the right-hand side converges to zero as $s \to 0$. On the other hand, since $y \mapsto \Gamma_u^\nu(x,t;y,0)$ is continuous and vanishes at infinity, and since $\omega(s) \rightharpoonup \mu$ as $s \to 0$, we can take the limit $s \to 0$ in the first integral and we obtain the useful formula

$$\omega(x,t) \;=\; \int_{\mathbb{R}^2} \Gamma_u^\nu(x,t;y,0)\,\mathrm{d}\mu_y\;, \qquad x \in \mathbb{R}^2\;, \quad 0 < t < T\;, \tag{3.10}$$

which shows that any solution $\omega(x,t)$ of (2.2) which is uniformly bounded in $L^1(\mathbb{R}^2)$ can be represented in terms of its trace μ at $t = 0$, using the corresponding fundamental solution Γ_u^ν.

In particular, since $\Gamma_u^\nu(x,t;y,0)$ is positive and satisfies (3.8), it follows from (3.10) that $\|\omega(t)\|_{L^1} \le \|\mu\|_{\mathrm{tv}}$ for all $t \in (0,T)$. Thus we can assume that $K_0 = \|\mu\|_{\mathrm{tv}}$ without loss of generality. Moreover, using the upper bound (3.6) we deduce from (3.10) that

$$\sup_{0<t<T} (\nu t)^{1-\frac{1}{p}}\|\omega(t)\|_{L^p} \;\le\; C\|\mu\|_{\mathrm{tv}}\;, \qquad 1 \le p \le \infty\;, \tag{3.11}$$

where C is a positive constant depending only on K_0. In fact, a more careful argument shows that the constant $C > 0$ in (3.11) is independent of K_0, see [3, 42, 10]. Similarly, repeating the proof of Lemma 2.6, we find

$$\limsup_{t\to 0}(\nu t)^{1-\frac{1}{p}}\|\omega(t)\|_{L^p} \;\le\; C\|\mu_{pp}\|_{\mathrm{tv}}\;, \qquad 1 < p \le \infty\;. \tag{3.12}$$

Taking $p = 4/3$, we deduce from (3.11), (3.12) that the solution ω of (2.2) lies in the space X_T defined in (2.23), and that $\|\omega\|_{X_T}$ is small if $\|\mu\|_{\mathrm{tv}}$ is small, or if $\|\mu_{pp}\|_{\mathrm{tv}}$ and T are small. In both cases, ω necessarily coincides with the solution of (2.2) constructed by the fixed point argument.

3.2 Self-Similar Variables

Let $\omega \in C^0((0,\infty), L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$ be a solution of (2.2). Given $x_0 \in \mathbb{R}^2$, $t_0 > 0$, and $T > 0$, we introduce the *self-similar variables*

$$\xi \;=\; \frac{x - x_0}{\sqrt{\nu(t+t_0)}}\;, \qquad \tau \;=\; \log\Bigl(\frac{t+t_0}{T}\Bigr)\;, \tag{3.13}$$

and we transform the vorticity and velocity fields as follows:

$$\begin{aligned} \omega(x,t) \;&=\; \frac{1}{t+t_0}\, w\left(\frac{x-x_0}{\sqrt{\nu(t+t_0)}}\,,\, \log\Bigl(\frac{t+t_0}{T}\Bigr)\right)\;, \\ u(x,t) \;&=\; \sqrt{\frac{\nu}{t+t_0}}\; v\left(\frac{x-x_0}{\sqrt{\nu(t+t_0)}}\,,\, \log\Bigl(\frac{t+t_0}{T}\Bigr)\right)\;. \end{aligned} \tag{3.14}$$

Then the rescaled vorticity $w(\xi,\tau)$ satisfies the evolution equation

$$\partial_\tau w + v\cdot\nabla_\xi\, w \;=\; \Delta_\xi\, w + \frac{1}{2}\,\xi\cdot\nabla_\xi\, w + w\;, \tag{3.15}$$

and the rescaled velocity v is again expressed in terms of w via the Biot-Savart law (2.4): $v = K * w$. The initial data of (2.2) and (3.15) are related via

$$w(\xi,\tau_0) \;=\; t_0\,\omega(x_0 + \xi\sqrt{\nu t_0}, 0)\;, \qquad \xi \in \mathbb{R}^2\;,$$

where $\tau_0 = \log(t_0/T)$. Note that $w(\xi,\tau)$ and $v(\xi,\tau)$ are now dimensionless quantities, as are the new space and time variables ξ, τ. If $w(\xi,\tau) = \alpha G(\xi)$ for some $\alpha \in \mathbb{R}$, then by (3.14) $\omega(x,t)$ is Oseen's vortex with circulation $\gamma = \alpha\nu$ located at point $x_0 \in \mathbb{R}^2$ and originating from time $-t_0$. Thus Oseen vortices are now *equilibria* of the rescaled system (3.15). In what follows, we shall use the change of variables (3.14) with $x_0 = 0$ and $t_0 = T$, so that $\tau_0 = 0$, but other choices of x_0 or t_0 are sometimes more appropriate, see e.g. [29].

The main effect of the transformation (3.14) is to replace the diffusion operator $\nu\Delta$ on the right-hand side of (2.2) by the more complicated operator

$$\mathcal{L} \,=\, \Delta + \frac{1}{2}\xi\cdot\nabla + 1\;, \tag{3.16}$$

which appears in (3.15). As we shall see in Section 4, the operator $\mathcal{L}$ has better spectral properties than the Laplacian, when acting on appropriate function spaces. Moreover, the semigroup $e^{\tau\mathcal{L}}$ generated by $\mathcal{L}$ is *asymptotically confining*, a property that does not hold for the heat semigroup $e^{\nu t\Delta}$. In fact, using (2.18) and (3.14), it is straightforward to derive the explicit formula

$$(e^{\tau\mathcal{L}}w_0)(\xi) \,=\, \frac{1}{4\pi a(\tau)}\int_{\mathbb{R}^2}\exp\Bigl(-\frac{|\xi-\eta\, e^{-\tau/2}|^2}{4a(\tau)}\Bigr)w_0(\eta)\,\mathrm{d}\eta\;, \tag{3.17}$$

for all $\xi \in \mathbb{R}^2$ and all $\tau > 0$, where $a(\tau) = 1-e^{-\tau}$. If $w_0 \in L^1(\mathbb{R}^2)$, then Lebesgue's dominated convergence theorem shows that $(e^{\tau\mathcal{L}}w_0)(\xi)$ converges to $\alpha G(\xi)$ as $\tau \to \infty$, where $\alpha = \int_{\mathbb{R}^2} w_0\,\mathrm{d}\xi$ and G is given by (3.2). Thus the solution of the linear equation $\partial_\tau w = \mathcal{L}w$ is asymptotically confined like a Gaussian function provided the initial data are integrable.

A similar property holds for the nonlinear equation (3.15), and allows to show that the trajectories of this system are compact.

Lemma 3.3 *For any $w_0 \in L^1(\mathbb{R}^2)$, the solution $\{w(\tau)\}_{\tau\ge 0}$ of (3.15) with initial data w_0 is relatively compact in $L^1(\mathbb{R}^2)$.*

Proof. We first observe that the Cauchy problem for Eq. (3.15) is globally well-posed in the space $L^1(\mathbb{R}^2)$. This can be deduced, via the change of variables (3.14), from the corresponding statement for the original equation (2.2), which is a particular case of Theorem 2.5. This argument also shows that the L^1 norm of the solution $w(\tau)$ is nonincreasing with time, and that the circulation parameter $\alpha = \int_{\mathbb{R}^2} w(\xi,\tau)\,\mathrm{d}\xi$ is a conserved quantity. Now, using the representation (3.10) of the solution of (2.2) in terms of the initial data, and the bound (3.6) on the corresponding fundamental solution, we obtain the following estimate for the solution of (3.15) :

$$|w(\xi,\tau)| \,\le\, \frac{K_1}{a(\tau)}\int_{\mathbb{R}^2}\exp\Bigl(-\beta\frac{|\xi-\eta e^{-\tau/2}|^2}{4a(\tau)}\Bigr)|w_0(\eta)|\,\mathrm{d}\eta\;, \tag{3.18}$$

for $\xi \in \mathbb{R}^2$ and $\tau > 0$, where again $a(\tau) = 1-e^{-\tau}$. Here $\beta \in (0,1)$ and $K_1 > 0$ depend only on $\|w_0\|_{L^1}$. Using this bound, it is rather easy to show that

$$\sup_{\tau\ge 1}\int_{|\xi|\ge R}|w(\xi,\tau)|\,\mathrm{d}\xi \;\xrightarrow[R\to\infty]{}\; 0\;, \tag{3.19}$$

see [31, Lemma 2.5]. On the other hand, classical estimates for the derivatives of solutions of (2.2) ensure that

$$\sup_{\tau\ge 1}\sup_{\xi\in\mathbb{R}^2}|\nabla w(\xi,\tau)| \,<\, \infty\;,$$

see [3, 43]. This together with (3.19) implies that

$$\sup_{\tau\ge 1}\sup_{|\eta|\le\delta}\int_{\mathbb{R}^2}|w(\xi-\eta,\tau)-w(\xi,\tau)|\,\mathrm{d}\xi \;\xrightarrow[\delta\to 0]{}\; 0\;. \tag{3.20}$$

By a well-known criterion of Riesz [66], (3.19) and (3.20) together imply that the trajectory $\{w(\tau)\}_{\tau\ge 1}$ is relatively compact in $L^1(\mathbb{R}^2)$. Since we also know that $w\in C^0([0,1],L^1(\mathbb{R}^2))$, the conclusion follows. □

If $w_0\in L^1(\mathbb{R}^2)$, it follows from Lemma 3.3 that the solution $w(\tau)$ of (3.15) with initial data w_0 converges as $\tau\to\infty$, in the L^1 topology, to the ω-limit set

$$\Omega(w_0) \;=\; \bigcap_{T\ge 0}\overline{\{w(\tau)\,|\,\tau\ge T\}}\;,$$

where the overline denotes here the closure in $L^1(\mathbb{R}^2)$. This is a nonempty compact and connected subset of $L^1(\mathbb{R}^2)$ which is positively and negatively invariant under the evolution defined by (3.15), in the following sense: for any $\bar w_0\in\Omega(w_0)$, there exists a complete trajectory $\bar w\in C^0(\mathbb{R},L^1(\mathbb{R}^2))$ of (3.15) such that $\bar w(\tau)\in\Omega(w_0)$ for all $\tau\in\mathbb{R}$ and $\bar w(0)=\bar w_0$. In the next section, we shall prove that $\Omega(w_0)$ consists of a single point $\{\alpha G\}$, where $\alpha=\int_{\mathbb{R}^2}w_0\,\mathrm{d}\xi$.

3.3 Lyapunov Functions and Liouville's Theorem

Lemma 3.3 shows that positive trajectories of (3.15) in $L^1(\mathbb{R}^2)$ are relatively compact. The same conclusion does not apply to negative trajectories in general. In fact, the following proposition, which is the main result of this section, implies that the equilibria associated to Oseen vortices are the only negative trajectories of (3.15) which are relatively compact in $L^1(\mathbb{R}^2)$.

Proposition 3.4 ("Liouville's theorem") *If $\{w(\tau)\}_{\tau\in\mathbb{R}}$ is a complete trajectory of* (3.15) *which is relatively compact in $L^1(\mathbb{R}^2)$, then there exists $\alpha\in\mathbb{R}$ such that $w(\tau)=\alpha G$ for all $\tau\in\mathbb{R}$.*

Proof. Let $\mathcal{A}$ and Ω denote the α-limit set and the ω-limit set of the trajectory $\{w(\tau)\}_{\tau\in\mathbb{R}}$ in $L^1(\mathbb{R}^2)$:

$$\mathcal{A} \;=\; \bigcap_{T\le 0}\overline{\{w(\tau)\,|\,\tau\le T\}}\;,\qquad \Omega \;=\; \bigcap_{T\ge 0}\overline{\{w(\tau)\,|\,\tau\ge T\}}\;.$$

Both $\mathcal{A}$, Ω are nonempty compact sets in $L^1(\mathbb{R}^2)$. Moreover, they attract $w(\tau)$ in the sense that $\mathrm{dist}_{L^1}(w(\tau),\mathcal{A})\to 0$ as $\tau\to-\infty$, while $\mathrm{dist}_{L^1}(w(\tau),\Omega)\to 0$ as $\tau\to+\infty$. Following [31], to characterize these limit sets more precisely, we use a pair of Lyapunov functions for Eq. (3.15).

