Asymptotic behavior for a viscous Hamilton-Jacobi equation with critical exponent

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Abstract

The large time behavior of non-negative solutions to the viscous Hamilton-Jacobi equation $\partial_t u - \Delta u + |\nabla u|^q = 0$ in $(0,\infty) \times \mathbb{R}^N$ is investigated for the critical exponent $q = (N+2)/(N+1)$. Convergence towards a rescaled self-similar solution to the linear heat equation is shown, the rescaling factor being $(\ln t)^{-(N+1)}$. The proof relies on the construction of a one-dimensional invariant manifold for a suitable truncation of the equation written in self-similar variables.

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1 Introduction

The dynamics of integrable non-negative solutions to the viscous Hamilton-Jacobi equation

$$\partial_t u - \Delta u + |\nabla u|^q \;=\; 0\,, \quad (t,x) \in (0,\infty) \times \mathbb{R}^N\,, \tag{1}$$

depends strongly on the value of the parameter $q \in (0,\infty)$ and results from the competition between the linear diffusion term Δu and the nonlinear absorption term $|\nabla u|^q$. An important issue is therefore to determine which mechanism (diffusion or absorption) is dominant for large times. A first indication is given by the behavior of the L^1 norm $\|u(t)\|_{L^1}$, which is time-independent for non-negative solutions of the heat equation and strictly decreasing for nontrivial non-negative solutions of (1). For such solutions, it is proved in [1, 5, 11] that

$$I_\infty \;:=\; \lim_{t\to\infty} \|u(t)\|_{L^1} \begin{cases} > 0 & \text{if} \quad q > q_\star := \dfrac{N+2}{N+1}\,, \\ \\ = 0 & \text{if} \quad q \in (0, q_\star]\,. \end{cases} \tag{2}$$

This suggests that diffusion dominates the large time behavior when $q > q_\star$, whereas absorption becomes effective for $q \le q_\star$. As a matter of fact, if $q > q_\star$ it is shown in [3, 13] that the nonlinear term $|\nabla u|^q$ becomes negligible for large times, and that the solution of (1) behaves as $t \to \infty$ like the self-similar solution $I_\infty\, g$ of the linear heat equation, where

$$g(t,x) \;=\; \frac{1}{t^{N/2}}\, G\left(\frac{x}{t^{1/2}}\right) \quad \text{and} \quad G(\xi) \;=\; \frac{1}{(4\pi)^{N/2}}\, \exp\left(-\frac{|\xi|^2}{4}\right). \tag{3}$$

On the other hand, if $q \in (1, q_\star)$, both diffusion and absorption play a role in the large time asymptotics. Indeed, if $u(0,x)$ decays faster than $|x|^{-\alpha}$ as $|x| \to \infty$ with $\alpha = (2-q)/(q-1) > N$, it is proved in [3] that the solution $u(t)$ converges as $t \to \infty$ to the so-called *very singular solution*, a self-similar solution of (1) whose existence and uniqueness have been established in [4, 6, 22]. In that case, the L^1-norm of $u(t)$ decays to zero like $t^{-(\alpha-N)/2}$ as $t \to \infty$. Finally, the influence of the absorption term $|\nabla u|^q$ is much stronger for $q \in (0,1]$: depending on the initial data, one might have exponential decay of the solution as $t \to \infty$ [1, 9, 10, 18], or even extinction in finite time if $q \in (0,1)$ [7, 8, 18]. For such values of the parameter, it is the diffusion term which is expected to be negligible for large times.

To summarize, precise asymptotic expansions show that the large time behavior of non-negative solutions to (1) with sufficiently localized initial data is determined by the sole diffusion if $q > q_\star$, whereas absorption plays an important role if $q < q_\star$. With this perspective in mind, it is interesting to investigate the critical case $q = q_\star = (N+2)/(N+1)$ where a transition between both regimes is expected to occur. Very few results are available in this situation: we only know that $\|u(t)\|_{L^1} \to 0$ as $t \to \infty$, as already stated in (2), and that $\|u(t)\|_{L^1}$ cannot decay faster than $(\ln t)^{-(N+1)}$ for large times [7, Proposition 3]. The purpose of this work is to fill this gap and to give an accurate description of the large time behavior of the non-negative solutions to

$$\partial_t u - \Delta u + |\nabla u|^{q_\star} = 0\,, \quad (t,x) \in (0,\infty)\times\mathbb{R}^N\,, \tag{4}$$
$$u(0) = u_0\,, \quad x \in \mathbb{R}^N\,, \tag{5}$$

when the initial data $u_0(x)$ decay to zero sufficiently rapidly as $|x|\to\infty$. More precisely, we assume that $u_0 \ge 0$ belongs to the weighted L^2 space

$$L^2_m(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) \,\middle|\, |u|_m := \|(1+|x|^{2m})^{1/2} u\|_{L^2} < \infty \right\}, \tag{6}$$

for some $m > N/2$. Then (by Hölder's inequality) $u_0 \in L^1(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)$ and it follows from [5, 12, 19] that the Cauchy problem (4), (5) has a unique global solution $u \in \mathcal{C}([0,\infty);L^1(\mathbb{R}^N))\cap\mathcal{C}((0,\infty);W^{1,\infty}(\mathbb{R}^N))$. Our main result describes the large time behavior of this solution:

Theorem 1 *Assume that the initial condition u_0 is non-negative, not identically zero, and belongs to $L^2_m(\mathbb{R}^N)$ for some $m > N/2$. Then the (unique) solution u to (4),(5) satisfies, for all $p \in [1,\infty]$,*

$$\lim_{t\to\infty} t^{\frac{N}{2}(1-\frac{1}{p})} (\ln t)^{N+1} \left\| u(t) - \frac{M_\star}{(\ln t)^{N+1}}\, g(t) \right\|_{L^p} = 0\,, \tag{7}$$

where $M_\star = (N+1)^{N+1}\, \|\nabla G\|_{L^{q_\star}}^{-(N+2)}$ and $g(t,x)$, $G(\xi)$ are defined in (3).

In other words, if the initial condition decays sufficiently rapidly at infinity, the solution u to (4) behaves asymptotically like a particular self-similar solution $M_\star\, g$ of the linear heat equation, with an extra logarithmic factor due to the effect of the absorption term. Such a logarithmic correction also appears in other parabolic equations with critical nonlinearity, for instance in the nonlinear diffusion equation $\partial_t u - \Delta u^m + u^{m+(2/N)} = 0$ with $m \ge 1$, see e.g. [15, 16, 21] and the references therein. It is interesting to observe that, in both situations, the leading order term in the long-time asymptotics is completely independent of the initial conditions. In the case of the viscous Hamilton-Jacobi equation (4), Theorem 1 indeed shows that neither the asymptotic profile $g(t,x)$ nor the prefactor $M_\star\,(\ln t)^{-(N+1)}$ depends on u_0. This universality property reflects the important role played here by the nonlinearity: when the large time behavior is driven by the linear part of the system, which is the case for (1) when $q > q_\star$, the leading term in the asymptotics does depend on the initial condition.

