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Theta functions, root systems and 3-manifold invariants

Louis Funar

Institute of Mathematics of the Romanian Academy, Bucharest, Romania
Mathématiques, Bat. 425, Univ. Paris-Sud, France
Institut Fourier, BP 74, Univ. Grenoble I, 38402 Saint-Martin-d'Hères cedex, France [1]

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Abstract

We describe a semi-abelian version of Witten's theory using the quantization of dimension g tori for a general gauge group G. We derive a family of invariants for closed oriented 3-manifolds which coincide with those defined by Witten for lens spaces and torus bundles.

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1. Introduction

The motivation of this paper is the attempt of understanding a semi-abelian version of Chern–Simons–Witten invariants using representations of mapping class groups. This has been done in the case when the gauge group G is $U(1)$ in [8,11,14] and for general G but only in genus 1 case in [19].

We follow the program outlined by Witten in [30] but we replace the Teichmuller space with the Siegel space. He associates vector spaces $Z(\Sigma_g, k)$ to every Riemann surface of genus g obtained from the quantization of M_{Σ_g} the space of representations of the fundamental group $\pi_1(\Sigma_g)$ in G, modulo conjugation. If $G = U(l)$ then a theorem of Narasimhan–Seshadri [28] identifies M_{Σ_g} with the moduli space of rank l semi-stable holomorphic bundles of degree 0 over Σ_g. The Picard group of M_{Σ_g} is generated by an ample line bundle L_{Σ_g} and it turns that $Z(\Sigma_g, k) = H^0(M_{\Sigma_g}, L^k_{\Sigma_g})$ are the fibres of a projectively flat holomorphic vector bundle over the Teichmuller space using the HADW

[1] Permanent address.

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connection (see [17,2]). It is clear (see [12]) that the (projective) representation of the mapping class group $\mathcal{M}_g$ arising as the monodromy of the natural action on this flat bundle will determine the topological field theory we are looking for. One way to understand $Z(\Sigma_g, k)$ was opened in [4,5] where it is identified with some space of theta functions on the jacobian variety $Jac(\Sigma_g)$ in the case when $G = SU(2)$ and $k = 1, 2$.

Our aim is to use a semi-abelian quantization in general gauge. Namely, instead of quantizing the Chern–Simons action on $\Sigma_g \times \mathbb{R}$ we shall quantize the Chern–Simons type action on the higher-dimensional jacobian tori $Jac(\Sigma_g) \times \mathbb{R}$. Since $\pi_1(Jac(\Sigma_g))$ is abelian the space of interest $N_{\Sigma_g} = Hom(\pi_1(Jac(\Sigma_g)), G)/G$ of the representations of $\pi_1(Jac(\Sigma_g))$ has a simple description. The associated bundle may be extended to a projectively flat bundle over the moduli space of principally polarized abelian varieties. This way a representation of the symplectic group $Sp(2g, \mathbb{Z})$ will be obtained.

One then obtains 3-manifold invariants by considering Heegaard decompositions as is done in [22] using also the p_1-structures from [6].

This is a reasonable "abelian" approximation of Witten's theory which can be viewed as a study of a simplified (but non-trivial) model mathematically justified. Notice that as an alternative for circumvent the problems with functional integration one may use the technique of quantum groups developed in [29].

2. The quantization of N_{Σ_g}

We choose G a compact Lie group which is assumed to be simple and connected. It has maximal torus T and Weyl group W. The usual alternating character on W is denoted by det and the rank of G (the dimension of T) is denoted by l. Let R be a reduced irreducible root system in the dual t^* of the Lie algebra t and let $R^\vee \subset t$ denote its dual. We write Q and $Q^\vee$ for the lattices generated by R and $R^\vee$ respectively. We denote their dual lattices by $P^\vee \subset t$ and $P \subset t^*$ and we have $Q \subset P$ and $Q^\vee \subset P^\vee$. We fix a basis $\alpha_1, \alpha_2, ..., \alpha_l$ for R and then $\alpha_1^\vee, \alpha_2^\vee, ..., \alpha_l^\vee$ is a basis for $R^\vee$. Let $\tilde{\alpha}$ be the highest root. We write

$$\tilde{\alpha}^\vee = \sum_{i=1}^{l} s_i \alpha_i^\vee$$

and we put $h = 1 + \sum_{i=1}^{l} s_i$. If G is simply laced (all the roots have the same length) then h will be the Coxeter number of G. We consider the positive definite, symmetric bilinear form I on t given by

$$I(x, y) = (2g)^{-1} \sum_{i=1}^{l} \langle \alpha_i, x \rangle \langle \alpha_i, y \rangle,$$

where $\langle , \rangle$ denotes the basic inner product. If $S^2 Q^\vee$ denotes the lattice of integral symmetric bilinear forms on $Q^\vee$ then $(S^2 Q^\vee)^W$ is infinite cyclic generated by I unless R is of type C_l ($l \geq 3$) in which case $\frac{1}{2} I$ is a generator. Now I determines a homomorphism $t \to t^*$ which we also denote by I. We set $M = I^{-1}(P)$.

We return now to the moduli space of representations N_{Σ_g}. Any representation will map the whole group $\mathbb{Z}^{2g}$ into a maximal torus of G and therefore the only conjugation freedom left is the diagonal action of W. Hence

$$N_{\Sigma_g} = T \times T \times \cdots \times T/W.$$

The tangent space to T^{2g} is $A = t \oplus t \oplus \cdots \oplus t$ and

$$T^{2g} = A/(Q^{\check{}})^{2g}.$$

The basic symplectic form ω on A is

$$\omega((\xi_1, \xi_2, ..., \xi_{2g})(\eta_1, \eta_2, ..., \eta_{2g})) = -2\pi I((\xi_1, \xi_2, ..., \xi_{2g}), S(\eta_1, \eta_2, ..., \eta_{2g})),$$

where I denotes the extension of the above considered bilinear form to $t \oplus t \oplus \cdots \oplus t$ by direct sum, and

$$S = \begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix} \in Sp(2g, \mathbb{Z}).$$

It is known that a connection ∇ on the trivial line bundle $A \times \mathbb{C}$ over the symplectic affine space (A, ω) with curvature $-\mathrm{i}\omega$ is given by

$$\nabla_X(a) = -\tfrac{1}{2}\mathrm{i}\omega(X - X_0, a)$$

for any $X_0 \in A$. Our task will be the construction of a line bundle $\mathcal{L}$ on T^{2g}, the prequantum line bundle, such that $c_1(\mathcal{L}) = (1/2\pi)\omega$, which must support a lift of the W action. In order to proceed we need to introduce a holomorphic structure on T^{2g}. As in the genus 1 case [2] a holomorphic structure on T^{2g} will be specified by a modular parameter Ω in the Siegel space $\mathcal{S}_g$ (of complex symmetric matrices of dimension g whose imaginary part is positive definite). To each such Ω there is a principally polarized abelian variety $Ab(\Omega)$ associated, namely the quotient of $\mathbb{C}^g$ by the lattice $L(\Omega)$ generated by the columns of the matrix $\left[1_g \Omega\right]$ with the Kähler polarization $\eta = \sum_{i=1}^{g} \mathrm{d}x_i \wedge \mathrm{d}x_{i+g}$. Here x_i are the coordinates on $\mathbb{C}^g$ duals to $L(\Omega)$. Now the product $J(\Omega) = Q^{\check{}} \otimes Ab(\Omega)$ is an abelian variety of dimension gl which is diffeomorphic to T^{2g}. Also the action of the Weyl group W is naturally extended to a diagonal action on $J(\Omega)$. We set for brevity $E = Ab(\Omega)$ and $J = J(\Omega)$.

Lemma 2.1. *The fixed point locus J^W is a finite subgroup of J, naturally isomorphic to $P^{\check{}}/Q^{\check{}} \otimes H_1(E, \mathbb{Z})$.*

Proof. We know that E is isomorphic as a group with $H_1(E, \mathbb{R})/H_1(E, \mathbb{Z})$. Then $z = (z_1, z_2, ..., z_g) \in Q^{\check{}} \otimes H_1(E, \mathbb{R})$ maps to J^W iff

$$z - t_j z = (\langle \alpha_j, z_i \rangle \alpha_j^{\check{}})_{i=1,g} \in Q^{\check{}} \otimes H_1(E, \mathbb{Z})$$

for all $j = 1, l$. Here t_j stands for the reflection of W which sends α_j to $-\alpha_j$. But this is equivalent to $z \in P^{\check{}} \otimes H_1(E, \mathbb{Z})$, hence the lemma. □

This implies that the geometric quotient J/W is a Cohen–Macaulay variety with a finite number of singular points. We may identify therefore $Pic(J/W)$ with $(Pic(J))^W$. We regard the last group as being the set Λ of isomorphism classes of holomorphic line bundle L over J with the property that w^*L and L are isomorphic for all $w \in W$.