First step. Our first Lyapunov function is just the L^1 norm of the solution. Let $\Phi: L^1(\mathbb{R}^2)\to\mathbb{R}_+$ be the continuous function defined by

$$\Phi(w) \;=\; \int_{\mathbb{R}^2}|w(\xi)|\,\mathrm{d}\xi\;,\qquad w\in L^1(\mathbb{R}^2)\;,$$

and let

$$\Sigma \;=\; \left\{w\in L^1(\mathbb{R}^2)\,\Big|\,\int_{\mathbb{R}^2}|w(\xi)|\,\mathrm{d}\xi=\Big|\int_{\mathbb{R}^2}w(\xi)\,\mathrm{d}\xi\Big|\right\}\;.$$

In other words, a function $w\in L^1(\mathbb{R}^2)$ belongs to Σ if and only if $w(\xi)$ has almost everywhere a constant sign. We have already mentioned that Φ is nonincreasing along trajectories of (3.15), as a consequence of (2.11). Now, using the strong maximum principle, it is straightforward to verify that Φ is in fact strictly decreasing, except along trajectories which lie in Σ: if $\{\bar w(\tau)\}_{\tau\ge 0}$

is a solution of (3.15) such that $\Phi(\bar{w}(\tau)) = \Phi(\bar{w}(0))$ for all $\tau \ge 0$, then $\bar{w}(0) \in \Sigma$ (hence $\bar{w}(\tau) \in \Sigma$ for all $\tau \ge 0$).

By LaSalle's invariance principle [71], the α-limit set of the trajectory $\{w(\tau)\}_{\tau\in\mathbb{R}}$ lies in the neutral set Σ, because $\mathcal{A}$ is positively invariant under the evolution defined by (3.15) and entirely contained in a level set of the Lyapunov function Φ. The same conclusion applies to the ω-limit set Ω. In view of the definition of Σ, it follows that $\Phi(\bar{w}) = |\alpha|$ for all $\bar{w} \in \mathcal{A}$ or Ω, where $\alpha = \int_{\mathbb{R}^2} w(\xi,\tau)\,\mathrm{d}\xi$ is the circulation of our solution (which is a conserved quantity). This in turn implies that $\Phi(w(\tau)) = |\alpha|$ for all $\tau \in \mathbb{R}$, which is possible only if the whole trajectory $\{w(\tau)\}_{\tau\in\mathbb{R}}$ lies in the neutral set Σ. Thus any relatively compact complete trajectory of (3.15) has necessarily a definite sign, which is the sign of the circulation parameter α. If $\alpha = 0$, this means that $w(\xi,\tau) \equiv 0$, in which case the proof is complete. If $\alpha < 0$, we observe that the substitution $w(\xi_1,\xi_2,\tau) \mapsto -w(\xi_2,\xi_1,\tau)$ leaves Eq. (3.15) unchanged, but reverses the sign of the circulation. Thus we can assume henceforth that $\alpha > 0$. In that case $w(\xi,\tau) > 0$ for all $\xi \in \mathbb{R}^2$ and all $\tau \in \mathbb{R}$, and proceeding as in Section 3.1 one can show that $w(\xi,\tau)$ is bounded from above and from below by time-independent Gaussian functions.

Second step. As a second Lyapunov function, we use the *relative entropy* of the vorticity distribution $w(\tau)$ with respect to the Gaussian G [79]. If $w : \mathbb{R}^2 \to \mathbb{R}_+$ is a (measurable) positive function with a Gaussian upper bound, we define

$$H(w) \;=\; \int_{\mathbb{R}^2} w(\xi)\log\Bigl(\frac{w(\xi)}{G(\xi)}\Bigr)\,\mathrm{d}\xi\;. \tag{3.21}$$

Then H is nonincreasing along trajectories of (3.15), because

$$\frac{\mathrm{d}}{\mathrm{d}\tau}H(w(\tau)) \;=\; -\int_{\mathbb{R}^2} w(\xi,\tau)\Bigl|\nabla\log\Bigl(\frac{w(\xi,\tau)}{G(\xi)}\Bigr)\Bigr|^2\,\mathrm{d}\xi \;\le\; 0\;. \tag{3.22}$$

To prove (3.22), we compute

$$\frac{\mathrm{d}}{\mathrm{d}\tau}H(w(\tau)) \;=\; \int_{\mathbb{R}^2}\Bigl(1+\log\frac{w}{G}\Bigr)\partial_\tau w\,\mathrm{d}\xi = \int_{\mathbb{R}^2}\Bigl(1+\log\frac{w}{G}\Bigr)(\mathcal{L}w - v\cdot\nabla w)\,\mathrm{d}\xi\;.$$

Using the identity $\mathcal{L}w = \mathrm{div}(G\nabla(\frac{w}{G}))$ and integrating by parts, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2}\Bigl(1+\log\frac{w}{G}\Bigr)(\mathcal{L}w)\,\mathrm{d}\xi \;&=\; -\int_{\mathbb{R}^2}\nabla\Bigl(\log\frac{w}{G}\Bigr)\cdot\frac{G}{w}\nabla\Bigl(\frac{w}{G}\Bigr)w\,\mathrm{d}\xi\\ &=\; -\int_{\mathbb{R}^2} w\Bigl|\nabla\Bigl(\log\frac{w}{G}\Bigr)\Bigr|^2\,\mathrm{d}\xi\;.\end{aligned}$$

On the other hand, using $v\cdot\nabla w = \mathrm{div}(vw)$ and integrating by parts, we find

$$\begin{aligned}\int_{\mathbb{R}^2}\Bigl(1+\log\frac{w}{G}\Bigr)(v\cdot\nabla w)\,\mathrm{d}\xi \;&=\; \int_{\mathbb{R}^2}(1+\log(4\pi w))(v\cdot\nabla w)\,\mathrm{d}\xi + \int_{\mathbb{R}^2}\frac{|\xi|^2}{4}(v\cdot\nabla w)\,\mathrm{d}\xi\\ &=\; -\int_{\mathbb{R}^2} v\cdot\nabla w\,\mathrm{d}\xi - \frac{1}{2}\int_{\mathbb{R}^2}(\xi\cdot v)w\,\mathrm{d}\xi \;=\; 0\;.\end{aligned}$$

Note that the last integral vanishes because $v = K * w$, see (2.10). This concludes the proof of inequality (3.22), which shows in addition that H is strictly decreasing along trajectories of (3.15), except on the line of equilibria corresponding to Oseen vortices. More precisely, if $\{\bar{w}(\tau)\}_{\tau\ge 0}$ is a positive solution of (3.15) (with a Gaussian upper bound) such that $H(\bar{w}(\tau)) = H(\bar{w}(0))$ for all $\tau \ge 0$, then $\bar{w}(0) \in \mathcal{T}$, where $\mathcal{T} = \{\beta G\}_{\beta>0}$ is the half-line of positive equilibria.

We now return to the analysis of our complete trajectory $\{w(\tau)\}_{\tau\in\mathbb{R}}$ of (3.15). We know from the first step that $w(\tau) \in \Sigma$ for all $\tau \in \mathbb{R}$, and without loss of generality we assume that the circulation parameter $\alpha = \int_{\mathbb{R}^2} w(\xi,\tau)\,\mathrm{d}\xi$ is positive, which implies that $w(\xi,\tau)$ is positive and has a Gaussian upper bound. By LaSalle's invariance principle, both the α-limit set $\mathcal{A}$ and the ω-limit set Ω are contained in the line of equilibria $\mathcal{T}$, because these sets are positively invariant and contained in a level set of the Lyapunov function H. The conservation of the total circulation then implies that $\mathcal{A} = \Omega = \{\alpha G\}$, hence $H = \alpha\log(\alpha)$ on $\mathcal{A}$ and Ω. It follows that $H(w(\tau)) = \alpha\log(\alpha)$ for all $\tau \in \mathbb{R}$, which is possible only if $w(\tau) \in \mathcal{T}$ for all $\tau \in \mathbb{R}$. But $\mathcal{T}$ is a line of equilibria, hence we must have $w(\tau) = \alpha G$ for all $\tau \in \mathbb{R}$, as stated in Proposition 3.4. □

As a first application of Proposition 3.4, we show that Oseen vortices describe the long-time asymptotics of all solutions of (2.2) in $L^1(\mathbb{R}^2)$.

Proof of Theorem 3.1. Assume that $\omega \in C^0((0,\infty), L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$ is a solution of (2.2) with initial data $\mu \in \mathcal{M}(\mathbb{R}^2)$, in the sense of Theorem 2.8, and let $w(\xi,\tau)$ be the rescaled vorticity obtained from $\omega(x,t)$ via the change of variables (3.14), with an arbitrary choice of the center $x_0 \in \mathbb{R}^2$ and the initial time $t_0 = T > 0$. Then $w \in C^0((0,\infty), L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$ is a solution of the rescaled vorticity equation (3.15) such that

$$\alpha = \int_{\mathbb{R}^2} w(\xi,\tau)\,\mathrm{d}\xi \;=\; \frac{\gamma}{\nu}\;, \qquad \text{where} \quad \gamma \;=\; \int_{\mathbb{R}^2} \omega(x,t)\,\mathrm{d}x \;=\; \int_{\mathbb{R}^2} \mathrm{d}\mu\;. \tag{3.23}$$

By Lemma 3.3, the trajectory $\{w(\tau)\}_{\tau\ge 1}$ is relatively compact in $L^1(\mathbb{R}^2)$. Let Ω denote the ω-limit set of this trajectory in $L^1(\mathbb{R}^2)$. Then Ω is a nonempty compact subset of $L^1(\mathbb{R}^2)$ which is positively and negatively invariant under the evolution defined by (3.15). In other words, Ω is a collection of complete, relatively compact trajectories of (3.15). By Liouville's theorem, all such trajectories are equilibria of the form βG, for some $\beta \in \mathbb{R}$. But the conservation of the total circulation implies that $\beta = \alpha$, so we conclude that $\Omega = \{\alpha G\}$. This exactly means that $w(\tau) \to \alpha G$ in $L^1(\mathbb{R}^2)$ as $\tau \to \infty$. Returning to the original variables, we obtain (3.3). □

Remark 3.5 *Under the assumptions of Theorem 3.1, the above proof shows that*

$$\lim_{t\to\infty} \Big\|\omega(x,t) - \frac{\gamma}{\nu(t+t_0)}\,G\Big(\frac{x-x_0}{\sqrt{\nu(t+t_0)}}\Big)\Big\|_{L^1} \;=\; 0\;, \tag{3.24}$$

for any $x_0 \in \mathbb{R}^2$ and any $t_0 > 0$. This might be surprising at first sight, but there is no contradiction because, as is easily verified, two Oseen vortices with different values of x_0 or t_0 converge to each other in the L^1 topology as $t \to \infty$. However, the choice of the vortex center and the initial time becomes important if one wants to determine the optimal decay rate in (3.24)*. This is possible if one assumes that the vorticity $\omega(x,t)$ decays to zero sufficiently fast as $|x| \to \infty$, see [31, 29] and Section 4 below.*

We next show that Proposition 3.4 implies the uniqueness of the solution of the vorticity equation (2.2) with a point vortex as initial data.

