Stability of Vortex Rings at High Reynolds Number

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Based on joint work with V. Šverák!

I Introduction

Throughout these lectures we consider the incompressible Navier-Stokes equations in the whole space $\mathbb{R}^3$:

$$\begin{cases} \partial_t u + (u\cdot\nabla)u = \nu\Delta u - \nabla p & x\in\mathbb{R}^3,\ t>0 \qquad \text{(NS)} \\ \operatorname{div} u = 0 \end{cases}$$

where:

$$\begin{cases} u = u(x,t)\in\mathbb{R}^3 : & \text{(Eulerian) velocity field} \\ p = p(x,t)\in\mathbb{R} : & \text{pressure field / density} \\ \nu = \text{const} : & \text{kinematic viscosity} \end{cases}$$

The fluid density is assumed to be constant. No boundaries are considered, and the fluid is supposed to be at rest at infinity. No forcing term is included in (NS), and we consider the initial value problem:

$$u(x,0) = u_0(x),\quad x\in\mathbb{R} \qquad (u_0 \text{ is given}).$$

The viscosity parameter $\nu>0$ is small in usual situations:

$$\nu \approx 10^{-6}\ m^2/s \quad \text{for the water at } 20\ °C$$

$\Rightarrow$ away from the boundaries one often considers the ideal case $\nu=0$, corresponding to the Euler equations.

The pressure p in (NS) is determined by solving the elliptic equation

$$-\Delta p(x,t) = \operatorname{div}\big((u\cdot\nabla)u\big)(x,t) \qquad \text{in } \mathbb{R}^3.$$

Alternatively, on can eliminate it by introducing the vorticity

$$\omega = \operatorname{curl} u = \nabla\wedge u \qquad \text{(Helmholtz, 1858)}$$

which satisfies:

$$\begin{cases} (u\cdot\nabla)u = \frac{1}{2}\nabla|u|^2 - u\wedge\omega \\ \operatorname{curl}(\omega\wedge u) = (u\cdot\nabla)\omega - (\omega\cdot\nabla)u \end{cases}$$

Taking the curl of (NS) we thus obtain:

$$\partial_t\omega + (u\cdot\nabla)\omega - (\omega\cdot\nabla)u = \nu\Delta\omega. \qquad \text{(NSV)}$$

The velocity u can be expressed in terms of ω by solving the elliptic system:

$$\begin{cases} \operatorname{div} u = 0 \\ \operatorname{curl} u = \omega \end{cases} \Rightarrow \quad u(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{(x-y)\wedge\omega(y)}{|x-y|^3}\,dy$$

Biot-Savart formula

There is a famous analogy with electromagnetism:

Fluid mechanics	Electromagnetism:
$\begin{cases} \operatorname{div} u = 0 \\ \operatorname{curl} u = \omega \end{cases}$	$\begin{cases} \operatorname{div} B = 0 \\ \operatorname{curl} B = \mu_0 j \end{cases}$

B = magnetic field, j = density of current.

Eq. (NSV) is an <u>advection equation</u> with <u>stretching</u>.

$(u\cdot\nabla)\omega$: advection term

$(\omega\cdot\nabla)u$: stretching term (which couples $\omega_1, \omega_2, \omega_3$).

To understand the action of the stretching term, it is instructive to make a Taylor expansion of the velocity field at frozen time:

$$\begin{aligned} u_i(x) &= u_i(0) + \partial_j u_i(0) x_j + O(|x|^2) \qquad (\text{assuming } u \in C^2!) \\ &= u_i(0) + \underbrace{\tfrac{1}{2}\big(\partial_j u_i(0) - \partial_i u_j(0)\big)}_{a_{ij}} x_j + \underbrace{\tfrac{1}{2}\big(\partial_j u_i(0) + \partial_i u_j(0)\big)}_{s_{ij}} x_j + O(|x|^2) \end{aligned}$$

- $u_i(0) \Rightarrow$ advection at constant speed (can be eliminated by Galilean transf.)
- $A = (a_{ij})_{ij} = \frac{1}{2}\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$, $\quad Ax = \frac{1}{2}\omega \wedge x$

$\Longrightarrow$ infinitesimal rotation, no effect on stretching: $(\omega(0)\cdot\nabla)Ax = 0$.

- $S = (s_{ij})_{ij} = \frac{1}{2}\big(\nabla u + (\nabla u)^t\big)$: symmetric part of ∇u.

In a suitable Euclidean frame: (<u>rate-of-strain tensor</u>)

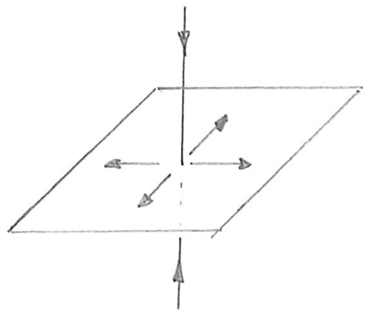
$$S = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad \lambda_1 + \lambda_2 + \lambda_3 = \operatorname{div}(u(0)) = 0.$$

$\lambda_1, \lambda_2, \lambda_3$ are called the <u>principal rates of strain</u>. One has

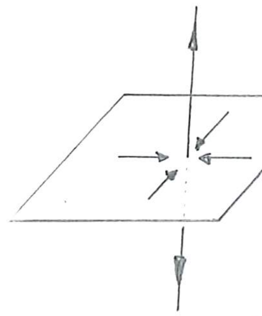
$$(\omega(0)\cdot\nabla)Sx = \begin{pmatrix} \lambda_1\omega_1 \\ \lambda_2\omega_2 \\ \lambda_3\omega_3 \end{pmatrix}$$

$$\Longrightarrow \begin{cases} \text{exponential decay of } \omega_i \text{ if } \lambda_i < 0 \\ \text{exponential growth of } \omega_i \text{ if } \lambda_i > 0 \end{cases}$$

Two generic pictures (locally, at frozen time):



$\lambda_1, \lambda_2 > 0, \; \lambda_3 < 0$
(vortex sheet)



$\lambda_1, \lambda_2 < 0, \; \lambda_3 > 0$
(vortex filament)

Vortex sheets are genuinely unstable (Kelvin-Helmholtz instability)
$\Longrightarrow$ vorticity stretching tends to produce vortex filaments.

Two simple geometric configurations (Helmholtz, 1858)

* 2D case
$$\begin{cases} u = u_1(x_1,x_2,t)\, e_1 + u_2(x_1,x_2,t)\, e_2 + 0 \cdot e_3 \\ \omega = \omega_3(x_1,x_2,t)\, e_3 \end{cases}$$

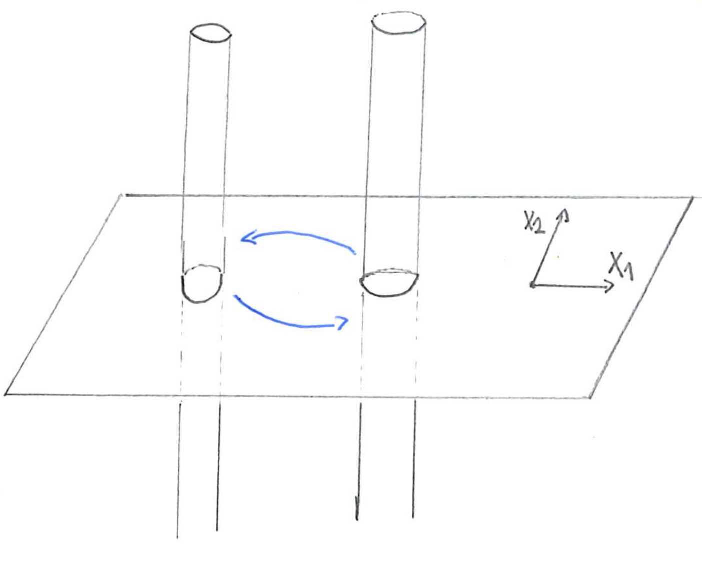
e_1, e_2, e_3 = canonical basis

Then: $\partial_t \omega_3 + (u \cdot \nabla)\omega_3 = \nu \Delta \omega_3$

No stretching at all!

$$\begin{cases} \partial_1 u_1 + \partial_2 u_2 = 0 \\ \partial_1 u_2 - \partial_2 u_1 = \omega_3 \end{cases} \Longrightarrow u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\, \omega(y)\, dy$$

All vortex tubes or filaments are straight and parallel to each other:



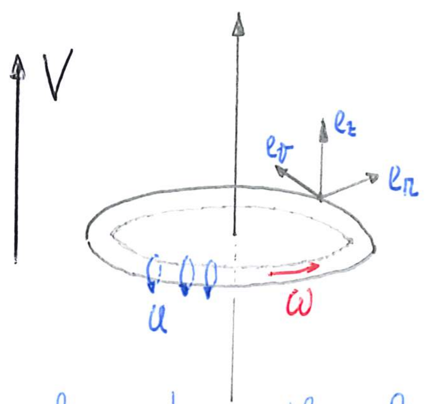
The motion of filaments is given by the point vortex system.

* Axisymmetric case (without swirl)

In cylindrical coordinates (r, ϑ, z) we have:

$$\begin{cases} u = u_r(r,z,t)\,e_r + \underbrace{0 \cdot e_\vartheta}_{\text{no swirl!}} + u_z(r,z,t)\,e_z \\ \omega = \omega_\vartheta(r,z,t)\,e_\vartheta \end{cases}$$

All vortex tubes are vortex rings symmetric about the vertical axis.



The motion of a vortex ring along its symmetry axis was studied by Kelvin (1867).

In this configuration, the velocity field is:

- invariant under rotation about the symmetry axis (axisymmetric)
- invariant under reflection by any plane containing the axis (no swirl)

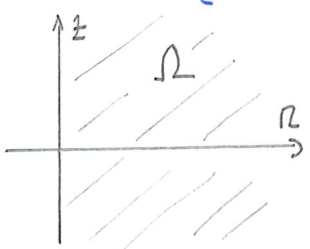
II The Cauchy problem for axisymmetric flows without swirl

The evolution equation for the axisymmetric vorticity $\omega_\vartheta(r,z,t)$ is:

$$\partial_t \omega_\vartheta + (u \cdot \nabla)\omega_\vartheta - \frac{u_r}{r}\omega_\vartheta = \nu\left(\Delta\omega_\vartheta - \frac{1}{r^2}\omega_\vartheta\right), \qquad (E\omega)$$

where $u \cdot \nabla = u_r \partial_r + u_z \partial_z$, $\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2$.

Spatial domain: $\Omega = \{(r,z) \in \mathbb{R}^2 \,;\, r > 0\}$



⚠ $r=0$ is not a material boundary
No boundary layers!

Formula for the velocity. The elliptic system is

$$\begin{cases} \partial_r u_r + \frac{1}{r} u_r + \partial_z u_z = 0 & \text{(divergence-free condition)} \\ \partial_z u_r - \partial_r u_z = \omega_\theta & \text{(definition of } \omega_\theta) \end{cases}$$

Boundary conditions: $u_r = \partial_r u_z = 0$ on $\partial\Omega$.