Proposition 2.2. *The exact sequence*

$$0 \to Pic^0(J) \xrightarrow{i} Pic(J) \xrightarrow{c} H^2(J, \mathbb{Z})$$

restricts to the following exact sequence:

$$0 \to P/Q \otimes H^1(E, \mathbb{Z}) \xrightarrow{i} \Lambda \xrightarrow{c} (S^2 Q^{\check{}})^W \otimes H^2(E, \mathbb{Z}) \cap H^{1,1}(E).$$

Proof. The proof goes as in the genus 1 case (see [24]): The theorem of Appell–Humbert [26] identifies $Pic^0(J)$ in a natural way with

$$Hom(H_1(J, \mathbb{Z}), \mathbb{R}/\mathbb{Z}) \cong P \otimes H^1(E, \mathbb{R}/\mathbb{Z}).$$

On the other hand, we have canonical isomorphisms

$$H^2(J, \mathbb{C}) \cong \Lambda^2 Hom_{\mathbb{R}}(H_1(J, \mathbb{R}), \mathbb{C}) \cong \Lambda^2 Hom_{\mathbb{R}}(Q^{\check{}} \otimes H_1(E, \mathbb{R}), \mathbb{C})$$

and the last term contains

$$S^2 Q^{\check{}} \otimes \Lambda^2 Hom_{\mathbb{R}}(H_1(E, \mathbb{R}), \mathbb{C}) \cong S^2 Q^{\check{}} \otimes H^2(E, \mathbb{C})$$

as a subspace. Another application of the theorem of Appell–Humbert shows that $c(Pic(J)) = S^2 Q^{\check{}} \otimes H^2(E, \mathbb{Z}) \cap H^{1,1}(E)$. Next an element $z \in P \otimes H^1(E, \mathbb{R})$ projects onto a W-invariant element of $P \otimes H^1(E, \mathbb{R}/\mathbb{Z})$ iff as in the previous lemma $z - t_j z \in P \otimes H^1(E, \mathbb{Z})$ for all $j = 1, l$. Therefore the map

$$z \to \sum_{i=1}^{l} (z - t_i z)$$

induces an isomorphism between $(P \otimes H^1(E, \mathbb{R}/\mathbb{Z}))^W$ and $(P/Q) \otimes H^1(E, \mathbb{Z})$. Since W acts transitively on the set of bases of the root system R and trivially on P/Q this isomorphism is canonical. We have now the exact sequence

$$0 \to P \otimes H^1(E, \mathbb{R}/\mathbb{Z}) \xrightarrow{i} Pic(J) \xrightarrow{c} (S^2 Q^{\check{}}) \otimes H^2(E, \mathbb{Z}) \cap H^{1,1}(E),$$

which will be (non-canonically) split as an exact sequence of W-modules because W is finite. Therefore, by taking the W-invariants the sequence remains exact and we are done. □

Now let $\mathcal{L}$ be a holomorphic line bundle over J whose isomorphism class belongs to Λ and $c(\mathcal{L}) = I \otimes \eta$. This will be the prequantum line bundle which we wanted.

We remark that there is also a natural product action of W^g on J. We may state the following proposition.

Proposition 2.3.
(1) *The prequantum line bundle $\mathcal{L}$ is ample.*
(2) *For any $w \in W^g$ the line bundles $w^*\mathcal{L}$ and $\mathcal{L}$ are isomorphic.*

Proof. We remark that the line bundle $\mathcal{L}$ is well defined modulo a translation in J. Now since I and η are positive definite the Lefschetz theorem on theta functions implies that $\mathcal{L}$ is ample ([15], p.317).

Secondly, we remark that the set Λ_g of isomorphisms classes of line bundles L over J which satisfy the condition stated at the second point may be inserted into an exact sequence similar to that appearing in Proposition 2.2, namely

$$0 \to P/Q \otimes H^1(E,\mathbb{Z}) \xrightarrow{i} \Lambda_g \xrightarrow{c} \bigoplus_{i=1}^{g} (S^2 Q^{\check{}})^W \otimes \mathbb{C}\langle \eta_i \rangle,$$

where η_i is the cohomology class of $dx_i \wedge dx_{i+g}$. The proof is quite similar. Since

$$I \otimes \eta \in \bigoplus_{i=1}^{g} ((S^2 Q^{\check{}})^W \otimes \mathbb{C}\langle \eta_i \rangle)$$

and $\mathcal{L}$ is uniquely defined up to a translation the claim will follow. □

We wish to construct explicitly such a line bundle $\mathcal{L}$. Remember that $E = Ab(\Omega)$. Let $e : Q^{\check{}g} \otimes L(\Omega) \times t_{\mathbb{C}}^g \to \mathbb{C}^*$ be defined by

$$e(u + \Omega v, z) = \exp(\pi i I(2z + \Omega u, v)), \quad \text{where}\, u, v \in Q^{\check{}g}$$

and I is extended to the complexification $t_{\mathbb{C}}^g$. We have an induced action F of $Q^{\check{}g} \otimes L(\Omega)$ on $\mathbb{C} \times t_{\mathbb{C}}^g$ given by

$$F(x)(a, z) = (a/e(x, z), z + x).$$

The orbit space of this action is in a natural way a line bundle which we call $\mathcal{L}$ over $t_{\mathbb{C}}^g / Q^{\check{}g} \otimes L(\Omega) \cong J$. Since I is W-invariant we have $F(wx)(a, wz) = (a, w(z + x))$ for any $w \in W$. Therefore, the action of W on $\mathbb{C} \times t_{\mathbb{C}}^g$ defined by $w(a, z) = (a, wz)$ induces one on $\mathcal{L}$, so $\mathcal{L} \in \Lambda$. Much more I is also W^g-invariant and we see that $\mathcal{L} \in \Lambda_g$. Following the theorem of Appell–Humbert we have $c(\mathcal{L}) = I \otimes \eta$, so that $\mathcal{L}$ is the required prequantum line bundle.

Now the orbit space of the action F_k defined by

$$F_k(x)(a, z) = (ae(x, z)^{-k}, z + x)$$

determines a line bundle over J which is naturally isomorphic to $\mathcal{L}^k$. The sections of $\mathcal{L}^k$ correspond to the level k theta functions on $J(\Omega)$ hence they are holomorphic functions θ on $t_{\mathbb{C}}^g$ satisfying

$$\theta(z + x) = e(x, z)^{-k} \theta(z).$$

We denote by $Th(k, g, R, \Omega) = H^0(J, \mathcal{L}^k)$. This space will support the W^g action coming from the action on $\mathcal{L}$, and in particular the diagonal W-action. The quantization space (in level k) for $J(\Omega)$ will be therefore the space of W-invariant sections $Th(k, g, R, \Omega)^W$.

3. Theta functions and Coxeter groups

The purpose of this section is to define some representations of the symplectic group arising from the study of W-invariant theta functions. Let $\Gamma(1, 2)$ be the so-called theta group consisting of elements $\gamma \in Sp(2g, \mathbb{Z})$ which preserve the orthogonal form

$$Q(n, m) = n^\top \cdot m \in \mathbb{Z}/2\mathbb{Z}.$$

We represent any element $\gamma \in Sp(2g, \mathbb{Z})$ as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A, B, C, D are $g \times g$ matrices. Then $\Gamma(1, 2)$ may be alternatively described as the set of those elements γ having the property that the diagonals of $A^\top C$ and $B^\top D$ are even.