Proposition 3.6 [31, 27]
Let $\omega \in C^0((0,T), L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$ be a solution of (2.2) *which is bounded in $L^1(\mathbb{R}^2)$ and satisfies $\omega \rightharpoonup \gamma\delta_0$ as $t \to 0$, for some $\gamma \in \mathbb{R}$. Then*

$$\omega(x,t) \;=\; \frac{\gamma}{\nu t}\,G\Big(\frac{x}{\sqrt{\nu t}}\Big)\;, \qquad x \in \mathbb{R}^2\;, \quad 0 < t < T\;. \tag{3.25}$$

Proof. Applying (3.10) in the particular case where $\mu = \gamma\delta_0$, we obtain $\omega(x,t) = \gamma\Gamma_u^\nu(x,t;0,0)$, where Γ_u^ν is the fundamental solution of (3.4) with advection field $u = K * \omega$ given by the Biot-Savart law (2.4). In view of (3.6), we thus have

$$|\omega(x,t)| \le \frac{|\gamma|K_1}{\nu t} \exp\Big(-\beta\frac{|x|^2}{4\nu t}\Big) , \qquad x \in \mathbb{R}^2 , \quad 0 < t < T . \tag{3.26}$$

for some $K_1 > 0$ and $\beta \in (0,1)$. We now apply the change of variables (3.14) with $x_0 = 0$ and $t_0 = 0$. Since $\tau = \log(t/T)$, the rescaled vorticity $w(\xi,\tau)$ is then defined for all $\xi \in \mathbb{R}^2$ and all $\tau < 0$, so that $w \in C^0((-\infty,0), L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$ is a negative trajectory of (3.15). Moreover, the bound (3.26) implies that $|w(\xi,\tau)| \le (K_1|\gamma|/\nu)e^{-\beta|\xi|^2/4}$, for all $\xi \in \mathbb{R}^2$ and all $\tau < 0$. As in the proof of Lemma 3.3, this bound implies that the trajectory $\{w(\tau)\}_{\tau<0}$ is relatively compact in $L^1(\mathbb{R}^2)$. Applying Proposition 3.4, we conclude that $w(\xi,\tau) = \alpha G(\xi)$ for some $\alpha \in \mathbb{R}$, and the conservation of the total circulation implies that $\alpha = \gamma/\nu$. Thus $w(\xi,\tau) = (\gamma/\nu)G(\xi)$ for all $\xi \in \mathbb{R}^2$ and all $\tau < 0$, and returning to the original variables we obtain (3.25). □

It is important to emphasize that there is no restriction on the circulation γ in Proposition 3.6. Thus the conclusion cannot be obtained by a standard application of Gronwall's lemma, which would require that the ratio $|\gamma|/\nu$ be sufficiently small. Intuitively, Proposition 3.6 and Theorem 2.5 together imply that the solution of (2.2) with arbitrary initial data $\mu \in \mathcal{M}(\mathbb{R}^2)$ should be unique, as asserted in Theorem 2.8. Indeed, given any $\epsilon > 0$, we can decompose $\mu = \mu_1 + \mu_2$, where μ_1 is a finite superposition of Dirac masses and $\|\mu_2\|_{\mathrm{tv}} \le \epsilon$. If ϵ is sufficiently small, we know from Theorem 2.5 that (2.2) has a unique solution with initial data μ_2, and Proposition 3.6 implies that each atom in μ_1 also generates a unique solution of (2.2). Although the the vorticity equation (2.2) is nonlinear, this decomposition can be used to prove uniqueness if we can show, as in the particular case considered in Lemma 2.9, that the various components of the solution do not strongly interact for small times. These rough ideas can be turned into a rigorous proof, and we refer the interested reader to [25] for more details.

4 Asymptotic Stability of Oseen Vortices

As was mentioned in Section 3.2, the rescaled vorticity equation

$$\partial_\tau w + v\cdot\nabla_\xi w \;=\; \mathcal{L}w , \qquad \text{where} \quad \mathcal{L} \;=\; \Delta + \frac{1}{2}\,\xi\cdot\nabla + 1 , \tag{4.1}$$

has a line of equilibria $\{\alpha G\}_{\alpha\in\mathbb{R}}$ which correspond, in the original variables, to the family of self-similar Oseen vortices (3.1). In fact, Proposition 3.4 implies that these are the only steady states of (4.1) in the space $L^1(\mathbb{R}^2)$, and Theorem 3.1 even shows that the family $\{\alpha G\}_{\alpha\in\mathbb{R}}$ is globally attracting in the sense that any solution of (4.1) in $L^1(\mathbb{R}^2)$ converges to αG as $\tau \to \infty$, where $\alpha = \int_{\mathbb{R}^2} w(\xi,\tau)\,\mathrm{d}\xi$ is the circulation parameter. This strongly suggests, but does not rigorously prove, that αG is a stable equilibrium of (4.1) for any $\alpha \in \mathbb{R}$.

4.1 Stability with Respect to Localized Perturbations

The aim of this section is to study the flow of (4.1) in a neighborhood of Oseen's vortex αG, and to show that this steady state is indeed stable for any value of the circulation parameter $\alpha \in \mathbb{R}$. Setting $w = \alpha G + \tilde{w}$, $v = \alpha v^G + \tilde{v}$, we obtain the perturbation equation

$$\partial_\tau \tilde{w} + \tilde{v}\cdot\nabla\tilde{w} \;=\; (\mathcal{L} - \alpha\Lambda)\tilde{w} , \tag{4.2}$$

where

$$\Lambda\tilde{w} \;=\; v^G\cdot\nabla\tilde{w} + \tilde{v}\cdot\nabla G . \tag{4.3}$$

Here $\tilde{v} = K * \tilde{w}$ is the velocity field obtained from $\tilde{w}$ via the Biot-Savart law (2.4). The perturbation equation (4.2) is globally well-posed in $L^1(\mathbb{R}^2)$, and it can be proved that the origin $\tilde{w} = 0$ is a stable equilibrium, but in such a large function space it is impossible to obtain a more precise description of the long-time behavior of the solutions. Indeed, it follows from the results of [30, 31] that the spectrum of the linearized operator $\mathcal{L} - \alpha\Lambda$ in $L^1(\mathbb{R}^2)$ is exactly the full left-half plane $\{z \in \mathbb{C} \,|\, \mathrm{Re}(z) \le 0\}$, for any value of $\alpha \in \mathbb{R}$. This means that there exist perturbations $\tilde{w}$ which converge to zero at an arbitrarily slow rate.

For a more precise stability analysis, we have to assume that the perturbations $\tilde{w}$ have a faster decay as $|\xi| \to \infty$ than what is strictly needed for integrability. From now on, we write w, v instead of $\tilde{w}, \tilde{v}$, and we consider the perturbation equation (4.2) in the weighted space $X = L^2(\mathbb{R}^2, G^{-1}\,\mathrm{d}\xi)$ equipped with the scalar product

$$\langle w_1\,,\, w_2\rangle \;=\; \int_{\mathbb{R}^2} G(\xi)^{-1}\, w_1(\xi) w_2(\xi)\,\mathrm{d}\xi\;, \tag{4.4}$$

and the associated norm $\|w\|_X = \langle w, w\rangle^{1/2}$. Functions in X are locally in L^2 but have a Gaussian decay at infinity; in particular $X \hookrightarrow L^p(\mathbb{R}^2)$ for all $p \in [1, 2]$. This function space is convenient because, as we shall see, the linearized operator $\mathcal{L} - \alpha\Lambda$ has very nice spectral properties in X. However, if we do not not want to restrict ourselves to perturbations with Gaussian decay at infinity, it is possible to use the polynomially weighted space $L^2(m) = L^2(\mathbb{R}^2, (1 + |\xi|^2)^m\,\mathrm{d}\xi)$, for some $m > 1$ [31].

The choice of the Gaussian space X is justified by the following result :

Lemma 4.1 [30, 31, 50]
i) The operator $\mathcal{L}$ is selfadjoint *in the space $X = L^2(\mathbb{R}^2, G^{-1}\,\mathrm{d}\xi)$, with compact resolvent and purely discrete spectrum :*

$$\sigma(\mathcal{L}) \;=\; \Big\{-\frac{n}{2}\,\Big|\, n = 0, 1, 2, \dots\Big\}\;. \tag{4.5}$$

ii) The operator Λ is skew-symmetric *in the same space X :*

$$\langle \Lambda w_1\,,\, w_2\rangle + \langle w_1\,,\, \Lambda w_2\rangle \;=\; 0\;, \qquad \textit{for all} \quad w_1, w_2 \in D(\Lambda) \subset X\;. \tag{4.6}$$

Proof. To prove i), we conjugate $\mathcal{L}$ with the square root of the weight G to obtain the operator

$$L \;=\; G^{-1/2}\,\mathcal{L}\,G^{1/2} \;=\; \Delta - \frac{|\xi|^2}{16} + \frac{1}{2}\;, \tag{4.7}$$

which we have to consider as acting on $L^2(\mathbb{R}^2)$. Clearly L is the two-dimensional harmonic oscillator, which is known to be self-adjoint in $L^2(\mathbb{R}^2)$ with compact resolvent and discrete spectrum given by (4.5). Returning to the original operator, we obtain the desired conclusions, together with the following characterization of the domain of $\mathcal{L}$:

$$D(\mathcal{L}) \;=\; \Big\{w \in X \,\Big|\, \Delta w \in X\,,\; \xi\cdot\nabla w \in X\Big\}\;.$$

The proof of ii) is a direct calculation. By (4.3) we have $\Lambda = \Lambda_1 + \Lambda_2$, where $\Lambda_1 w = v^G \cdot \nabla w$ and $\Lambda_2 w = v \cdot \nabla G = (K * w)\cdot\nabla G$. If $w_1, w_2 \in D(\mathcal{L}) \subset X$, then

$$\begin{aligned}\langle \Lambda_1 w_1, w_2\rangle + \langle w_1, \Lambda_1 w_2\rangle \;&=\; \int_{\mathbb{R}^2} G^{-1}\Big(w_2\, v^G\cdot\nabla w_1 + w_1\, v^G\cdot\nabla w_2\Big)\,\mathrm{d}\xi \\ &=\; \int_{\mathbb{R}^2} G^{-1}\, v^G\cdot\nabla(w_1 w_2)\,\mathrm{d}\xi \;=\; 0\;,\end{aligned}$$

because $G^{-1}v^G$ is divergence-free. Moreover, since $\nabla G = -\frac{1}{2}\xi G$, we have

$$\begin{aligned}\langle \Lambda_2 w_1, w_2\rangle + \langle w_1, \Lambda_2 w_2\rangle \;&=\; -\frac{1}{2}\int_{\mathbb{R}^2}\Bigl((\xi\cdot v_1)w_2 + (\xi\cdot v_2)w_1\Bigr)\,\mathrm{d}\xi \\ = -\frac{1}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}&\left\{\xi\cdot\frac{(\xi-\eta)^\perp}{|\xi-\eta|^2} + \eta\cdot\frac{(\eta-\xi)^\perp}{|\xi-\eta|^2}\right\} w_1(\eta)w_2(\xi)\,\mathrm{d}\eta\,\mathrm{d}\xi \;=\; 0\;,\end{aligned}$$

see also (2.10). Thus $\langle\Lambda w_1, w_2\rangle + \langle w_1, \Lambda w_2\rangle = 0$ for all $w_1, w_2 \in D(\mathcal{L}) \subset X$. By density, this relation holds for all w_1, w_2 in the domain of Λ, which can be characterized precisely [50]. One can also show that Λ is not only skew-symmetric, but also skew-adjoint in X, see [50]. □

It is useful to list the first few eigenfunctions of $\mathcal{L}$ in X, and to give explicit formulas for the corresponding spectral projections.

1) The first eigenvalue $\lambda_0 = 0$ is simple, and the associated eigenfunction is of course the Oseen vortex profile: $\mathcal{L}G = 0$. The corresponding spectral projection $P_0 : X \to X$ is given by $P_0 w = G\int_{\mathbb{R}^2} w\,\mathrm{d}\xi$. We denote by $X_0 \subset X$ the kernel of P_0, namely

$$X_0 \;=\; \Bigl\{w \in X \Bigm| \int_{\mathbb{R}^2} w\,\mathrm{d}\xi = 0\Bigr\} \;=\; \{G\}^\perp\;, \tag{4.8}$$

where $\{G\}^\perp$ denotes the set of all $w \in X$ which are orthogonal to G in X. Due to the conservation of the total circulation, this subspace of X is invariant under the evolution defined by the full equation (4.2).

2) The second eigenvalue $\lambda_1 = -1/2$ has multiplicity two, and the associated eigenfunctions are the first order derivatives $\partial_1 G$ and $\partial_2 G$. The spectral projection $P_1 : X \to X$ satisfies $P_1 w = -\partial_1 G\int_{\mathbb{R}^2}\xi_1 w\,\mathrm{d}\xi - \partial_2 G\int_{\mathbb{R}^2}\xi_2 w\,\mathrm{d}\xi$. Let $X_1 = \ker(P_0)\cap\ker(P_1)$, namely

$$X_1 \;=\; \Bigl\{w \in X_0 \Bigm| \int_{\mathbb{R}^2}\xi_i w\,\mathrm{d}\xi = 0 \quad \text{for } i = 1,2\Bigr\} \;=\; \{G;\partial_1 G;\partial_2 G\}^\perp\;.$$

Due to the conservation of the first-oder moments of the vorticity, see (2.8), the subspace X_1 is also invariant under the evolution defined by (4.2).

3) The third eigenvalue $\lambda_2 = -1$ has multiplicity three, and the associated eigenfunctions are the second order derivatives ΔG, $(\partial_1^2 - \partial_2^2)G$, and $\partial_1\partial_2 G$. Let $\tilde{P}_2$ be the spectral projection associated only with the symmetric eigenfunction ΔG, i.e. $\tilde{P}_2 w = \frac{1}{4}\Delta G\int_{\mathbb{R}^2}|\xi|^2 w\,\mathrm{d}\xi$. We denote $X_2 = \ker(P_0)\cap\ker(P_1)\cap\ker(\tilde{P}_2)$, namely

$$X_2 \;=\; \Bigl\{w \in X_1 \Bigm| \int_{\mathbb{R}^2}|\xi|^2 w\,\mathrm{d}\xi = 0\Bigr\} \;=\; \{G;\partial_1 G;\partial_2 G;\Delta G\}^\perp\;.$$

In view of (2.9), the subspace X_2 is also invariant under the evolution defined by (4.2).

