On the connected components of the space of codimension one foliations on a closed 3-manifold

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Abstract

In 1969, J. Wood showed that any plane field on a closed 3-manifold can be deformed into the tangent plane field to a foliation. In these notes, we outline the proof of a one-parameter version of this statement: if two $\mathcal{C}^\infty$ foliations have homotopic tangent plane fields, they can be connected by a path of $\mathcal{C}^1$ foliations. Under some restrictions on the holonomy of the initial foliations, one can actually remain in the $\mathcal{C}^\infty$ class. The full proof is to appear in an upcoming paper [Ey3] and is part of the author's PhD dissertation [Ey1].

Introduction

A *codimension one foliation* on a closed 3-manifold M is a partition of M into connected subsets, called the *leaves*, such that every point of M has a neighbourhood U with coordinates $(x, y, z)\colon U \to \mathbb{R}^3$ in which every connected component of $F \cap U$, for every leaf F, is defined by $z =$ constant. In other words, a foliation is a partition of M into immersed surfaces which locally pile up nicely like a family of parallel affine planes. Until further notice, everything – manifolds, coordinates, etc. – is assumed $\mathcal{C}^\infty$.

An important example is the Reeb foliation on the solid torus, which can be visualized by rotating the leftmost picture on Fig. 1 around a vertical axis and quotienting out by the unit vertical translation. It has only one compact leaf, the boundary torus, on which all the other leaves accumulate. When this foliation appears in a larger foliated manifold, it is called a *Reeb component*.

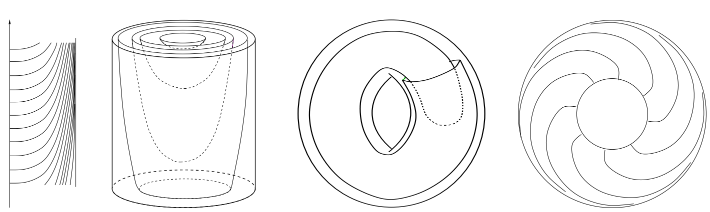

Figure 1: Reeb foliation of the solid torus

Whether any closed 3-manifold admits a codimension one foliation is not a priori obvious. It turns out however that they all do, according to Lickorish [Li] and Novikov and Zieschang [No] (independently). The next natural step is to try and classify these objects: how many "different" foliations are there on a given manifold M?

One (rather unexplored) way of tackling this question is to put a nice topology on the set $\mathcal{F}(M)$ of foliations on M and to try and describe the topological properties of this space. To that end, a key observation is that foliations can be viewed as "a special kind of plane fields". Indeed, each leaf has, at each point, a tangent plane, and the collection of those planes defines the so-called *tangent plane field to the foliation*, which determines the foliation completely. Thus, one can consider a foliation and its tangent plane field to be the same object, and endow $\mathcal{F}(M)$ with a natural topology, inherited from the space $\mathcal{P}(M)$ of plane fields on M.

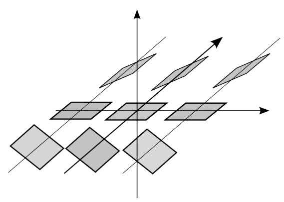

Figure 2: Nonintegrable plane field on $\mathbb{R}^3$

Note that not every plane field is tangent to a foliation. The ones that are are called *integrable*. The others actually form a dense open subset of $\mathcal{P}(M)$. The topology of the closed subset $\mathcal{F}(M)$ is therefore likely to be quite complicated. As a matter of fact, little is known about this topology, apart from the following fundamental theorem, proved first by J. Wood [Wo], and then by W. Thurston [Th2], whose original techniques underlie the work presented in these notes.

Theorem (Wood [Wo])**.** *Every plane field on a closed* 3*-manifold M is homotopic to an integrable plane field.*

In other words, every connected component of $\mathcal{P}(M)$ contains at least one component of $\mathcal{F}(M)$: the map $\pi_0\mathcal{F}(M) \to \pi_0\mathcal{P}(M)$ induced by the inclusion $\mathcal{F}(M) \overset{\iota}{\hookrightarrow} \mathcal{P}(M)$ is surjective. It is tempting to ask whether this inclusion is actually a homotopy equivalence – or equivalently if *foliations satisfy Gromov's h-principle* – all the more as this kind of result exists in the close field of 3-dimensional contact geometry: Y. Eliashberg [El] has generalized Thurston's ideas [Th2] to prove a similar equivalence for overtwisted contact structures (another subspace of $\mathcal{P}(M)$).

In these notes, we focus on the injectivity of the map between π_0's, though similar arguments give the surjectivity of the maps between π_k's, $k \geq 1$ (see [Ey3]). The concrete question we have to tackle is:

If two foliations have homotopic tangent plane fields, are they connected by a path of foliations?

The aim of these notes is to present the strategy leading to the following partial answer:

Theorem A. *Let M be a closed* 3*-manifold. Two transversely oriented $\mathcal{C}^\infty$ foliations with homotopic tangent plane fields can be connected by a continuous path of $\mathcal{C}^1$ foliations.*

The complete proof lies in the author's PhD dissertation [Ey1] and will be transcribed and improved in an upcoming paper [Ey3].