It is usual to introduce the Stokes stream function defined by

$$u_r = -\frac{1}{r}\partial_z \psi, \quad u_z = \frac{1}{r}\partial_r \psi \quad \Longrightarrow \quad \operatorname{div} u = 0.$$

$$\Longrightarrow \quad \omega_\theta = -\frac{1}{r}\left(\partial_r^2 \psi + \partial_z^2 \psi\right) + \frac{1}{r^2}\partial_r \psi. \qquad \text{(EE}\psi\text{)}$$

On $\partial\Omega$ one can impose $\psi = \partial_r \psi = 0$.

The potential vorticity. Unlike in 2D, the stretching term $\frac{u_r}{r}\omega_\theta$ is not identically zero for axisymmetric flows without swirl.
So the axisymmetric vorticity ω_θ does not satisfy a maximum principle.
However we can introduce the "potential vorticity":

$$\xi = \frac{\omega_\theta}{r}.$$

We compute the Poisson bracket:

$$\begin{aligned} \{\psi, \xi\} &= \partial_r \psi\, \partial_z \xi - \partial_z \psi\, \partial_r \xi \\ &= r u_z \partial_z \frac{\omega_\theta}{r} + r u_r \partial_r \frac{\omega_\theta}{r} \\ &= (u\cdot\nabla)\omega_\theta - \frac{u_r}{r}\omega_\theta. \end{aligned}$$

It follows that ξ satisfies the evolution equation

$$\partial_t \xi + \underbrace{\frac{1}{r}\{\psi, \xi\}}_{(u\cdot\nabla)\xi} = \nu\left(\Delta \xi + \frac{2}{r}\partial_r \xi\right). \qquad \text{(E}\xi\text{)}$$

The maximum principle applies to (Eξ), and gives a priori estimates for ξ in various L^p norms.

Conserved quantities and Lyapunov functions

1) Total impulse (in the vertical direction)

$$I = \int_\Omega r^2 \omega_\theta \, dr\,dz\,, \qquad \frac{dI}{dt} = 0\,.$$

2) Total circulation

$$\Gamma = \int_\Omega \omega_\theta \, dr\,dz\,, \qquad \frac{d\Gamma}{dt} = -\nu \int_{\partial\Omega} \partial_r \omega_\theta \, dz \leq 0$$

if $\omega_\theta \geq 0$

3) Kinetic energy

$$E = \frac{1}{2}\int_\Omega |u|^2 r\,dr\,dz = \frac{1}{2}\int_\Omega \psi\, \omega_\theta \, dr\,dz \qquad \geq 0$$

$$\frac{dE}{dt} = -\nu \int_\Omega \omega_\theta^2 \, r\,dr\,dz \leq 0\,.$$

Scaling and critical spaces

The NS equation is invariant under the rescaling:

$$\begin{cases} u(r,z,t) \longrightarrow \lambda u(\lambda r, \lambda z, \lambda^2 t) \\ \omega_\theta(r,z,t) \longrightarrow \lambda^2 \omega_\theta(\lambda r, \lambda z, \lambda^2 t) \\ \xi(r,z,t) \longrightarrow \lambda^3 \xi(\lambda r, \lambda z, \lambda^2 t) \end{cases} \qquad \lambda > 0$$

This suggests the natural critical space:

$$L^1(\Omega) = \left\{ \omega_\theta \,;\ \|\omega_\theta\|_{L^1} := \int_\Omega |\omega_\theta|\, dr\,dz < \infty \right\}.$$

The main result of this chapter is:

Thm 1 (ThG & V. Sverak, 2015)
For all initial data $\omega_0 \in L^1(\Omega)$, the axisymmetric vorticity eq. $(E\omega)$ has a unique global mild solution

$$\omega_\theta \in C^0([0,+\infty), L^1(\Omega)) \cap C^0((0,+\infty), L^\infty(\Omega))$$

such that $\omega_\theta(0) = \omega_0$. Moreover:

i) $\|\omega_\theta(t)\|_{L^1} \leq \|\omega_0\|_{L^1} \quad \forall t \geq 0$

ii) $\lim_{t \to +\infty} t^{1-1/p} \|\omega_\theta(t)\|_{L^p} = 0 \quad \forall p \in [1,+\infty]$.

Here $\|\omega_\theta\|_{L^p} = \left(\int_\Omega |\omega_\theta|^p \underbrace{dr\,dz} \right)^{1/p}$

↑ slice measure, not volume measure $r\,dr\,dz$!

Comments:

a) Local existence in $L^1(\Omega)$ is established by the same argument as for the vorticity equation in $L^1(\mathbb{R}^2)$, see Ben-Artzi (1994).
The argument dates back to Kato (1984), or Fujita-Kato (1962, 1964).

b) Global well-posedness was first established by Ladyzhenskaia (1968) and Ukhovskii & Yudovich (1968), for finite-energy solutions. The argument in Thm 1 is a little bit different, but also relies on a priori estimates for the potential vorticity ξ.

c) According to ii) all solutions converge to zero in scale-invariant norms as $t \to +\infty$. This is in contrast with the 2D case, where $\|\omega(t)\|_{L^1(\mathbb{R}^2)}$ is constant if ω has a definite sign. Here the decay originates from cancellations that are inherent to axisymmetric solutions.

In the rest of this section we give a sketch of the proof of Thm 1.

Step 1: The axisymmetric Biot-Savart law

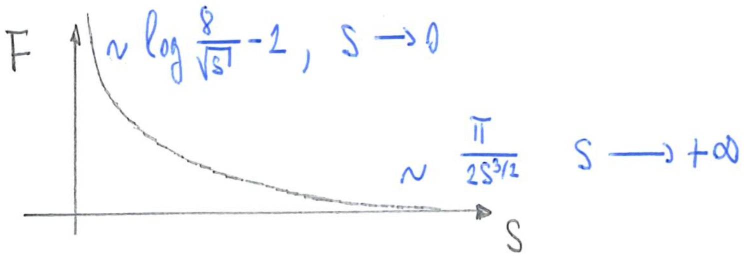
The solution of the elliptic equation

$$-\frac{1}{r}\left(\partial_r^2 \psi + \partial_z^2 \psi\right) + \frac{1}{r^2}\partial_r \psi = \omega_\theta \quad \text{in } \Omega$$

with $\psi = \partial_n \psi = 0$ on $\partial\Omega$ is given by:

$$\psi(r,z) = \frac{1}{2\pi}\int_\Omega \sqrt{r\bar{r}}\, F\left(\frac{(r-\bar{r})^2+(z-\bar{z})^2}{r\bar{r}}\right)\omega_\theta(\bar{r},\bar{z})\, d\bar{r}\, d\bar{z} \qquad (1)$$

$$F(s) = \int_0^{\pi/2} \frac{\cos(2\varphi)}{(\sin^2\varphi + s/4)^{1/2}}\, d\varphi\,, \quad s > 0. \qquad \text{(elliptic integral)}$$



This formula can be found in Maxwell's book on electromagnetism (1875). The singularity of $F(s)$ as $s \to 0$ is the same as the one of the Green function of Δ in $\mathbb{R}^2$. The decay of $F(s)$ as $s \to +\infty$ is specific to the axisymmetric case, and reflects the cancellations that occur in the far field.

Differentiating (1) with respect to r and z, we obtain a representation of the velocity field in the form:

$$u(r,z) = \int_\Omega G(r,\bar{r};z-\bar{z})\,\omega_\theta(\bar{r},\bar{z})\, d\bar{r}\, d\bar{z}\,. \qquad (2)$$

Lemma 1: $\exists\, C > 0$ such that

$$|G(r,\bar{r},z-\bar{z})| \le \frac{C}{\left((r-\bar{r})^2+(z-\bar{z})^2\right)^{1/2}} \qquad \forall (r,z)\in\Omega,\ \forall(\bar{r},\bar{z})\in\Omega$$

Thus the axisymmetric Biot-Savart kernel G satisfies an upper bound of the same form as the 2D Biot-Savart kernel $\frac{(x-y)^\perp}{|x-y|^2}\frac{1}{2\pi}$.
Using the Hardy-Littlewood-Sobolev inequality, we thus obtain:

<u>Lemma 2:</u> $\|u\|_{L^q(\Omega)} \leq C\|\omega_\theta\|_{L^p(\Omega)}$ if $1 < p < 2 < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$.

Also $\|u\|_{L^\infty} \leq C\|\omega_\theta\|_{L^1}^{1/2}\,\|\omega_\theta\|_{L^\infty}^{1/2}$ etc...

In fact, the velocity field u given by (2) also satisfies <u>weighted estimates</u> (involving powers of r) which have no analog in 2D. An example is:

<u>Lemma 3:</u> $\left\|\frac{u_r}{r}\right\|_{L^\infty} \leq C\|\omega_\theta\|_{L^1}^{1/3}\left\|\frac{\omega_\theta}{r}\right\|_{L^\infty}^{2/3}$.

<u>Step 2:</u> <u>The axisymmetric Stokes semigroup</u>

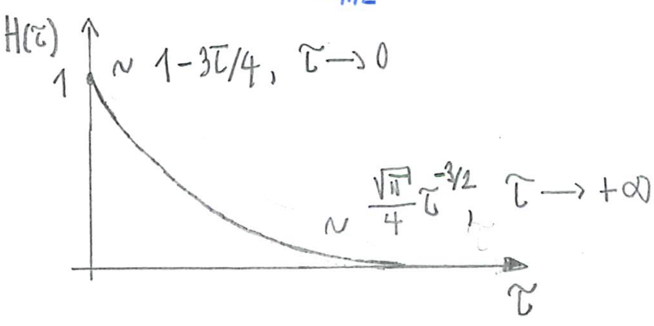
We consider the linearization of $(E\omega)$ at the rest state:

$$\begin{cases} \partial_t \omega_\theta = \nu\left(\Delta\omega_\theta - \frac{1}{r^2}\omega_\theta\right) & \text{in } \Omega \\ \omega_\theta|_{\partial\Omega} = 0 \end{cases} \qquad (E\omega\,\mathrm{lin})$$

To simplify the subsequent formulas, we assume w.l.o.g that <u>$\nu = 1$</u>.
The solution of $(E\omega\,\mathrm{lin})$ with initial data ω_0 is $\omega_\theta(t) = S(t)\omega_0$ where

$$(S(t)\omega_0)(r,z) = \frac{1}{4\pi t}\int_\Omega \underbrace{\left(\frac{r}{\bar r}\right)^{1/2} H\left(\frac{t}{r\bar r}\right)}_{\neq 2D} e^{-\frac{(r-\bar r)^2+(z-\bar z)^2}{4t}}\,\omega_0(\bar r,\bar z)\,d\bar r\,d\bar z$$

$$H(\tau) = \frac{1}{\sqrt{\pi\tau}}\int_{-\pi/2}^{\pi/2} e^{-\frac{\sin^2\varphi}{\tau}}\cos(2\varphi)\,d\varphi, \quad \tau > 0.$$



$$H(\tau) = \sqrt{\frac{\pi}{\tau}}\, e^{-\frac{1}{2\tau}}\, I_1\left(\frac{1}{2\tau}\right)$$

$\uparrow$ (modified Bessel)

<u>Lemma 4:</u> $\forall \varepsilon > 0 \ \exists C_\varepsilon > 0$ such that

$$\left(\frac{r}{\bar r}\right)^{1/2} H\left(\frac{t}{r\bar r}\right) e^{-\frac{(r-\bar r)^2 + (z-\bar z)^2}{4t}} \leq C_\varepsilon \, e^{-\frac{(r-\bar r)^2 + (z-\bar z)^2}{(4+\varepsilon)t}}$$

$\forall t > 0 \quad \forall (r,z) \in \Omega \quad \forall (\bar r, \bar z) \in \Omega.$

So the axisymmetric Stokes kernel is bounded from above by a multiple of the usual 2D heat kernel, with slightly deteriorated viscosity. A similar result holds for the first order derivatives too. Applying Young's inequality we obtain the usual L^p-L^q estimates:

<u>Lemma 5:</u> If $1 \leq p \leq q \leq \infty$ we have:

$$\| S(t)\omega_0 \|_{L^q} \leq \frac{C}{t^{1/p - 1/q}} \|\omega_0\|_{L^p}, \quad t > 0$$

$$\| S(t)\nabla\omega_0 \|_{L^q} \leq \frac{C}{t^{1/2 + 1/p - 1/q}} \|\omega_0\|_{L^p}, \quad t > 0$$

In addition, $S(t)$ satisfies some weighted L^p-L^q estimates involving powers of r, which do not exist in 2D.

