Long-Time Asymptotics for the Navier-Stokes Equation in a Two-Dimensional Exterior Domain

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Abstract. We study the long-time behavior of infinite-energy solutions to the incompressible Navier-Stokes equations in a two-dimensional exterior domain, with no-slip boundary conditions. The initial data we consider are finite-energy perturbations of a smooth vortex with small circulation at infinity, but are otherwise arbitrarily large. Using a logarithmic energy estimate and some interpolation arguments, we prove that the solution approaches a self-similar Oseen vortex as $t \to \infty$. This result was obtained in collaboration with Y. Maekawa (Kobe University).

Comportement asymptotique en temps des solutions de l'équation de Navier-Stokes dans un domaine extérieur du plan

Résumé. Nous étudions le comportement asymptotique en temps des solutions de l'équation de Navier-Stokes incompressible dans un domaine extérieur du plan, avec condition de non-glissement à la frontière. Les données initiales que nous considérons sont des perturbations d'énergie finie d'un tourbillon régulier dont la circulation à l'infini est petite, mais nous n'imposons aucune autre restriction à leur taille. En utilisant une estimation d'énergie logarithmique et des arguments d'interpolation, nous montrons que la solution converge lorsque $t \to \infty$ vers un tourbillon d'Oseen autosimilaire. Ce résultat a été obtenu en collaboration avec Y. Maekawa (Université de Kobe).

1. Introduction

We consider the free motion of an incompressible viscous fluid in a two-dimensional exterior domain $\Omega = \mathbb{R}^2 \setminus K$, where $K \subset \mathbb{R}^2$ is a compact obstacle with a smooth boundary. We do not assume that K is connected, hence we include the case where the fluid moves around a finite collection of obstacles. As for the boundary conditions, we suppose that the velocity of the fluid vanishes on $\partial\Omega$ and decays to zero at infinity. The evolution of our system is thus governed by the Navier-Stokes equations

$$\begin{cases} \partial_t u + (u \cdot \nabla) u \,=\, \Delta u - \nabla p \,, \quad \operatorname{div} u \,=\, 0 \,, & \text{for } x \in \Omega \,, \quad t > 0 \,, \\ u(x,t) \,=\, 0 \,, & \text{for } x \in \partial\Omega \,, \quad t > 0 \,, \\ u(x,0) \,=\, u_0(x) \,, & \text{for } x \in \Omega \,, \end{cases} \tag{1.1}$$

where $u(x,t) \in \mathbb{R}^2$ and $p(x,t) \in \mathbb{R}$ denote, respectively, the velocity and the pressure of the fluid at a space-time point $(x,t) \in \Omega \times \mathbb{R}_+$. As can be seen from the first equation in (1.1), we assume that the fluid density is constant and that the kinematic viscosity is equal to 1. Since (1.1) includes no forcing, the motion of the fluid originates entirely from the initial data $u_0 : \Omega \to \mathbb{R}^2$, which we assume to be divergence-free and tangent to the boundary on $\partial\Omega$.

The behavior of the solutions of (1.1) depends in a crucial way on the decay rate of the velocity field $u(x,t)$ as $|x| \to \infty$. If the initial velocity u_0 belongs to the energy space

$$L^2_\sigma(\Omega) \,=\, \Big\{ u \in L^2(\Omega)^2 \,\Big|\, \operatorname{div} u = 0 \text{ in } \Omega \,,\ u \cdot n = 0 \text{ on } \partial\Omega \Big\} \,,$$

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where n denotes the interior unit normal on $\partial\Omega$, we have the following classical result:

Theorem 1. *For all initial data $u_0 \in L^2_\sigma(\Omega)$, Eq. (1.1) has a unique global solution*

$$u \in C^0([0,\infty), L^2_\sigma(\Omega)) \cap C^1((0,\infty), L^2_\sigma(\Omega)) \cap C^0((0,\infty), H^1_0(\Omega)^2 \cap H^2(\Omega)^2) \ ,$$

which satisfies for all $t \geq 0$ the energy equality:

$$\frac{1}{2}\|u(\cdot,t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u(\cdot,s)\|^2_{L^2(\Omega)}\,\mathrm{d}s \ = \ \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} \ . \tag{1.2}$$

Global well-posedness for the Navier-Stokes equations was first established by Leray [25] in the particular case where $\Omega = \mathbb{R}^2$. When $\Omega \subset \mathbb{R}^2$ is bounded, the first results also go back to Leray [26], but global existence of large solutions was shown only later by Ladyzhenskaia [24], see also [27, 19, 9]. To prove Theorem 1, one can use a regularization or a discretization procedure to construct global *weak* solutions of (1.1) which satisfy the energy inequality, and then prove that these solutions are unique and have the desired regularity. Alternatively, one can construct local *mild* solutions by transforming (1.1) into an integral equation and solving it by a fixed point argument, and then use the energy equality (1.2) to show that all solutions can be extended to the whole time interval $[0,\infty)$. Although most of the literature is devoted to the situation where Ω is either a bounded domain or the whole plane $\mathbb{R}^2$, the case of an exterior domain can be treated without essential modifications, see e.g. [22].

It follows from (1.2) that the kinetic energy $E(t) = \frac{1}{2}\|u(\cdot,t)\|^2_{L^2(\Omega)}$ is nonincreasing in time, and a result of Masuda [28] shows that $E(t)$ converges to zero as $t \to \infty$. Moreover, under additional assumptions on the initial data, it is possible to specify a decay rate in time. For instance, if $u_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$ for some $q \in (1,2)$, the solution of (1.1) lies in the same space for all $t > 0$ and

$$\|u(\cdot,t)\|_{L^2(\Omega)} \ = \ o\Big(t^{\frac{1}{2}-\frac{1}{q}}\Big) \qquad \text{as} \quad t \to \infty \ , \tag{1.3}$$

see [4, 21, 1]. It is interesting to notice that (1.3) fails in the limiting case $q = 1$. Indeed, if $u_0 \in L^2_\sigma(\Omega) \cap L^1(\Omega)^2$, then in general the velocity field $u(x,t)$ decays like $|x|^{-2}$ as $|x| \to \infty$, so that $u(\cdot,t) \notin L^1(\Omega)^2$ for $t > 0$.

As an aside, we mention that this loss of spatial decay is related to the net force F exerted by the fluid on the obstacle K. To see this, we first observe that any velocity field $u \in L^2_\sigma(\Omega) \cap L^1(\Omega)^2$ satisfies $\int_\Omega u\,\mathrm{d}x = 0$. Indeed, if u is smooth and compactly supported, then using Gauss' theorem and the fact that $u \cdot n = 0$ on $\partial\Omega$ we find

$$\int_\Omega u_j\,\mathrm{d}x \ = \ \int_\Omega (u\cdot\nabla)x_j\,\mathrm{d}x \ = \ \int_\Omega \mathrm{div}(u\,x_j)\,\mathrm{d}x \ = \ 0 \ , \qquad \text{for } j = 1,2 \ .$$

The general easily case follows by a density argument [20]. Now, if $u \in C^1([0,T], L^1(\Omega)^2)$ is a solution of the Navier-Stokes equation (1.1), then

$$0 \ = \ \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\,\mathrm{d}x \ = \ \int_\Omega \Big(\Delta u - \nabla p - (u\cdot\nabla)u\Big)\,\mathrm{d}x \ = \ -\int_{\partial\Omega}(Tn)\,\mathrm{d}\sigma \ = \ -F \ ,$$

because $\Delta u_i - \partial_i p = \partial_j T_{ij}$ where $T_{ij} = \partial_i u_j + \partial_j u_i - p\delta_{ij}$ is the stress tensor (we recall that all physical parameters have been normalized to 1). The formal calculation above can be made rigorous [20] and shows that, no matter how localized the initial data may be, the velocity field $u(\cdot,t)$ does not stay integrable for positive times, unless the net force F vanishes identically. Of course this is not the case in general, but in highly symmetric situations it is possible to construct solutions of (1.1) for which $F \equiv 0$, and which decay faster as $t \to \infty$ than what is indicated in (1.3), see [15, 16].

