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Cohomologie des algèbres de Fomin-Kirillov

Cohomology of Fomin-Kirillov algebras

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Contents

Abstract

After recalling the basics in quadratic algebras, Hochschild (co)homology, Gröbner bases for noncommutative algebras, and Fomin-Kirillov algebras $\mathrm{FK}(n)$ over a field for $n \geqslant 2$, we compute the Hochschild (co)homology of the Fomin-Kirillov algebra $\mathrm{FK}(3)$ on three generators over a field of characteristic different from 2 and 3, and the cyclic homology of $\mathrm{FK}(3)$ in case the characteristic of the field is zero. Moreover, we compute the algebra structure and Gerstenhaber bracket of the Hochschild cohomology of $\mathrm{FK}(3)$ over a field of characteristic different from 2 and 3. The latter is in part based on a general method we introduce to easily compute the Gerstenhaber bracket between elements of $\mathrm{HH}^0(A)$ and elements of $\mathrm{HH}^n(A)$, the method by M. Suárez-Álvarez in [24] to calculate the Gerstenhaber bracket between elements of $\mathrm{HH}^1(A)$ and elements of $\mathrm{HH}^n(A)$ for any $n \in \mathbb{N}_0$ and any algebra A over a field, as well as an elementary result that allows computing the remaining brackets from the previous ones. We also show that the Gerstenhaber bracket of the Hochschild cohomology of $\mathrm{FK}(3)$ over a field of characteristic different from 2 and 3 is not induced by any Batalin-Vilkovisky generator.

Keywords: Fomin-Kirillov algebra, Hochschild cohomology, Gerstenhaber bracket.

Résumé

Après avoir rappelé les notions fondamentales en théorie des algèbres quadratiques, la homologie et cohomologie de Hochschild des algèbres associatives, les bases de Gröbner pour les algèbres non commutatives, et les algèbres de Fomin-Kirillov $\mathrm{FK}(n)$ sur un corps pour $n \geqslant 2$, on calcule la (co)homologie de Hochschild de l'algèbre de Fomin-Kirillov $\mathrm{FK}(3)$ à trois générateurs sur un corps de caractéristique différente de 2 et 3, et l'homologie cyclique de $\mathrm{FK}(3)$ dans le cas d'un corps de caractéristique nulle. De plus, nous calculons la structure algébrique et de Gerstenhaber sur la cohomologie de Hochschild de $\mathrm{FK}(3)$ pour un corps de caractéristique différente de 2 et 3. Le calcul de la structure de Gerstenhaber est en partie basée sur une méthode générale que nous introduisons pour calculer facilement la parenthèse de Gerstenhaber entre les éléments de $\mathrm{HH}^0(A)$ et les éléments de $\mathrm{HH}^n(A)$ pour tout $n \in \mathbb{N}_0$ et toute algèbre A sur un corps, la méthode par M. Suárez-Álvarez dans [24] pour calculer la parenthèse de Gerstenhaber entre les éléments de $\mathrm{HH}^1(A)$ et les éléments de $\mathrm{HH}^n(A)$, ainsi qu'une résultat qui permet de calculer les cas restantes à partir des précédentes. Nous montrons aussi que le crochet de Gerstenhaber de la cohomologie de Hochschild de $\mathrm{FK}(3)$ sur un corps de caractéristique différent de 2 et 3 n'est induit par aucun générateur de Batalin-Vilkovisky.

Mot-clés: Algèbre de Fomin-Kirillov, Cohomologie de Hochschild, Crochet de Gerstenhaber.

Brève présentation

On sait que l'anneau de cohomologie $\mathrm{H}^\bullet(Fl_n,\mathbb{Z})$ de la variété de drapeaux complète complexe Fl_n est isomorphe à $\mathbb{Z}[X_1,\dots,X_n]/I_n$, où I_n est l'idéal engendré par les polynômes symétriques. La cohomologie $\mathrm{H}^\bullet(Fl_n,\mathbb{Z})$ a une base formée par les classes de Schubert σ_w, indexées par les éléments w du groupe symétrique $\mathbb{S}_n$. Sous l'isomorphisme ci-dessus, les polynômes de Schubert $\mathfrak{S}_w, w \in \mathbb{S}_n$ représentent les classes de Schubert. Pour étudier la combinatoire de l'anneau de cohomologie de la variété de drapeaux, S. Fomin et A. Kirillov ont introduit une famille d'algèbres quadratiques, maintenant appelée **algèbres de Fomin-Kirillov** $\mathrm{FK}(n)$, indexé par les entiers positifs $n \in \mathbb{N}$ (voir [8, 14, 15]). Ils prouvent que le sous-anneau commutatif de $\mathrm{FK}(n)$ généré par les éléments de Dunkl θ_i pour $i \in [\![1,n]\!]$ est isomorphe à $\mathrm{H}^\bullet(Fl_n,\mathbb{Z})$, et les évaluations des polynômes de Schubert $\mathfrak{S}_w(\theta_1,\dots,\theta_n)$ aux éléments de Dunkl agissent sur l'anneau de cohomologie $\mathrm{H}^\bullet(Fl_n,\mathbb{Z})$ par la multiplication à gauche de la classe de Schubert σ_w. Ils conjecturent que chacune de ces évaluations est une combinaison linéaire non négative de monômes dans les générateurs $x_{i,j}, i<j$ de $\mathrm{FK}(n)$, et montrent que cette conjecture implique la non négativité des constantes c^w_{uv}, où $\sigma_u\sigma_v = \sum_{w\in\mathbb{S}_n} c^w_{uv}\sigma_w$.

L'algèbre de Fomin-Kirillov $\mathrm{FK}(n)$ pour $n \in [\![3,5]\!]$ est une algèbre de Nichols de dimension finie (voir [10, 17]), qui apparaît dans la classification des algèbres de Hopf pointées de dimension finie dont les groupes d'éléments de type groupe sont abéliens (voir [3]). L'algèbre de Fomin-Kirillov $\mathrm{FK}(n)$ pour $n \in [\![3,5]\!]$ produit une algèbre de Hopf de dimension finie par bosonisation, avec un groupe non abélien d'éléments de type groupe. La conjecture de P. Etingof et V. Ostrik affirme que l'algèbre de Yoneda $\mathrm{H}^\bullet(H,\Bbbk) = \mathrm{Ext}^\bullet_H(\Bbbk,\Bbbk)$ de toute algèbre de Hopf (tréssée) H de dimension finie est de type fini. N. Andruskiewitsch, I. Angiono, J. Pevtsova et S. Witherspoon ont prouvé la conjecture pour toute algèbre de Hopf complexe de dimension finie pointée avec un groupe abélien d'éléments de type groupe (voir [1]). L'algèbre de Yoneda de l'algèbre de Fomin-Kirillov $\mathrm{FK}(3)$ à trois générateurs a d'abord été calculée par D. Ştefan et C. Vay dans [23], à l'aide de plusieurs calculs assez lourds avec des séquences spectrales. L'algèbre de Yoneda de $\mathrm{FK}(3)$ a été plus récemment obtenue dans [11] par des méthodes plus directes, à savoir en calculant explicitement la résolution projective minimale du module trivial $\Bbbk$ dans la catégorie des modules graduées et inférieurement bornés.

Le but de cette thèse est de calculer explicitement la (co)homologie de Hochschild de $\mathrm{FK}(3)$ sur un corps $\Bbbk$ de caractéristique différente de 2 et 3 (voir [12, 13]). En utilisant la description explicite de la résolution projective minimale du bimodule standard de $\mathrm{FK}(3)$, nous calculons la dimension de la (co)homologie de Hochschild de $\mathrm{FK}(3)$ sur un corps $\Bbbk$ de caractéristique différente de 2 et 3, et l'homologie cyclique si le corps $\Bbbk$ est de caractéristique nulle. La structure algébrique (pour le produit cup) et le crochet de Gerstenhaber sur la cohomologie de Hochschild de $\mathrm{FK}(3)$ sont également entièrement calculés. Pour cela, on fournit une méthode générale de nature homologique pour calculer facilement le crochet de Gerstenhaber entre les éléments de $\mathrm{HH}^0(A)$ et les éléments de $\mathrm{HH}^n(A)$ pour tout $n \in \mathbb{N}_0$ et toute algèbre A sur un corps $\Bbbk$.

La thèse est organisée de la façon suivante. Dans le chapitre 1, nous rappelons les fondaments de la théorie des algèbres quadratiques et de l'homologie et cohomologie de Hochschild. On introduit une résolution projective du module trivial de toute algèbre quadratique satisfaisant certaines hypothèses (voir Théorème 1.2.5). Dans la Sous-section 1.4.1 nous introduisons une méthode générale pour calculer le crochet de Gerstenhaber entre les éléments de $\mathrm{HH}^0(A)$ et les éléments de $\mathrm{HH}^n(A)$ pour tout $n \in \mathbb{N}_0$ et toute algèbre A sur un corps $\Bbbk$ (voir le Théorème 1.4.1). Dans la Sous-section 1.4.2 on rappelle brièvement la méthode introduite par M. Suárez-Álvarez dans [24] pour calculer la parenthèse de Gerstenhaber entre les éléments de $\mathrm{HH}^1(A)$ et $\mathrm{HH}^n(A)$ pour $n \in \mathbb{N}_0$. Dans le Chapitre 2, nous rappelons les notions fondamentales de la théorie de bases de Gröbner pour les algèbres non commutatives, ainsi que le lemme du losange de Bergman. Nous montrons également la base de Gröbner de FK(3) comme exemple. Au chapitre 3, nous rappelons les modules de Yetter-Drinfeld sur une algèbre de groupe et la définition des algèbres de Fomin-Kirillov $\mathrm{FK}(n)$ pour $n \geqslant 2$ sur un corps $\Bbbk$, qui sont des modules de Yetter-Drinfeld sur l'algèbre de groupe $\Bbbk\mathbb{S}_n$. Au chapitre 4, après avoir rappelé quelques résultats fondamentaux et la structure du module de Yetter-Drinfeld de l'algèbre de Fomin-Kirillov FK(3) à trois générateurs, nous construisons explicitement la résolution projective minimale du bimodule standard de FK(3) dans la catégorie des bimodules gradués et inférieurement bornés (voir la Proposition 4.1.15), en nous appuyant sur la résolution projective minimale du module trivial $\Bbbk$ dans la catégorie des bimodules gradués et inférieurement bornés dans [11]. En utilisant cette résolution nous calculons ensuite des bases explicites pour les groupes d'homologie et de cohomologie de Hochschild de FK(3) sur un corps $\Bbbk$ de caractéristique différente de 2 et 3. En particulier, on prouve le résultat suivant.

Proposition (voir Proposition 4.2.7)**.** *Soit* $A = \mathrm{FK}(3)$*. La dimension de* $\mathrm{HH}_n(A)$ *est donnée par*

$$\dim \mathrm{HH}_n(A) = \begin{cases} 6, & \text{si } n = 0, \\ \frac{5}{2}n + 5, & \text{si } n = 4r \text{ pour } r \in \mathbb{N}, \\ \frac{5n+13}{2}, & \text{si } n = 4r+1 \text{ pour } r \in \mathbb{N}_0, \\ \frac{5}{2}n + 6, & \text{si } n = 4r+2 \text{ pour } r \in \mathbb{N}_0, \\ \frac{5n+9}{2}, & \text{si } n = 4r+3 \text{ pour } r \in \mathbb{N}_0. \end{cases}$$

On calcule aussi les séries de Hilbert par rapport au degré interne de FK(3) (voir Corollaire 4.2.8). De plus, la série de Hilbert de l'homologie cyclique est immédiatement obtenue à partir de l'homologie de Hochschild au moyen du théorème de Goodwillie (voir Corollaire 4.2.10) dans le cas où la caractéristique du corps est nulle. Nous calculons également des bases explicites pour les groupes de cohomologie de Hochschild de FK(3) sur un corps $\Bbbk$ de caractéristique différente de 2 et 3. En particulier, on prouve le résultat suivant.

Proposition (voir Proposition 4.2.19). *Soit* $A = \mathrm{FK}(3)$*. La dimension de* $\mathrm{HH}^n(A)$ *est donnée par*

$$\dim \mathrm{HH}^n(A) = \begin{cases} \frac{5}{2}n + 4, & \text{si } n = 4r \text{ pour } r \in \mathbb{N}_0, \\ \frac{5}{2}n + 5, & \text{si } n = 4r+2 \text{ pour } r \in \mathbb{N}_0, \\ \frac{5n+9}{2}, & \text{if } n = 2r+1 \text{ for } r \in \mathbb{N}_0. \end{cases}$$

La série complète de Hilbert de la cohomologie de Hochschild par rapport au degré interne de FK(3) est dans le Corollaire 4.2.20. Dans le chapitre 5, en calculant les produits cup et en utilisant des techniques issues des bases de Gröbner, nous prouvons que la cohomologie de Hochschild de FK(3) sur un corps $\Bbbk$ de caractéristique différente de 2 et 3 est donné comme un quotient d'une algèbre commutative graduée libre (pour le degré cohomologique) avec 14 générateurs homogènes (voir Proposition 5.1.5) modulo l'idéal homogène engendré par les 63 relations listées dans (5.1.5) (voir Corollaire 5.1.11). En utilisant les méthodes générales introduites dans la section 1.4, on calcule les crochets de Gerstenhaber sur la cohomologie de Hochschild de FK(3) sur un corps $\Bbbk$ de caractéristique différente de 2 et 3 entre éléments de degré de cohomologie m pour $m \in [\![0,1]\!]$ et éléments de degré de cohomologie $n \in \mathbb{N}_0$. Enfin, nous présentons un résultat simple qui nous permet de calculer les crochets de Gerstenhaber restants sous certaines hypothèses sur la structure algébrique de la cohomologie de Hochschild d'une algèbre (voir Lemme 5.2.12), qui sont vérifiées dans le cas de l'algèbre de Fomin-Kirillov FK(3) à trois générateurs sur un corps de caractéristique différente de 2 et 3. Nous résumons tous les crochets de Gerstenhaber de la cohomologie de Hochschild de FK(3) dans le tableau 5.2.1. En utilisant l'expression explicite ci-dessus du crochet de Gerstenhaber, nous montrons que la structure de Gerstenhaber sur la cohomologie de Hochschild de FK(3) n'est induite par aucun générateur de Batalin-Vilkovisky (voir la proposition 5.2.15). Le résultat principal du chapitre 6 est que FK(4) possède une donnée de résolution, ce qui nous permet de calculer une résolution projective du module trivial.

Introduction

It is known that the cohomology ring $\mathrm{H}^\bullet(Fl_n,\mathbb{Z})$ of the complex complete flag manifold Fl_n is isomorphic to $\mathbb{Z}[X_1,\dots,X_n]/I_n$, where I_n is the ideal generated by symmetric polynomials. The cohomology $\mathrm{H}^\bullet(Fl_n,\mathbb{Z})$ has a basis formed by Schubert classes σ_w, indexed by the elements w of symmetric group $\mathbb{S}_n$. Under the above isomorphism, the Schubert polynomials $\mathfrak{S}_w, w \in \mathbb{S}_n$ represent the Schubert classes. To study the combinatorics of the cohomology ring of the flag manifold, S. Fomin and A. Kirillov introduced a family of quadratic algebras, now called the **Fomin-Kirillov algebras** $\mathrm{FK}(n)$, indexed by the positive integers $n \in \mathbb{N}$ (see [8, 14, 15]). They prove that the commutative subring of $\mathrm{FK}(n)$ generated by Dunkl elements θ_i for $i \in [\![1,n]\!]$ is isomorphic to $\mathrm{H}^\bullet(Fl_n,\mathbb{Z})$, and the evaluations of Schubert polynomials $\mathfrak{S}_w(\theta_1,\dots,\theta_n)$ at Dunkl elements acts on the cohomology ring $\mathrm{H}^\bullet(Fl_n,\mathbb{Z})$ by the left multiplication of the Schubert class σ_w. They conjecture that each of these evaluations is a nonnegative linear combination of monomials in the generators $x_{i,j}, i < j$ of $\mathrm{FK}(n)$, and show that this conjecture implies the nonnegativity of constants c^w_{uv}, where $\sigma_u\sigma_v = \sum_{w\in\mathbb{S}_n} c^w_{uv}\sigma_w$.

The Fomin-Kirillov algebra $\mathrm{FK}(n)$ for $n \in [\![3,5]\!]$ is a finite-dimensional Nichols algebra (see [10, 17]), which appears in the classification of finite-dimensional pointed Hopf algebras with abelian groups of group-like elements (see [3]). The Fomin-Kirillov algebra $\mathrm{FK}(n)$ for $n \in [\![3,5]\!]$ produces a finite-dimensional Hopf algebra by bosonisation, with a non-abelian group of group-like elements. The conjecture by P. Etingof and V. Ostrik claims that the Yoneda algebra $\mathrm{H}^\bullet(H,\Bbbk) = \mathrm{Ext}^\bullet_H(\Bbbk,\Bbbk)$ of every finite-dimensional (braided) Hopf algebra H is finitely generated. N. Andruskiewitsch, I. Angiono, J. Pevtsova and S. Witherspoon have proved the conjecture for finite-dimensional complex pointed Hopf algebra with an abelian group of group-like elements (see [1]). The Yoneda algebra of the Fomin-Kirillov algebra $\mathrm{FK}(3)$ on three generators was first computed by D. Ştefan and C. Vay in [23], using several calculations involving spectral sequences. The Yoneda algebra of $\mathrm{FK}(3)$ was more recently obtained in [11] by more direct methods, namely by explicitly computing the minimal projective resolution of the trivial module $\Bbbk$ in the category of bounded below graded modules.

The aim of this thesis is to explicitly compute the Hochschild (co)homology of $\mathrm{FK}(3)$ over a field $\Bbbk$ of characteristic different from 2 and 3 (see [12, 13]). Using the explicit projective bimodule resolution of $\mathrm{FK}(3)$, we compute the dimension of the Hochschild (co)homology of $\mathrm{FK}(3)$ over a field $\Bbbk$ of characteristic different from 2 and 3, and the cyclic homology if the field $\Bbbk$ has characteristic zero. The algebraic structure (for cup product) and Gerstenhaber bracket on the Hochschild cohomology of $\mathrm{FK}(3)$ are also entirely computed. To do this, we provide a general method of homological flavour to easily compute the Gerstenhaber bracket between elements of $\mathrm{HH}^0(A)$ and elements of $\mathrm{HH}^n(A)$ for any $n \in \mathbb{N}_0$ and any algebra A over a field $\Bbbk$.

The thesis is organised as follows. In Chapter 1, we recall the basics about quadratic algebras and Hochschild (co)homology. We introduce a new construction of a projective resolution of the trivial module of a quadratic algebra satisfying some assumptions (see Theorem 1.2.5). We also introduce a general method to compute the Gerstenhaber bracket between elements of $\mathrm{HH}^0(A)$ and elements of $\mathrm{HH}^n(A)$ for any $n \in \mathbb{N}_0$ and any algebra A over a field $\Bbbk$ (see Theorem 1.4.1) in Subsection 1.4.1. We also briefly recall the method introduced by M. Suárez-Álvarez in [24] to compute the Gerstenhaber bracket between elements of $\mathrm{HH}^1(A)$ and $\mathrm{HH}^n(A)$ for $n \in \mathbb{N}_0$ in Subsection 1.4.2. In Chapter 2, we recall the Gröbner bases and Bergman's diamond lemma for noncommutative algebras. We also show the Gröbner basis of $\mathrm{FK}(3)$ as an example. In Chapter 3, we recall the Yetter-Drinfeld modules over a group algebra and the definition of Fomin-Kirillov algebras $\mathrm{FK}(n)$ for $n \geqslant 2$ over a field $\Bbbk$, which are Yetter-Drinfeld modules over the group algebra $\Bbbk\mathbb{S}_n$. In Chapter 4, after recalling some basic facts and Yetter-Drinfeld module structure about the Fomin-Kirillov algebra $\mathrm{FK}(3)$ on three generators, we explicitly construct the minimal projective resolution of the $\mathrm{FK}(3)$ in the category of bounded below graded bimodules (see Proposition 4.1.15), building upon the minimal projective resolution of the trivial module $\Bbbk$ in the category of bounded below graded modules in [11]. Using this resolution we then compute explicit bases for the Hochschild homology groups of $\mathrm{FK}(3)$ over a field $\Bbbk$ of characteristic different from 2 and 3. In particular, we prove the following result.

Proposition (see Proposition 4.2.7)**.** *Let* $A = \mathrm{FK}(3)$. *The dimension of* $\mathrm{HH}_n(A)$ *is given by*

$$\dim \mathrm{HH}_n(A) = \begin{cases} 6, & \text{if } n = 0, \\ \frac{5}{2}n + 5, & \text{if } n = 4r \text{ for } r \in \mathbb{N}, \\ \frac{5n+13}{2}, & \text{if } n = 4r+1 \text{ for } r \in \mathbb{N}_0, \\ \frac{5}{2}n + 6, & \text{if } n = 4r+2 \text{ for } r \in \mathbb{N}_0, \\ \frac{5n+9}{2}, & \text{if } n = 4r+3 \text{ for } r \in \mathbb{N}_0. \end{cases}$$

We also compute their full Hilbert series with respect to the internal degree of $\mathrm{FK}(3)$ (see Corollary 4.2.8). Moreover, the Hilbert series of the cyclic homology is immediately obtained from the Hochschild homology by means of Goodwillie's theorem (see Corollary 4.2.10) in case the characteristic of the field is zero. We also compute explicit bases for the Hochschild cohomology groups of $\mathrm{FK}(3)$ over a field $\Bbbk$ of characteristic different from 2 and 3. In particular, we prove the following result.

Proposition (see Proposition 4.2.19)**.** *Let* $A = \mathrm{FK}(3)$. *The dimension of* $\mathrm{HH}^n(A)$ *is given by*

$$\dim \mathrm{HH}^n(A) = \begin{cases} \frac{5}{2}n + 4, & \text{if } n = 4r \text{ for } r \in \mathbb{N}_0, \\ \frac{5}{2}n + 5, & \text{if } n = 4r+2 \text{ for } r \in \mathbb{N}_0, \\ \frac{5n+9}{2}, & \text{if } n = 2r+1 \text{ for } r \in \mathbb{N}_0. \end{cases}$$

The full Hilbert series of the Hochschild cohomology with respect to the internal degree of $\mathrm{FK}(3)$ is in Corollary 4.2.20. In Chapter 5, by computing the cup products and using techniques from Gröbner bases, we prove that the Hochschild cohomology of $\mathrm{FK}(3)$ over a field $\Bbbk$ of characteristic different from 2 and 3 is given as a quotient of a free graded-commutative algebra (for the cohomological degree) with 14 homogeneous generators (see Proposition 5.1.5) modulo the homogeneous ideal generated by the 63 relations listed in (5.1.5) (see Corollary 5.1.11). Using the general methods introduced in Section 1.4, we compute the Gerstenhaber brackets on Hochschild cohomology of $\mathrm{FK}(3)$ over a field $\Bbbk$ of characteristic different from 2 and 3 between elements of cohomology degree m for $m \in [\![0,1]\!]$ and elements of cohomology degree $n \in \mathbb{N}_0$. Finally, we present a simple result that allows us to compute the remaining Gerstenhaber brackets under some assumptions on the algebra structure of the Hochschild cohomology of an algebra (see Lemma 5.2.12), which are verified in the case of the Fomin-Kirillov algebra $\mathrm{FK}(3)$ on three generators over a field of characteristic different from 2 and 3. We summarize all Gerstenhaber brackets of Hochschild cohomology of $\mathrm{FK}(3)$ in Table 5.2.1. By using the above explicit expression of the Gerstenhaber bracket, we show that the Gerstenhaber bracket on the Hochschild cohomology of $\mathrm{FK}(3)$ is not induced by any Batalin-Vilkovisky generator (see Proposition 5.2.15). The main result of Chapter 6 is that the Fomin-Kirillov algebra $\mathrm{FK}(4)$ of index 4 has a resolving datum, which allows us to construct a projective resolution of the trivial module.

Notations

We denote by $\mathbb{N}$ (resp., $\mathbb{N}_0$) the set of positive (resp., nonnegative) integers, and $\mathbb{Z}$ the set of integers. Given $i \in \mathbb{Z}$, we will denote by $\mathbb{Z}_{\leqslant i}$ the set $\{m \in \mathbb{Z} | m \leqslant i\}$. Given $i,j \in \mathbb{Z}$ with $i \leqslant j$, we will denote by $[\![i,j]\!] = \{m \in \mathbb{Z} | i \leqslant m \leqslant j\}$ the integer interval, and we define $\chi_n = 0$ if n is an odd integer and $\chi_n = 1$ if n is an even integer. Moreover, given $r \in \mathbb{R}$, we set $\lfloor r \rfloor = \sup\{n \in \mathbb{Z} | n \leqslant r\}$ the usual **floor function**.

In the whole thesis, $\Bbbk$ is a field and $\Bbbk^\times = \Bbbk \setminus \{0\}$. All maps between $\Bbbk$-vector spaces will be $\Bbbk$-linear and all unadorned tensor products $\otimes$ will be over $\Bbbk$. We denote by $v^* \in V^* = \mathrm{Hom}_\Bbbk(V,\Bbbk)$ the dual element of v, for an element v in a finite-dimensional $\Bbbk$-vector space V.

To reduce space in the expressions of the article we will typically denote the composition $f \circ g$ of maps f and g, the left action $f \cdot g$, where f is an element in a group and g is an element in a set, or the right action $f \cdot g$, where g is an element in a group and f is an element in a set, simply by their juxtaposition fg.

Chapter 1

Quadratic algebras and Hochschild (co)homology

In this chapter, we are going to recall the definitions and basic properties of quadratic algebras and modules, as well as the definition of Hochschild (co)homology. We also present a new construction of a projective resolution of the trivial module of a quadratic algebra satisfying some assumptions, and some methods to compute the Gerstenhaber bracket of Hochschild cohomology.

1.1 Koszul complex of quadratic algebras and modules

All the following results can be found in [21]. Let A be a unitary associative $\Bbbk$-algebra. The field $\Bbbk$ embeds in A via the ring homomorphism $\eta : \Bbbk \hookrightarrow A$. Suppose that A has an augmentation, i.e. there is a ring homomorphism $\epsilon : A \to \Bbbk$ such that $\epsilon\eta = \mathrm{id}_{\Bbbk}$. The field $\Bbbk$ is an A-module via the map ϵ. The algebra A is called **graded** if there are vector subspaces $\{A_n | n \in \mathbb{Z}\}$ of A such that $A = \oplus_{n\in\mathbb{Z}} A_n$ and $A_m \cdot A_n \subseteq A_{m+n}$ for all $m,n \in \mathbb{Z}$. A graded algebra A is called **connected** if $A_0 = \Bbbk$, $A_n = 0$ for $n \in \mathbb{Z} \setminus \mathbb{N}_0$ and $\mathrm{Ker}(\epsilon) = \oplus_{n=1}^{\infty} A_n$. We denote $\oplus_{n=1}^{\infty} A_n$ by A_+. A right module M over a graded algebra $A = \oplus_{n\in\mathbb{Z}} A_n$ is called **graded** if there are vector subspaces $\{M_n | n \in \mathbb{Z}\}$ of M such that $M = \oplus_{n\in\mathbb{Z}} M_n$ and $M_n \cdot A_m \subseteq M_{n+m}$ for all $m,n \in \mathbb{Z}$. Analogously, we will have the definition of a graded left A-module. The elements of A_n or M_n are called **homogeneous**, and n is called the **Adams degree** or the **internal degree** of the elements in A_n or M_n. A graded A-module M is called **bounded below** if $M_n = 0$ for $n \ll 0$. A graded algebra A (resp., graded A-module M) is called **locally finite dimensional** if every vector space A_n (resp., M_n) for $n \in \mathbb{Z}$ is finite dimensional over $\Bbbk$. For a graded A-module M and $i \in \mathbb{Z}$, we denote by $M(i)$ the same module with shifted grading $M(i)_n = M_{i+n}$ for $n \in \mathbb{Z}$. Given two graded A-modules M and N, a morphism $f : M \to N$ of A-modules is called **homogeneous** of degree $d \in \mathbb{Z}$ if $f(M_n) \subseteq N_{n+d}$ for all $n \in \mathbb{Z}$. In particular, a morphism of graded modules will be a homogeneous morphism of A-modules of degree zero. We assume that all graded algebras are connected and locally finite-dimensional, and all graded modules M are bounded below and locally finite-dimensional.

Let V be a vector space. The **tensor algebra** $\mathbb{T}(V)$ generated by V is given by $\mathbb{T}(V) = \oplus_{n=0}^{\infty} \mathbb{T}^n(V)$ where $\mathbb{T}^0(V) = \Bbbk$ and $\mathbb{T}^n(V) = V^{\otimes n}$ for $n \in \mathbb{N}$. The multiplication in $\mathbb{T}(V)$ is given by the tensor product $\mathbb{T}^m(V) \otimes \mathbb{T}^m(V) \cong \mathbb{T}^{m+n}(V)$ for $m,n \in \mathbb{N}_0$. The tensor algebra $\mathbb{T}(V)$ is a graded algebra.

Definition 1.1.1. *A graded $\Bbbk$-algebra $A = \oplus_{n\in\mathbb{N}_0} A_n$ is called **quadratic** if A is generated by $V = A_1$ as a $\Bbbk$-algebra and the kernel of the natural surjection $\mathbb{T}(V) \to A$ is generated as a two-sided ideal of $\mathbb{T}(V)$ by a subspace $R \subseteq \mathbb{T}^2(V) = V \otimes V$.*

More generally, given an integer $N \geqslant 2$, an algebra A is said to be **N-homogeneous** if it is of the form $\mathbb{T}(V)/(R)$, where V is a $\Bbbk$-vector space and $R \subseteq V^{\otimes N}$. An N-homogeneous algebra with $N = 2$ is quadratic.

For a graded algebra $A = \oplus_{n\in\mathbb{N}_0} A_n$, the **quadratic part** of A is a quadratic algebra ${}_{\mathrm{q}}A = \mathbb{T}(A_1)/(I \cap \mathbb{T}^2(A_1))$, where I is the kernel of the natural morphism $\mathbb{T}(A_1) \to A$. There is a morphism ${}_{\mathrm{q}}A \to A$ of algebras, which is bijective in degree 1 and injective in degree 2.

Let V^* be the dual vector space of V and define the linear map $\gamma_n : (V^*)^{\otimes n} \otimes V^{\otimes n} \to \Bbbk$ by

$$\gamma_n(f_1 \otimes \cdots \otimes f_n, v_1 \otimes \cdots \otimes v_n) = f_1(v_1) \dots f_n(v_n) \tag{1.1.1}$$

for all $f_1, \dots, f_n \in V^*$ and $v_1, \dots, v_n \in V$. Let $A = \mathbb{T}(V)/(R)$ be a quadratic algebra. The **quadratic dual algebra** of A defined by $A^! = \mathbb{T}(V^*)/(R^\perp) = \oplus_{n \in \mathbb{N}_0} A^!_{-n}$ is a quadratic algebra, where $R^\perp$ is the subspace of $V^* \otimes V^*$ defined by

$$R^\perp = \{\alpha \in V^* \otimes V^* \mid \gamma_2(\alpha, r) = 0 \text{ for all } r \in R\}.$$

Note that $A^!_0 = \Bbbk$, $A^!_{-1} = V^*$, and the isomorphism $(V^*)^{\otimes n} \cong (V^{\otimes n})^*$ induced by γ_n induces an isomorphism of vector spaces

$$A^!_{-n} = \frac{(V^*)^{\otimes n}}{\sum\limits_{i=0}^{n-2} (V^*)^{\otimes i} \otimes R^\perp \otimes (V^*)^{\otimes (n-2-i)}} \xrightarrow{\sim} \left(\bigcap_{i=0}^{n-2} V^{\otimes i} \otimes R \otimes V^{\otimes (n-2-i)} \right)^*$$

for $n \geqslant 2$. The space $A^!_{-n}$ is concentrated in Adams degree $-n$ for $n \in \mathbb{N}_0$, and we consider $A^!$ to be $\mathbb{Z}$-graded with $A^!_n = 0$ for $n \in \mathbb{N}$. Let $e = \sum_{i \in I} e_i^* \otimes e_i \in A^!_1 \otimes A_1$, where $\{e_i | i \in I\}$ is a basis of V and $\{e_i^* | i \in I\}$ is the dual basis. Note that $e^2 = 0$ in the algebra $A^! \otimes A$. Indeed, by the isomorphism of vector spaces $\mathrm{Hom}_\Bbbk(V_1, V_2) \cong V_1^* \otimes V_2$ for any finite dimensional vector spaces V_1, V_2, the multiplication map $(V^* \otimes V)^{\otimes 2} \to A^!_{-2} \otimes A_2$ can be identified with the map $\mathrm{Hom}_\Bbbk(V^{\otimes 2}, V^{\otimes 2}) \to \mathrm{Hom}_\Bbbk(R, V^{\otimes 2}/R)$. Then $e \otimes e$ corresponds to the identity element in $\mathrm{Hom}_\Bbbk(V^{\otimes 2}, V^{\otimes 2})$, and above map sends it to zero.