<u>Step 3</u> <u>Local well-posedness</u>

We write $(E\omega)$ in the equivalent form

$$\partial_t \omega_\theta + \mathrm{div}_*(u\omega_\theta) = \nu\left(\Delta\omega_\theta - \frac{1}{r^2}\omega_\theta\right)$$

where $\mathrm{div}_*(u\omega_\theta) = \partial_r(u_r\omega_\theta) + \partial_z(u_z\omega_\theta)$. Taking again $\nu = 1$, we consider the associated <u>integral equation</u>:

$$\omega_\theta(t) = S(t)\omega_0 - \int_0^t S(t-s)\,\mathrm{div}_*(u(s)\omega_\theta(s))\,ds, \quad t > 0. \qquad (E\mathrm{int})$$

We plan to solve $(E\mathrm{int})$ by a <u>fixed point argument</u>.

Fix $T>0$. Following Kato (1984) we introduce the space

$$\begin{cases} X_T = \{ \omega_\vartheta \in C^0((0,T], L^{4/3}(\Omega))\ ;\ \|\omega_\vartheta\|_{X_T} < \infty \} \\ \|\omega_\vartheta\|_{X_T} = \sup_{0<t\leq T} t^{1/4} \|\omega_\vartheta(t)\|_{L^{4/3}(\Omega)} . \end{cases}$$

- <u>Linear term</u>: $\omega_{lin}(t) := S(t)\omega_0$

lemma 5

$$t^{1/4}\|\omega_{lin}(t)\|_{L^{4/3}} = t^{1/4}\|S(t)\omega_0\|_{L^{4/3}} \leqslant \frac{C\,t^{1/4}}{t^{1-3/4}}\|\omega_0\|_{L^1} = C\|\omega_0\|_{L^1}$$

$\Longrightarrow$ $\omega_{lin} \in X_T$ $\forall T>0$ and $\|\omega_{lin}\|_{X_T} \leq C\|\omega_0\|_{L^1}$.

Moreover a density argument shows that $\|\omega_{lin}\|_{X_T} \xrightarrow[T\to 0]{} 0$.

- <u>Nonlinear term</u>: if $\omega_\vartheta \in X_T$ we have

$$t^{1/4}\left\| \int_0^t S(t-s)\,\mathrm{div}_*(u(s)\omega_\vartheta(s))\,ds \right\|_{L^{4/3}}$$

$$\leqslant t^{1/4}\int_0^t \frac{C}{(t-s)^{1/2+1/4}}\|u(s)\omega_\vartheta(s)\|_{L^1}\,ds \qquad \text{(Lemma 5 with } p=1, q=4/3)$$

$$\leqslant t^{1/4}\int_0^t \frac{C}{(t-s)^{3/4}}\|u(s)\|_{L^4}\|\omega_\vartheta(s)\|_{L^{4/3}}\,ds \qquad \text{(Hölder)}$$

$$\leqslant t^{1/4}\int_0^t \frac{C}{(t-s)^{3/4}}\|\omega_\vartheta(s)\|^2_{L^{4/3}}\,ds \qquad \text{(Lemma 2 with } p=4/3, q=4)$$

$$\leqslant C\|\omega_\vartheta\|^2_{X_T}\underbrace{\int_0^t \frac{t^{1/4}}{(t-s)^{3/4}s^{1/2}}\,ds}_{=C} \leqslant \tilde{C}\|\omega_\vartheta\|^2_{X_T}$$

$$\Longrightarrow \left\| \int_0^t S(t-s)\,\mathrm{div}_*(u(s)\omega_\vartheta(s))\,ds \right\|_{X_T} \leq \tilde{C}\|\omega_\vartheta\|^2_{X_T}. \quad \text{(Idem for the Lipschitz bound)}$$

Standard fixed point argument in the ball $B(0,R) \subset X_T$ with $2\tilde{C}R<1$ yields existence and uniqueness of the solution of (Eint) in that ball.

To ensure that $\omega_{lin} \in B(0,R/2)$, two cases:

- if $\|\omega_0\|_{L^1}$ is small enough, $\|\omega_{\theta\,in}\|_{X_T} \leq C\|\omega_0\|_{L^1} \leq R/2 \quad \forall T>0$

  $\Longrightarrow$ Global well-posedness for small data;

- if $\|\omega_0\|_{L^1}$ is large, one takes $T>0$ small enough so that $\|\omega_{\theta\,in}\|_{X_T} \leq R/2$

  $\Longrightarrow$ Local well-posedness for small data.

In both cases one verifies that $\omega_\theta \in C^0([0,T], L^1(\Omega)) \cap C^0((0,T], L^\infty(\Omega))$, and is unique in that space.

Step 4: A priori estimates and global existence $\quad (\nu = 1)$

All global existence results for axisymmetric solutions without swirl of (NS) are based on a priori estimates for the potential vorticity

$$\xi(r,z,t) = \frac{1}{r}\,\omega_\theta(r,z,t).$$

In the original variable $x = (r\cos\theta, r\sin\theta, z)$ we have the evolution eq.

$$\partial_t \xi + (u\cdot\nabla)\xi = \Delta\xi + \frac{2}{r}\partial_r \xi$$

where $\frac{1}{r}\partial_r = \frac{x_1\partial_{x_1} + x_2\partial_{x_2}}{x_1^2+x_2^2}$. Equivalently:

$$\partial_t \xi + \hat{u}\cdot\nabla\xi = \Delta\xi \quad , \quad \hat{u} = u - \frac{2}{x_1^2+x_2^2}\begin{pmatrix} x_1 \\ x_2 \\ 0\end{pmatrix}. \qquad (E\xi')$$

The "velocity field" is not divergence-free:

$$\operatorname{div}\hat{u} = -4\pi\,\delta_{x_1=x_2=0} \leq 0.$$

Probabilistic interpretation: Brownian particles are advected by the velocity field u and killed (with some probability) when they hit the symmetry axis $x_1 = x_2 = 0$.

A standard consequence of these observations is:

Lemma 6: The solution of $(E\rho')$ satisfies, if $\rho_0 \in L^1(\mathbb{R}^3)$:

- $\|\rho(t)\|_{L^1(\mathbb{R}^3)} \leq \|\rho_0\|_{L^1(\mathbb{R}^3)} \quad \forall t \geq 0$
- $\|\rho(t)\|_{L^p(\mathbb{R}^3)} \leq \dfrac{C_p}{t^{\frac{3}{2}(1-1/p)}} \|\rho_0\|_{L^1(\mathbb{R}^3)} \quad \forall t > 0 \qquad (1 \leq p \leq \infty)$

where C_p depends only on p (not on u).

By the maximum principle it is enough to prove lemma 6 for ρ_0 nonnegative. The L^1 estimate is a direct calculation, the L^p bound can be established using Nash's inequality (see below).

Returning to the cylindrical coordinates (r, ϑ, z), we deduce from lemma 6:

$$\|\omega_\vartheta(t)\|_{L^1(\Omega)} \leq \|\omega_0\|_{L^1(\Omega)}, \qquad \left\|\frac{\omega_\vartheta(t)}{r}\right\|_{L^\infty(\Omega)} \leq \frac{C}{t^{3/2}} \|\omega_0\|_{L^1(\Omega)}. \qquad (*)$$

Proposition 1: Any solution $\omega_\vartheta \in C^0([0,T], L^1(\Omega)) \cap C^0((0,T], L^\infty(\Omega))$ of $(E\omega)$ satisfies, for $1 \leq p \leq \infty$:

$$\|\omega_\vartheta(t)\|_{L^p(\Omega)} \leq \frac{C(\|\omega_0\|_{L^1(\Omega)})}{t^{1-1/p}}, \qquad 0 < t \leq T \qquad (**)$$

where $C(s) = O(s)$ as $s \to 0$.

Proof: The case $p=1$ is covered by $(*)$. Let's prove $(**)$ for $p=2$, the other cases are similar. We denote $M = \|\omega_0\|_{L^1(\Omega)}$. We have:

$$\frac{d}{dt}\int_\Omega \omega_\vartheta^2 \, dr\,dz = \underbrace{-2\int_\Omega |\nabla\omega_\vartheta|^2 \, dr\,dz}_{\text{bulk dissipation}} + \int_\Omega \left(\frac{u_r}{r} - \frac{1}{r^2}\right)\omega_\vartheta^2 \, dr\,dz$$

stretching   negative term!

- Nash's inequality:

$$\|\omega_\vartheta\|_{L^2}^2 \leq C\|\omega_\vartheta\|_{L^1}\|\nabla\omega_\vartheta\|_{L^2} \leq CM\|\nabla\omega_\vartheta\|_{L^2}$$

- Lemma 3 + estimate (*):

$$\left\|\frac{u_r}{r}\right\|_{L^\infty} \leq C\|\omega_\vartheta\|_{L^1}^{1/3}\left\|\frac{\omega_\vartheta}{r}\right\|_{L^\infty}^{2/3} \leq \frac{CM}{t}$$

We thus arrive at the differential inequality:

$$\frac{d}{dt}\|\omega_\vartheta\|_{L^2}^2 \leq -\frac{K_1}{M^2}\|\omega_\vartheta\|_{L^2}^4 + \frac{K_2 M}{t}\|\omega_\vartheta\|_{L^2}^2, \quad K_1, K_2 > 0.$$

Integrating we obtain

$$\|\omega_\vartheta(t)\|_{L^2(\Omega)}^2 \leq \frac{K_2 M + 1}{K_1}\,\frac{M^2}{t}, \quad 0 < t \leq T$$

where $M = \|\omega_0\|_{L^1(\Omega)}$. This is (**) with $p = 2$.

- Case $1 \leq p \leq 2$: follows by interpolation
- case $2 \leq p \leq \infty$: further analysis of the integral equation. □

<u>Rem:</u> The uniform bound (**) with $p = 4/3$ implies a lower bound on the local existence time in Step 3 $\Rightarrow$ global existence.