The classical theta function in dimension g is defined by the formula (see [7,18,27]):

$$\theta(z, \Omega) = \sum_{l \in \mathbb{Z}^g} \exp(\pi \mathrm{i} \langle l, \Omega l \rangle + 2\pi \mathrm{i} \langle l, z \rangle)$$

for $z \in \mathbb{C}^g$, $\Omega \in \mathcal{S}_g$, where $\langle , \rangle$ is the usual hermitian product on $\mathbb{C}^g$. There is a natural $Sp(2g, \mathbb{Z})$ action on $\mathbb{C}^g \times \mathcal{S}_g$ given by

$$\gamma \cdot (z, \Omega) = (((C\Omega + D)^{\top^{-1}})z, (A\Omega + B)(C\Omega + D)^{-1}). \tag{1}$$

The behaviour of the theta function under this action is described by the following functional equation (going back to Jacobi):

$$\begin{aligned} &\theta((C\Omega + D)^{\top^{-1}} z, (A\Omega + B)(C\Omega + D)^{-1}) \\ &\quad = \zeta_\gamma det(C\Omega + D)^{1/2} \exp(\pi \langle \mathrm{i}z, (C\Omega + D)^{-1} Cz \rangle) \theta(z, \Omega), \end{aligned} \tag{2}$$

where $\gamma \in \Gamma(1, 2)$, and ζ_γ is an 8th root of unity.

For $g = 1$ we suppose that $c > 0$ or $c = 0$ and $d > 0$ so the imaginary part $\mathrm{Im}(c\Omega + d) \geq 0$ for Ω in the upper half plane. Then we shall choose the square root $(c\Omega + d)^{1/2}$ in the first quadrant. Now we can express the dependence of ζ_γ on γ as follows:

(1) for even c and odd d $\zeta_\gamma = \mathrm{i}^{(d-1)/2}(c/|d|)$,
(2) for odd c and even d $\zeta_\gamma = \exp(-\pi \mathrm{i} c/4)(d/c)$,
where (x/y) is the usual Jacobi symbol [16].

For $g > 1$ we suppose that D is invertible. Firstly, we fix the choice of the square root of $det(C\Omega + D)$ in the following manner: Let $det^{1/2}(Z/\mathrm{i})$ be the unique holomorphic function on $\mathcal{S}_g$ satisfying

$$(det^{1/2}(Z/\mathrm{i}))^2 = det(Z/\mathrm{i})$$

and taking in $\mathrm{i}1_g$ the value 1. Next define

$$det^{1/2}(C\Omega + D) = det^{1/2}(D)det^{1/2}\left(\frac{\Omega}{\mathrm{i}}\right)det^{1/2}\left(\frac{-\Omega^{-1} - D^{-1}C}{\mathrm{i}}\right),$$

where the square root of $det(D)$ is taken to lie in the first quadrant. Using this convention we may express ζ_γ as a Gauss sum:

$$\zeta_\gamma = det^{-1/2}(D) \sum_{l \in \mathbb{Z}^g / D\mathbb{Z}^g} \exp(\pi \mathrm{i} \langle l, BD^{-1}l \rangle)$$

and in particular we recover the formula from above for $g = 1$.

There is also an interesting connection between the multiplier system ζ_γ and the Maslov index for lagrangian subspaces. Let $\mathbb{R}^{2g}$ be endowed with the usual symplectic structure $s = \sum_{i=1}^{g} \mathrm{d}x_i \wedge \mathrm{d}x_{i+g}$, and let l_i, $i = 1, 3$ be lagrangian subspaces of dimension g. We may define a quadratic form on $l_1 \oplus l_2 \oplus l_3$ by

$$B(x_1 + x_2 + x_3) = s(x_1, x_2) + s(x_2, x_3) + s(x_3, x_1)$$

for $x_i \in l_i$, $i = 1, 3$. The signature of this quadratic form is the so-called Maslov index of the triple (l_1, l_2, l_3) and is denoted by $m(l_1, l_2, l_3)$. The failure of the multiplier system ζ_γ to be a homomorphism is expressed via a 2-cocycle. Specifically set $\mu(\gamma_1, \gamma_2) = m(l, \gamma_1 l, \gamma_1\gamma_2 l)$, where l is the lagrangian space $l = \{x_{i+g} = 0, \text{ for } i = 1, g\}$. Therefore we have [23]:

$$\zeta_{\gamma_1\gamma_2} = \exp(-\tfrac{1}{4}\pi \mathrm{i} \mu(\gamma_1, \gamma_2))\zeta_{\gamma_1}\zeta_{\gamma_2}.$$

We come back now to level k theta functions which are defined by

$$\theta_m(z, \Omega) = \sum_{l \in m + k\mathbb{Z}^g} \exp\left(\frac{\pi \mathrm{i}}{k}\langle l, \Omega l \rangle + 2\pi \mathrm{i} \langle l, z \rangle\right) \tag{3}$$

for $m \in (\mathbb{Z}/k\mathbb{Z})^g$. The functional equation above stated is generalized to level k theta functions (see [8,11]) as follows: Let

$$\Theta_k(z, \Omega) = (\theta_m(z, \Omega))_{m \in (\mathbb{Z}/k\mathbb{Z})^g}$$

be the theta vector of level k. Then the following equation is fulfilled:

$$\begin{aligned} \Theta\ (\gamma \cdot (z, \Omega)) = {} & \zeta_\gamma det(C\Omega + D)^{1/2} \\ & \times \exp(k\pi \mathrm{i} \langle z, (C\Omega + D)^{-1}Cz \rangle)\rho_k(\gamma)(\Theta_k(z, \Omega)), \end{aligned} \tag{4}$$

where

(1) γ belongs to the theta group $\Gamma(1, 2)$ if k is odd and to $Sp(2g, \mathbb{Z})$ elsewhere.
(2) $\zeta_\gamma \in R_8$ is the multiplier system from above.
(3) $\rho_k : Sp(2g, \mathbb{Z}) \to U(k^g)$ is a mapping which becomes a group homomorphism (denoted also by ρ_k when no confusion arises) when passing to the quotient $U(k^g)/R_8$ for even k (or equivalently it gives rise to a representation of the central extension of $Sp(2g, \mathbb{Z})$ determined by the 2-cocycle $\exp(-\frac{1}{4}\pi \mathrm{i}\mu(*, *))$; a similar assertion holds for odd k when $Sp(2g, \mathbb{Z})$ is replaced by $\Gamma(1, 2)$.

We computed explicitly ρ_k for a system of generators:

(1) $$\rho_k \begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix} = diag\left(\exp\left(\frac{\pi \mathrm{i}}{k}\langle m, Bm\rangle\right)\right) \tag{5}$$

for $B = B^\top$ a matrix with integer entries.

(2) $$\rho_k \begin{bmatrix} A & 0 \\ 0 & (A^\top)^{-1} \end{bmatrix} = (\delta_{A^\top m,n})_{m,n\in(\mathbb{Z}/k\mathbb{Z})^g} \tag{6}$$

for $A \in GL(g, \mathbb{Z})$

(3) $$\rho_k \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = k^{-g/2}\exp(2\pi \mathrm{i} k^{-1}\langle m, l\rangle)_{m,l\in(\mathbb{Z}/k\mathbb{Z})^g}, \tag{7}$$

where $\langle , \rangle$ is the inner product on $\mathbb{R}^g$.

We wish now that a Coxeter group enter in our picture. Remember that the sections of the prequantum line bundle are theta functions which may be given explicitly. For $\Omega \in \mathcal{S}_g$ and $x \in t_{\mathbb{C}}^g$ we put $\Omega x = (\sum_{j=1}^{g} \Omega_{ij} x_j)_{i=1,g} \in t_{\mathbb{C}}^g$. Consider

$$\theta_\lambda(z, \Omega) = \sum_{x\in Q^g + k^{-1}\lambda} \exp(k\pi \mathrm{i}\langle x, \Omega x\rangle + 2k\pi \mathrm{i}\langle x, z\rangle), \tag{8}$$

where $\lambda \in M^g = (I^{-1}(P))^g$, $z \in t_{\mathbb{C}}^g$. It is clear that $\theta_\lambda(z, \Omega)$ lies in $Th(k, g, R, \Omega)$. We may extract moreover a $\mathbb{C}$-basis of theta functions:

Proposition 3.1. *Consider $X \subset M^g$ be a set of representatives for M^g / kQ^g. Therefore $\{\theta_\lambda(z, \Omega); \lambda \in X\}$ is a $\mathbb{C}$-basis for $Th(k, g, R, \Omega)$.*

The proof is analogous to the classical case (see [20,27,26,24]).