Much less is known about the solutions of (1.1) if the initial data u_0 are not square integrable. Although the physical relevance of infinite-energy solutions can be questioned, we believe that such solutions naturally occur when studying the dynamics of (1.1) in a

two-dimensional exterior domain Ω. One way to realize that is to consider the relation between the velocity field u and the associated vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$. Given $p \in [1,2)$, we denote

$$\dot{W}^{1,p}_{0,\sigma}(\Omega) \,=\, \Big\{ u \in L^{\frac{2p}{2-p}}(\Omega)^2 \;\Big|\; \nabla u \in L^p(\Omega)^4 \,,\; \operatorname{div} u = 0 \text{ in } \Omega \,,\; u = 0 \text{ on } \partial\Omega \Big\} \,.$$

In other words $\dot{W}^{1,p}_{0,\sigma}(\Omega)$ is the completion with respect to the norm $u \mapsto \|\nabla u\|_{L^p}$ of the space of all smooth, divergence-free vector fields with compact support in Ω, see [10]. We then have the following result:

Lemma 2. *If $u \in \dot{W}^{1,p}_{0,\sigma}(\Omega)$ for some $p \in [1,2)$, and if $\omega = \partial_1 u_2 - \partial_2 u_1$, then*

$$u(x) \,=\, \frac{1}{2\pi} \int_\Omega \frac{(x-y)^\perp}{|x-y|^2}\, \omega(y)\, \mathrm{d}y \;, \tag{1.4}$$

for almost every $x \in \Omega$. Here $x^\perp = (-x_2, x_1)$ and $|x|^2 = x_1^2 + x_2^2$ if $x = (x_1, x_2) \in \mathbb{R}^2$.

The proof of Lemma 2 is very simple: if $\bar{u} : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the extension of u by zero outside Ω, then $\bar{u} \in L^{2p/(2-p)}(\mathbb{R}^2)^2$, $\nabla \bar{u} \in L^p(\mathbb{R}^2)^4$, and $\operatorname{div} \bar{u} = 0$. Moreover $\bar{\omega} = \partial_1 \bar{u}_2 - \partial_2 \bar{u}_1$ is the extension of ω by zero outside Ω. Thus $\bar{u}$ can be expressed in terms of $\bar{\omega}$ using the classical Biot-Savart law in $\mathbb{R}^2$, and restricting that relation to Ω we obtain (1.4). We emphasize that the representation (1.4) is only valid if ω is the curl of a divergence-free velocity field u which *vanishes on* $\partial\Omega$. In contrast, if $\omega : \Omega \to \mathbb{R}$ is an arbitrary smooth function with compact support, the velocity field *defined by* (1.4) does not even satisfy $u \cdot n = 0$ on $\partial\Omega$!

We now assume that the vorticity distribution ω is sufficiently localized so that $\omega \in L^1(\Omega)$, and we define the *total circulation*

$$\alpha \,=\, \int_\Omega \omega(x)\, \mathrm{d}x \,=\, \lim_{R\to\infty} \oint_{|x|=R} u_1\, \mathrm{d}x_1 + u_2\, \mathrm{d}x_2 \;,$$

where the second equality follows from Green's theorem, since $\omega = \partial_1 u_2 - \partial_2 u_1$ and u vanishes on $\partial\Omega$. Using the vorticity formulation of the Navier-Stokes equation (1.1), it is not difficult to verify that the total circulation is a conserved quantity. But it follows from (1.4) that

$$u(x) \;\sim\; \frac{\alpha}{2\pi} \frac{x^\perp}{|x|^2} \;, \qquad \text{as} \quad |x| \to \infty \;, \tag{1.5}$$

hence $u \notin L^2(\Omega)^2$ as soon as $\alpha \neq 0$. This shows that finite-energy solutions of (1.1) necessarily have zero total circulation. On the other hand, in many important examples of two-dimensional flows such as vortex patches, vortex sheets, or point vortices, the vorticity distribution typically has a constant sign, hence the total circulation is necessarily nonzero. In our opinion, it is thus important to enlarge the class of admissible solutions of (1.1), so as to allow for velocity fields which decay like $|x|^{-1}$ as $|x| \to \infty$.

A possible framework for the study of infinite-energy solutions of the Navier-Stokes equation (1.1) is the *weak energy space*

$$L^{2,\infty}_\sigma(\Omega) \,=\, \Big\{ u \in L^{2,\infty}(\Omega)^2 \;\Big|\; \operatorname{div} u \,=\, 0 \text{ in } \Omega \,,\; u \cdot n = 0 \text{ on } \partial\Omega \Big\} \;,$$

where $L^{2,\infty}(\Omega)$ is the weak L^2 space on Ω, see [3]. We recall that

$$\|u\|_{L^{2,\infty}(\Omega)} \,\approx\, \sup_{\lambda>0} \lambda \Big(\operatorname{meas}\{x \in \Omega \,|\, |u(x)| > \lambda\} \Big)^{1/2} \;, \tag{1.6}$$

in the sense that the norm $\|u\|_{L^{2,\infty}}$ is equivalent to the quantity in the right-hand side of (1.6). Clearly $L^2_\sigma(\Omega) \hookrightarrow L^{2,\infty}_\sigma(\Omega)$, but the weak energy space is large enough to include velocity fields which decay slowly at infinity, as in (1.5). Concerning the solvability of (1.1) in $L^{2,\infty}_\sigma(\Omega)$, the following general result was obtained by Kozono and Yamazaki:

Theorem 3. [23] *There exists $\epsilon > 0$ such that, for all initial data $u_0 \in L^{2,\infty}_\sigma(\Omega)$ satisfying*

$$\limsup_{\lambda\to+\infty} \lambda\Big(\mathrm{meas}\{x \in \Omega \,|\, |u_0(x)| > \lambda\}\Big)^{1/2} \le \epsilon \ , \tag{1.7}$$

Eq. (1.1) *has a unique global solution such that, for all $T > 0$,*

$$\sup_{0<t<T} \|u(\cdot,t)\|_{L^{2,\infty}(\Omega)} + \sup_{0<t<T} t^{1/4}\|u(\cdot,t)\|_{L^4(\Omega)} < \infty \ ,$$

and such that $u(\cdot,t) \to u_0$ as $t \to 0$ in the weak-$*$ topology of $L^{2,\infty}_\sigma(\Omega)$.

Theorem 3 shows that the Cauchy problem for the Navier-Stokes equations (1.1) is globally well-posed in the weak energy space $L^{2,\infty}_\sigma(\Omega)$, provided that the local singularity of the initial data u_0 is sufficiently small, in the sense of (1.7). To illustrate the meaning of this smallness condition, we consider the simple situation where the initial flow is just a point vortex of circulation $\alpha \in \mathbb{R}$ located at $x_0 \in \Omega$. The initial vorticity is thus given by $\omega_0(x) = \alpha\delta(x - x_0)$, and using the classical Biot-Savart law in the exterior domain Ω (see e.g. [17]) it is easy to verify that the corresponding velocity field u_0 lies in $L^{2,\infty}_\sigma(\Omega)$, is smooth in $\Omega \setminus \{x_0\}$, and satisfies

$$u_0(x) \approx \frac{\alpha}{2\pi}\frac{(x-x_0)^\perp}{|x-x_0|^2} \qquad \text{as} \quad x \to x_0 \ ,$$

so that (1.7) is fulfilled if and only if $|\alpha| \le \sqrt{4\pi}\epsilon$. This example shows that, if the initial vorticity ω_0 is a finite measure, condition (1.7) implies a restriction on the size of the atomic part of ω_0. Such a restriction also arises in the analysis of the two-dimensional vorticity equation in the whole space $\mathbb{R}^2$, see [14], but in that particular case the uniqueness of the solution can be established when the initial vorticity is an arbitrary finite measure [11, 2].