More generally, for any $n \in \mathbb{N}$, the n^{th} eigenvalue $\lambda_n = -n/2$ of $\mathcal{L}$ has multiplicity $n+1$, and the corresponding eigenspace is spanned by the Hermite functions of degree n which can be expressed as homogeneous n^{th} order derivatives of the Gaussian profile G.

We now consider the spectrum of the linearized operator $\mathcal{L} - \alpha\Lambda$ in the (complexified) Hilbert space X, for any fixed $\alpha \in \mathbb{R}$. As is easily verified, the operator Λ is a relatively compact perturbation of $\mathcal{L}$ in X. Indeed, if we decompose $\Lambda = \Lambda_1 + \Lambda_2$ as in the proof of Lemma 4.1, we see that Λ_2 is a compact operator in X, while $\Lambda_1 = v^G\cdot\nabla$ is relatively compact with respect to $\mathcal{L}$ because this is a first order differential operator whose coefficients decay to zero as $|\xi| \to \infty$. By a well-known perturbation argument [41], this implies that $\mathcal{L} - \alpha\Lambda$ has itself a compact resolvent in X. In particular, its spectrum is a sequence of (complex) eigenvalues $\{\lambda_n(\alpha)\}_{n\in\mathbb{N}}$ with finite multiplicities, which can accumulate only at infinity. In fact, it is not difficult to verify that $\mathrm{Re}(\lambda_n(\alpha)) \to -\infty$ as $n \to \infty$, for any $\alpha \in \mathbb{R}$.

In general, the eigenvalues of $\mathcal{L}-\alpha\Lambda$ depend in a nontrivial way on the circulation parameter α and are not explicitly known. However, if $w \in X$ is a radially symmetric eigenfunction of $\mathcal{L}$, then obviously $\Lambda w = 0$, hence w is also an eigenfunction of $\mathcal{L} - \alpha\Lambda$ for any $\alpha \in \mathbb{R}$, and the corresponding eigenvalue does not depend on α. In particular, $\lambda_0 = 0$ is an eigenvalue of $\mathcal{L} - \alpha\Lambda$ with eigenfunction G, and the orthogonal subspace $X_0 = \{G\}^\perp$ is invariant under the linear evolution generated by $\mathcal{L} - \alpha\Lambda$. Moreover, if we differentiate the identity $v^G \cdot \nabla G = 0$ with respect to ξ_1 and ξ_2, we see that $\Lambda(\partial_i G) = 0$ for $i = 1, 2$. This implies that $\lambda_1 = -1/2$ is a double eigenfunction of $\mathcal{L} - \alpha\Lambda$ for any $\alpha \in \mathbb{R}$, with eigenfunctions $\partial_1 G$ and $\partial_2 G$. In addition, the orthogonal subspace $X_1 = \{G; \partial_1 G; \partial_2 G\}^\perp$ is invariant under the linear evolution generated by $\mathcal{L} - \alpha\Lambda$. Finally, since $\Lambda(\Delta G) = 0$ by symmetry, we see that $\lambda_2 = -1$ is an eigenvalue of $\mathcal{L} - \alpha\Lambda$ for any $\alpha \in \mathbb{R}$, and the subspace $X_2 = \{G; \partial_1 G; \partial_2 G; \Delta G\}^\perp$ is also invariant under the linearized evolution.

The following simple but important result shows that the equilibria of (4.1) corresponding to Oseen vortices are *spectrally stable* with respect to perturbations in the Gaussian space X.

Proposition 4.2 [31]
For any $\alpha \in \mathbb{R}$, the spectrum of the linearized operator $\mathcal{L} - \alpha\Lambda$ in the space X satisfies

$$\sigma(\mathcal{L} - \alpha\Lambda) \subset \Big\{ z \in \mathbb{C} \,\Big|\, \mathrm{Re}(z) \le 0 \Big\} . \tag{4.9}$$

Moreover,

$$\begin{aligned} \sigma(\mathcal{L} - \alpha\Lambda) &\subset \Big\{ z \in \mathbb{C} \,\Big|\, \mathrm{Re}(z) \le -\frac{1}{2} \Big\} \quad \textit{in} \quad X_0 , \\ \sigma(\mathcal{L} - \alpha\Lambda) &\subset \Big\{ z \in \mathbb{C} \,\Big|\, \mathrm{Re}(z) \le -1 \Big\} \quad \textit{in} \quad X_1 . \end{aligned} \tag{4.10}$$

Proof. We already know that the spectrum of $\mathcal{L} - \alpha\Lambda$ consists entirely of eigenvalues. Assume that $(\mathcal{L} - \alpha\Lambda)w = \lambda w$ for some $\lambda \in \mathbb{C}$ and some normalized vector $w \in D(\mathcal{L}) \subset X$. Then, using Lemma 4.1, we find

$$\mathrm{Re}(\lambda) \;=\; \mathrm{Re}\langle(\mathcal{L} - \alpha\Lambda)w, w\rangle \;=\; \langle \mathcal{L}w, w\rangle \;\le\; 0 , \tag{4.11}$$

which proves (4.9). Moreover $\langle \mathcal{L}w, w\rangle \le -1/2$ if $w \in X_0$ and $\langle \mathcal{L}w, w\rangle \le -1$ if $w \in X_1$, and (4.10) follows. ☐

Remark 4.3 *If we restrict ourselves to the invariant subspace X_2, it is easy to show, as in the proof of Proposition 4.2, that all eigenvalues of the linearized operator $\mathcal{L}-\alpha\Lambda$ satisfy* $\mathrm{Re}(\lambda) < -1$ *if $\alpha \neq 0$, but it is difficult to give an optimal upper bound on the real part of the spectrum.*

As an immediate consequence, we can show that the Oseen vortices are *linearly stable* with respect to perturbations in X:

Corollary 4.4 *For all $\alpha \in \mathbb{R}$, the semigroup generated by the linearized operator $\mathcal{L}-\alpha\Lambda$ satisfies*

$$\|e^{\tau(\mathcal{L}-\alpha\Lambda)}\|_{Z\to Z} \;\le\; e^{-\mu\tau} , \qquad \textit{for all} \quad \tau \ge 0 , \tag{4.12}$$

where $\mu = 0$ if $Z = X$, $\mu = 1/2$ if $Z = X_0$, and $\mu = 1$ if $Z = X_1$.

Proof. We know that the self-adjoint operator $\mathcal{L}$ is the generator of an analytic semigroup in X, and since Λ is a relatively compact perturbation of $\mathcal{L}$ it is clear that $\mathcal{L} - \alpha\Lambda$ is also the generator of an analytic semigroup, for any $\alpha \in \mathbb{R}$. Now, estimate (4.11) means that $\mathcal{L} - \alpha\Lambda$ is m-dissipative in X [41]. By the Lumer-Philips theorem [64], the associated semigroup satisfies the bound (4.12) with $Z = X$ and $\mu = 0$. Applying the same argument to the operator $\mathcal{L}-\alpha\Lambda+\mu$ restricted to X_0 (if $\mu = 1/2$) or X_1 (if $\mu = 1$), we obtain the desired result in the other cases. □

With some additional work, one can also prove that the Oseen vortices are *nonlinearly stable* with respect to pertubations in X. Here we can restrict ourselves, without loss of generality, to perturbations with zero mean; i.e., we can study Eq. (4.2) in the invariant subspace $X_0 \subset X$ defined by (4.8). Indeed, adding a perturbation with nonzero mean to the equilibrium αG is equivalent to adding a perturbation with zero mean to some modified equilibrium $\tilde{\alpha} G$, with $\tilde{\alpha}$ close to α. The result is:

Proposition 4.5 [31, 29]
There exists $\epsilon > 0$ such that, for any $\alpha \in \mathbb{R}$, we have the following result. If $w_0 \in X_0$ satisfies $\|w_0\|_X \le \epsilon$, then the unique solution of (4.2) with initial data w_0 satisfies

$$\|w(\tau)\|_X \;\le\; \min(1, 2e^{-\mu\tau})\|w_0\|_X \;, \qquad \textit{for all} \quad \tau \ge 0 \;, \tag{4.13}$$

where $\mu = 1$ if $w_0 \in X_1$ and $\mu = 1/2$ if $w_0 \in X_0 \setminus X_1$.

Proof. Arguing as in Section 2.3, it is not difficult to verify that the perturbation equation (4.2) is locally well-posed in the Gaussian weighted space X. Given $w_0 \in X_0$ with $\|w_0\|_X \le \epsilon$, let $w \in C^0([0, T_*), X)$ be the maximal solution of (4.2) with initial data w_0. We know that $w(\tau) \in X_0$ for all $\tau \in [0, T_*)$, and that $w(\tau) \in X_1$ if $w_0 \in X_1$. To control the time evolution of $w(\tau)$, we use the energy estimate

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\|w(\tau)\|_X^2 \;&=\; \langle w(\tau)\,,\, \mathcal{L}w(\tau)\rangle - \int_{\mathbb{R}^2} G^{-1}(\xi) w(\xi,\tau) v(\xi,\tau)\cdot\nabla w(\xi,\tau)\,\mathrm{d}\xi \\ &=\; \langle w(\tau)\,,\, \mathcal{L}w(\tau)\rangle + \frac{1}{4}\int_{\mathbb{R}^2} G^{-1}(\xi)(\xi\cdot v(\xi,\tau)) w(\xi,\tau)^2\,\mathrm{d}\xi \;, \end{aligned}$$

where in the first equality we used the skew-symmetry of the operator Λ, and in the second one we integrated by parts and used the identity $\nabla G^{-1} = (\xi/2)G^{-1}$.

Fix $\tau > 0$ and let $f(\xi) = G^{-1/2}(\xi) w(\xi,\tau)$, for all $\xi \in \mathbb{R}^2$. Then $\|f\|_{L^2} = \|w(\tau)\|_X$ and, in view of (4.7),

$$E(f) \;:=\; \|\nabla f\|_{L^2}^2 + \frac{1}{16}\,\|\xi f\|_{L^2}^2 - \frac{1}{2}\|f\|_{L^2}^2 \;=\; -\langle w(\tau)\,,\, \mathcal{L}w(\tau)\rangle \;.$$

Since $w(\tau) \in X_0$, we have $E(f) \ge \mu\|f\|_{L^2}^2$ where $\mu = 1$ if $w_0 \in X_1$ and $\mu = 1/2$ if $w_0 \in X_0 \setminus X_1$. In particular, there exists $C > 0$ such that

$$E(f) \;\ge\; C(\|\nabla f\|_{L^2}^2 + \|\xi f\|_{L^2}^2 + \|f\|_{L^2}^2) \;.$$

On the other hand,

$$\begin{aligned} \Big|\int_{\mathbb{R}^2} G^{-1}(\xi)(\xi\cdot v(\xi,\tau)) w(\xi,\tau)^2\,\mathrm{d}\xi\Big| \;&\le\; \int_{\mathbb{R}^2} |\xi\cdot v(\xi,\tau)| f(\xi)^2\,\mathrm{d}\xi \\ &\le\; \|v(\tau)\|_{L^4}\,\|f\|_{L^4}\,\|\xi f\|_{L^2} \;. \end{aligned} \tag{4.14}$$

By (2.5) we have $\|v\|_{L^4} \le C\|w\|_{L^{4/3}} \le C\|w\|_X$, hence the right-hand side of (4.14) is bounded by $C\|w(\tau)\|_X E(f)$. As a consequence, there exists $C_1 > 0$ such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\|w(\tau)\|_X^2 \;\le\; -E(f)(1 - C_1\|w(\tau)\|_X) \;\le\; -\mu\|w(\tau)\|_X^2(1 - C_1\|w(\tau)\|_X)\;, \tag{4.15}$$

where the second inequality is valid provided $C_1\|w(\tau)\|_X \le 1$.