Thus $Th(k, g, R, \Omega)$ are the fibres of a vector bundle, say $Th(k, g, R)$ over the Siegel space $\mathcal{S}_g$. We have moreover a hermitian structure on this bundle given by

$$\langle \theta_\lambda(z, \Omega), \theta_\mu(z, \Omega)\rangle = 2k^{-lg/2} det^{1/2}(\mathrm{Im}(\Omega_R))\delta_{\lambda,\mu},$$

where Ω_R states for the matrix with each $\Omega_{i,j}$ replaced by a block $\Omega_{i,j} 1_l$. Obviously $\Omega_R \in \mathcal{S}_{lg}$ and $J(\Omega) \cong Ab(\Omega_R)$.

We can get this hermitian structure geometrically using the construction of Gocho [14]. Specifically set $j_R : \mathcal{S}_g \to \mathcal{S}_{lg}$ for the holomorphic embedding $j_R(\Omega) = \Omega_R$. Then $Th(k, g, R)$ is a subbundle of the trivial bundle $L^2\Theta$ of L^2-sections:

$$L^2(H^0(Ab(\Omega_R), \mathcal{L}^k)) \times \mathcal{S}_g \to \mathcal{S}_g.$$

Proposition 3.2. *The L^2-metric and the trivial connection on the trivial L^2-bundle induce the above hermitian structure and a projectively flat connection on $Th(k, g, R)$.*

The proof is essentially contained in [14].

Moreover $\{\theta_\lambda(z, \Omega); \lambda \in X\}$ will be a basis of covariant constant sections with respect to the induced connection.

We remark further that the W^g-action on $Th(k, g, R, \Omega)$ takes a particularly simple form, namely

$$w\theta_\lambda(z, \Omega) = \theta_{w\lambda}(z, \Omega),$$

where $w = (w_1, w_2, \ldots, w_g) \in W^g$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_g) \in M^g/kQ^g$ and $w\lambda = (w_1\lambda_1, w_2\lambda_2, \ldots, w_g\lambda_g) \in M^g/kQ^g$. The diagonal action of W is the induced one.

We shall consider now the anti-invariant theta functions with respect to these two actions, namely

$$\psi^-_{\lambda,k}(z, \Omega) = \sum_{w \in W^g} det(w)\theta_{w\lambda}(z, \Omega),$$

where $det(w) = det(w_1)det(w_2)\cdots det(w_g)$ and $det : W \to \{-1, 1\}$ is the usual alternating character of the Coxeter group W, and

$$\varphi^-_{\lambda,k}(z, \Omega) = \sum_{w \in W} det(w)\theta_{w\lambda}(z, \Omega).$$

Set

$$P_k = \{\lambda \in M, \text{such that } 0 < \langle\lambda, \alpha\rangle \leq k, \text{for all positive roots } \alpha\}.$$

We may describe therefore the anti-invariant subspace $Th(k, g, R, \Omega)^{-W^g}$.

Proposition 3.3. *We have:*
(1) $Th(k, g, R, \Omega)^{-W^g} = 0$, *for* $k < h$.
(2) $Th(h, g, R, \Omega)^{-W^g} = \mathbb{C}\langle\psi^-_{r_g,h}(z, \Omega)\rangle$, *where*

$$r_g = \underbrace{(r, r, \ldots, r)}_{g}$$

and r is determined as follows: set $f_j \in t$ such that $I(f_j, \alpha_i^\vee) = \delta_{i,j}$. Then $r = h^{-1}(d_1 + d_2 + \cdots + d_l)$. If G is simply laced then r is the half sum of the positive roots.
(3) $Th(k + h, g, R, \Omega)^{-W^g} = \mathbb{C}\langle\psi^-_{\lambda+r_g,k+h}(z, \Omega); \lambda \in P_k^g\rangle$.

Proof. The W^g-action splits into g copies of independent W-actions, so

$$Th(k, g, R, \Omega)^{-W^g} \cong (Th(k, 1, R, \Omega)^{-W})^{\otimes g}$$

and the case when $g = 1$ is treated in [24] (see also [2,20]). □

Now we can deal with the spaces of W^g-invariant theta functions $Th(k, g, R, \Omega)^{W^g}$ which will be naturally a subspace of the quantization space $Th(k, g, R, \Omega)^W$. We state the following proposition.

Proposition 3.4. *The following theta functions*

$$\psi_{\lambda,k}(z, \Omega) = \psi^-_{\lambda+r_g,k+h}(z, \Omega)/\psi^-_{r_g,h}(z, \Omega)$$

with $\lambda \in P_k^g$ *form a* $\mathbb{C}$*-basis of the space* $Th(k, g, R, \Omega)^{W^g}$ *of invariant theta functions.*

Proof. We have an injective homomorphism

$$Th(k, g, R, \Omega)^{W^g} \to Th(k+h, g, R, \Omega)^{-W^g}$$

given by $\theta \to \theta\psi^-_{r_g,h}$. The inverse of this homomorphism will associate to $\theta \in Th(k + h, g, R, \Omega)^{-W^g}$ the meromorphic theta function $\theta/\psi^-_{r_g,h}$. What remains to be proved is that $\theta/\psi^-_{r_g,h}$ is actually an holomorphic function.

To every root α there is an associated morphism of abelian varieties

$$r_\alpha : J \cong Q^\vee \otimes E \to \mathbb{Z} \otimes E \cong E.$$

Consider Θ the theta divisor on E which passes through zero. Therefore, $c_1(\mathcal{O}(\Theta)) = \eta$. Next we consider the divisor Δ on J defined as $\Delta = \sum_{\alpha\in R^+} r^*_\alpha(\Theta)$, the sum being taken over the positive roots. Then we have the following lemma.

Lemma 3.5. *The divisor* $(\psi^-_{r_g,h})$ *associated to the section* $\psi^-_{r_g,h}$ *is* Δ.

Proof. Observe first that $(\psi_{r_g,h}) \geq \Delta$. Indeed if $z \in r^*_{\alpha_j}\Theta$ then the element

$$w_\alpha = \begin{cases} (t_j, t_j, \ldots, t_j) \in W^g & \text{for odd } g, \\ (t_j, \ldots, t_j, 1) \in W^g & \text{for even } g, \end{cases}$$

leaves the fibre over z fixed so that

$$\psi^-_{r_g,h}(wz, \Omega) = w\psi^-_{r_g,h}(z, \Omega) = -\psi^-_{r_g,h}(z, \Omega)$$

because $\psi^-_{r_g,h}$ is an anti-invariant theta function. Furthermore, $\psi^-_{r_g,h}$ is a section of $\mathcal{L}^h$ hence the Chern class $c_1(\mathcal{O}(\psi^-_{r_g,h})) = hI \otimes \eta$. Next the Chern class of $r_\alpha(\Theta)$ is

$$\alpha \otimes \alpha \otimes \eta \in S^2 Q^\vee \otimes H^2(E),$$

so that

$$c_1(\Delta) = \frac{1}{2}\sum_{j=1}^{l} \alpha_j \otimes \alpha_j \otimes \eta = hI \otimes \eta.$$

Therefore, $(\psi^-_{r_g,h}) - \Delta$ is a non-negative divisor of vanishing Chern class so our claim follows.

Next we remark that the same proof as above will give

$$(\psi^-_{\lambda+r_g,k+h}) \geq \Delta,$$

which implies that $\psi^-_{\lambda+r_g,k+h}/\psi^-_{r_g,h}(z, *)$ is a holomorphic function on z and we are done. □

Now the projectively flat connection ∇ on $Th(k, g, R)$ will induce a projectively flat connection on the subbundle of anti-invariant sections $Th(k, g, R)^{-W^g}$. We identify the vector bundle of invariant sections $Th(k, g, R)^{W^g}$ with $Th(k+h, g, R)^{-W^g} \otimes (Th(h, g, R)^{-W^g})^*$ and we shall derive an induced connection ∇^{W^g} on $Th(k, g, R)^{W^g}$. Our aim is to compute the monodromy of the symplectic action with respect to this connection. We set

$$\Psi_k(z, \Omega) = (\psi_{\lambda,k}(z, \Omega))_{\lambda \in P_k^g}$$

for the (k, W^g)-theta vector.