Let us make a few remarks about this statement. First of all, by $\mathcal{C}^1$ foliations, we mean $\mathcal{C}^1$ integrable plane fields, which is slightly stronger than the usual terminology but makes the notion of "continuous path" much simpler. Integrability has a meaning in any regularity class: for all $r \geq 1$, a $\mathcal{C}^r$ plane field is said to be *integrable* if every point of the manifold belongs to a $\mathcal{C}^r$ surface which is everywhere tangent to this plane field. Just like $\mathcal{F}(M)(= \mathcal{F}^\infty(M))$, the space $\mathcal{F}^1(M)$ of $\mathcal{C}^1$ foliations inherits a natural topology from the space $\mathcal{P}^1(M)$ of $\mathcal{C}^1$ plane fields on M.

As a matter of fact, the foliations we build are $\mathcal{C}^\infty$ outside finitely many regions of the form $\mathbb{T}^2 \times [0,1]$, on which they are transverse to the $[0,1]$ factor, and they can even be made $\mathcal{C}^\infty$ everywhere if we put some restriction on the holonomy of the initial foliations (cf. Theorem 7).

Let us note that a positive answer to the above question had already been obtained by A. Larcanché [La] in two specific cases: when M is a circle bundle and the foliations under scrutiny are transverse to the fibres, and, for a general M, when both foliations are *taut* and sufficiently close to each other (we will recall later what a taut foliation is). Her main tool plays a key role in the proof of Theorem A.

Let us finally dwell for a while on the perhaps misleading notion of "deformation" or "homotopy" of foliations. A continuous path of foliations/integrable plane fields is *not* necessarily induced by an isotopy! In particular, the topology of the leaves can change completely under such a deformation. Let us illustrate this via two crucial examples.

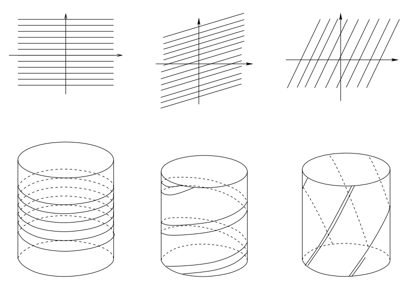

Figure 3: Linear foliations on the torus[1]

For the first example (cf. Fig. 3), we drop one dimension and consider codimension one foliations on the torus $\mathbb{T}^2$, obtained by quotienting out linear foliations of $\mathbb{R}^2$ (that is foliations by parallel straight lines). A one parameter family of such foliations on $\mathbb{R}^2$ (obtained by increasing the slope continuously) induces a continuous family of foliations on $\mathbb{T}^2$, which are clearly not pairwise isotopic: if the slope is rational, all the leaves are compact, whereas if the slope is irrational, all the leaves are dense.

The second example is the creation of a Reeb component: on the solid torus, the foliation by meridian disks can be homotoped (among foliations and relatively to the boundary) into a foliation with a Reeb component along the core and cylinder leaves accumulating on it. This homotopy can be visualized by rotating each picture of the central

[1]In most figures, tori are represented as cylinders for a better visibility. The top and bottom should be identified to get the real picture.

sequence of Fig. 4 around a vertical axis. This sequence represents a continuous path of dimension one foliations invariant under vertical translations, the continuous deformation of their tangent line fields being sketched above.

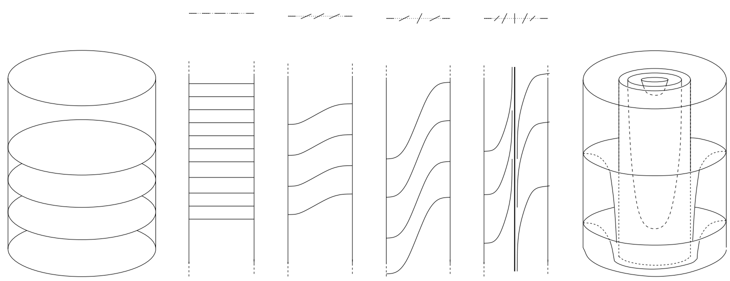

Figure 4: Adding a Reeb component

Strategy of the proof

The main idea to prove Theorem A is to give a parametric version of the process carried out by Thurston in [Th2] to deform any given plane field into a foliation. But we must also deal with the relative aspect of the problem: the ends of the one parameter family of plane fields we want to deform must remain unchanged along the process.

Thurston's construction, outlined in Section 1, consists of three steps: first make the initial plane field integrable outside ball-shaped holes, then enlarge the holes into solid toric ones by digging out tunnels along arcs transverse to the newly defined foliation, and finally make the plane field integrable inside the enlarged holes. Recently, A. Larcanché found a new way of carrying out the last step so that, in addition, the foliations inside the holes depend continuously on the foliations already given on their boundaries, which will be very useful for our problem. We recall her construction in Section 2. In Section 3, we start the parametric adaptation of Thurston's construction by perturbing a given one-parameter family of plane fields into a family of foliations with holes. This step relies on works by Eliashberg in 3-dimensional contact geometry. In Section 4, we complete the proof of Theorem A in a specific case: when the initial foliations (the ends of the initial one-parameter family) have "sufficiently many" transversals. Eventually, in Section 5, we explain how to reduce to this specific case. Basically, the idea is: given any foliation, eliminate as many torus leaves as possible by a homotopy of foliations, and in particular get rid of those which do not bound solid tori. In Sections 1 to 4, all plane fields and foliations are $\mathcal{C}^\infty$. The regularity loss occurs in Section 5, while trying to eliminate torus leaves.