<u>Rem:</u> Different arguments are needed to study the long-time behavior. The first step is to show that the solution of $(E\xi)$ with $\xi_0 \in L^1(\mathbb{R}^3)$ satisfies:

$$\|\xi(t)\|_{L^1(\mathbb{R}^3)} \xrightarrow[t\to+\infty]{} 0$$

i.e. $\|\omega_\vartheta(t)\|_{L^1(\Omega)} \xrightarrow[t\to\infty]{} 0$ (unlike in 2D!). The influence of the boundary condition at $r = 0$ is strong enough to drive the solution to zero.

III Measure-valued initial data

Thm 1 asserts that the vorticity equation $(E\omega)$ is globally well-posed in the critical Lebesgue space $L^1(\Omega)$. Can we enlarge the set of possible initial data, so as to include vortex sheets or vortex filaments?

Define $\mathcal{M}(\Omega)$ as the set of all finite Radon measures on Ω equipped with the total variation norm:

$$\|\mu\|_{TV} = \sup\left\{ \int_\Omega \varphi \, d\mu \; ; \; \varphi \in C_0(\Omega), \|\varphi\|_{L^\infty} \leq 1 \right\}.$$

Here:

- $\varphi \in C_0(\Omega) \iff \varphi$ is continuous on Ω, φ vanishes on the axis $r=0$ and at ∞
- Radon measure = regular Borel measure (in this context)
- μ finite $\iff \|\mu\|_{TV} < \infty$
- $\mathcal{M}(\Omega) = C_0(\Omega)'$ is a Banach space.

If $d\mu = \omega_0 \, dr\, dz$ for some $\omega_0 \in L^1(\Omega)$, then $\mu \in \mathcal{M}(\Omega)$ and $\|\mu\|_{TV} = \|\omega_0\|_{L^1(\Omega)}$: $L^1(\Omega)$ is a closed subspace of $\mathcal{M}(\Omega)$.

General decomposition: If $\mu \in \mathcal{M}(\Omega)$ we have

$$\begin{cases} \mu = \mu_{ac} + \mu_{sc} + \mu_{pp} \\ \|\mu\|_{TV} = \|\mu_{ac}\|_{TV} + \|\mu\|_{TV} + \|\mu_{pp}\|_{TV} \end{cases}$$ where

- μ_{ac} is absolutely continuous with respect to Lebesgue's measure $dr\,dz$:
  $d\mu_{ac} = \omega_0(r,z)\, dr\, dz$ for some $\omega_0 \in L^1(\Omega)$.

  Ex: μ_{ac} describes a vortex ring

- μ_{sc} is singular w.r.t. Lebesgue, yet has no atoms:

$$\mu_{sc}(\{(r,z)\}) = 0 \quad \forall (r,z) \in \Omega.$$

Ex of singularly continuous measure: vortex sheet.

- μ_{pp} is pure point: $\quad \mu_{pp} = \sum_{i=1}^{\infty} \Gamma_i \, \delta_{(r_i,z_i)} \qquad \Gamma_i \in \mathbb{R},\ (r_i,z_i) \in \Omega$

$$\|\mu_{pp}\|_{tv} = \sum_{i=1}^{\infty} |\Gamma_i| < \infty.$$

Ex: each atom of μ_{pp} corresponds to a vortex filament.

Can we solve the Cauchy problem when the initial measure μ is not absolutely continuous w.r.t. Lebesgue?

Lemma 7 (see Giga, Miyakawa, Osada 1988; Kato 1994)

If $\mu \in \mathcal{M}(\Omega)$ then

i) $\displaystyle \sup_{t>0} t^{1-1/p} \|S(t)\mu\|_{L^p(\Omega)} \leq C_p \|\mu\|_{tv} \qquad 1 \leq p \leq \infty$

ii) $\displaystyle \limsup_{t\to 0} t^{1-1/p} \|S(t)\mu\|_{L^p(\Omega)} \leq \tilde{C}_p \|\mu_{pp}\|_{tv} \qquad 1 < p \leq \infty.$

According to i), we have the same L^1-L^p estimates when $\mu \in \mathcal{M}(\Omega)$ as in the particular case where μ is absolutely continuous.
According to ii), all what matters for small times is the atomic part of the initial measure!
Returning to the proof of Thm 1 (local existence, step 3), all we need is that $\displaystyle \|\omega_{lin}\|_{X_T} = \sup_{0<t\leq T} t^{1/4} \|S(t)\mu\|_{L^{4/3}} \leq C_0 \nu$
universal constant.
if $T>0$ is small enough. In view of Lemma 7 this gives:

$$\| \mu_{pp} \|_{TV} \leq \tilde{C}_0 \, \nu$$

for some universal constant $\tilde{C}_0$. The only possible obstruction to local existence is the presence of large atoms (compared to viscosity) in the initial vorticity.

⚠ If the above condition is not met, the fixed point argument fails and no conclusion can be drawn on local existence, even for short times.

In what follows, we concentrate on vortex filaments as initial data. The following result gives a very satisfactory answer in the case of a single filament:

Thm 2 (Feng & Sverak 2015; ThG & Sverak 2019)

Fix $\Gamma \in \mathbb{R}$, $(r_0, z_0) \in \Omega$, and $\nu > 0$. The axisymmetric vorticity eq. $(E\omega)$ has a unique global solution

$$\omega_\theta \in C^0\big((0,+\infty), L^1(\Omega) \cap L^\infty(\Omega)\big)$$

such that:

- $\sup_{t>0} \| \omega_\theta(t) \|_{L^1(\Omega)} < \infty$, (A)
- $\omega_\theta(r,z,t)\,dr\,dz \xrightarrow[t \to 0+]{} \Gamma \, \delta_{(r_0,z_0)}$. (B)

In addition there exists a constant $C = C(|\Gamma|/\nu)$ such that

$$\int_\Omega \Big| \omega_\theta(r,z,t) - \frac{\Gamma}{4\pi\nu t}\, e^{-\frac{(r-r_0)^2 + (z-z_0)^2}{4\nu t}} \Big|\, dr\,dz \leq C |\Gamma| \frac{\sqrt{\nu t}}{r_0} \log \frac{r_0}{\sqrt{\nu t}} \qquad (C)$$

as long as $\frac{\sqrt{\nu t}}{r_0} \leq 1/2$.

circulation Reynolds number

⚠ Here no assumption is made on the size of $Re := \dfrac{|\Gamma|}{\nu}$!

Remarks:

- Assumption (A) is essential, otherwise the conclusion fails even for the linear heat equation. Note in particular that (C) $\Rightarrow$ (A).
- (B) is perhaps the weakest way of specifying initial data!
- Estimate (C) holds for fixed Γ, ν in the small time limit $t \to 0$. It gives no information for fixed $t > 0$ as $\nu \to 0$! Note that $\omega_\theta(r, z, t)$ is compared to a vortex ring located at initial position (r_0, z_0).
- Uniqueness of the solution is only asserted within the class of axisymmetric flows without swirl!

Possible extensions:

a) A similar result holds if the initial measure is $\mu = \sum_{i=1}^{N} \Gamma_i \delta_{(r_i, z_i)}$ with $\Gamma_i > 0 \ \forall i$ (or $\Gamma_i < 0 \ \forall i$), see Lévy & Liu (2018).

b) So far the case of a general measure $\mu \in \mathcal{M}(\Omega)$ is open.
The corresponding question has been completely solved in $\mathbb{R}^2$:
Cottet 1986, Giga-Miyakawa-Osada 1988, Kato 1984, ThG & Wayne 2005, Gallagher & ThG 2005, Bedrossian-Masmoudi 2014.

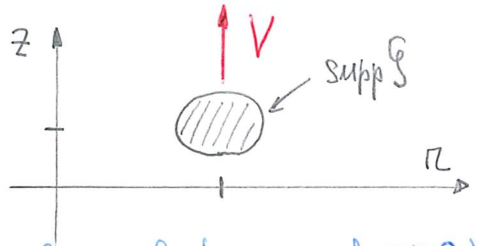
c) The vanishing viscosity limit for initial data in $\mathcal{M}(\Omega)$ is a very difficult problem, see below for the case of a single vortex filament.

IV Inviscid vortex rings

In this section we consider the inviscid case $\nu = 0$, corresponding to Euler's equation. The evolution equation for the potential vorticity reduces to

$$\partial_t \xi + \frac{1}{r}\{\psi, \xi\} = 0. \qquad (E\xi_0)$$

We look for travelling wave solutions of (Eξ_0), where the vorticity ξ has compact support and moves with constant speed V in the vertical direction



Looking for solutions of (Eξ_0) of the form $\xi(r, z-Vt)$, we obtain for the profile ξ the eq:

$$0 = -V\partial_z \xi + \frac{1}{r}\{\psi, \xi\} = \frac{1}{r}\left\{\psi - \frac{1}{2}Vr^2, \xi\right\}$$

so that
$$\left\{\psi - \frac{1}{2}Vr^2, \xi\right\} = 0. \qquad \text{(PBE)}$$

We recall that ψ solves the elliptic equation

$$\begin{cases} \mathcal{L}\psi := -\frac{1}{r^2}(\partial_r^2 + \partial_z^2)\psi + \frac{1}{r^3}\partial_r\psi = \xi & \text{in } \Omega \\ \psi(0,z) = \partial_r\psi(0,z) = 0, & z \in \mathbb{R} \\ \psi(r,z) \longrightarrow 0 \text{ as } r^2+z^2 \longrightarrow +\infty. \end{cases}$$

$\Longrightarrow$ $\psi := \mathcal{L}^{-1}\xi$ is given by formula (1) on p.9 with $\omega_\theta = r\xi$.

The Poisson bracket equation (PBE) means that the level sets of $\psi - \frac{1}{2}Vr^2$ and ξ coincide. It is usually solved by postulating a functional relation of the form

$$\xi = f\left(\psi - \frac{1}{2}Vr^2\right)$$

for some function $f : \mathbb{R} \to \mathbb{R}$ (which may be discontinuous at some pt).

The problem is thus reformulated as a <u>nonlinear elliptic equation</u>:

$$\mathcal{L}\psi = f\left(\psi - \tfrac{1}{2}V r^2\right) \quad \text{in } \Omega \qquad \text{(NEE)}$$

with the boundary conditions specified above.

Existence of vortex rings has been proved by <u>various methods</u>:

1) <u>Perturbative approach</u> (Fraenkel 1970, Cao, Lin, Yu, Zhan & Zou 2022)

- Fix f, for instance $f = \lambda \mathbb{1}_{[K,+\infty)}$ $\lambda > 0, K > 0$ (Rankine case)
- Construct an approximate solution in the regime $\varepsilon \to 0$
  ε = aspect ratio = diameter of supp(ξ) / distance to the axis.
- Solve the equation $\psi = \mathcal{L}^{-1} f\left(\psi - \tfrac{1}{2}Vr^2\right)$ near the approximate solution, by a fixed point argument.