Definition 1.1.2. *Let $A = \mathbb{T}(V)/(R)$ be a quadratic algebra. A graded right A-module $M = \oplus_{n \in \mathbb{Z}} M_n$ is called* ***quadratic*** *if $M_n = 0$ for $n \in \mathbb{Z} \setminus \mathbb{N}_0$, M is generated by $W = M_0$ as a right A-module, and the kernel of the natural surjection $W \otimes A \to M$ is generated as an A-submodule of $W \otimes A$ by a subspace $J \subseteq W \otimes V$.*

For a graded algebra $A = \oplus_{n \in \mathbb{N}_0} A_n$ and a graded A-module $M = \oplus_{n \in \mathbb{N}_0} M_n$, the **quadratic part** of M is a quadratic module ${}_{\mathrm{q}}M = (M_0 \otimes {}_{\mathrm{q}}A)/(I \cap (M_0 \otimes A_1))$ over the quadratic algebra ${}_{\mathrm{q}}A$, where I is the kernel of the natural morphism $M_0 \otimes {}_{\mathrm{q}}A \to M$. There is a morphism ${}_{\mathrm{q}}M \to M$ of ${}_{\mathrm{q}}A$-modules, which is bijective in degree 0 and injective in degree 1.

Let $M = (W \otimes A)/(J)$ be a quadratic module over a quadratic algebra A. The **quadratic dual module** of M defined by $M^! = (W^* \otimes A^!)/(J^\perp) = \oplus_{n \in \mathbb{N}_0} M^!_{-n}$ is a quadratic module over $A^!$, where $J^\perp$ is the subspace of $W^* \otimes V^*$ given by

$$J^\perp = \{\alpha \in W^* \otimes V^* \mid \gamma'(\alpha, r) = 0 \text{ for all } r \in J\},$$

and the linear map $\gamma' : W^* \otimes V^* \otimes W \otimes V \to \Bbbk$ is defined by $\gamma'(f_1 \otimes f_2, v_1 \otimes v_2) = f_1(v_1) f_2(v_2)$ for all $f_1 \in W^*$, $f_2 \in V^*$, $v_1 \in W$ and $v_2 \in V$. Note that $M^!_0 = W^*$, $M^!_{-1} \cong J^*$, and

$$M^!_{-n} \cong \left(J \otimes V^{\otimes (n-1)} \cap \Big(\bigcap_{i=0}^{n-2} W \otimes V^{\otimes i} \otimes R \otimes V^{\otimes (n-2-i)} \Big) \right)^* \tag{1.1.2}$$

for $n \geqslant 2$. The space $M^!_{-n}$ is concentrated in Adams degree $-n$ for $n \in \mathbb{N}_0$, and we consider $M^!$ to be $\mathbb{Z}$-graded with $M^!_n = 0$ for $n \in \mathbb{N}$.

Proposition 1.1.3 ([21], Proposition 3.1 of chapter 1)**.** *For a graded algebra A and graded A-module $M = \oplus_{n \in \mathbb{N}_0} M_n$, the cohomology spaces $\mathrm{Ext}_A^{i,-j}(M, \Bbbk)$ concentrated in homological degree i and internal degree $-j$ are zero for all $i > j$. Moreover, the diagonal subalgebra $\bigoplus_{i \in \mathbb{N}_0} \mathrm{Ext}_A^{i,-i}(\Bbbk, \Bbbk)$ of the algebra $\bigoplus_{i,j \in \mathbb{N}_0} \mathrm{Ext}_A^{i,-j}(\Bbbk, \Bbbk)$ is always quadratic. The diagonal submodule $\bigoplus_{i \in \mathbb{N}_0} \mathrm{Ext}_A^{i,-i}(M, \Bbbk)$ of the module $\bigoplus_{i,j \in \mathbb{N}_0} \mathrm{Ext}_A^{i,-j}(M, \Bbbk)$ is always a quadratic module over $\bigoplus_{i \in \mathbb{N}_0} \mathrm{Ext}_A^{i,-i}(\Bbbk, \Bbbk)$. More precisely,*

$$\bigoplus_{i \in \mathbb{N}_0} \mathrm{Ext}_A^{i,-i}(\Bbbk, \Bbbk) \cong ({}_{\mathrm{q}}A)^!, \quad \bigoplus_{i \in \mathbb{N}_0} \mathrm{Ext}_A^{i,-i}(M, \Bbbk) \cong ({}_{\mathrm{q}}M)^!.$$

Lemma 1.1.4 ([21], Corollary 5.3 of chapter 1)**.** *Let A be a graded algebra and $M = \oplus_{n\in\mathbb{N}_0} M_n$ a graded A-module.*

(1) A is quadratic if and only if $\mathrm{Ext}_A^{i,-j}(\Bbbk,\Bbbk) = 0$ for $i < j$ and $i = 1,2$.

(2) Assume that A is quadratic. Then M is quadratic if and only if $\mathrm{Ext}_A^{i,-j}(M,\Bbbk) = 0$ for $i < j$ and $i = 0,1$.

Let $A = \mathbb{T}(V)/(R)$ be a quadratic algebra and $M = (W \otimes A)/(J)$ a quadratic right module over A. The graded dual $(M^!)^\# = \oplus_{n\in\mathbb{N}_0}(M^!_{-n})^*$ is a graded left $A^!$-module via the action $(uf)(v) = f(vu)$ for $u \in A^!$, $v \in M^!$ and $f \in (M^!)^\#$. Then $(M^!)^\# \otimes A$ is a graded left module over the algebra $A^! \otimes A$ by the above action and the multiplication in A. Let $\mathrm{K}_n(M) = (M^!_{-n})^* \otimes A$ for $n \in \mathbb{N}_0$ and the differential $d_n : \mathrm{K}_n(M) \to \mathrm{K}_{n-1}(M)$ for $n \in \mathbb{N}$ be the morphism defined by $d_n(u \otimes v) = e(u \otimes v)$ for $u \in (M^!_{-n})^*$ and $v \in A$. Since $e^2 = 0$, we have $d_{n+1}d_n = 0$ for $n \in \mathbb{N}$. The complex $(\mathrm{K}_\bullet(M),d_\bullet)$

$$\cdots \longrightarrow \mathrm{K}_2(M) \xrightarrow{d_2} \mathrm{K}_1(M) \xrightarrow{d_1} \mathrm{K}_0(M) \longrightarrow 0$$

of free (bounded-below) graded right A-modules is called the (right) **Koszul complex** of the quadratic module M over A. As usual, we can consider the Koszul complex as a complex indexed by $\mathbb{Z}$, with $\mathrm{K}_n(M) = 0$ for $n \in \mathbb{Z} \setminus \mathbb{N}_0$ and $d_n = 0$ for $n \in \mathbb{Z} \setminus \mathbb{N}$. By the composition of the canonical isomorphism $V^{\otimes n} \xrightarrow{\sim} (V^{\otimes n})^{**}$ and the dual of (1.1.2) for $M^!_{-n}$, the differential $d_n : \mathrm{K}_n(M) \to \mathrm{K}_{n-1}(M)$ is the restriction of the map $\tilde{d}_n : W \otimes V^{\otimes n} \otimes A \to W \otimes V^{\otimes(n-1)} \otimes A$ determined by

$$v_0 \otimes (v_1 \otimes \cdots \otimes v_n) \otimes a \mapsto v_0 \otimes (v_1 \otimes \cdots \otimes v_{n-1}) \otimes v_n a$$

for all $v_0 \in W$, $v_1, \ldots, v_n \in V$, $a \in A$ and $n \in \mathbb{N}$. Let $\epsilon' : W \otimes A \to M$ be the natural surjection.

Fact 1.1.5. *We have $\mathrm{Ker}(\epsilon') = \mathrm{Im}(d_1)$, and in fact $(\mathrm{K}_\bullet(M),d_\bullet)$ coincides with the minimal projective resolution of M in the category of bounded below graded right A-modules, up to homological degree 1.*

Recall that the Koszul complex $(\mathrm{K}_\bullet(M),d_\bullet)$ is **minimal**, *i.e.* the induced map $d_n \otimes \mathrm{id}_\Bbbk$: $\mathrm{K}_n(M) \otimes_A \Bbbk \to \mathrm{K}_{n-1}(M) \otimes_A \Bbbk$ vanishes for all $n \in \mathbb{N}$. Note that the trivial module $\Bbbk = (W \otimes A)/(J)$ is a quadratic module over A with $W = \Bbbk$ and $J = \Bbbk \otimes V$. Since $J^\perp = 0$, the quadratic dual module of $\Bbbk$ is $\Bbbk^! = \Bbbk^* \otimes A^! \cong A^!$. The **Koszul complex** $(K_\bullet(A),d_\bullet)$ of a quadratic algebra A is defined as the Koszul complex of the trivial module $\Bbbk$ over A. Then $K_n(A) = (A^!_{-n})^* \otimes A$ and $(K_\bullet(A),d_\bullet)$ has the following form

$$\cdots \longrightarrow K_2(A) \xrightarrow{d_2} K_1(A) \xrightarrow{d_1} K_0(A) \longrightarrow 0.$$

Fact 1.1.6. *We have $\mathrm{Ker}(\epsilon) = \mathrm{Im}(d_1)$ and $\mathrm{Ker}(d_1) = \mathrm{Im}(d_2)$, and in fact $(K_\bullet(A),d_\bullet)$ coincides with the minimal projective resolution of the trivial right A-module $\Bbbk$ in the category of bounded below graded right A-modules, up to homological degree 2.*

Let $A = \mathbb{T}(V)/(R)$ be a quadratic algebra, and let $M = (V_M \otimes A)/(R_M)$ and $N = (V_N \otimes A)/(R_N)$ be two quadratic right A-modules. Let us denote by $\hom_A(M,N)$ the vector space formed by all homogeneous morphisms $f : M \to N$ of A-modules of degree zero, and by $\mathrm{Hom}((V_M,R_M),(V_N,R_N))$ the vector space formed by all linear morphisms $g : V_M \to V_N$ satisfying that $(g \otimes \mathrm{id}_V)(R_M) \subseteq R_N$. Then, it is clear that the map

$$\hom_A(M,N) \to \mathrm{Hom}\big((V_M,R_M),(V_N,R_N)\big)$$

sending f to its restriction $f|_{V_M} : V_M \to V_N$ is an isomorphism. This tells us that $f : M \to N$ is a monomorphism (resp., epimorphism) in the category of quadratic right A-modules with homogeneous morphisms of A-modules of degree zero if and only if $f|_{M_0} : M_0 \to N_0$ is injective (resp., surjective). In particular, a morphism of the category of quadratic right A-modules with homogeneous morphisms of A-modules of degree zero is an epimorphism if and only if it is a surjection.

Remark 1.1.7. *Assume the space of generators of the quadratic algebra A has a nonzero dimension. Then, the category of quadratic right A-modules with homogeneous morphisms of A-modules of degree zero is not abelian, since the canonical projection $A \to \Bbbk$ is a monomorphism and an epimorphism but it is not an isomorphism. In particular, the example shows that monomorphisms of the category of quadratic* *modules are not necessarily injective. For a less trivial example, consider* $\Bbbk$ *of characteristic different from* 2, $A = \Bbbk\langle x,y\rangle/(xy - yx) = \Bbbk[x,y]$, $M = e.A$, $M' = (e_1.A \oplus e_2.A)/(e_1.x + e_2.x, e_1.y - e_2.y)$ *and the morphism* $f : M \to M'$ *of* A*-modules sending* e *to* e_1 *is a non-injective monomorphism of quadratic modules, since*

$$f(e.xy + e.yx) = (e_1.x + e_2.x).y + (e_1.y - e_2.y).x$$

vanishes, but $f|_{M_0}$ *and* $f|_{M_1}$ *are injective.*

Given $f \in \hom_A(M,N)$, define the homogeneous morphism $f^{!_m} : N^{!_m} \to M^{!_m}$ of right $A^!$-modules of degree zero whose restriction to V_N^* is precisely the dual $(f|_{V_M})^*$ of $f|_{V_M} : V_M \to V_N$. Since $((f|_{V_M})^* \otimes \mathrm{id}_{V^*})(R_N^\perp) \subseteq R_M^\perp$, the map $f^{!_m}$ is well defined. By taking the graded dual $(f^{!_m})^\# : (M^{!_m})^\# \to (N^{!_m})^\#$ we obtain a homogeneous morphism of left $A^!$-modules of degree zero.

Let $\mathrm{K}_\bullet(M)$ and $\mathrm{K}_\bullet(N)$ be the Koszul complex of M and N respectively. We finally define the morphism

$$\mathrm{K}_\bullet(f) : \mathrm{K}_\bullet(M) \to \mathrm{K}_\bullet(N)$$

of complexes of right A-modules by $\mathrm{K}_\bullet(f) = (f^{!_m})^\# \otimes \mathrm{id}_A$. It is clear that $\mathrm{K}_\bullet(fg) = \mathrm{K}_\bullet(f)\mathrm{K}_\bullet(g)$ and $\mathrm{K}_\bullet(\mathrm{id}_M) = \mathrm{id}_{\mathrm{K}_\bullet(M)}$, for $f \in \hom_A(M,N)$, $g \in \hom_A(N',M)$ and N' a quadratic right A-module.

Remark 1.1.8. *If* f *is injective, then* $f|_{V_M}$ *is also injective, which implies that its dual* $(f|_{V_M})^*$ *is surjective, so* $f^{!_m}$ *is surjective as well, which in turn implies that* $(f^{!_m})^\#$ *and* $\mathrm{K}_\bullet(f)$ *are injective.*

Definition 1.1.9. *A graded algebra* A *is called* ***Koszul*** *if the following equivalent conditions hold:*

(1) $\mathrm{Ext}_A^{i,-j}(\Bbbk,\Bbbk) = 0$ *for* $i \neq j$.

(2) A *is quadratic and* $\mathrm{Ext}_A^\bullet(\Bbbk,\Bbbk) \cong A^!$.

(3) $\Bbbk$ *has a graded projective resolution* $(P_\bullet, d_\bullet)$ *such that* P_n *as a graded* A*-module is generated by homogeneous elements of degree* n.

Definition 1.1.10. *Let* A *be a Koszul algebra. A graded* A*-module* M *is called* ***Koszul*** *if the following equivalent conditions hold:*

(1) $\mathrm{Ext}_A^{i,-j}(M,\Bbbk) = 0$ *for* $i \neq j$.

(2) M *is quadratic and* $\mathrm{Ext}_A^\bullet(M,\Bbbk) \cong M^!$.

(3) M *has a graded projective resolution* $(P_\bullet, d_\bullet)$ *such that* P_n *as a graded* A*-module is generated by homogeneous elements of degree* n.

Lemma 1.1.11 ([21], Corollary 3.2 of chapter 2)**.** *(1) A quadratic algebra* A *is Koszul if and only if its Koszul complex* $(K_\bullet(A),d_\bullet)$ *is exact in positive homological degrees.*

(2) Let A *be a Koszul algebra. A quadratic module* M *over* A *is Koszul if and only if its Koszul complex* $(\mathrm{K}_\bullet(M),d_\bullet)$ *is exact in positive homological degrees.*

1.2 Resolving data on quadratic algebras

In this section, we introduce a new construction of a projective resolution of the trivial module of a quadratic algebra satisfying some assumptions (see Theorem 1.2.5).

A **resolving datum** on a quadratic algebra A is a finite set $\mathscr{M} = \{M^0, \dots, M^N\}$ of pairwise non-isomorphic quadratic (right) A-modules with $N \in \mathbb{N}_0$ such that $M^0 = \Bbbk$ is the trivial module and a map

$$\hbar : [\![0,N]\!]^2 \times \mathbb{N}^2 \to \mathbb{N}_0^2$$

such that

(R.1) $\hbar$ has finite support,

(R.2) there are short exact sequences of right A-modules

$$0 \to \bigoplus_{j=0}^{N} \bigoplus_{\ell\in\mathbb{N}} \left(M^j(-\ell)\right)^{\pi_1(\hbar(i,j,k,\ell))} \to \mathrm{H}_k(\mathrm{K}_\bullet(M^i)) \to \bigoplus_{j=0}^{N} \bigoplus_{\ell\in\mathbb{N}} \left(M^j(-\ell)\right)^{\pi_2(\hbar(i,j,k,\ell))} \to 0 \qquad (1.2.1)$$

with homogeneous morphisms of degree zero for all $(i,k) \in [\![0,N]\!] \times \mathbb{N}$, where $\pi_i : \mathbb{N}_0^2 \to \mathbb{N}_0$ is the canonical projection on the i-th component for $i \in \{1,2\}$, (R.3) If (1.2.1) splits for some $i_0 \in [\![0, N]\!]$ and $k_0 \in \mathbb{N}$, then $\pi_1(\hbar(i_0,j,k_0,\ell)) = 0$ for all $j \in [\![0, N]\!]$ and $\ell \in \mathbb{N}$.

Recall that a **quiver** is the datum of a set Q_0, called **set of vertices**, and set Q_1, called **set of arrows**, together with maps $s, t : Q_1 \to Q_0$ called the **source** and **target** maps of the quiver. We say the quiver is **bigraded** if we further have a map $\mathrm{bideg} : Q_1 \to \mathbb{Z}^2$. We will denote the bidegree of an arrow α of Q_1 by $\mathrm{bideg}(\alpha) = (\mathrm{bideg}_1(\alpha),\mathrm{bideg}_2(\alpha)) \in \mathbb{Z}^2$. The **difference degree** of an arrow α is defined as $\mathrm{dfdeg}(\alpha) = \mathrm{bideg}_2(\alpha) - \mathrm{bideg}_1(\alpha) \in \mathbb{Z}$.

We also recall that, given a quiver with a set of vertices Q_0 and set of arrows Q_1, a **path** of length $n \in \mathbb{N}_0$ is a vertex if $n = 0$, and a tuple $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$ in Q_1^n for $n \in \mathbb{N}$ such that $t(\alpha_i) = s(\alpha_{i+1})$ for all $i \in [\![1, n-1]\!]$. As usual, we define $s(e) = t(e) = e$ for any vertex e, $s(\alpha_1, \dots, \alpha_n) = s(\alpha_1)$ and $t(\alpha_1, \dots, \alpha_n) = t(\alpha_n)$ for every path $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$ of length $n \in \mathbb{N}$. Furthermore, if the quiver is bigraded, given a path $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$ of length $n \in \mathbb{N}$, we define its bidegree $\mathrm{bideg}(\bar{\alpha}) = (\mathrm{bideg}_1(\bar{\alpha}),\mathrm{bideg}_2(\bar{\alpha})) \in \mathbb{Z}^2$ by $(\sum_{i=1}^{n} \mathrm{bideg}_1(\alpha_i), \sum_{i=1}^{n} \mathrm{bideg}_2(\alpha_i))$. The bidegree of a path of length zero given by a vertex e is defined as $\mathrm{bideg}(e) = (\mathrm{bideg}_1(e),\mathrm{bideg}_2(e)) = (0,0)$. The **difference degree** of a path $\bar{\alpha}$ is defined as $\mathrm{dfdeg}(\bar{\alpha}) = \mathrm{bideg}_2(\bar{\alpha}) - \mathrm{bideg}_1(\bar{\alpha}) \in \mathbb{Z}$.

Given a quadratic algebra together with a resolving data as in the first paragraph of this subsection, we define the associated **resolving quiver** $\mathcal{R}\mathbb{Q}_A$ as the unique bigraded quiver with set of vertices $\{M^0, \dots, M^N\}$, and whose set of arrows of degree (d',d'') from M^i to M^j has cardinality $\pi_1(\hbar(i,j,d'-1,d'')) + \pi_2(\hbar(i,j,d'-1,d''))$. To be able to manipulate these arrows, assume we have chosen a fixed set $\mathcal{A}r'_{i,j,d',d''}$ of arrows of degree (d',d'') from M^i to M^j of cardinality $\pi_1(\hbar(i,j,d'-1,d''))$ and another fixed set $\mathcal{A}r''_{i,j,d',d''}$ of arrows of degree (d',d'') from M^i to M^j of cardinality $\pi_2(\hbar(i,j,d'-1,d''))$, such that $\mathcal{A}r'_{i,j,d',d''}$ and $\mathcal{A}r''_{i,j,d',d''}$ are disjoint. For every $i \in [\![0,N]\!]$ and $d' \in \mathbb{N}$, we also set a strict partial order on the set of all arrows α of $\mathcal{R}\mathbb{Q}_A$ such that $s(\alpha) = M^i$ and $\mathrm{bideg}_1(\alpha) = d'$ by setting precisely that every arrow of $\mathcal{A}r''_{i,j,d',d''}$ is strictly less than every arrow of $\mathcal{A}r'_{i,j',d',d'''}$ for all $j,j' \in [\![0,N]\!]$ and $d'',d''' \in \mathbb{N}$. Note that this quiver is finite by (R.1). We will say that the resolving datum is **connected** if the associated resolving quiver is connected.

As we will see, the resolving quiver $\mathcal{R}\mathbb{Q}_A$ contains some homological information about the algebra A. The first clues in this direction are given by the following results, the first of which is trivial.

Proposition 1.2.1. *A quadratic algebra A is Koszul if and only if the resolving quiver associated to a (equivalently, to every) connected resolving datum on A has no arrows.*

Proposition 1.2.2. *Let $p, q \geqslant 2$ be integers. A quadratic algebra A is (p,q)-Koszul (in the sense introduced by S. Brenner, M. Butler and A. King in [5]) if and only if it is finite-dimensional with $\dim(A_p) \neq 0$ and $\dim(A_{p+1}) = 0$, the Koszul complex of A has finite length q and the resolving quiver associated to a (equivalently, to every) connected resolving datum on A has only one vertex and $\dim(A_p) \cdot \dim(A^!_q)$ arrows of bidegree $(q+1,q+p)$.*

Proof. This is precisely Prop. 3.9 of [5]. □

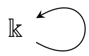
We also have the following two examples of resolving quivers.

Example 1.2.3. *Let $m \geqslant 5$ be an integer and let C be the quadratic algebra defined in Section 2 of [7], which depends on m. We will follow the notation of that article. Let $\mathcal{M} = \{\Bbbk, M^1\}$ where M^1 is the standard right module C, and let $\hbar : \{0,1\}^2 \times \mathbb{N}^2 \to \mathbb{N}_0^2$ be the map given by $\hbar(0,1,m-1,m+1) = (0,1)$ and $\hbar(i,j,k,\ell) = (0,0)$ if $(i,j,k,\ell) \neq (0,1,m-1,m+1)$. Then, [7], Theorem 2.7, tells us that this gives us a connected resolving datum on C, whose associated resolving quiver is*

$$\Bbbk \longrightarrow C$$

such that its arrow has bidegree $(m,m+1)$.

Example 1.2.4. *Consider the Fomin-Kirillov algebra* $\mathrm{FK}(3)$ *on three generators. Let $\mathcal{M} = \{\Bbbk\}$ and let $\hbar : \{0\}^2 \times \mathbb{N}^2 \to \mathbb{N}_0^2$ be the map given by $\hbar(0,0,3,6) = (0,1)$ and $\hbar(i,j,k,\ell) = (0,0)$ if $(i,j,k,\ell) \neq (0,0,3,6)$. Then, Proposition 3.1 of [11], tells us that this gives a resolving datum on* $\mathrm{FK}(3)$ *whose associated resolving quiver is*



such that its unique arrow has bidegree (4,6).

From the resolving quiver associated to a resolving datum of the form given in the first paragraph of this subsection, we can define the **set of paths** $\mathscr{P}a_{M^i}$ given as the set formed by all paths $\bar{\alpha}$ of the quiver $\mathscr{R}\mathbb{Q}_A$ such that $s(\bar{\alpha}) = M^i$.

Moreover, we will define the following strict partial order on $\mathscr{P}a_{M^i}$ for every $i \in [\![0,N]\!]$ as follows. First, we set the vertex at M^i to be strictly greater than any other path of $\mathscr{P}a_{M^i}$. Given $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\bar{\beta} = (\beta_1, \dots, \beta_m)$ in $\mathscr{P}a_{M^i}$ with $n, m \in \mathbb{N}$, we say that $\bar{\alpha} < \bar{\beta}$ if $\alpha_j = \beta_j$ for all $j \in [\![1,j_0]\!]$ for some $j_0 \in [\![0, \min(n,m)]\!]$, and one of the following possibilities holds:

(O.1) $n,m > j_0$, $\mathrm{bideg}_1(\alpha_{j_0+1}) = \mathrm{bideg}_1(\beta_{j_0+1})$ and $\alpha_{j_0+1} < \beta_{j_0+1}$;

(O.2) $n,m > j_0$, $\mathrm{bideg}_1(\alpha_{j_0+1}) < \mathrm{bideg}_1(\beta_{j_0+1})$;

(O.3) $j_0 = m < n$.

It is clear that this defines a strict partial order on $\mathscr{P}a_{M^i}$.

Given a connected resolving datum we have the following result, which gives a description of a projective resolution of every quadratic module M^i.

Theorem 1.2.5. *Assume we have a connected resolving datum on a quadratic algebra A with a set of quadratic modules $\mathscr{M} = \{M^0, \dots, M^N\}$ and whose resolving quiver is denoted by $\mathscr{R}\mathbb{Q}_A$. Then, there exists a projective resolution $P_\bullet^{M^i}$ of M^i in the category of bounded below graded right A-modules such that*

$$P_n^{M^i} = \bigoplus_{\substack{\bar{\alpha} \,\in\, \mathscr{P}a_{M^i}, \\ \mathrm{bideg}_1(\bar{\alpha}) \leqslant n}} \bar{\alpha}.\mathrm{K}_{n-\mathrm{bideg}_1(\bar{\alpha})}\big(t(\bar{\alpha})\big)\big(-\mathrm{bideg}_2(\bar{\alpha})\big) \tag{1.2.2}$$

for all $n \in \mathbb{N}_0$ and $i \in [\![0,N]\!]$, where the symbol $\bar{\alpha}$ multiplying the Koszul complex on the left is only a formal symbol used as a simple bookkeeping device. Moreover, if

$$\mathrm{dfdeg}(\bar{\alpha}) \neq \mathrm{dfdeg}(\bar{\beta}) - 1 \tag{1.2.3}$$

for all $\bar{\alpha}, \bar{\beta} \in \mathscr{P}a_{M^i}$ such that $\bar{\alpha} < \bar{\beta}$ (e.g. if $\mathrm{dfdeg}(\alpha)$ is even for all arrows α of $\mathscr{R}\mathbb{Q}_A$), then the previous projective resolution is minimal.

Proof. We are going to use the following notation. Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a short exact sequence of right A-modules and let $P'_\bullet \to M'$ and $P''_\bullet \to M''$ be two projective resolutions (resp., up to homological degree $m \in \mathbb{N}$) with differentials $d'_\bullet$ and $d''_\bullet$, respectively. Then, we will note by $P_\bullet = P'_\bullet \hat{\oplus} P''_\bullet \to M$ the projective resolution (resp., up to homological degree $m \in \mathbb{N}$) given by the Horseshoe lemma (see [29], Lemma 2.2.8). We recall that $P_n = P'_n \oplus P''_n$ for all $n \in \mathbb{N}_0$ (resp., for all $n \in [\![0,m]\!]$) with differential $d_\bullet$ satisfying that $d_\bullet|_{P'_\bullet} = d'_\bullet$ and $d_\bullet|_{P''_\bullet} = d''_\bullet + f_\bullet$ for some family $\{f_n : P''_n \to P'_{n-1} \mid n \in \mathbb{N}\}$ (resp., $\{f_n : P''_n \to P'_{n-1} \mid n \in [\![1,m]\!]\}$) of morphisms of A-modules.

Given $i \in [\![0, N]\!]$, let $m_i \in \mathbb{N}$ be the largest positive integer such that $\mathrm{H}_{m_i}(\mathrm{K}_\bullet(M^i)) \neq 0$ and $\mathrm{H}_k(\mathrm{K}_\bullet(M^i)) = 0$ for all integers $k > m_i$. If $\mathrm{H}_k(\mathrm{K}_\bullet(M^i)) = 0$ for all $k \in \mathbb{N}$, then we set $m_i = 0$.

We will denote by $d_{k+1}^i : \mathrm{K}_{k+1}(M^i) \to \mathrm{K}_k(M^i)$ the differential of the Koszul complex of M^i for $k \in \mathbb{N}_0$ and $i \in [\![0, N]\!]$. For every $i \in [\![0, N]\!]$, we will construct a projective resolution $P_\bullet^i$ of M^i. By Fact 1.1.5 we will assume that $P_n^i = \mathrm{K}_n(M^i)$ for $i \in [\![0, N]\!]$ and $n \in \{0, 1\}$. In particular, P_n^i coincides with (1.2.2) for all $i \in [\![0, N]\!]$ and $n \in \{0, 1\}$. We will in fact prove that P_n^i coincides with (1.2.2) for all $i \in [\![0, N]\!]$ and $n \in \mathbb{N}_0$ by induction on the homological degree n. If $m_i = 0$, we set $P_\bullet^i = \mathrm{K}_\bullet(M^i)$ for all $\bullet \in \mathbb{N}_0$. It is straightforward to see that the resolutions $P_\bullet^i$ and (1.2.2) coincide.