We now assume that $\epsilon > 0$ is small enough so that $C_1\epsilon \le \frac{1}{2}$. Then the differential inquality (4.15) implies that the norm $\|w(\tau)\|_X$ is nonincreasing with time, hence the solution of (4.2) is global (i.e., $T_* = \infty$) and satisfies (4.15) for all $\tau \ge 0$. Integrating (4.15), we obtain

$$\frac{\|w(\tau)\|_X}{1 - C_1\|w(\tau)\|_X} \;\le\; \frac{\|w_0\|_X}{1 - C_1\|w_0\|_X}\,e^{-\mu\tau}\;, \qquad \text{for all } \tau \ge 0\;.$$

This implies (4.13), since $C_1\|w_0\|_X \le \frac{1}{2}$ by assumption. □

Proposition 4.5 shows that the steady state $w = \alpha G$ of the rescaled vorticity equation (4.1) is stable with respect to perturbations in the space X, for any value of the circulation parameter $\alpha \in \mathbb{R}$. Moreover, we have a lower bound on the size of the local basin of attraction which is uniform in α. Here we mean by “local basin of attraction” a neighborhood of the equilibrium αG which is small enough so that the time evolution of the perturbations in that neighborhood can be controlled by the dissipative properties of the linearized operator, as in the above proof. Of course, Theorem 3.1 shows that all trajectories of (4.1) in $L^1(\mathbb{R}^2)$ converge to the line of equilibria $\{\alpha G\}_{\alpha\in\mathbb{R}}$ as $\tau \to \infty$, hence one can argue that the basin of attraction of these equilibria has, actually, infinite size.

If the initial data are sufficiently localized, Proposition 4.5 allows to improve the conclusion of Theorem 3.1 by specifying sharp convergence rates. Assume for instance that the initial vorticity $\omega_0 : \mathbb{R}^2 \to \mathbb{R}$ satisfies

$$\int_{\mathbb{R}^2} \omega_0(x)^2 \,\exp\Bigl(\frac{|x|^2}{4\nu t_0}\Bigr)\,\mathrm{d}x \;<\; \infty\;, \tag{4.16}$$

for some $t_0 > 0$, and let $\gamma = \int_{\mathbb{R}^2} \omega_0(x)\,\mathrm{d}x$. If $\gamma \neq 0$, we define the center of vorticity as

$$x_0 \;=\; \frac{1}{\gamma}\int_{\mathbb{R}^2} x\,\omega_0(x)\,\mathrm{d}x \;\in\; \mathbb{R}^2\;,$$

and we set $x_0 = 0$ otherwise. Using the change of variables (3.14) with x_0 as above and $T = t_0$, we transform (2.2) into the rescaled vorticity equation (4.1), and hypothesis (4.16) means that the initial data w_0 for (4.1) belong to the Gaussian space X. Then estimate (3.18) (with $\beta > 1/2$) implies that the solution $w(\tau)$ of (4.1) is uniformly bounded in X for all $\tau \ge 0$, and arguing as in Section 3.3 it is not difficult to show that $\|w(\tau) - \alpha G\|_X \to 0$ as $\tau \to \infty$, where $\alpha = \gamma/\nu$. Thus we can apply Proposition 4.5 when τ is sufficiently large, and we conclude that $\|w(\tau) - \alpha G\|_X = \mathcal{O}(e^{-\mu\tau})$ as $\tau \to \infty$, for some $\mu \ge 1/2$. More precisely, when $\alpha \neq 0$, the choice of x_0 above implies that $w_0 - \alpha G \in X_1$, hence we can take $\mu = 1$; otherwise, we take $\mu = 1/2$. Returning to the original variables, we obtain in particular the estimate

$$\Bigl\|\omega(x,t) - \frac{\gamma}{\nu(t+t_0)}G\Bigl(\frac{x - x_0}{\sqrt{\nu(t+t_0)}}\Bigr)\Bigr\|_{L^1} \;=\; \mathcal{O}(t^{-\mu})\;, \quad \text{as} \quad t \to \infty\;, \tag{4.17}$$

which improves (3.24). The decay rate in (4.17) is optimal in general, but can sometimes be improved by an appropriate choice of t_0, see [31, 29]. We also mention that the convergence (4.17) still holds if, instead of (4.16), we assume that the initial vorticity satisfies $\int_{\mathbb{R}^2} \omega_0(x)^2(1 + |x|^2)^m\,\mathrm{d}x < \infty$ for some $m > 3$, see [31].

To conclude our discussion of Proposition 4.5, we would like to emphasize that the stability analysis of Oseen vortices lead to conclusions which differ radically from what is known for other classical examples in fluid mechanics, such as the Poiseuille flow in a cylindrical pipe or the Couette-Taylor flow between two rotating cylinders [23, 78]. Indeed, in the case of Oseen vortices, no hydrodynamic instability, neither of spectral nor of pseudospectral nature, develops as the circulation parameter $\alpha = \gamma/\nu$ (which plays the role of the Reynolds number) is increased. On the contrary, it is possible to show that a fast rotation has even a *stabilizing effect* on the vortex, as we shall see below.

4.2 Improved Stability Estimates for Rapidly Rotating Vortices

For a more detailed study of the linearized operator $\mathcal{L} - \alpha\Lambda$, it is useful to observe that both operators $\mathcal{L}$ and Λ are invariant under rotations about the origin in $\mathbb{R}^2$. This symmetry is fully exploited if we introduce polar coordinates (r, θ) in $\mathbb{R}^2$ and expand the vorticity distribution in Fourier series with respect to the angular variable θ. Let us decompose

$$X = \bigoplus_{n\in\mathbb{N}} \tilde{X}_n \ , \tag{4.18}$$

where $\tilde{X}_n$ denotes the subspace of all $w \in X$ such that

$$w(r\cos\theta, r\sin\theta) = a(r)\cos(n\theta) + b(r)\sin(n\theta) \ , \qquad r > 0 \ , \quad \theta \in [0, 2\pi] \ ,$$

for some radial functions $a, b : \mathbb{R}_+ \to \mathbb{R}$. The invariance under rotations implies the following result, which can be obtained by a direct calculation :

Lemma 4.6 [31]
For each $n \in \mathbb{N}$ the subspace $\tilde{X}_n$ is invariant under the action of both linear operators $\mathcal{L}$ and Λ. The restriction $\mathcal{L}_n$ of $\mathcal{L}$ to $\tilde{X}_n$ is the one-dimensional operator

$$\mathcal{L}_n \ = \ \partial_r^2 + \Bigl(\frac{r}{2} + \frac{1}{r}\Bigr)\partial_r + \Bigl(1 - \frac{n^2}{r^2}\Bigr) \ .$$

The restriction Λ_n of Λ to $\tilde{X}_n$ satisfies $\Lambda_n\omega \ = \ in(\phi\omega - g\Omega_n)$, where

$$\phi(r) \ = \ \frac{1}{2\pi r^2}(1 - e^{-r^2/4}) \ , \qquad g(r) \ = \ \frac{1}{4\pi}\, e^{-r^2/4} \ , \qquad r > 0 \ , \tag{4.19}$$

and (for $n \ge 1$) Ω_n is the regular solution of $-\Omega_n'' - \frac{1}{r}\Omega_n' + \frac{n^2}{r^2}\Omega_n = \omega$, namely

$$\Omega_n(r) \ = \ \frac{1}{4n}\left(\int_0^r \Bigl(\frac{r'}{r}\Bigr)^n r'\omega(r')\,\mathrm{d}r' + \int_r^\infty \Bigl(\frac{r}{r'}\Bigr)^n r'\omega(r')\,\mathrm{d}r'\right) \ , \quad r > 0 \ . \tag{4.20}$$

In particular, Lemma 4.6 implies that the subspace $\tilde{X}_0 \subset X$, which consists of all radially symmetric functions, is entirely contained in the kernel of the operator Λ. As was already observed, the first order derivatives $\partial_1 G, \partial_2 G$ also belong to $\ker(\Lambda)$, and the following result shows that these are the only non-radially symmetric functions in the kernel of Λ :

Lemma 4.7 [50] $\ker(\Lambda) = \tilde{X}_0 \oplus \{\beta_1\partial_1 G + \beta_2\partial_2 G \,|\, \beta_1, \beta_2 \in \mathbb{R}\}$.

Proof. We already know that $\tilde{X}_0 \subset \ker(\Lambda)$. Suppose now that $w \in \ker(\Lambda) \cap \tilde{X}_n$ for some $n \ge 1$. Without loss of generality, we can assume that $w(r\cos\theta, r\sin\theta) = \omega(r)\cos(n\theta)$ or $\omega(r)\sin(n\theta)$

for some $\omega : \mathbb{R}_+ \to \mathbb{R}$, and using the notations (4.19), (4.20) we have $\phi\omega - g\Omega_n = 0$. This means that Ω_n satisfies the differential equation

$$-\Omega_n''(r) - \frac{1}{r}\Omega_n'(r) + \Bigl(\frac{n^2}{r^2} - \frac{g(r)}{\phi(r)}\Bigr)\Omega_n(r) \;=\; 0 \;, \qquad r > 0 \;. \tag{4.21}$$

Now it is easy to verify that $r^2 g(r)/\phi(r) < 4$ for all $r > 0$. Thus Eq. (4.21) satisfies the maximum principle if $n \ge 2$, hence has no nontrivial solution vanishing at the origin and at infinity. This shows that $\ker(\Lambda) \cap \tilde{X}_n = \{0\}$ for $n \ge 2$. When $n = 1$, Eq. (4.21) has a (unique) nontrivial solution given by $\Omega_1(r) = r\phi(r)$, which satisfies (4.20) with $\omega(r) = rg(r)$. This shows that $\ker(\Lambda) \cap \tilde{X}_1$ is spanned by $\partial_1 G$ and $\partial_2 G$. □

In the rest of this section, to investigate the effect of the rotation on the stability of Oseen's vortex, we restrict ourselves to the orthogonal complement of $\ker(\Lambda)$ in X. This subspace, denoted by $\ker(\Lambda)^\perp$, is invariant under the evolution defined by the linearized equation $\partial_\tau w = (\mathcal{L} - \alpha\Lambda)w$, but not under the full equation (4.2). Let $\mathcal{L}_\perp$ and $\Lambda_\perp$ denote the restrictions of the operators $\mathcal{L}$ and Λ to the invariant subspace $\ker(\Lambda)^\perp$. For any $\alpha \in \mathbb{R}$ we define the *spectral lower bound*

$$\Sigma(\alpha) \;=\; \inf\Bigl\{\mathrm{Re}(z)\,\Big|\, z \in \sigma(-\mathcal{L}_\perp + \alpha\Lambda_\perp)\Bigr\} \;,$$

and the *pseudospectral bound*

$$\Psi(\alpha) \;=\; \Bigl(\sup_{\lambda\in\mathbb{R}} \|(\mathcal{L}_\perp - \alpha\Lambda_\perp - i\lambda)^{-1}\|_{X\to X}\Bigr)^{-1} \;.$$

Note that, in the definition of $\Sigma(\alpha)$, we have changed the sign of the linearized operator, in order to get a positive quantity.

Lemma 4.8 *For any $\alpha \in \mathbb{R}$ one has $\Sigma(\alpha) \ge \Psi(\alpha) \ge 1$.*

Proof. Fix $\alpha \in \mathbb{R}$. Since $\ker(\Lambda)^\perp \subset X_1$, it follows from (4.10) that $\Sigma(\alpha) \ge 1$. Moreover, if $(\mathcal{L} - \alpha\Lambda + z)w = 0$ for some $z \in \mathbb{C}$ and some normalized $w \in \ker(\Lambda)^\perp$, then $(\mathcal{L} - \alpha\Lambda + i\,\mathrm{Im}(z))w = -\mathrm{Re}(z)w$, so that

$$\mathrm{Re}(z) \;\ge\; \|(\mathcal{L}_\perp - \alpha\Lambda_\perp + i\,\mathrm{Im}(z))^{-1}\|^{-1} \;\ge\; \Psi(\alpha) \;.$$

This proves that $\Sigma(\alpha) \ge \Psi(\alpha)$. On the other, the proof of Proposition 4.2 shows that the operator $\mathcal{L}_\perp - \alpha\Lambda_\perp + 1$ is m-dissipative. This in particular implies that $\|(\mathcal{L}_\perp - \alpha\Lambda_\perp - i\lambda)^{-1}\| \le 1$ for all $\lambda \in \mathbb{R}$, hence $\Psi(\alpha) \ge 1$. □

Numerical observations due to Prochazka and Pullin [65] indicate that $\Sigma(\alpha) = \mathcal{O}(|\alpha|^{1/2})$ as $|\alpha| \to \infty$. This has not been proved yet, but a recent result by Maekawa [50] shows that $\Sigma(\alpha) \to +\infty$ as $|\alpha| \to \infty$. Although the proof is not constructive and does not give any information on the asymptotic behavior of $\Sigma(\alpha)$, this is the first rigorous result which shows that a fast rotation stabilizes Oseen's vortex, in the sense that the decay rate of the perturbations in $\ker(\Lambda)^\perp$ becomes arbitrarily large as $|\alpha| \to \infty$. In fact, the argument of [50] can be easily modified to yield the stronger result $\Psi(\alpha) \to \infty$ as $|\alpha| \to \infty$. This indicates that the size of the local basin of attraction of Oseen's vortex αG becomes arbitrarily large (in the Gaussian space X) when $|\alpha| \to \infty$.