Remark 2 *As $\|g(t)\|_{L^1} = 1$ for all $t > 0$, the L^1-norm of the solution $u(t)$ behaves exactly like $M_\star(\ln t)^{-(N+1)}$ for large times under the assumptions of Theorem 1. This has to be compared with [7, Proposition 3], where it is shown that there is no nontrivial non-negative solution of (4) such that $\|u(t)\|_{L^1} \le C(\ln t)^{-\gamma}$ for $\gamma > N+1$. In fact, using Theorem 1 and a comparison argument, it is straightforward to verify that, for all nontrivial non-negative integrable data, the solution of (4) satisfies*

$$\liminf_{t\to\infty}\, (\ln t)^{N+1} \|u(t)\|_{L^1} \ge M_\star\,.$$

Our analysis of the large time behavior of solutions to (4), (5) relies on an alternative formulation of (4) in terms of the so-called "scaling variables" or "similarity variables"

$$\xi \;=\; \frac{x}{(1+t)^{1/2}} \quad \text{and} \quad \tau \;=\; \ln(1+t)\,. \tag{8}$$

Introducing the new unknown function v defined by

$$u(t,x) \;=\; \frac{1}{(1+t)^{N/2}}\, v\left(\ln(1+t), \frac{x}{(1+t)^{1/2}}\right)\,, \quad (t,x)\in[0,\infty)\times\mathbb{R}^N\,, \tag{9}$$

we deduce from (4), (5) that $v(\tau,\xi)$ solves the initial-value problem

$$\partial_\tau v \;=\; \mathcal{L}v - |\nabla v|^{q_\star}\,, \quad (\tau,\xi)\in(0,\infty)\times\mathbb{R}^N\,, \tag{10}$$

$$v(0) \;=\; u_0\,, \quad \xi\in\mathbb{R}^N\,, \tag{11}$$

where the linear operator $\mathcal{L}$ is given by

$$\mathcal{L}v(\xi) \;=\; \Delta v(\xi) + \frac{1}{2}\,\xi\cdot\nabla v(\xi) + \frac{N}{2}\,v(\xi)\,, \quad \xi\in\mathbb{R}^N\,. \tag{12}$$

Observe that equation (10) is still autonomous, although it was obtained from (4) through the time-dependent transformation (9). This crucial property follows from the fact that (4) is invariant under the rescaling $u(t,x)\mapsto\lambda^N u(\lambda^2 t,\lambda x)$, because $q_\star=(N+2)/(N+1)$. Remark also that $\mathcal{L}G=0$, where G is defined in (3).

At this stage, we follow the approach of [17, 23] and prove that the large time behavior of the solutions of (10), (11) is governed, up to exponentially decaying terms, by an ordinary differential equation which results from restricting the dynamics of (10) to a one-dimensional invariant manifold. This manifold is tangent at the origin to the kernel $\mathbb{R}G$ of $\mathcal{L}$, and solutions to (10) which lie on this manifold satisfy $v(\tau,\xi)\approx M(\tau)\,G(\xi)$ for large times. Inserting this ansatz into (10) and integrating over $\mathbb{R}^N$ we obtain the ordinary differential equation $dM/d\tau + \|\nabla G\|_{L^{q_\star}}^{q_\star}\,M^{q_\star} = 0$ for $M(\tau)$, from which we deduce that $M(\tau)\approx M_\star\,\tau^{-(N+1)}$ for large times. Returning to the original variables (t,x), we then conclude that $u(t)\approx M_\star(\ln t)^{-(N+1)}g(t)$ as $t\to\infty$, and Theorem 1 follows.

To construct the center manifold, it is necessary to assume that the solutions we consider decay a little bit faster as $|x|\to\infty$ than what is needed to be integrable. Indeed, using the results of [17, Appendix A], it is easy to see that the spectrum of the linear operator $\mathcal{L}$ in $L^1(\mathbb{R}^N)$ is just the left half-plane $\{z\in\mathbb{C}\,|\,\Re(z)\le 0\}$ (no spectral gap). In contrast, the spectrum of the same operator in $L^2_m(\mathbb{R}^N)=L^2(\mathbb{R}^N;(1+|\xi|^{2m})\,\mathrm{d}\xi)$ is given by

$$\sigma(\mathcal{L};L^2_m(\mathbb{R}^N)) \;=\; \left\{z\in\mathbb{C}\,\middle|\,\Re(z)\le\frac{N}{4}-\frac{m}{2}\right\}\bigcup\left\{-\frac{k}{2}\,\middle|\,k\in\mathbb{N}\right\},$$

see [17, Theorem A.1]. Thus, if $m>N/2$, the operator $\mathcal{L}$ has a simple isolated eigenvalue at the origin and the rest of the spectrum is strictly contained in the interior of the left half-plane, a spectral configuration which allows to construct the center manifold. This explains the choice of the weighted Lebesgue space $L^2_m(\mathbb{R}^N)$ in Theorem 1. In fact, since the nonlinearity in (4) involves the gradient of the solution u, we shall rather use the corresponding Sobolev space $H^1_m(\mathbb{R}^N)$ (defined in (16) below) in the proof.

The rest of this paper is organized as follows. In the next section, we recall existence and uniqueness results for (10), (11) and establish the convergence to zero of the solution $v(\tau)$ in $H^1_m(\mathbb{R}^N)$ as $\tau \to \infty$. In Section 3, we study a suitable truncated version of (10), (11) to which we can apply an abstract result of [14] to construct the invariant manifold. The proof of Theorem 1 is then performed in the final section.

2 Global existence and convergence to zero

In this section we briefly discuss the Cauchy problem for the rescaled equation (10) and we show that the solutions converge to zero as $\tau \to \infty$. We first consider initial data in the Lebesgue space $L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$.

Proposition 3 *Let u_0 be a non-negative function in $L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. Then (10), (11) have a unique non-negative (mild) solution*

$$v \in \mathcal{C}([0,\infty); L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)) \,\cap\, L^\infty_{\mathrm{loc}}((0,\infty); W^{1,\infty}(\mathbb{R}^N))\,,$$

which moreover satisfies

$$\lim_{\tau\to\infty} \{\|v(\tau)\|_{L^1} + \|v(\tau)\|_{L^\infty} + \|\nabla v(\tau)\|_{L^\infty}\} \;=\; 0\,. \tag{13}$$

Proof: For such initial data u_0, the results of [5, 12, 19] imply that the original system (4), (5) has a unique (mild) solution

$$u \in \mathcal{C}([0,\infty); L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)) \,\cap\, \mathcal{C}((0,\infty); W^{1,\infty}(\mathbb{R}^N))\,.$$

For all $t > 0$, the function $u(t,x)$ is $\mathcal{C}^1$ with respect to t, $\mathcal{C}^2$ with respect to x, and (4) is satisfied in the classical sense. In addition, the following bounds hold for all $t > 0$:

$$\|u(t)\|_{L^1} + t^{N/2}\,\|u(t)\|_{L^\infty} + t^{(N+1)/2}\,\|\nabla u(t)\|_{L^\infty} \;\le\; C\,\|u(t/2)\|_{L^1} \;\le\; C\,\|u_0\|_{L^1}\,. \tag{14}$$

Since $\|u(t)\|_{L^1} \to 0$ as $t \to \infty$ by [5, 11], we deduce from (14) that

$$\lim_{t\to\infty} \left\{\|u(t)\|_{L^1} + t^{N/2}\,\|u(t)\|_{L^\infty} + t^{(N+1)/2}\,\|\nabla u(t)\|_{L^\infty}\right\} \;=\; 0\,. \tag{15}$$

The conclusions of Proposition 3 are straightforward consequences of these results, since (10) is obtained from (4) via the simple transformation (9). In particular, $\|v(\tau)\|_{L^1} = \|u(t)\|_{L^1}$, $\|v(\tau)\|_{L^\infty} = (1{+}t)^{N/2}\|u(t)\|_{L^\infty}$, $\|\nabla v(\tau)\|_{L^\infty} = (1{+}t)^{(N+1)/2}\|\nabla u(t)\|_{L^\infty}$, hence (13) follows immediately from (15). Finally, as the transformation (9) involves a time-dependent dilation which is not continuous in $L^\infty(\mathbb{R}^N)$, the fact that $u \in \mathcal{C}((0,\infty); W^{1,\infty}(\mathbb{R}^N))$ only implies that $v \in L^\infty_{\mathrm{loc}}((0,\infty); W^{1,\infty}(\mathbb{R}^N))$. $\square$

We next study the properties of solutions to (10), (11) in the weighted Lebesgue space $L^2_m(\mathbb{R}^N)$ defined in (6), and in the corresponding Sobolev space

$$H^1_m(\mathbb{R}^N) \;=\; \Big\{ v \in H^1(\mathbb{R}^N) \,\Big|\, \|v\|_m := (|v|_m^2 + |\nabla v|_m^2)^{1/2} < \infty \Big\}\,, \tag{16}$$

where

$$|v|_m \;=\; \left(\int_{\mathbb{R}^N} \left(1 + |\xi|^{2m}\right) |v(\xi)|^2 \, \mathrm{d}\xi \right)^{1/2}.$$