We will now construct $P_\bullet^i$ for all $\bullet \in \mathbb{N}_0$ for $i \in [\![0, N]\!]$ such that $m_i > 0$. Let $m \in \mathbb{N}$. Assume that we have defined P_n^i for all $i \in [\![0, N]\!]$ such that $m_i > 0$ and $n \in [\![0, m]\!]$ such that P_n^i coincides with (1.2.2) for all $n \in [\![0, m]\!]$. Using the Horseshoe lemma for (1.2.1), we get a projective resolution of $\mathrm{H}_k(\mathrm{K}_\bullet(M^i))$ of the form

$$^{m}Q_\bullet^{i,k} = \Big(\bigoplus_{j=0}^{N} \bigoplus_{\ell \in \mathbb{N}} \Big(P_\bullet^j(-\ell)\Big)^{\pi_1(\hbar(i,j,k,\ell))} \Big) \hat{\oplus} \Big(\bigoplus_{j=0}^{N} \bigoplus_{\ell \in \mathbb{N}} \Big(P_\bullet^j(-\ell)\Big)^{\pi_2(\hbar(i,j,k,\ell))} \Big)$$

defined for homological degrees $\bullet \in [\![0, m]\!]$, $i \in [\![1,N]\!]$ and $k \in [\![1,m_i]\!]$. We will construct by induction on the index $k \in [\![0, m_i]\!]$ a family of complexes of right A-modules ${}^m R^{i,k}_\bullet$ for $\bullet \in [\![0, m + 1]\!]$ such that ${}^m R^{i,k}_\bullet$ is a projective resolution of $\mathrm{Im}(d^i_{m_i-k+1})$ up to homological degree $m + 1$. For $k = 0$, we set ${}^m R^{i,0}_\bullet$ as the complex of right A-modules given by $(\mathrm{K}_{\bullet+m_i+1}(M^i), d^i_{\bullet+m_i+1})_{\bullet\in\mathbb{N}_0}$. Note that ${}^m R^{i,0}_\bullet$ is a projective resolution of $\mathrm{Im}(d^i_{m_i+1})$ for $i \in [\![0, N]\!]$ such that $m_i > 0$, and it is independent of m. Assume now we have defined a complex of right A-modules ${}^m R^{i,k-1}_\bullet$ for some $k \in [\![1, m_i]\!]$ and $\bullet \in [\![0, m + 1]\!]$ such that ${}^m R^{i,k-1}_\bullet$ is a projective resolution of $\mathrm{Im}(d^i_{m_i-k+2})$ up to homological degree $m+1$. Then, we define the complex of right A-modules ${}^m R^{i,k}_\bullet$ by

$$ {}^m R^{i,k}_0 = \mathrm{K}_{m_i-k+1}(M^i) \text{ and } {}^m R^{i,k}_\bullet = {}^m R^{i,k-1}_{\bullet-1} \hat{\oplus} {}^m Q^{i,m_i-k+1}_{\bullet-1} $$

for $\bullet \in [\![1, m + 1]\!]$, the differential $d^{i,k}_\bullet$ for $\bullet \geqslant 2$ is induced by that of ${}^m R^{i,k-1}_{\bullet-1} \hat{\oplus} {}^m Q^{i,m_i-k+1}_{\bullet-1}$ and $d^{i,k}_1 : {}^m R^{i,k}_1 \to {}^m R^{i,k}_0$ is given as the composition of the augmentation ${}^m R^{i,k-1}_\bullet \hat{\oplus} {}^m Q^{i,m_i-k+1}_\bullet \to \mathrm{Ker}(d^i_{m_i-k+1})$ and the inclusion $\mathrm{Ker}(d_{m_i-k+1}) \hookrightarrow \mathrm{K}_{m_i-k+1}(M^i)$. Using the Horseshoe lemma for

$$ 0 \longrightarrow \mathrm{Im}(d^i_{m_i-k+2}) \longrightarrow \mathrm{Ker}(d^i_{m_i-k+1}) \longrightarrow \mathrm{H}_{m_i-k+1}\left(\mathrm{K}_\bullet(M^i)\right) \longrightarrow 0 $$

together with the projective resolutions ${}^m R^{i,k-1}_\bullet$ and ${}^m Q^{i,m_i-k+1}_\bullet$ for for $\bullet \in [\![0, m]\!]$, we obtain that the complex ${}^m R^{i,k-1}_\bullet \hat{\oplus} {}^m Q^{i,m_i-k+1}_\bullet$ for $\bullet \in [\![0, m]\!]$ is a projective resolution of $\mathrm{Ker}(d^i_{m_i-k+1})$ up to homological degree m, and thus ${}^m R^{i,k}_\bullet$ for for $\bullet \in [\![0, m + 1]\!]$ is a projective resolution of $\mathrm{Im}(d^i_{m_i-k+1})$ up to homological degree $m + 1$, as was to be shown. In particular, ${}^m R^{i,m_i}_\bullet$ for $\bullet \in [\![0, m + 1]\!]$ is a projective resolution of $\mathrm{Im}(d^i_1)$ up to homological degree $m + 1$. Let ${}^m R^i_\bullet = \mathrm{K}_0(M^i)$ and ${}^m R^i_\bullet = {}^m R^{i,m_i}_{\bullet-1}$ for $\bullet \in [\![1,m+2]\!]$. Then ${}^m R^i_\bullet$ for $\bullet \in [\![0, m+2]\!]$ is a projective resolution of M^i up to homological degree $m + 2$. A long but straightforward computation shows that ${}^m R^i_\bullet$ coincides with (1.2.2) for $\bullet \in [\![0, m + 2]\!]$, and that we can take the complexes ${}^m R^i_\bullet$ and ${}^{m-1} R^i_\bullet$ to coincide up to homological degree $m + 1$. Hence, if $i \in [\![0, N]\!]$ such that $m_i > 0$, we define the complex $P^i_\bullet$ to be equal to ${}^m R^i_\bullet$ up to homological degree $m + 2$. Since this holds for every $m \in \mathbb{N}$, the first part of the theorem is proved.

To prove the last one, let us denote by $P^i_{n,\bar{\alpha}}$ the direct summand in (1.2.2) indexed by $\bar{\alpha} \in \mathscr{P}\!\alpha_{M^i}$. The construction of the projective resolution $P^i_\bullet$ given in the first part of the proof tells us that, given $\bar{\alpha}, \bar{\beta} \in \mathscr{P}\!\alpha_{M^i}$, if the component

$$ d^{\bar{\alpha},\bar{\beta}}_{n+1} \otimes_A \mathrm{id}_\Bbbk : P^i_{n+1,\bar{\alpha}} \otimes_A \Bbbk \to P^i_{n,\bar{\beta}} \otimes_A \Bbbk $$

of the differential of $P^i_\bullet \otimes_A \Bbbk$ is nonzero, then $\bar{\alpha} < \bar{\beta}$ and $\mathrm{dfdeg}(\bar{\alpha}) = \mathrm{dfdeg}(\bar{\beta})-1$. The minimality result then follows. □

Remark 1.2.6. *It is easy to check that conditions* (1.2.3) *are verified in the case of Proposition 1.2.2, as well as in Examples 1.2.3 and 1.2.4, so the corresponding projective resolution* (1.2.2) *is minimal, coinciding with the resolutions constructed in those references.*

1.3 Hochschild (co)homology

All the following results can be found in [30]. Let A be a unitary associative $\Bbbk$-algebra. We denote by A^{op} the **opposite algebra** of A, which is the $\Bbbk$-module A with the multiplication defined by $a \cdot_{A^{\mathrm{op}}} b = ba$ for $a,b \in A$. An A-**bimodule** is a left and right A-module M satisfying $(a_1 m)a_2 = a_1(m a_2)$ for all $a_1,a_2 \in A$ and $m \in M$, and the left and right actions of $\Bbbk$ agree. We recall that an A-bimodule M can be viewed as a left A^e-module, where $A^e = A \otimes A^{\mathrm{op}}$ is the **enveloping algebra** of A with the multiplication defined by tensor product $(a_1 \otimes b_1)\cdot(a_2 \otimes b_2) = a_1 a_2 \otimes b_2 b_1$ for all $a_1,a_2,b_1,b_2 \in A$, and the action of A^e on M is $(a \otimes b)m = amb$ for $a,b \in A$ and $m \in M$. Analogously, an A-bimodule M can also be regarded as a right A^e-module, where the action is $m(a \otimes b) = bma$ for $a,b \in A$ and $m \in M$. The tensor product $A^{\otimes n}$ of A for $n \in \mathbb{N}$ is an A-bimodule under left and right multiplications.

Definition 1.3.1. *Let M be an A-bimodule. The* ***Hochschild homology*** $\mathrm{HH}_\bullet(A,M)$ *of A with coefficients in M is defined by* $\mathrm{HH}_n(A,M) = \mathrm{Tor}_n^{A^e}(M,A)$ *for $n \in \mathbb{N}_0$. The* ***Hochschild cohomology*** $\mathrm{HH}^\bullet(A,M)$ *of A with coefficients in M is defined by* $\mathrm{HH}^n(A,M) = \mathrm{Ext}_{A^e}^n(A,M)$ *for $n \in \mathbb{N}_0$. If $M = A$, we denote $\mathrm{HH}_\bullet(A,A)$ by $\mathrm{HH}_\bullet(A)$, and $\mathrm{HH}^\bullet(A,A)$ by $\mathrm{HH}^\bullet(A)$.*

The **bar resolution** $(B_\bullet(A), d_\bullet)$ of A in the category of A-bimodules is given by $B_n(A) = A^{\otimes(n+2)}$ for $n \in \mathbb{N}_0$, with the differentials $d_n : B_n(A) \to B_{n-1}(A)$ given by

$$d_n(a_0|\dots|a_{n+1}) = \sum_{j=0}^{n}(-1)^j a_0|\dots|a_{j-1}|a_j a_{j+1}|a_{j+2}|\dots|a_{n+1}$$

for $a_0,\dots,a_{n+1} \in A$ and $n \in \mathbb{N}$, and the augmentation $\pi : B_0(A) = A \otimes A \to A$ defined by the multiplication of A. We will typically use vertical bars instead of the tensor product symbols $\otimes$ for simplicity. The exactness of the bar resolution comes from the existence of the contracting homotopy given by the $\Bbbk$-linear map $s_n(a_0|\dots|a_{n+1}) = 1|a_0|\dots|a_{n+1}$ for $a_0,\dots,a_{n+1} \in A$ and $n \geqslant -1$.

Let M be an A-bimodule. There is an isomorphism of $\Bbbk$-vector spaces

$$M \otimes_{A^e} B_n(A) \xrightarrow{\sim} M \otimes A^{\otimes n}$$

given by $m|a_0|\dots|a_{n+1} \mapsto a_{n+1}ma_0|a_1|\dots|a_n$ for $m \in M$ and $a_0,\dots,a_{n+1} \in A$. The inverse isomorphism is given by $m|a_1|\dots|a_n \mapsto m|1|a_1|\dots|a_n|1$ for $m \in M$ and $a_1,\dots,a_n \in A$. Accordingly, we will get the induced differential on the complex $M \otimes A^{\otimes\bullet}$ corresponding to the map $\mathrm{id}_M \otimes d_n$ on $M \otimes_{A^e} B_n(A)$. There is also an isomorphism of $\Bbbk$-vector spaces

$$\tilde{F} : \mathrm{Hom}_{A^e}\big(B_n(A), M\big) \longrightarrow \mathrm{Hom}_{\Bbbk}(A^{\otimes n},M)$$

given by $\tilde{F}(f)(a_1|\dots|a_n) = f(1|a_1|\dots|a_n|1)$ for $f \in \mathrm{Hom}_{A^e}(B_n(A), M)$ and $a_1,\dots,a_n \in A$. The inverse map

$$\tilde{G} : \mathrm{Hom}_{\Bbbk}(A^{\otimes n},M) \longrightarrow \mathrm{Hom}_{A^e}\big(B_n(A), M\big)$$

of $\tilde{F}$ is explicitly given by $\tilde{G}(g)(a_0|\dots|a_{n+1}) = a_0 g(a_1|\dots|a_n)a_{n+1}$ for $g \in \mathrm{Hom}_{\Bbbk}(A^{\otimes n},M)$ and $a_0,\dots,a_{n+1} \in A$. We will get the induced differential on the complex $\mathrm{Hom}_{\Bbbk}(A^{\otimes\bullet},M)$ corresponding to the map d_{n+1}^* on $\mathrm{Hom}_{A^e}(B_n(A), M)$. The Hochschild homology $\mathrm{HH}_n(A,M)$ is given by the homology $\mathrm{H}_n(M \otimes_{A^e} B_\bullet(A)) \cong \mathrm{H}_n(M \otimes A^{\otimes\bullet})$, and the Hochschild cohomology $\mathrm{HH}^n(A,M)$ is given by the cohomology $\mathrm{H}^n(\mathrm{Hom}_{A^e}(B_\bullet(A),M)) \cong \mathrm{H}^n(\mathrm{Hom}_{\Bbbk}(A^{\otimes\bullet},M))$.

There are some interpretations for Hochschild (co)homology in low degrees (see [30], Section 1.2).

Remark 1.3.2. *(1)* $\mathrm{HH}^0(A,M) \cong \{m \in M \mid am = ma \text{ for all } a \in A\}$. *In particular,* $\mathrm{HH}^0(A) \cong \mathcal{Z}(A)$*, the center of A.*

(2) $\mathrm{HH}^1(A,M) \cong \mathrm{Der}(A,M)/\mathrm{InnDer}(A,M)$*, where*

$$\mathrm{Der}(A,M) = \{f : A \to M \mid f(ab) = f(a)b + af(b) \text{ for all } a,b \in A\}$$

is the space of ***derivations*** *from A to M, and*

$$\mathrm{InnDer}(A,M) = \{f : A \to M \mid \exists m \in M \text{ such that } f(a) = am - ma \text{ for all } a \in A\}$$

is the space of ***inner derivations*** *from A to M.*

Definition 1.3.3. *Let $m,n \in \mathbb{N}_0$, $f \in \mathrm{Hom}_{\Bbbk}(A^{\otimes m},A)$ and $g \in \mathrm{Hom}_{\Bbbk}(A^{\otimes n},A)$, the* ***cup product*** *$f \smile g$ is the element of $\mathrm{Hom}_{\Bbbk}(A^{\otimes(m+n)},A)$ defined by*

$$(f \smile g)(a_1|\dots|a_{m+n}) = f(a_1|\dots|a_m)g(a_{m+1}|\dots|a_{m+n})$$

for $a_1,\dots,a_{m+n} \in A$. If $m = 0$, then $(f \smile g)(a_1|\dots|a_n) = f(1)g(a_1|\dots|a_n)$. Similarly if $n = 0$, then $(f \smile g)(a_1|\dots|a_m) = g(1)f(a_1|\dots|a_m)$.

Remark 1.3.4. *To avoid any confusion, we remark that the definition of cup product on Hochschild cohomology in the previous definition is the one given in [9], Section 7. A different convention, in the spirit of Koszul's sign rule, includes a sign $(-1)^{mn}$ (see [30], Definition 1.3.1 and Remark 1.3.3).*

Let ∂ be the differential in the complex $\mathrm{Hom}_{\Bbbk}(A^{\otimes \bullet},A)$. Then

$$\partial(f \smile g) = \partial(f) \smile g + (-1)^m f \smile \partial(g)$$

for $f \in \mathrm{Hom}_{\Bbbk}(A^{\otimes m},A)$ and $g \in \mathrm{Hom}_{\Bbbk}(A^{\otimes n},A)$. The cup product on the complex $\mathrm{Hom}_{\Bbbk}(A^{\otimes \bullet},A)$ induces a graded associative product $\smile$: $\mathrm{HH}^m(A) \times \mathrm{HH}^n(A) \to \mathrm{HH}^{m+n}(A)$ on Hochschild cohomology of A for all $m,n \in \mathbb{N}_0$, that we also call the **cup product**. The cup product on $\mathrm{HH}^\bullet(A)$ is graded commutative, *i.e.*

$$f \smile g = (-1)^{mn} g \smile f$$

for $f \in \mathrm{HH}^m(A)$, $g \in \mathrm{HH}^n(A)$ and $m,n \in \mathbb{N}_0$ (see [30], Theorem 1.4.4). Then the Hochschild cohomology with the multiplication given by the cup product is a graded commutative associative algebra.

We will use the following definition of cup product (see [30], Sections 2.1 and 2.2) for later computations in Section 5.1. Let $(P_\bullet,\partial_\bullet)$ be a projective bimodule resolution over A with augmentation $\mu : P_0 \to A$. Let $f \in \mathrm{Hom}_{A^e}(P_m,A)$ and $g \in \mathrm{Hom}_{A^e}(P_n,A)$ be cocycles for $m,n \in \mathbb{N}_0$. Extend g to a chain map $\{g_i : P_{n+i} \to P_i\}_{i\in\mathbb{N}_0}$ such that $\mu g_0 = g$ and $\partial_i g_i = g_{i-1}\partial_{n+i}$ for $n \in \mathbb{N}$. The **cup product** $f \smile g \in \mathrm{Hom}_{A^e}(P_{m+n},A)$ is defined by the composition

$$f \smile g = f g_m.$$

This cup product at the chain level induces the **cup product** on the Hochschild cohomology. At the level of cohomology, this definition does not depend on the choice of resolution $(P_\bullet,\partial_\bullet)$. If we take $(P_\bullet,\partial_\bullet)$ to be the bar resolution $(B_\bullet(A), d_\bullet)$, this definition coincides with the Definition 1.3.3.

Definition 1.3.5. *Let $m,n \in \mathbb{N}_0$, $f \in \mathrm{Hom}_{\Bbbk}(A^{\otimes m},A)$ and $g \in \mathrm{Hom}_{\Bbbk}(A^{\otimes n},A)$. The* ***Gerstenhaber bracket*** *$[f,g]$ is defined at the chain level as the element of $\mathrm{Hom}_{\Bbbk}(A^{\otimes(m+n-1)},A)$ given by*

$$[f,g] = f \circ_G g - (-1)^{(m-1)(n-1)} g \circ_G f,$$

where $f \circ_G g$ is defined by

$$(f \circ_G g)(a_1|\dots|a_{m+n-1}) = \sum_{i=1}^{m} (-1)^{(n-1)(i-1)} f\big(a_1|\dots|a_{i-1}|g(a_i|\dots|a_{i+n-1})|a_{i+n}|\dots|a_{m+n-1}\big).$$

Moreover, if $m = 0$, then $f \circ_G g = 0$, while if $n = 0$, then the formula should be interpreted by taking the value $g(1)$ in place of $g(a_i|\dots|a_{i+n-1})$.

Recall that there is an isomorphism

$$F : \mathrm{Hom}_{A^e}\big(B_n(A), A\big) \longrightarrow \mathrm{Hom}_{\Bbbk}(A^{\otimes n},A) \tag{1.3.1}$$

given by $F(f)(a_1|\dots|a_n) = f(1|a_1|\dots|a_n|1)$ for $f \in \mathrm{Hom}_{A^e}(B_n(A), A)$ and $a_1,\dots,a_n \in A$. The inverse map

$$G : \mathrm{Hom}_{\Bbbk}(A^{\otimes n},A) \longrightarrow \mathrm{Hom}_{A^e}\big(B_n(A), A\big) \tag{1.3.2}$$

of F is explicitly given by $G(g)(a_0|\dots|a_{n+1}) = a_0 g(a_1|\dots|a_n)a_{n+1}$ for $g \in \mathrm{Hom}_{\Bbbk}(A^{\otimes n},A)$ and $a_0,\dots,a_{n+1} \in A$. Using the isomorphisms F and G of chain complexes given above, one defines the **Gerstenhaber bracket** in $\mathrm{Hom}_{A^e}(B_\bullet(A),A)$ by

$$[f,g] = G([F(f),F(g)]) \in \mathrm{Hom}_{A^e}(B_{m+n-1}(A),A)$$

for $f \in \mathrm{Hom}_{A^e}(B_m(A),A)$, $g \in \mathrm{Hom}_{A^e}(B_n(A),A)$ and $m,n \in \mathbb{N}_0$. Since

$$\partial([f,g]) = (-1)^{n-1}[\partial(f),g] + [f,\partial(g)]$$

for $f \in \mathrm{Hom}_{\Bbbk}(A^{\otimes m},A)$ and $g \in \mathrm{Hom}_{\Bbbk}(A^{\otimes n},A)$, the Gerstenhaber bracket given before induces a well-defined bilinear map

$$[\,,] : \mathrm{H}^m\big(\mathrm{Hom}_{A^e}(B_\bullet(A),A)\big) \times \mathrm{H}^n\big(\mathrm{Hom}_{A^e}(B_\bullet(A),A)\big) \to \mathrm{H}^{m+n-1}\big(\mathrm{Hom}_{A^e}(B_\bullet(A),A)\big)$$

for all $m, n \in \mathbb{N}_0$, that we also call the **Gerstenhaber bracket**.

More generally, let $(P_\bullet,\partial_\bullet)$ be a projective bimodule resolution over A with augmentation $\mu : P_0 \to A$. Let $i_\bullet : P_\bullet \to B_\bullet(A)$ and $p_\bullet : B_\bullet(A) \to P_\bullet$ be morphisms of complexes of A-bimodules lifting id_A, so $p_\bullet i_\bullet$ is homotopic to $\mathrm{id}_{P_\bullet}$ and $i_\bullet p_\bullet$ is homotopic to $\mathrm{id}_{B_\bullet(A)}$. We also recall that the morphisms $i_\bullet$ and $p_\bullet$ induce the quasi-isomorphisms $i_\bullet^* : \mathrm{Hom}_{A^e}(B_\bullet(A),A) \to \mathrm{Hom}_{A^e}(P_\bullet,A)$ and $p_\bullet^* : \mathrm{Hom}_{A^e}(P_\bullet,A) \to \mathrm{Hom}_{A^e}(B_\bullet(A),A)$ given by $i_\bullet^*(f) = fi_\bullet$ and $p_\bullet^*(g) = gp_\bullet$ for $f \in \mathrm{Hom}_{A^e}(B_\bullet(A),A)$ and $g \in \mathrm{Hom}_{A^e}(P_\bullet,A)$, respectively. Moreover, $\mathrm{H}(i_\bullet^*), \mathrm{H}(p_\bullet^*) : \mathrm{HH}^\bullet(A) \to \mathrm{HH}^\bullet(A)$ are independent of the choice of $i_\bullet$ and $p_\bullet$. The **Gerstenhaber bracket**

$$[\,,] : \mathrm{H}^m\left(\mathrm{Hom}_{A^e}(P_\bullet(A),A)\right) \times \mathrm{H}^n\left(\mathrm{Hom}_{A^e}(P_\bullet(A),A)\right) \to \mathrm{H}^{m+n-1}\left(\mathrm{Hom}_{A^e}(P_\bullet(A),A)\right)$$

for all $m, n \in \mathbb{N}_0$ is then defined by transport of structures. More generally, given cocycles $f \in \mathrm{Hom}_{A^e}(P_m,A)$ and $g \in \mathrm{Hom}_{A^e}(P_n,A)$, we define the Gerstenhaber bracket

$$[f, g] \in \mathrm{H}^{m+n-1}(\mathrm{Hom}_{A^e}(P_\bullet,A)) \cong \mathrm{HH}^{m+n-1}(A)$$

as the cohomology class of $i_\bullet^*([p_\bullet^*(f), p_\bullet^*(g)])$.

The following properties of the Gerstenhaber bracket are classical (see for instance [9], equation (2), *cf.* [30], Lemmas 1.4.3 and 1.4.7).

Lemma 1.3.6. *Let $\Bbbk$ be a field and A a $\Bbbk$-algebra. Then*

$$[x,y] = -(-1)^{(m-1)(n-1)}[y,x] \text{ and } [x, [y,z]] = [[x,y],z] + (-1)^{(m-1)(n-1)}[y,[x,z]], \tag{1.3.3}$$

and

$$[x \smile y,z] = [x,z] \smile y + (-1)^{m(p-1)}x \smile [y,z] \tag{1.3.4}$$

for all $x \in \mathrm{HH}^m(A)$, $y \in \mathrm{HH}^n(A)$ and $z \in \mathrm{HH}^p(A)$.

The previous result is typically rephrased by stating that the Hochschild cohomology is a **Gerstenhaber algebra**, *i.e.* a graded-commutative algebra $H = \oplus_{n\in\mathbb{N}_0} H^n$ endowed with a bracket $[\,,] : H \otimes H \to H$ satisfying $[H^m, H^n] \subseteq H^{m+n-1}$ for $m,n \in \mathbb{N}_0$, (1.3.3) and (1.3.4).

Assume that A is a graded $\Bbbk$-algebra. For graded A-modules M,N, let $\mathrm{Hom}_A(M,N)$ be the $\Bbbk$-vector space consisting of all morphisms of A-modules from M to N. Let

$$\mathcal{H}om_A(M,N) = \bigoplus_{d\in\mathbb{Z}} \mathcal{H}om_A(M,N)_d,$$

be the graded $\Bbbk$-vector space, where $\mathcal{H}om_A(M,N)_d$ is the subspace of $\mathrm{Hom}_A(M,N)$ consisting of all homogeneous morphisms of degree d.

The following result is classical (see [19], Corollary 2.4.4).

Lemma 1.3.7. *If M is a finitely generated graded module over a graded algebra A, then $\mathrm{Hom}_A(M,N) = \mathcal{H}om_A(M,N)$.*

Corollary 1.3.8. *Let $(P_\bullet,\partial_\bullet)$*

$$\cdots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\mu} A \longrightarrow 0$$

be a projective bimodule resolution of a graded $\Bbbk$-algebra A, where P_i is finitely generated as left A^e-module for $i \in \mathbb{N}_0$, and μ and ∂_i are homogeneous of degree 0 for $i \in \mathbb{N}$. Then $\mathrm{Hom}_{A^e}(P_i,A) = \mathcal{H}om_{A^e}(P_i,A)$ for $i \in \mathbb{N}_0$. Hence, the Hochschild cohomology $\mathrm{HH}^\bullet(A) \cong \oplus_{i\in\mathbb{N}_0} \mathrm{H}^i(\mathrm{Hom}_{A^e}(P_\bullet,A))$ of A is a bigraded algebra, for the cohomological degree i and the internal degree induced by that of A and $P_\bullet$. Moreover, the cup product and the Gerstenhaber bracket on $\mathrm{HH}^\bullet(A)$ preserve the internal degree.

Remark 1.3.9. *The existence of a projective bimodule resolution of the graded $\Bbbk$-algebra A satisfying the conditions of the previous corollary clearly holds if the graded $\Bbbk$-algebra A^e is noetherian (e.g. if A is finite-dimensional over $\Bbbk$).*

1.4 Methods computing the Gerstenhaber bracket on Hochschild cohomology

In this section, we will consider A to be a unital associative $\Bbbk$-algebra. Let $(B_\bullet(A), d_\bullet)$ be the bar resolution of A with the augmentation $\pi : B_0(A) \to A$ defined in Section 1.3. We will typically write $a_0|\dots|a_{n+1}$ instead of $a_0 \otimes \cdots \otimes a_{n+1}$ for simplicity. The maps F and G are given in (1.3.1) and (1.3.2).

1.4.1 Method computing the bracket between $\mathrm{HH}^0(A)$ and $\mathrm{HH}^n(A)$

In this subsection, we introduce an elementary method to compute the Gerstenhaber bracket between the cohomology groups $\mathrm{HH}^0(A)$ and $\mathrm{HH}^n(A)$ for $n \in \mathbb{N}_0$ of any algebra A using any projective bimodule resolution of A (see Theorem 1.4.1). We were unable to explicitly find this method in the existing literature (see Remark 1.4.6), although we suspect it could be well-known to the experts. These results were published in [13].

Let ρ be an element of the center $\mathcal{Z}(A) \cong \mathrm{HH}^0(A)$ of A and $\ell_\rho \in \mathrm{Hom}_{A^e}(B_0(A),A)$ be the morphism defined by $\ell_\rho(1|1) = \rho$. Let $(P_\bullet,\partial_\bullet)$ be a projective bimodule resolution over A with augmentation $\mu : P_0 \to A$, and let $i_0 : P_0 \to B_0(A)$ be the 0-th component of a morphism $i_\bullet : P_\bullet \to B_\bullet(A)$ of complexes of A-bimodules lifting id_A. The main aim of this subsection is to prove the following theorem, which tells us that we can compute the Gerstenhaber bracket between $\mathrm{HH}^0(A)$ and $\mathrm{HH}^n(A)$ for $n \in \mathbb{N}_0$ using a simple homological procedure on any projective bimodule resolution of A.

Theorem 1.4.1. *Consider the same assumptions as in the previous paragraph. Let $\eta_n : P_n \to P_n$ be the map given by $\eta_n(v) = \rho v - v\rho$ for $v \in P_n$ and $n \in \mathbb{N}_0$. Since $\eta_\bullet = \{\eta_n : P_n \to P_n\}_{n\in\mathbb{N}_0}$ is a lifting of the zero morphism from A to itself, $\eta_\bullet$ is null-homotopic, i.e. there is a family $h^\rho_\bullet = \{h^\rho_n : P_n \to P_{n+1}\}_{n\in\mathbb{N}_0}$ of morphisms of A-bimodules such that*

$$\eta_0 = \partial_1 h^\rho_0 \quad \text{and} \quad \eta_n = h^\rho_{n-1}\partial_n + \partial_{n+1}h^\rho_n \tag{1.4.1}$$

for $n \in \mathbb{N}$. Then, if $\phi \in \mathrm{Hom}_{A^e}(P_n,A)$ is a cocycle for some $n \in \mathbb{N}$, the Gerstenhaber bracket $[\phi, \ell_\rho i_0] \in \mathrm{HH}^{n-1}(A)$ is given by the cohomology class of ϕh^ρ_{n-1}.

Remark 1.4.2. *It is easy to see that if $\phi \in \mathrm{Hom}_{A^e}(P_n,A)$ is a cocycle (resp., coboundary), then ϕh^ρ_{n-1} is a cocycle (resp., coboundary) by applying (1.4.1). On the other hand, in general we have $P_0 = B_0 = A \otimes A$ and $i_0 = \mathrm{id}_{A\otimes A}$, so we can forget about i_0 in Theorem 1.4.1.*

The rest of this subsection is devoted to proving Theorem 1.4.1. In order to do that, we first need to prove some preliminary results.

Let $t_n : B_n(A) \to B_{n+1}(A)$ be the morphism of A-bimodules given by

$$t_n(a_0|\dots|a_{n+1}) = \sum_{j=0}^{n} (-1)^j a_0|\dots|a_j|\rho|a_{j+1}|\dots|a_{n+1}$$

for $a_0, \dots, a_{n+1} \in A$ and $n \in \mathbb{N}_0$. Let $\xi_\bullet = \{\xi_n : B_n(A) \to B_n(A)\}_{n\in\mathbb{N}_0}$ be the family of morphisms of A-bimodules defined by $\xi_n(u) = \rho u - u\rho$ for $u \in B_n(A)$ and $n \in \mathbb{N}_0$.