Although Theorem 3 provides the existence of a large class of infinite-energy solutions, very little is known about the asymptotic behavior of these solutions as $t \to \infty$. In fact, we do not even know whether they stay bounded in the weak energy space $L^{2,\infty}_\sigma(\Omega)$, because we are lacking a priori estimates. Indeed, if $u_0 \notin L^2_\sigma(\Omega)$ the energy equality (1.2) does not make sense, and because of the no-slip boundary condition on $\partial\Omega$ it is quite difficult to obtain estimates on the vorticity distribution if $\Omega \neq \mathbb{R}^2$. In the rest of this paper, however, we consider a particular class of infinite-energy solutions of the Navier-Stokes equations (1.1), for which the asymptotic behavior in time can be accurately described.

2. Main Results

In the particular case where $\Omega = \mathbb{R}^2$, the Navier-Stokes equations (1.1) have a family of self-similar solutions of the form $u(x,t) = \alpha\Theta(x,t)$, $p(x,t) = \alpha^2\Pi(x,t)$, where $\alpha \in \mathbb{R}$ is a free parameter (the total circulation) and

$$\Theta(x,t) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2}\Big(1 - e^{-\frac{|x|^2}{4(1+t)}}\Big) \ , \qquad \nabla\Pi(x,t) = \frac{x}{|x|^2}|\Theta(x,t)|^2 \ . \tag{2.1}$$

These solutions are usually called the *Lamb-Oseen vortices*. If $u(x,t) = \alpha\Theta(x,t)$, the corresponding vorticity distribution is $\omega(x,t) = \alpha\Xi(x,t)$, where

$$\Xi(x,t) = \partial_1\Theta_2(x,t) - \partial_2\Theta_1(x,t) = \frac{1}{4\pi(1+t)}\, e^{-\frac{|x|^2}{4(1+t)}} \ . \tag{2.2}$$

Note that $\Xi(x,t) > 0$ and $\int_{\mathbb{R}^2}\Xi(x,t)\,\mathrm{d}x = 1$ for all $t \ge 0$. Oseen vortices play an important role in the dynamics of the Navier-Stokes equations in $\mathbb{R}^2$. In particular, we have the following result :

Theorem 4. [13]. *For all initial data $u_0 \in L^{2,\infty}_\sigma(\mathbb{R}^2)$ such that the vorticity distribution ω_0 is integrable, the solution of the Navier-Stokes equation in $\mathbb{R}^2$ satisfies*

$$\int_{\mathbb{R}^2} |\omega(x,t) - \alpha\Xi(x,t)|\,\mathrm{d}x \xrightarrow[t\to\infty]{} 0 \ , \qquad \textit{where} \quad \alpha = \int_{\mathbb{R}^2} \omega_0\,\mathrm{d}x \ .$$

In other words, Oseen vortices describe the leading order asymptotics of all solutions of the Navier-Stokes equations in $\mathbb{R}^2$ with integrable vorticity distribution and nonzero total circulation.

In the case of an exterior domain $\Omega = \mathbb{R}^2 \setminus K$, approximate Oseen vortices can be constructed in the following way. Let $\chi : \mathbb{R}^2 \to [0,1]$ be a smooth, radially symmetric cut-off function such that χ is nondecreasing along rays, $\chi = 0$ on a neighborhood of K, and $\chi(x) = 1$ when $|x|$ is sufficiently large. The *truncated Oseen vortex* with unit circulation is defined as follows:

$$u^\chi(x,t) = \chi(x)\Theta(x,t) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2}\Big(1 - e^{-\frac{|x|^2}{4(1+t)}}\Big)\chi(x) \ . \tag{2.3}$$

For any $t \geq 0$, it is clear that $u^\chi(\cdot,t)$ is a smooth divergence-free vector field which vanishes in a neighborhood of K. The corresponding vorticity distribution $\omega^\chi = \partial_1 u^\chi_2 - \partial_2 u^\chi_1$ has the explicit expression

$$\omega^\chi(x,t) = \chi(x)\Xi(x,t) + \frac{1}{2\pi}\frac{1}{|x|^2}\Big(1 - e^{-\frac{|x|^2}{4(1+t)}}\Big)x\cdot\nabla\chi(x) \ , \tag{2.4}$$

where $\Xi(x,t)$ is defined in (2.2). In particular $\omega^\chi(x,t) \geq 0$ and $\int_{\mathbb{R}^2}\omega^\chi(x,t)\,\mathrm{d}x = 1$ for all $t \geq 0$. Moreover, a direct calculation shows that

$$(u^\chi\cdot\nabla)u^\chi = \frac{1}{2}\nabla|u^\chi|^2 + (u^\chi)^\perp\omega^\chi = -\frac{x}{|x|^2}|u^\chi|^2 \ , \tag{2.5}$$

hence there exists a radially symmetric function $p^\chi(x,t)$ such that $-\nabla p^\chi = (u^\chi\cdot\nabla)u^\chi$.

Now, given $\alpha \in \mathbb{R}$, we consider solutions of (1.1) of the particular form

$$u(x,t) = \alpha u^\chi(x,t) + v(x,t) \ , \qquad p(x,t) = \alpha^2 p^\chi(x,t) + q(x,t) \ , \tag{2.6}$$

where $u^\chi(x,t)$ is the truncated Oseen vortex defined in (2.3), and $v(x,t)$ is a *finite-energy* perturbation. In this situation, we expect that $v(\cdot,t)$ converges to zero in energy norm as $t \to \infty$, so that the long-time behavior of $u(\cdot,t)$ is described, to leading order, by the Oseen vortex $\alpha\Theta(\cdot,t)$. Our main result, which was obtained in collaboration with Y. Maekawa, shows that this is indeed the case, provided the total circulation α is sufficiently small.

Theorem 5. [12] *Fix $q \in (1,2)$, and let $\mu = 1/q - 1/2$. There exists a constant $\epsilon = \epsilon(q) > 0$ such that, for any smooth exterior domain $\Omega \subset \mathbb{R}^2$ and for all initial data of the form $u_0 = \alpha u^\chi(\cdot,0) + v_0$ with $|\alpha| \leq \epsilon$ and $v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$, the solution of the Navier-Stokes equations* (1.1) *satisfies*

$$\|u(\cdot,t) - \alpha\Theta(\cdot,t)\|_{L^2(\Omega)} + t^{1/2}\|\nabla u(\cdot,t) - \alpha\nabla\Theta(\cdot,t)\|_{L^2(\Omega)} = \mathcal{O}(t^{-\mu}) \ , \tag{2.7}$$

as $t \to \infty$.

To understand the scope and the limitations of this statement, a few comments are in order. First of all, Theorem 5 is a *global stability result* for Oseen vortices with small circulation at infinity, because we do not impose any restriction on the size of the perturbation $v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$. In the particular case where $\alpha = 0$, there is no vortex at all and we just recover the asymptotics (1.3) with $\mathcal{O}(t^{-\mu})$ instead of $o(t^{-\mu})$ in the right-hand side. Also, in the simple situation where $\Omega = \mathbb{R}^2$, our result is comparable to that of Carpio [5], although the proof is very different.

The main limitation of Theorem 5 is of course the restriction on the size of the circulation α, which we believe is purely technical. In this respect, the fact that $\epsilon(q)$ can

be taken independent of the domain Ω is quite significant, because we know that there is no restriction on the circulation in the particular case where $\Omega = \mathbb{R}^2$, see Theorem 4. Obviously $\epsilon(q)$ is a decreasing function of q, and the proof shows that $\epsilon(q) = \mathcal{O}(\sqrt{2-q})$ as $q \to 2$. Thus the limiting case $q = 2$ is not included, which means that we are not able to control arbitrary finite-energy perturbations of the Oseen vortex (see however [18] for a partial result in that direction). On the other hand the limit of $\epsilon(q)$ as $q \to 1$ can be estimated and is found to be approximately $\epsilon_* = 5.306$, see [12].