1 Thurston's process

Let M be a closed 3-manifold. In this section, we recall how Thurston deforms a given plane field ξ on M into a foliation.

Step 1. First, he constructs a triangulation "in good position" with respect to ξ, and makes ξ integrable in a neighbourhood V of its 2-skeleton. More precisely, he requires all faces and edges to be transverse to ξ, and the direction of ξ to be almost constant on each 3-simplex. Then he makes ξ integrable in a neighbourhood of every vertex, then every edge, and finally every face (cf. Fig. 5). The key point is that, in a neighbourhood of every simplex σ of the 2-skeleton, there exists a nonsingular vector field ν tangent to ξ and transverse to σ. The deformation consists in making ξ invariant under ν in a neighbourhood of σ. Since ξ is already integrable near $\partial\sigma$, it is already invariant under ν there and thus remains unchanged. This guarantees the global coherence of these local perturbations. The neighbourhood V of the 2-skeleton can be chosen so that, at the end of this step, every component of ∂V is a sphere S enclosed in a 3-simplex on which ξ is almost horizontal, meaning that the foliation traced by ξ on S has only two singularities, the *poles*, which are connected (on S) by a vector field transverse to ξ (see Fig. 5).

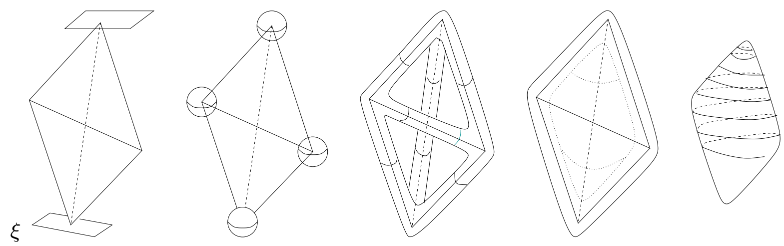

Figure 5: Making ξ integrable near the 2-skeleton

Step 2. At that stage, we have a "foliation with holes" (the new $\xi\,|\,_V$), and we want to extend it to all of M. However, because of the Reeb Stability Theorem [Re], this is only possible if $\xi\,|\,_S$ is a foliation by circles away from the poles for every component S of ∂V. So the newly built foliation $\xi\,|\,_V$ needs to be modified in some places (outside the holes). A nice situation, Thurston observes, is when, for every S, there exists a properly imbedded arc $A \subset V$ transverse to $\xi\,|\,_V$ joining the poles of S. In that case, the union of the ball bounded by S and a tube around A foliated by disks forms a solid torus $W \simeq \mathbb{D}^2 \times \mathbb{S}^1$ on the boundary of which ξ induces a foliation transverse to $\mathbb{S}^1$. Now Thurston shows that such a foliation is always the trace of a foliation of the solid torus (homotopic to ξ rel. boundary), using the simplicity of the group $\mathcal{D}^\infty_+(\mathbb{S}^1)$, proved by M. Herman and J. Mather. A different argument, due to A. Larcanché and outlined in Section 2, provides an extension which, in addition, depends continuously on the foliation given on the boundary.

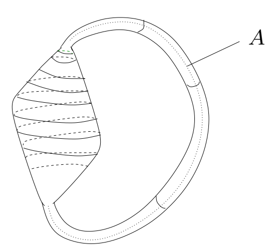

Figure 6: Enlarging the holes

Step 3. The required transverse arcs A do not always exist. A classical sufficient condition for them to exist is that $\xi\,|\,_V$ has no compact leaf. Thurston artfully reduces to this situation by digging a number of new ball-shaped holes in V and breaking the integrability on these balls. More precisely, assume that the grey surface on Fig. 7 is a small piece of compact leaf and that the foliation is horizontal in the neighbourhood represented on the picture. Then consider two little balls centered on this leaf and linked by a "cubic tunnel". Perturb the foliation continuously inside the cube like the picture suggests. Because of the Reeb stability theorem again, this does not extend into a homotopy of foliations inside the balls, but it can be extended into a homotopy of plane fields. Now (what remains of) the grey leaf is not compact anymore since its boundary spirals around the balls and eventually accumulates on circles. Repeating this trick as many times as necessary (finitely many times since the manifold is compact), Thurston gets rid of all compact leaves.

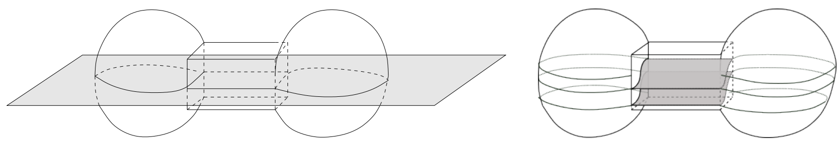

Figure 7: Killing all compact leaves

2 The key construction of Larcanché

As we said before, A. Larcanché [La] invented a *continuous* process to extend to $\mathbb{D}^2 \times \mathbb{S}^1$ any foliation of $\partial\mathbb{D}^2 \times \mathbb{S}^1$ transverse to $\mathbb{S}^1$.