Lemma 1.4.3. *We have that $\xi_0 = d_1 t_0$ and $\xi_n = t_{n-1}d_n + d_{n+1}t_n$ for $n \in \mathbb{N}$.*

Proof. For $a_0, \dots, a_{n+1} \in A$ and $n \in \mathbb{N}$,

$$d_1 t_0(a_0|a_1) = d_1(a_0|\rho|a_1) = a_0\rho|a_1 - a_0|\rho a_1 = \rho a_0|a_1 - a_0|a_1\rho = \xi_0(a_0|a_1),$$

and

$$d_{n+1}t_n(a_0|\dots|a_{n+1}) = d_{n+1}\left(\sum_{j=0}^{n} (-1)^j a_0|\dots|a_j|\rho|a_{j+1}|\dots|a_{n+1}\right) = S_1 + S_2,$$

where

$$S_1 = \sum_{j=0}^{n} (-1)^j \Big\{ (-1)^j a_0|\dots|a_{j-1}|a_j\rho|a_{j+1}|\dots|a_{n+1} + (-1)^{j+1} a_0|\dots|a_j|\rho a_{j+1}|a_{j+2}|\dots|a_{n+1} \Big\}$$

$$
\begin{aligned}
&= a_0\rho|\dots|a_{n+1} - a_0|\dots|a_n|\rho a_{n+1}\\
&= \xi_n(a_0|\dots|a_{n+1}),
\end{aligned}
$$

and

$$
\begin{aligned}
S_2 = \sum_{j=0}^{n}(-1)^j \Bigg\{ &\sum_{r=0}^{j-2}(-1)^r a_0|\dots|a_{r-1}|a_r a_{r+1}|a_{r+2}|\dots|a_j|\rho|a_{j+1}|\dots|a_{n+1}\\
&+ (-1)^{j-1}a_0|\dots|a_{j-2}|a_{j-1}a_j|\rho|a_{j+1}|\dots|a_{n+1}\\
&+ (-1)^{j+2}a_0|\dots|a_j|\rho|a_{j+1}a_{j+2}|a_{j+3}|\dots|a_{n+1}\\
&+ \sum_{r=j+2}^{n}(-1)^{r+1}a_0|\dots|a_j|\rho|a_{j+1}|\dots|a_{r-1}|a_r a_{r+1}|a_{r+2}|\dots|a_{n+1}\Bigg\}\\
= \sum_{i=0}^{n}\Bigg\{ &\sum_{j=i+2}^{n}(-1)^j(-1)^i a_0|\dots|a_{i-1}|a_i a_{i+1}|a_{i+2}|\dots|a_j|\rho|a_{j+1}|\dots|a_{n+1}\\
&- a_0|\dots|a_{i-1}|a_i a_{i+1}|\rho|a_{i+2}|\dots|a_{n+1} + a_0|\dots|a_{i-1}|\rho|a_i a_{i+1}|a_{i+2}|\dots|a_{n+1}\\
&+ \sum_{j=0}^{i-2}(-1)^j(-1)^{i+1}a_0|\dots|a_j|\rho|a_{j+1}|\dots|a_{i-1}|a_i a_{i+1}|a_{i+2}|\dots|a_{n+1}\Bigg\}\\
= -\sum_{i=0}^{n}(-1)^i\Bigg\{ &\sum_{j=0}^{i-2}(-1)^j a_0|\dots|a_j|\rho|a_{j+1}|\dots|a_{i-1}|a_i a_{i+1}|a_{i+2}|\dots|a_{n+1}\\
&+ (-1)^{i-1}a_0|\dots|a_{i-1}|\rho|a_i a_{i+1}|a_{i+2}|\dots|a_{n+1} + (-1)^i a_0|\dots|a_{i-1}|a_i a_{i+1}|\rho|a_{i+2}|\dots|a_{n+1}\\
&+ \sum_{j=i+2}^{n}(-1)^{j-1}a_0|\dots|a_{i-1}|a_i a_{i+1}|a_{i+2}|\dots|a_j|\rho|a_{j+1}|\dots|a_{n+1}\Bigg\}\\
= -t_{n-1}&d_n(a_0|\dots|a_{n+1}).
\end{aligned}
$$

Hence, $\xi_n = t_{n-1}d_n + d_{n+1}t_n$. □

Lemma 1.4.4. *The Gerstenhaber bracket $[\varphi,\ell_\rho] \in \mathrm{Hom}_{A^e}(B_{n-1}(A),A)$ is given by $[\varphi,\ell_\rho] = \varphi t_{n-1}$ for $\varphi \in \mathrm{Hom}_{A^e}(B_n(A),A)$ and $n \in \mathbb{N}$.*

Proof. For $a_0,\dots,a_n \in A$,

$$
\begin{aligned}
[\varphi,\ell_\rho](a_0|\dots|a_n) &= a_0[F(\varphi),F(\ell_\rho)](a_1|\dots|a_{n-1})a_n\\
&= a_0(F(\varphi)\circ_G F(\ell_\rho))(a_1|\dots|a_{n-1})a_n\\
&= a_0\left(\sum_{i=1}^{n}(-1)^{i-1}F(\varphi)(a_1|\dots|a_{i-1}|\rho|a_i|\dots|a_{n-1})\right)a_n\\
&= a_0\left(\sum_{i=1}^{n}(-1)^{i-1}\varphi(1|a_1|\dots|a_{i-1}|\rho|a_i|\dots|a_{n-1}|1)\right)a_n\\
&= \sum_{i=1}^{n}(-1)^{i-1}\varphi(a_0|a_1|\dots|a_{i-1}|\rho|a_i|\dots|a_{n-1}|a_n)\\
&= (\varphi t_{n-1})(a_0|a_1|\dots|a_{n-1}|a_n).
\end{aligned}
$$

Hence, $[\varphi,\ell_\rho] = \varphi t_{n-1}$. □

Lemma 1.4.5. *We assume the same hypotheses as those of Theorem 1.4.1. Then, there exists a family $s_\bullet = \{s_n : P_n \to B_{n+2}(A)\}_{n\in\mathbb{N}_0}$ of morphisms of A-bimodules such that $i_1h_0^\rho - t_0i_0 = d_2s_0$ and $i_{n+1}h_n^\rho - t_ni_n = d_{n+2}s_n - s_{n-1}\partial_n$ for $n \in \mathbb{N}$.*

Proof. Since $d_1(i_1h_0^\rho - t_0i_0) = i_0\partial_1h_0^\rho - d_1t_0i_0 = i_0\eta_0 - \xi_0i_0 = 0$, where we used that i_0 is a morphism of A-bimodules in the last equality, there exists a morphism $s_0 : P_0 \to B_2(A)$ of A-bimodules such that $d_2s_0 = i_1h_0^\rho - t_0i_0$. We now claim that there exists a family $s_\bullet = \{s_n :$

$P_n \to B_{n+2}(A)\}_{n\in\mathbb{N}_0}$ of morphisms of A-bimodules such that $d_{n+2}s_n = i_{n+1}h^\rho_n - t_n i_n + s_{n-1}\partial_n$ by induction on $n \in \mathbb{N}_0$ (where $s_{-1} = 0$). Indeed,

$$\begin{aligned} d_{n+1}(i_{n+1}h^\rho_n - t_n i_n + s_{n-1}\partial_n) &= d_{n+1}i_{n+1}h^\rho_n - d_{n+1}t_n i_n + (i_n h^\rho_{n-1} - t_{n-1}i_{n-1} + s_{n-2}\partial_{n-1})\partial_n \\ &= i_n\partial_{n+1}h^\rho_n - (\xi_n - t_{n-1}d_n)i_n + i_n h^\rho_{n-1}\partial_n - t_{n-1}i_{n-1}\partial_n \\ &= i_n(\partial_{n+1}h^\rho_n + h^\rho_{n-1}\partial_n) - \xi_n i_n = i_n\eta_n - \xi_n i_n = 0, \end{aligned}$$

where we used the inductive assumption in the first equality, Lemma 1.4.3 in the second equality, the definition of η_n in the third equality and the fact that i_n is a morphism of A-bimodules in the last equality. The result thus follows. □

Proof of Theorem 1.4.1. Let $\varphi = \phi p_n \in \mathrm{Hom}_{A^e}(B_n(A),A)$. Then φ is a cocycle and $[\phi, i^*_\bullet(\ell_\rho)] = i^*_\bullet([p^*_\bullet(\phi), \ell_\rho]) = [\varphi, \ell_\rho]i_{n-1} = \varphi t_{n-1}i_{n-1}$ by Lemma 1.4.4. Since $p_\bullet i_\bullet$ is homotopic to the identity of $P_\bullet$, there exists $\phi_1 \in \mathrm{Hom}_{A^e}(P_{n-1}, A)$ such that $\phi - \varphi i_n = \phi - \phi p_n i_n = \phi_1\partial_n$. Then,

$$\begin{aligned} \phi h^\rho_{n-1} - \varphi t_{n-1}i_{n-1} &= (\varphi i_n + \phi_1\partial_n)h^\rho_{n-1} - \varphi t_{n-1}i_{n-1} \\ &= \varphi(d_{n+1}s_{n-1} - s_{n-2}\partial_{n-1}) + \phi_1(\eta_{n-1} - h^\rho_{n-2}\partial_{n-1}) \\ &= -\varphi s_{n-2}\partial_{n-1} - \phi_1 h^\rho_{n-2}\partial_{n-1} \in \mathrm{Hom}_{A^e}(P_{n-1}, A) \end{aligned}$$

is a boundary, where we used Lemma 1.4.5 and the definition of η_{n-1} in the second equality and the fact that ϕ_1 is a morphism of A-bimodules in the last identity. Hence, $[\phi, i^*_\bullet(\ell_\rho)] \in \mathrm{HH}^{n-1}(A)$ coincides with the cohomology class of ϕh^ρ_{n-1}, as was to be shown. □

Remark 1.4.6. *The homotopy maps $h^\rho_\bullet$ in Theorem 1.4.1 are presumably homotopy liftings in the sense of* [27]. *However, our maps $h^\rho_\bullet$ do not directly follow the scheme of that definition –as well as being far simpler, for they are restricted to a much easier situation– since they do not require the computation of any map $\Delta : P_\bullet \to P_\bullet \otimes_A P_\bullet$ lifting the isomorphism $A \to A \otimes_A A$, which is also the case in* [20].

1.4.2 Method computing the bracket between $\mathrm{HH}^1(A)$ and $\mathrm{HH}^n(A)$ (after M. Suárez-Álvarez)

In this subsection, we will briefly recall the method introduced by M. Suárez-Álvarez in [24] to compute the Gerstenhaber bracket between $\mathrm{HH}^1(A)$ and $\mathrm{HH}^n(A)$ for $n \in \mathbb{N}_0$.

Recall that $\mathrm{HH}^1(A)$ is isomorphic to the quotient of the space of derivations of A modulo the subspace of inner derivations. Let $\rho : A \to A$ be a derivation of A, *i.e.* $\rho(xy) = \rho(x)y + x\rho(y)$ for all $x,y \in A$. For a left A-module M, a **ρ-operator** on M is a map $f : M \to M$ such that $f(am) = \rho(a)m + af(m)$ for all $a \in A$ and $m \in M$. It is direct to see that the map $\rho^e = \rho\otimes\mathrm{id}_A + \mathrm{id}_A\otimes\rho : A^e \to A^e$ defined by $\rho^e(x\otimes y) = \rho(x)\otimes y + x\otimes\rho(y)$ for $x,y \in A$ is a derivation of the enveloping algebra A^e and ρ is a ρ^e-operator on A.

Let $(P_\bullet,\partial_\bullet)$ be a projective bimodule resolution over A with augmentation $\mu : P_0 \to A$. A **ρ^e-lifting** of ρ to $(P_\bullet,\partial_\bullet)$ is a family of ρ^e-operators $\rho_\bullet = \{\rho_n : P_n \to P_n\}_{n\in\mathbb{N}_0}$ such that $\mu\rho_0 = \rho\mu$ and $\partial_n\rho_n = \rho_{n-1}\partial_n$ for $n \in \mathbb{N}$. The morphism of complexes

$$\rho^\sharp_{\bullet,P_\bullet} : \mathrm{Hom}_{A^e}(P_\bullet,A) \to \mathrm{Hom}_{A^e}(P_\bullet,A)$$

defined by $\rho^\sharp_{n,P_\bullet}(\phi) = \rho\phi - \phi\rho_n$ for $\phi \in \mathrm{Hom}_{A^e}(P_n,A)$ and $n \in \mathbb{N}_0$ is independent of the ρ^e-lifting up to homotopy (see [24], Lemma 1.6) and it thus induces a morphism on cohomology that we will denote by the same symbol. Let $i_\bullet : P_\bullet \to B_\bullet(A)$ and $p_\bullet : B_\bullet(A) \to P_\bullet$ be morphisms of complexes of A-bimodules lifting id_A. Then the diagram

$$\begin{array}{ccc} \mathrm{H}^n\big(\mathrm{Hom}_{A^e}(B_\bullet(A),A)\big) & \xrightarrow{\mathrm{H}(\rho^\sharp_{\bullet,B_\bullet(A)})} & \mathrm{H}^n\big(\mathrm{Hom}_{A^e}(B_\bullet(A),A)\big) \\ \downarrow{\scriptstyle \mathrm{H}(i^*_\bullet)} & & \downarrow{\scriptstyle \mathrm{H}(i^*_\bullet)} \\ \mathrm{H}^n\big(\mathrm{Hom}_{A^e}(P_\bullet,A)\big) & \xrightarrow{\mathrm{H}(\rho^\sharp_{\bullet,P_\bullet})} & \mathrm{H}^n\big(\mathrm{Hom}_{A^e}(P_\bullet,A)\big) \end{array} \tag{1.4.2}$$

commutes (see [24], Lemma 1.6). On the other hand, as noted in [24], Sections 2.1 and 2.2, using the ρ^e-lifting of ρ to the bar resolution defined by

$$\rho_n(a_0|\dots|a_{n+1}) = \sum_{j=0}^{n+1} a_0|\dots|a_{j-1}|\rho(a_j)|a_{j+1}|\dots|a_{n+1}$$

for $a_0, \ldots, a_{n+1} \in A$ and $n \in \mathbb{N}_0$, it is easy to check that the diagram

$$\begin{array}{ccc} \operatorname{Hom}_{A^e}\big(B_n(A),A\big) & \xrightarrow{\rho^{\sharp}_{n,B_\bullet(A)}} & \operatorname{Hom}_{A^e}\big(B_n(A),A\big) \\ \downarrow F & & \downarrow F \\ \operatorname{Hom}_{\Bbbk}(A^{\otimes n},A) & \xrightarrow{[\rho,-]} & \operatorname{Hom}_{\Bbbk}(A^{\otimes n},A) \end{array}$$

commutes. As a consequence, the Gerstenhaber bracket between the cohomology classes of $G(\rho) \in \operatorname{Hom}_{A^e}(B_1(A),A)$ and $\varphi \in \operatorname{Hom}_{A^e}(B_n(A),A)$ is given by the cohomology class of $[G(\rho), \varphi] = G([\rho,F(\varphi)]) = \rho^{\sharp}_{n,B_\bullet(A)}(\varphi)$.

We finally recall one of the main results of [24], which tells us that we can compute the Gerstenhaber bracket between $\mathrm{HH}^1(A)$ and $\mathrm{HH}^n(A)$ for $n \in \mathbb{N}_0$ using any projective bimodule resolution of A (see [24], Theorem A and Section 2.2). The proof just follows from observing that, on cohomology, (1.4.2) gives us the identities

$$\big[i^*_\bullet(G(\rho)), \phi\big] = i^*_\bullet\big([G(\rho),p^*_\bullet(\phi)]\big) = i^*_\bullet\big(\rho^{\sharp}_{n,B_\bullet(A)}(p^*_\bullet(\phi))\big) = \rho^{\sharp}_{n,P_\bullet}(\phi).$$

Theorem 1.4.7. *Let $(P_\bullet,\partial_\bullet)$ be a projective bimodule resolution over the algebra A with augmentation $\mu : P_0 \to A$, and let $i_1 : P_1 \to B_1(A)$ be the first component of the morphism $i_\bullet : P_\bullet \to B_\bullet(A)$ of complexes of A-bimodules lifting id_A. Given a cocycle $\phi \in \operatorname{Hom}_{A^e}(P_n,A)$ and $n \in \mathbb{N}_0$, the Gerstenhaber bracket $[G(\rho)i_1, \phi] \in \mathrm{HH}^n(A)$ is given by the cohomology class of $\rho^{\sharp}_{n,P_\bullet}(\phi)$.*

Remark 1.4.8. *Note that in our Theorem 1.4.1, as well as in the result proved in [24] that was recalled before as Theorem 1.4.7, we need at least some component(s) of the comparison map from the generic projective resolution $(P_\bullet,\partial_\bullet)$ to the bar resolution.*

Chapter 2

Gröbner bases

We will present in this chapter the basic theory of non-commutative Gröbner bases. We will mainly follow Ufnarovskij [25] and Varadarajan [26].

2.1 Normal words and Gröbner bases in noncommutative algebras

Let us first recall the definition of a totally ordered set.

Definition 2.1.1. *Let X be a non-empty set. A binary relation $\succeq$ on X is called a* ***total order*** *on X if the following statements hold for all $x,y,z \in X$:*
(1) Antisymmetry: if $x \succeq y$ and $y \succeq x$, then $x = y$.
(2) Transitivity: if $x \succeq y$ and $y \succeq z$, then $x \succeq z$.
(3) Connexity: $x \succeq y$ or $y \succeq x$.

Definition 2.1.2. *Let $(X, \succeq)$ be a totally ordered set. Then we define $x \succ y$ if $x \succeq y$ and $x \neq y$ for $x,y \in X$. We define $x \preceq y$ if $y \succeq x$ and define $x \prec y$ if $y \succ x$.*

Definition 2.1.3. *Let X be a non-empty set. A total order on X is called a* ***well order*** *on X if every non-empty subset of X has a least element in this ordering.*

Remark 2.1.4. *In a well-ordered set $(X, \succeq)$, any sequence $x_1 \succeq x_2 \succeq x_3 \succeq \cdots$ stabilises, i.e. there exists $n \in \mathbb{N}$ such that $x_m = x_n$ for all $m \geqslant n$.*

Lemma 2.1.5 (Transfinite induction)**.** *Let $(X, \succeq)$ be a well-ordered set and $x_0 \in X$ the least element of X. Let $P(x)$ be a property defined for all elements $x \in X$. Assume that the following conditions hold:*
(1) $P(x_0)$ is true.
(2) Let $x \in X$. If $P(y)$ is true for any $x \succ y$, then $P(x)$ is true.
Then $P(x)$ is true for all $x \in X$.

Proof. Suppose that the set $A = \{x \in X | P(x) \text{ is not true}\}$ is non-empty. Then there is the least element $\alpha \in A$. We have $\alpha \neq x_0$. Then for any $y \in X$ with $\alpha \succ y$, $P(y)$ is true, which implies $P(\alpha)$ is true. Contradiction! □

Let X be a non-empty set whose elements are called **letters** and W the free non-commutative monoid with unit generated by X. Specifically, W is the set of all the finite sequences of zero or more elements from X with concatenation operation: $(x_1 \cdots x_m)\cdot(y_1 \cdots y_n) = x_1 \cdots x_m y_1 \cdots y_n$, where $x_i,y_j \in X$. The unique sequence of zero elements is the identity element $1 \in W$. The elements of W are called **words**. The **length** of a word $w \in W$ is the number of the letters inside w. The length of $1 \in W$ is zero.

Definition 2.1.6. *For $w_1,w_2 \in W$, we say w_2 is a* ***subword*** *of w_1, denoted by $w_2 \subseteq w_1$, if $w_2 = 1$ or $w_1 = x_{i_1} x_{i_2} \cdots x_{i_m}$ and $w_2 = x_{i_l} x_{i_{l+1}} \cdots x_{i_r}$ for some $1 \leqslant l \leqslant r \leqslant m$ and $x_{i_j} \in X$. We define $w_2 \subsetneq w_1$ if $w_2 \subseteq w_1$ and $w_2 \neq w_1$.*

We assume that the set X is well-ordered. We may define an order $\succeq$ on W, called **homogeneous lexicographic order**, as follows: for w_1,$w_2 \in W$, if the length of w_1 is strictly larger than w_2, we define $w_1 \succ w_2$; if the length of w_1 equals to w_2, we sort them in lexicographic order induced by the well order on X. Then the homogeneous lexicographic order $\succeq$ is a well order on W.

Remark 2.1.7. *The homogeneous lexicographic order on W has the following properties:*

(1) For every $w \in W$ which is not 1, we have $w \succ 1$.

(2) For all w_1,w_2,u,$v \in W$, if $w_1 \succ w_2$, then $uw_1v \succ uw_2v$.

Let $\Bbbk\langle X\rangle$ be the free (non-commutative) associative $\Bbbk$-algebra generated by X. It is a fact that the algebra $\Bbbk\langle X\rangle$ is a $\Bbbk$-vector space spanned by all words. For any non-zero element x in $\Bbbk\langle X\rangle$ with the form $x = \sum_{i=1}^{r} c_i w_i$ where $c_i \in \Bbbk \setminus \{0\}, w_i \in W$ and $w_1 \succ w_2 \succ \cdots \succ w_r$, we call c_1w_1 the **leading term** of x, w_1 the **leading word** of x, and c_1 the **leading coefficient** of x.

Let I be a non-zero two-sided ideal of $\Bbbk\langle X\rangle$.

Definition 2.1.8. *A word $w \in W$ is called* ***normal*** *with respect to I if w is not the leading term of any element in I.*

Theorem 2.1.9. *Let N be the $\Bbbk$-vector space spanned by all normal words. Then we have a decomposition $\Bbbk\langle X\rangle = N \oplus I$ as vector spaces.*

Proof. Since $N \cap I = 0$, we only need to prove $\Bbbk\langle X\rangle = N + I$. It is sufficient to prove that $W \subseteq N+I$. Let $w \in W$ be a word. If the word w is normal, we have $w = w+0$. Otherwise, w is the leading term of an element y in I. Let $y = w + w_1$, where $w_1 \in \Bbbk\langle X\rangle$. Let $w_1 = \sum_{i=1}^{r_1} a_i w_{i1}$, where $a_i \in \Bbbk$,$w_{i1} \in W$ and $w \succ w_{i1}$. We have $w = -w_1 + y$. If all w_{i_1} are normal words, we obtain $w \in N+I$. Otherwise, say w_{11} is not normal, w_{11} is the leading term of an element $z \in I$. Let $z = w_{11}+w_2$, where $w_2 \in \Bbbk\langle X\rangle$. Let $w_2 = \sum_{i=1}^{r_2} b_i w_{i2}$, where $b_i \in \Bbbk$, $w_{i2} \in W$ and $w_{11} \succ w_{i2}$. If all w_{i2} are normal words, we have $w_{11} \in N + I$. Otherwise, say w_{12} is not normal, repeat the above process, then we get a sequence $w \succ w_{11} \succ w_{12} \succ \cdots$, which must be a finite sequence as W is well-ordered. Finally, we get $w \in N + I$. □

Definition 2.1.10. *There is a natural projection map $p : \Bbbk\langle X\rangle \to N$. For every $x \in \Bbbk\langle X\rangle$, we call $p(x)$ the* ***normal form*** *of x.*

Remark 2.1.11. *Let $A = \Bbbk\langle X\rangle/I$ be the quotient algebra. Then $A \cong N$ as vector spaces.*

Definition 2.1.12. *A subset G of I is called a* ***Gröbner basis*** *of I in $\Bbbk\langle X\rangle$ if the leading word of any non-zero element in I contains the leading word of some element in G as a subword. Moreover, if we require that no proper subset of G is a Gröbner basis, G is called a* ***minimal*** *Gröbner basis. A Gröbner basis G is called* ***reduced*** *if it is minimal and every element $x \in G$ has the form $w - p(w)$, where w is the leading term of x and the coefficient of w is 1.*

Remark 2.1.13. *(1) Many Gröbner bases exist. For example, the ideal I itself is a Gröbner basis of I.*

(2) A Gröbner basis G is minimal if and only if $0 \notin G$ and the leading word of any element in G doesn't contain the leading word of any other element in G as a subword.

Proposition 2.1.14. Let G be a Gröbner basis of I in $\Bbbk\langle X\rangle$. Then the set G generates the two-sided ideal I.

Proof. Since $G \subseteq I$, it is clear that $(G) \subseteq I$, where (G) is the two-sided ideal generated by G. We shall prove the converse. Let y be a non-zero element in I. We want to prove $y \in (G)$. Let $y = aw + z$, where $a \in \Bbbk \setminus \{0\}, w \in W, z \in \Bbbk\langle X\rangle$ and aw is the leading term of y. There exists $x \in G$ with the leading term x_1 and exist $c \in \Bbbk \setminus \{0\}, u$,$v \in W$ such that $aw = cux_1v$. Then $y = cuxv + y_1$, where $cuxv \in (G)$, $y_1 = z - cu(x - x_1)v$. Let $y_1 = a_1w_1 + z_1$, where $a_1 \in \Bbbk \setminus \{0\}, w_1 \in W, z_1 \in \Bbbk\langle X\rangle$ and a_1w_1 is the leading term of y_1. We have $w \succ w_1$. By repeating the above process for y_1, y_2,$y_3, \ldots$, we get a sequence $w \succ w_1 \succ w_2 \succ \cdots$. As the set W is well-ordered, the process will be terminated in a finite number of steps. Finally, we obtain $y \in (G)$. □

Theorem 2.1.15. Let G be a Gröbner basis of I in $\Bbbk\langle X\rangle$. Then a word $w \in W$ is normal if and only if w doesn't contain the leading word of any element in G as a subword.

Proof. (⇒) Let w be a normal word. Suppose there exist $x \in G$ with leading word x_1 and $w_1, w_2 \in W$ such that $w = w_1 x_1 w_2$. Then w is the leading word of $w_1 x w_2 \in I$. Contradiction!

(⇐) Suppose the word $w \in W$ is not normal. Then w is the leading word of some element in I. By the definition of Gröbner basis, w contains the leading word of some element in G as a subword. □

2.2 Bergman's diamond lemma

Now we introduce Bergman's diamond lemma according to Varadarajan [26], Section 7.2 and give an example from Ufnarovskij [25], Section 2.6 about how to find a Gröbner basis.

Let X be a well-ordered set and $A = \Bbbk\langle X\rangle/I$ an associative $\Bbbk$-algebra where I is a non-zero two-sided ideal of the free associative $\Bbbk$-algebra $\Bbbk\langle X\rangle$. The free monoid W generated by X is equipped with the homogeneous lexicographic order $\succeq$ induced by the well order on X. Suppose that the ideal I is generated by the set

$$\{w_\sigma - f_\sigma \in I | w_\sigma \in W, f_\sigma \in \Bbbk\langle X\rangle, \sigma \in \Sigma\}. \tag{2.2.1}$$

We also assume that the following conditions hold:

(1) For any $\sigma \neq \tau$ in Σ, we have $w_\sigma \neq w_\tau$.

(2) For all $\sigma \in \Sigma$, we have $f_\sigma = 0$ or the leading word of f_σ is strictly less than w_σ in the homogeneous lexicographic order.

Definition 2.2.1. *A word $w \in W$ is called* ***standard*** *with respect to* (2.2.1) *if w doesn't contain any word w_σ $(\sigma \in \Sigma)$ as a subword.*

Remark 2.2.2. *Let S be the vector space spanned by all standard words with respect to* (2.2.1)*, N the vector space spanned by all normal words with respect to I. Then $N \subseteq S$.*

Definition 2.2.3. *For $\sigma \in \Sigma$ and $u, v \in W$, we define the* ***elementary reduction operator*** *as the $\Bbbk$-linear map $R_{(u,w_\sigma,v)} : \Bbbk\langle X\rangle \to \Bbbk\langle X\rangle$ which maps the word $uw_\sigma v$ to $uf_\sigma v$, and fixes other words. The composition of finite elementary reduction operators is called a* ***reduction operator****.*

Remark 2.2.4. *$R_{(au,w_\sigma,vb)}(axb) = a(R_{(u,w_\sigma,v)}x)b$ for $a, b \in W$ and $x \in \Bbbk\langle X\rangle$. Moreover, for any reduction operator R and $a, b \in W$, there exists a reduction operator $\widetilde{R}$ such that $\widetilde{R}(axb) = a(Rx)b$ for any $x \in \Bbbk\langle X\rangle$.*

Remark 2.2.5. *The vector space S is exactly the set of elements which are fixed by all reduction operators in $\Bbbk\langle X\rangle$.*

Lemma 2.2.6. *For every $x \in \Bbbk\langle X\rangle$ and reduction operator R, we have $x - Rx \in I$.*

Proof. Let $R = R_n R_{n-1} \cdots R_1$ where R_i are elementary reduction operators. We will prove the lemma by induction on n. When $n = 1$, $R = R_{(u,w_\sigma,v)}$ is an elementary reduction operator, where $\sigma \in \Sigma$ and $u, v \in W$, we write $x = cuw_\sigma v + x'$, where $c \in \Bbbk$ and $x' \in \Bbbk\langle X\rangle$ is a linear combination of words not equal to $uw_\sigma v$. Then we have $Rx = cuf_\sigma v + x'$, so $x - Rx = cu(w_\sigma - f_\sigma)v \in I$. Suppose that $x - Rx \in I$ for every $x \in \Bbbk\langle X\rangle$ and every reduction operator R which can be written as a composition of $n-1$ elementary reduction operators. Then for $R = R_n R_{n-1} \cdots R_1$, we have $x - Rx = (x - R_1 x) + (R_1 x - R_n R_{n-1} \cdots R_2 R_1 x) \in I$. □

Definition 2.2.7. *An element $x \in \Bbbk\langle X\rangle$ is called* ***reduction finite*** *if for every sequence $\{R_i | i \in \mathbb{N}\}$ of elementary reduction operators, the sequence*

$$R_1 x, R_2 R_1 x, \ldots, R_i R_{i-1} \cdots R_1 x, \ldots$$

stabilizes, i.e. there exists $n \in \mathbb{N}$, such that $R_m R_{m-1} \cdots R_1 x = R_n R_{n-1} \cdots R_1 x$ for all $m \geqslant n$. Let F be the set of all reduction finite elements.

Remark 2.2.8. *An element $x \in \Bbbk\langle X\rangle$ is reduction finite if and only if for every sequence $\{R_i | i \in \mathbb{N}\}$ of reduction operators, the sequence $R_1 x, R_2 R_1 x, \ldots, R_i R_{i-1} \cdots R_1 x, \ldots$ stabilizes.*

Lemma 2.2.9. *We have $F = \Bbbk\langle X\rangle$.*

Proof. The set F is a vector space. It is sufficient to prove $W \subseteq F$. Suppose $W \setminus F \neq \emptyset$. Since W is well-ordered, we can take the least element w in $W \setminus F$. Let $\{R_i | i \in \mathbb{N}\}$ be a sequence of elementary reduction operators. Assume without loss of generality $R_1 w \neq w$, then $R_1 w$ is a linear combination of words strictly less than w in the homogeneous lexicographic order. Thus, $R_1 w \in F$. This implies that the sequence $R_1 w, R_2 R_1 w, \ldots$ stabilizes. Hence, $w \in F$. Contradiction! □

Remark 2.2.10. *For every $x \in \Bbbk\langle X\rangle$, there exists a reduction operator R such that $Rx \in S$.*

Definition 2.2.11. *We call Rx in last remark a* ***reduced form*** *of x. If all reduced forms of x are same, x is called* ***reduction unique****. Let U be the set of reduction unique elements.*

Remark 2.2.12. *(1) We have $S \subseteq U$ and U is a vector space which is stable under all reduction operators, i.e. $R(U) \subseteq U$ for every reduction operator R.*

(2) We have a map $\mathcal{R} : U \to S$ which maps $x \in U$ to its reduced form. This is a linear map satisfying $\mathcal{R}(Rx) = \mathcal{R}(x)$ for every $x \in U$ and every reduction operator R. Moreover, $\mathcal{R}|_S = \mathrm{id}_S$.

Definition 2.2.13. *An* ***overlap ambiguity*** *is a triple (w_1,w_2,w_3), where $w_i \in W$ and there are $\sigma, \tau \in \Sigma$ such that $w_1 w_2 = w_\sigma, w_2 w_3 = w_\tau$. An* ***inclusion ambiguity*** *is a triple (w_1,w_2,w_3), where $w_i \in W$ and there are $\sigma,\tau \in \Sigma$ such that $w_2 = w_\sigma, w_1 w_2 w_3 = w_\tau$. An overlap ambiguity (w_1,w_2,w_3) is called* ***resolvable*** *if there are reduction operators R_1,R_2 such that $R_1(f_\sigma w_3) = R_2(w_1 f_\tau) \in S$. An inclusion ambiguity (w_1,w_2,w_3) is called* ***resolvable*** *if there are reduction operators R_1,R_2 such that $R_1(w_1 f_\sigma w_2) = R_2(f_\tau) \in S$.*

Theorem 2.2.14 (Bergman's diamond lemma)**.** *The following statements are equivalent:*

(1) $S = N$.

(2) Every element in $\Bbbk\langle X\rangle$ is reduction unique.

(3) All ambiguities are resolvable.

(4) The set $\{w_\sigma - f_\sigma | \sigma \in \Sigma\}$ is a Gröbner basis of I in $\Bbbk\langle X\rangle$.

Proof. (1) ⇒ (2) Let x be an element in $\Bbbk\langle X\rangle$ and $s = Rx \in S$ a reduced form of x, where R is a reduction operator. Then $x - s = x - Rx \in I$ by Lemma 2.2.6. Since $S = N$, we have $\Bbbk\langle X\rangle = S \oplus I$ as vector spaces by Theorem 2.1.9. Then the decomposition $x = s + (x - s)$ is unique. This implies that the reduced form of x is unique.

(2) ⇒ (1) Suppose $U = \Bbbk\langle X\rangle$. Then we have a linear map $\mathcal{R} : \Bbbk\langle X\rangle \to S$. Let K be the kernel of $\mathcal{R}$. Since $\mathcal{R}|_S = \mathrm{id}_S$, we obtain $\Bbbk\langle X\rangle = S \oplus K$. We want to prove $I = K$. Let $x \in K$. Then $\mathcal{R}(x) = 0$. There is a reduction operator R such that $Rx = \mathcal{R}(x) = 0$. By Lemma 2.2.6, we have $x = x - Rx \in I$. Conversely, let $x \in I$. Then x is a linear combination of elements of the form $u(w_\sigma - f_\sigma)v$, where $\sigma \in \Sigma$ and $u,v \in W$. We have $\mathcal{R}(u(w_\sigma - f_\sigma)v) = \mathcal{R}(R_{(u,w_\sigma,v)}(u(w_\sigma - f_\sigma)v)) = \mathcal{R}(0) = 0$, which implies $u(w_\sigma - f_\sigma)v \in K$. Then $x \in K$. We obtain $\Bbbk\langle X\rangle = S \oplus I$. Let $s \in S$. Then $s = s + 0 = n + y$ for some $n \in N$ and $y \in I$. As $N \subseteq S$, we get $s = n$. Hence, $S = N$.