Proposition 4 *Let u_0 be a non-negative function in $L^2_m(\mathbb{R}^N)$ for some $m > N/2$. Then the solution v to (10), (11) given by Proposition 3 satisfies*

$$v \in \mathcal{C}([0,\infty); L^2_m(\mathbb{R}^N)) \,\cap\, \mathcal{C}((0,\infty); H^1_m(\mathbb{R}^N))\,,$$

and

$$\lim_{\tau\to\infty} \|v(\tau)\|_m \;=\; 0\,. \tag{17}$$

Proof: The fact that $v \in \mathcal{C}([0,T); L^2_m(\mathbb{R}^N)) \cap \mathcal{C}((0,T); H^1_m(\mathbb{R}^N))$ for some $T > 0$ can be established by a classical fixed point argument, which will be implemented in Section 3 for a truncated version of (10). Here we just obtain differential inequalities for the norms $|v(\tau)|_m$ and $|\nabla v(\tau)|_m$ which imply (in view of the local existence theory) that $T > 0$ can be chosen arbitrarily large and that (17) holds.

We first multiply (10) by v and integrate over $\mathbb{R}^N$. Using the non-negativity of v and integrating by parts, we find

$$\frac{1}{2}\frac{d}{d\tau}\int_{\mathbb{R}^N} v^2 \,\mathrm{d}\xi \;=\; \int_{\mathbb{R}^N} v\,\partial_\tau v \,\mathrm{d}\xi \;\le\; \int_{\mathbb{R}^N} v\mathcal{L}v \,\mathrm{d}\xi \;=\; -\int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}\xi + \frac{N}{4}\int_{\mathbb{R}^N} v^2 \,\mathrm{d}\xi\,. \tag{18}$$

Similarly, multiplying (10) by $|\xi|^{2m}\, v$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{d\tau}\int_{\mathbb{R}^N} |\xi|^{2m}\, v^2 \;\mathrm{d}\xi \;&\le\; \int_{\mathbb{R}^N} |\xi|^{2m}\, v\Big(\Delta v + \frac{1}{2}\,\xi\cdot\nabla v + \frac{N}{2} v\Big)\,\mathrm{d}\xi \\ &= -\int_{\mathbb{R}^N} |\xi|^{2m}\, |\nabla v|^2 \,\mathrm{d}\xi \,-\, 2m\int_{\mathbb{R}^N} |\xi|^{2m-2}\, v\,\xi\cdot\nabla v \,\mathrm{d}\xi \,-\, \frac{2m{-}N}{4}\int_{\mathbb{R}^N} |\xi|^{2m}\, v^2 \,\mathrm{d}\xi\,. \end{aligned} \tag{19}$$

The only difficulty is to bound the integral involving $\xi\cdot\nabla v$. If $m \ge 1$, we have by Young's inequality

$$2m|\xi|^{2m-1} v|\nabla v| \;\le\; \frac{1}{2}|\xi|^{2m}|\nabla v|^2 + 2m^2|\xi|^{2m-2}v^2\,, \quad \text{and} \quad |\xi|^{2m-2} \;\le\; \varepsilon|\xi|^{2m} + C(\varepsilon)\,,$$

where $\varepsilon > 0$ is arbitrary. If $1/2 < m < 1$ (which is possible only if $N = 1$) we find similarly

$$2m|\xi|^{2m-1} v|\nabla v| \;\le\; \frac{1}{2}|\nabla v|^2 + 2m^2|\xi|^{4m-2}v^2\,, \quad \text{and} \quad |\xi|^{4m-2} \;\le\; \varepsilon|\xi|^{2m} + C(\varepsilon)\,.$$

In both cases, summing up (18) and (19), we obtain the inequality

$$\frac{d}{d\tau}\,|v(\tau)|_m^2 + |\nabla v(\tau)|_m^2 + \frac{2m{-}N{-}8m^2\varepsilon}{2}\,|v(\tau)|_m^2 \;\le\; (m+4m^2C(\varepsilon))\,\|v(\tau)\|_{L^2}^2\,. \tag{20}$$

We now choose $\varepsilon>0$ sufficiently small so that $2m-N-8m^2\varepsilon>0$. Since $\|v(\tau)\|_{L^2}\to 0$ by (13), the differential inequality (20) implies that $|v(\tau)|_m\to 0$ as $\tau\to\infty$.

We next control the evolution of ∇v. Multiplying (10) by $-\Delta v$ and integrating by parts, we obtain

$$\frac{1}{2}\,\frac{d}{d\tau}\int_{\mathbb{R}^N}|\nabla v|^2\,\mathrm{d}\xi \;=\; -\int_{\mathbb{R}^N}|\Delta v|^2\,\mathrm{d}\xi + \frac{N{+}2}{4}\int_{\mathbb{R}^N}|\nabla v|^2\,\mathrm{d}\xi + \int_{\mathbb{R}^N}\Delta v\,|\nabla v|^{q_\star}\,\mathrm{d}\xi\,. \tag{21}$$

Similarly,

$$\begin{aligned}\frac{1}{2}\,\frac{d}{d\tau}\int_{\mathbb{R}^N}|\xi|^{2m}\,|\nabla v|^2\,\mathrm{d}\xi \;=\; &-\int_{\mathbb{R}^N}|\xi|^{2m}\,|\Delta v|^2\,\mathrm{d}\xi + \frac{N{+}2-2m}{4}\int_{\mathbb{R}^N}|\xi|^{2m}\,|\nabla v|^2\,\mathrm{d}\xi\\ -2m\int_{\mathbb{R}^N}|\xi|^{2m-2}\,\Delta v\,\xi\cdot\nabla v\,\mathrm{d}\xi &+ \int_{\mathbb{R}^N}\Big(|\xi|^{2m}\,\Delta v + 2m|\xi|^{2m-2}\,\xi\cdot\nabla v\Big)|\nabla v|^{q_\star}\,\mathrm{d}\xi\,.\end{aligned} \tag{22}$$

Using the crude estimate $|\xi|^{2m-1}\le 1+|\xi|^{2m}$, we find

$$2m\int_{\mathbb{R}^N}|\xi|^{2m-1}\,|\Delta v|\,|\nabla v|\,\mathrm{d}\xi \;\le\; \frac{1}{4}\,|\Delta v|_m^2 + C|\nabla v|_m^2\,.$$

Moreover, as $\|\nabla v\|_{L^\infty}$ is uniformly bounded for all $\tau\ge 1$ by (13), we have for such times

$$\begin{aligned}\int_{\mathbb{R}^N}(1{+}|\xi|^{2m})\,|\Delta v|\,|\nabla v|^{q_\star}\,\mathrm{d}\xi \;&\le\; \frac{1}{4}\,|\Delta v|_m^2 + C|\nabla v|_m^2\,,\\ 2m\int_{\mathbb{R}^N}|\xi|^{2m-1}\,|\nabla v|\,|\nabla v|^{q_\star}\,\mathrm{d}\xi \;&\le\; C|\nabla v|_m^2\,.\end{aligned}$$

Thus adding up (21) and (22), we obtain

$$\frac{d}{d\tau}\,|\nabla v(\tau)|_m^2 + |\Delta v(\tau)|_m^2 + |\nabla v(\tau)|_m^2 \;\le\; K\,|\nabla v(\tau)|_m^2\,,\quad \tau\ge 1\,, \tag{23}$$

for some $K>0$. Now, if we combine (20) and (23), we see that $h(\tau):=K|v(\tau)|_m^2+|\nabla v(\tau)|_m^2$ satisfies a differential inequality of the form

$$h'(\tau)+\varepsilon_0\,h(\tau) \;\le\; C\,\|v(\tau)\|_{L^2}^2\,,\quad \tau\ge 1\,,$$

for some positive constants ε_0 and C. Thus $h(\tau)\to 0$ as $\tau\to\infty$, and (17) follows. This concludes the proof of Proposition 4. □

3 Construction of the center manifold

We now proceed along the lines of [17, Section 3] to describe the large time behavior of the non-negative solutions to (10), (11) in the space $L^2_m(\mathbb{R}^N)$. By Proposition 4 these solutions converge to zero in $H^1_m(\mathbb{R}^N)$ as $\tau \to \infty$, hence the large time asymptotics remain unchanged if we truncate the nonlinearity in (10) outside a neighborhood of the origin. This modification will allow us to apply the center manifold theorem as stated in [14].