To clarify this statement and give an idea of her construction, we first need to define the *holonomy* of a foliation φ of $\partial\mathbb{D}^2 \times \mathbb{S}^1$ transverse to $\mathbb{S}^1$. The transversality condition implies that, for every x in $\mathbb{S}^1$, the leaf through $(1, x) \in \partial\mathbb{D}^2 \times \mathbb{S}^1$ goes all the way around the solid torus, alternately intersecting every fiber $\{e^{2\pi it}\} \times \mathbb{S}^1$, $t \in [0,1]$, at a point $(e^{2\pi it}, f_t(x))$. This defines a one-parameter family $(f_t)_{t\in[0,1]}$ of smooth orientation-preserving diffeomorphisms of the circle, which has a unique lift $(\widetilde{f}_t)_{t\in[0,1]}$ in $\widetilde{\mathcal{D}^\infty_+(\mathbb{S}^1)}$ – the group of orientation-preserving diffeomorphisms of $\mathbb{R}$ commuting to the unit translation – satisfying $\widetilde{f}_0 = \mathrm{id}_\mathbb{R}$. What we call *holonomy* of the foliation φ, and denote by $hol(\varphi)$, is the diffeomorphism $\widetilde{f}_1$.

If $\mathcal{F}(\mathbb{D}^2 \times \mathbb{S}^1)$ denotes the set of foliations on $\mathbb{D}^2 \times \mathbb{S}^1$ transverse to the boundary (and homotopic rel. boundary to a plane field transverse to the $\mathbb{S}^1$ factor), Larcanché's key result can be stated as follows:

Theorem (Larcanché [La])**.** *There exists a continuous map*

$$\ell\colon \begin{cases} \widetilde{\mathcal{D}^\infty_+(\mathbb{S}^1)} \to \mathcal{F}(\mathbb{D}^2 \times \mathbb{S}^1) \\ \qquad f \;\mapsto\; \ell_f \end{cases}$$

satisfying $hol(\ell_f \mid_{\partial\mathbb{D}^2 \times \mathcal{S}^1}) = f$.

Her proof relies on two important facts. First, when the foliation on the boundary is linear – in which case the holonomy is a translation $T_\lambda\colon x \mapsto x + \lambda$ – there is a classical way of extending it: put a Reeb component along the core of the solid torus, and wrap the external leaves around it as shown on Fig. 8 (explicit formulas can be written). We will call the resulting foliation a *Reeb filling of* T_λ, or *of slope* λ. This also works when the holonomy is conjugate to a translation, but not in the general case.

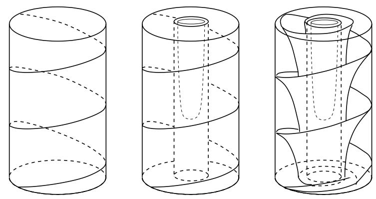

Figure 8: Reeb filling of a linear foliation

To deal with the general case, Larcanché's idea is to combine the above with a decomposition theorem of Herman:

Theorem (Herman [He], p.123). *Let $\mu = (1+\sqrt{5})/2$ denote the Golden Number*[2]*. There is a continuous map*

$$\begin{cases} \widetilde{\mathcal{D}^\infty_+(\mathbb{S}^1)} \to \mathbb{R} \times \widetilde{\mathcal{D}^\infty_+(\mathbb{S}^1)} \\ \qquad f \mapsto (\lambda_f, g_f) \end{cases}$$

such that $f = T_{\lambda_f} \circ (g_f^{-1} \circ T_\mu \circ g_f)$ for all $f \in \widetilde{\mathcal{D}^\infty_+(\mathbb{S}^1)}$, and $(\lambda_{\mathrm{id}}, g_{\mathrm{id}}) = (-\mu, \mathrm{id})$.

Now take any holonomy f. Roughly speaking, the foliation ℓ_f is obtained by taking the Reeb fillings of T_{λ_f} and $g_f^{-1} \circ T_\mu \circ g_f$, gluing them together as Fig. 9 suggests, and inflating the result a little (to remove the angles). The holonomy on the boundary of the resulting bigger solid torus is exactly the composition of the holonomies on the smaller tori, *i.e* precisely f, and this construction depends continuously on f since the decomposition of f does.

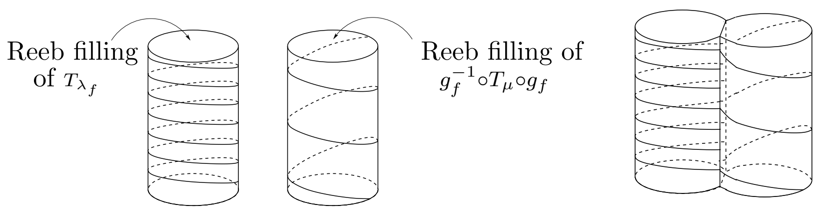

Figure 9: Larcanché filling

Note that when f is the identity, Larcanché's extension is far from being the most natural one (i.e. a foliation by disks), since it consists of two Reeb fillings (of slope μ and $-\mu$ respectively) glued together. The "inflated" picture is depicted on Fig. 10.