Theorem 3.6. *The (k, W^g)-theta vector satisfies the functional equation:*

$$\Psi_k(\gamma(z, \Omega)) = \exp(lk\pi \mathrm{i} \langle z, (C\Omega + D)^{-1} Cz \rangle) \rho_k^{W^g}(\gamma) \Psi_k(z, \Omega), \tag{9}$$

where, for even k

$$\rho_k^{W^g} : Sp(2g, \mathbb{Z}) \to U(Th(k, g, R, \Omega)^{W^g})$$

is a representation of the symplectic group given by

(1)
$$\rho_k^{W^g} \begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix} = diag \left(\exp \left(\frac{\pi \mathrm{i}}{k+h} \langle \lambda + r_g, B(\lambda + r_g) \rangle - \frac{\pi \mathrm{i}}{k} \langle r_g, B r_g \rangle \right) \right) \tag{10}$$

for $B = B^\top$ a matrix with integer entries.

(2)
$$\rho_k^{W^g} \begin{bmatrix} A & 0 \\ 0 & (A^\top)^{-1} \end{bmatrix} = (\delta_{A^\top \lambda, \mu})_{\lambda, \mu \in M^g / k Q^{\check{}g} \rtimes W^g} \tag{11}$$

for $A \in GL(g, \mathbb{Z})$.

(3)
$$\rho_k^{W^g} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \mathrm{i}^{gp} (k+h)^{-lg/2} \left(\frac{vol(M)}{vol(Q\check{})} \right)^{g/2} \times \sum_{w \in W^g} det(w) \exp \left(\frac{2\pi \mathrm{i}}{k+h} \langle w(\lambda + r_g), \mu + r_g \rangle \right), \tag{12}$$

where p is the number of positive roots;
where $\langle , \rangle$ is the natural extension of the inner product I on $R^{gl} \cong t^g$.
For odd k the same formulas define a representation of the theta group $\Gamma(1, 2)$.

Proof. We consider first the symplectic action on anti-invariant theta functions

$$\psi^-_{\lambda,k}(\gamma(z, \Omega)) = \sum_{w \in W^g} \theta_{w\lambda}(\gamma(z, \Omega)).$$

But we may write

$$\theta_{w\lambda}(\gamma(z,\Omega)) = \zeta_{\gamma_R} det(C_R + \Omega_R D_R)^{1/2} \exp(lk\pi i\langle z, (C+\Omega D)^{-1}Cz\rangle) \times \sum_{\lambda\in M^g/kQ^{\vee g}} \rho_k(\gamma_R)^{\mu}_{w\lambda}\theta_\mu(z,\Omega),$$

since $\theta_\lambda(z,\Omega)$ are theta functions for $Ab(\Omega_R)$. Because the inner product $\langle,\rangle$ is W^g-invariant it can be checked on the generators that

$$\rho_k(\gamma_R)^{w\mu}_{w\lambda} = \rho_k(\gamma_R)^{\mu}_{\lambda}.$$

It follows

$$\psi^-_{\lambda,k}(\gamma(z,\Omega)) = \zeta_{\gamma_R} det(C_R + \Omega_R D_R)^{1/2} \exp(lk\pi i\langle z, (C+\Omega D)^{-1}Cz\rangle) \times \sum_{\mu\in P^g_{k-h}} \left(\sum_{w\in W^g} det(w)\rho_k(\gamma_R)^{\mu}_{w\lambda}\right) \psi^-_{\mu,k}(z,\Omega).$$

We derive

$$\rho_k^{W^g}(\gamma)^{\mu}_{\lambda} = \left(\sum_{w\in W^g} det(w)\rho_h(\gamma_R)^{r_g}_{wr_g}\right)^{-1} \left(\sum_{w\in W^g} det(w)\rho_{k+h}(\gamma_R)^{\mu+r_g}_{w(\lambda+r_g)}\right).$$

Using the calculations performed in [19,20] for the transformation rules of $\psi^-_{r_g,h}$ we get our claim for the generators considered. Finally, we remark that the map

$$\gamma \to \exp(lk\pi i\langle z, (C+\Omega D)^{-1}Cz\rangle)$$

is a character for $Sp(2g,\mathbb{Z})$ which implies that $\rho_k^{W^g}$ is a group representation and our claim follows. □

Remark 3.7. It is interesting to note that in the non-abelian case $W \neq 1$ the messy factor ζ_γ is cancelled out. This comes from the fact that the connection ∇^{W^g} is actually flat not only projectively flat.

We come back now to the invariant theta functions arising from the diagonal W-action. This time we do not have such an explicit description for the space $Th(k,g,R,\Omega)^W$. However, we can state the following proposition.

Proposition 3.8.

(1) *Consider* $B^0_{k,g}$ *be a set of representatives for* $M^g/kQ^{\vee G} \rtimes W$. *Set* $B_{k,g} \subset B^0_{k,g}$ *be the subset of those* λ *having an even isotropy group* $Stab(\lambda) = \{w \in W; w\lambda = \lambda\}$ *(i.e. the character det on* $Stab(\lambda)$ *is identically one for* $\lambda \in B_{k,g}$*). Therefore, we have*

$$Th(k,g,R,\Omega)^{-W} = \mathbb{C}\langle\varphi^-_{\lambda,k}(z,\Omega); \lambda \in B_{k,g}\rangle.$$

(2) *The W-invariant theta functions*

$$\{\varphi_{\lambda-r_g,k}(z,\Omega)=\varphi^-_{\lambda,k+h}(z,\Omega)/\varphi_{r_g,h}(z,\Omega)\},\quad \text{with } \lambda\in B_{k+h,g}$$

form a $\mathbb{C}$*-basis for the space* $Th(k,g,R,\Omega)^W$.

Proof. It is clear that $\varphi^-_{\lambda,k}$ are W-anti-invariant. These theta functions will generate the space $Th(k,g,R,\Omega)$ from the general theory of invariants of finite group actions. It remains to prove the linear independence. We make first a little digression on formal theta functions (see [24]). Let F denote the lattice of affine linear functions on $V=t^g_{\mathbb{C}}$ which takes integral values on $Q^{\vee g}$ and let $e(F)$ denote the subgroup of $\mathbb{Z}^F$ whose elements are of the form

$$\xi=\sum_{f(\Omega r_g)\geq n} c_f e(f)$$

for some real number n. Here $e(f)$ stands for the element of $\mathbb{Z}^F$ which is one on f and zero on $F-\{f\}$. The order of ξ is $o(\xi)=\inf\{f(\Omega r_g);\, c_f\neq 0\}$ and the initial part of ξ is by definition

$$in(\xi)=\sum_{f(\Omega r_g)=o(\xi)} c_f e(f).$$

Now V acts on F by translation and hence on $\mathbb{Z}^F$. We call $\xi\in e(F)$ a formal theta function of level k if for any $v\in Q^{\vee g}\otimes L(\Omega)$ we have

$$(u+\Omega v)^*\xi=e(-kI(v)-\tfrac{1}{2}I(\Omega v,v))\xi.$$

The set of theta functions of level k will be denoted by Th^k. Any element of Th^0 has the form

$$\sum_{n\geq n_0} c_n e(n),\quad \text{with } n,n_0\in\mathbb{Z},$$

where $e(n)$ is the constant function n. We put for any $\lambda\in M^g$

$$\theta_\lambda=\sum_{\mu\in k^{-1}\lambda+Q^{\vee g}} e(-kI(v)+\tfrac{1}{2}k(I(\Omega v,v)-I(\Omega\lambda,\lambda))).$$

It follows that $\{\theta_\lambda;\,\lambda\in S\}$, for S a system of representatives for $M^g/kQ^{\vee g}$ is a Th^0-basis for Th^k. Next we take into account the diagonal W-action which is given by

$$w\theta_\lambda=\theta_{w\lambda}.$$

Define the anti-invariant (formal) theta functions by

$$\theta^-_\lambda=\sum_{w\in W} det(w)\theta_{w\lambda}.$$

To any $\lambda\in B_{k,g}$ we associate some $\tilde{\lambda}\in(Q^{\vee g}\bowtie W)\lambda$ with the property that the (convex) function $I(\Omega(x-r_g),x-r_g)$ for $x\in(Q^{\vee g}\bowtie W)\lambda$ has a minimum in $x=\tilde{\lambda}$. Therefore, it will follow that, for real and positive definite Ω

$$in(\theta^-_{\tilde{\lambda}})=card(Stab(\lambda))e(-kI(\tilde{\lambda})).$$

Indeed we have for $w \in Q^{\vee g} \bowtie W$, and $m = \tilde{\lambda}$ the following relations:

$$\begin{aligned}&-kI(wm, \Omega r_g) + \tfrac{1}{2}k(I(\Omega wm, wm) - I(\Omega m, m))\\&= \tfrac{1}{2}k(I(\Omega(wm - r_g), wm - r_g) - I(\Omega(m - r_g), m - r_g)) - kI(m, \Omega r_g)\\&\geq -kI(m, \Omega r_g).\end{aligned}$$

But now for generic Ω the convex function $I(\Omega(x - r_g), x - r_g)$ has exactly one minimum on the orbit of λ under the affine Weyl group. Therefore equality can hold before only if $wm = m$. If $\lambda \in B_{k,g}$ then our claim follows. Otherwise there exists some $w \in Stab(\lambda)$ with $det(w) = -1$. Then

$$\theta_{\lambda}^{-} = -\theta_{w\lambda}^{-} = -\theta_{\lambda}^{-},$$

hence $\theta_{\lambda}^{-} = 0$.