2) <u>Stream function method</u>

Fraenkel & Berger 1974, 1980; Ni 1980; Ambrosetti & Struwe 1989; De Valeriola & van Schaftingen 2013.

Consider the functionals:

$$E(\psi) = \pi \int_\Omega \psi (\mathcal{L}\psi)\, r\, dr\, dz = \pi \int_\Omega \frac{1}{r^2} |\nabla\psi|^2 r\, dr\, dz$$

$$= \text{kinetic energy} \quad \text{(conserved)}$$

$$J(\psi) = \pi \int_\Omega F\left(\psi - \tfrac{1}{2}Vr^2\right) r\, dr\, dz, \quad F(s) = \int_0^s f(t)\,dt \quad \text{(not conserved!)}$$

Then (NEE) is equivalent to $dE(\psi) = dJ(\psi)$. One can thus:

- maximize J for a fixed value of E, or
- minimize E for a fixed value of J, or
- find a critical point of $J - E$ by a mountain pass argument.

3) Vorticity method

Benjamin 1976, Friedman & Turkington 1981, Cao, Wan, Wang & Zhan (to appear).

Consider the functionals:

$$\begin{cases} \mathcal{E}(\zeta) = \pi \int_\Omega \zeta (\mathcal{L}^{-1}\zeta)\, r\,dr\,dz & \text{(kinetic energy)} \\ I(\zeta) = \pi \int_\Omega \zeta\, r^3\, dr\,dz & \text{(impulse)} \end{cases}$$

(conserved!)

A solution of (NEE) is obtained by maximizing $\mathcal{E}$ for fixed I on the set of measure preserving rearrangements of a given function ζ_0.

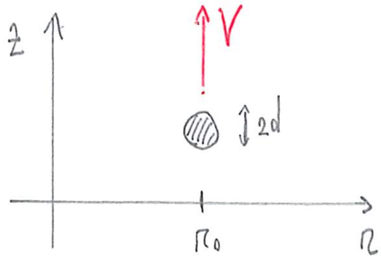
This approach is difficult to implement (due to the ∞-dimensional constraint) but can give orbital stability results because both functionals $\mathcal{E}, I$ are conserved quantities.

Rem: The profile ζ of the travelling wave is related to the structure function f in 1) and 2), and to the reference profile ζ_0 in 3).

Asymptotics of concentrated vortex rings (Kelvin's formula).

Consider a vortex ring of radius $r_0 > 0$ and thickness $d > 0$ such that

$$\varepsilon := \frac{d}{r_0} \ll 1.$$



The following asymptotic formula is known for the translation speed:

$$V = \frac{\Gamma}{4\pi r_0}\left(\log\frac{8r_0}{d} - C + O\left(\varepsilon^2 \log\frac{1}{\varepsilon}\right)\right), \quad \varepsilon \to 0 \qquad \text{(KF)}$$

where $\Gamma = \int_\Omega \zeta\, r\,dr\,dz$ is the circulation of the vortex ring.

The leading term $\frac{\Gamma}{4\pi r_0}\log\frac{8}{\varepsilon}$ is universal, and represents the binormal motion due to the local induction approximation (Da Rios, 1906). The constant C depends on the distribution of vorticity in the cross section. For Rankine's vortex (ξ = const. inside) the value $C = 1/4$ was obtained by Kelvin in 1867. The higher order corrections $O(\varepsilon^2 \log 1/\varepsilon)$ are obtained by heuristic calculations, see Fukumoto & Moffatt 2000.

V Vortex rings at high Reynolds number

We now return to the viscous case. We know from Thm 2 that the axisymmetric vorticity equation (EW) has a unique global solution with initial data

$$\omega_0 = \Gamma\, \delta_{(r_0,z_0)}, \qquad \Gamma \in \mathbb{R}, \ (r_0,z_0) \in \Omega.$$

The solution $\omega_\theta(r,z,t)$ is smooth at positive time >0, and when $\sqrt{\nu t} \ll r_0$ we expect it to describe a viscous vortex ring of thickness $\approx \sqrt{\nu t}$, with Gaussian distribution of vorticity in any cross-section. According to Kelvin's formula (KF), this vortex ring is supposed to move upwards (if $\Gamma > 0$) at the time-dependent velocity

$$V(t) = \frac{\Gamma}{4\pi r_0}\left(\log\frac{r_0}{\sqrt{\nu t}} + V_0 + O\!\left(\varepsilon^2 \log\frac{1}{\varepsilon}\right)\right) \qquad \text{(KSF)}$$

where $\varepsilon = \sqrt{\nu t}/r_0$ and V_0 is the constant corresponding to a Gaussian vorticity distribution:

$$V_0 = \frac{3}{2}\log 2 + \frac{1}{2}(\gamma_E - 1), \qquad \gamma_E \cong 0{,}5772\ldots$$

We call (KSF) the Kelvin-Saffman formula, see Saffman (1970).

We thus expect that the vertical location $\bar{Z}(t)$ of the viscous vortex ring will satisfy

$$\bar{Z}(t) = z_0 + \int_0^t V(s)\,ds, \quad \text{with } V \text{ as in (KSF).}$$

To formulate our main result, given $\Gamma > 0$, $\nu > 0$, and $(r_0, z_0) \in \Omega$, we denote:

- $T_{adv} = \dfrac{r_0^2}{\Gamma}$ (advection time)
- $T_{dif} = \dfrac{r_0^2}{\nu}$ (diffusion time)
- $\delta = \dfrac{\nu}{\Gamma}$ (inverse Reynolds number).

Note that $T_{dif} \gg T_{adv}$ when $\delta \ll 1$.

Thm 3: There exist $K > 0$, $\delta_0 > 0$, and $\sigma > 0$ such that the following holds. Given $\Gamma > 0$, $\nu > 0$ such that $\delta = \nu/\Gamma \leq \delta_0$ and $(r_0, z_0) \in \Omega$, let ω_θ be the unique solution of $(E\omega)$ satisfying

$$\sup_{t>0} \|\omega_\theta(t)\|_{L^1(\Omega)} < \infty, \quad \omega_\theta(t)\,dr\,dz \xrightarrow[t \to 0+]{} \Gamma\,\delta_{(r_0, z_0)}.$$

Then we have the estimate

$$\int_\Omega \left| \omega_\theta(r,z,t) - \frac{\Gamma}{4\pi\nu t}\, e^{-\frac{(r-\bar{r}(t))^2 + (z-\bar{Z}(t))^2}{4\nu t}} \right| dr\,dz \leq K\Gamma\, \frac{\sqrt{\nu t}}{r_0}, \quad \underline{0 < t < T_{adv}\,\delta^{-\sigma}};$$

where $\bar{r}(t), \bar{Z}(t)$ are smooth functions satisfying $\bar{r}(0) = r_0$, $\bar{Z}(0) = z_0$ and

$$\bar{r}'(t) = O\Big(\frac{\nu}{r_0}\Big), \quad \bar{Z}'(t) = \frac{\Gamma}{4\pi r_0}\Big(\log\frac{1}{\varepsilon(t)} + V_0 + O\Big(\varepsilon(t)^2 \log\frac{1}{\varepsilon(t)}\Big)\Big),$$

with $\varepsilon(t) = \sqrt{\nu t}/r_0$.

⚠ The constant $K > 0$ is independent of $\delta \in (0, \delta_0]$, unlike in Thm 2!

Summarizing, Thm 3 asserts that:

- The initial vortex filament diffuses according to the heat equation; in particular, at time t, it looks like a viscous vortex ring of aspect ratio $\varepsilon(t) = \sqrt{\nu t}/r_0$ with Gaussian vorticity distribution in the cross section.
- The viscous vortex ring translates along the symmetry axis at a speed given by the Kelvin-Saffman formula.
- The approximation above is valid up to time $\underbrace{T_{adv}\,\delta^{-\sigma}}_{=T}$ $(0<\sigma\ll 1)$, which is intermediate between T_{adv} $(\sigma=0)$ and T_{dif} $(\sigma=1)$. On this time interval the vertical displacement satisfies:

$$\bar{z}(T) - \bar{z}(0) = \int_0^T \bar{z}'(t)\,dt \approx \frac{r_0}{4\pi}\frac{1}{\delta^\sigma}\Big(\log \delta^{-\frac{1-\sigma}{2}} + C\Big).$$

In particular: $\dfrac{\bar{z}(T)-\bar{z}(0)}{r_0} \xrightarrow[\delta\to 0]{} +\infty$.

The rest of these lectures is devoted to a sketch of proof of Thm 3.

Modulated self-similar variables

The solution we consider is highly concentrated near a point $(\bar{r}(t),\bar{z}(t))\in\Omega$ when $\nu t \ll r_0^2$. To desingularize the initial value problem, we introduce the following self-similar variables:

$$R = \frac{r-\bar{r}(t)}{\sqrt{\nu t}}, \quad Z = \frac{z-\bar{z}(t)}{\sqrt{\nu t}}, \quad \varepsilon = \frac{\sqrt{\nu t}}{\bar{r}(t)}, \quad \delta = \frac{\nu}{\Gamma}$$

where $(\bar{r}(t),\bar{z}(t))\in\Omega$ is the vortex location (or center of vorticity) which is unknown at this stage and will be determined in the course of the proof. All the quantities above are dimensionless.

For the dependent variables, we make the transformation:

$$\begin{cases} \omega_\theta(r,z,t) = \dfrac{\Gamma}{\nu t}\,\eta\left(\dfrac{r-\bar{r}(t)}{\sqrt{\nu t}}, \dfrac{z-\bar{z}(t)}{\sqrt{\nu t}}, t\right) \\[2ex] \psi(r,z,t) = \Gamma\,\bar{r}(t)\,\varphi\left(\dfrac{r-\bar{r}(t)}{\sqrt{\nu t}}, \dfrac{z-\bar{z}(t)}{\sqrt{\nu t}}, t\right) \end{cases}$$

The dimensionless quantities η, φ are the vorticity and the stream function expressed in the new coordinates.

Evolution equation:

$$t\,\partial_t \eta + \frac{1}{\delta}\left\{\varphi, \frac{\eta}{1+\varepsilon R}\right\} - \sqrt{\frac{t}{\nu}}\left(\dot{\bar{r}}\,\partial_R + \dot{\bar{z}}\,\partial_Z\right)\eta = \mathcal{L}\eta + \varepsilon\,\partial_R \frac{\eta}{1+\varepsilon R} \qquad (E\eta)$$

where $\{f,g\} = \partial_R f\,\partial_Z g - \partial_Z f\,\partial_R g$: Poisson bracket

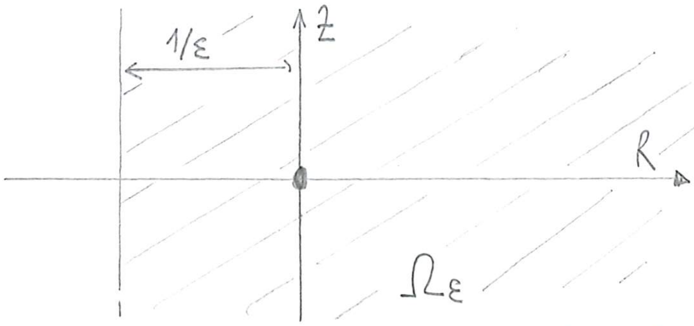
$\mathcal{L} = \partial_R^2 + \partial_Z^2 + \frac{1}{2}(R\partial_R + Z\partial_Z) + 1$: rescaled diff. operator.