It is crucial, however, to realize that ℓ_{id} and the foliation by meridian disks are homotopic *among foliations*. Indeed, let us start with the foliation by meridian disks. By a homotopy of foliations (cf. Introduction), we can create two Reeb components (or, in other words, two Reeb fillings of slope 0) in two smaller solid tori within the big one. In these small tori, we simply increase the slopes of the Reeb fillings continuously, from 0 to $\pm\mu$ respectively. Using some basic knowledge of *suspension foliations* on "*Pair of pants*" $\times \mathbb{S}^1$ (that is foliations transverse to the $\mathbb{S}^1$ factor), we extend this deformation to the complement without affecting the trivial foliation on the boundary of the big torus.

[2]actually any diophantine number would do according to [Yo]

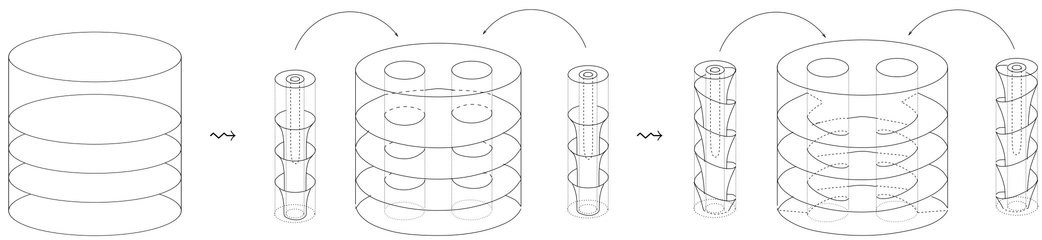

Figure 10: From the trivial foliation to ℓ_{id}

3 Making foliations with holes

We are now ready to try and adapt Thurston's construction to the one-parameter situation. We start with a family ξ_t, $t \in [0,1]$, of plane fields on M, with ξ_0 and ξ_1 already integrable, and we want to make all the ξ_t's integrable. Like Thurston, we first want to construct a nice triangulation and make all plane fields integrable near its 2-skeleton. However, we have *a priori* no chance of finding a triangulation whose edges and faces are transverse to *every* ξ_t, the direction of the latter varying with the parameter in an uncontrollable way. Still, we can find a triangulation such that the direction of each ξ_t is almost constant on every 3-simplex; but we will have to deal with the situation in which, for some parameter t and some point x of the 2-skeleton, the plane $\xi_t(x)$ is tangent to the face or edge through x.

Fortunately, this problem has been studied by Eliashberg [El] in the case of contact structures, and his arguments work the same for foliations (see [Ey1] for the adaptation). As in Thurston's construction, the idea is to find near each simplex a continuous family ν_t, $t \in [0,1]$, of nonsingular vector fields tangent to their respective ξ_t, and to make ξ_t invariant under ν_t. But this time, ν_t will not necessarily be transverse to the simplex, and to ensure the coherence of the local deformations, we will have to start with the "problematic" simplices. Eliashberg also deals with the perhaps bigger issue of ensuring the almost horizontality of the resulting plane fields on the boundary spheres of the newly foliated area V. This requires a careful control of the amplitude of the deformations *and* of the size of the neighbourhood of the 2-skeleton on which the plane fields are made integrable.

Note that this process leaves ξ_0 and ξ_1 unchanged, since they were integrable from the beginning.

4 Enlarging and filling the holes in the taut case

We are now dealing with a (new) family ξ_t, $t \in [0,1]$, of plane fields which are integrable outside a number of ball-shaped holes on the boundary of which they are almost horizontal, and we want to make them integrable everywhere on M. Note that the poles of the boundary spheres vary continuously with the parameter t. Let V denote the complement of the holes. As in [Th2], a very nice situation is when one can find, for every sphere S of ∂V and every $t \in [0,1]$, an arc A_t in V transverse to $\xi_t \mid_V$ connecting the poles of S and depending continuously on t. In that case, Larcanché's construction (cf. Section 2) is the perfect tool to fill the time-dependent toric holes obtained by the union of a ball-shaped one with a moving tubular neighbourhood of a moving transversal. Note that even in this ideal case, we are not quite done yet, because this last manipulation affects ξ_0 and ξ_1, which were supposed to remain unchanged. Indeed, the new ξ_0 and ξ_1 differ from the initial ones by a number of Larcanché ℓ_{id} foliations inserted in solid tori (toric holes)which initially contained simple foliations by meridian disks. But this is not a real problem according to the last remark of Section 2.

However, two real problems can prevent the situation from being ideal:

- though we can apply a parametric version of Thurston's trick (cf. Section 1) to the family $(\xi_t)_{t\in(0,1)}$ to ensure the existence of transverse arcs A_t for all $t \in (0,1)$, the ends ξ_0 and ξ_1 must remain unchanged (and one can easily come up with examples of ξ_0 and ξ_1 which do not have "enough" transversals);
- even when it is possible to find A_t's for all $t \in [0,1]$, it is in general impossible to make them depend continuously on t.