Now we remark that the initial parts we obtained $in(\theta^{-}\tilde{\lambda})$ will be linear independent over Th^0 since the family $e(-kI(\lambda))$ fulfills this property. This will prove the linear independence of the corresponding family of formal theta functions. The same proof will work if we take $e(\mathrm{i}f)$ in the place of $e(f)$ and $\mathrm{i}\Omega$ in place of Ω. But if we replace $e(f)$ by $\exp(2\pi \mathrm{i} f)$ and Ω by $\mathrm{i}\Omega$ we derive some multiples of the usual theta functions. Therefore for generic and purely imaginary $\Omega \in \mathcal{S}_g$ the usual anti-invariant theta functions which we considered will be linear independent over $\mathbb{C}$. Since the independence is an open condition this will be true for Ω in a Zariski open subset of $\mathcal{S}_g$. Since $Th(k, g, R, \Omega)^{-W} \subset Th(k, g, R, \Omega)$ and the second family of spaces is a vector bundle endowed with a W-invariant projectively flat connection we obtain that the dimension of $Th(k, g, R, \Omega)$ is constant. This will prove our first claim.

We consider first the case of odd g. Then $\psi_{r_g,h}^{-}(z, \Omega)$ is a W-anti-invariant theta function. Then for any $k \geq 0$ we have

$$(\varphi_{\lambda,k+h}^{-}) \geq (\Delta)$$

as in the proof of Lemma 3.5. It will follow that

$$\{\varphi_{\lambda,k+h}^{-}(z, \Omega)/\psi_{r_g,h}^{-}(z, \Omega);\ \lambda \in B_{k,g}\}$$

is a basis for $Th(k, g, R, \Omega)^W$. The proof is similar. We may consider the induced $Sp(2g, \mathbb{Z})$-action on the associated vector bundle $Th(k, g, R)^W$. Essentially, the same computation as in Theorem 3.6 (remark that $\mathbb{C}\langle\psi_{r_g,h}^{-}\rangle$ is $Sp(2g, \mathbb{Z})$-invariant !) will give that

$$\gamma(\varphi_{r_g,k+h}^{-}/\psi_{r_g,h}^{-}) = \chi(\gamma)\varphi_{r_g,k+h}^{-}/\psi_{r_g,h}^{-},$$

where $\gamma \in Sp^{+}(2g, \mathbb{Z})$ and χ is a character for $Sp^{+}(2g, \mathbb{Z})$. Moreover, this vector is the only (projectively) invariant vector of $Sp^{+}(2g, \mathbb{Z})$. On the other hand, $\psi_{0_g,k+h} \in Th(k, g, R, \Omega)^W$ and has the same property. We derive that

$$\varphi_{r_g,k+h}^{-}(z, \Omega) = s(\Omega)\psi_{r_g,k+h}^{-}(z, \Omega),$$

where $s : \mathcal{S}_g \longrightarrow \mathbb{C}$ is a holomorphic $Sp(2g, \mathbb{Z})$-invariant function. This will prove the claim in case of odd g.

Further, we have

$$\varphi^-_{r_g,h}(z,\Omega)=\varphi^-_{r_{g+1},h}((z,0),(\Omega\oplus \mathrm{i}1))$$

so $\varphi^-_{r_g,h}(z,\Omega)$ is $Sp(2g,\mathbb{Z})$-invariant also for even g. Next the proof proceeds as in Theorem 3.6 and we are done. □

Denote now by $\Phi_k(z,\Omega)=(\varphi_{\lambda,k}(z,\Omega))_{\lambda\in B_{k,g}}$ the (k,W)-theta vector. Then we may compute the monodromy of the symplectic action actually using the connection ∇^W on the vector bundle $Th(k,g,R)^W$ which comes from its identification with $Th(k+h,g,R)^{-W}\otimes \mathbb{C}\langle\varphi^-_{r_g,h}\rangle^*$.

Theorem 3.9. *The (k,W)-theta vector satisfies the functional equation*

$$\Phi_k(\gamma(z,\Omega))=\exp(lk\pi\mathrm{i}\langle z,(C\Omega+D)^{-1}Cz\rangle)\rho_k^W(\gamma)\Phi_k(z,\Omega),$$

where for even k

$$\rho_k^W : Sp(2g,\mathbb{Z})\to U(Th(k,g,R,\Omega)^W)$$

is a group representation determined by

(1)
$$\rho_k^W\begin{bmatrix}1 & B\\ 0 & 1\end{bmatrix}=diag\left(\exp\left(\frac{\pi\mathrm{i}}{k+h}\langle\lambda+r_g,B(\lambda+r_g)\rangle-\frac{\pi\mathrm{i}}{k}\langle r_g,Br_g\rangle\right)\right)\tag{13}$$

for $B=B^\top$ a matrix with integer entries.

(2)
$$\rho_k^W\begin{bmatrix}A & 0\\ 0 & (A^\top)^{-1}\end{bmatrix}=(\delta_{A^\top\lambda,\mu})_{\lambda,\mu\in B_{k+h,g}}\tag{14}$$

for $A\in GL(g,\mathbb{Z})$.

(3)
$$\rho_k^W\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}=\mathrm{i}^{gp}(k+h)^{-lg/2}\left(\frac{vol(M)}{vol(Q^\vee)}\right)^{g/2}\times\sum_{w\in W}det(w)\exp\left(\frac{2\pi\mathrm{i}}{k+h}\langle w(\lambda+r_g),\mu+r_g\rangle\right),\tag{15}$$

where p is the number of positive roots.

For odd k the same formulas define a representation of the theta group $\Gamma(1,2)$.

Proof. Since $\varphi^-_{r_g,h}$ is $Sp(2g,\mathbb{Z})$-invariant the proof goes as in the previous theorem. □

We remark that the natural map induced by $A\in Gl(g,\mathbb{Z})$:

$$A: M^g/Q^{\vee g}\rtimes W\to M^g/Q^{\vee g}\rtimes W$$

maps $B_{k,g}$ onto itself so that formula 2 makes sense.

4. Invariants for framed 3-manifolds

We wish to define some invariants for closed orientable 3-manifolds using the method of [8] for the representations ρ_k^W.

We start with the $\rho_k^{W^g}$ which parallels the $W = 1$ case. We identify $Th(k, g_1+g_2, R, \Omega_1 \oplus \Omega_2)^{W^g}$ with $Th(k, g_1, R, \Omega_1)^{W^{g_1}} \otimes Th(k, g_2, R, \Omega_2)^{W^{g_2}}$ via the map

$$\psi_{\lambda_1,k} \otimes \psi_{\lambda_2,k} \to \psi_{(\lambda_1,\lambda_2),k}.$$

Set $c_k = k\langle r, r\rangle / h(k+h)$ for the central charge in level k, and $\zeta_k = \exp(2\pi i c_k)$. We define the symplectic sum of two matrices

$$\oplus_c : Sp(2g, \mathbb{Z}) \times Sp(2h, \mathbb{Z}) \to Sp(2(g+h), \mathbb{Z})$$

by the formula

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \otimes \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} A \oplus A' & B \oplus B' \\ C \oplus C' & D \oplus D' \end{bmatrix}.$$

Therefore we can state the following proposition.

Proposition 4.1.

(1) *The representation $\rho_k^{W^g}$ is a tensor representation, i.e.*

$$\rho_k^{W^g}(\gamma_1 \oplus_c \gamma_2) = \rho_k^{W^g}(\gamma_1) \otimes \rho_k^{W^g}(\gamma_2)$$

holds.