Equation $(E\eta)$ has to be solved in the time-dependent domain

$$\Omega_\varepsilon = \left\{(R,Z) \in \mathbb{R}^2 ;\ Z \in \mathbb{R},\ 1+\varepsilon R > 0\right\}$$

with homogeneous Dirichlet boundary condition. Note that

$$\Omega_0 = \mathbb{R}^2, \quad \Omega_{+\infty} = \Omega, \qquad \Omega \subset \Omega_\varepsilon \subset \mathbb{R}^2$$



In the new coordinates, the vortex is located near the origin.

Time-dependent Biot-Savart law

The rescaled stream function φ solves the elliptic equation

$$-\partial_R \frac{\partial_R \varphi}{1+\varepsilon R} - \frac{\partial_Z^2 \varphi}{1+\varepsilon R} \quad \text{in } \Omega_\varepsilon$$

with boundary conditions:

$$\varphi(-1/\varepsilon, Z) = \partial_R\varphi(-1/\varepsilon, Z) = 0, \qquad \varphi(R,Z) \xrightarrow[R^2+Z^2\to\infty]{} 0 .$$

Explicit formula: $\qquad \varphi = BS_\varepsilon[\eta]$

$$\varphi(R,Z) = \frac{1}{2\pi}\int_{\Omega_\varepsilon} K_\varepsilon(R,R',Z-Z')\,\eta(R',Z')\,dR'\,dZ'$$

$$K_\varepsilon = \sqrt{(1+\varepsilon R)(1+\varepsilon R')}\; F\left(\frac{\varepsilon^2 D^2}{(1+\varepsilon R)(1+\varepsilon R')}\right), \qquad D^2 = (R-R')^2 + (Z-Z')^2 .$$

Asymptotic expansion:

$$K_\varepsilon = (\beta_\varepsilon + L)\sum_{m=0}^{\infty} \varepsilon^m P_m + \sum_{m=0}^{\infty} \varepsilon^m Q_m$$

where $\beta_\varepsilon = \log\frac{1}{\varepsilon}$, $L = \log\frac{8}{D}$ $\quad$ (= 2D Biot-Savart kernel)

P_m, Q_m are homogeneous polynomials of degree m in $R, R', Z-Z'$:

$$P_0 = 1 \qquad\qquad Q_0 = -2$$

$$P_1 = \frac{1}{2}(R+R') \qquad\qquad Q_1 = -\frac{1}{2}(R+R')$$

$$P_2 = \frac{1}{16}(R-R')^2 + \frac{3}{16}(Z-Z')^2 \qquad\qquad Q_2 = \frac{1}{4}(R^2+R'^2) - \frac{1}{16}D^2 \quad \text{etc...}$$

Neglecting irrelevant constants: $\quad BS_\varepsilon[\eta] = BS_0[\eta] + \varepsilon\, BS_1[\eta] + \dots$

$$\begin{cases} BS_0[\eta] = \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}^2} L\,\eta\,dR'\,dZ' & \text{(2D Biot-Savart law)} \\[2ex] BS_1[\eta] = \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}^2} (\beta_\varepsilon - 1 + L)\,\dfrac{R+R'}{2}\,\eta\,dR'\,dZ' & \dots \end{cases}$$

Initial data:

Eq. $(E\eta)$ cannot be solved for arbitrary data at $t = 0$!

Setting formally $t = 0$ (hence $\varepsilon = 0$) in $(E\eta)$ one obtains the following constraints for the initial data $\eta_0, \varphi_0 = BS_0[\eta_0]$:

$$\frac{1}{\delta}\{\varphi_0, \eta_0\} = \mathcal{L}\eta_0 .$$

- If η_0 is radially symmetric, then $\varphi_0 = BS_0[\eta_0]$ is also symmetric, hence $\{\varphi_0, \eta_0\} = 0$ (Euler condition)
- Among radially symmetric functions, one has $\mathcal{L}\eta_0 = 0$ iff η_0 is a multiple of the Gaussian

$$\eta_0(R,Z) = \frac{1}{4\pi} e^{-(R^2+Z^2)/4}, \quad (R,Z) \in \mathbb{R}^2$$

This vorticity is normalized so that $\int_{\mathbb{R}^2} \eta_0 \, dR\, dZ = 1$.

The corresponding stream function is:

$$\varphi_0(R,Z) = C - \frac{1}{4\pi} \operatorname{Ein}\left(\frac{R^2+Z^2}{4}\right), \quad C = \frac{\log 2}{\pi} + \frac{\gamma_E}{4\pi}$$

where

$$\operatorname{Ein}(x) = \int_0^x \frac{1-e^{-t}}{t}\, dt \sim \log x \text{ as } x \to +\infty .$$

Thm 2 (which deals with the particular case $\bar{r}(t) = r_0$, $\bar{z}(t) = z_0$) asserts that equation $(E\eta)$ has a unique global solution with initial data η_0, φ_0, in suitable function spaces. Our goal is to compute an asymptotic expansion of that solution in the vanishing viscosity limit where

$$\varepsilon \longrightarrow 0 \text{ and } \delta \longrightarrow 0 .$$

VI Construction of an approximate solution

The first key step in the proof of Thm 3 is the construction of an approximate solution of $(E\eta)$ of the form

$$\begin{cases} \eta_{app}(R,z,t) = \sum_{m=0}^{4} \varepsilon^m \eta_m(R,z,\beta_\varepsilon) \\ \varphi_{app}(R,z,t) = \sum_{m=0}^{4} \varepsilon^m \varphi_m(R,z,\beta_\varepsilon) \end{cases}$$

- This is an expansion in $\varepsilon = \sqrt{\nu t}/\bar{r}(t)$, but the profiles η_m, φ_m may also depend on $\underline{\delta = \nu/\Gamma}$ and $\underline{\log 1/\varepsilon = \beta_\varepsilon}$.
- The expansion can be carried out to <u>arbitrary order</u> in ε, but 4^{th} order will be sufficient for our purposes.
- The vortex speed $\bar{r}', \bar{z}'$ is not known a priori. We also construct it perturbatively:

$$\bar{r}'(t) = \sum_{m=0}^{3} \varepsilon^m \bar{r}_m'(t), \qquad \bar{z}'(t) = \sum_{m=0}^{3} \varepsilon^m \bar{z}_m'(t).$$

The zeroth order for $\eta_{app}, \varphi_{app}$ is given by the initial data η_0, φ_0, see previous page. At order $O(\varepsilon)$, Equation $(E\eta)$ gives the relation:

$$\Lambda\eta_1 + \delta\left(\tfrac{1}{2} - \mathcal{L}\right)\eta_1 = g_1 \qquad (\Lambda)$$

where $\mathcal{L} = \partial_R^2 + \partial_z^2 + \frac{1}{2}(R\partial_R + z\partial_z) + 1$, Λ is the linear operator

$$\Lambda\eta_1 = \frac{1}{2\pi}\left(\{L\eta_0, \eta_1\} + \{L\eta_1, \eta_0\}\right) \qquad \text{(linearization of Euler)}$$

and

$$g_1 = \left(\frac{\bar{r}_0\bar{r}_0'}{\Gamma} + \delta\right)\partial_R\eta_0 + \left(\frac{\bar{r}_0\bar{z}_0'}{\Gamma} - \frac{\beta_\varepsilon - 1}{4\pi}\right)\partial_z\eta_0 - \frac{3}{2}(\partial_z\varphi_0)\eta_0 - \frac{1}{2}\varphi_0\partial_z\eta_0.$$

The operators $\mathcal{L}, \Lambda$ were studied by ThG & Wayne (2005) and by Maekawa (2011) in connection with the stability of Oseen vortices in $\mathbb{R}^2$. In the function space

$$Y = \left\{ \eta \in L^2(\mathbb{R}^2)\,;\ \int_{\mathbb{R}^2} |\eta(X)|^2 e^{|X|^2/4}\, dX < \infty \right\} \qquad X = (R, Z)$$

we have the following result:

Proposition 2

i) The operator $\mathcal{L}$ is self-adjoint on Y with discrete spectrum

$$\sigma(\mathcal{L}) = \left\{ -\frac{n}{2}\,;\ n = 0, 1, 2, \dots \right\}. \qquad \text{(harmonic oscillator!)}$$

The kernel of $\mathcal{L}$ is one-dimensional and spanned by η_0.
The eigenvalue $-1/2$ is double, with eigenfunctions $\partial_R \eta_0, \partial_Z \eta_0$ etc...

ii) The operator Λ is skew-adjoint on Y, i.e. $\Lambda^* = -\Lambda$, and

$$\mathrm{Ker}(\Lambda) = \underbrace{Y_0}_{\text{radially symmetric}} \oplus \underbrace{\left\{ \beta_1 \partial_R \eta_0 + \beta_2 \partial_Z \eta_0\,;\ \beta_1, \beta_2 \in \mathbb{R} \right\}}_{\text{2-dimensional}}.$$

It follows in particular that the operator $\Lambda + \delta(\frac{1}{2} - \mathcal{L})$ is invertible, so that (Λ) has a unique solution (since $g_1 \in Y$). But that solution is not regular as $\delta \to 0$, unless $g_1 \in \mathrm{Range}(\Lambda)$. So to solve (Λ) we must impose that $g_1 \perp \mathrm{Ker}(\Lambda)$. It is obvious that g_1 is orthogonal to Y_0, so we are left with two solvability conditions:

1) $\int_{\mathbb{R}^2} g_1 R \, dR\, dZ = 0 \quad \Longrightarrow \quad \bar{r}_0' = -\dfrac{\delta \Gamma}{\bar{r}_0}$

2) $\int_{\mathbb{R}^2} g_1 Z \, dR\, dZ = 0$: this determines the vertical speed $\bar{z}_0'$.

After some calculations one obtains:

$$\overline{Z}'_0 = \frac{\Gamma}{4\pi R_0}\left(\beta_\varepsilon + V_0\right), \quad V_0 = \frac{3}{2}\log 2 + \frac{1}{2}(\gamma_E - 1)$$

which agrees with the Kelvin-Saffman formula.

Now solving (Λ) we obtain a regular solution of the form

$$\eta_1(R,Z) = R\,\tilde{\eta}_1(\sqrt{R^2+Z^2}) + O(\delta),$$

and applying the Biot-Savart law we determine the first order correction to the stream function:

$$\varphi_1(R,Z,\beta_\varepsilon) = \frac{\beta_\varepsilon - 1}{4\pi} R + \frac{R}{2}\varphi_0 - \partial_R \varphi_0 + R\,\tilde{\varphi}_1(\sqrt{R^2+Z^2}) + O(\delta).$$

The construction can be carried out at arbitrary order in ε. At order m, one obtains a relation of the form

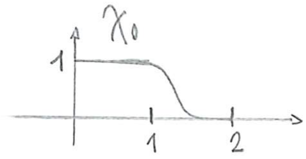
$$\boxed{\Lambda \eta_m + \delta\left(\frac{m}{2} - \mathcal{L}\right)\eta_m = g_m} \qquad (\Lambda')$$

which can be solved uniformly in δ as δ → 0. Indeed:

- The projections of g_m onto $\partial_R \eta_0$ and $\partial_Z \eta_0$ can be removed by choosing appropriately the speeds $\overline{r}'_{m-1}$ and $\overline{z}'_{m-1}$ (this occurs for m odd only).
- The projection of g_m onto Y_0 (which is nonzero for m even) is always of order δ, due to a symmetry of (E_η) when $\delta = 0$.