We also mention that the decomposition $u_0 = \alpha u^\chi(\cdot, 0) + v_0$ of the initial data is automatically satisfied if we assume that the initial vorticity is sufficiently localized. More precisely, we have the following auxiliary result, which follows quite easily from Lemma 2.

Proposition 6. [12] *Given $q \in (1,2)$, assume that $u_0 \in \dot{W}^{1,p}_{0,\sigma}(\Omega)$ for some $p \in [1,2)$ and that the associated vorticity $\omega_0 = \operatorname{curl} u_0$ satisfies*

$$\int_\Omega (1+|x|^2)^m |\omega_0(x)|^2 \,\mathrm{d}x \,<\, \infty \;, \tag{2.8}$$

for some $m > 2/q$. If we denote $\alpha = \int_\Omega \omega_0(x)\,\mathrm{d}x$, then $u_0 = \alpha u^\chi(\cdot,0) + v_0$ for some $v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$. In particular, if $|\alpha| \le \epsilon$, the conclusion of Theorem 5 holds.

In view of Proposition 6, it would be nice to extend the conclusion Theorem 5 so as to include a convergence result for the vorticity distribution in the critical space $L^1(\Omega)$. This is not immediately obvious, because the classical L^p–L^q estimates for the Stokes semigroup in an exterior domain do not include the limiting case $p = 1$, see [7, 8]. However, combining Theorem 5 with a relatively standard estimate, which shows that the L^1 norm of the vorticity cannot leak to infinity, we obtain the following result which is the main original contribution of the present paper:

Proposition 7. *Under the assumptions of Theorem 5, if we suppose in addition that* (2.8) *holds for some $m \ge 2/q$, then the vorticity $\omega = \operatorname{curl} u$ satisfies*

$$\int_\Omega |\omega(x,t) - \alpha\Xi(x,t)|\,\mathrm{d}x \,=\, \mathcal{O}(t^{-\mu}\log t) \;, \qquad \text{as} \quad t \to \infty \;, \tag{2.9}$$

where $\Xi(x,t)$ is defined in (2.2).

In the rest of this paper, we give a simplified proof of Theorem 5, which does not yield the optimal conclusion. In particular, we shall find a suboptimal convergence rate in (2.7), and our limitation on the size of the circulation will (a priori) depend on the domain Ω. We refer the reader to [12] for a complete proof, including all details. In the last section, we briefly show how Proposition 7 follows from Theorem 5, using some additional information on the vorticity near infinity.

3. Energy estimates

Given $\alpha \in \mathbb{R}$ we consider solutions of (1.1) of the form (2.6). The perturbation $v(x,t)$ vanishes on the boundary $\partial\Omega$ and satisfies the equation

$$\partial_t v + \alpha(u^\chi \cdot \nabla)v + \alpha(v\cdot\nabla)u^\chi + (v\cdot\nabla)v \,=\, \Delta v + \alpha R^\chi - \nabla q \;, \qquad \operatorname{div} v \,=\, 0 \;, \tag{3.1}$$

where R^χ is the remainder term given by (6.5) below. It is not difficult to verify that the Cauchy problem for equation (3.1) is globally well-posed in the energy space $L^2_\sigma(\Omega)$. The goal of this section is to control the long-time evolution of the perturbation $v(t) \equiv v(\cdot,t)$ using energy estimates.

First of all, we multiply both sides of (3.1) by v and integrate by parts over Ω. Taking into account the no-slip boundary condition, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2_{L^2} + \|\nabla v(t)\|^2_{L^2} \,=\, \alpha\langle v(t), R^\chi(t)\rangle - \alpha\langle v(t), (v(t)\cdot\nabla)u^\chi(t)\rangle \;, \tag{3.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product in $L^2_\sigma(\Omega)$, so that $\|\cdot\|_{L^2} = \langle\cdot\,,\cdot\rangle^{1/2}$. To estimate the right-hand side of (3.2), we use the results of Section 6 below. First, in view of (6.7), we have

$$|\alpha\langle v(t), R^\chi(t)\rangle| \;\le\; \frac{\kappa_3\,|\alpha|}{1+t}\|\nabla v(t)\|_{L^2} \;\le\; \frac{1}{2}\|\nabla v(t)\|^2_{L^2} + \frac{\kappa_3^2\,\alpha^2}{2(1+t)^2}\;.$$

Moreover, applying (6.2) with $p=\infty$, we see that

$$|\langle v(t), (v(t)\cdot\nabla)u^\chi(t)\rangle| \;\le\; \frac{b_\infty}{1+t}\,\|v(t)\|^2_{L^2}\;.$$

We thus obtain the energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2_{L^2} + \|\nabla v(t)\|^2_{L^2} \;\le\; \frac{2b_\infty|\alpha|}{1+t}\|v(t)\|^2_{L^2} + \frac{\kappa_3^2\,\alpha^2}{(1+t)^2}\;, \qquad t>0\;.$$

Using Gronwall's lemma, we deduce that

$$\|v(t)\|^2_{L^2} + \int_{t_0}^t \|\nabla v(s)\|^2_{L^2}\,\mathrm{d}s \;\le\; \Big(\frac{1+t}{1+t_0}\Big)^{2b_\infty|\alpha|}\Big(\|v(t_0)\|^2_{L^2} + \frac{\kappa_3^2\,\alpha^2}{1+t_0}\frac{t-t_0}{1+t}\Big)\;, \tag{3.3}$$

for $t\ge t_0\ge 0$. This simple estimate shows that the energy of the perturbation $v(x,t)$ grows at most polynomially in time as $t\to\infty$. Such a conclusion is rather pessimistic, however, because by a relatively simple modification of the previous argument it is possible to establish a logarithmic bound, which is clearly superior for large times.

Proposition 8. *There exists a constant $K_1>0$ such that, for any circulation $\alpha\in\mathbb{R}$ and any $v_0\in L^2_\sigma(\Omega)$, the solution of (3.1) with initial data v_0 satisfies*

$$\|v(t)\|^2_{L^2} + \int_0^t \|\nabla v(s)\|^2_{L^2}\,\mathrm{d}s \;\le\; 2^{c|\alpha|}K_1\Big(\|v_0\|^2_{L^2} + \alpha^2\log(1+t)\Big)\;, \tag{3.4}$$

for all $t\ge 0$, where $c=2b_\infty>0$.

Proof. If $t\le 1$ then (3.4) follows from (3.3) with $t_0=0$, hence we can assume that $t\ge 1$. Given any $\tau\ge 0$, we denote

$$\tilde v(x,t) \;=\; u(x,t) - \alpha u^\chi(x,t+\tau) \;=\; v(x,t) + \alpha\Big(u^\chi(x,t) - u^\chi(x,t+\tau)\Big)\;, \tag{3.5}$$

for all $x\in\Omega$ and all $t\ge 0$. Then $\tilde v$ satisfies (3.1) where $u^\chi(x,t)$ and $R^\chi(x,t)$ are replaced by $u^\chi(x,t+\tau)$ and $R^\chi(x,t+\tau)$, respectively. Proceeding exactly as above, we thus obtain the energy estimate

$$\|\tilde v(t)\|^2_{L^2} + \int_0^t \|\nabla\tilde v(s)\|^2_{L^2}\,\mathrm{d}s \;\le\; \Big(\frac{1+t+\tau}{1+\tau}\Big)^{c|\alpha|}\Big(\|\tilde v(0)\|^2_{L^2} + C\alpha^2\Big)\;, \qquad t\ge 0\;. \tag{3.6}$$

Now, we fix $t\ge 1$ and choose $\tau = t-1$. From (6.3), (3.5), we have

$$\|v(t)\|^2_{L^2} \;\le\; 2\|\tilde v(t)\|^2_{L^2} + 2\alpha^2\|u^\chi(t) - u^\chi(2t-1)\|^2_{L^2} \;\le\; 2\|\tilde v(t)\|^2_{L^2} + 2\kappa_1\alpha^2\log 2\;.$$