(2) *If $Sp^+(2g, \mathbb{Z})$ denotes the subgroup of symplectic matrices of the form $\left[\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right]$ then $\phi_{0_g,k}$ is a projective weight vector for $Sp^+(2g, \mathbb{Z})$, i.e.*

$$\rho_k^{W^g}(\gamma)\psi_{0_g,k} = \chi(\gamma)\psi_{0_g,k}$$

for $\gamma \in Sp^+(2g, \mathbb{Z})$, where $\chi : Sp^+(2g, \mathbb{Z}) \to U_W$ is a character taking values in the group of roots of unity generated by ζ_k. This character is determined by

$$\chi\left(\begin{bmatrix} A & 0 \\ 0 & (A^\top)^{-1} \end{bmatrix}\right) = 1, \qquad \chi\left(\begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix}\right) = (\zeta_k)^{\left(\sum_{i,j} B_{i,j}\right)}.$$

Proof. It is known that ρ_k is tensorial (see [8]). Now the W^g-action being split we can pass to the W^g-anti-invariant part and we are done. Otherwise this property can be checked directly on the generators. The second part is a corollary of Theorem 3.6. □

So we obtained a tensor representation of $(Sp(2g, \mathbb{Z}), Sp^+(2g, \mathbb{Z}))$ in the terminology of [8,11]. Now there is a standard way to derive invariants for closed 3-manifolds: Let M^3 be a closed orientable 3-manifold and $M^3 = T_g \cup \bar{T}_g$ be a Heegaard splitting into two handlebodies of genus g. The gluing homeomorphism induces an automorphism in homology $H_1(\partial T_g)$ which we may identify with an element $h(M) \in Sp(2g, \mathbb{Z})$. This identification corresponds to the choice of a canonical basis in the homology of a genus g surface. We set

$$I_W(M^3, k) = (k+h)^{-lg/2} \langle \rho_k^{W_g}(h(M^3)) \psi_{0_g,k}, \psi_{0_g,k} \rangle .$$

We have then the following proposition.

Proposition 4.2.

(1) *The class of equivalence $I_W(M^3, k) \in \mathbb{C}/U_W$ does not depend upon the various choices made and defines therefore a topological invariant of M^3.*

(2) *The invariant $I_W(*, k)$ behaves multiplicatively under connected sums.*

Proof. The proof is standard (see also [8]): the ambiguities in the choices of $h(M^3)$ come from the non-uniqueness of a canonical basis in homology and that of the Heegaard splitting. But choosing another canonical basis in the homology $h(M^3)$ changes into $ch(M^3)d$ with $c, d \in Sp^+(2g, \mathbb{Z})$. Since $\psi_{0_g,k}$ is a projective weight vector the invariant I_W is not affected. Also by stabilizing an Heegaard splitting changes $h(M^3)$ into $h(M^3) \oplus \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and I_W takes the same value. But any two Heegaard splittings are stably equivalent by Reidemester–Singer theorem and our claim follows. □

We wish now to pass to the representation ρ_k^W. The only point here is that $Th(k, g_1, R, \Omega)^W \otimes Th(k, g_2, R, \Omega)^W$ is a proper subspace of $Th(k, g_1+g_2, R\Omega)$. Also there is no canonical inclusion mapping

$$Th(k, g_1, R, \Omega)^W \otimes Th(k, g_2, R, \Omega)^W \to Th(k, g_1+g_2, R, \Omega)$$

as for the usual tensor structures (see [9,13]) but a surjective mapping:

$$\pi : Th(k, g_1+g_2, R\Omega) \to Th(k, g_1, R, \Omega)^W \otimes Th(k, g_2, R, \Omega)^W ,$$

which is defined as follows: we take

$$\pi_i : M^{g_1+g_2}/kQ^{\vee g_1+g_2} \bowtie W \to M^{g_i}/kQ^{\vee g_i} \bowtie W$$

be the canonical projections and set also π_i for the induced maps

$$\tilde{\pi}_i : B_{k,g_1+g_2} \to B_{k,g_i} \cup \{\phi\},$$

which are given by

$$\tilde{\pi}_i(x) = \begin{cases} \pi_i(x) & \text{if } \pi_i(x) \in B_{k,g_i}, \\ \phi & \text{otherwise.} \end{cases}$$

We put formally $\theta_\phi = 0$. Therefore the mapping π is given by

$$\pi(\varphi_\lambda) = \varphi_{\pi_1(\lambda)} \otimes \varphi_{\pi_2(\lambda)} .$$

Furthermore, we have

$$\theta_{(\lambda_1,\lambda_2)}((z_1, z_2), \Omega_1 \oplus \Omega_2) = \theta_{\lambda_1}(z_1, \Omega_1) \otimes \theta_{\lambda_2}(z_2, \Omega_2).$$

If ρ denotes the symplectic action on $Th^-(k, g, R, \Omega)$ then it follows that

$$\rho(\gamma_1 \oplus_s \gamma_2)\varphi_\lambda^- = \sum_{\mu_1=\pi_1(\mu),\mu_2=\pi_2(\mu)} \rho(\gamma_1)_{\pi_1(\lambda)}^{\pi_1(\mu)} \rho(\gamma_2)_{\pi_2(\lambda)}^{\pi_2(\mu)} \varphi_\mu^-,$$

where $\oplus_s$ denotes the symplectic direct sum of matrices and the coefficients of the matrices on the right-hand side are zero if some index is ϕ. This implies that

$$\begin{aligned}&\langle \rho_k^W(\gamma_1 \oplus \gamma_2)\varphi_{k,\lambda}, \varphi_{k,\mu}\rangle \\ &\quad = \langle \rho_k^W(\gamma_1)\varphi_{\pi_1(\lambda)}, \varphi_{k,\pi_1(\mu)}\rangle \langle \rho_k^W(\gamma_2)\varphi_{k,\pi_2(\lambda)}, \varphi_{k,\pi_1(\mu)}\rangle.\end{aligned}$$

Then if we define

$$I'_W(M^3, k) = (k+h)^{-lg/2} \langle \rho_k^W(h(M^3))\varphi_{0_g,k}, \varphi_{0_g,k}\rangle$$

it will follow that I'_W is a topological invariant as above. Since $\varphi_{0_g,k}$ is the only one projective weight vector associated to the character χ and $Th(k, g, R, \Omega)^{W^g}$ is a $Sp(2g, \mathbb{Z})$-submodule of $Th(k, g, R, \Omega)^W$ we find that in fact

$$I'_W(M^3, k) = I_W(M^3, k),$$

so nothing new appears. This is a particular case of the following more general principle which is used in [9,13] for mapping class groups: if we have a tensor representation of $Sp(2g, \mathbb{Z})$ in the unitary automorphisms of the hermitian vector space V_g which define topological invariants for 3-manifolds then we may restrict to the subrepresentations on $V' = Span(Sp(2g, \mathbb{Z})v_g)$ where v_g is the projective $Sp^+(2g, \mathbb{Z})$ weight vector. This implies that we may restrict ourselves to the full symplectic submodule, i.e. of type $V_g = V_1^{\otimes g}$.

We want now to remove the ambiguity U_W in the definition of our invariants. This will be done by adding some structure on the manifold M^3, namely a framing. For technical reasons we shall consider a p_1-structure on M^3 (see [6]) which is a notion equivalent to Atiyah's [1] 2-framings. Let X denote the homotopy fibre of the map $p_1 : BO \to K(\mathbb{Z}, 4)$ corresponding to the first Pontryaghin class of the tautological bundle τ of BO. Then a p_1-structure on a manifold M is fibre map from τ_M the stable tangent bundle of M^3 to $p_1^*\tau$ the pull-back of τ over X. Actually, we shall consider only homotopy classes of p_1-structures. If M^3 is an oriented closed 3-manifold then M^3 bounds a 4-manifold Y. If α is a p_1-structure on M^3 then let $p_1(Y, \alpha) \in H^4(Y, M, \mathbb{Z})$ denote the obstruction to extending it to Y. Set

$$\sigma(\alpha) = 3signature(Y) - \langle p_1(Y, \alpha), [Y]\rangle \in \mathbb{Z},$$

which does not depend on Y according to Hirzebruch's signature theorem and is equal to 3 times Atiyah's σ. It is known that the set of homotopy classes of p_1-structures on M^3 is affine isomorphic to $\mathbb{Z}$, the isomorphism being given by σ. A similar statement holds for the set of homotopy classes of p_1-structures on an oriented, compact, connected 3-manifold with boundary which restrict to a given p_1-structure on the boundary. We shall be concerned only with homotopy classes of p_1-structures below. The canonical p_1-structure on M^3 is that on which σ vanishes.