Explicit bounds on $\Sigma(\alpha)$ or $\Psi(\alpha)$ are more difficult to obtain, and were first established in [26] for a model problem in one space dimension, and in [21] for a simplified version of the linearized operator $\mathcal{L} - \alpha\Lambda$ where the nonlocal term Λ_2 in Λ is omitted. In both cases one gets $\Psi(\alpha) = \mathcal{O}(|\alpha|^{1/3})$ as $|\alpha| \to \infty$, and for the one-dimensional model we also have $\Sigma(\alpha) = \mathcal{O}(|\alpha|^{1/2})$. These partial results suggest that $\Sigma(\alpha) \gg \Psi(\alpha)$ when $|\alpha|$ is large, a property which reflects the fact that the linearized operator is highly non-selfadjoint in the large circulation regime. Indeed, it is easily verified that the spectral and pseudospectral bounds always coincide for selfadjoint operators.

Although the analysis of the linearized operator $\mathcal{L}-\alpha\Lambda$ is still the object of ongoing research, we formulate the following conjecture which should be proved soon.

Conjecture 4.9 *There exist constants* $\delta > \gamma > 0$ *such that* $\Psi(\alpha) = \mathcal{O}(|\alpha|^\gamma)$ *and* $\Sigma(\alpha) = \mathcal{O}(|\alpha|^\delta)$ *as* $|\alpha| \to \infty$.

On the basis of the analysis of simpler models [26], we further conjecture that $\gamma = 1/3$ and $\delta = 1/2$, but these optimal values may be difficult to reach. To conclude this discussion, we mention that, if the linearized operator $\mathcal{L} - \alpha\Lambda$ is restricted to the invariant subspace

$$\bigoplus_{n\geq N} \tilde{X}_n \subset \ker(\Lambda)^\perp ,$$

for some sufficiently large integer $N \geq 2$, then it is possible to show that the pseudospectral bound in that subspace satisfies $\Psi_N(\alpha) = \mathcal{O}(|\alpha|^{1/3})$ as $|\alpha| \to \infty$, see [22].

5 Interaction of Vortices in Weakly Viscous Flows

Theorem 2.8 shows that the two-dimensional vorticity equation (2.2) is globally well-posed in the space of finite measures, for any value of the viscosity parameter ν. It is therefore natural to investigate how the solutions of (2.2) behave in the vanishing viscosity limit. This question is very difficult, because the formal limit of (2.2) as $\nu \to 0$ is the two-dimensional inviscid vorticity equation

$$\partial_t\omega(x,t) + u(x,t)\cdot\nabla\omega(x,t) \;=\; 0 \;, \tag{5.1}$$

which is certainly not well-posed in the space $\mathcal{M}(\mathbb{R}^2)$. In fact, a celebrated result of Yudovich [81] shows that (5.1) has a unique global solution if the initial vorticity μ belongs to $L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$, see also [82, 80]. If $\mu \in \mathcal{M}(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2)$ and if the singular part of μ has a definite sign, then (5.1) has at least one global weak solution [20, 51], but this result does not cover the case where the initial data contain point vortices.

A careful treatment of the inviscid limit is beyond the scope of these notes, but the following brief discussion will be useful to put our results into perspective. As a general rule, it is known that the Navier-Stokes equation is nicely approximated by the Euler equation in the vanishing viscosity limit if we restrict ourselves to smooth solutions in a domain without boundary [74, 42, 2]. Thus the main difficulties come either from the presence of boundaries or from singularities in the initial data. Since we chose to work in the whole plane $\mathbb{R}^2$, we do not mention here the (hard) problems related to boundary layers, and refer to [69, 70, 35] for a discussion of that point. But even in the absence of boundaries, the inviscid limit may be nontrivial to understand if one considers non-smooth initial data. In the case of *vortex patches*, the situation is well understood : the Euler flow is well defined due to Yudovich's theorem, and approximates the Navier-Stokes flow in every reasonable sense as $\nu \to 0$, see [14, 15, 11, 18, 19, 1, 37, 38, 58, 73]. The situation is much less clear for *vortex sheets* : although an inviscid solution can still be constructed, the Euler flow is unstable due to the discontinuity of the velocity field (Kelvin-Helmholtz instability), and the vanishing viscosity limit is as difficult to study as in the case of Prandtl boundary layers [7]. Finally, the case of *point vortices* is somewhat paradoxical : one the one hand, these are the most singular initial data for which we can solve the 2D Navier-Stokes equation, and strictly speaking the Euler flow is not even defined for such initial data. On the other hand, we have a nice substitute for the Euler flow in that particular case, namely the Helmholtz-Kirchhoff point vortex system, which does not exhibit any dynamical instability. As a result, the inviscid limit can be rigorously studied in the case of point vortices. Early results in this direction were obtained by Marchioro [54, 55], and will be described more precisely below.

5.1 The Viscous N-Vortex Solution

From now on, we fix an integer $N \ge 1$, and we assume that the initial vorticity is a collection of N point vortices characterized by their (pairwise distinct) positions $x_1, \dots, x_N \in \mathbb{R}^2$ and their (nonzero) circulations $\gamma_1, \dots, \gamma_N \in \mathbb{R}$:

$$\mu = \sum_{i=1}^{N} \gamma_i \, \delta_{x_i} \ . \tag{5.2}$$

Given a viscosity $\nu > 0$, we denote by $\omega^\nu(x,t)$, $u^\nu(x,t)$ the unique solution of the vorticity equation (2.2) with initial data μ. Existence of such a solution was first proved in [4], and uniqueness is guaranteed by Theorem 2.8. Note that Theorem 2.5 applies if $\sum |\gamma_i| \le C_0 \nu$, but such an assumption is totally inappropriate here because we want to study the limit $\nu \to 0$ for fixed initial data.

If $N = 1$, the solution of (2.2) is just the Oseen vortex with circulation γ_1 centered at the point x_1:

$$\omega^\nu(x,t) = \frac{\gamma_1}{\nu t}\, G\Bigl(\frac{x-x_1}{\sqrt{\nu t}}\Bigr) \ , \qquad u^\nu(x,t) = \frac{\gamma_1}{\sqrt{\nu t}}\, v^G\Bigl(\frac{x-x_1}{\sqrt{\nu t}}\Bigr) \ .$$

When $N \ge 2$, the viscous N-vortex solution is no longer explicit, but as long as the diffusive length $\sqrt{\nu t}$ is small compared to the distance between the vortex centers we expect that $\omega^\nu(x,t)$ can be approximated by a superposition of Oseen vortices:

$$\omega^\nu(x,t) \approx \sum_{i=1}^{N} \frac{\gamma_i}{\nu t}\, G\Bigl(\frac{x-z_i(t)}{\sqrt{\nu t}}\Bigr) \ , \qquad u^\nu(x,t) \approx \sum_{i=1}^{N} \frac{\gamma_i}{\sqrt{\nu t}}\, v^G\Bigl(\frac{x-z_i(t)}{\sqrt{\nu t}}\Bigr) \ , \tag{5.3}$$

where $z_1(t), \dots, z_N(t)$ denote the positions of the vortex centers at time t. In the same regime, we also expect that the positions $z_i(t)$ will be determined by the Helmholtz-Kirchhoff system [36, 44]:

$$z_i'(t) = \frac{1}{2\pi} \sum_{j \ne i} \gamma_j \, \frac{(z_i(t)-z_j(t))^\perp}{|z_i(t)-z_j(t)|^2} \ , \qquad z_i(0) = x_i \ . \tag{5.4}$$

Indeed, we know from the work of Marchioro and Pulvirenti that the system of ordinary differential equations (5.4) approximates rigorously the motion of localized vortex patches in two-dimensional inviscid [53, 56, 57] or slightly viscous [54, 55] fluids.

The point vortex system (5.4) is globally well-posed for all initial data if $N = 2$ or if all circulations γ_i have the same sign. In the general case, however, vortex collisions can occur in finite time for some exceptional initial configurations [57, 62]. To eliminate this potential problem, we assume in what follows that the solution $\{z_1(t), \dots, z_N(t)\}$ of (5.4) is defined on some time interval $[0,T]$, and we denote by d the minimal distance between the vortex centers:

$$d = \min_{t \in [0,T]} \min_{i \ne j} |z_i(t) - z_j(t)| > 0 \ . \tag{5.5}$$

We also introduce the turnover time $T_0 = d^2/|\gamma|$, where $|\gamma| = |\gamma_1| + \cdots + |\gamma_N|$, which is a natural time scale for the inviscid dynamics described by (5.4). For instance, for a pair of vortices with the same circulation γ separated by a distance d, one can check that the rotation period of each vortex around the midpoint is $4\pi^2 T_0$.

With these notations, we can formulate now the main result of this section:

Theorem 5.1 [28]
Assume that the point vortex system (5.4) *is well-posed on the time interval* $[0,T]$*. Then the unique solution of the two-dimensional vorticity equation* (2.2) *with initial data* $\mu = \sum_{i=1}^N \gamma_i\,\delta_{x_i}$ *satisfies*

$$\frac{1}{|\gamma|}\int_{\mathbb{R}^2}\Big|\omega^\nu(x,t) - \sum_{i=1}^N \frac{\gamma_i}{\nu t}\,G\Big(\frac{x-z_i(t)}{\sqrt{\nu t}}\Big)\Big|\,\mathrm{d}x \;\le\; K\,\frac{\nu t}{d^2}\;, \qquad t\in(0,T]\;, \tag{5.6}$$

where $\{z_1(t),\dots,z_N(t)\}$ *is the solution of* (5.4) *and* K *is a dimensionless constant depending only on the ratio* T/T_0*.*

Theorem 5.1 describes the viscous N-vortex solution to leading order in our expansion parameter $\sqrt{\nu t}/d$, which is the ratio of the typical size of the vortex cores to the minimal distance between the vortex centers. In other words, the approximation (5.3) is accurate whenever the vortices are widely separated compared to their size. Another dimensionless quantity that is present in our analysis is the ratio of the observation time T to the turnover time T_0. It is important to remark that no smallness assumption is made on this ratio in Theorem 5.1, although the constant K in (5.6) becomes very large if $T \gg T_0$.

Since each Oseen vortex converges weakly to a Dirac mass in the inviscid limit, an immediate consequence of Theorem 5.1 is:

Corollary 5.2 *Under the assumptions of Theorem 5.1, the viscous* N*-vortex solution* $\omega^\nu(x,t)$ *satisfies*

$$\omega^\nu(\cdot,t) \;\xrightarrow[\nu\to 0]{}\; \sum_{i=1}^N \gamma_i\,\delta_{z_i(t)}\;, \quad \textit{for all } t\in[0,T]\;, \tag{5.7}$$

where $\{z_1(t),\dots,\, z_N(t)\}$ *is the solution of* (5.4)*.*

While weaker than Theorem 5.1, this corollary provides a rigorous derivation of the point vortex system (5.4) using the Navier-Stokes equation in the inviscid limit. In contrast, the classical approach of Marchioro and Pulvirenti [56, 57] allows to justify (5.4) within the framework of Euler's equation, but requires an approximation argument which is not needed in Corollary 5.2.

5.2 Decomposition into Vorticity Profiles

In the rest of this section, we give a sketch of the proof of Theorem 5.1. Our starting point is a natural decomposition of the viscous N-vortex solution, which follows immediately from the representation formula (3.10). Since the initial measure μ is a sum of Dirac masses, we have by (3.10)

$$\omega^\nu(x,t) \;=\; \sum_{i=1}^N \omega_i^\nu(x,t)\;, \qquad u^\nu(x,t) \;=\; \sum_{i=1}^N u_i^\nu(x,t)\;, \tag{5.8}$$

where

$$\omega_i^\nu(x,t) \;=\; \gamma_i\Gamma_u^\nu(x,t;x_i,0)\;, \qquad x\in\mathbb{R}^2\;, \quad 0<t\le T\;, \quad i=1,\dots,N\;,$$

and $u_i^\nu(x,t)$ is the velocity field corresponding to $\omega^\nu(x,t)$ via the Biot-Savart law (2.4). Here $\Gamma_u^\nu(x,t;y,s)$ is the fundamental solution of the convection-diffusion equation (3.4) with advection field $U(x,t) = u^\nu(x,t)$. In view of (3.6), we have the bound

$$|\omega_i^\nu(x,t)| \le K_1 \frac{|\gamma_i|}{\nu t} \exp\Bigl(-\beta\frac{|x-x_i|^2}{4\nu t}\Bigr)~, \tag{5.9}$$

for all $i \in \{1,\dots,N\}$, all $x \in \mathbb{R}^2$, and all $t \in (0,T]$, where $\beta \in (0,1)$ and the dimensionless constant $K_1 > 0$ depends only on the total circulation $|\gamma|$ and on the viscosity ν. Estimate (5.9) gives a precise information on the N-vortex solution when $\nu > 0$ is fixed and $t \to 0$, but cannot be used to control $\omega_i^\nu(x,t)$ when $t > 0$ is fixed and $\nu \to 0$, because the constant K_1 blows up in that regime.