Similarly, using (6.4), we find

$$\begin{aligned}\int_0^t \|\nabla v(s)\|^2_{L^2}\,\mathrm{d}s \;&\le\; 2\int_0^t \|\nabla\tilde v(s)\|^2_{L^2}\,\mathrm{d}s + 2\alpha^2\int_0^t \|\nabla u^\chi(s) - \nabla u^\chi(s+t-1)\|^2_{L^2}\,\mathrm{d}s\\ &\le\; 2\int_0^t \|\nabla\tilde v(s)\|^2_{L^2}\,\mathrm{d}s + 2\kappa_2\alpha^2\,\log\frac{1+t}{2}\;.\end{aligned}$$

Thus it follows from (3.6) (with $\tau=t-1$) that

$$\|v(t)\|^2_{L^2} + \int_0^t \|\nabla v(s)\|^2_{L^2}\,\mathrm{d}s \;\le\; 2^{c|\alpha|+1}\Big(\|\tilde v(0)\|^2_{L^2} + C\alpha^2\Big) + \kappa\alpha^2\log(1+t)\;, \tag{3.7}$$

for all $t>0$, where $\kappa = 2\max(\kappa_1,\kappa_2)$. Finally, we have by (6.3)

$$\|\tilde v(0)\|^2_{L^2} \;\le\; 2\|v_0\|^2_{L^2} + 2\alpha^2\|u^\chi(0) - u^\chi(t-1)\|^2_{L^2} \;\le\; 2\|v_0\|^2_{L^2} + 2\kappa_1\alpha^2\log t\;,$$

hence (3.4) easily follows from (3.7). □

Remark 9. The logarithmic energy estimate (3.4) is the main new ingredient in the proof of Theorem 5. To a certain extent, we use it as a substitute for the energy equality (1.2), which does not make sense for the solutions we consider. As is clear from the proof, the logarithmic energy estimate relies on the fact that Oseen's vortex (2.1) has "nearly finite energy", in the sense that the integral defining $\|\Theta(\cdot,t)\|_{L^2}^2$ diverges only logarithmically at infinity.

4. Fractional interpolation

Let P be the Leray-Hopf projection in Ω, and $A = -P\Delta$ be the Stokes operator, see e.g. [6]. We recall that A is self-adjoint and nonnegative in $L^2_\sigma(\Omega)$, so that the fractional power A^β can be defined for all $\beta > 0$. The following result shows that the range of A^μ contains the (dense) subspace $L^2_\sigma(\Omega) \cap L^q(\Omega)^2$.

Lemma 10. [4, 21] *Let $q \in (1,2)$ and $\mu = \frac{1}{q} - \frac{1}{2}$. For all $v \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$, there exists a unique $w \in D(A^\mu) \subset L^2_\sigma(\Omega)$ such that $v = A^\mu w$. Moreover, there exists a constant $C > 0$ (independent of v) such that $\|w\|_{L^2(\Omega)} \le C\|v\|_{L^q(\Omega)}$.*

Remark 11. If v, w are as in the above statement, we denote $w = A^{-\mu}v$. Roughly speaking, the proof of Lemma 10 argues as follows. By classical Sobolev embedding, we know that the domain of A^μ is contained in $L^2_\sigma(\Omega) \cap L^{q'}(\Omega)^2$, where $\frac{1}{q'} = \frac{1}{2} - \mu = 1 - \frac{1}{q}$, and by duality we deduce that the range of $A^{-\mu}$ contains $L^2_\sigma(\Omega) \cap L^q(\Omega)^2$.

We go back to the study of the perturbation equation (3.1), which can be written in the equivalent form

$$\partial_t v + Av + \alpha P\Big((u^\chi \cdot \nabla)v + (v \cdot \nabla)u^\chi\Big) + P(v \cdot \nabla)v \;=\; \alpha R^\chi \;. \tag{4.1}$$

So far we only considered solutions in the energy space $L^2_\sigma(\Omega)$, but now we assume in addition that $v_0 \in L^q(\Omega)^2$, for some fixed $q \in (1,2)$, and we denote $\mu = \frac{1}{q} - \frac{1}{2} \in (0, \frac{1}{2})$. Then it is not difficult to verify that the solution $v(t)$ of (4.1) lies in $L^2_\sigma(\Omega) \cap L^q(\Omega)^2$ for all $t \ge 0$. In particular, invoking Lemma 10, we can define $w(t) = A^{-\mu}v(t)$ for all $t \ge 0$. This quantity solves the modified equation

$$\partial_t w + Aw + \alpha F_\mu(u^\chi, v) + \alpha F_\mu(v, u^\chi) + F_\mu(v,v) \;=\; \alpha A^{-\mu} R^\chi \;, \tag{4.2}$$

where $F_\mu(u,v)$ is the bilinear term formally defined by

$$F_\mu(u,v) \;=\; A^{-\mu}P(u \cdot \nabla)v \;. \tag{4.3}$$

We refer to [21, Section 2] for a rigorous definition and a list of properties of the bilinear map F_μ. Our goal here is to establish the following estimate:

Proposition 12. *There exists $K_3 > 0$ and, for all $\alpha \in \mathbb{R}$, there exist positive constants $K_2(\alpha)$ and $k(\alpha)$ such that, if v is any solution of (4.1) with initial data $v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$, the function $w(t) = A^{-\mu}v(t)$ satisfies*

$$\|w(t)\|_{L^2}^2 + \int_0^t \|\nabla w(s)\|_{L^2}^2 \,\mathrm{d}s \;\le\; (1+t)^{\alpha^2 k(\alpha)} \exp\Big(K_2(\alpha)\|v_0\|_{L^2}^2 + K_3\Big)(\|v_0\|_{L^q}^2 + \alpha^2) \;, \tag{4.4}$$

for all $t \ge 0$. Moreover $K_2(\alpha)$ and $k(\alpha)$ are $\mathcal{O}(1)$ as $\alpha \to 0$.

Proof. Taking the scalar product of both sides of (4.2) by w, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_{L^2}^2 + \|A^{1/2}w(t)\|_{L^2}^2 + \alpha\langle F_\mu(u^\chi(t), v(t)), w(t)\rangle + \alpha\langle F_\mu(v(t), u^\chi(t)), w(t)\rangle \\ + \langle F_\mu(v(t), v(t)), w(t)\rangle \;=\; \alpha\langle A^{-\mu}R^\chi(t), w(t)\rangle \;.\end{aligned} \tag{4.5}$$

It is well known that $\|A^{1/2}w\|_{L^2} = \|\nabla w\|_{L^2}$ for all $w \in D(A^{1/2}) = L^2_\sigma(\Omega) \cap H^1_0(\Omega)^2$. To bound the other terms, we observe that

$$\begin{aligned}|\langle F_\mu(u^\chi, v), w\rangle| &= |\langle (u^\chi \cdot \nabla)v, A^{-\mu}w\rangle| = |\langle (u^\chi \cdot \nabla)A^{-\mu}w, v\rangle| \\ &\le \|u^\chi\|_{L^\infty}\|A^{\frac12-\mu}w\|_{L^2}\|v\|_{L^2} = \|u^\chi\|_{L^\infty}\|A^{\frac12-\mu}w\|_{L^2}\|A^\mu w\|_{L^2} \\ &\le \|u^\chi\|_{L^\infty}\|A^{1/2}w\|_{L^2}\|w\|_{L^2} \ ,\end{aligned}$$

where in the last inequality we used the interpolation inequality for fractional powers of A. The same argument shows that $|\langle F_\mu(v, u^\chi), w\rangle| \le \|u^\chi\|_{L^\infty}\|A^{1/2}w\|_{L^2}\|w\|_{L^2}$. In a similar way,

$$\begin{aligned}|\langle F_\mu(v, v), w\rangle| &= |\langle (v \cdot \nabla)v, A^{-\mu}w\rangle| = |\langle (v \cdot \nabla)A^{-\mu}w, v\rangle| \\ &\le \|v\|^2_{L^4}\|A^{\frac12-\mu}w\|_{L^2} \le C\|\nabla v\|_{L^2}\|v\|_{L^2}\|A^{\frac12-\mu}w\|_{L^2} \\ &\le C\|\nabla v\|_{L^2}\|A^{1/2}w\|_{L^2}\|w\|_{L^2} \ .\end{aligned}$$