(2) ⇒ (1) Suppose that (w_1,w_2,w_3) is an overlap ambiguity with $w_1 w_2 = w_\sigma$, $w_2 w_3 = w_\tau$, $\sigma,\tau \in \Sigma$ and $w_1,w_2,w_3 \in W$. There are reduction operators R_1, R_2 such that $R_1(f_\sigma w_3) \in S$ and $R_2(w_1 f_\tau) \in S$. The elements $R_1(f_\sigma w_3)$ and $R_2(w_1 f_\tau)$ are both reduced forms of the word $w_1 w_2 w_3$, hence they are same. For the same reason, all inclusion ambiguities are resolvable.

(3) ⇒ (2) We will prove that every word is reduction unique by transfinite induction. The least word 1 is reduction unique. Let $w \in W$. Suppose that all words strictly less than w are reduction unique. We want to prove that w is reduction unique. Let R_1 and R_2 be two elementary reduction operators such that $R_1 w \neq w$ and $R_2 w \neq w$. It is sufficient to prove that $R_1 w$ and $R_2 w$ are reduction unique, and they have the same reduced form. Let $R_1 = R_{(u_1,w_\sigma,v_1)}$ and $R_2 = R_{(u_2,w_\tau,v_2)}$, where $\sigma, \tau \in \Sigma$ and u_1,v_1,u_2,v_2 are words. There are three cases.

Assume first that $w = u w_1 w_2 w_3 v$, where $w_1 w_2 = w_\sigma$, $w_2 w_3 = w_\tau$, $u = u_1$, $w_3 v = v_1$, $u w_1 = u_2$, $v = v_2$ and $u,w_1,w_2,w_3,v \in W$. Since all overlap ambiguities are resolvable, there are reduction operators L_1 and L_2 such that $L_1(f_\sigma w_3) = L_2(w_1 f_\tau) \in S$. By Remark 2.2.4, there exist reduction operators $\widetilde{L_1}$ and $\widetilde{L_2}$ such that $\widetilde{L_1}(u f_\sigma w_3 v) = u L_1(f_\sigma w_3) v = u L_2(w_1 f_\tau) v = \widetilde{L_2}(u w_1 f_\tau v)$. The element $u f_\sigma w_3 v$ is a linear combination of words strictly less than w. Thus, $u f_\sigma w_3 v \in U$. Similarly, $u w_1 f_\tau v \in U$, and $\mathcal{R}(u f_\sigma w_3 v) = \mathcal{R}(u w_1 f_\tau v)$. In other words, $R_1 w$ and $R_2 w$ are reduction unique, and they have the same reduced form.

Assume now that $w = u_2w_1w_\sigma w_2v_2$, where $w_1w_\sigma w_2 = w_\tau$, $u_2w_1 = u_1$, $w_2v_2 = v_1$ and $w_1, w_2 \in W$. Since all inclusion ambiguities are resolvable, there are reduction operators L_1 and L_2 such that $L_1(w_1f_\sigma w_2) = L_2f_\tau \in S$. By Remark 2.2.4, there exist reduction operators $\widetilde{L_1}$ and $\widetilde{L_2}$ such that $\widetilde{L_1}(u_2w_1f_\sigma w_2v_2) = u_2L_1(w_1f_\sigma w_2)v_2 = u_2(L_2f_\tau)v_2 = \widetilde{L_2}(u_2f_\tau v_2)$. By the induction hypothesis, $u_2w_1f_\sigma w_2v_2$ and $u_2f_\tau v_2$ are reduction unique and have the same reduced form.

Assume finally that $w = u_1w_\sigma ww_\tau v_2$, where $ww_\tau v_2 = v_1$, $u_1w_\sigma w = u_2$ and $w \in W$. Then we have $R_1w = u_1f_\sigma ww_\tau v_2$ and $R_2w = u_1w_\sigma wf_\tau v_2$. There exist reduction operators L_1 and L_2 such that $L_1R_1w = u_1f_\sigma wf_\tau v_2 = L_2R_2w$. Then R_1w and R_2w are reduction unique and have the same reduced form.

(1) $\Rightarrow$ (4) The leading word of any element in I is not normal, hence not standard by $S = N$, then contains w_σ as a subword for some $\sigma \in \Sigma$. Then $\{w_\sigma - f_\sigma | \sigma \in \Sigma\}$ is a Gröbner basis.

(4) $\Rightarrow$ (1) If $\{w_\sigma - f_\sigma | \sigma \in \Sigma\}$ is a Gröbner basis, we have $N = S$ by Theorem 2.1.15 and that's all. □

Let G_0 be a generating set of the ideal I. In order to get a Gröbner basis starting from G_0, we apply a procedure consisting of the following steps. Assume that G' is an intermediate set with $G_0 \subseteq G' \subseteq I$.

Step 1. (*Normalization*) By multiplying a non-zero coefficient, the leading coefficient of every element in G' becomes 1. Then we get a new intermediate set G'.

Step 2. (*Reduction*) Take two normalized elements x and y in G' and two words $u,v \in W$. Let y_1 be the leading word of y. Compute $R_{(u,y_1,v)}(x)$. There are three cases: if $R_{(u,y_1,v)}(x) = 0$, we remove x from G'; if $R_{(u,y_1,v)}(x) \neq 0$ and $R_{(u,y_1,v)}(x) \neq x$, the leading word of $R_{(u,y_1,v)}(x) = R_{(u,y_1,v)}(x' - uy'v)$ is strictly less than the leading word of x in the homogeneous lexicographic order in which case we replace x by $R_{(u,y_1,v)}(x)$ in G'; if $R_{(u,y_1,v)}(x) = x$, we do nothing. This process is denoted by $x \to R_{(u,y_1,v)}(x)$.

Repeat Step 1 and Step 2 until G' does not change. Then we go to Step 3.

Step 3. (*Composition*) Take two normalized elements x and y in G' with the leading words x_1 and y_1 respectively. If there is a triple (w_1,w_2,w_3), where $w_i \in W$, such that $x_1 = w_1w_2$, $y_1 = w_2w_3$ and $w_2 \neq 1$, we compute $w_1y - xw_3$. If $w_1y - xw_3$ is not zero, it should be added to G'.

Repeat Step 1 to Step 3 until G' does not change, which may be an infinite number of repetitions. Finally, we obtain a set G, which is a minimal Gröbner basis.

Here we give an example of computing Gröbner bases in [25].

Example 2.2.15. *Let $A = \Bbbk\langle x,y\rangle/(x^2 - yx)$ be an algebra with the order $x \succ y$. In order to get a Gröbner basis, we start from the set $G' = \{x^2 - yx\}$. Applying Step 1 to Step 3, we have*

$$(x,x,x) : (x^2 - yx)x - x(x^2 - yx) = xyx - yx^2 \to xyx - y^2x.$$

The element $xyx - y^2x$ should be added to the set G', so, $G' = \{x^2 - yx, xyx - y^2x\}$. By the reduction

$$(x,x,yx) : (x^2 - yx)yx - x(xyx - y^2x) = xy^2x - yxyx \to xy^2x - y^3x,$$

we have $G' = \{x^2 - yx, xyx - y^2x, xy^2x - y^3x\}$. By the reductions

$$(xy,x,x) : (xyx - y^2x)x - xy(x^2 - yx) = xy^2x - y^2x^2 \to xy^2x - y^3x \to 0,$$
$$(xy,x,yx) : (xyx - y^2x)yx - xy(xyx - y^2x) = xy^3x - y^2xyx \to xy^3x - y^4x,$$

we have $G' = \{x^2 - yx, xyx - y^2x, xy^2x - y^3x, xy^3x - y^4x\}$. By the reductions

$$(x, x, y^2x) : (x^2 - yx)y^2x - x(xy^2x - y^3x) = xy^3x - yxy^2x \to xy^3x - y^4x \to 0,$$
$$(xy^2,x,x) : (xy^2x - y^3x)x - xy^2(x^2 - yx) = xy^3x - y^3x^2 \to xy^3x - y^4x \to 0,$$
$$(xy,x,y^2x) : (xyx - y^2x)y^2x - xy(xy^2x - y^3x) = xy^4x - y^2xy^2x \to xy^4x - y^5x,$$
$$\cdots\cdots$$

Reasoning inductively, we claim that the set $G = \{xy^nx - y^{n+1}x | n \in \mathbb{N}_0\}$ is a Gröbner basis of the ideal $(x^2 - yx)$ in $\Bbbk\langle x,y\rangle$. Indeed, for $k,l \in \mathbb{N}_0$ we have

$$xy^k \cdot x \cdot y^lx : (xy^kx - y^{k+1}x)y^lx - xy^k(xy^lx - y^{l+1}x) = xy^{k+l+1}x - y^{k+1}xy^lx$$

$$\to xy^{k+l+1}x - y^{k+l+2}x \to 0.$$

This shows that all ambiguities are resolvable. Moreover, the set $\{y^n, y^nxy^m | m,n \in \mathbb{N}_0\}$ *is a* $\Bbbk$*-basis of* A.

If we define $y \succ x$*, then* $\{yx - x^2\}$ *is a Gröbner basis of the ideal* $(x^2 - yx)$*, and* $\{x^ny^m | m,n \in \mathbb{N}_0\}$ *is a* $\Bbbk$*-basis of* A.

Example 2.2.16. *Let* $A = \Bbbk\langle X\rangle/I$*, where the set* $X = \{a,b,c\}$ *is equipped an ordering by setting* $c \succ b \succ a$ *and* I *is a two-sided ideal of* $\Bbbk\langle X\rangle$ *generated by the elements* $a^2,b^2,c^2,ca+bc+ab,cb+ba+ac$*. This is the Fomin-Kirillov algebra on* 3 *generators introduced in Section 3.2. In order to get a Gröbner basis of A, we start from the set* $G' = \{a^2,b^2,c^2,ca + bc + ab,cb + ba + ac\}$*. By the reduction*

$$(c, a, a) : (ca + bc + ab)a - ca^2 = bca + aba \to b(-bc - ab) + aba \to bab - aba,$$

the element $bab - aba$ *should be adjoined to* G'*. Do the following reductions*

$$\begin{aligned}
(c, b,b) &: (cb + ba + ac)b - cb^2 = bab + acb \to bab + a(-ba - ac) \to bab - aba \to 0,\\
(c, c,a) &: c(ca + bc + ab) - c^2a = cbc + cab \to (-ba - ac)c + (-bc - ab)b\\
&\quad \to bac + bcb \to bac + b(-ba - ac) \to 0,\\
(c,c, b) &: c(cb + ba + ac) - c^2b = cba + cac \to (-ba - ac)a + (-bc - ab)c \to aca + abc\\
&\quad \to a(-bc - ab) + abc = -a^2b \to 0.
\end{aligned}$$

Now we check that all ambiguities with respect to $G = \{a^2,b^2,c^2,ca + bc + ab,cb + ba + ac,bab - aba\}$ *are resolvable.*

$$\begin{aligned}
(b, b, ab) &: b^2ab - b(bab - aba) = baba \to (aba)a \to 0,\\
(ba, b, b) &: bab^2 - (bab - aba)b = abab \to a(aba) \to 0,\\
(c, b,ab) &: (cb + ba + ac)ab - c(bab - aba) = ba^2b + acab + caba \to a(-bc - ab)b + (-bc - ab)ba\\
&\quad \to ab(ba + ac) + b(ba + ac)a \to abac + ba(-bc - ab) \to abac - babc \to 0,\\
(ba,b,ab) &: (bab - aba)ab - ba(bab - aba) = ba^2ba - aba^2b \to 0.
\end{aligned}$$

So, the set $G = \{a^2,b^2,c^2,ca+bc+ab,cb+ba+ac,bab-aba\}$ *is a minimal Gröbner basis of* I*. Moreover, a* $\Bbbk$*-basis of* A *is* $\{1,a,b,c,ab,bc,ba,ac,aba,abc,bac,abac\}$.

Chapter 3

Fomin-Kirillov algebras

In this chapter, we will recall the definitions of Yetter-Drinfeld modules and Fomin-Kirillov algebras.

3.1 Yetter-Drinfeld modules over a group algebra

Recall that a **(linear) representation** ρ of a group G on a vector space V over a field $\Bbbk$ is a group homomorphism $\rho : G \to \mathrm{Aut}_{\Bbbk}(V)$, where $\mathrm{Aut}_{\Bbbk}(V)$ is the general linear group on V. The **dimension** of ρ is the dimension of V. Let G be a group and V a $\Bbbk$-vector space. Then $\rho : G \to \mathrm{Aut}_{\Bbbk}(V)$ is a linear representation if and only if V is a $\Bbbk G$-module, where $\Bbbk G$ is the group algebra. We also call V a **(linear) representation** of G if V is a $\Bbbk G$-module.

Example 3.1.1. *Let G be a group. The representation $\rho : G \to \mathrm{Aut}_{\Bbbk}(\Bbbk)$ given by $\rho(g) = \mathrm{id}_{\Bbbk}$ for all $g \in G$ is the* ***trivial representation****.*

Example 3.1.2. *Let G be a group and X a finite set with $\#X = n$. Assume that there is a left action of G on X. Let V be an n-dimensional $\Bbbk$-vector space with a basis $\{e_x | x \in X\}$. The* ***permutation representation*** *$\rho : G \to \mathrm{Aut}_{\Bbbk}(V)$ is given by $\rho(g)(e_x) = e_{g(x)}$ for $g \in G$ and $x \in X$. In particular, let $G = \mathbb{S}_n$ and $X = [\![1,n]\!]$, we get the permutation representation of $\mathbb{S}_n$.*

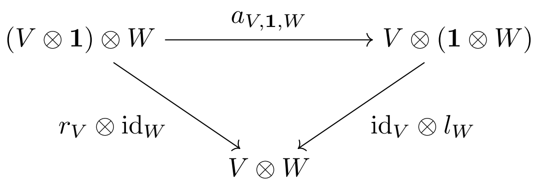
Example 3.1.3. *Let G be a cyclic group generated by g. Assume that $\#G = n \in \mathbb{N}$, and the field $\Bbbk$ is algebraically closed with $\mathrm{char}(\Bbbk) \nmid n$. Then G has n irreducible representations $\rho_i : G \to \mathrm{Aut}_{\Bbbk}(\Bbbk) \cong \Bbbk^{\times}$ of dimension 1 given by $\rho_i(g) = \xi_n^{i-1}$ for $i \in [\![1,n]\!]$, where ξ_n is a primitive root of unity of order n in $\Bbbk$.*

Example 3.1.4. *Let $G = \mathbb{S}_3$. Assume that the characteristic of $\Bbbk$ is different from 2 and 3. The irreducible representations of $\mathbb{S}_3$ are ρ_i for $i \in [\![1,3]\!]$, where ρ_1 is the trivial representation, $\rho_2 : \mathbb{S}_3 \to \Bbbk^{\times}$ is given by $\rho_2(\sigma) = 1$ if σ is an even permutation, and $\rho_2(\sigma) = -1$ if σ is an odd permutation, and ρ_3 is the unique 2-dimensional irreducible representation of $\mathbb{S}_3$. Recall that $\pi \cong \rho_1 \oplus \rho_3$, where π is the permutation representation of $\mathbb{S}_3$. Indeed, let $\pi : \mathbb{S}_3 \to \mathrm{Aut}_{\Bbbk}(V)$ and $\{e_1,e_2,e_3\}$ is a basis of V, then the subspace spanned by $e_1 + e_2 + e_3$ is the trivial representation, and the subspace $\mathrm{span}_{\Bbbk}\{\sum_{i=1}^{3} k_i e_i | \sum_{i=1}^{3} k_i = 0\}$ is ρ_3.*

Definition 3.1.5. *A* ***monoidal category*** *is a category $\mathcal{C}$ together with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object $\mathbf{1}$ (called* ***unit****), and natural isomorphisms $a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$, $r_V : V \otimes \mathbf{1} \to V$ and $l_V : \mathbf{1} \otimes V \to V$ such that the diagrams*

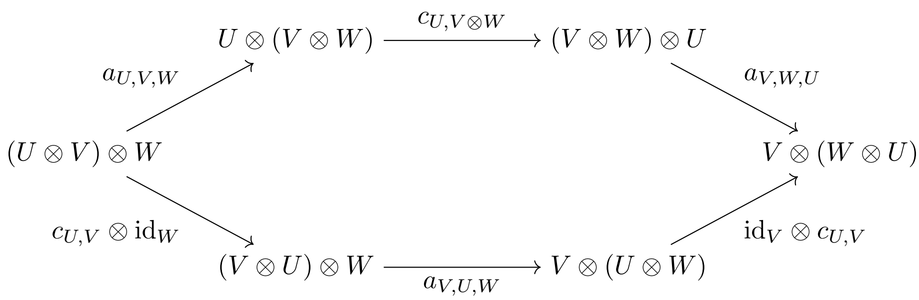
$$
\begin{array}{ccccc}
 & & ((U \otimes V) \otimes W) \otimes X & & \\
 & \swarrow^{a_{U,V,W} \otimes \mathrm{id}_X} & & \searrow^{a_{U \otimes V,W,X}} & \\
(U \otimes (V \otimes W)) \otimes X & & & & (U \otimes V) \otimes (W \otimes X) \\
\downarrow{\scriptstyle a_{U,V \otimes W,X}} & & & & \downarrow{\scriptstyle a_{U,V,W \otimes X}} \\
U \otimes ((V \otimes W) \otimes X) & & \xrightarrow[\mathrm{id}_U \otimes a_{V,W,X}]{} & & U \otimes (V \otimes (W \otimes X))
\end{array}
$$

and

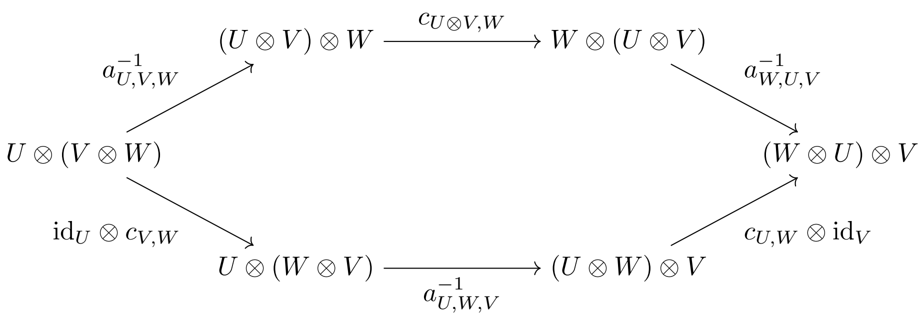


commute for objects U,V,W,X in $\mathcal{C}$.

Definition 3.1.6. *A* ***braided monoidal category*** *is a monoidal category $\mathcal{C}$ with a natural isomorphism $c_{V,W} : V \otimes W \to W \otimes V$, called* ***braiding****, such that the diagrams*



and



commute for objects U,V,W in $\mathcal{C}$.

Let G be a group. There are $\Bbbk$-linear maps $\Delta : \Bbbk G \to \Bbbk G \otimes \Bbbk G$ given by $\Delta(g) = g \otimes g$ for $g \in G$, and $\varepsilon : \Bbbk G \to \Bbbk$ given by $\varepsilon(g) = 1$ for $g \in G$. The group algebra $\Bbbk G$ is a **Hopf algebra** with the **antipode** $S : \Bbbk G \to \Bbbk G$ mapping $g \in G$ to g^{-1}. We refer the reader to [18] for more information on Hopf algebras.

Definition 3.1.7. *A left* ***comodule*** *over $\Bbbk G$ is a $\Bbbk$-vector space M together with a linear map $\delta : M \to \Bbbk G \otimes M$, called* ***coaction****, such that $(\mathrm{id}_{\Bbbk G} \otimes \delta)\delta = (\Delta \otimes \mathrm{id}_M)\delta$ and $(\varepsilon \otimes \mathrm{id}_M)\delta = \mathrm{id}_M$, where we identify $\Bbbk \otimes M$ with M in the second identity. Let M and N be two $\Bbbk G$-comodules, a* ***morphism*** *of comodules is a $\Bbbk$-linear map $f : M \to N$ such that $(\mathrm{id}_{\Bbbk G} \otimes f)\delta_M = \delta_N f$, where δ_M and δ_N are the coactions on M and N respectively.*

Remark 3.1.8. *If $\delta(m) = e \otimes m$ for all $m \in M$, where e is the identity element of the group G, then the coaction δ is called* ***trivial****.*

Definition 3.1.9. *Let M be a $\Bbbk G$-comodule with the coaction $\delta : M \to \Bbbk G \otimes M$. A $\Bbbk G$-**subcomodule** of M is a subspace N of M such that $\delta(N) \subseteq \Bbbk G \otimes N$. If N is a subcomodule of M, the quotient space M/N is also a comodule.*

Lemma 3.1.10. *If M is a left $\Bbbk G$-comodule, then there is a G-decomposition $M = \oplus_{g \in G} M_g$, where $M_g = \{m \in M | \delta(m) = g \otimes m\}$ for $g \in G$.*

Proof. The facts $M_g \cap M_h = 0$ for $g \neq h \in G$ and $M \supseteq \oplus_{g \in G} M_g$ are obvious. Let $m \in M$ and $\delta(m) = \sum_{i=1}^n g_i \otimes m_i$, where $n \in \mathbb{N}$, $g_i \in G$ distinct, and $m_i \in M$ for $i \in [\![1,n]\!]$. By the identities in Definition 3.1.7, we obtain $\sum_{i=1}^n g_i \otimes \delta(m_i) = \sum_{i=1}^n g_i \otimes g_i \otimes m_i$, implying $\delta(m_i) = g_i \otimes m_i$ for all $i \in [\![1,n]\!]$, and $\sum_{i=1}^n m_i = m$. Thus, $M \subseteq \oplus_{g \in G} M_g$. $\square$

Remark 3.1.11. *If $(M_g)_{g\in G}$ is a family of $\Bbbk$-vector spaces, then $M = \oplus_{g\in G} M_g$ is a left $\Bbbk G$-comodule, where the coaction $\delta : M \to \Bbbk G \otimes M$ is linearly extended by $\delta(m_g) = g \otimes m_g$ for all $g \in G$ and $m_g \in M_g$. Hence, a left comodule over a group algebra $\Bbbk G$ is equivalent to a G-graded vector space.*

Remark 3.1.12. *Given two comodules M and N over $\Bbbk G$, the tensor product $M \otimes N$ of vector spaces is a comodule with the G-decomposition $(M \otimes N)_g = \oplus_{h\in G}(M_h \otimes N_{h^{-1}g})$ for $g \in G$.*

Definition 3.1.13. *A left **Yetter-Drinfeld module** M over a group algebra $\Bbbk G$ is a left $\Bbbk G$-module and a left $\Bbbk G$-comodule satisfying the compatibility condition $\delta(gm) = ghg^{-1} \otimes gm$ for all $g,h \in G$ and $m \in M_h$. A **morphism** of Yetter-Drinfeld modules is a morphism of modules and comodules.*

Given two Yetter-Drinfeld modules M and N over $\Bbbk G$, the tensor product $M \otimes N$ is again a Yetter-Drinfeld module with the action given by $g(x \otimes y) = gx \otimes gy$ for $g \in G$, $x \in M$ and $y \in N$, and the coaction given by Remark 3.1.12. The category ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ of left Yetter-Drinfeld modules over $\Bbbk G$ is a **braided monoidal category**, the unit of which is $\Bbbk$ with the trivial action and coaction, and the braiding $c_{M,N} : M \otimes N \to N \otimes M$ is given by $m \otimes n \mapsto (gn) \otimes m$ for $g \in G$, $m \in M_g$ and $n \in N$.

If a Yetter-Drinfeld module M is finite-dimensional, then the dual $M^* = \mathrm{Hom}_{\Bbbk}(M,\Bbbk)$ is also a Yetter-Drinfeld module, where the action is given by $gf(m) = f(g^{-1}m)$ for $g \in G$, $f \in M^*$ and $m \in M$, and the coaction is given by the G-decomposition $M^* = \oplus_{g\in G}(M^*)_g$ with $(M^*)_g = (M_{g^{-1}})^*$. Given two finite-dimensional Yetter-Drinfeld modules M and N, the isomorphism $(M \otimes N)^* \cong N^* \otimes M^*$, as Yetter-Drinfeld modules, is induced by the usual pairing.

Let G be a finite group with $\#G = n \in \mathbb{N}$, and $\Bbbk$ a field with $\mathrm{char}(\Bbbk) \nmid n$. Then all Yetter-Drinfeld modules over $\Bbbk G$ are semi-simple. The following classification of irreducible Yetter-Drinfeld modules is introduced in [2], Section 1.1.

Remark 3.1.14. *Any irreducible Yetter-Drinfeld module $M(\mathfrak{C},\rho)$ over $\Bbbk G$ is parameterized by pairs $(\mathfrak{C},\rho)$, where $\mathfrak{C}$ is a conjugacy class in G and ρ is an irreducible representation of the isotropy subgroup $G^s = \{g \in G | gsg^{-1} = s\}$ of a fixed point $s \in \mathfrak{C}$ on a vector space V. A precise description is as follows. Let $\mathfrak{C} = \{t_i | i \in [\![1,m]\!]\}$ and $s = t_1$. Let $g_i \in G$ such that $g_i s g_i^{-1} = t_i$ for $i \in [\![1,m]\!]$. Define $M(\mathfrak{C},\rho) = \oplus_{i\in[\![1,m]\!]} g_i \otimes V$. The action is given by $g(g_i \otimes v) = g_j \otimes (\gamma v)$, where $gg_i = g_j\gamma$ for some unique $j \in [\![1,m]\!]$ and $\gamma \in G^s$. The coaction $\delta : M(\mathfrak{C},\rho) \to \Bbbk G \otimes M(\mathfrak{C},\rho)$ is given by $\delta(g_i \otimes v) = t_i \otimes (g_i \otimes v)$.*

Let $s,\tilde{s} \in \mathfrak{C}$, and $g \in G$ such that $gsg^{-1} = \tilde{s}$. Then $\varphi : G^{\tilde{s}} \to G^s$ given by $\phi(x) = g^{-1}xg$ for $x \in G^{\tilde{s}}$ is an isomorphism of groups. Let $\tilde{\rho} = \rho\varphi$, the pull back of ρ, which is an irreducible representation of $G^{\tilde{s}}$. Then $M(\mathfrak{C},\rho) = M(\mathfrak{C},\tilde{\rho})$.

3.2 Fomin-Kirillov algebras

We refer the reader to [8,17] for more information on Fomin-Kirillov algebras. Define the vector space

$$V_n = \Bbbk\{x_{i,j} | i \neq j \in [\![1,n]\!]\}/\Bbbk\{x_{i,j} + x_{j,i} | i \neq j \in [\![1,n]\!]\}$$

for $n \geqslant 2$. We denote the class of $x_{i,j}$ also by $x_{i,j}$. Let $\mathbb{S}_n$ be the group of permutations of $\{1,\ldots,n\}$ and $(i,j) \in \mathbb{S}_n$ the unique transposition interchanging i and j. There is a left action of $\Bbbk\mathbb{S}_n$ on V_n given by $\sigma x_{i,j} = x_{\sigma(i),\sigma(j)}$ for $\sigma \in \mathbb{S}_n$, $x_{i,j} \in V_n$, and a left coaction $\delta : V_n \to \Bbbk\mathbb{S}_n \otimes V_n$ defined by $\delta(x_{i,j}) = (i,j) \otimes x_{i,j}$ for $x_{i,j} \in V_n$. The space V_n is a Yetter-Drinfeld module over $\mathbb{S}_n$ for the previous structures. The braiding $V_n \otimes V_n \to V_n \otimes V_n$ on V_n is given by $x_{i,j} \otimes x_{k,l} \mapsto ((i,j)x_{k,l}) \otimes x_{i,j}$ for $x_{i,j},x_{k,l} \in V_n$.

The dual V_n^* is also a Yetter-Drinfeld module over $\Bbbk\mathbb{S}_n$. We denote by $y_{i,j}$ the dual element of $x_{i,j}$. The left action of $\Bbbk\mathbb{S}_n$ on V_n^* is given by $\sigma y_{i,j} = y_{\sigma(i),\sigma(j)}$, and the left coaction $\delta' : V_n^* \to \Bbbk\mathbb{S}_n \otimes V_n^*$ is given by $\delta'(y_{i,j}) = (i,j) \otimes y_{i,j}$.

Definition 3.2.1. *Let $n \geqslant 2$ be an integer. The **Fomin-Kirillov algebra** of index n, denoted as $\mathrm{FK}(n)$, is defined as the quotient of $\mathbb{T}(V_n)$ modulo the two-sided ideal (R_n) generated by R_n, where R_n is the subspace of $V_n \otimes V_n$ spanned by*

$x_{i,j}^2$ *for* $i \neq j \in [\![1,n]\!]$,

$x_{i,j}x_{j,k} + x_{j,k}x_{k,i} + x_{k,i}x_{i,j}$ *for* $i,j,k \in [\![1,n]\!]$ *with* $\#\{i,j,k\} = 3$,

$x_{i,j}x_{k,l} - x_{k,l}x_{i,j}$ *for* $i,j,k,l \in [\![1,n]\!]$ *with* $\#\{i,j,k,l\} = 4$.

Recall that the **Hilbert series** of a graded vector space $W = \oplus_{i\in\mathbb{Z}} W_i$ is the formal series $h_W(t) = \sum_{i\in\mathbb{Z}} \dim(W_i)t^i$. Let $[k] = \sum_{i=0}^{k-1} t^i$ for $k \in \mathbb{N}$. The Hilbert series of $\mathrm{FK}(2)$ is $[2]$, the Hilbert series of $\mathrm{FK}(3)$ is $[2]^2[3]$, the Hilbert series of $\mathrm{FK}(4)$ is $[2]^2[3]^2[4]^2$, and the Hilbert series of $\mathrm{FK}(5)$ is $[4]^4[5]^2[6]^4$. Then, the dimension of $\mathrm{FK}(2)$ is 2, the dimension of $\mathrm{FK}(3)$ is 12, the dimension of $\mathrm{FK}(4)$ is 576, and the dimension of $\mathrm{FK}(5)$ is 8294400. However, it is not known if $\mathrm{FK}(6)$ is finite-dimensional.

The quadratic dual algebra $\mathrm{FK}(n)^!$ of $\mathrm{FK}(n)$ is given as the quotient of $\mathbb{T}(V_n^*)$ modulo the two-sided ideal $(R_n^\perp)$, where $R_n^\perp$ is the subspace of $V_n^* \otimes V_n^*$ spanned by

$$y_{i,j}y_{j,k} + y_{j,k}y_{i,k} \text{ for } i,j,k \in [\![1,n]\!] \text{ with } \#\{i,j,k\} = 3,$$
$$y_{i,j}y_{k,l} + y_{k,l}y_{i,j} \text{ for } i,j,k,l \in [\![1,n]\!] \text{ with } \#\{i,j,k,l\} = 4.$$

The subspace $\Bbbk$ of $\mathrm{FK}(n)$ is the trivial Yetter-Drinfeld module. The spaces R_n, $R_n^\perp$, and the algebras $\mathrm{FK}(n)$, $\mathrm{FK}(n)^!$ are all Yetter-Drinfeld modules over $\Bbbk\mathbb{S}_n$. Note that the isomorphism $(V^*)^{\otimes n} \cong (V^{\otimes n})^*$ induced by γ_n defined in (1.1.1) is not of $\Bbbk\mathbb{S}_n$-comodules. So, we have the following remark.

Remark 3.2.2 ([11], Remark 2.6)**.** *If M is a Yetter-Drinfeld module over $\Bbbk G$, where G is a group, then the* ***inverse*** *Yetter-Drinfeld module M^{inv} is given by the same action but the coaction is defined by the grading $(M^{\mathrm{inv}})_g = M_{g^{-1}}$ for $g \in G$. Let A be the Fomin-Kirillov algebra $\mathrm{FK}(3)$. Note that $K_i(A) = ((A^!_{-i})^*)^{\mathrm{inv}} \otimes A$ is a Yetter-Drinfeld module over $\Bbbk\mathbb{S}_n$, and the differentials $d_i : K_i(A) \to K_{i-1}(A)$ in the Koszul complex are morphisms of Yetter-Drinfeld modules. We will omit the superscript* inv *from now on to simplify the notation. As a consequence, the homology $\mathrm{H}_\bullet(K_\bullet(A))$ is also a Yetter-Drinfeld modules over $\Bbbk\mathbb{S}_n$.*

Let us recall a result in [28].