The goal of the present section is to verify that our problem fits into the general framework considered in [14]. First, we introduce a truncated version of system (10) and we show that it generates a $\mathcal{C}^1$-smooth and globally Lipschitz continuous semiflow $(\varphi_\tau)_{\tau\ge 0}$ in $H^1_m(\mathbb{R}^N)$ (Proposition 5). Using [14, Theorem 1.1], we then prove the existence of a one-dimensional center manifold $W_c \subset H^1_m(\mathbb{R}^N)$, which is tangent at the origin to the kernel of the linear operator $\mathcal{L}$, and which attracts all trajectories of the semiflow $(\varphi_\tau)_{\tau\ge 0}$ as $\tau \to \infty$ (Theorem 9). Thanks to this construction, proving Theorem 1 is reduced to determining the large time behavior of the solutions on the center manifold, a relatively simple task that is postponed to Section 4. The reader who is mainly interested in computing the large time asymptotics may just read the beginning of Sections 3.1 and 3.2, including the statements of Proposition 5 and Theorem 9, and then proceed directly to Section 4.

3.1 The semiflow of a truncated system

Throughout this section, we fix a function $\chi \in \mathcal{C}^\infty([0,\infty))$ such that $0 \le \chi \le 1$, $\chi(r) = 0$ if $r \ge 4$ and $\chi(r) = 1$ if $r \le 1$. For $\varrho > 0$ and $r \ge 0$, we denote $\chi_\varrho(r) = \chi(r/\varrho^2)$. Given $\varrho \in (0,1)$ and $m > N/2$, we consider the initial-value problem

$$\begin{aligned} \partial_\tau v &= \mathcal{L}v - F_\varrho(v)\,, \qquad (\tau,\xi) \in (0,\infty)\times\mathbb{R}^N\,, && (24)\\ v(0) &= v_0 \in H^1_m(\mathbb{R}^N)\,, \quad \xi \in \mathbb{R}^N\,, && (25) \end{aligned}$$

where $\mathcal{L}$ is the linear operator (12) and F_ϱ is the truncated nonlinearity

$$F_\varrho(v) = \chi_\varrho\left(\|v\|_m^2\right)\ |\nabla v|^{q_\star}\,, \quad v \in H^1_m(\mathbb{R}^N)\,. \tag{26}$$

We first establish the well-posedness of (24), (25) and show that this system generates a $\mathcal{C}^1$-smooth semiflow in $H^1_m(\mathbb{R}^N)$.

Proposition 5 *Fix $\varrho \in (0,1)$ and $m > N/2$. For each $v_0 \in H^1_m(\mathbb{R}^N)$, the initial-value problem (24), (25) has a unique global solution $v \in \mathcal{C}([0,\infty); H^1_m(\mathbb{R}^N))$. Moreover, the map $\varphi_\tau : H^1_m(\mathbb{R}^N) \to H^1_m(\mathbb{R}^N)$ defined for $\tau \ge 0$ by $\varphi_\tau(v_0) = v(\tau)$ is globally Lipschitz continuous, uniformly in τ on compact intervals. Finally φ_τ is $\mathcal{C}^1$-smooth for all $\tau \ge 0$, so that the family $(\varphi_\tau)_{\tau\ge 0}$ defines a $\mathcal{C}^1$ semiflow in $H^1_m(\mathbb{R}^N)$.*

Before proving Proposition 5, we recall that the linear operator $\mathcal{L}$ defined by (12) is the generator of a strongly continuous semigroup $\left(e^{\tau\mathcal{L}}\right)_{\tau\ge 0}$ in $L^2_m(\mathbb{R}^N)$, see e.g. [17, Appendix A].

If $m > N/2$, this semigroup is uniformly bounded, i.e. there exists $C_1 > 0$ such that, for all $w \in L^2_m(\mathbb{R}^N)$,

$$\left|e^{\tau\mathcal{L}}w\right|_m \le C_1 \, |w|_m\,, \quad \tau \ge 0\,. \tag{27}$$

More generally, let $b(\xi) = (1+|\xi|^2)^{1/2}$ and assume that $b^m w \in L^p(\mathbb{R}^N)$ for some $p \in [1,2]$. Then $e^{\tau\mathcal{L}}w \in L^2_m(\mathbb{R}^N)$ for all $\tau > 0$, and there exists $C_2 > 0$ such that

$$\left|e^{\tau\mathcal{L}}w\right|_m \le \frac{C_2}{a(\tau)^{\frac{N}{2}(\frac{1}{p}-\frac{1}{2})}} \, \|b^m w\|_{L^p}\,, \quad \tau > 0\,, \tag{28}$$

where $a(\tau) = 1 - e^{-\tau}$, see [17, Proposition A.5] and [17, Proposition A.2]. Similar estimates hold for the spatial derivatives of $e^{\tau\mathcal{L}}w$. For instance, as $\nabla e^{\tau\mathcal{L}} = e^{\tau/2}\, e^{\tau\mathcal{L}}\nabla$, it follows from (27) that $|\nabla e^{\tau\mathcal{L}}w|_m \le C_1\, e^{\tau/2}|\nabla w|_m$ for all $w \in H^1_m(\mathbb{R}^N)$. In addition, if $b^m w \in L^p(\mathbb{R}^N)$ for some $p \in [1,2]$, we have the analog of (28):

$$\left|\nabla e^{\tau\mathcal{L}}w\right|_m \le \frac{C_3}{a(\tau)^{\frac{N}{2}(\frac{1}{p}-\frac{1}{2})+\frac{1}{2}}} \, \|b^m w\|_{L^p}\,, \quad \tau > 0\,. \tag{29}$$

In the rest of this section, we fix $p \in (1,2)$ such that

$$\frac{2(N+1)}{N+3} < p < \frac{2(N+1)}{N+2}\,. \tag{30}$$

Given $T > 0$ and $w \in \mathcal{C}([0,T]; H^1_m(\mathbb{R}^N))$, we define

$$(\mathcal{N}_\varrho w)(\tau) = \int_0^\tau e^{(\tau-s)\mathcal{L}}\, F_\varrho(w(s))\,\mathrm{d}s\,, \quad \tau \in [0,T]\,. \tag{31}$$

Then $\mathcal{N}_\varrho w$ belongs to $\mathcal{C}([0,T]; H^1_m(\mathbb{R}^N))$ and enjoys the following property:

Lemma 6 *There exists a constant $C_4 > 0$ such that, for all $T > 0$, all $\varrho \in (0,1)$, and all $w_1, w_2 \in \mathcal{C}([0,T]; H^1_m(\mathbb{R}^N))$, the following inequality holds:*

$$\|(\mathcal{N}_\varrho w_1 - \mathcal{N}_\varrho w_2)(\tau)\|_m \le C_4\, Z_p(\tau)\, \varrho^{q_\star - 1} \sup_{s\in[0,\tau]} \|(w_1-w_2)(s)\|_m\,, \quad \tau \in [0,T]\,,$$

where

$$Z_p(\tau) = \int_0^\tau \frac{1}{a(s)^{\frac{N}{2}(\frac{1}{p}-\frac{1}{2})}}\Big(1 + \frac{1}{a(s)^{1/2}}\Big)\,\mathrm{d}s\,.$$

Proof: We first observe that our choice of p in (30) guarantees that $\frac{N}{2}(\frac{1}{p}-\frac{1}{2}) < \frac{1}{2}$, so that $Z_p(\tau)$ is well-defined and finite for every $\tau \ge 0$. The following inequality will also be useful: if $f, g \in L^2_m(\mathbb{R}^N)$, then $b^m|f|^{q_\star-1}g \in L^p(\mathbb{R}^N)$ and

$$\left\|b^m\, |f|^{q_\star-1}\, g\right\|_{L^p} \le C_5\, |f|_m^{q_\star-1}\, |g|_m\,. \tag{32}$$

Indeed, by Hölder's inequality,

$$\|b^m |f|^{q_\star -1} g\|_{L^p} \le \|b^m g\|_{L^2} \|f\|_{L^r}^{q_\star -1}, \quad \text{where} \quad r = (q_\star - 1)\,\frac{2p}{2-p}.$$

But $\|b^m g\|_{L^2} \le C|g|_m$, and $1 < r < 2$ by (30), hence $\|f\|_{L^r} \le C(\|f\|_{L^1} + \|f\|_{L^2}) \le C|f|_m$ because $L^2_m(\mathbb{R}^N) \hookrightarrow L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ for $m > N/2$. This proves (32).