Let us forget about the first problem for a while by assuming that ξ_0 and ξ_1 are *taut* foliations, that is foliations for which each transverse arc can be extended into a closed transversal (note that not every manifold admits such a foliation, so this temporary hypothesis is a very restrictive one).

Now, for all t in $[0,1]$ and all S in ∂V, there is an arc A_t in V transverse to $\xi_t \mid_V$ connecting the poles of S. For s close enough to t, A_t is still transverse to ξ_s, so one can actually make the family A_t piecewise continuous. In other words, for some values of the parameter t, we have two transverse arcs A_- and A_+ joining the poles of some $S \subset \partial V$, which means two ways of digging out a tunnel in which to replace $\xi := \xi_t$ by a Larcanché foliation. So what we need to check is:

Lemma 1 (Communicating Vessels Lemma)**.** *The two foliations built from ξ by Larcanché's construction using the arcs A_- and A_+ respectively are homotopic among foliations.*

To see this, let us define two foliations ξ_- and ξ_+ which are not exactly the foliations mentioned above but isotopic ones. First, we perturb A_+ and A_- near both ends to make them disjoint (see Fig. 11). Then, we cut the ball B bounded by S along a disk D through the poles and, to each half-ball $B_\pm$, we add a small tubular neighbourhood of $A_\pm$ so as to obtain two (slightly angular) solid tori W_+ and W_- intersecting along D. Outside B, both solid tori are foliated by disks.

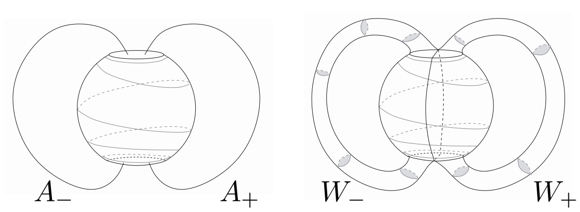

Figure 11: $A_\pm$ and $W_\pm$

The foliation $\xi_\pm$ is obtained by filling $W_\mp$ with a trivial foliation by meridian disks, which traces a new foliation on D, and then putting a "Larcanché foliation" in $W_\pm$ −

the “only one” inducing the new foliation traced on $\partial W_\pm$. The key observation is that the holonomies of $\xi_+ \mid_{\partial W_+}$ and $\xi_- \mid_{\partial W_-}$ are the same. This diffeomorphism f essentially describes the holonomy of ξ around the ball.

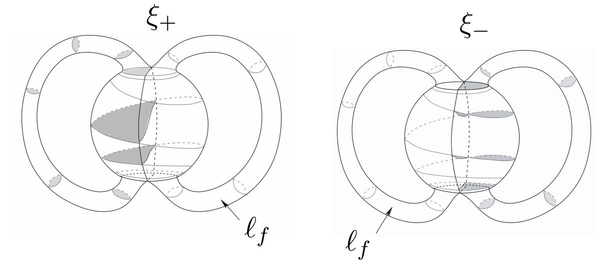

Figure 12: ξ_+ and ξ_-

Now to homotope ξ_+ into ξ_-, we start with ξ_+, we perturb the trivial foliation on W_- into ℓ_{id}, then we perturb the foliation on D so that the resulting holonomy on ∂W_+ (resp. ∂W_-) varies from f to id (resp. id to f) and we extend this perturbation to $W_\pm$ (rel. their complement) by continuous paths of Larcanché foliations. At the end, W_+ is foliated by ℓ_{id}. To get ξ_-, we only need to homotope ℓ_{id} back into a foliation by meridian disks (cf. Section 2).

At this point, we have:

Theorem 2. *Two $\mathcal{C}^\infty$ taut foliations homotopic among plane fields can be connected by a path of $\mathcal{C}^\infty$ foliations.*

Actually, little extra work is needed to generalize this to a larger class of foliations, namely the kind we just used to build the path, which will be called *malleable* in these notes. A smooth foliation is *malleable* if it is taut outside finitely many solid tori on which it is of the form ℓ_f, for an f having a whole interval of fixed points.

Theorem 3. *Two malleable foliations homotopic among plane fields can be connected by a path of $\mathcal{C}^\infty$ foliations.*

5 General case

To obtain the full force of Theorem A, the last thing to prove is that any $\mathcal{C}^\infty$ foliation can be deformed into a malleable one by a homotopy of foliations. This is where the loss of regularity occurs.

Theorem 4. *Any $\mathcal{C}^\infty$ foliation can be connected to a malleable one by a continuous path of $\mathcal{C}^1$ foliations.*

The general idea is to try and make the given foliation “as taut as possible”, by a homotopy of foliations. To that end, note that a foliation on some region is taut if and only if every leaf meets a closed transversal in this region. So basically, we want to alter the given foliation so that in the end, as many leaves as possible meet closed transversals. Now according to works by Novikov [No] and Goodman [Go], only torus leaves do not necessarily satisfy this condition. So our first task is to get rid of as many torus leaves as possible (we call this *cleaning*, cf. 5.1). Then (cf. 5.2), we will have to see to it that the remaining "problematic" ones all lie in "Larcanché foliated" solid tori (in particular, we will need to get rid of the incompressible ones).