We come back to our representation $\rho_k^{W^g}$. The ambiguity comes from the fact that $\psi_{0_g,k}$ is only a projective weight vector for $Sp^+(2g,\mathbb{Z})$. Now we consider the central extension of $Sp(2g,\mathbb{Z})$ corresponding to the 2-cocycle signature (or cocycle de Meyer [3]) $c : Sp(2g,\mathbb{Z}) \times Sp(2g,\mathbb{Z}) \to \mathbb{Z}$. This may be constructed as follows [1]. Let Γ_g be the mapping class group of genus g surfaces and set $\tilde{\Gamma}_g$ for the set of isomorphism classes of fibrations $Y \to S^1$ with fibre a surface of genus g, which are endowed with a p_1-structure. There is a natural group law on $\tilde{\Gamma}_g$. For $f, g \in \Gamma_g$ we construct a 4-manifold T which is fibred (with fibre the genus g surface) over the pants $D^2 - D_1^2 - D_2^2$ and has the monodromies fg, f, g on the circles $\partial D^2, \partial D_1^2, \partial D_2^2$ respectively. Set X_f for the boundary component which fibres over ∂D_1^2.Given two p_1-structures α, β on X_f, X_g respectively, then there is a unique p_1-structure γ on X_{fg} which extends the p_1-structure on boundary to T. Since $\tilde{\Gamma}_g$ is essentially the set of pairs (f,α) with α a p_1-structure on X_f we may define the group law on $\tilde{\Gamma}_g$ by

$$(f,\alpha)(g,\beta) = (fg,\gamma).$$

We obtain this way a central extension of Γ_g

$$0 \to \mathbb{Z} \to \tilde{\Gamma}_g \to \Gamma_g \to 0$$

with a canonical section s given by

$$s(f) = (f,\alpha), \quad \text{where } \sigma(\alpha) = 0.$$

The canonical 2-cocycle for this extension will be therefore

$$c(f,g) = s(f)s(g)s(fg)^{-1} = signature(T).$$

Now the cohomology of T depends only on the elements f_*, g_* in $Sp(2g,\mathbb{Z})$ induced by the action of f, g in the homology of the fibre (see [1,25]) therefore we have an induced central extension of the symplectic group

$$0 \to \mathbb{Z} \to \widetilde{Sp}(2g,\mathbb{Z}) \to Sp(2g,\mathbb{Z}) \to 0,$$

which is also endowed with a canonical section denoted also by s. Then Meyer's function $\Phi : \widetilde{Sp}(2g,\mathbb{Z}) \to \mathbb{Z}$ which lifts to $\widetilde{Sp}(2g,R)$ is the quasi-morphism defined by the equation $c(f,g) = \Phi(s(fg)) - \Phi(s(f)) - \Phi(s(g))$. There exists exactly one quasi-morphism on $\widetilde{Sp}(2g,\mathbb{Z})$ which satisfies the previous relation (see [3]). We shall consider now the associated homogeneous quasi-morphism, namely

$$\Psi(f) = \lim_{n\to\infty} \frac{1}{n}\Phi((s(f))^n)$$

on $Sp(2g,\mathbb{Z})$ and also

$$\Psi(f) = \lim_{n\to\infty} \frac{1}{n}\Phi((f)^n)$$

for $f \in \widetilde{Sp}(2g,\mathbb{Z})$. Then set

$$\rho_{k,W} : \widetilde{Sp}(2g,\mathbb{Z}) \to U(Th(k,g,R,\Omega)^{W^g})$$

defined by

$$\rho_{k,W}(s(f)+m) = (\zeta_k)^{(\Psi(f)+m)}\rho_k^{W^g}(f).$$

Here $m \in \mathbb{Z}$ makes sense since we may alter a p_1-structure with an integer. It is clear that $\rho_{k,W}$ is a projective representation of $\widetilde{Sp}(2g,\mathbb{Z})$. Consider now an oriented closed 3-manifold M^3 presented by a Heegaard splitting $M^3 = T_g \cup \bar{T}_g$ with gluing homeomorphism $h(M^3)$. Set $h_*(M)$ for the corresponding element of $Sp(2g,\mathbb{Z})$. Suppose that a p_1-structure α is chosen on M^3. Then α differs from the canonical p_1-structure by an integer m. We define

$$Z_W((M^3,\alpha),k) = \langle \rho_{k,W}(s(h_*(M))+m)\psi_{0_g,k}, \psi_{0_g,k}\rangle.$$

Our main result is the following theorem.

Theorem 4.3. *The complex number $Z_W(*,k)$ is a topological invariant for closed 3-manifolds with p_1-structure which behaves multiplicatively under connected sums and pass to the conjugate when the orientation is changed. If the p_1-structure is altered by an integer m then the invariant is multiplied by ζ_k^m.*

Proof. We remark that it is sufficient to prove the following lemma.

Lemma 4.4. *We have:*
(1) *The 2-cocycle $\tilde{c}$ associated to Ψ satisfies*

$$\tilde{c}(\gamma_1,\gamma_2) = 0 \quad if\ \gamma_1 \in Sp^+(2g,\mathbb{Z}).$$

(2) $\chi(f) = \exp(2\pi i c_k \Psi(f))$ *if* $f \in Sp^+(2g,\mathbb{Z})$.

In fact $\tilde{c}$ could be obtained as follows:

$$\tilde{c}(f_1,f_2) = \lim_{n\to\infty}\frac{1}{n}\mu(f_1^n, f_2^n),$$

where μ is the Maslov 2-cocycle from the 3rd paragraph (see [3]). Therefore the first claim follows since the Maslov cocycle verifies the required relation.

In particular Ψ is a character on $Sp^+(2g,\mathbb{Z})$. Also Ψ is constant on conjugation classes hence

$$\Psi\left(\begin{bmatrix} A & 0 \\ 0 & (A^\top)^{-1}\end{bmatrix}\right) = 0.$$

On the other hand,

$$\Psi(ab) = \Psi(a) + \Psi(b)$$

if a and b commute according to [3]. Thus it remains to compute

$$\Psi\left(\begin{bmatrix} 1 & B \\ 0 & 1\end{bmatrix}\right)$$

in the case when B has only one non-zero entry. But

$$\Psi\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = 1$$

and Ψ is constant under direct sum with identity. Every concerned element is conjugate to a stabilization of $\left[\begin{smallmatrix} 1 & B \\ 0 & 1 \end{smallmatrix}\right]$ and therefore

$$\Psi\left(\begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix}\right) = \sum_{i,j} B_{ij}.$$

This proves our lemma. □

Since Ψ takes integer values on $Sp(2g, \mathbb{Z})$ the claim of the theorem follows.

We can do something also in the case when $G = U(1)$ (hence $W = 1$) by taking into account the spin structures. Let us consider that M^3 has a spin structure α. Then the Heegaard splitting will be one in the context of spin manifolds. But the spin structure on the surface ∂T_g induces a quadratic form

$$q_\alpha : H_1(\partial T_g, \mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$$

defined as follows. Let $x \in H_1(\partial T_g, \mathbb{Z})$ and x' be a circle representing the homology class x. If the spin structure induced on x' is the bounding spin structure of the circle then we set $q_\alpha(x) = 0$ otherwise q_α equals one (see [21]). Now the gluing homeomorphism $h(M)$ will be compatible with q_α so $h_*(M)$ may be identified with an element of $\Gamma(1, 2)$. It follows that $Z_W(M, \alpha) \in \mathbb{C}$ is well defined.

Remark 4.5. If we should use the quantization procedure for the space $Hom(\pi_1(\Sigma_g), T)/T$ (which amounts to consider a space of sections of a line bundle directly over T^{2g}) we should obtain in a very similar manner some numerical invariant I_T.

Now even if our starting point was a simple Lie group G, all computations may be carried out for a semi-simple Lie group. Furthermore, the invariant associated to the Lie group $G \times H$ is nothing but the product of the two invariants associated to G and H. In particular, I_T is a power of the abelian invariant for $W = 1$ considered in [8], hence a homotopical invariant. However for general W the invariants Z_W are no more homotopical invariants from the computations carried by Jeffrey [19] in the case of lens spaces.

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