Finally, since $|\langle A^{-\mu}R^\chi, w\rangle| = |\langle R^\chi, A^{-\mu}w\rangle| \le \kappa_3(1+t)^{-1}\|A^{\frac12-\mu}w\|_{L^2}$ by (6.7), we can use interpolation and Young's inequality to obtain

$$|\alpha\langle A^{-\mu}R^\chi, w\rangle| \le \frac{\kappa_3|\alpha|}{1+t}\|A^{1/2}w\|^{1-2\mu}_{L^2}\|w\|^{2\mu}_{L^2} \le \frac14\|A^{1/2}w\|^2_{L^2} + \frac{\|w\|^2_{L^2}}{(1+t)^{\gamma_1}} + \frac{C\alpha^2}{(1+t)^{\gamma_2}} \ ,$$

for some $\gamma_1, \gamma_2 > 1$ satisfying $\gamma_2 + 2\mu\gamma_1 = 2$. Thus (4.5) implies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2_{L^2} + \|\nabla w\|^2_{L^2} &\le -\|\nabla w\|^2_{L^2} + C\|\nabla w\|_{L^2}\|w\|_{L^2}(|\alpha|\|u^\chi\|_{L^\infty} + \|\nabla v\|_{L^2}) \\ &\quad + \frac12\|\nabla w\|^2_{L^2} + \frac{2\|w\|^2_{L^2}}{(1+t)^{\gamma_1}} + \frac{2C\alpha^2}{(1+t)^{\gamma_2}} \\ &\le C_1\|w\|^2_{L^2}\Big(\alpha^2\|u^\chi\|^2_{L^\infty} + \|\nabla v\|^2_{L^2} + \frac{1}{(1+t)^{\gamma_1}}\Big) + \frac{C_2\alpha^2}{(1+t)^{\gamma_2}} \ ,\end{aligned} \tag{4.6}$$

for some positive constants C_1, C_2.

Now, using (6.1) with $p = \infty$ and the logarithmic energy estimate (3.4), we obtain

$$\begin{aligned}C_1\int_0^t\Big(\alpha^2\|u^\chi(s)\|^2_{L^\infty} + \|\nabla v(s)\|^2_{L^2} + \frac{1}{(1+s)^{\gamma_1}}\Big)\,\mathrm{d}s \\ \le \alpha^2 k(\alpha)\log(1+t) + K_2(\alpha)\|v_0\|^2_{L^2} + C_3 \ ,\end{aligned}$$

where $K_2(\alpha) = 2^{c|\alpha|}C_1K_1$, $k(\alpha) = C_1a^2_\infty + K_2(\alpha)$, and $C_3 = C_1(\gamma_1 - 1)^{-1}$. Applying Gronwall's lemma to (4.6), we thus find

$$\|w(t)\|^2_{L^2} + \int_0^t \|\nabla w(s)\|^2_{L^2}\,\mathrm{d}s \le (1+t)^{\alpha^2k(\alpha)}\exp\Big(K_2(\alpha)\|v_0\|^2_{L^2} + C_3\Big)(\|w_0\|^2_{L^2} + C_4\alpha^2) \ ,$$

where $C_4 = C_2(\gamma_2 - 1)^{-1}$, and (4.4) follows since $\|w_0\|_{L^2} \le C\|v_0\|_{L^q}$ by Lemma 10. □

Corollary 13. *Under the assumptions of Proposition 12, there exists a positive constant K_4 depending on $|\alpha|$ and $\|v_0\|_{L^2\cap L^q}$ such that, for any $t \ge 2$, there exists a time $t_0 \in [t/2, t]$ for which*

$$\|v(t_0)\|^2_{L^2} \le K_4(1+t_0)^{\alpha^2k(\alpha)-2\mu} \ . \tag{4.7}$$

Proof. Fix $t \ge 2$. In view of (4.4), there exists a time $t_0 \in [t/2, t]$ such that

$$\|\nabla w(t_0)\|^2_{L^2} \le \frac2t\int_{t/2}^t \|\nabla w(s)\|^2_{L^2}\,\mathrm{d}s \le \frac2t K_0(1+t)^{\alpha^2k(\alpha)} \le 2^{\alpha^2k(\alpha)+2}K_0(1+t_0)^{\alpha^2k(\alpha)-1} \ ,$$

where $K_0 = \exp(K_2\|v_0\|_{L^2}^2 + K_3)(\|v_0\|_{L^q}^2 + \alpha^2)$. Moreover, $\|w(t_0)\|_{L^2}^2 \le K_0(1+t_0)^{\alpha^2 k(\alpha)}$ by (4.4). Thus, we obtain (4.7) using the interpolation inequality $\|v(t_0)\|_{L^2} = \|A^\mu w(t_0)\|_{L^2} \le \|\nabla w(t_0)\|_{L^2}^{2\mu}\, \|w(t_0)\|_{L^2}^{1-2\mu}$. □

We are now able to conclude our sketch of the proof of Theorem 5. Given $\alpha \in \mathbb{R}$ and $v_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$, let $v(x,t)$ be the solution of the perturbation equation (4.1). For any $t \ge 2$, we choose $t_0 \in [t/2, t]$ as in Corollary 13, and we apply estimate (3.3). We thus obtain

$$\|v(t)\|_{L^2}^2 \;\le\; 2^{2b_\infty|\alpha|}\Big(\|v(t_0)\|_{L^2}^2 + \frac{\kappa_3^2\,\alpha^2}{1+t_0}\Big) \;\le\; C_\alpha(1+t)^{\alpha^2 k(\alpha)-2\mu}\;, \tag{4.8}$$

where $C_\alpha > 0$ is $\mathcal{O}(1)$ as $\alpha \to 0$. Now, if $|\alpha|$ is small enough to that $\alpha^2 k(\alpha) < 2\mu$, the right-hand side of (4.8) converges to zero (at a suboptimal rate) as $t \to \infty$. In particular, the perturbation $v(\cdot,t)$ becomes very small in energy norm for large times. In that regime, the perturbation equation (4.1) can be solved by a global fixed point argument, which allows to show that

$$\|v(\cdot,t)\|_{L^2(\Omega)} + t^{1/2}\|\nabla v(\cdot,t)\|_{L^2(\Omega)} \;=\; \mathcal{O}(t^{-\mu})\;, \tag{4.9}$$

as $t \to \infty$, see [12, Section 3]. Finally (2.7) follows from (4.9), because $v(x,t) = u(x,t) - \alpha u^\chi(x,t)$ and $\|u^\chi(\cdot,t) - \Theta(\cdot,t)\|_{L^2} + \|\nabla u^\chi(\cdot,t) - \nabla\Theta(\cdot,t)\|_{L^2} \le C(1+t)^{-1}$ for all $t \ge 0$. □

5. Convergence of the vorticity

This final section is devoted to the proof of Proposition 7. We first show that, under the assumptions of Theorem 5, one can control the L^1 norm of the vorticity sufficiently far away from the obstacle K.