5.1 Cleaning

We start with any foliation and we want to reduce to a finite number of torus leaves by a homotopy of foliations. To do so, we first note (thanks to Novikov [No] and Thurston [Th1]) that the problematic torus leaves always arise in "bundles": there are finitely many disjoint saturated sets of the form $\mathbb{T}^2 \times J$, where J is a segment, on which the foliation is transverse to J and outside which every leaf is cut by a closed transversal. The idea then is to shrink these foliated bundles into isolated compact leaves. By holonomy, this is in fact equivalent to proving that the space of representations (homomorphisms) of $\mathbb{Z}^2 = \pi_1(\mathbb{T}^2)$ in $\mathcal{D}^\infty_+[0,1]$ is connected. This is a difficult problem, but the following result, based on classical works by Szekeres [Sz] and Kopell [Ko] on commuting interval diffeomorphisms, at least allows us to shrink the bundles via a homotopy of $\mathcal{C}^1$ foliations.

Theorem 5 ([Ey2])**.** *Every representation of $\mathbb{Z}^2$ in $\mathcal{D}^\infty_+[0,1]$ can be linked to the trivial representation through a continuous path of representations of $\mathbb{Z}^2$ in $\mathcal{D}^1_+[0,1]$.*

With similar arguments (related to commuting germs of interval diffeomorphisms), we can actually reduce to foliations whose torus leaves all have a neighbourhood parametrized by $\mathbb{T}^2\times]-\varepsilon,\varepsilon[$ where the foliation has an equation of the form:

$$dz - u(z)(a^+dx_1 + b^+dx_2) \quad \text{on} \quad \mathbb{T}^2 \times [0,\varepsilon[$$

and

$$dz - u(z)(a^-dx_1 + b^-dx_2) \quad \text{on} \quad \mathbb{T}^2\times]-\varepsilon,0]$$

with $(a^\pm, b^\pm) \in \mathbb{R}^2 \setminus \{0\}$ and u a smooth function vanishing only at 0 (see Fig. 13). We call them *clean* foliations. In the next step, we use the flexibility of this simple model to "kill" the problematic torus leaves of clean foliations, or, more precisely, to replace them by compressible ones enclosed in Reeb fillings.

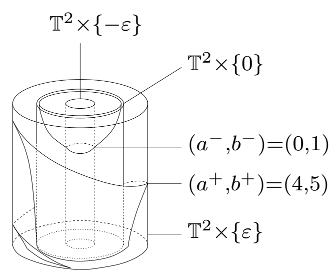

Figure 13: Model foliation near a torus leaf

5.2 Killing incompressible torus leaves

There is a nice case in which getting rid of the torus leaf $\mathbb{T}^2 \times \{0\} =: T_0$ in the above model is easy: when $(a^+, b^+) = (a^-, b^-)$ and u has a constant sign outside 0. In that case, we only need to perturb u locally around 0 into a nonvanishing function.

Now let us deal with the general case, represented on Fig. 13. In order to reduce to the previous case, we want to deform the foliation on one side of the compact leaf to make its slope match the slope on the other side. But this deformation has to be local – say between the compact leaf T_0 and some slightly larger parallel torus T – because the foliation near the outermost tori T_ε and $T_{-\varepsilon}$ must remain unchanged.

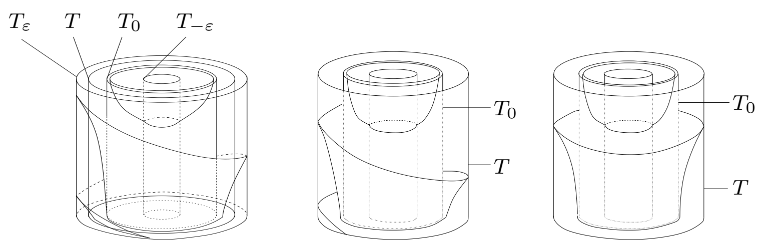

Figure 14: Desired deformation between T and the compact leaf T_0

But this will trace on T a path of linear foliations of slope varying continuously from $\lambda \neq 0$ to 0, which must then be extended between T and T_ε rel. T_ε. To make this possible, we first need to add a Reeb component in this external region, along a "vertical" transverse circle (see Fig. 15). We denote by T' the boundary torus of the affected region. T, T_ε and T' bound a region of the form "*Pair of pants*"$\times \mathbb{S}^1$. In this region, it is easy (again using some basic knowledge on suspension foliations) to define a path of foliations which is invariant on T_ε and varies from the linear foliation of slope 0 (resp. λ) to the one of slope λ (resp. 0) on T' (resp. T) (cf. Fig. 15). We extend this by Reeb fillings on the solid torus bounded by T' and by the desired path of spiralling foliations between T and the compact leaf T_0 (cf. Fig. 14), and we are done.