So projecting (Λ′) onto Y_0 and dividing both sides by δ, we obtain a regular solution by inverting the second order operator $\left(\frac{m}{2} - \mathcal{L}\right)$.

The approximate solution $\eta_{app}(R,z,t)$ is defined on the whole plane $\mathbb{R}^2$ and does not satisfy the Dirichlet condition at the boundary $\partial\Omega_\varepsilon$. It is therefore necessary to truncate it sufficiently far away from the origin.

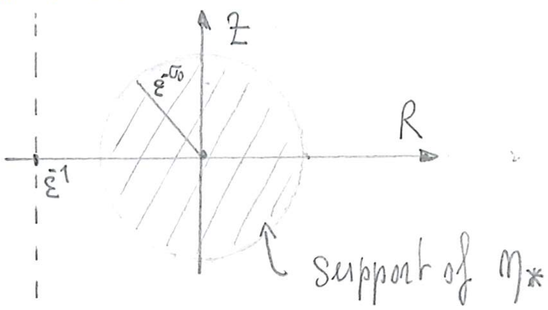


Given $0<\sigma_0<1$ and a cut-off function χ_0:
we define

$$\eta_*(R,z,t) = \chi_0\big(\varepsilon^{\sigma_0}(R^2+z^2)^{1/2}\big)\,\eta_{app}(R,z,t),$$

$$\varphi_*(R,z,t) = BS_\varepsilon[\eta_*](R,z,t).$$

The errors introduced by the truncation are $O(\varepsilon^\infty)$, because η_{app} has a Gaussian decay as $R^2+z^2 \longrightarrow \infty$. By construction the approximate solution η_* vanishes when $(R^2+z^2)^{1/2} \geq 2\varepsilon^{-\sigma_0}$, in particular on $\partial\Omega_\varepsilon$.



Remainder of the approximation:

$$\mathrm{Rem} = \mathcal{L}\eta_* + \varepsilon\,\partial_R\frac{\eta_*}{1+\varepsilon R} - t\,\partial_t\eta_* - \frac{1}{\delta}\Big\{\varphi_*,\frac{\eta_*}{1+\varepsilon R}\Big\} - \sqrt{\frac{t}{\nu}}\Big(\bar{r}'\partial_R\eta_* + \bar{z}'\partial_z\eta_*\Big).$$

Proposition 3 For $\gamma_0<1$ and $\gamma_5<5$ we have

$$\sup_{(R,z)\in\Omega_\varepsilon} e^{\gamma_0(R^2+z^2)/4}\,|\mathrm{Rem}(R,z,t)| \leq C\Big(\varepsilon\delta + \frac{\varepsilon^{\gamma_5}}{\delta}\Big).$$

Rem: $\varepsilon^2 \approx \dfrac{\delta t}{T_{adv}}$ thus $\varepsilon^2 \lesssim \delta$ when $\dfrac{t}{T_{adv}} \leq 1$.

VII Stability estimates

We look for an exact solution of $(E\eta)$ in the form

$$\begin{cases} \eta(R,z,t) = \eta_*(R,z,t) + \delta\,\tilde{\eta}(R,z,t) \\ \varphi(R,z,t) = \varphi_*(R,z,t) + \delta\,\tilde{\varphi}(R,z,t) \end{cases}$$

where η_*, φ_* is the approximate solution constructed in the previous section.

N.B. We anticipate that the correction must be at least $O(\delta)$ so that the nonlinear terms can be controlled.

The evolution eq. for $\tilde{\eta}$ reads:

nonlinear

$$t\,\partial_t\tilde{\eta} + \frac{1}{\delta}\left\{\varphi_*, \frac{\tilde{\eta}}{1+\varepsilon R}\right\} + \frac{1}{\delta}\left\{\tilde{\varphi}, \frac{\eta_*}{1+\varepsilon R}\right\} + \left\{\tilde{\varphi}, \frac{\tilde{\eta}}{1+\varepsilon R}\right\}$$
$$- \sqrt{\frac{t}{\nu}}\left(\bar{\Pi}'\partial_R\tilde{\eta} + \bar{Z}'\partial_z\tilde{\eta}\right) = \mathcal{L}\tilde{\eta} + \varepsilon\,\partial_R\frac{\tilde{\eta}}{1+\varepsilon R} + \frac{1}{\delta}\,\mathrm{Rem}. \qquad (E\tilde{\eta})$$

This equation is to be solved with zero initial data: $\tilde{\eta}|_{t=0} = 0$.

Everything is driven by the source term $\frac{1}{\delta}\mathrm{Rem}$, which satisfies

$$\left|\frac{1}{\delta}\mathrm{Rem}\right| \leq C\left(\varepsilon + \frac{\varepsilon^{\gamma_5}}{\delta^2}\right) \ll 1.$$

This is the reason for constructing an accurate approximation in section VI!

The nonlinear term and the diffusive terms (involving $\mathcal{L}$) are regular.

The danger comes from the linear terms:

$$\frac{1}{\delta}\left\{\varphi_*, \frac{\tilde{\eta}}{1+\varepsilon R}\right\} + \frac{1}{\delta}\left\{\tilde{\varphi}, \frac{\eta_*}{1+\varepsilon R}\right\}\left(-\sqrt{\frac{t}{\nu}}\,\bar{Z}'\partial_z\tilde{\eta}\right) \qquad (DLT)$$

which involve the small factor $\delta = \nu/\Gamma$ in the denominator.

The problem is to control the solution of $(E\tilde{\eta})$ uniformly in the vanishing viscosity limit $\varepsilon \to 0$, $\delta \to 0$.

To this end we introduce the energy functional

$$E_\varepsilon(t) = \frac{1}{2}\int_{\Omega_\varepsilon} W_\varepsilon\, \tilde{\eta}^2\, dR\, dz - \frac{1}{2}\int_{\Omega_\varepsilon} \tilde{\varphi}\,\tilde{\eta}\, dR\, dz \qquad (E_\varepsilon)$$

where $W_\varepsilon(R,z,t)$ is a coercive weight function described below.
The second term is the kinetic energy of the fluid:

$$\frac{1}{2}\int_{\Omega_\varepsilon} \tilde{\varphi}\,\tilde{\eta}\, dR\, dz = \frac{1}{2}\int_{\Omega_\varepsilon} \frac{|\nabla\tilde{\varphi}|^2}{1+\varepsilon R}\, dR\, dz = \frac{1}{2}\int_{\Omega_\varepsilon} |\underbrace{\tilde{u}}_{\text{velocity}}|^2 (1+\varepsilon R)\, dR\, dz \geqslant 0.$$

The weight W_ε is chosen so as to minimize the effect of the dangerous linear terms involving δ^{-1}. To construct it we fix:

$$0 < \sigma_1 < \sigma_0 < 1 < \sigma_2$$

and we consider three different regions in the space domain Ω_ε:

i) Inner region $\quad \rho := \sqrt{R^2+z^2} \leqslant \varepsilon^{-\sigma_1}$. Here we choose

$$W_\varepsilon(R,z,t) = \frac{1}{1+\varepsilon R}\, \Phi_\varepsilon'\left(\frac{\eta_*(R,z,t)}{1+\varepsilon R}\right)$$

where Φ_ε describes the functional relation between the stream function and the potential vorticity:

$$\psi_* - \frac{\bar{r}\,\bar{z}'}{2\Gamma}(1+\varepsilon R)^2 = \Phi_\varepsilon\left(\frac{\eta_*}{1+\varepsilon R}\right) + O(\varepsilon\delta + \varepsilon^{\gamma_3}) \qquad (*)$$

$\gamma_3 < 3$

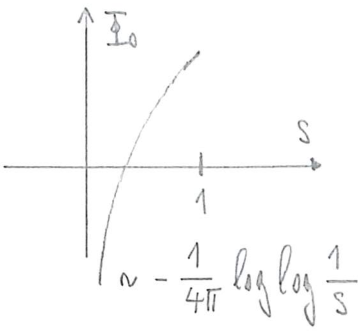
Comment: If $(*)$ was true without correction terms, this relation would mean that η_* is a stationary solution of Euler in a frame moving with speed $\bar{z}' e_z$ (expressed in non-dimensional variables).
This is not exactly the case, but the construction in section VI also provides a function Φ_ε such that $(*)$ holds. One finds that

$$\Phi_*(s) = \Phi_0(s) + \varepsilon^2\,\Phi(s),$$

Where $\Phi_0 : \mathbb{R}_+ \to \mathbb{R}$ satisfies:

$$\Phi_0(\eta_0(R,z)) = \varphi_0(R,z) - \frac{1}{8\pi}\left(\log\frac{1}{\varepsilon} + V_0\right)$$

This gives $\Phi_0(s) = C - \frac{1}{8\pi}\log\frac{1}{\varepsilon} - \frac{1}{4\pi}\,\mathrm{Eint}\left(\log\frac{1}{4\pi s}\right), \quad s > 0.$



(Φ_0 is increasing and concave)

Arnold's formula!