To analyze more precisely each term in the decomposition (5.8) we introduce, for each $i \in \{1,\dots,N\}$, the self-similar variable

$$\xi = \frac{x - z_i(t)}{\sqrt{\nu t}}~,$$

where $\{z_1(t),\dots,\,z_N(t)\}$ is the solution of (5.4). As in (3.14), we define rescaled vorticities $w_i^\nu(\xi,t) \in \mathbb{R}$ and velocities $v_i^\nu(\xi,t) \in \mathbb{R}^2$ by setting

$$\omega_i^\nu(x,t) = \frac{\gamma_i}{\nu t}\, w_i^\nu\Bigl(\frac{x-z_i(t)}{\sqrt{\nu t}}\,,\,t\Bigr)~, \qquad u_i^\nu(x,t) = \frac{\gamma_i}{\sqrt{\nu t}}\, v_i^\nu\Bigl(\frac{x-z_i(t)}{\sqrt{\nu t}}\,,\,t\Bigr)~. \tag{5.10}$$

Our goal is to describe the vorticity profiles $w_i^\nu(\xi,t)$ as precisely as possible, in the regime where $\sqrt{\nu t} \ll d$. According to (5.6), we have $w_i^\nu(\xi,t) = G(\xi) + \mathcal{O}(\nu t/d^2)$ for each $i \in \{1,\dots,N\}$, where G is the profile of Oseen's vortex. However, it is very important to realize that this leading order approximation is *not sufficient* to control the inviscid limit of the viscous N-vortex solution. The reason is that the profile G is radially symmetric, and thus does not take into account the deformations of the vortices due to mutual interactions.

If we go back to the explicit formula (3.1), we see that the velocity field of Oseen's vortex is very large near the center if the viscosity ν is small, with a maximal angular speed of the order of $|\gamma|/(\nu t)$. As long as the vortex stays isolated, this large velocity does not affect the vorticity distribution, which is perfectly radially symmetric. Now, if the same vortex is advected by a non-homogenous external field, which in our case will be the velocity field produced by the other $N-1$ vortices, its vorticity profile gets deformed under the external strain and the vortex starts feeling the effect of its own velocity field. As is easily verified, this *self-interaction* effect is very strong if the circulation Reynolds number $|\gamma|/\nu$ is large, even for a moderate deformation of the vortex core. This gives a clear indication that self-interaction effects must be taken into account if one wants to control the vanishing viscosity limit in presence of point vortices.

The deformation of a rapidly rotating Oseen vortex advected by an external velocity field can be computed using the properties of the linearized operator $\mathcal{L} - \alpha\Lambda$, which was defined in Section 4. Remarkably enough, one finds that Oseen's vortex adapts its shape in such a way that the self-interaction *counterbalances* the effect of the external field [77, 76], except for radially symmetric corrections and for a rigid translation. This is related to Lemma 4.7, which shows that the kernel of the operator Λ consists of all radially symmetric functions and of the two-dimensional subspace spanned by $\partial_1 G, \partial_2 G$. This fundamental fact explains why one can observe, in turbulent two-dimensional flows, stable asymmetric vortices which in a first approximation are simply advected by the main stream [5].

In Section 5.3, we implement this idea by constructing a higher order approximation of the viscous N-vortex solution which takes into account the self-interaction effects. For each

$i \in \{1, \ldots, N\}$ and all $t \in [0,T]$, the approximated vortex profile will be of the form

$$w_i^{\mathrm{app}}(\xi,t) \;=\; G(\xi) + \Big(\frac{\nu t}{d^2}\Big)\Big\{\bar F_i(\xi,t) + F_i^\nu(\xi,t)\Big\}~, \qquad \xi \in \mathbb{R}^2~, \tag{5.11}$$

where $\bar F_i(\xi,t)$ is a radially symmetric function of ξ which we will not describe precisely because it only represents a small perturbation of the Gaussian profile $G(\xi)$. The important correction at this order is the nonsymmetric function $F_i^\nu(\xi,t)$ which is given by

$$F_i^\nu(\xi,t) \;=\; \frac{d^2}{4\pi}\,\omega(|\xi|)\sum_{j\neq i}\frac{\gamma_j}{\gamma_i}\,\frac{1}{|z_{ij}(t)|^2}\Big(2\frac{|\xi\cdot z_{ij}(t)|^2}{|\xi|^2|z_{ij}(t)|^2} - 1\Big) + \mathcal{O}\Big(\frac{\nu}{|\gamma|}\Big)~, \tag{5.12}$$

where we denote $z_{ij}(t) = z_i(t) - z_j(t)$. Here $\omega : (0,\infty) \to \mathbb{R}$ is a smooth, positive function satisfying $\omega(r) \approx C_1 r^2$ as $r \to 0$ and $\omega(r) \approx C_2 r^4 e^{-r^2/4}$ as $r \to \infty$ for some $C_1, C_2 > 0$.

The right-hand side of (5.11) is the beginning of an asymptotic expansion of the rescaled vortex patch $w_i^\nu(\xi,t)$ in powers of the non-dimensional parameter $\sqrt{\nu t}/d$, which is the ratio of the size of the vortex cores to the minimal distance between the centers. Each term in this expansion can in turn be developed in powers of the inverse circulation Reynolds number $\nu/|\gamma|$. The most important effect is due to the nonsymmetric term $F_i^\nu(\xi,t)$, which describes to leading order the deformation of the i^{th} vortex due to the influence of the other vortices. Keeping only that term and using polar coordinates $\xi = (r\cos\theta, r\sin\theta)$, we can rewrite (5.11) in the following simplified form

$$w_i^{\mathrm{app}}(\xi,t) \;=\; g(r) + \frac{\omega(r)}{4\pi}\sum_{j\neq i}\frac{\gamma_j}{\gamma_i}\,\frac{\nu t}{|z_{ij}(t)|^2}\cos\Big(2(\theta-\theta_{ij}(t))\Big) \;+\; \ldots~, \tag{5.13}$$

where $g(|\xi|) = G(\xi)$ and $\theta_{ij}(t)$ is the argument of the planar vector $z_{ij}(t) = z_i(t) - z_j(t)$. This formula allows us to compute the principal axes and the eccentricities of the vorticity contours, which are elliptical at this level of approximation.

To formulate our result in its final form, we introduce a function space that is appropriate to control the vorticity profiles. Given a small $\beta \in (0,1)$, which will be specified later, we define the weighted L^2 space X_β equipped with the norm

$$\|w\|_{X_\beta} \;=\; \Big(\int_{\mathbb{R}^2}|w(\xi)|^2\, e^{\beta|\xi|/4}\,\mathrm{d}\xi\Big)^{1/2}~. \tag{5.14}$$

In particular, we have $X_\beta \hookrightarrow L^1(\mathbb{R}^2)$ for any $\beta > 0$. As is explained in [28], it would be more natural to use here the Gaussian space X defined in (4.4), but for technical reasons it appears necessary to replace the Gaussian weight by an exponential one, at least if the ratio T/T_0 is large.

Proposition 5.3 *Assume that the point vortex system* (5.4) *is well-posed on the time interval* $[0,T]$*, and let* $\omega^\nu(x,t)$ *be the solution of* (2.2) *with initial data* (5.2)*. Then there exist positive constants* β *and* K_2*, depending only on the ratio* T/T_0*, such that, if* $\omega^\nu(x,t)$ *is decomposed as in* (5.8)*, then the rescaled profiles* $w_i^\nu(\xi,t)$ *defined by* (5.10) *satisfy*

$$\max_{i=1,\ldots,N}\|w_i^\nu(\cdot,t) - w_i^{\mathrm{app}}(\cdot,t)\|_{X_\beta} \;\le\; K_2\Big(\frac{\nu t}{d^2}\Big)^{3/2}~, \tag{5.15}$$

for all $t \in (0,T]$*, provided* $\nu T/d^2 \le K_2^{-1}$.

As is clear from (5.15), (5.11), and (5.10), Proposition 5.3 implies immediately Theorem 5.1. Note that the error term on the right-hand side of (5.15) is smaller than in (5.6), and in particular smaller than the first order corrections to the Gaussian profile in (5.11). This means that the deformations of the interacting vortices are indeed described, to leading order, by (5.13). According to that formula, each vortex adapts its shape *instantaneously* to the relative positions of the other vortices, without oscillations or inertia. By this we mean that, for each $t \in (0, T]$, the angular factor $\cos(2(\theta - \theta_{ij}))$ in (5.13), which gives the leading order deformation up to a time-dependent prefactor $\nu t / |z_{ij}|^2$, is entirely determined by the instantaneous positions of the vortex centers. In contrast, it is shown in [28] that the first order radially symmetric corrections $\bar{F}_i(\xi, t)$ do not only depend on the instantaneous vortex positions, but on the whole history of the system.

It is instructive at this point to recall what is known in the more general situation where the initial data are not point vortices, but finite size vortex patches. For simplicity, we consider the case of two identical, radially symmetric vortex patches of size $R > 0$ initially located at points $x_1, x_2 \in \mathbb{R}^2$ with $|x_1 - x_2| \gg R$. Under the time evolution defined by (2.2), the centers of the patches will rotate with approximatively constant angular velocity around the mid-point $(x_1 + x_2)/2$. To obtain a more precise description of the solution, we go to a rotating frame where the centers of the patches stay fixed, and we concentrate on the deformations of the vortex cores. What is observed in numerical simulations [45] is that the interaction begins with a fast relaxation process, during which each vortex adapts its shape to the velocity field generated by the other vortex. This first step depends on the details of the initial data, and is characterized by temporal oscillations of the vortex cores which disappear on a non-viscous time scale. In a second step, the vortices relax to a Gaussian-like profile at a diffusive rate, and the system reaches a "metastable state" which is independent of the initial data, and will persist until two vortices get sufficiently close to start a merging process. In this metastable regime, the vortex centers move in the plane according to the Helmholtz-Kirchhoff dynamics, and the vortex profiles are uniquely determined, up to a scaling factor, by the relative positions of the centers. This is exactly the situation described by Proposition 5.3. In fact, if we start with point vortices, the system is immediately in the metastable state which, for more general initial data, is reached only after the transients steps described above. In this sense, point vortices can be considered as *well-prepared initial data* for the vortex interaction problem.

5.3 Perturbation Expansion and Error Estimates

In this final section, we briefly indicate how to construct an asymptotic expansion of the viscous N-vortex solution and to control the error terms. We first write the evolution equation satisfied by the rescaled vorticity profiles defined in (5.10). Since $\partial_t \omega_i^\nu + u^\nu \cdot \nabla \omega_i^\nu = \nu \Delta \omega_i^\nu$ for $i = 1, \dots, N$, where $u^\nu = \sum_{j=1}^N u_j^\nu$, we obtain

$$t\partial_t w_i^\nu(\xi, t) + \left\{ \sum_{j=1}^N \frac{\gamma_j}{\nu} v_j^\nu\Big(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\,, t\Big) - \sqrt{\frac{t}{\nu}}\, z_i'(t) \right\} \cdot \nabla w_i^\nu(\xi, t) \;=\; (\mathcal{L} w_i^\nu)(\xi, t) \;,$$

where $\mathcal{L}$ is the linear operator (4.1). The left-hand side is clearly singular in the limit $\nu \to 0$, but this difficulty can be partially eliminated by an appropriate choice of the speeds $z_i'(t)$. To this purpose, we set

$$z_i'(t) \;=\; \sum_{j=1}^N \frac{\gamma_j}{\sqrt{\nu t}}\, v^G\Big(\frac{z_{ij}(t)}{\sqrt{\nu t}}\Big) \;, \qquad i = 1, \dots, N \;, \tag{5.16}$$

where $z_{ij}(t) = z_i(t) - z_j(t)$ and v^G is the velocity profile defined in (3.2). This is a viscous approximation of the original point vortex system (5.4), which has its own interest and was recently studied in [61, 40]. It is shown in [28, Lemma 2] that the solutions of (5.16) stay extremely close to those of (5.4) in our perturbative regime where the vortices are widely separated from each other. In what follows, we thus make no distinction between the solutions of (5.4) and (5.16).