Proposition 14. *Under the assumptions of Proposition 7, the vorticity* $\omega = \operatorname{curl} u$ *satisfies*

$$\int_{|x|\ge t^{1/2}\log t} |\omega(x,t)|\,\mathrm{d}x \;=\; \mathcal{O}(t^{-\mu})\;, \qquad \textit{as} \quad t \to \infty\;. \tag{5.1}$$

Proof. Since by (2.8) the initial vorticity is assumed to be square integrable, the solution $u(x,t)$ of (1.1) given by Theorem 5 satisfies

$$\|u(\cdot,t) - \alpha\Theta(\cdot,t)\|_{L^2(\Omega)} + (1+t)^{1/2}\|\nabla u(\cdot,t) - \alpha\nabla\Theta(\cdot,t)\|_{L^2(\Omega)} \;\le\; \frac{C_0}{(1+t)^\mu}\;, \tag{5.2}$$

for all $t \ge 0$, where $C_0 > 0$ depends only on the initial data. The associated vorticity $\omega = \operatorname{curl} u$ is a solution of the advection-diffusion equation

$$\partial_t\omega + u\cdot\nabla\omega \;=\; \Delta\omega\;, \qquad x \in \Omega\;, \quad t > 0\;, \tag{5.3}$$

but the no-slip boundary condition becomes very complicated when expressed in terms of ω. It is thus difficult to use (5.3) to obtain estimates in the whole domain Ω, and in particular near the boundary $\partial\Omega$. Here, however, our goal is to bound ω near infinity, so we can avoid that problem using localized energy estimates and invoking (5.2) to control the flux terms in the regions where the localization function is not constant.

Given $T \ge 4$ and $R \ge 1$, we define the cut-off function

$$\psi(x,t) \;=\; \phi\Big(\frac{|x|}{r(t+T)}\Big)\Big(1 - \phi\Big(\frac{|x|}{R}\Big)\Big)\;, \qquad x \in \mathbb{R}^2\;, \quad t \ge 0\;, \tag{5.4}$$

where $r(t) = 2^{-3/2}t^{1/2}\log(t/2)$, and $\phi : [0,\infty) \to [0,1]$ is a smooth, nondecreasing function satisfying $\phi(r) = 0$ for $r \le 1$ and $\phi(r) = 1$ for $r \ge 2$. We always assume that $T \ge 4$ is large enough so that the support of $\psi(\cdot,t)$ is contained in Ω, and that $R \ge 1$ is large enough (depending on t and T) so that $\psi(\cdot,t)$ is not identically zero. Given $\lambda > 0$, we also denote

$$\Phi_\lambda(\omega) \;=\; (\lambda^2 + \omega^2)^{1/2} - \lambda\;,$$

and we observe that $0 \le \Phi_\lambda(\omega) \le |\omega|$ and $\Phi''_\lambda(\omega) \ge 0$ for all $\omega \in \mathbb{R}$.

Now, using (5.3), we obtain by a direct calculation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \psi\Phi_\lambda(\omega)\,\mathrm{d}x &= \int_\Omega \Big(\psi_t + \Delta\psi + (u\cdot\nabla)\psi\Big)\Phi_\lambda(\omega)\,\mathrm{d}x - \int_\Omega \psi\Phi''_\lambda(\omega)|\nabla\omega|^2\,\mathrm{d}x \\ &\le \int_\Omega \Big(\Delta\psi + (u\cdot\nabla)\psi\Big)\Phi_\lambda(\omega)\,\mathrm{d}x\ ,\end{aligned}$$

because $\psi_t \le 0$ and $\Phi''_\lambda(\omega) \ge 0$. If we integrate this inequality over $t \in [0,T]$, we find

$$\int_\Omega \psi(x,T)\Phi_\lambda(\omega(x,T))\,\mathrm{d}x \le \int_\Omega \psi(x,0)\Phi_\lambda(\omega_0(x))\,\mathrm{d}x + \int_0^T\!\!\int_\Omega Q_{\psi,u}(x,t)\Phi_\lambda(\omega(x,t))\,\mathrm{d}x\,\mathrm{d}t\ ,$$

where $Q_{\psi,u} = |\Delta\psi + (u\cdot\nabla)\psi|$. Using Lebesgue's monotone convergence theorem, we can take the limit $\lambda \to 0$ in both sides, and we arrive at the simpler estimate

$$\int_\Omega \psi(x,T)|\omega(x,T)|\,\mathrm{d}x \le \int_\Omega \psi(x,0)|\omega_0(x)|\,\mathrm{d}x + \int_0^T\!\!\int_\Omega Q_{\psi,u}(x,t)|\omega(x,t)|\,\mathrm{d}x\,\mathrm{d}t\ . \tag{5.5}$$

Our next task is to take the limit $R \to \infty$ in (5.5). Again, we use the monotone convergence theorem, except in the last integral where it does not apply. To treat that term, we observe that $Q_{\psi,u}(\cdot,t)$ vanishes identically except in the region $D_R \cup D_{r(t+T)}$, where for any $\rho > 0$ we denote $D_\rho = \{x \in \mathbb{R}^2 \,|\, \rho \le |x| \le 2\rho\}$. Taking $R > 0$ sufficiently large and using (5.2), we easily obtain

$$\int_{D_R} Q_{\psi,u}|\omega|\,\mathrm{d}x \le \int_{D_R}\Big(|\Delta\psi| + |u||\nabla\psi|\Big)|\omega|\,\mathrm{d}x \le \frac{C_1}{R}\ ,$$

for some $C_1 > 0$ independent of $t \in [0,T]$. The contribution of the annulus D_R is therefore negligible for large R, hence taking the limit $R \to \infty$ in (5.5) we arrive at

$$\int_\Omega \psi_1(x,T)|\omega(x,T)|\,\mathrm{d}x \le \int_\Omega \psi_1(x,0)|\omega_0(x)|\,\mathrm{d}x + \int_0^T\!\!\int_\Omega Q_{\psi_1,u}(x,t)|\omega(x,t)|\,\mathrm{d}x\,\mathrm{d}t\ ,$$

where $\psi_1(x,t) = \phi(|x|/r(t+T))$. In particular $\psi_1(x,T) = 1$ for $|x| \ge T^{1/2}\log T$ and $\psi_1(x,0) = 0$ for $|x| \le r(T)$, hence the last inequality implies

$$\int_{|x|\ge T^{1/2}\log T} |\omega(x,T)|\,\mathrm{d}x \le \int_{|x|\ge r(T)} |\omega_0(x)|\,\mathrm{d}x + \int_0^T\!\!\int_\Omega Q_{\psi_1,u}(x,t)|\omega(x,t)|\,\mathrm{d}x\,\mathrm{d}t\,. \tag{5.6}$$

To conclude the proof of Proposition 14, it remains to estimate both terms in the right-hand side of (5.6). First, using (2.8) and Hölder's inequality, we easily find

$$\begin{aligned}\int_{|x|\ge r(T)} |\omega_0(x)|\,\mathrm{d}x &\le \left(\int_\Omega (1+|x|^2)^m|\omega_0(x)|^2\,\mathrm{d}x\right)^{1/2}\left(\int_{|x|\ge r(T)} \frac{1}{(1+|x|^2)^m}\,\mathrm{d}x\right)^{1/2} \\ &\le Cr(T)^{-(m-1)} \le C_2T^{-\mu}\ ,\end{aligned} \tag{5.7}$$

for some $C_2 > 0$ independent of T. In the last inequality, we used the hypothesis $m \ge 2/q = 1 + 2\mu$ and the fact that $r(T) \ge CT^{1/2}$. On the other hand, since $\psi_1(x,t)$ is given by (5.4) with $R = \infty$, there exists $C_3 > 0$ such that

$$|\nabla\psi_1(x,t)| \le \frac{C_3}{r(t+T)}\mathbf{1}_{D'_t}\ , \qquad |\Delta\psi_1(x,t)| \le \frac{C_3}{r(t+T)^2}\mathbf{1}_{D'_t}\ ,$$

where $D'_t = D_{r(t+T)} = \{x \in \mathbb{R}^2 \,|\, r(t+T) \le |x| \le 2r(t+T)\}$. It follows that

$$\int_\Omega Q_{\psi_1,u}(x,t)|\omega(x,t)|\,\mathrm{d}x \le C_3\int_{D'_t}\left(\frac{1}{r(t+T)^2} + \frac{|u(x,t)|}{r(t+T)}\right)|\omega(x,t)|\,\mathrm{d}x\ . \tag{5.8}$$