To leading order in ε we thus have:

$$W_0(R,z) = \Phi_0'(\eta_0(R,z)) = \frac{\partial_z \varphi_0(R,z)}{\partial_z \eta_0(R,z)} = \frac{4}{\rho^2}\left(e^{\rho^2/4} - 1\right).$$

The weight W_ε is chosen so that the dangerous linear terms (DLT) involving δ^{-1} or $\sqrt{t/\nu}$ produce regular contributions to the energy balance. More precisely we find

$$t\,E_\varepsilon'(t) = I_1 + I_2 + \text{more regular terms},$$

where:

$$\cdot\ I_1 = -\frac{1}{\delta}\int W_\varepsilon \tilde\eta \left\{\varphi_*, \frac{\tilde\eta}{1+\varepsilon R}\right\} dX + \sqrt{\frac{t}{\nu}}\,\bar z' \int W_\varepsilon \tilde\eta\, \partial_z \tilde\eta \, dX$$

$$= -\frac{1}{\delta}\int W_\varepsilon \tilde\eta \left\{\varphi_* - \frac{\bar r \bar z'}{2\Gamma}(1+\varepsilon R)^2, \frac{\tilde\eta}{1+\varepsilon R}\right\} dX$$

$$= -\frac{1}{2\delta}\int W_\varepsilon (1+\varepsilon R) \left\{\varphi_* - \frac{\bar r \bar z'}{2\Gamma}(1+\varepsilon R)^2, \left(\frac{\tilde\eta}{1+\varepsilon R}\right)^2\right\} dX$$

$$= -\frac{1}{2\delta}\int \left\{W_\varepsilon(1+\varepsilon R), \varphi_* - \frac{\bar r \bar z'}{2\Gamma}(1+\varepsilon R)^2\right\} \left(\frac{\tilde\eta}{1+\varepsilon R}\right)^2 dX$$

because formally: $\int f\{g,h\}\,dX = \int \{f,g\}\,h\,dX.$

But using (∗) and the definition of W_ε we find

$$\frac{1}{\delta}\left\{ W_\varepsilon(1+\varepsilon R), \varphi_* - \frac{\bar\Omega \bar Z'}{2\Gamma}(1+\varepsilon R)^2 \right\}$$

$$= \frac{1}{\delta}\left\{ \Phi_\varepsilon'\Big(\frac{\eta_*}{1+\varepsilon R}\Big), \Phi_\varepsilon\Big(\frac{\eta_*}{1+\varepsilon R}\Big) + O(\varepsilon\delta + \varepsilon^{\gamma_3}) \right\}$$

$$= O\Big(\varepsilon + \frac{\varepsilon^{\gamma_3}}{\delta}\Big) \ : \ \text{small as } \varepsilon \to 0.$$

- $$I_2 = \frac{1}{\delta}\int \tilde\varphi \Big\{\varphi_*, \frac{\tilde\eta}{1+\varepsilon R}\Big\} dX - \sqrt{\frac{t}{\nu}}\,\bar Z' \int \tilde\varphi\, \partial_z \tilde\eta\, dX - \frac{1}{\delta}\int (W_\varepsilon \tilde\eta - \tilde\varphi)\Big\{\tilde\varphi, \frac{\eta_*}{1+\varepsilon R}\Big\} dX$$

$$= \frac{1}{\delta}\int \tilde\varphi \Big\{\varphi_* - \frac{\bar\Omega \bar Z'}{2\Gamma}(1+\varepsilon R)^2, \frac{\tilde\eta}{1+\varepsilon R}\Big\} dX - \frac{1}{\delta}\int W_\varepsilon \tilde\eta \Big\{\tilde\varphi, \frac{\eta_*}{1+\varepsilon R}\Big\} dX$$

$$= \frac{1}{\delta}\int \underbrace{\Big\{\tilde\varphi, \varphi_* - \frac{\bar\Omega \bar Z'}{2\Gamma}(1+\varepsilon R)^2 - \Phi_\varepsilon\Big(\frac{\eta_*}{1+\varepsilon R}\Big)\Big\}}_{= O(\varepsilon\delta + \varepsilon^{\gamma_3})} \frac{\tilde\eta}{1+\varepsilon R}\, dX .$$

The other terms are regular in the vanishing viscosity limit, and the diffusion terms even give negative contributions to the second variation of the energy, which is not obvious at all (ThG and Sverak, 2021).

ii) Intermediate region $\quad \varepsilon^{-\sigma_1} \le \rho \le \varepsilon^{-\sigma_1}$

Outside the inner region the kinetic energy term in $E_\varepsilon(t)$ is extremely small $\Rightarrow$ $E_\varepsilon(t)$ is essentially a weighted enstrophy.

We take $W_\varepsilon \approx \exp(\varepsilon^{-2\sigma_1}/4)$ = const. in this region, so that the advection terms involving δ^{-1} make small contributions too.

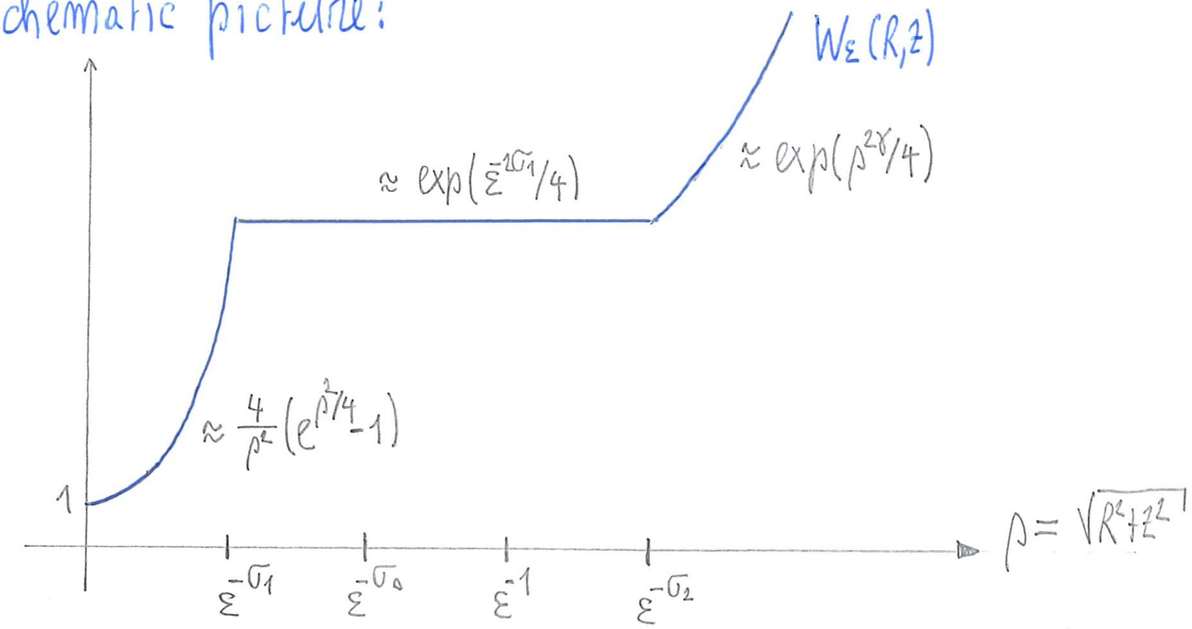
(the nonlocal terms in DLT are small anyway).

iii) Far field region $\quad \rho \geq \varepsilon^{-\sigma_2}$

Here the dominant effect is the advection $\frac{1}{2}(R\partial_R + Z\partial_Z)$ in $\mathcal{L}$.

We take $W_\varepsilon \approx \exp(\rho^{2\gamma}/4)$ where $\gamma = \sigma_1/\sigma_2 \quad (0<\gamma<1)$.

Schematic picture:



⚠ The true weight W_ε is not exactly radially symmetric!

Summarizing: The proof of Thm 3 consists in using the energy functional $E_\varepsilon(t)$ to control the solution $\tilde{\eta}$ of $(E\tilde{\eta})$ in the time-dependent space X_ε defined by the norm

$$\|\tilde{\eta}\|_{X_\varepsilon} = \left(\int_{\Omega_\varepsilon} W_\varepsilon(R,z)\,|\tilde{\eta}(R,z)|^2\,dR\,dz\right)^{1/2}.$$

⚠ Main problem: The functional $E_\varepsilon(t)$ is not coercive in the whole space X_ε! It is coercive for functions $\tilde{\eta}$ satisfying the moment conditions:

a) $\int_{\Omega_\varepsilon} \tilde{\eta}(R,z)\,dR\,dz = 0$

b) $\int_{\Omega_\varepsilon} \tilde{\eta}(R,z)\,R\,dR\,dz = 0$

c) $\int_{\Omega_\varepsilon} \tilde{\eta}(R,z)\,z\,dR\,dz = 0$

a) The total circulation $\int \eta\,dX$ is not exactly conserved by $(E\eta)$, due to boundary contributions that are extremely small for the solutions we consider. It follows that:

$$\int_{\Omega_\varepsilon} \tilde{\eta} \, dR\, dz = O(\varepsilon^\infty) \qquad \text{(sufficient for our purposes)}$$

b) Using the conservation of the impulse I one can show that

$$\int_{\Omega_\varepsilon} \tilde{\eta} \, R \, dR\, dz = O\left(\varepsilon + \frac{\varepsilon^{\gamma_5}}{\delta^2}\right), \quad \gamma_5 < 5.$$

This is again sufficient for us.

c) Due to the translation invariance in the vertical direction, it is possible to ensure that $\int_{\Omega_\varepsilon} \tilde{\eta}\, z \, dR\, dz = 0 \ \forall t$ by modifying slightly the vertical position $\bar{Z}(t)$ ("modulation technique").
This does not alter the final result.

Differential inequality for the energy: Using $(E\tilde{\eta})$ one finds

$$t\, E_\varepsilon'(t) \leq -k\, D_\varepsilon(t) + C\, R_\varepsilon \qquad 0 < t \leq T_{adv}$$

where $R_\varepsilon = \left(\varepsilon + \frac{\varepsilon^{\gamma_3}}{\delta}\right)^2$ and

$$D_\varepsilon(t) = \|\nabla \tilde{\eta}\|^2_{X_\varepsilon} + \|\rho^\gamma \tilde{\eta}\|^2_{X_\varepsilon} + \|\tilde{\eta}\|^2_{X_\varepsilon}, \quad \rho_\gamma = \begin{cases} \rho & \text{if } \rho \leq \varepsilon^{-\sigma_1} \\ \varepsilon^{-\sigma_1} & \text{if } \varepsilon^{-\sigma_1} \leq \rho \leq \varepsilon^{-\sigma_2} \\ \rho^\gamma & \text{if } \rho \geq \varepsilon^{-\sigma_2} \end{cases}.$$

Integrating and using the moments conditions we obtain:

$$\|\tilde{\eta}(t)\|^2_{X_\varepsilon} \leq C\, E_\varepsilon(t) + C\varepsilon^2 \leq C\left(\varepsilon + \frac{\varepsilon^{\gamma_3}}{\delta}\right)^2,$$

hence:

$$\|\tilde{\eta}(t)\|_{X_\varepsilon} \leq C\left(\varepsilon + \frac{\varepsilon^{\gamma_3}}{\delta}\right), \quad 0 < t \leq T_{adv}.$$

N.B. Still works if $t \leq T_{adv}\, \delta^{-\sigma}$ ($\sigma > 0$ small) because in that case

$$\varepsilon^2 \leq \delta \frac{t}{T_{adv}} \leq \delta^{1-\sigma} \implies \frac{1}{\delta} \leq \frac{1}{\varepsilon^{\frac{2}{1-\sigma}}} \quad \text{with } \frac{2}{1-\sigma} < \gamma_3.$$

VIII Final discussion

The above proof shows that, if $0 < \delta \leq \delta_0$,

$$\| \eta(t) - \eta_*(t) \|_{X_\varepsilon} \leq K(\varepsilon\delta + \varepsilon^{\gamma_3}) \,, \quad 0 < t \leq T_{adv}\,\delta^{-\sigma} \qquad (**)$$

where $\gamma_3 < 3$. Since $X_\varepsilon \hookrightarrow L^1(\Omega_\varepsilon)$ uniformly in ε, this implies the conclusion of Thm 3, but $(**)$ is considerably stronger. In particular we see that $\eta(t)$ stays close to the approximate solution $\eta_*(t)$ in a L^2 norm with strong weight W_ε.

So far, we only considered a single vortex filament, but we fully control the vanishing viscosity limit, and give the first rigorous derivation of the binormal flow for the NS equations when the filament has nonzero curvature.

We hope that, in a near future, the techniques presented here can be adapted to study the interaction of several filaments with common symmetry axis, as in the leapfrogging phenomenon.