Now, let $T > 0$, $\varrho \in (0,1)$, and $w_1, w_2 \in \mathcal{C}([0,T]; H^1_m(\mathbb{R}^N))$. For all $\tau \in [0,T]$ we have by (31), (28)

$$\begin{aligned}
|(\mathcal{N}_\varrho w_1 - \mathcal{N}_\varrho w_2)(\tau)|_m &\le \int_0^\tau \left| e^{(\tau-s)\mathcal{L}} \left(F_\varrho(w_1(s)) - F_\varrho(w_2(s))\right)\right|_m \mathrm{d}s \\
&\le \int_0^\tau \frac{C_2}{a(\tau - s)^{\frac{N}{2}(\frac{1}{p}-\frac{1}{2})}} \,\|b^m \left(F_\varrho(w_1(s)) - F_\varrho(w_2(s))\right)\|_{L^p}\, \mathrm{d}s\,. \qquad (33)
\end{aligned}$$

For a fixed $s \in [0,\tau]$, we can assume for instance that $\|w_1(s)\|_m \ge \|w_2(s)\|_m$. Then

$$\begin{aligned}
\|b^m \left(F_\varrho(w_1(s)) - F_\varrho(w_2(s))\right)\|_{L^p} &\le \left|\chi_\varrho\left(\|w_1(s)\|_m^2\right) - \chi_\varrho\left(\|w_2(s)\|_m^2\right)\right| \, \|b^m\,|\nabla w_2(s)|^{q_\star}\|_{L^p} \\
&+ \ \chi_\varrho\left(\|w_1(s)\|_m^2\right) \|b^m\, \left(|\nabla w_1(s)|^{q_\star} - |\nabla w_2(s)|^{q_\star}\right)\|_{L^p}\,.
\end{aligned}$$

Obviously, the right-hand side vanishes if $\|w_2(s)\|_m \ge 2\varrho$, hence we can suppose that $\|w_2(s)\|_m \le 2\varrho$. To bound the first term, we apply (32) with $f = g = |\nabla w_2|$ and obtain $\|b^m\,|\nabla w_2(s)|^{q_\star}\|_{L^p} \le C_5 |\nabla w_2(s)|_m^{q_\star} \le C\varrho^{q_\star}$. Moreover, if $\|w_1(s)\|_m \le 4\varrho$, we have

$$\left|\chi_\varrho\left(\|w_1(s)\|_m^2\right) - \chi_\varrho\left(\|w_2(s)\|_m^2\right)\right| \le \frac{C}{\varrho^2}\left(\|w_1(s)\|_m^2 - \|w_2(s)\|_m^2\right) \le \frac{C}{\varrho}\|(w_1 - w_2)(s)\|_m\,,$$

and the same estimate holds if $\|w_1(s)\|_m \ge 4\varrho$ because $\|(w_1 - w_2)(s)\|_m \ge 2\varrho$ in that case. Thus

$$\left|\chi_\varrho\left(\|w_1(s)\|_m^2\right) - \chi_\varrho\left(\|w_2(s)\|_m^2\right)\right| \, \|b^m\,|\nabla w_2(s)|^{q_\star}\|_{L^p} \le C\,\varrho^{q_\star -1}\,\|(w_1 - w_2)(s)\|_m\,.$$

On the other hand, using (32) and the inequality $\big|\,|y|^{q_\star} - |z|^{q_\star}\,\big| \le q_\star(|y|^{q_\star -1} + |z|^{q_\star-1})|y - z|$, we obtain

$$\begin{aligned}
&\chi_\varrho\left(\|w_1(s)\|_m^2\right) \|b^m\,\left(|\nabla w_1(s)|^{q_\star} - |\nabla w_2(s)|^{q_\star}\right)\|_{L^p} \\
&\quad \le C\,\chi_\varrho\left(\|w_1(s)\|_m^2\right) \left(|\nabla w_1(s)|_m^{q_\star -1} + |\nabla w_2(s)|_m^{q_\star-1}\right) |\nabla(w_1 - w_2)(s)|_m \\
&\quad \le C\,\chi_\varrho\left(\|w_1(s)\|_m^2\right) \|w_1(s)\|_m^{q_\star-1}\, \|(w_1-w_2)(s)\|_m \\
&\quad \le C\,\varrho^{q_\star-1}\,\|(w_1 - w_2)(s)\|_m\,.
\end{aligned}$$

Therefore $\|b^m(F_\varrho(w_1(s)) - F_\varrho(w_2(s)))\|_{L^p} \le C\,\varrho^{q_\star-1}\|(w_1 - w_2)(s)\|_m$, and inserting this bound into (33) we conclude that

$$|(\mathcal{N}_\varrho w_1 - \mathcal{N}_\varrho w_2)(\tau)|_m \le C\,\varrho^{q_\star -1} \int_0^\tau \frac{1}{a(\tau-s)^{\frac{N}{2}(\frac{1}{p}-\frac{1}{2})}}\, \|(w_1 - w_2)(s)\|_m\, \mathrm{d}s\,. \qquad (34)$$

Finally, using (29) and proceeding in the same way, we also obtain

$$\begin{aligned} |\nabla(\mathcal{N}_\varrho w_1 - \mathcal{N}_\varrho w_2)(\tau)|_m &\le \int_0^\tau \frac{C_3}{a(\tau-s)^{\frac{N}{2}(\frac{1}{p}-\frac{1}{2})+\frac{1}{2}}} \, \|b^m (F_\varrho(w_1(s)) - F_\varrho(w_2(s)))\|_{L^p} \, \mathrm{d}s \\ &\le C\,\varrho^{q_\star-1} \int_0^\tau \frac{1}{a(\tau-s)^{\frac{N}{2}(\frac{1}{p}-\frac{1}{2})+\frac{1}{2}}} \, \|(w_1-w_2)(s)\|_m \, \mathrm{d}s\,. \end{aligned} \tag{35}$$

Lemma 6 is now a immediate consequence of (34) and (35). □

Proof of Proposition 5: Given $v_0 \in H^1_m(\mathbb{R}^N)$, we choose $K > 2C_1\|v_0\|_m$ and $T > 0$ sufficiently small so that

$$C_1\,\|v_0\|_m\, e^{T/2} \le \frac{K}{2}\,, \quad \text{and} \quad C_4\,\varrho^{q_\star-1} Z_p(T) \le \frac{1}{2}\,, \tag{36}$$

where C_1 is as in (27) and C_4, Z_p are defined in Lemma 6. We introduce the set

$$X_{K,T} = \Big\{ w \in \mathcal{C}([0,T]; H^1_m(\mathbb{R}^N)) \;\Big|\; \sup_{\tau\in[0,T]} \|w(\tau)\|_m \le K \Big\}\,,$$

which is a complete metric space for the distance d_T defined by

$$d_T(w_1,w_2) = \sup_{\tau\in[0,T]} \|(w_1-w_2)(\tau)\|_m\,, \quad (w_1,w_2) \in X_{K,T}\times X_{K,T}\,.$$

Using (27) and Lemma 6 it is straightforward to verify that, if $w \in X_{K,T}$, then the function $\mathcal{T}_\varrho w : [0,T] \to H^1_m(\mathbb{R}^N)$ defined by

$$(\mathcal{T}_\varrho w)(\tau) = e^{\tau\mathcal{L}} v_0 - (\mathcal{N}_\varrho w)(\tau)\,, \quad \tau \in [0,T]\,,$$

belongs to $X_{K,T}$, and that the map $w \mapsto \mathcal{T}_\varrho w$ is a strict contraction in $X_{K,T}$. By the Banach fixed point theorem, $\mathcal{T}_\varrho$ has thus a unique fixed point v in $X_{K,T}$. This proves that the Cauchy problem (24), (25) is locally well-posed in $H^1_m(\mathbb{R}^N)$.