As was already mentioned, we expect the Gaussian function G to represent the leading order approximation of each vorticity profile. Substituting G for $w_i^\nu(\xi,t)$ in the evolution equation above, and using (5.16) together with $\partial_t G = \mathcal{L}G = 0$, we obtain the following expression for the residuum of this naive approximation :

$$R_i^{(0)}(\xi,t) \,=\, \sum_{j\neq i} \frac{\gamma_j}{\nu}\Bigl\{v^G\Bigl(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\Bigr) - v^G\Bigl(\frac{z_{ij}(t)}{\sqrt{\nu t}}\Bigr)\Bigr\}\cdot\nabla G(\xi)\ .$$

This expression is not singular in the limit $\nu \to 0$, and a direct calculation (see [28, Proposition 1]) yields the asymptotic expansion

$$R_i^{(0)}(\xi,t) \,=\, \frac{\gamma_i t}{d^2}\Bigl\{A_i(\xi,t) + \Bigl(\frac{\nu t}{d^2}\Bigr)^{1/2}B_i(\xi,t) + \Bigl(\frac{\nu t}{d^2}\Bigr)C_i(\xi,t) + \tilde R_i^{(0)}(\xi,t)\Bigr\}\ , \tag{5.17}$$

for all $\xi \in \mathbb{R}^2$ and all $t \in (0,T]$, where

$$\begin{aligned}
A_i(\xi,t) \,&=\, \frac{d^2}{2\pi}\sum_{j\neq i}\frac{\gamma_j}{\gamma_i}\,\frac{(\xi\cdot z_{ij}(t))(\xi\cdot z_{ij}(t)^\perp)}{|z_{ij}(t)|^4}\,G(\xi)\ ,\\
B_i(\xi,t) \,&=\, \frac{d^3}{4\pi}\sum_{j\neq i}\frac{\gamma_j}{\gamma_i}\,\frac{(\xi\cdot z_{ij}(t)^\perp)}{|z_{ij}(t)|^6}\,\Bigl(|\xi|^2|z_{ij}(t)|^2 - 4(\xi\cdot z_{ij}(t))^2\Bigr)\,G(\xi)\ ,\\
C_i(\xi,t) \,&=\, \frac{d^4}{\pi}\sum_{j\neq i}\frac{\gamma_j}{\gamma_i}\,\frac{(\xi\cdot z_{ij}(t))(\xi\cdot z_{ij}(t)^\perp)}{|z_{ij}(t)|^8}\Bigl(2(\xi\cdot z_{ij}(t))^2 - |\xi|^2|z_{ij}(t)|^2\Bigr)\,G(\xi)\ .
\end{aligned}$$

Moreover, given any $\delta < 1$, the last term $\tilde R_i^{(0)}$ in (5.17) can be estimated as follows :

$$|\tilde R_i^{(0)}(\xi,t)| \,\le\, C\Bigl(\frac{\nu t}{d^2}\Bigr)^{3/2} e^{-\delta|\xi|^2/4}\ , \qquad \xi\in\mathbb{R}^2\ , \quad 0 < t \le T\ .$$

The formula (5.17) shows in particular that $R_i^{(0)}(\xi,t) = \mathcal{O}(1)$ as $\nu \to 0$. Thus, if we decompose $w_i^\nu(\xi,t) = G(\xi) + \tilde w_i(\xi,t)$, the equation for $\tilde w_i(\xi,t)$ will contain a source term of size $\mathcal{O}(1)$ as $\nu \to 0$, and we therefore expect that the remainder $\tilde w_i(\xi,t)$ itself will be of size $\mathcal{O}(1)$ after a short time. But, as is easily verified, the equation for $\tilde w_i(\xi,t)$ contains nonlinear terms involving negative powers of ν, and such terms cannot be controlled in the vanishing viscosity limit if $\tilde w_i(\xi,t)$ is $\mathcal{O}(1)$. This is the reason why it is necessary to construct a more precise approximate solution of our vorticity profiles, in order to "desingularize" the equation for the remainder.

To this end, we first consider approximate solutions of the form

$$w_i^{\rm app}(\xi,t) \,=\, G(\xi) + \Bigl(\frac{\nu t}{d^2}\Bigr)F_i(\xi,t)\ , \qquad v_i^{\rm app}(\xi,t) \,=\, v^G(\xi) + \Bigl(\frac{\nu t}{d^2}\Bigr)v^{F_i}(\xi,t)\ ,$$

where $F_1(\xi,t),\dots,F_n(\xi,t)$ are smooth vorticity profiles to be determined, and the corresponding velocity profiles $v^{F_1}(\xi,t),\dots,v^{F_n}(\xi,t)$ are obtained via the Biot-Savart law (2.4). The residuum of this improved approximation can be computed [28], and has the form

$$R_i^{(1)}(\xi,t) \,=\, \frac{\gamma_i t}{d^2}\Bigl(A_i(\xi,t) + \Lambda F_i(\xi,t)\Bigr) + \mathcal{O}\left(\frac{\sqrt{\nu t}}{d}\right)\ ,$$

where Λ is the linear operator defined in (4.3). To minimize the residuum, we want to choose F_i so as to satisfy the linear equation $\Lambda F_i + A_i = 0$. This is indeed possible, because A_i belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^2)$ and to the subspace $\tilde{X}_2 \subset X$, see (4.18), and it is shown in [60, 32, 28] that $\mathcal{S}(\mathbb{R}^2) \cap \tilde{X}_2 \subset \mathrm{ran}(\Lambda)$. The equation $\Lambda F_i + A_i = 0$ has therefore a unique solution in $\tilde{X}_2$, which coincides with the leading term in the expression (5.12) of $F_i^\nu(\xi,t)$. Note however that the equation above determines F_i only up to an element of $\ker(\Lambda)$. As a matter of fact, in (5.11), a radially symmetric correction $\bar{F}_i(\xi,t)$ is added to $F_i^\nu(\xi,t)$ in order to compensate for higher order terms (which will not be considered here). In any case, we can choose F_i in such a way that $R_i^{(1)}(\xi,t) = \mathcal{O}(\sqrt{\nu t}/d)$.

Following the same procedure, we next construct a higher order approximate solution of the form

$$\begin{aligned} w_i^{\mathrm{app}}(\xi,t) &= G(\xi) + \Bigl(\frac{\nu t}{d^2}\Bigr) F_i(\xi,t) + \Bigl(\frac{\nu t}{d^2}\Bigr)^{3/2} H_i(\xi,t) + \Bigl(\frac{\nu t}{d^2}\Bigr)^2 K_i(\xi,t)~, \\ v_i^{\mathrm{app}}(\xi,t) &= v^G(\xi) + \Bigl(\frac{\nu t}{d^2}\Bigr) v^{F_i}(\xi,t) + \Bigl(\frac{\nu t}{d^2}\Bigr)^{3/2} v^{H_i}(\xi,t) + \Bigl(\frac{\nu t}{d^2}\Bigr)^2 v^{K_i}(\xi,t)~. \end{aligned} \tag{5.18}$$

Again, the vorticity profiles F_i, H_i, and K_i are obtained by solving under-determined linear equations involving the operator Λ. More precisely, we have $\Lambda F_i + A_i = 0$ and $\Lambda H_i + B_i = 0$, but the equation for K_i is more complicated. The residuum of that approximation now satisfies

$$|R_i^{(3)}(\xi,t)| \le C \Bigl(\frac{\nu t}{d^2}\Bigr)^{3/2} e^{-\delta|\xi|^2/4}~, \qquad \xi \in \mathbb{R}^2~, \quad 0 < t \le T~. \tag{5.19}$$

Finally, once the approximate solution (5.18) has been constructed, we decompose the vorticity profiles as follows:

$$w_i^\nu(\xi,t) = w_i^{\mathrm{app}}(\xi,t) + \Bigl(\frac{\nu t}{d^2}\Bigr) \tilde{w}_i(\xi,t)~, \quad v_i^\nu(\xi,t) = v_i^{\mathrm{app}}(\xi,t) + \Bigl(\frac{\nu t}{d^2}\Bigr) \tilde{v}_i(\xi,t)~, \tag{5.20}$$

and we consider the evolution system satisfied by the remainder $\tilde{w}_i(\xi,t)$, $\tilde{v}_i(\xi,t)$. This system reads

$$\begin{aligned} & t\partial_t \tilde{w}_i(\xi,t) - (\mathcal{L}\tilde{w}_i)(\xi,t) + \tilde{w}_i(\xi,t) \\ &\quad + \frac{\gamma_i}{\nu}\Bigl(v_i^{\mathrm{app}}(\xi,t)\cdot\nabla\tilde{w}_i(\xi,t) + \tilde{v}_i(\xi,t)\cdot\nabla w_i^{\mathrm{app}}(\xi,t)\Bigr) \\ &\quad + \sum_{j\neq i} \frac{\gamma_j}{\nu}\Bigl\{ v_j^{\mathrm{app}}\Bigl(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\,,\, t\Bigr) - v^G\Bigl(\frac{z_{ij}(t)}{\sqrt{\nu t}}\Bigr)\Bigr\}\cdot\nabla\tilde{w}_i(\xi,t) \\ &\quad + \sum_{j\neq i} \frac{\gamma_j}{\nu}\, \tilde{v}_j\Bigl(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\,,\, t\Bigr)\cdot\nabla w_i^{\mathrm{app}}(\xi,t) \\ &\quad + \sum_{j=1}^N \frac{\gamma_j t}{d^2}\, \tilde{v}_j\Bigl(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\,,\, t\Bigr)\cdot\nabla\tilde{w}_i(\xi,t) + \tilde{R}_i(\xi,t) = 0~, \end{aligned} \tag{5.21}$$

and using (5.19), (5.20) we see that the residuum $\tilde{R}_i$ is $\mathcal{O}(\sqrt{\nu t}/d)$ as $\nu \to 0$. Also, due to the ansatz (5.20), the nonlinear terms in (5.21) are now regular in the limit $\nu \to 0$. The linear terms still contain negative powers of ν, but this difficulty can be avoided by using appropriate norms. In a much simpler setting, the same idea was already present in the proof of Proposition 4.5, where the skew-symmetry of the operator Λ in the Gaussian space X was used to obtain a stability estimate that was uniform in the circulation parameter α.

To control the solution of (5.21), we introduce a weighted energy functional of the form

$$E(t) \;=\; \sum_{i=1}^N \int_{\mathbb{R}^2} p_i(\xi,t)|\tilde{w}_i(\xi,t)|^2 \,\mathrm{d}\xi\;, \qquad t \in (0,T]\;.$$

If the observation time $T > 0$ is small with respect to the turnover time $T_0 = d^2/|\gamma|$, we can take $p_i(\xi,t) = p_{a(t)}(\xi)$ for $i = 1,\dots,N$, where $a(t) = d/(3\sqrt{\nu t})$ and

$$p_a(\xi) \;=\; \begin{cases} e^{|\xi|^2/4} & \text{if} \quad |\xi| \le a\;, \\ e^{a^2/4} & \text{if} \quad a \le |\xi| \le Ka\;, \\ e^{|\xi|^2/(4K^2)} & \text{if} \quad |\xi| \ge Ka\;, \end{cases}$$

for some $K \gg 1$. We then have $e^{|\xi|^2/(4K^2)} \le p_i(\xi,t) \le e^{|\xi|^2/4}$ for all $\xi \in \mathbb{R}^2$ and $t \in (0,T)$. With this choice, we obtain from (5.21) a differential inequality for the weighted energy $E(t)$, which can be integrated using Gronwall's lemma and yields the bound:

$$\int_{\mathbb{R}^2} e^{\frac{|\xi|^2}{4K^2}} \Bigl(|\tilde{w}_1(\xi,t)|^2 + \dots + |\tilde{w}_N(\xi,t)|^2\Bigr)\,\mathrm{d}\xi \;\le\; E(t) \;\le\; C\,\frac{\nu t}{d^2}\;.$$

This concludes the proof of Proposition 5.3 if $T \ll T_0$. In the general case, one has to introduce more complicated weights, which can be constructed using the same procedure as the approximate solution itself. These weights satisfy $e^{\beta|\xi|/4} \le p_i(\xi,t) \le e^{|\xi|^2/4}$, for some small $\beta > 0$ depending only on T/T_0. We thus obtain the weaker estimate:

$$\int_{\mathbb{R}^2} e^{\frac{\beta|\xi|}{4}} \Bigl(|\tilde{w}_1(\xi,t)|^2 + \dots + |\tilde{w}_N(\xi,t)|^2\Bigr)\,\mathrm{d}\xi \;\le\; E(t) \;\le\; C\,\frac{\nu t}{d^2}\;,$$

which still implies the desired conclusion.

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