Lemma 3.2.3 ([28], Lemmas 2.3 and 2.4)**.** *Order the generators $y_{i,j}$ for $i < j \in [\![1,n]\!]$ such that $y_{i,j} \prec y_{k,l}$ if $j < l$, or if $j = l$ and $i < k$. Then every element in $\mathrm{FK}(n)^!$ is a linear combination of the monomials of the form $y_{1,2}^{r_{1,2}} y_{1,3}^{r_{1,3}} y_{2,3}^{r_{2,3}} y_{1,4}^{r_{1,4}} \cdots y_{n-2,n}^{r_{n-2,n}} y_{n-1,n}^{r_{n-1,n}}$ for $r_{i,j} \in \mathbb{N}_0$. Moreover, the element $y_{i,j}^2$ is central in $\mathrm{FK}(n)^!$.*

Finally, we recall that $\mathrm{FK}(n)$ is not Koszul for $n \geqslant 3$ (see [22]).

Chapter 4

Fomin-Kirillov algebra on 3 generators

For simplicity, we will denote the Fomin-Kirillov algebra $\mathrm{FK}(3)$ on 3 generators simply by A in this chapter. We will construct the minimal projective bimodule resolution of A, and compute the linear structure of Hochschild homology and cohomology using this resolution.

4.1 The projective bimodule resolution of $\mathrm{FK}(3)$

In this section, we will explicitly construct the minimal projective resolution of the standard bimodule A in the category of bounded below graded A-bimodules.

4.1.1 Generalities

Let $a = x_{1,2}$, $b = x_{2,3}$, $c = x_{3,1}$. By Definition 3.2.1, A is the quadratic $\Bbbk$-algebra generated by the $\Bbbk$-vector space V spanned by three elements a,b,c, modulo the ideal generated by the vector space $R \subseteq V^{\otimes 2}$ spanned by

$$\{a^2, b^2, c^2, ab + bc + ca, ba + ac + cb\}.$$

This is a connected graded $\Bbbk$-algebra by setting the generators a, b and c in internal degree 1. As usual, we will omit the tensor symbol $\otimes$ when denoting the product of the elements of the tensor algebra $\mathbb{T}(V)$. Assume that the set $\{a,b,c\}$ is equipped with an ordering by setting $c \succ b \succ a$. A Gröbner basis is given in Example 2.2.16. Recall that the set

$$\mathscr{B} = \{1, a, b, c, ab, bc, ba, ac, aba, abc, bac, abac\} \tag{4.1.1}$$

is a basis of A (see [8]). Note that $A = \oplus_{m \in [\![0,4]\!]} A_m$, where A_m is the subspace of A concentrated in internal degree m. Given $m \in [\![0,4]\!]$, we will denote by $\mathscr{B}_m$ the subset of (4.1.1) that is a basis of A_m.

Let us briefly denote by $\mathscr{B}_1^! = \{\mathcal{A},\mathcal{B},\mathcal{C}\}$ the basis of V^* dual to the basis $\mathscr{B}_1 = \{a,b,c\}$ of V, where the former are concentrated in internal degree -1. The quadratic dual $A^! = \oplus_{n \in \mathbb{N}_0} A^!_{-n}$ of A is then given by

$$A^! = \Bbbk\langle \mathcal{A},\mathcal{B},\mathcal{C}\rangle / (\mathcal{B}\mathcal{A} - \mathcal{A}\mathcal{C}, \mathcal{C}\mathcal{A} - \mathcal{A}\mathcal{B}, \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{C}, \mathcal{C}\mathcal{B} - \mathcal{B}\mathcal{A}),$$

where $A^!_{-n}$ is the subspace of $A^!$ concentrated in internal degree $-n$. Notice that $A^!_0 = \Bbbk$ and $A^!_{-1} = V^*$. By the relations in $A^!$, we immediately have the following fact.

Fact 4.1.1. *The identities*

$$\begin{aligned}
&X^{2n}Y = Z^{2n}Y, \quad X^{2n+1}Y = Z^{2n+1}X, \\
&\mathcal{B}^{2n+2}\mathcal{A} = \mathcal{A}\mathcal{B}^{2n+2} = \mathcal{A}^{2n+1}\mathcal{B}^2, \quad \mathcal{A}\mathcal{B}^{2n+1} = \mathcal{A}^{2n+1}\mathcal{B}, \quad \mathcal{B}^{2n+1}\mathcal{A} = \mathcal{A}^{2n+1}\mathcal{C}, \\
&\mathcal{C}^{2n+2}\mathcal{A} = \mathcal{A}\mathcal{C}^{2n+2} = \mathcal{A}^{2n+1}\mathcal{B}^2, \quad \mathcal{A}\mathcal{C}^{2n+1} = \mathcal{A}^{2n+1}\mathcal{C}, \quad \mathcal{C}^{2n+1}\mathcal{A} = \mathcal{A}^{2n+1}\mathcal{B},
\end{aligned}$$

holds for $n \in \mathbb{N}_0$ and $\{X,Y,Z\} = \{\mathcal{A},\mathcal{B},\mathcal{C}\}$ (see [11], Fact 3.6), and the identities

$$XY^n = \mathcal{A}^n X \qquad \text{and} \qquad X\mathcal{A}^n = \begin{cases} \mathcal{A}^n X, & \text{if } n \text{ is even,} \\ \mathcal{A}^n Y, & \text{if } n \text{ is odd,} \end{cases}$$

holds for $n \in \mathbb{N}$ and $\{X,Y\} = \{\mathcal{B},\mathcal{C}\}$ in $A^!$.

Lemma 4.1.2. *The set $\mathscr{B}^!_n = \{\mathcal{A}^n, \mathcal{B}^n, \mathcal{C}^n, \mathcal{A}^{n-1}\mathcal{B}, \mathcal{A}^{n-1}\mathcal{C}, \mathcal{A}^{n-2}\mathcal{B}^2\}$ is a basis of $A^!_{-n}$ for all integers $n \geqslant 2$, where we follow the convention that $\mathcal{A}^0\mathcal{B}^2 = \mathcal{B}^2$ (see [23], Lemma 4.4). Note that $\#\mathscr{B}^!_2 = 5$ and $\#\mathscr{B}^!_n = 6$ for $n \geqslant 3$.*

Let $(A^!_{-n})^*$ be the dual space of $A^!_{-n}$ and $\mathscr{B}^{!*}_n = \{\alpha_n, \beta_n, \gamma_n, \alpha_{n-1}\beta, \alpha_{n-1}\gamma, \alpha_{n-2}\beta_2\} \setminus \{\mathbf{0}\}$ the dual basis to $\mathscr{B}^!_n$ for $n \in \mathbb{N}$, where we will follow the convention that if the index of some letter in an element of the previous sets is less than or equal to zero, this element is zero $\mathbf{0}$. We will omit the index 1 for the elements of the previous bases and write $\mathscr{B}^{!*}_0 = \{\epsilon^!\}$, where $\epsilon^!$ is the basis element of $(A^!_0)^*$. The previous bases for the homogeneous components of A, $A^!$ or $(A^!)^\# = \oplus_{n\in\mathbb{N}_0}(A^!_{-n})^*$ will be called **usual**.
Recall that $(A^!)^\#$ is a graded bimodule over $A^!$ via $(ufv)(w) = f(vwu)$ for $u,v,w \in A^!$ and $f \in (A^!)^\#$. Using this definition of action of $A^!$ together with Fact 4.1.1, we immediately get

$$\begin{aligned} &\mathcal{A}\alpha = \mathcal{B}\beta = \mathcal{C}\gamma = \alpha\mathcal{A} = \beta\mathcal{B} = \gamma\mathcal{C} = \epsilon^!, \\ &\mathcal{A}\beta = \mathcal{A}\gamma = \mathcal{B}\alpha = \mathcal{B}\gamma = \mathcal{C}\alpha = \mathcal{C}\beta = \beta\mathcal{A} = \gamma\mathcal{A} = \alpha\mathcal{B} = \gamma\mathcal{B} = \alpha\mathcal{C} = \beta\mathcal{C} = 0, \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}\alpha_n &= \alpha_n\mathcal{A} = \alpha_{n-1}, & \mathcal{A}\beta_n &= \beta_n\mathcal{A} = \mathcal{A}\gamma_n = \gamma_n\mathcal{A} = 0, \\ \mathcal{A}\alpha_{n-1}\beta &= \chi_n\gamma_{n-1} + \alpha_{n-2}\gamma, & \alpha_{n-1}\beta\mathcal{A} &= \chi_n\beta_{n-1} + \alpha_{n-2}\beta, \\ \mathcal{A}\alpha_{n-1}\gamma &= \chi_n\beta_{n-1} + \alpha_{n-2}\beta, & \alpha_{n-1}\gamma\mathcal{A} &= \chi_n\gamma_{n-1} + \alpha_{n-2}\gamma, \\ \mathcal{A}\alpha_{n-2}\beta_2 &= \chi_{n+1}(\beta_{n-1} + \gamma_{n-1}) + \alpha_{n-3}\beta_2, & \alpha_{n-2}\beta_2\mathcal{A} &= \chi_{n+1}(\beta_{n-1} + \gamma_{n-1}) + \alpha_{n-3}\beta_2, \\ \mathcal{B}\beta_n &= \beta_n\mathcal{B} = \beta_{n-1}, & \mathcal{B}\alpha_n &= \alpha_n\mathcal{B} = \mathcal{B}\gamma_n = \gamma_n\mathcal{B} = 0, \\ \mathcal{B}\alpha_{n-1}\beta &= \alpha_{n-1} + \chi_{n+1}\gamma_{n-1} + \alpha_{n-3}\beta_2, & \alpha_{n-1}\beta\mathcal{B} &= \gamma_{n-1} + \chi_n\alpha_{n-2}\gamma + \chi_{n+1}(\alpha_{n-1} + \alpha_{n-3}\beta_2), \\ \mathcal{B}\alpha_{n-1}\gamma &= \chi_n\gamma_{n-1} + \alpha_{n-2}\gamma, & \alpha_{n-1}\gamma\mathcal{B} &= \chi_{n+1}\alpha_{n-2}\beta + \chi_n(\alpha_{n-1} + \alpha_{n-3}\beta_2), \\ \mathcal{B}\alpha_{n-2}\beta_2 &= \alpha_{n-2}\beta, & \alpha_{n-2}\beta_2\mathcal{B} &= \chi_n\alpha_{n-2}\beta + \chi_{n+1}\alpha_{n-2}\gamma, \\ \mathcal{C}\gamma_n &= \gamma_n\mathcal{C} = \gamma_{n-1}, & \mathcal{C}\alpha_n &= \alpha_n\mathcal{C} = \mathcal{C}\beta_n = \beta_n\mathcal{C} = 0, \\ \mathcal{C}\alpha_{n-1}\beta &= \chi_n\beta_{n-1} + \alpha_{n-2}\beta, & \alpha_{n-1}\beta\mathcal{C} &= \chi_{n+1}\alpha_{n-2}\gamma + \chi_n(\alpha_{n-1} + \alpha_{n-3}\beta_2), \\ \mathcal{C}\alpha_{n-1}\gamma &= \alpha_{n-1} + \chi_{n+1}\beta_{n-1} + \alpha_{n-3}\beta_2, & \alpha_{n-1}\gamma\mathcal{C} &= \beta_{n-1} + \chi_n\alpha_{n-2}\beta + \chi_{n+1}(\alpha_{n-1} + \alpha_{n-3}\beta_2), \\ \mathcal{C}\alpha_{n-2}\beta_2 &= \alpha_{n-2}\gamma, & \alpha_{n-2}\beta_2\mathcal{C} &= \chi_{n+1}\alpha_{n-2}\beta + \chi_n\alpha_{n-2}\gamma, \end{aligned}$$

for integers $n \geqslant 2$.
Finally, the following elementary result, whose proof is immediate, will be useful in the sequel to establish the linear independence of several sets of (co)boundaries and (co)cycles.

Fact 4.1.3. *Let V be a $\Bbbk$-vector space of dimension $n \in \mathbb{N}$ and $\{v_1, \cdots, v_n\}$ a basis of V. Let $r \leqslant n$ be a positive integer and $U = \{\sum_{j=1}^n c_j^k v_j \in V | c_j^k \in \Bbbk, k \in [\![1, r]\!]\}$ a set of r elements. If there is an injective map $\varphi : [\![1,r]\!] \to [\![1, n]\!]$ such that for all $k \in [\![1,r]\!]$, $c^k_{\varphi(k)} \neq 0$, but $c^i_{\varphi(k)} = 0$ for $i \in [\![1, k-1]\!]$, then the elements in U are linearly independent.*

Instead of writing the specific map φ, in the cases of (ordered) sets U we will consider in the sequel we will simply underline the corresponding term $c^k_{\varphi(k)}v_{\varphi(k)}$. This will be the case in particular in Subsubsections 4.2.1.3 - 4.2.1.5 and 4.2.2.3 - 4.2.2.5. In that situation, the basis $\{v_1, \cdots, v_n\}$ of the larger vector space will be a usual basis and the condition on φ is tantamount to the fact that the underlined term of an element does not appear (with nonzero coefficient) in the expressions of the previous elements of the same set.

4.1.2 The Yetter-Drinfeld structures

Let $\Bbbk$ be an algebraically closed field of characteristic different from 2 and 3, and $G = \mathbb{S}_3$ in this subsection. We will introduce the Yetter-Drinfeld structures on A and $(A^!)^*$.

First, we explicitly present the irreducible Yetter-Drinfeld modules over $\Bbbk\mathbb{S}_3$ by Remark 3.1.14.

Recall that the conjugacy classes in $G = \mathbb{S}_3$ are $\mathfrak{C}_1 = \{(1)\}$, $\mathfrak{C}_2 = \{(12),(23),(13)\}$ and $\mathfrak{C}_3 = \{(123),(132)\}$. If $s = (1) \in \mathfrak{C}_1$, then $G^s = \mathbb{S}_3$. Let ρ_i for $i \in [\![1,3]\!]$ be the irreducible representations given in Example 3.1.4. If $s = (12) \in \mathfrak{C}_2$, then $G^s = \{(1),(12)\}$, the cyclic group of order 2. Let ρ_i for $i \in [\![4,5]\!]$ be the irreducible representations of the group $\{(1),(12)\}$. Here ρ_4 is the trivial representation and $\rho_5 : \{(1),(12)\} \to \Bbbk^\times$ is given by $\rho_5((12)) = -1$. If $s = (123) \in \mathfrak{C}_3$, then $G^s = \{(1),(123),(132)\}$, the cyclic group of order 3. Let ρ_i for $i \in [\![6,8]\!]$ be the irreducible representations of the group $\{(1),(123),(132)\}$. Here ρ_6 is the trivial representation, $\rho_7 : \{(1),(123),(132)\} \to \Bbbk^\times$ is given by $\rho_7((123)) = w$ and $\rho_8 : \{(1),(123),(132)\} \to \Bbbk^\times$ is given by $\rho_8((123)) = w^2$, where w is a primitive root of unity of order 3 in $\Bbbk$. Note that $w^2+w+1 = 0$. Let $M_i = M(\mathfrak{C}_j,\rho_i)$ be the irreducible Yetter-Drinfeld modules over $\Bbbk G$ defined in Remark 3.1.14 parameterized by $(\mathfrak{C}_j,\rho_i)$ for $(j,i) \in (\{1\} \times [\![1,3]\!]) \cup (\{2\} \times [\![4,5]\!]) \cup (\{3\} \times [\![6,8]\!])$. A precise description of M_i for $i \in [\![1,8]\!]$ is as follows. The module $M_1 = \Bbbk$ is the trivial Yetter-Drinfeld module, *i.e.* the action is given by $gx = x$ for all $g \in G$ and $x \in M_1$, and the coaction $M_1 \to G \otimes M_1$ is given by $x \mapsto (1) \otimes x$ for all $x \in M_1$. The action on $M_2 = \Bbbk$ is given by $gx = x$ for $g \in \mathcal{C}_1 \cup \mathcal{C}_3$, and $gx = -x$ for $g \in \mathcal{C}_2$ and $x \in M_2$. The coaction on M_2 is the trivial coaction. Let V be a 3-dimensional $\Bbbk$-vector space with a basis $\{e_1,e_2,e_3\}$. The action on $M_3 = V/\operatorname{span}_{\Bbbk}\{e_1+e_2+e_3\}$ is given by $(12)\bar{e}_1 = \bar{e}_2$, $(12)\bar{e}_2 = \bar{e}_1$, $(23)\bar{e}_1 = \bar{e}_1$, $(23)\bar{e}_2 = -\bar{e}_1-\bar{e}_2$, $(13)\bar{e}_1 = -\bar{e}_1 - \bar{e}_2$, $(13)\bar{e}_2 = \bar{e}_2$, $(123)\bar{e}_1 = \bar{e}_2$, $(123)\bar{e}_2 = -\bar{e}_1 - \bar{e}_2$, $(132)\bar{e}_1 = -\bar{e}_1 - \bar{e}_2$ and $(132)\bar{e}_2 = \bar{e}_1$, where $\bar{e}_i$ is the class of e_i in the quotient space. The coaction on M_3 is the trivial coaction. The action on $M_4 = (1) \otimes \Bbbk \oplus (13) \otimes \Bbbk \oplus (23) \otimes \Bbbk$ is given by

$$
\begin{aligned}
&(12)((1) \otimes x) = (1) \otimes x, && (12)((13) \otimes x) = (23) \otimes x, && (12)((23) \otimes x) = (13) \otimes x,\\
&(23)((1) \otimes x) = (23) \otimes x, && (23)((13) \otimes x) = (13) \otimes x, && (23)((23) \otimes x) = (1) \otimes x,\\
&(13)((1) \otimes x) = (13) \otimes x, && (13)((13) \otimes x) = (1) \otimes x, && (13)((23) \otimes x) = (23) \otimes x,\\
&(123)((1) \otimes x) = (13) \otimes x, && (123)((13) \otimes x) = (23) \otimes x, && (123)((23) \otimes x) = (1) \otimes x,\\
&(132)((1) \otimes x) = (23) \otimes x, && (132)((13) \otimes x) = (1) \otimes x, && (132)((23) \otimes x) = (13) \otimes x,
\end{aligned}
$$

for $x \in \Bbbk$. The action on $M_5 = (1) \otimes \Bbbk \oplus (13) \otimes \Bbbk \oplus (23) \otimes \Bbbk$ is given by

$$
\begin{aligned}
&(12)((1) \otimes x) = -(1) \otimes x, && (12)((13) \otimes x) = -(23) \otimes x, && (12)((23) \otimes x) = -(13) \otimes x,\\
&(23)((1) \otimes x) = (23) \otimes x, && (23)((13) \otimes x) = -(13) \otimes x, && (23)((23) \otimes x) = (1) \otimes x,\\
&(13)((1) \otimes x) = (13) \otimes x, && (13)((13) \otimes x) = (1) \otimes x, && (13)((23) \otimes x) = -(23) \otimes x,\\
&(123)((1) \otimes x) = -(13) \otimes x, && (123)((13) \otimes x) = (23) \otimes x, && (123)((23) \otimes x) = -(1) \otimes x,\\
&(132)((1) \otimes x) = -(23) \otimes x, && (132)((13) \otimes x) = -(1) \otimes x, && (132)((23) \otimes x) = (13) \otimes x,
\end{aligned}
$$

for $x \in \Bbbk$. The coaction on M_4 and M_5 is given by $(1) \otimes x \mapsto (12) \otimes ((1) \otimes x)$, $(13) \otimes x \mapsto (23) \otimes ((13) \otimes x)$ and $(23) \otimes x \mapsto (13) \otimes ((23) \otimes x)$ for $x \in \Bbbk$. The action on $M_6 = (1) \otimes \Bbbk \oplus (12) \otimes \Bbbk$ is given by $g((1) \otimes x) = (12) \otimes x$, $g((12) \otimes x) = (1) \otimes x$ for $g \in \{(12),(23),(13)\}$, and $g((1) \otimes x) = (1) \otimes x$, $g((12) \otimes x) = (12) \otimes x$ for $g \in \{(1),(123),(132)\}$ and $x \in \Bbbk$. The action on $M_7 = (1) \otimes \Bbbk \oplus (12) \otimes \Bbbk$ is given by

$$
\begin{aligned}
&(12)((1) \otimes x) = (12) \otimes x, && (12)((12) \otimes x) = (1) \otimes x,\\
&(23)((1) \otimes x) = (12) \otimes wx, && (23)((12) \otimes x) = (1) \otimes w^2x,\\
&(13)((1) \otimes x) = (12) \otimes w^2x, && (13)((12) \otimes x) = (1) \otimes wx,\\
&(123)((1) \otimes x) = (1) \otimes wx, && (123)((12) \otimes x) = (12) \otimes w^2x,\\
&(132)((1) \otimes x) = (1) \otimes w^2x, && (132)((12) \otimes x) = (12) \otimes wx,
\end{aligned}
$$

for $x \in \Bbbk$. The action on $M_8 = (1) \otimes \Bbbk \oplus (12) \otimes \Bbbk$ is given by

$$
\begin{aligned}
&(12)((1) \otimes x) = (12) \otimes x, && (12)((12) \otimes x) = (1) \otimes x,\\
&(23)((1) \otimes x) = (12) \otimes w^2x, && (23)((12) \otimes x) = (1) \otimes wx,\\
&(13)((1) \otimes x) = (12) \otimes wx, && (13)((12) \otimes x) = (1) \otimes w^2x,\\
&(123)((1) \otimes x) = (1) \otimes w^2x, && (123)((12) \otimes x) = (12) \otimes wx,
\end{aligned}
$$

$$(132)((1) \otimes x) = (1) \otimes wx, \qquad (132)((12) \otimes x) = (12) \otimes w^2 x,$$

for $x \in \Bbbk$. The coaction on M_i for $i \in [\![6,8]\!]$ is given by $(1) \otimes x \mapsto (123) \otimes ((1) \otimes x)$ and $(12) \otimes x \mapsto (132) \otimes ((12) \otimes x)$ for $x \in \Bbbk$.

Recall that Yetter-Drinfeld modules over $\Bbbk G$ are semi-simple. The tensor product of two Yetter-Drinfeld modules is also a Yetter-Drinfeld module.

Lemma 4.1.4. *There are the following isomorphisms*

$$\begin{aligned}
&M_5 \otimes M_5 \cong M_1 \oplus M_3 \oplus M_6 \oplus M_7 \oplus M_8,\\
&M_5 \otimes M_7 \cong M_5 \otimes M_8 \cong M_6 \otimes M_5 \cong M_3 \otimes M_5 \cong M_4 \oplus M_5,\\
&M_6 \otimes M_7 \cong M_3 \oplus M_8,\ M_6 \otimes M_8 \cong M_3 \oplus M_7,\ M_3 \otimes M_7 \cong M_6 \oplus M_8,\ M_3 \otimes M_8 \cong M_6 \oplus M_7,
\end{aligned}$$

of Yetter-Drinfeld modules over $\Bbbk\mathbb{S}_3$.

Proof. Let $x = (1) \otimes 1$, $y = (13) \otimes 1$ and $z = (23) \otimes 1$ given in the definition of M_5. The isomorphism $M_5 \otimes M_5 \cong M_1 \oplus M_3 \oplus M_6 \oplus M_7 \oplus M_8$ comes from the obvious isomorphisms $\mathrm{span}_\Bbbk\{x|x + y|y + z|z\} \cong M_1$, $\mathrm{span}_\Bbbk\{x|x - 2y|y + z|z, x|x + y|y - 2z|z\} \cong M_3$, $\mathrm{span}_\Bbbk\{-x|y + y|z - z|x, -x|z - y|x + z|y\} \cong M_6$, $\mathrm{span}_\Bbbk\{-w^2x|y + wy|z - z|x, -w^2x|z - y|x + wz|y\} \cong M_7$ and $\mathrm{span}_\Bbbk\{-x|y + wy|z - w^2z|x, -x|z - w^2y|x + wz|y\} \cong M_8$, where we use vertical bars instead of the tensor product symbols $\otimes$. Let $u = (1) \otimes 1$ and $v = (12) \otimes 1$ given in the definition of M_i for $i \in [\![6,8]\!]$. Then $M_5 \otimes M_7 \cong M_4 \oplus M_5$ since $\mathrm{span}_\Bbbk\{-w^2y|v + w^2z|u, -x|u - wz|v, x|v + wy|u\} \cong M_4$ and $\mathrm{span}_\Bbbk\{w^2y|v + w^2z|u, x|u - wz|v, x|v - wy|u\} \cong M_5$, $M_5 \otimes M_8 \cong M_4 \oplus M_5$ since $\mathrm{span}_\Bbbk\{-wy|v + wz|u, -x|u - w^2z|v, x|v + w^2y|u\} \cong M_4$ and $\mathrm{span}_\Bbbk\{wy|v + wz|u, x|u - w^2z|v, x|v - w^2y|u\} \cong M_5$, $M_6 \otimes M_5 \cong M_4 \oplus M_5$ since $\mathrm{span}_\Bbbk\{-u|y + v|z, -v|x - u|z, u|x + v|y\} \cong M_4$ and $\mathrm{span}_\Bbbk\{u|y + v|z, v|x - u|z, u|x - v|y\} \cong M_5$, $M_6 \otimes M_7 \cong M_3 \oplus M_8$ since $\mathrm{span}_\Bbbk\{w^2v|u + u|v, v|u + w^2u|v\} \cong M_3$ and $\mathrm{span}_\Bbbk\{v|v, u|u\} \cong M_8$, and $M_6 \otimes M_8 \cong M_3 \oplus M_7$ since $\mathrm{span}_\Bbbk\{wv|u + u|v, v|u + wu|v\} \cong M_3$ and $\mathrm{span}_\Bbbk\{v|v, u|u\} \cong M_7$. Finally, $M_3 \otimes M_5 \cong M_4 \oplus M_5$ since $\mathrm{span}_\Bbbk\{-\bar{e}_1|x + \bar{e}_2|x, \bar{e}_1|y + 2\bar{e}_2|y, -2\bar{e}_1|z - \bar{e}_2|z\} \cong M_4$ and $\mathrm{span}_\Bbbk\{\bar{e}_1|x + \bar{e}_2|x, -\bar{e}_1|y, -\bar{e}_2|z\} \cong M_5$, $M_3 \otimes M_7 \cong M_6 \oplus M_8$ since $\mathrm{span}_\Bbbk\{-\bar{e}_1|u + w^2\bar{e}_2|u, w^2\bar{e}_1|v - \bar{e}_2|v\} \cong M_6$ and $\mathrm{span}_\Bbbk\{w\bar{e}_1|u - w^2\bar{e}_2|u, -w^2\bar{e}_1|v + w\bar{e}_2|v\} \cong M_8$, and $M_3 \otimes M_8 \cong M_6 \oplus M_7$ since $\mathrm{span}_\Bbbk\{w^2\bar{e}_1|u - \bar{e}_2|u, -\bar{e}_1|v + w^2\bar{e}_2|v\} \cong M_6$ and $\mathrm{span}_\Bbbk\{-\bar{e}_1|u + w^2\bar{e}_2|u, w^2\bar{e}_1|v - \bar{e}_2|v\} \cong M_7$. ☐

Next, we decompose A and $(A^!)^\#$ as a direct sum of irreducible Yetter-Drinfeld modules.

Fact 4.1.5. *The action of G on A is given by*

$$\begin{aligned}
&(12)a = -a, && (23)a = -c, && (13)a = -b, && (123)a = b, && (132)a = c,\\
&(12)b = -c, && (23)b = -b, && (13)b = -a, && (123)b = c, && (132)b = a,\\
&(12)c = -b, && (23)c = -a, && (13)c = -c, && (123)c = a, && (132)c = b,
\end{aligned}$$

and

$$\begin{aligned}
&(12)ab = ac,\ (23)ab = -ba - ac,\ (13)ab = ba,\ (123)ab = bc,\ (132)ab = -ab - bc,\\
&(12)bc = -ba - ac,\ (23)bc = ba,\ (13)bc = ac,\ (123)bc = -ab - bc,\ (132)bc = ab,\\
&(12)ba = -ab - bc,\ (23)ba = bc,\ (13)ba = ab,\ (123)ba = -ba - ac,\ (132)ba = ac,\\
&(12)ac = ab,\ (23)ac = -ab - bc,\ (13)ac = bc,\ (123)ac = ba,\ (132)ac = -ba - ac,
\end{aligned}$$

together with

$$\begin{aligned}
&(12)aba = abc, && (23)aba = bac, && (13)aba = -aba, && (123)aba = -bac, && (132)aba = -abc,\\
&(12)abc = aba, && (23)abc = -abc, && (13)abc = -bac, && (123)abc = -aba, && (132)abc = bac,\\
&(12)bac = -bac, && (23)bac = aba, && (13)bac = -abc, && (123)bac = abc, && (132)bac = -aba,
\end{aligned}$$

as well as $g(abac) = abac$ for all $g \in G$.

The coaction of G on A is given by the G-decomposition $A = \oplus_{g \in G} A_g$, where $A_{(1)}$ is spanned by $\{1, abac\}$, $A_{(12)}$ is spanned by $\{a, bac\}$, $A_{(23)}$ is spanned by $\{b, abc\}$, $A_{(13)}$ is spanned by $\{c, aba\}$, and $A_{(123)}$ is spanned by $\{ab, bc\}$, $A_{(132)}$ is spanned by $\{ac, ba\}$.

Lemma 4.1.6. *We have*

$$A \cong M_1^{\oplus 2} \oplus M_5^{\oplus 2} \oplus M_7 \oplus M_8$$

as Yetter-Drinfeld modules.

Proof. By Fact 4.1.5, it is easy to check the following. The subspaces $\Bbbk$ and $\mathrm{span}_{\Bbbk}\{abac\}$ are trivial Yetter-Drinfeld modules. An isomorphism $M_5 \to \mathrm{span}_{\Bbbk}\{a,b,c\}$ of Yetter-Drinfeld modules is given by $(1)\otimes 1 \mapsto -a$, $(13)\otimes 1 \mapsto b$ and $(23)\otimes 1 \mapsto c$. An isomorphism $M_5 \to \mathrm{span}_{\Bbbk}\{aba,abc,bac\}$ of Yetter-Drinfeld modules is given by $(1)\otimes 1 \mapsto bac$, $(13)\otimes 1 \mapsto -abc$ and $(23)\otimes 1 \mapsto aba$. An isomorphism $M_7 \to \mathrm{span}_{\Bbbk}\{w^2ab - bc, ba - wac\}$ of Yetter-Drinfeld modules is given by $(1)\otimes 1 \mapsto w^2ab - bc$ and $(12)\otimes 1 \mapsto ba - wac$. An isomorphism $M_8 \to \mathrm{span}_{\Bbbk}\{-w^2ab + wbc, ac - wba\}$ of Yetter-Drinfeld modules is given by $(1)\otimes 1 \mapsto -w^2ab + wbc$ and $(12)\otimes 1 \mapsto ac - wba$. We get the decomposition in this lemma immediately. $\square$

Fact 4.1.7. *The action of G on $(A^!)^{\#}$ is given by*

$$\begin{aligned}
&(12)\alpha_n = (-1)^n\alpha_n, \quad (23)\alpha_n = (-1)^n\gamma_n, \quad (13)\alpha_n = (-1)^n\beta_n, \quad (123)\alpha_n = \beta_n, \quad (132)\alpha_n = \gamma_n,\\
&(12)\beta_n = (-1)^n\gamma_n, \quad (23)\beta_n = (-1)^n\beta_n, \quad (13)\beta_n = (-1)^n\alpha_n, \quad (123)\beta_n = \gamma_n, \quad (132)\beta_n = \alpha_n,\\
&(12)\gamma_n = (-1)^n\beta_n, \quad (23)\gamma_n = (-1)^n\alpha_n, \quad (13)\gamma_n = (-1)^n\gamma_n, \quad (123)\gamma_n = \alpha_n, \quad (132)\gamma_n = \beta_n,
\end{aligned}$$

for $n \in \mathbb{N}$, and

$$\begin{aligned}
&g\xi = \eta, \textit{ for } g \in \{(12),(23),(13)\} \textit{ and } \{\xi,\eta\} = \{\alpha_{n-1}\beta, \alpha_{n-1}\gamma\},\\
&g\xi = \xi, \textit{ for } g \in G \textit{ with } \xi = \alpha_{n-2}\beta_2, \textit{ or } g \in \{(123),(132)\} \textit{ with } \xi \in \{\alpha_{n-1}\beta, \alpha_{n-1}\gamma\},
\end{aligned}$$

for $n \geqslant 2$ with n even, together with

$$\begin{aligned}
&(12)\alpha_{n-1}\beta = -\alpha_{n-1}\gamma, && (12)\alpha_{n-1}\gamma = -\alpha_{n-1}\beta, && (12)\alpha_{n-2}\beta_2 = -\alpha_{n-2}\beta_2,\\
&(23)\alpha_{n-1}\beta = -\alpha_{n-1}\beta, && (23)\alpha_{n-1}\gamma = -\alpha_{n-2}\beta_2, && (23)\alpha_{n-2}\beta_2 = -\alpha_{n-1}\gamma,\\
&(13)\alpha_{n-1}\beta = -\alpha_{n-2}\beta_2, && (13)\alpha_{n-1}\gamma = -\alpha_{n-1}\gamma, && (13)\alpha_{n-2}\beta_2 = -\alpha_{n-1}\beta,\\
&(123)\alpha_{n-1}\beta = \alpha_{n-1}\gamma, && (123)\alpha_{n-1}\gamma = \alpha_{n-2}\beta_2, && (123)\alpha_{n-2}\beta_2 = \alpha_{n-1}\beta,\\
&(132)\alpha_{n-1}\beta = \alpha_{n-2}\beta_2, && (132)\alpha_{n-1}\gamma = \alpha_{n-1}\beta, && (132)\alpha_{n-2}\beta_2 = \alpha_{n-1}\gamma,
\end{aligned}$$

for $n \geqslant 3$ with n odd.