Let $T_*(v_0) \in (0,\infty]$ be the maximal existence time for the solution of (24), (25) in $H^1_m(\mathbb{R}^N)$. For all $\tau < T_*(v_0)$, it follows from (27), (34), and (35) (with $w_1 = v$ and $w_2 = 0$) that

$$\|v(\tau)\|_m \le C_1\, e^{\tau/2}\, \|v_0\|_m + C_4\, \varrho^{q_\star-1} \int_0^\tau \frac{\|v(s)\|_m}{a(\tau-s)^{\frac{N}{2}(\frac{1}{p}-\frac{1}{2})+\frac{1}{2}}} \, \mathrm{d}s\,.$$

Using a version of Gronwall's lemma (see e.g. [20, Lemma 7.1.1]), we deduce that $\|v(\tau)\|_m$ cannot blow up in finite time, hence $T_*(v_0) = \infty$. Thus (24) has a unique global solution $v \in \mathcal{C}([0,\infty); H^1_m(\mathbb{R}^N))$ for all $v_0 \in H^1_m(\mathbb{R}^N)$, and we may define a semiflow $(\varphi_\tau)_{\tau\ge0}$ by the relation $\varphi_\tau(v_0) = v(\tau)$ for $\tau \ge 0$.

By construction, the map $v_0 \mapsto \varphi_\tau(v_0)$ is globally Lipschitz continuous, uniformly in time on compact intervals: for each $T > 0$, there exists $L(T) > 0$ such that

$$\|\varphi_\tau(v_0) - \varphi_\tau(\hat{v}_0)\|_m \le L(T)\, \|v_0 - \hat{v}_0\|_m\,, \tag{37}$$

for all $\tau \in [0,T]$ and all $(v_0, \hat{v}_0) \in H^1_m(\mathbb{R}^N) \times H^1_m(\mathbb{R}^N)$. Indeed, by the semigroup property, it is sufficient to prove (37) for a $T > 0$ satisfying (36), in which case (37) follows immediately from the fixed point argument above, with $L(T) = 2C_1\, e^{T/2}$. This proof also shows that $L(T)$ can be chosen independent of ϱ if $\varrho \in (0,1)$. Finally, the fact that the map φ_τ is $\mathcal{C}^1$ for each $\tau \geq 0$ is obtained by classical arguments which we omit here. We only mention that, given $v_0 \in H^1_m(\mathbb{R}^N)$, $\tau \geq 0$, and $h \in H^1_m(\mathbb{R}^N)$, the differential $D\varphi_\tau(v_0)h$ of φ_τ at v_0 applied to h is equal to $V(\tau)$, where V denotes the solution of the linear non-autonomous equation

$$\begin{aligned} \partial_\tau V &= \mathcal{L}V - q_\star \chi_\varrho\left(\|v\|_m^2\right) |\nabla v|^{q_\star - 2}\, \nabla v \cdot \nabla V - 2\chi'_\varrho\left(\|v\|_m^2\right)\ |\nabla v|^{q_\star} \ll v, V \gg_m, \\ V(0) &= h\,. \end{aligned}$$

Here $v(\tau) = \varphi_\tau(v_0)$ and $\ll \cdot,\cdot \gg_m$ denotes the scalar product in $H^1_m(\mathbb{R}^N)$. In particular, since $\varphi_\tau(0) = 0$ for all $\tau \geq 0$, this formula shows that $D\varphi_\tau(0) = e^{\tau\mathcal{L}}$ for each $\tau \geq 0$. □

Remark 7 *It can actually be shown that the differential $D\varphi_\tau : H^1_m(\mathbb{R}^N) \to \mathscr{L}(H^1_m(\mathbb{R}^N))$ is Hölder continuous with exponent $q_\star - 1$ for any $\tau \geq 0$.*

For later use, we also point out the following properties of the time-one map φ_1:

Corollary 8 *The map $\mathcal{R} = \varphi_1 - e^{\mathcal{L}}$ belongs to $\mathcal{C}^1(H^1_m(\mathbb{R}^N); H^1_m(\mathbb{R}^N))$ and satisfies $\mathcal{R}(0) = 0$, $D\mathcal{R}(0) = 0$. Moreover $\mathcal{R}$ is globally Lipschitz continuous and there exists $C_6 > 0$ (independent of ϱ) such that its Lipschitz constant satisfies $\mathrm{Lip}(\mathcal{R}) \leq C_6\, \varrho^{q_\star - 1}$.*

Proof: We know from Proposition 5 that $\mathcal{R}$ is indeed a $\mathcal{C}^1$-map from $H^1_m(\mathbb{R}^N)$ into itself, and it was observed at the end of the proof that $\varphi_1(0) = 0$ and $D\varphi_1(0) = e^{\mathcal{L}}$, hence $\mathcal{R}(0) = 0$ and $D\mathcal{R}(0) = 0$. Now, given $v_0, \hat{v}_0$ in $H^1_m(\mathbb{R}^N)$ we define $v(\tau) = \varphi_\tau(v_0)$ and $\hat{v}(\tau) = \varphi_\tau(\hat{v}_0)$ for $\tau \geq 0$. Using Lemma 6 and estimate (37) we find

$$\begin{aligned} \|\mathcal{R}(v_0) - \mathcal{R}(\hat{v}_0)\|_m &= \|(\mathcal{N}_\varrho v)(1) - (\mathcal{N}_\varrho \hat{v})(1)\|_m \leq C_4\, \varrho^{q_\star - 1} \sup_{s\in[0,1]} \|(v - \hat{v})(s)\|_m \\ &\leq C_4\, L(1)\, \varrho^{q_\star - 1}\, \|v_0 - \hat{v}_0\|_m\,, \end{aligned}$$

which is the desired bound. □

3.2 Existence of the center manifold

Having associated a $\mathcal{C}^1$-semiflow to the truncated system (24), we now turn to the construction of a center manifold for this semiflow at the origin. If $m > N/2$, we can decompose $H^1_m(\mathbb{R}^N) = E_c \oplus E_s$, where $E_c = \{\alpha G \,|\, \alpha \in \mathbb{R}\}$ is the kernel of the operator $\mathcal{L}$ and

$$E_s = \left\{ w \in H^1_m(\mathbb{R}^N) \;\Big|\; \int_{\mathbb{R}^N} w(\xi)\, \mathrm{d}\xi = 0 \right\}. \tag{38}$$

We recall that G is the Gaussian function defined in (3). Let P_0 be the continuous projection onto E_c along E_s, namely

$$P_0 w \;=\; \left(\int_{\mathbb{R}^N} w(\xi)\,\mathrm{d}\xi\right) G\,, \quad w \in H^1_m(\mathbb{R}^N)\,,$$

and let $Q_0 = \mathbf{1} - P_0$. It is easily verified that P_0 and Q_0 commute with $\mathcal{L}$, so that the subspaces E_c and E_s are invariant under the action of $\mathcal{L}$. Moreover, we know from [17, Appendix A] that the spectrum of the restriction of $\mathcal{L}$ to the invariant subspace E_s is strictly contained in the left-half plane in $\mathbb{C}$, because the associated semigroup $e^{\tau\mathcal{L}}$ decreases exponentially in E_s. More precisely, if $\mu_0 \in (0, 1/2)$ satisfies $2\mu_0 < m - (N/2)$, there exists $C_7 > 0$ such that

$$\left|e^{\tau\mathcal{L}} Q_0 w\right|_m + a(\tau)^{1/2} \left|\nabla e^{\tau\mathcal{L}} Q_0 w\right|_m \;\le\; C_7\, e^{-\mu_0 \tau}\, |w|_m\,, \tag{39}$$

for all $w \in L^2_m(\mathbb{R}^N)$ and all $\tau > 0$, see [17, Proposition A.2].