But using (5.8) and (5.2) we easily find,

$$\begin{aligned}\int_{D_t'} |\omega(x,t)|\,\mathrm{d}x &\le \int_{D_t'} \Big|\omega(x,t)-\alpha\Xi(x,t)\Big|\,\mathrm{d}x + |\alpha|\int_{D_t'}\Xi(x,t)\,\mathrm{d}x \\ &\le \mathrm{meas}(D_t')^{1/2}\|\omega(\cdot,t)-\alpha\Xi(\cdot,t)\|_{L^2(\Omega)} + |\alpha|\|\Xi(\cdot,t)\|_{L^1(D_t')} \\ &\le C\,\frac{r(t+T)}{(1+t)^{\mu+1/2}} + C\,\exp\Big(-\frac{r(t+T)^2}{4(1+t)}\Big) \le C\,\frac{r(t+T)}{(1+t)^{\mu+1/2}}\ ,\end{aligned}$$

where in the last inequality we used the fact that $r(t+T)\ge C(t+T)^{1/2}\log(t+T)$. In a similar way,

$$\begin{aligned}\int_{D_t'} |u(x,t)||\omega(x,t)|\,\mathrm{d}x &\le \int_{D_t'} |u(x,t)||\omega(x,t)-\alpha\Xi(x,t)|\,\mathrm{d}x + |\alpha|\int_{D_t'}|u(x,t)||\Xi(x,t)|\,\mathrm{d}x \\ &\le \|u\|_{L^2(D_t')}\Big(\|\omega(\cdot,t)-\alpha\Xi(\cdot,t)\|_{L^2(\Omega)} + \|\Xi(\cdot,t)\|_{L^2(D_t')}\Big) \le \frac{C}{(1+t)^{\mu+1/2}}\ .\end{aligned}$$

Inserting these estimates in the right-hand side of (5.8), we obtain

$$\int_0^T\!\!\int_\Omega Q_{\psi_1,u}(x,t)|\omega(x,t)|\,\mathrm{d}x\,\mathrm{d}t \le \int_0^T \frac{C}{r(t+T)\,(1+t)^{\mu+1/2}}\,\mathrm{d}t \le \frac{C_4}{T^\mu}\ , \tag{5.9}$$

for some $C_4>0$ independent of T. Thus, if we combine (5.6), (5.7), and (5.9), we conclude that

$$\int_{|x|\ge T^{1/2}\log T} |\omega(x,T)|\,\mathrm{d}x \le \frac{C_2+C_4}{T^\mu}\ ,$$

for all sufficiently large $T>0$, which is the desired result. □

It is now easy to conclude the proof of Proposition 7. For $t>0$ sufficiently large, we denote $\Omega_t = \{x\in\Omega\,|\,|x|\le t^{1/2}\log t\}$ and we decompose

$$\int_\Omega |\omega(x,t)-\alpha\Xi(x,t)|\,\mathrm{d}x \le \int_{\Omega_t} |\omega(x,t)-\alpha\Xi(x,t)|\,\mathrm{d}x + \int_{\Omega\setminus\Omega_t}\Big(|\omega(x,t)|+|\alpha|\Xi(x,t)\Big)\,\mathrm{d}x\ .$$

The last integral in the right-hand side is controlled using Proposition 14 and the explicit expression (2.2) of $\Xi(x,t)$. To estimate the first integral, we use (5.2) and Hölder's inequality:

$$\int_{\Omega_t} |\omega(x,t)-\alpha\Xi(x,t)|\,\mathrm{d}x \le \sqrt{\pi}\,t^{1/2}\log t\,\|\omega(\cdot,t)-\alpha\Xi(\cdot,t)\|_{L^2(\Omega)} \le C\frac{\log t}{(1+t)^\mu}\ .$$

Summarizing, we find

$$\int_\Omega |\omega(x,t)-\alpha\Xi(x,t)|\,\mathrm{d}x \le C\frac{\log t}{(1+t)^\mu}\ ,$$

for all sufficiently large $t>0$. This concludes the proof. □

6. Appendix: estimates for truncated Oseen vortices

In this appendix, we collect a few estimates on the truncated Oseen vortices (2.3) which are used throughout the paper. We first list a few bounds which follow from (2.3) and (2.4) by rather straightforward calculations, see [12, Lemma 2.1].

Lemma 15.
1. *For any $p\in(2,\infty]$, there exists a constant $a_p>0$ such that*

$$\|u^\chi(\cdot,t)\|_{L^p(\mathbb{R}^2)} \le \frac{a_p}{(1+t)^{\frac12-\frac1p}}\ ,\qquad t\ge 0\ . \tag{6.1}$$

2. *For any $p \in (1,\infty]$, there exists a constant $b_p > 0$ such that*

$$\|\nabla u^\chi(\cdot,t)\|_{L^p(\mathbb{R}^2)} \le \frac{b_p}{(1+t)^{1-\frac{1}{p}}} , \qquad t \ge 0 . \tag{6.2}$$

3. *There exists a constant $\kappa_1 > 0$ such that, for all $t, s \ge 0$,*

$$\|u^\chi(\cdot,t) - u^\chi(\cdot,s)\|^2_{L^2(\mathbb{R}^2)} \le \kappa_1 \left| \log \frac{1+t}{1+s} \right| . \tag{6.3}$$

4. *There exists a constant $\kappa_2 > 0$ such that, for all $t, s \ge 0$,*

$$\|\nabla u^\chi(\cdot,t) - \nabla u^\chi(\cdot,s)\|^2_{L^2(\mathbb{R}^2)} \le \kappa_2 \left| \frac{1}{1+t} - \frac{1}{1+s} \right| . \tag{6.4}$$

Since the truncated Oseen vortex is not a solution of the Navier-Stokes equation, we also need a control on the remainder term $R^\chi = \Delta u^\chi - \partial_t u^\chi = (\Delta\chi)\Theta + 2(\nabla\chi\cdot\nabla)\Theta$, which has the explicit expression

$$R^\chi(x,t) = \Theta(x,t)\Delta\chi(x) + 2\frac{x\cdot\nabla\chi(x)}{|x|^2}\Big(x^\perp \Xi(x,t) - \Theta(x,t)\Big) . \tag{6.5}$$

Lemma 16. *There exists a constant $\kappa_3 > 0$ such that, for any $p \in [1,\infty]$,*

$$\|R^\chi(\cdot,t)\|_{L^p(\mathbb{R}^2)} \le \frac{\kappa_3}{1+t} , \qquad t \ge 0 . \tag{6.6}$$

Moreover, for any vector field $u \in H^1_{\mathrm{loc}}(\mathbb{R}^2)^2$, we have

$$\Big| \int_{\mathbb{R}^2} R^\chi(x,t)\cdot u(x)\,\mathrm{d}x \Big| \le \frac{\kappa_3}{1+t}\,\|\nabla u\|_{L^2(D)} , \qquad t \ge 0 , \tag{6.7}$$

where $D \subset \Omega$ is a compact annulus containing the support of $\nabla\chi$.

Proof. It is clear from (6.5) that $|R^\chi(x,t)| \le C(1+t)^{-1}\mathbf{1}_D(x)$ for all $x \in \mathbb{R}^2$ and all $t \ge 0$, and (6.6) follows immediately. Moreover, we have $R^\chi(x,t) = x^\perp Q^\chi(x,t)$ for some radially symmetric scalar function $Q(x,t)$, hence $R^\chi(\cdot,t)$ has zero mean over the annulus D. If $u \in H^1_{\mathrm{loc}}(\mathbb{R}^2)^2$ and if we denote by $\bar{u}$ the average of u over D, the Poincaré-Wirtinger inequality implies

$$\Big| \int_{\mathbb{R}^2} R^\chi(x,t)\cdot u(x)\,\mathrm{d}x \Big| = \Big| \int_D R^\chi(x,t)\cdot (u(x)-\bar{u})\,\mathrm{d}x \Big| \le C\|R^\chi(\cdot,t)\|_{L^2(\mathbb{R}^2)}\|\nabla u\|_{L^2(D)} ,$$

and using (6.6) with $p = 2$ we obtain (6.7). □

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