The coaction of G on $(A^!)^{\#}$ is given by the G-decomposition $(A^!)^{\#} = \oplus_{g\in G}((A^!)^{\#})_g$, where

$$\begin{aligned}
&((A^!)^{\#})_{(1)} = \mathrm{span}_{\Bbbk}\{1, \alpha_n, \beta_n, \gamma_n, \alpha_{n-2}\beta_2 \mid n \geqslant 2 \textit{ even}\}, && ((A^!)^{\#})_{(12)} = \mathrm{span}_{\Bbbk}\{\alpha_n, \alpha_{n-2}\beta_2 \mid n \in \mathbb{N} \textit{ odd}\},\\
&((A^!)^{\#})_{(23)} = \mathrm{span}_{\Bbbk}\{\beta_n, \alpha_{n-1}\beta \mid n \in \mathbb{N} \textit{ odd}\}, && ((A^!)^{\#})_{(13)} = \mathrm{span}_{\Bbbk}\{\gamma_n, \alpha_{n-1}\gamma \mid n \in \mathbb{N} \textit{ odd}\},\\
&((A^!)^{\#})_{(123)} = \mathrm{span}_{\Bbbk}\{\alpha_{n-1}\beta \mid n \geqslant 2 \textit{ even}\}, && ((A^!)^{\#})_{(132)} = \mathrm{span}_{\Bbbk}\{\alpha_{n-1}\gamma \mid n \geqslant 2 \textit{ even}\}.
\end{aligned}$$

Lemma 4.1.8. *As Yetter-Drinfeld modules over $\Bbbk\mathbb{S}_3$, $(A^!_{-1})^* \cong M_5$, $(A^!_{-2})^* \cong M_1 \oplus M_3 \oplus M_6$, and*

$$(A^!_{-n})^* \cong \begin{cases} M_5 \oplus M_5, & \textit{if } n \textit{ is odd,}\\ M_1^{\oplus 2} \oplus M_3 \oplus M_6, & \textit{if } n \textit{ is even,}\end{cases}$$

for $n \geqslant 3$.

Proof. By Fact 4.1.7, it is to check the following statements. The subspace $\Bbbk$ and the subspace $\mathrm{span}_{\Bbbk}\{\alpha_{n-2}\beta_2\}$ for $n \geqslant 4$ even, are trivial Yetter-Drinfeld modules. An isomorphism $M_5 \to \mathrm{span}_{\Bbbk}\{\alpha_n,\beta_n,\gamma_n\}$ for $n \in \mathbb{N}$ odd is given by $(1)\otimes 1 \mapsto -\alpha_n$, $(13)\otimes 1 \mapsto \beta_n$ and $(23)\otimes 1 \mapsto \gamma_n$. An isomorphism $M_5 \to \mathrm{span}_{\Bbbk}\{\alpha_{n-1}\beta,\alpha_{n-1}\gamma,\alpha_{n-2}\beta_2\}$ for $n \geqslant 3$ odd is given by $(1)\otimes 1 \mapsto -\alpha_{n-2}\beta_2$, $(13)\otimes 1 \mapsto \alpha_{n-1}\beta$ and $(23)\otimes 1 \mapsto \alpha_{n-1}\gamma$. An isomorphism $M_6 \to \mathrm{span}_{\Bbbk}\{\alpha_{n-1}\beta,\alpha_{n-1}\gamma\}$ for $n \geqslant 2$ even is given by $(1)\otimes 1 \mapsto \alpha_{n-1}\beta$ and $(12)\otimes 1 \mapsto \alpha_{n-1}\gamma$. The subspace $\mathrm{span}_{\Bbbk}\{\alpha_n,\beta_n,\gamma_n\}$ for $n \geqslant 2$ even is the permutation representation of G. With the trivial coaction, $\mathrm{span}_{\Bbbk}\{\alpha_n,\beta_n,\gamma_n\} \cong M_1 \oplus M_3$ as Yetter-Drinfeld modules. $\square$

4.1.3 The projective bimodule resolution

In this subsection, we will explicitly describe the minimal projective resolution of A in the category of bounded below graded A-bimodules. These results were published in [12].

4.1.3.1 The bimodule Koszul complex

In the article [4] R. Berger and N. Marconnet introduced the **bimodule Koszul complex** for any N-homogeneous algebra. We will recall this for the special case of the Fomin–Kirillov algebra on 3 generators (so $N = 2$). Given $n \in \mathbb{N}_0$, let $K_n^b = A \otimes (A^!_{-n})^* \otimes A$ be the bimodule over A for the outer action. Define the maps

$$i_\ell, i_r : A \otimes (A^!)^\# \otimes A \to A \otimes (A^!)^\# \otimes A$$

given by $i_\ell(x \otimes u \otimes y) = xa \otimes u\mathcal{A} \otimes y + xb \otimes u\mathcal{B} \otimes y + xc \otimes u\mathcal{C} \otimes y$ and $i_r(x \otimes u \otimes y) = x \otimes \mathcal{A}u \otimes ay + x \otimes \mathcal{B}u \otimes by + x \otimes \mathcal{C}u \otimes cy$ for $x,y \in A$ and $u \in (A^!)^\#$. Note that $i_\ell^2 = 0$, $i_r^2 = 0$ and $i_\ell i_r = i_r i_\ell$. Indeed, the first identity follows from the fact that

$$(a \otimes \mathcal{A} + b \otimes \mathcal{B} + c \otimes \mathcal{C})^2 = ba \otimes \mathcal{BA} + ca \otimes \mathcal{CA} + ab \otimes \mathcal{AB} + cb \otimes \mathcal{CB} + ac \otimes \mathcal{AC} + bc \otimes \mathcal{BC}$$

is trivially zero by applying the relations in A and $A^!$ and the fact that $i_\ell^2(x \otimes u \otimes y) = (x \otimes u \otimes y)(a \otimes \mathcal{A} \otimes 1 + b \otimes \mathcal{B} \otimes 1 + c \otimes \mathcal{C} \otimes 1)^2$. The identity $i_r^2 = 0$ is proved in the same way. Since the left and right actions of $A^!$ on $(A^!)^\#$ are compatible, the maps i_ℓ and i_r commute.

Fact 4.1.9. *Take $x, y \in A$. To reduce space, we will typically use vertical bars instead of the tensor product symbols $\otimes$. The map $i_\ell|_{A\otimes(A^!_{-1})^*\otimes A} : A \otimes (A^!_{-1})^* \otimes A \to A \otimes (A^!_0)^* \otimes A$ sends $x|\alpha|y$ to $xa|\epsilon^!|y$, $x|\beta|y$ to $xb|\epsilon^!|y$, and $x|\gamma|y$ to $xc|\epsilon^!|y$. For $n \geqslant 2$, $i_\ell|_{A\otimes(A^!_{-n})^*\otimes A} : A \otimes (A^!_{-n})^* \otimes A \to A \otimes (A^!_{-(n-1)})^* \otimes A$ is given by*

$$\begin{aligned}
x|\alpha_n|y &\mapsto xa|\alpha_{n-1}|y, \quad x|\beta_n|y \mapsto xb|\beta_{n-1}|y, \quad x|\gamma_n|y \mapsto xc|\gamma_{n-1}|y,\\
x|\alpha_{n-1}\beta|y &\mapsto xa|(\chi_n\beta_{n-1} + \alpha_{n-2}\beta)|y + xb|(\gamma_{n-1} + \chi_n\alpha_{n-2}\gamma + \chi_{n+1}(\alpha_{n-1} + \alpha_{n-3}\beta_2))|y\\
&\quad + xc|(\chi_{n+1}\alpha_{n-2}\gamma + \chi_n(\alpha_{n-1} + \alpha_{n-3}\beta_2))|y,\\
x|\alpha_{n-1}\gamma|y &\mapsto xa|(\chi_n\gamma_{n-1} + \alpha_{n-2}\gamma)|y + xb|(\chi_{n+1}\alpha_{n-2}\beta + \chi_n(\alpha_{n-1} + \alpha_{n-3}\beta_2))|y\\
&\quad + xc|(\beta_{n-1} + \chi_n\alpha_{n-2}\beta + \chi_{n+1}(\alpha_{n-1} + \alpha_{n-3}\beta_2))|y,\\
x|\alpha_{n-2}\beta_2|y &\mapsto xa|(\chi_{n+1}(\beta_{n-1} + \gamma_{n-1}) + \alpha_{n-3}\beta_2)|y + xb|(\chi_n\alpha_{n-2}\beta + \chi_{n+1}\alpha_{n-2}\gamma)|y\\
&\quad + xc|(\chi_{n+1}\alpha_{n-2}\beta + \chi_n\alpha_{n-2}\gamma)|y.
\end{aligned}$$

The map $i_r|_{A\otimes(A^!_{-1})^*\otimes A} : A \otimes (A^!_{-1})^* \otimes A \to A \otimes (A^!_0)^* \otimes A$ sends $x|\alpha|y$ to $x|\epsilon^!|ay$, $x|\beta|y$ to $x|\epsilon^!|by$, and $x|\gamma|y$ to $x|\epsilon^!|cy$. For $n \geqslant 2$, $i_r|_{A\otimes(A^!_{-n})^*\otimes A} : A \otimes (A^!_{-n})^* \otimes A \to A \otimes (A^!_{-(n-1)})^* \otimes A$ is given by

$$\begin{aligned}
x|\alpha_n|y &\mapsto x|\alpha_{n-1}|ay, \quad x|\beta_n|y \mapsto x|\beta_{n-1}|by, \quad x|\gamma_n|y \mapsto x|\gamma_{n-1}|cy,\\
x|\alpha_{n-1}\beta|y &\mapsto x|(\chi_n\gamma_{n-1} + \alpha_{n-2}\gamma)|ay + x|(\alpha_{n-1} + \chi_{n+1}\gamma_{n-1} + \alpha_{n-3}\beta_2)|by\\
&\quad + x|(\chi_n\beta_{n-1} + \alpha_{n-2}\beta)|cy,\\
x|\alpha_{n-1}\gamma|y &\mapsto x|(\chi_n\beta_{n-1} + \alpha_{n-2}\beta)|ay + x|(\chi_n\gamma_{n-1} + \alpha_{n-2}\gamma)|by\\
&\quad + x|(\alpha_{n-1} + \chi_{n+1}\beta_{n-1} + \alpha_{n-3}\beta_2)|cy,\\
x|\alpha_{n-2}\beta_2|y &\mapsto x|(\chi_{n+1}(\beta_{n-1} + \gamma_{n-1}) + \alpha_{n-3}\beta_2)|ay + x|\alpha_{n-2}\beta|by + x|\alpha_{n-2}\gamma|cy.
\end{aligned}$$

Following [4], we now set $d_n^b : K_n^b \to K_{n-1}^b$ by $d_n^b = (-1)^n i_\ell + i_r$ for $n \in \mathbb{N}$. It is easy to see that $d_n^b d_{n+1}^b = -i_\ell^2 + i_r^2 = 0$ for $n \in \mathbb{N}$. Then $(K_\bullet^b, d_\bullet^b)$ is a complex in the category of bounded below graded A-bimodules, called the **bimodule Koszul complex** over A. It is clear that $\Bbbk \otimes_A (K_\bullet^b, d_\bullet^b) \cong (K_\bullet, d_\bullet)$, where $(K_\bullet, d_\bullet)$, introduced in Section 1.1, is the Koszul complex of the trivial right A-module $\Bbbk$ in the category of graded right A-modules.

Remark 4.1.10. *The bimodule Koszul complex $(K_\bullet^b, d_\bullet^b)$ is minimal, since the complex $\Bbbk \otimes_{A^e} (K_\bullet^b, d_\bullet^b) \cong \Bbbk \otimes_A (K_\bullet^b, d_\bullet^b) \otimes_A \Bbbk \cong (K_\bullet, d_\bullet) \otimes_A \Bbbk$ has zero differentials.*

We recall the following result.

Proposition 4.1.11 ([4], Proposition 4.1). *Let B be a nonnegatively graded connected $\Bbbk$-algebra, and let*

$$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$$

be a sequence of graded-free B-modules, with M_1 bounded below and $gf = 0$. Then this sequence is exact if

$$\Bbbk \otimes_B M_1 \xrightarrow{\mathrm{id}_\Bbbk \otimes_B f} \Bbbk \otimes_B M_2 \xrightarrow{\mathrm{id}_\Bbbk \otimes_B g} \Bbbk \otimes_B M_3$$

is exact.

Corollary 4.1.12. *We have $\mathrm{H}_n(K^b_\bullet, d^b_\bullet) = 0$ for n different from 0 and 3.*

Proof. Recall that $\mathrm{H}_n(K_\bullet, d_\bullet) = 0$ for $n \neq 0,3$, by [11], Proposition 3.1. Applying Proposition 4.1.11, we get the result. □

4.1.3.2 The minimal projective bimodule resolution

We recall the following result (see [11], Proposition 3.3).

Proposition 4.1.13. *Let $(K_\bullet, d_\bullet)$ be the Koszul complex of the trivial right A-module $\Bbbk$ in the category of graded right A-modules. The minimal projective resolution $(P_\bullet, \delta_\bullet)$ of $\Bbbk$ in the category of bounded below graded right A-modules is given as follows. For $n \in \mathbb{N}_0$, set*

$$P_n = \bigoplus_{i \in [\![0, \lfloor n/4 \rfloor]\!]} \omega_i K_{n-4i},$$

where ω_i is a symbol of internal degree $6i$ for all $i \in \mathbb{N}_0$, and the differential $\delta_n : P_n \to P_{n-1}$ for $n \in \mathbb{N}$ is given by

$$\delta_n\Bigg(\sum_{i \in [\![0, \lfloor n/4 \rfloor]\!]} \omega_i \rho_{n-4i}\Bigg) = \sum_{i \in [\![0, \lfloor n/4 \rfloor]\!]} \big(\omega_i d_{n-4i}(\rho_{n-4i}) + \omega_{i-1} f_{n-4i}(\rho_{n-4i})\big),$$

where $\rho_j \in K_j$ for $j \in \mathbb{N}_0$, $\omega_{-1} = 0$ and $f_j : K_j \to K_{j+3}$ are morphisms of graded right A-modules of internal degree 6 such that $d_{j+4} f_{j+1} = -f_j d_{j+1}$ for $j \in \mathbb{N}_0$, $d_3 f_0 = 0$ and $\mathrm{Im}(f_0) \nsubseteq \mathrm{Im}(d_4)$. This gives a minimal projective resolution of the trivial right A-module $\Bbbk$ by means of the augmentation $\epsilon : P_0 = K_0 \to \Bbbk$ of the Koszul complex. We usually omit ω_0 for simplicity. Furthermore, if the characteristic of $\Bbbk$ is different from 2 and 3, then the maps $\{f_\bullet\}_{\bullet \in \mathbb{N}_0}$ can further be chosen so that $(P_\bullet, \delta_\bullet)$ is a projective resolution of $\Bbbk$ in the category of bounded below graded A-modules provided with a Yetter-Drinfeld module structure over $\Bbbk\mathbb{S}_3$.

We further provide an explicit family of morphisms $\{f_\bullet\}_{\bullet \in \mathbb{N}_0}$ satisfying the above conditions, since we will need it for the calculations. Indeed, a lengthy but straightforward computation shows that

$$\begin{aligned}
f_0(\epsilon^!|1) &= 2\alpha_3|bac + 2\beta_3|abc - 2\gamma_3|aba - \alpha_2\beta|abc + \alpha_2\gamma|aba - \alpha\beta_2|bac, \\
f_n(\alpha_n|1) &= (2\alpha_{n+3} - \alpha_{n+1}\beta_2)|bac + \chi_n\beta_{n+3}|abc - \chi_n\gamma_{n+3}|aba, \\
f_n(\beta_n|1) &= (2\beta_{n+3} - \chi_n\alpha_{n+2}\beta - \chi_{n+1}\alpha_{n+1}\beta_2)|abc + \chi_n\alpha_{n+3}|bac - \chi_n\gamma_{n+3}|aba, \\
f_n(\gamma_n|1) &= (-2\gamma_{n+3} + \chi_n\alpha_{n+2}\gamma + \chi_{n+1}\alpha_{n+1}\beta_2)|aba + \chi_n\alpha_{n+3}|bac + \chi_n\beta_{n+3}|abc, \quad (4.1.2) \\
f_n(\alpha_{n-1}\beta|1) &= (n-1)\chi_{n+1}\beta_{n+3}|abc, \\
f_n(\alpha_{n-1}\gamma|1) &= -(n-1)\chi_{n+1}\gamma_{n+3}|aba, \\
f_n(\alpha_{n-2}\beta_2|1) &= ((n-2) + \chi_{n+1})\alpha_{n+3}|bac + (n-2)\chi_n\beta_{n+3}|abc - (n-2)\chi_n\gamma_{n+3}|aba,
\end{aligned}$$

for $n \in \mathbb{N}$, satisfy the conditions of Proposition 4.1.13. Note that f_0 already appeared in [11].

Given $n \in \mathbb{N}_0$, we now define the morphisms of A-bimodules $f^b_n : K^b_n \to K^b_{n+3}$ by

$$\begin{aligned}
f^b_0(1|\epsilon^!|1) = {} & 2|\alpha_3|bac + 2|\beta_3|abc - 2|\gamma_3|aba - 1|\alpha_2\beta|abc + 1|\alpha_2\gamma|aba - 1|\alpha\beta_2|bac \\
& + a|\alpha_2\beta|(ba+ac) - c|\alpha_2\beta|ab - a|\alpha_2\gamma|bc - b|\alpha_2\gamma|ac + b|\alpha\beta_2|(ab+bc) - c|\alpha\beta_2|ba \\
& - 2b|\alpha_3|(ab+bc) + 2c|\alpha_3|ba - 2a|\beta_3|(ba+ac) + 2c|\beta_3|ab + 2a|\gamma_3|bc + 2b|\gamma_3|ac \\
& - bc|\alpha_2\beta|a - ba|\alpha_2\beta|c + (ab+bc)|\alpha_2\gamma|b + (ba+ac)|\alpha_2\gamma|a - ab|\alpha\beta_2|c - ac|\alpha\beta_2|b \\
& + 2ab|\alpha_3|c + 2ac|\alpha_3|b + 2bc|\beta_3|a + 2ba|\beta_3|c - 2(ab+bc)|\gamma_3|b - 2(ba+ac)|\gamma_3|a \\
& + 2bac|\alpha_3|1 + 2abc|\beta_3|1 - 2aba|\gamma_3|1 - abc|\alpha_2\beta|1 + aba|\alpha_2\gamma|1 - bac|\alpha\beta_2|1, \\
f^b_n(1|\alpha_n|1) = {} & 2|\alpha_{n+3}|bac + \chi_n|\beta_{n+3}|abc - \chi_n|\gamma_{n+3}|aba - 1|\alpha_{n+1}\beta_2|bac - \chi_n c|\alpha_{n+2}\beta|ab \\
& - \chi_n b|\alpha_{n+2}\gamma|ac + \chi_{n+1} b|\alpha_{n+1}\beta_2|ac + \chi_{n+1} c|\alpha_{n+1}\beta_2|ab - \chi_n b|\alpha_{n+3}|(ab+bc)
\end{aligned}$$

$$
\begin{aligned}
&+ \chi_n c|\alpha_{n+3}|ba + (-1)^n 2c|\beta_{n+3}|ab - \chi_n a|\beta_{n+3}|ac - \chi_n a|\gamma_{n+3}|ab + (-1)^n 2b|\gamma_{n+3}|ac \\
&- \chi_n ba|\alpha_{n+2}\beta|c + \chi_n (ab+bc)|\alpha_{n+2}\gamma|b + \chi_{n+1}(ab+bc)|\alpha_{n+1}\beta_2|b \\
&- \chi_{n+1} ba|\alpha_{n+1}\beta_2|c + \chi_n ac|\alpha_{n+3}|b + \chi_n ab|\alpha_{n+3}|c + 2ba|\beta_{n+3}|c + \chi_n(ab+bc)|\beta_{n+3}|a \\
&- \chi_n ba|\gamma_{n+3}|a - 2(ab+bc)|\gamma_{n+3}|b + (-1)^n 2bac|\alpha_{n+3}|1 + \chi_n abc|\beta_{n+3}|1 \\
&- \chi_n aba|\gamma_{n+3}|1 + (-1)^{n+1} bac|\alpha_{n+1}\beta_2|1, \\
f_n^b(1|\beta_n|1) = {}& 2|\beta_{n+3}|abc - \chi_n|\gamma_{n+3}|aba + \chi_n|\alpha_{n+3}|bac - \chi_n|\alpha_{n+2}\beta|abc - \chi_{n+1}|\alpha_{n+1}\beta_2|abc \\
&- \chi_n a|\alpha_{n+2}\gamma|bc + (-1)^{n+1} c|\alpha_{n+1}\beta_2|ba + \chi_{n+1} a|\alpha_{n+1}\beta_2|bc + \chi_n c|\beta_{n+3}|ab \\
&- \chi_n a|\beta_{n+3}|(ba+ac) + (-1)^n 2a|\gamma_{n+3}|bc - \chi_n b|\gamma_{n+3}|ba - \chi_n b|\alpha_{n+3}|bc \\
&+ (-1)^n 2c|\alpha_{n+3}|ba + \chi_n(ba+ac)|\alpha_{n+2}\gamma|a - ab|\alpha_{n+1}\beta_2|c + \chi_{n+1}(ba+ac)|\alpha_{n+1}\beta_2|a \\
&+ \chi_n ba|\beta_{n+3}|c + \chi_n bc|\beta_{n+3}|a - 2(ba+ac)|\gamma_{n+3}|a - \chi_n ab|\gamma_{n+3}|b \\
&+ \chi_n(ba+ac)|\alpha_{n+3}|b + 2ab|\alpha_{n+3}|c + (-1)^n 2abc|\beta_{n+3}|1 - \chi_n aba|\gamma_{n+3}|1 \\
&+ \chi_n bac|\alpha_{n+3}|1 - \chi_n abc|\alpha_{n+2}\beta|1 + \chi_{n+1} abc|\alpha_{n+1}\beta_2|1, \\
f_n^b(1|\gamma_n|1) = {}& -2|\gamma_{n+3}|aba + \chi_n|\alpha_{n+3}|bac + \chi_n|\beta_{n+3}|abc + \chi_n|\alpha_{n+2}\gamma|aba + \chi_{n+1}|\alpha_{n+1}\beta_2|aba \\
&+ \chi_n a|\alpha_{n+2}\beta|(ba+ac) - \chi_{n+1} a|\alpha_{n+1}\beta_2|(ba+ac) + (-1)^n b|\alpha_{n+1}\beta_2|(ab+bc) \\
&+ \chi_n a|\gamma_{n+3}|bc + \chi_n b|\gamma_{n+3}|ac + (-1)^{n+1} 2b|\alpha_{n+3}|(ab+bc) + \chi_n c|\alpha_{n+3}|(ba+ac) \\
&+ \chi_n c|\beta_{n+3}|(ab+bc) + (-1)^{n+1} 2a|\beta_{n+3}|(ba+ac) - \chi_n bc|\alpha_{n+2}\beta|a \\
&- \chi_{n+1} bc|\alpha_{n+1}\beta_2|a - ac|\alpha_{n+1}\beta_2|b - \chi_n(ba+ac)|\gamma_{n+3}|a - \chi_n(ab+bc)|\gamma_{n+3}|b \\
&+ 2ac|\alpha_{n+3}|b - \chi_n bc|\alpha_{n+3}|c - \chi_n ac|\beta_{n+3}|c + 2bc|\beta_{n+3}|a + (-1)^{n+1} 2aba|\gamma_{n+3}|1 \\
&+ \chi_n bac|\alpha_{n+3}|1 + \chi_n abc|\beta_{n+3}|1 + \chi_n aba|\alpha_{n+2}\gamma|1 - \chi_{n+1} aba|\alpha_{n+1}\beta_2|1, \\
f_n^b(1|\alpha_{n-1}\beta|1) = {}& \chi_{n+1}[(n-1)|\beta_{n+3}|abc + a|\alpha_{n+3}|ab - (n-2)c|\alpha_{n+3}|ba + c|\alpha_{n+3}|ac - a|\beta_{n+3}|ab \\
&+ c|\beta_{n+3}|(ba+ac) - a|\gamma_{n+3}|ab - c|\gamma_{n+3}|(ba+ac) - (n-1)a|\gamma_{n+3}|bc - ba|\alpha_{n+3}|a \\
&+ (n-1)ab|\alpha_{n+3}|c + bc|\alpha_{n+3}|c + ba|\beta_{n+3}|a + bc|\beta_{n+3}|c - (n-2)ba|\gamma_{n+3}|a \\
&- (n-1)ac|\gamma_{n+3}|a - bc|\gamma_{n+3}|c - (n-1)abc|\beta_{n+3}|1], \\
f_n^b(1|\alpha_{n-1}\gamma|1) = {}& \chi_{n+1}[-(n-1)|\gamma_{n+3}|aba + b|\beta_{n+3}|bc + (n-1)a|\beta_{n+3}|ba + (n-2)a|\beta_{n+3}|ac \\
&- b|\gamma_{n+3}|bc - a|\gamma_{n+3}|ac + a|\alpha_{n+3}|ac + (n-1)b|\alpha_{n+3}|ab + (n-2)b|\alpha_{n+3}|bc \\
&+ (ba+ac)|\beta_{n+3}|b + (n-2)bc|\beta_{n+3}|a - ab|\beta_{n+3}|a - (ba+ac)|\gamma_{n+3}|b \\
&- (ab+bc)|\gamma_{n+3}|a + (n-2)ac|\alpha_{n+3}|b - ba|\alpha_{n+3}|b + (ab+bc)|\alpha_{n+3}|a \\
&+ (n-1)aba|\gamma_{n+3}|1], \\
f_n^b(1|\alpha_{n-2}\beta_2|1) = {}& \chi_{n+1}[(n-1)|\alpha_{n+3}|bac - c|\gamma_{n+3}|(ab+bc) - (n-1)b|\gamma_{n+3}|ac - b|\gamma_{n+3}|ba \\
&+ c|\alpha_{n+3}|(ab+bc) - b|\alpha_{n+3}|ba + b|\beta_{n+3}|ba - (n-2)c|\beta_{n+3}|ab + c|\beta_{n+3}|bc \\
&- ac|\gamma_{n+3}|c - (n-1)bc|\gamma_{n+3}|b - (n-2)ab|\gamma_{n+3}|b + ac|\alpha_{n+3}|c + ab|\alpha_{n+3}|b \\
&+ ac|\beta_{n+3}|c + (n-1)ba|\beta_{n+3}|c - ab|\beta_{n+3}|b - (n-1)bac|\alpha_{n+3}|1] \\
&+ \chi_n(n-2)[1|\alpha_{n+3}|bac + 1|\beta_{n+3}|abc - 1|\gamma_{n+3}|aba - b|\alpha_{n+3}|(ab+bc) \\
&+ c|\alpha_{n+3}|ba + c|\beta_{n+3}|ab - a|\beta_{n+3}|(ba+ac) + a|\gamma_{n+3}|bc + b|\gamma_{n+3}|ac \\
&+ ac|\alpha_{n+3}|b + ab|\alpha_{n+3}|c + ba|\beta_{n+3}|c + bc|\beta_{n+3}|a - (ba+ac)|\gamma_{n+3}|a \\
&- (ab+bc)|\gamma_{n+3}|b + bac|\alpha_{n+3}|1 + abc|\beta_{n+3}|1 - aba|\gamma_{n+3}|1],
\end{aligned}
\tag{4.1.3}
$$

where $n \in \mathbb{N}$.

The proof of the following result is a tedious but straightforward computation, that we leave to the reader.

Lemma 4.1.14. *The A-bimodule morphisms $f_n^b : K_n^b \to K_{n+3}^b$ defined above are homogeneous morphisms of internal degree 6, such that $d_3^b f_0^b = 0$, $d_{n+4}^b f_{n+1}^b + f_n^b d_{n+1}^b = 0$ and $\mathrm{id}_{\Bbbk} \otimes_A f_n^b = f_n$ for $n \in \mathbb{N}_0$, where f_n are the specific morphisms given in (4.1.2). Furthermore, $\{f_\bullet^b\}_{\bullet\in\mathbb{N}_0}$ preserves $\mathbb{S}_3$-action and coaction.*

Using the previous lemma, we can now prove the main result of this section.

Proposition 4.1.15. *The minimal projective resolution $(P_\bullet^b, \delta_\bullet^b)$ of A in the category of bounded below graded A-bimodules is given as follows. For $n \in \mathbb{N}_0$, set*

$$
P_n^b = \bigoplus_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_i K_{n-4i}^b = \bigoplus_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_i A \otimes (A^!_{-(n-4i)})^* \otimes A,
$$

where ω_i is a symbol of internal degree $6i$ for all $i \in \mathbb{N}_0$, the A-bimodule structure of P_n^b is given by $x'(\omega_i x \otimes u \otimes y)y' = \omega_i x'x \otimes u \otimes yy'$ for all $x,x',y,y' \in A$ and $u \in (A^!_{-(n-4i)})^*$, and the differential $\delta_n^b : P_n^b \to P_{n-1}^b$ for $n \in \mathbb{N}$ is given by

$$\delta_n^b\Big(\sum_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_i \rho_{n-4i}\Big) = \sum_{i\in[\![0,\lfloor n/4\rfloor]\!]} \big(\omega_i d_{n-4i}^b(\rho_{n-4i}) + \omega_{i-1} f_{n-4i}^b(\rho_{n-4i})\big),$$

where $\rho_j \in K_j^b$ for $j \in \mathbb{N}_0$, $\omega_{-1} = 0$ and $f_j^b : K_j^b \to K_{j+3}^b$ are the morphisms in (4.1.3). This gives a minimal projective resolution of A by means of the augmentation $\epsilon^b : P_0^b = A \otimes (A_0^!)^* \otimes A \to A$, where $\epsilon^b(x|\epsilon^!|y) = xy$ for $x,y \in A$. Furthermore, if the characteristic of $\Bbbk$ is different from 2 and 3, then $(P_\bullet^b,\delta_\bullet^b)$ is a projective resolution of A in the category of bounded below graded A-bimodules provided with a Yetter-Drinfeld module structure over $\Bbbk\mathbb{S}_3$.