We are now in a position to apply the invariant manifold theorem as stated in [14, Theorem 1.1]. The main result of this section reads:

Theorem 9 *Fix $\mu \in (0, 1/2)$ such that $2\mu < m - (N/2)$. If $\varrho > 0$ is sufficiently small, there exists a globally Lipschitz continuous map $f \in \mathcal{C}^1(E_c; E_s)$ with $f(0) = 0$ and $Df(0) = 0$ such that the submanifold $W_c = \{\alpha G + f(\alpha G) \mid \alpha \in \mathbb{R}\} \subset H^1_m(\mathbb{R}^N)$ enjoys the following properties:*

- **(a)** *$\varphi_\tau(W_c) = W_c$ for every $\tau \ge 0$,*
- **(b)** *for every $v_0 \in H^1_m(\mathbb{R}^N)$ there exist a unique $w_0 \in W_c$ and a positive constant $C_8(v_0)$ such that*

$$\|\varphi_\tau(v_0) - \varphi_\tau(w_0)\|_m \;\le\; C_8(v_0)\, e^{-\mu\tau} \quad \textit{for} \quad \tau \ge 0\,. \tag{40}$$

Proof: Theorem 9 readily follows from [14, Theorem 1.1] once we have checked that the assumptions (**H.1**)–(**H.4**) of [14] are fulfilled in our case. By Proposition 5, $(\varphi_\tau)_{\tau \ge 0}$ is a $\mathcal{C}^1$ semiflow in $H^1_m(\mathbb{R}^N)$ and φ_τ is globally Lipschitz continuous, uniformly for $\tau \in [0, 1]$. Therefore, [14, (**H.1**)] is verified. Next, assumption [14, (**H.2**)] is nothing but the decomposition $\varphi_1 = e^{\mathcal{L}} + \mathcal{R}$ described in Corollary 8. To check [14, (**H.3**)], we remark that $P_0\, e^{\mathcal{L}}\, P_0 = \mathbf{1}$, hence

$$\left\|\left(P_0\, e^{\mathcal{L}}\, P_0\right)^{-k} P_0\right\|_{\mathscr{L}(E_c)} \;=\; \|P_0\|_{\mathscr{L}(E_c)}\,, \quad \text{for all } k \in \mathbb{N}\,.$$

On the other hand, if we choose $\mu_0 \in (\mu, 1/2)$ such that $2\mu_0 < m - (N/2)$, it follows from (39) that $\|e^{k\mathcal{L}} Q_0 w\|_m \le C\, e^{-k\mu_0}\, \|w\|_m$ for all $k \in \mathbb{N}$. Since Q_0 and $e^{\mathcal{L}}$ commute, this inequality is equivalent to

$$\left\|\left(Q_0\, e^{\mathcal{L}}\, Q_0\right)^{k} Q_0\right\|_{\mathscr{L}(E_s)} \;\le\; C\, e^{-k\mu_0}\,, \quad \text{for all } k \in \mathbb{N}\,.$$

As $e^{-\mu_0} < 1$, we have thus checked that [14, (**H.3**)] is fulfilled. Finally [14, (**H.4**)] is automatically satisfied if the Lipschitz constant of $\mathcal{R}$ is sufficiently small. By Corollary 8, this is easily achieved by choosing ϱ appropriately small.

Therefore, by [14, Theorem 1.1], there exist $\mu_1 \in (0, \mu_0)$ and a globally Lipschitz continuous map $f \in \mathcal{C}^1(E_c; E_s)$ such that the submanifold

$$W_c \;=\; \{\alpha G + f(\alpha G) \,|\, \alpha \in \mathbb{R}\} \subset H^1_m(\mathbb{R}^N)$$

enjoys the following properties:

Invariance: $\varphi_\tau(W_c) = W_c$ for all $\tau \geq 0$, and the restriction to W_c of the semiflow $(\varphi_\tau)_{\tau\geq 0}$ can be extended to a Lipschitz continuous flow on W_c.

Invariant foliation: There is a continuous map $h : H^1_m(\mathbb{R}^N) \times E_s \to E_c$ such that, for each $v \in W_c$, one has $h(v, Q_0 v) = P_0 v$ and the manifold

$$\mathcal{M}_v \;=\; \{h(v, w) + w \,|\, w \in E_s\} \subset H^1_m(\mathbb{R}^N)$$

passing through v satisfies $\varphi_\tau(\mathcal{M}_v) \subset \mathcal{M}_{\varphi_\tau(v)}$ for $\tau \geq 0$ and is characterized by

$$\mathcal{M}_v \;=\; \Big\{ w \in H^1_m(\mathbb{R}^N) \;\Big|\; \limsup_{\tau\to\infty} \frac{1}{\tau} \ln\left(\|\varphi_\tau(w) - \varphi_\tau(v)\|_m\right) \leq -\mu_1 \Big\}\,.$$

Completeness: For every $v \in W_c$, $\mathcal{M}_v \cap W_c = \{v\}$. In particular, $\mathcal{M}_v \cap \mathcal{M}_w = \emptyset$ if $(v, w) \in W_c \times W_c$ and $v \neq w$, and $H^1_m(\mathbb{R}^N) = \cup_{v\in W_c}\mathcal{M}_v$.

Moreover, we can assume that $\mu_1 \in (\mu, \mu_0)$ if $\varrho > 0$ is sufficiently small.

We can now conclude the proof of Theorem 9. Assertion **(a)** is nothing but the invariance property of W_c. To prove **(b)**, let $v_0 \in H^1_m(\mathbb{R}^N)$. By the completeness property of W_c, there is a unique $w_0 \in W_c$ such that $v_0 \in \mathcal{M}_{w_0}$. Since $\mu < \mu_1$, we deduce from the invariant foliation property of W_c that there is $\tau_0 > 0$ such that

$$\|\varphi_\tau(v_0) - \varphi_\tau(w_0)\|_m \;\leq\; e^{-\mu\tau}\,, \quad \text{for all } \tau \geq \tau_0\,.$$

Using in addition (37), we obtain (40). $\square$

4 Large time behavior

This final section is entirely devoted to the proof of Theorem 1. Assume that u_0 is a non-negative function in $L^2_m(\mathbb{R}^N)$, $m > N/2$, such that $\|u_0\|_{L^1} > 0$. Let $u(t, x)$ be the corresponding solution of (4), (5) and $v(\tau, \xi)$ the corresponding solution of (10), (11). By the strong maximum principle [19, Corollary 4.2], we know that $u(t, x) > 0$ for all $t > 0$ and all $x \in \mathbb{R}^N$. Choose $\mu \in (0, 1/2)$ such that $2\mu < m - (N/2)$ and $\varrho \in (0, 1)$ sufficiently small so that Theorem 9 applies.