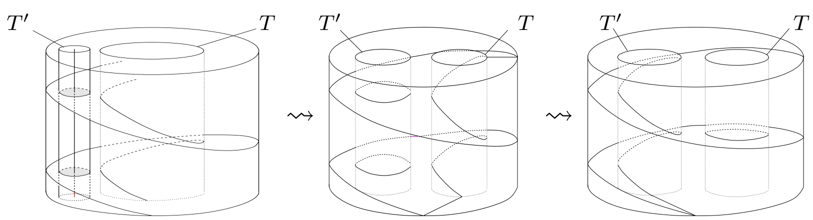

Figure 15: Deformation between T_ε and T

5.3 Holonomy fragmentation

After the above step, we end up with a foliation which is taut outside finitely many solid tori foliated by Reeb fillings. The last thing to do is to replace each Reeb filling of slope λ by a collection of Larcanché foliations whose holonomies on the boundary have intervals of fixed points and have T_λ for product. To that end, we first homotope the Reeb filling into the Larcanché foliation ℓ_{T_λ} (adapting the last remark of Section 2). Then we conclude with a combination of Larcanché's theorem, the following (elementary) fragmentation lemma and, once again, the flexibility of suspension foliations on "*Punctured disk*"$\times \mathbb{S}^1$.

Lemma 6. *Every element of $\widetilde{\mathcal{D}^\infty_+(\mathbb{S}^1)}$ a (finite) composition of elements of $\widetilde{\mathcal{D}^\infty_+(\mathbb{S}^1)}$ which all have an interval of fixed points.*

Note that 5.2 and 5.3, together with Theorem 3, yield the following intermediate result:

Theorem 7. *Two clean foliations homotopic among plane fields can be connected by a continuous path of clean foliations.*

Conclusion

The surjectivity of the map $\pi_1\mathcal{F}(M) \to \pi_1\mathcal{P}(M)$ induced by the inclusion is actually much easier to prove than the injectivity of the map between π_0's, since one avoids the relative aspect of the problem. All the necessary tools lie in Sections 3 and 4. As a matter of fact, most techniques described in these sections adapt to any number of parameters, yielding the surjectivity of the maps between π_k's for all $k \geq 1$ (see [Ey3]). Injectivity, on the other hand, requires, among other things, a better understanding of the topology of the space of $\mathcal{C}^\infty$ orientation-preserving actions of $\mathbb{Z}^2$ on $[0,1]$, of which little is known.

Acknowledgements. I am deeply grateful to my advisor, Emmanuel Giroux, for trusting me with such a beautiful problem, and for his amazing guidance and support through all the stages of this work. I also wish to thank Pr. Tsuboi who gave me the opportunity to come back to Japan, after a decisive first stay, to present and pursue my research on the subject (with the financial support of the Japanese Society for the Promotion of Science).

References

[El] Y. Eliashberg — *Classification of overtwisted contact structures on 3-manifolds.* Invent. Math. **98** (1989), 623–637.

[Ey1] H. Eynard — *Sur deux questions connexes de connexité concernant les feuilletages et leurs holonomies.* Ph. D. dissertation (ENS-Lyon), available at `http://tel.archives-ouvertes.fr/tel-00436304/fr/`.

[Ey2] H. Eynard — *A connectedness result for commuting diffeomorphisms of the interval.* Ergod. Th. & Dynam. Sys. (First published online 2010).

[Ey3] H. Eynard — *On the homotopy type of the space of codimension one foliations on a closed 3-manifold.* In preparation.

[Go] S. Goodman — *Closed leaves in foliations of codimension one.* Comment. Math. Helvetici **50** (1975), 383–388.

[He] M. R. Herman — *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations.* Inst. Hautes Études Sci. Publ. Math. **49** (1979), 5–233.

[Ko] N. Kopell — *Commuting diffeomorphisms.* In *Global Analysis*, Proc. Sympos. Pure Math. XIV, Amer. Math. Soc. (1968), 165–184.

[La] A. Larcanché — *Topologie locale des espaces de feuilletages en surfaces des variétés fermées de dimension 3.* Comment. Math. Helvetici **82** (2007), 385–411.

[Li] W. B. R. Lickorish — *A foliation dor* 3*-manifolds.* Ann. of Math., **82** (1965), 414–420.

[No] S. P. Novikov — *Topology of foliations.* Trans. Moscow Math. Soc. **14** (1965), 248–278 (Russian), A.M.S Translation (1967), 268–304.

[Re] G. Reeb — *Sur certaines propriétés topologiques des variétés feuilletées.* Actual. Sci. Ind. **1183**, Hermann, Paris (1977).

[Sz] G. Szekeres — *Regular iteration of real and complex functions.* Acta Math. **100** (1958), 203–258.

[Th1] W. P. Thurston — *Foliations of Three-Manifolds Which are Circle Bundles.* Ph. D. Dissertation, University of California, Berkeley (1967).

[Th2] W. P. Thurston — *A local construction of foliations for three-manifolds.* Differential topology (Proc. Sympos. Pure Math. **27**, Stanford Univ., California, 1973), Amer. Math. Soc. (1975), 315–319.

[Wo] J. Wood — *Foliations on 3-manifolds.* Ann. of Math. **89** (1969), 336–358.

[Yo] J-C. Yoccoz — *Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition diophantienne.* Ann. Sci. École Norm. Sup. (4) **17** (1984), 333–359.