By Proposition 4, the solution v of (10) converges to zero in $H^1_m(\mathbb{R}^N)$ as $\tau \to \infty$, hence there exists $\tau_0 \geq 0$ such that $\|v(\tau)\|_m \leq \varrho$ for all $\tau \geq \tau_0$. Setting $v_0 = v(\tau_0)$ and $\hat{v}(\tau) = v(\tau+\tau_0)$, we obtain a solution $\hat{v}(\tau)$ of (10) with initial condition $v_0 \in H^1_m(\mathbb{R}^N)$ which satisfies

$$\|\hat{v}(\tau)\|_m \;\leq\; \varrho \quad \text{for all } \tau \geq 0\,. \tag{41}$$

Using the notations of Section 3, it follows that $\hat{v}(\tau) = \varphi_\tau(v_0)$ for $\tau \geq 0$, because (41) implies that $\chi_\varrho\left(\|\hat{v}(\tau)\|_m^2\right) = 1$. Thus, in view of Theorem 9, there exist $w_0 \in W_c$ and $C_9 > 0$ such that

$$\|\hat{v}(\tau) - \varphi_\tau(w_0)\|_m \leq C_9\, e^{-\mu\tau}, \quad \tau \geq 0. \tag{42}$$

To simplify the notations, we set $w(\tau) = \varphi_\tau(w_0)$ and

$$M(\tau) = \int_{\mathbb{R}^N} w(\tau,\xi)\,\mathrm{d}\xi, \quad \tau \geq 0.$$

We claim that

$$M(\tau) > 0 \quad \text{for all } \tau \geq 0, \quad \text{and} \quad \lim_{\tau\to\infty} M(\tau) = 0. \tag{43}$$

Indeed, since $H_m^1(\mathbb{R}^N) \hookrightarrow L^1(\mathbb{R}^N)$, it follows from (42) that

$$\left|\int_{\mathbb{R}^N} \hat{v}(\tau,\xi)\,\mathrm{d}\xi - M(\tau)\right| \leq C\,\|\hat{v}(\tau) - w(\tau)\|_m \leq C_{10}\, e^{-\mu\tau}, \tag{44}$$

for all $\tau \geq 0$. Assume by contradiction that there exists $\tau_1 \geq 0$ such that $M(\tau_1) \leq 0$. Since w is a solution of (24), (25) and $F_\varrho \geq 0$, it is clear that $\tau \mapsto M(\tau)$ is non-increasing, hence $M(\tau) \leq M(\tau_1) \leq 0$ for $\tau \geq \tau_1$. Using (44) and recalling that $\hat{v}$ is non-negative, we thus find

$$\|\hat{v}(\tau)\|_{L^1} = \int_{\mathbb{R}^N} \hat{v}(\tau,\xi)\,\mathrm{d}\xi \leq M(\tau) + C_{10}\, e^{-\mu\tau} \leq C_{10}\, e^{-\mu\tau},$$

for $\tau \geq \tau_1$. As a consequence,

$$\|u(t)\|_{L^1} = \|\hat{v}(\ln(1+t) - \tau_0)\|_{L^1} \leq C_{10}\, e^{\mu\tau_0}\,(1+t)^{-\mu} \quad \text{for } t \geq e^{\tau_1+\tau_0} - 1.$$

By (14), we also have $\|u(t)\|_{L^\infty} \leq C\,t^{-\mu-(N/2)}$ for t sufficiently large, a property which implies that $u \equiv 0$ by [7, Proposition 3] and [19, Corollary 4.2]. This contradicts the fact that $u(t,x) > 0$ for $t > 0$, hence we have proved the first assertion in (43). As for the second claim, it is a straightforward consequence of (13) and (44).

Now, since $\|\hat{v}(\tau)\|_m \to 0$ as $\tau \to \infty$, it follows from (42) that there exists $\tau_2 \geq 0$ such that $\|w(\tau)\|_m \leq \varrho$ for all $\tau \geq \tau_2$. On the other hand, as $w(\tau) \in W_c$ for each $\tau \geq 0$, we have $w(\tau,\xi) = M(\tau)\,G(\xi) + f(M(\tau)\,G(\xi))$ for $(\tau,\xi) \in [0,\infty) \times \mathbb{R}^N$, where f is as in Theorem 9. In view of (24) and (26) we deduce that, for $\tau \geq \tau_2$,

$$\frac{dM}{d\tau}(\tau) = -\int_{\mathbb{R}^N} |\nabla w(\tau,\xi)|^{q_\star}\,\mathrm{d}\xi = -\|\nabla G\|_{L^{q_\star}}^{q_\star}\, M(\tau)^{q_\star} - \omega(\tau), \tag{45}$$

where

$$\omega(\tau) = \int_{\mathbb{R}^N} \Big(|\nabla w(\tau,\xi)|^{q_\star} - M(\tau)^{q_\star}\,|\nabla G(\xi)|^{q_\star}\Big)\,\mathrm{d}\xi.$$

To bound $\omega(\tau)$, we remark that $|\,|y+z|^{q_\star} - |y|^{q_\star}\,| \le q_\star(|y|+|z|)^{q_\star-1}|z|$ for all $y,z\in\mathbb{R}$. Also, since

$$\frac{1}{2} + \frac{q_\star - 1}{2} + \frac{N}{2(N+1)} \;=\; 1\,, \quad \text{and} \quad 2mq_\star\,\frac{N+1}{N} \;=\; 2m\,\frac{N+2}{N} \;>\; N+2\,,$$

it follows from Hölder's inequality that, for all $g,h \in L^2_m(\mathbb{R}^N)$,

$$\|\,|h|^{q_\star-1}g\|_{L^1} \;\le\; \|b^m h\|_{L^2}^{q_\star-1}\,\|b^m g\|_{L^2}\,\|b^{-mq_\star}\|_{L^{2(N+1)/N}} \;\le\; C\,|h|_m^{q_\star-1}|g|_m\,,$$

where $b(\xi) = (1+|\xi|^2)^{1/2}$. Thus

$$\begin{aligned}
|\omega(\tau)| \;&\le\; q_\star \int_{\mathbb{R}^N} \Big(M(\tau)\,|\nabla G(\xi)| + |\nabla f(M(\tau)\,G(\xi))|\Big)^{q_\star-1}\,|\nabla f(M(\tau)\,G(\xi))|\,\mathrm{d}\xi \\
&\le\; C\,(M(\tau)\,\|G\|_m + \|f(M(\tau)\,G)\|_m)^{q_\star-1}\,\|f(M(\tau)\,G)\|_m \\
&\le\; C\,M(\tau)^{q_\star-1}\,\|f(M(\tau)\,G)\|_m\,,
\end{aligned}$$

where in the last inequality we have used the fact that $f(0)=0$ and f is globally Lipschitz continuous. Since $f\in\mathcal{C}^1(E_c;E_s)$ with $f(0)=0$ and $Df(0)=0$, the above inequality and (43) imply that

$$\lim_{\tau\to\infty} \frac{\omega(\tau)}{M(\tau)^{q_\star}} \;=\; 0\,. \tag{46}$$

Combining (43), (45), (46), and recalling that $q_\star - 1 = 1/(N{+}1)$, we conclude that

$$\lim_{\tau\to\infty} \tau M(\tau)^{q_\star-1} \;=\; \frac{1}{(q_\star{-}1)\|\nabla G\|_{L^{q_\star}}^{q_\star}}\,, \quad \text{i.e.} \quad \lim_{\tau\to\infty} \tau^{N+1}M(\tau) \;=\; M_\star\,, \tag{47}$$

where $M_\star$ is as in Theorem 1. As $w(\tau,\xi) = M(\tau)\,G(\xi) + f(M(\tau)\,G(\xi))$, we deduce from (43), (47) and the properties of f that $\tau^{N+1}\,\|w(\tau) - M(\tau)\,G\|_m \to 0$ as $\tau\to\infty$. Combining this result with (42), (47), we arrive at

$$\lim_{\tau\to\infty} \tau^{N+1}\,\Big\|\hat{v}(\tau) - \frac{M_\star}{\tau^{N+1}}\,G\Big\|_{L^1} \;=\; 0\,. \tag{48}$$

Of course, the same result holds for $v(\tau) = \hat{v}(\tau{-}\tau_0)$. If we now return to the original function $u(t,x)$ via the transformation (9), we obtain exactly (7) for $p=1$. The case $p\in(1,\infty]$ then follows from (14) by a classical interpolation argument. □

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