Proof. It is clear that $P_\bullet^b \to A \to 0$ is a complex of graded-free (left) A-modules by Lemma 4.1.14, $(\Bbbk \otimes_A P_\bullet^b, \mathrm{id}_\Bbbk \otimes_A \delta_\bullet^b) \cong (P_\bullet, \delta_\bullet)$ and $\mathrm{id}_\Bbbk \otimes_A \epsilon^b \cong \epsilon$. Proposition 4.1.13 tells us that $\Bbbk \otimes_A P_\bullet^b \to \Bbbk \otimes_A A \to 0$ is exact. Proposition 4.1.11 in turn shows that the complex $P_\bullet^b \to A \to 0$ is also exact. Moreover, the bimodule resolution $(P_\bullet^b,\delta_\bullet^b)$ is minimal since $\mathrm{id}_\Bbbk \otimes_A \delta_\bullet^b \otimes_A \mathrm{id}_\Bbbk = 0$. □

We follow the convention that $P_n^b = 0$, $K_n^b = 0$ for $n \in \mathbb{Z}\setminus\mathbb{N}_0$, and $\delta_n^b = 0$, $d_n^b = 0$ for $n \in \mathbb{Z}\setminus\mathbb{N}$ in the following sections.

4.2 Hochschild (co)homology and cyclic homology of $\mathrm{FK}(3)$

In this section, we will compute the linear structure of Hochschild homology and cohomology of A. These results were published in [12].

4.2.1 Hochschild and cyclic homology

Using the minimal projective bimodule resolution $(P_\bullet^b, \delta_\bullet^b)$ of A in Proposition 4.1.15, we will compute the linear structure of the Hochschild homology

$$\mathrm{HH}_\bullet(A) = \mathrm{Tor}_\bullet^{A^e}(A,A) = \mathrm{H}_\bullet(A \otimes_{A^e} P_\bullet^b).$$

For further information about Hochschild and cyclic homology, we refer the reader to [16].

4.2.1.1 Recursive description of the spaces

Let $\tilde{K}_n = A \otimes (A^!_{-n})^*$ for $n \in \mathbb{N}_0$ and $\tilde{K}_n = 0$ for $n \in \mathbb{Z}\setminus\mathbb{N}_0$. We have $A \otimes_{A^e} P_n^b \cong \tilde{P}_n$ as $\Bbbk$-vector spaces, where $\tilde{P}_n = \oplus_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_i \tilde{K}_{n-4i}$ for $n \in \mathbb{N}_0$ and $\tilde{P}_n = 0$ for $n \in \mathbb{Z}\setminus\mathbb{N}_0$. We will denote by $\partial_n : \tilde{P}_n \to \tilde{P}_{n-1}$ the differential $\mathrm{id}_A \otimes_{A^e} \delta_n^b$, $\tilde{\partial}_n : \tilde{K}_n \to \tilde{K}_{n-1}$ the differential $\mathrm{id}_A \otimes_{A^e} d_n^b$ for $n \in \mathbb{Z}$, and $\tilde{f}_n$ the map $\mathrm{id}_A \otimes_{A^e} f_n^b$ for $n \in \mathbb{N}_0$. Then the differential ∂_n for $n \in \mathbb{N}$ is given by

$$\partial_n\Big(\sum_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_i \rho_{n-4i}\Big) = \sum_{i\in[\![0,\lfloor n/4\rfloor]\!]} \big(\omega_i \tilde{\partial}_{n-4i}(\rho_{n-4i}) + \omega_{i-1} \tilde{f}_{n-4i}(\rho_{n-4i})\big),$$

where $\rho_j \in \tilde{K}_j$ for $j \in \mathbb{N}_0$. Note that $\partial_n = \tilde{\partial}_n = 0$ for $n \in \mathbb{Z}\setminus\mathbb{N}$.

The aim of this subsection is to compute the homology of the complex $(\tilde{P}_\bullet, \partial_\bullet)$. Let $\tilde{K}_{n,m} = A_m \otimes (A^!_{-n})^*$ for $(n,m) \in \mathbb{N}_0 \times [\![0,4]\!]$ and $\tilde{K}_{n,m} = 0$ for $(n,m) \in \mathbb{Z}^2 \setminus (\mathbb{N}_0 \times [\![0,4]\!])$. Let $\tilde{P}_{n,m} = \oplus_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_i \tilde{K}_{n-4i,m-2i}$ for $m,n \in \mathbb{N}_0$ and $\tilde{P}_{n,m} = 0$ for $(n,m) \in \mathbb{Z}^2 \setminus \mathbb{N}_0^2$, where the symbol ω_i has homological degree $4i$ and internal degree $6i$ for $i \in \mathbb{N}_0$, and we usually omit ω_0 for simplicity. The spaces $\tilde{K}_{n,m}$ and $\tilde{P}_{n,m}$ are concentrated in homological degree n and internal degree $m+n$. We have $\tilde{P}_n = \oplus_{m\in\mathbb{N}_0} \tilde{P}_{n,m}$. Let $\partial_{n,m} = \partial_n|_{\tilde{P}_{n,m}} : \tilde{P}_{n,m} \to \tilde{P}_{n-1,m+1}$, and $\tilde{\partial}_{n,m} = \tilde{\partial}_n|_{\tilde{K}_{n,m}} : \tilde{K}_{n,m} \to \tilde{K}_{n-1,m+1}$. Let $D_{n,m} = \mathrm{Ker}(\partial_{n,m})$, $B_{n,m} = \mathrm{Im}(\partial_{n+1,m-1})$ for $m,n \in \mathbb{N}_0$, and $\tilde{D}_{n,m} = \mathrm{Ker}(\tilde{\partial}_{n,m})$, $\tilde{B}_{n,m} = \mathrm{Im}(\tilde{\partial}_{n+1,m-1})$ for $(n,m) \in \mathbb{N}_0 \times [\![0,4]\!]$. Notice that $D_{n,m} = B_{n,m} = 0$ for $(n,m) \in \mathbb{Z}^2 \setminus \mathbb{N}_0^2$, and $\tilde{D}_{n,m} = \tilde{B}_{n,m} = 0$ for $(n,m) \in \mathbb{Z}^2 \setminus (\mathbb{N}_0 \times [\![0,4]\!])$.

Proposition 4.2.1. *For integers $m \geqslant 3$ and $n \in \mathbb{N}_0$, we have*

$$B_{n,m} = \begin{cases} \omega_{\frac{m-3}{2}} B_{n-2m+6,3}, & \text{if } m \text{ is odd,} \\ \omega_{\frac{m}{2}-2} B_{n-2m+8,4}, & \text{if } m \text{ is even,} \end{cases} \tag{4.2.1}$$

and

$$D_{n,m} = \begin{cases} \omega_{\frac{m-3}{2}} D_{n-2m+6,3}, & \text{if } m \text{ is odd,} \\ \omega_{\frac{m}{2}-2} D_{n-2m+8,4}, & \text{if } m \text{ is even,} \end{cases} \tag{4.2.2}$$

where we follow the convention that $\omega_i \omega_j = \omega_{i+j}$ for $i,j \in \mathbb{N}_0$ and $\omega_i = 0$ for $i \in \mathbb{Z} \setminus \mathbb{N}_0$.

Proof. Consider $\tilde{P}_{n,m} = \oplus_{i \in [\![0,\lfloor n/4 \rfloor]\!]} \omega_i \tilde{K}_{n-4i,m-2i}$ for $m,n \in \mathbb{N}_0$. For the index $m-2i$ of $\tilde{K}_{n-4i,m-2i}$, we have $m - 2i \in [\![0,4]\!]$. If m is odd, then $m-2i = 1$ or 3, *i.e.* $i = (m-1)/2$ or $(m-3)/2$. Since $n - 4i \in \mathbb{N}_0$, we have

$$\tilde{P}_{n,m} = \begin{cases} \omega_{\frac{m-3}{2}} \tilde{K}_{n-2m+6,3} \oplus \omega_{\frac{m-1}{2}} \tilde{K}_{n-2m+2,1}, & \text{if } n \geqslant 2m-2, \\ \omega_{\frac{m-3}{2}} \tilde{K}_{n-2m+6,3}, & \text{if } 2m-6 \leqslant n < 2m-2, \\ 0, & \text{if } 0 \leqslant n < 2m-6. \end{cases}$$

If m is even, then $m - 2i = 0,2$ or 4, *i.e.* $i = m/2$, $m/2-1$ or $m/2-2$. Then

$$\tilde{P}_{n,m} = \begin{cases} \omega_{\frac{m}{2}-2} \tilde{K}_{n-2m+8,4} \oplus \omega_{\frac{m}{2}-1} \tilde{K}_{n-2m+4,2} \oplus \omega_{\frac{m}{2}} \tilde{K}_{n-2m,0}, & \text{if } n \geqslant 2m, \\ \omega_{\frac{m}{2}-2} \tilde{K}_{n-2m+8,4} \oplus \omega_{\frac{m}{2}-1} \tilde{K}_{n-2m+4,2}, & \text{if } 2m-4 \leqslant n < 2m, \\ \omega_{\frac{m}{2}-2} \tilde{K}_{n-2m+8,4}, & \text{if } 2m-8 \leqslant n < 2m-4, \\ 0, & \text{if } 0 \leqslant n < 2m-8. \end{cases} \tag{4.2.3}$$

Hence,

$$\tilde{P}_{n,m} = \begin{cases} \omega_{\frac{m-3}{2}} \tilde{P}_{n-2m+6,3}, & \text{if } m \geqslant 3 \text{ is odd,} \\ \omega_{\frac{m}{2}-2} \tilde{P}_{n-2m+8,4}, & \text{if } m \geqslant 4 \text{ is even.} \end{cases} \tag{4.2.4}$$

Since the identities (4.2.1) and (4.2.2) for $m = 3$ are immediate, we suppose $m \geqslant 4$ from now on.

Assume that m is even. Then (4.2.4) tells us that the sequence

$$\tilde{P}_{n+1,m-1} \xrightarrow{\partial n+1,m-1} \tilde{P}_{n,m} \xrightarrow{\partial_{n,m}} \tilde{P}_{n-1,m+1}$$

of graded $\Bbbk$-vector spaces is of the form

$$\omega_{\frac{m}{2}-2} \tilde{P}_{n-2m+9,3} \xrightarrow{\partial_{n+1,m-1}} \omega_{\frac{m}{2}-2} \tilde{P}_{n-2m+8,4} \xrightarrow{\partial_{n,m}} \omega_{\frac{m}{2}-1} \tilde{P}_{n-2m+3,3}.$$

Since $\tilde{P}_{n-2m+7,5} = \omega_1 \tilde{P}_{n-2m+3,3}$ by (4.2.4), the above sequence is of the form

$$\omega_{\frac{m}{2}-2} \tilde{P}_{n-2m+9,3} \xrightarrow{\partial_{n+1,m-1}} \omega_{\frac{m}{2}-2} \tilde{P}_{n-2m+8,4} \xrightarrow{\partial_{n,m}} \omega_{\frac{m}{2}-2} \tilde{P}_{n-2m+7,5}.$$

Note further that $\partial_{n,m} = \omega_{\frac{m}{2}-2} \partial_{n-2m+8,4}$ and $\partial_{n+1,m-1} = \omega_{\frac{m}{2}-2} \partial_{n-2m+9,3}$, where the differential $\omega_j \partial_{n',m'} : \omega_j \tilde{P}_{n',m'} \to \omega_j \tilde{P}_{n'-1,m'+1}$ maps $\omega_j x$ to $\omega_j \partial_{n',m'}(x)$ for all $x \in \tilde{P}_{n',m'}$ and $j,m',n' \in \mathbb{N}_0$. Hence, $B_{n,m} = \omega_{\frac{m}{2}-2} B_{n-2m+8,4}$ and $D_{n,m} = \omega_{\frac{m}{2}-2} D_{n-2m+8,4}$.

Assume that m is odd (so $m \geqslant 5$). Then (4.2.4) tells us that the sequence

$$\tilde{P}_{n+1,m-1} \xrightarrow{\partial_{n+1,m-1}} \tilde{P}_{n,m} \xrightarrow{\partial_{n,m}} \tilde{P}_{n-1,m+1}$$

of graded $\Bbbk$-vector spaces is of the form

$$\omega_{\frac{m-5}{2}} \tilde{P}_{n-2m+11,4} \xrightarrow{\partial_{n+1,m-1}} \omega_{\frac{m-3}{2}} \tilde{P}_{n-2m+6,3} \xrightarrow{\partial_{n,m}} \omega_{\frac{m-3}{2}} \tilde{P}_{n-2m+5,4}.$$

Note that $\partial_{n,m} = \omega_{\frac{m-3}{2}}\partial_{n-2m+6,3}$ and $\partial_{n+1,m-1} = \omega_{\frac{m-5}{2}}\partial_{n-2m+11,4}$. Since

$$\tilde{P}_{n-2m+11,4} = \omega_0\tilde{K}_{n-2m+11,4} \oplus \omega_1\tilde{P}_{n-2m+7,2}$$

by (4.2.3),

$$\partial_{n-2m+11,4}(\omega_0 x + \omega_1 y) = \omega_0\tilde{\partial}_{n-2m+11,4}(x) + \omega_1\partial_{n-2m+7,2}(y)$$

for all $x \in \tilde{K}_{n-2m+11,4}$ and $y \in \tilde{P}_{n-2m+7,2}$, and $\tilde{\partial}_{n-2m+11,4}(\tilde{K}_{n-2m+11,4}) = 0$, it is sufficient to consider the following sequence

$$\omega_{\frac{m-3}{2}}\tilde{P}_{n-2m+7,2} \xrightarrow{\omega_{\frac{m-3}{2}}\partial_{n-2m+7,2}} \omega_{\frac{m-3}{2}}\tilde{P}_{n-2m+6,3} \xrightarrow{\omega_{\frac{m-3}{2}}\partial_{n-2m+6,3}} \omega_{\frac{m-3}{2}}\tilde{P}_{n-2m+5,4}.$$

Hence, $B_{n,m} = \omega_{\frac{m-3}{2}}B_{n-2m+6,3}$ and $D_{n,m} = \omega_{\frac{m-3}{2}}D_{n-2m+6,3}$, as was to be shown. □

Proposition 4.2.2. *For $n \in \mathbb{N}_0$, we have $D_{n,4} = \tilde{K}_{n,4} \oplus \omega_1 D_{n-4,2}$.*

Proof. This follows directly from the facts that $\tilde{P}_{n,4} = \tilde{K}_{n,4} \oplus \omega_1\tilde{P}_{n-4,2}$, $\tilde{P}_{n-1,5} = \omega_1\tilde{P}_{n-5,3}$ and $\partial_{n,4}(\tilde{K}_{n,4}) = 0$. □

In order to compute $B_{n,m}$ and $D_{n,m}$, it is sufficient to compute the case $m \in [\![0,4]\!]$ according to Proposition 4.2.1. First, we will compute the boundaries, and then we will compute the cycles. Since this will require handling elements of $\tilde{K}_{n,m}$ and $\tilde{P}_{n,m}$ for $n \in \mathbb{N}_0$ and $m \in [\![0,4]\!]$, we will use the basis $\{x \otimes y | x \in \mathscr{B}_m, y \in \mathscr{B}^{!*}_n\}$ of $\tilde{K}_{n,m}$ and the basis $\{\omega_i x \otimes y | i \in [\![0, \lfloor n/4 \rfloor]\!], x \in \mathscr{B}_{m-2i}, y \in \mathscr{B}^{!*}_{n-4i}\}$ of $\tilde{P}_{n,m}$, both of which will be called **usual** bases, constructed from the usual bases of the homogeneous components of A and $(A^!)^{\#}$, introduced in Subsection 4.1.1.

4.2.1.2 Explicit description of the differentials

Recall the isomorphism $A \otimes_{A^e} (A \otimes (A^!_{-n})^* \otimes A) \to A \otimes (A^!_{-n})^*$ given by $x \otimes_{A^e} (y|u|z) \mapsto zxy|u$, and its inverse $A \otimes (A^!_{-n})^* \to A \otimes_{A^e} (A \otimes (A^!_{-n})^* \otimes A)$ given by $x|u \mapsto x \otimes_{A^e} (1|u|1)$ for all $x,y,z \in A$, $u \in (A^!_{-n})^*$ and $n \in \mathbb{N}_0$. We will use them together with Proposition 4.1.15 to explicitly describe $\tilde{\partial}_n$ and $\tilde{f}_n$, which were defined at the beginning of Subsubsection 4.2.1.1.

Let $x \in A$. It is then straightforward to see that the differential $\tilde{\partial}_1 : A \otimes (A^!_{-1})^* \to A \otimes (A^!_0)^*$ is given by $\tilde{\partial}_1(x|\alpha) = (ax - xa)|\epsilon^!$, $\tilde{\partial}_1(x|\beta) = (bx - xb)|\epsilon^!$ and $\tilde{\partial}_1(x|\gamma) = (cx - xc)|\epsilon^!$. Analogously, for $n \geqslant 2$ and n even, $\tilde{\partial}_n : A \otimes (A^!_{-n})^* \to A \otimes (A^!_{-(n-1)})^*$ is given by

$$\begin{aligned}
x|\alpha_n &\mapsto (xa + ax)|\alpha_{n-1},\ x|\beta_n \mapsto (xb + bx)|\beta_{n-1},\ x|\gamma_n \mapsto (xc + cx)|\gamma_{n-1},\\
x|\alpha_{n-1}\beta &\mapsto (xa + cx)|(\beta_{n-1} + \alpha_{n-2}\beta) + (xb + ax)|(\gamma_{n-1} + \alpha_{n-2}\gamma)\\
&\quad + (xc + bx)|(\alpha_{n-1} + \alpha_{n-3}\beta_2),\\
x|\alpha_{n-1}\gamma &\mapsto (xa + bx)|(\gamma_{n-1} + \alpha_{n-2}\gamma) + (xb + cx)|(\alpha_{n-1} + \alpha_{n-3}\beta_2)\\
&\quad + (xc + ax)|(\beta_{n-1} + \alpha_{n-2}\beta),\\
x|\alpha_{n-2}\beta_2 &\mapsto (xa + ax)|\alpha_{n-3}\beta_2 + (xb + bx)|\alpha_{n-2}\beta + (xc + cx)|\alpha_{n-2}\gamma,
\end{aligned}$$

whereas, for $n \geqslant 3$ and n odd, $\tilde{\partial}_n : A \otimes (A^!_{-n})^* \to A \otimes (A^!_{-(n-1)})^*$ is given by

$$\begin{aligned}
x|\alpha_n &\mapsto (ax - xa)|\alpha_{n-1},\quad x|\beta_n \mapsto (bx - xb)|\beta_{n-1},\quad x|\gamma_n \mapsto (cx - xc)|\gamma_{n-1},\\
x|\alpha_{n-1}\beta &\mapsto (cx - xa)|\alpha_{n-2}\beta + (ax - xc)|\alpha_{n-2}\gamma + (bx - xb)|(\alpha_{n-1} + \gamma_{n-1} + \alpha_{n-3}\beta_2),\\
x|\alpha_{n-1}\gamma &\mapsto (ax - xb)|\alpha_{n-2}\beta + (bx - xa)|\alpha_{n-2}\gamma + (cx - xc)|(\alpha_{n-1} + \beta_{n-1} + \alpha_{n-3}\beta_2),\\
x|\alpha_{n-2}\beta_2 &\mapsto (ax - xa)|(\beta_{n-1} + \gamma_{n-1} + \alpha_{n-3}\beta_2) + (bx - xc)|\alpha_{n-2}\beta + (cx - xb)|\alpha_{n-2}\gamma.
\end{aligned}$$

For the reader's convenience, we list the images of the differentials $\tilde{\partial}_n$ evaluated at elements of the usual $\mathbb{k}$-basis of the respective domain. In the following tables, $\tilde{\partial}_{n,m}(x|y)$ is the entry appearing in the column indexed by y and the row indexed by x, where m is the internal degree of x and n is the internal degree of y. The differential $\tilde{\partial}_1$ is given by

$x \backslash y$	α	β	γ
1	0	0	0
a	0	$(ba-ab)\|\epsilon^!$	$(-ab-bc-ac)\|\epsilon^!$
b	$(ab-ba)\|\epsilon^!$	0	$(-ba-ac-bc)\|\epsilon^!$
c	$(ab+bc+ac)\|\epsilon^!$	$(ba+ac+bc)\|\epsilon^!$	0
ab	$-aba\|\epsilon^!$	$aba\|\epsilon^!$	$(bac-abc)\|\epsilon^!$
bc	$(aba+abc)\|\epsilon^!$	$bac\|\epsilon^!$	$-bac\|\epsilon^!$
ba	$aba\|\epsilon^!$	$-aba\|\epsilon^!$	$(abc-bac)\|\epsilon^!$
ac	$abc\|\epsilon^!$	$(aba+bac)\|\epsilon^!$	$-abc\|\epsilon^!$
aba	0	0	$-2abac\|\epsilon^!$
abc	0	$2abac\|\epsilon^!$	0
bac	$2abac\|\epsilon^!$	0	0
$abac$	0	0	0

Table 4.2.1: Images of $\tilde{\partial}_1$.

For $n \geqslant 2$ and n even, $\tilde{\partial}_n$ is given by

$x \backslash y$	α_n	β_n	γ_n
1	$2a\|\alpha_{n-1}$	$2b\|\beta_{n-1}$	$2c\|\gamma_{n-1}$
a	0	$(ab+ba)\|\beta_{n-1}$	$(ac-ab-bc)\|\gamma_{n-1}$
b	$(ab+ba)\|\alpha_{n-1}$	0	$(bc-ba-ac)\|\gamma_{n-1}$
c	$(ac-ab-bc)\|\alpha_{n-1}$	$(bc-ba-ac)\|\beta_{n-1}$	0
ab	$aba\|\alpha_{n-1}$	$aba\|\beta_{n-1}$	$(abc+bac)\|\gamma_{n-1}$
bc	$(abc-aba)\|\alpha_{n-1}$	$-bac\|\beta_{n-1}$	$-bac\|\gamma_{n-1}$
ba	$aba\|\alpha_{n-1}$	$aba\|\beta_{n-1}$	$(abc+bac)\|\gamma_{n-1}$
ac	$-abc\|\alpha_{n-1}$	$(bac-aba)\|\beta_{n-1}$	$-abc\|\gamma_{n-1}$
aba	0	0	0
abc	0	0	0
bac	0	0	0
$abac$	0	0	0

Table 4.2.2: Images of $\tilde{\partial}_n$ for $n \geqslant 2$ and n even.

together with

$x \backslash y$	$\alpha_{n-1}\beta$
1	$(a+c)\|(\beta_{n-1}+\alpha_{n-2}\beta)+(b+a)\|(\gamma_{n-1}+\alpha_{n-2}\gamma)+(c+b)\|(\alpha_{n-1}+\alpha_{n-3}\beta_2)$
a	$-(ab+bc)\|(\beta_{n-1}+\alpha_{n-2}\beta)+ab\|(\gamma_{n-1}+\alpha_{n-2}\gamma)+(ba+ac)\|(\alpha_{n-1}+\alpha_{n-3}\beta_2)$
b	$-ac\|(\beta_{n-1}+\alpha_{n-2}\beta)+ab\|(\gamma_{n-1}+\alpha_{n-2}\gamma)+bc\|(\alpha_{n-1}+\alpha_{n-3}\beta_2)$
c	$-(ab+bc)\|(\beta_{n-1}+\alpha_{n-2}\beta)-ba\|(\gamma_{n-1}+\alpha_{n-2}\gamma)+bc\|(\alpha_{n-1}+\alpha_{n-3}\beta_2)$
ab	$(aba+bac)\|(\beta_{n-1}+\alpha_{n-2}\beta)+(aba+abc)\|(\alpha_{n-1}+\alpha_{n-3}\beta_2)$
bc	$(-aba-bac)\|(\beta_{n-1}+\alpha_{n-2}\beta)+(abc-bac)\|(\gamma_{n-1}+\alpha_{n-2}\gamma)$
ba	$abc\|(\beta_{n-1}+\alpha_{n-2}\beta)+2aba\|(\gamma_{n-1}+\alpha_{n-2}\gamma)+bac\|(\alpha_{n-1}+\alpha_{n-3}\beta_2)$
ac	$-2abc\|(\beta_{n-1}+\alpha_{n-2}\beta)-aba\|(\gamma_{n-1}+\alpha_{n-2}\gamma)+bac\|(\alpha_{n-1}+\alpha_{n-3}\beta_2)$
aba	$abac\|(-\beta_{n-1}-\alpha_{n-2}\beta+\alpha_{n-1}+\alpha_{n-3}\beta_2)$
abc	$abac\|(-\gamma_{n-1}-\alpha_{n-2}\gamma+\alpha_{n-1}+\alpha_{n-3}\beta_2)$
bac	$abac\|(-\beta_{n-1}-\alpha_{n-2}\beta+\gamma_{n-1}+\alpha_{n-2}\gamma)$
$abac$	0

Table 4.2.3: Images of $\tilde{\partial}_n$ for $n \geqslant 2$ and n even.

and

$x \backslash y$	$\alpha_{n-1}\gamma$
1	$(a+b)\|(\gamma_{n-1}+\alpha_{n-2}\gamma)+(b+c)\|(\alpha_{n-1}+\alpha_{n-3}\beta_2)+(c+a)\|(\beta_{n-1}+\alpha_{n-2}\beta)$
a	$ba\|(\gamma_{n-1}+\alpha_{n-2}\gamma)-bc\|(\alpha_{n-1}+\alpha_{n-3}\beta_2)+ac\|(\beta_{n-1}+\alpha_{n-2}\beta)$
b	$ba\|(\gamma_{n-1}+\alpha_{n-2}\gamma)-(ba+ac))\|(\alpha_{n-1}+\alpha_{n-3}\beta_2)+(ab+bc)\|(\beta_{n-1}+\alpha_{n-2}\beta)$
c	$-ab\|(\gamma_{n-1}+\alpha_{n-2}\gamma)-(ba+ac)\|(\alpha_{n-1}+\alpha_{n-3}\beta_2)+ac\|(\beta_{n-1}+\alpha_{n-2}\beta)$
ab	$2aba\|(\gamma_{n-1}+\alpha_{n-2}\gamma)+bac\|(\alpha_{n-1}+\alpha_{n-3}\beta_2)+abc\|(\beta_{n-1}+\alpha_{n-2}\beta)$
bc	$-aba\|(\gamma_{n-1}+\alpha_{n-2}\gamma)-2bac\|(\alpha_{n-1}+\alpha_{n-3}\beta_2)+abc\|(\beta_{n-1}+\alpha_{n-2}\beta)$
ba	$(aba+abc)\|(\alpha_{n-1}+\alpha_{n-3}\beta_2)+(aba+bac)\|(\beta_{n-1}+\alpha_{n-2}\beta)$
ac	$(bac-abc)\|(\gamma_{n-1}+\alpha_{n-2}\gamma)-(aba+abc)\|(\alpha_{n-1}+\alpha_{n-3}\beta_2)$
aba	$abac\|(-\alpha_{n-1}-\alpha_{n-3}\beta_2+\beta_{n-1}+\alpha_{n-2}\beta)$
abc	$abac\|(\gamma_{n-1}+\alpha_{n-2}\gamma-\alpha_{n-1}-\alpha_{n-3}\beta_2)$
bac	$abac\|(-\gamma_{n-1}-\alpha_{n-2}\gamma+\beta_{n-1}+\alpha_{n-2}\beta)$
$abac$	0

Table 4.2.4: Images of $\tilde{\partial}_n$ for $n \geqslant 2$ and n even.

as well as

$x \backslash y$	$\alpha_{n-2}\beta_2$
1	$2a\|\alpha_{n-3}\beta_2+2b\|\alpha_{n-2}\beta+2c\|\alpha_{n-2}\gamma$
a	$(ab+ba)\|\alpha_{n-2}\beta+(ac-ab-bc)\|\alpha_{n-2}\gamma$
b	$(ab+ba)\|\alpha_{n-3}\beta_2+(bc-ba-ac)\|\alpha_{n-2}\gamma$
c	$(ac-ab-bc)\|\alpha_{n-3}\beta_2+(bc-ba-ac)\|\alpha_{n-2}\beta$
ab	$aba\|\alpha_{n-3}\beta_2+aba\|\alpha_{n-2}\beta+(abc+bac)\|\alpha_{n-2}\gamma$
bc	$(abc-aba)\|\alpha_{n-3}\beta_2-bac\|\alpha_{n-2}\beta-bac\|\alpha_{n-2}\gamma$
ba	$aba\|\alpha_{n-3}\beta_2+aba\|\alpha_{n-2}\beta+(abc+bac)\|\alpha_{n-2}\gamma$
ac	$-abc\|\alpha_{n-3}\beta_2+(bac-aba)\|\alpha_{n-2}\beta-abc\|\alpha_{n-2}\gamma$
aba	0
abc	0
bac	0
$abac$	0

Table 4.2.5: Images of $\tilde{\partial}_n$ for $n \geqslant 4$ and n even.

For $n \geqslant 3$ and n odd, $\tilde{\partial}_n$ is given by

$x \backslash y$	α_n	β_n	γ_n
1	0	0	0
a	0	$(ba-ab)\|\beta_{n-1}$	$(-ab-bc-ac)\|\gamma_{n-1}$
b	$(ab-ba)\|\alpha_{n-1}$	0	$(-ba-ac-bc)\|\gamma_{n-1}$
c	$(ab+bc+ac)\|\alpha_{n-1}$	$(ba+ac+bc)\|\beta_{n-1}$	0
ab	$-aba\|\alpha_{n-1}$	$aba\|\beta_{n-1}$	$(bac-abc)\|\gamma_{n-1}$
bc	$(aba+abc)\|\alpha_{n-1}$	$bac\|\beta_{n-1}$	$-bac\|\gamma_{n-1}$
ba	$aba\|\alpha_{n-1}$	$-aba\|\beta_{n-1}$	$(abc-bac)\|\gamma_{n-1}$
ac	$abc\|\alpha_{n-1}$	$(aba+bac)\|\beta_{n-1}$	$-abc\|\gamma_{n-1}$
aba	0	0	$-2abac\|\gamma_{n-1}$
abc	0	$2abac\|\beta_{n-1}$	0
bac	$2abac\|\alpha_{n-1}$	0	0
$abac$	0	0	0

Table 4.2.6: Images of $\tilde{\partial}_n$ for $n \geqslant 3$ and n odd.

together with

$x \diagdown y$	$\alpha_{n-1}\beta$
1	$(c-a)\|\alpha_{n-2}\beta + (a-c)\|\alpha_{n-2}\gamma$
a	$-(ab+bc)\|\alpha_{n-2}\beta - ac\|\alpha_{n-2}\gamma + (ba-ab)\|(\alpha_{n-1}+\gamma_{n-1}+\alpha_{n-3}\beta_2)$
b	$(-2ba-ac)\|\alpha_{n-2}\beta + (ab-bc)\|\alpha_{n-2}\gamma$
c	$(ab+bc)\|\alpha_{n-2}\beta + ac\|\alpha_{n-2}\gamma + (ba+ac+bc)\|(\alpha_{n-1}+\gamma_{n-1}+\alpha_{n-3}\beta_2)$
ab	$(bac-aba)\|\alpha_{n-2}\beta - abc\|\alpha_{n-2}\gamma + aba\|(\alpha_{n-1}+\gamma_{n-1}+\alpha_{n-3}\beta_2)$
bc	$(aba-bac)\|\alpha_{n-2}\beta + abc\|\alpha_{n-2}\gamma + bac\|(\alpha_{n-1}+\gamma_{n-1}+\alpha_{n-3}\beta_2)$
ba	$abc\|\alpha_{n-2}\beta + (aba-bac)\|\alpha_{n-2}\gamma - aba\|(\alpha_{n-1}+\gamma_{n-1}+\alpha_{n-3}\beta_2)$
ac	$(aba+bac)\|(\alpha_{n-1}+\gamma_{n-1}+\alpha_{n-3}\beta_2)$
aba	$-abac\|(\alpha_{n-2}\beta+\alpha_{n-2}\gamma)$
abc	$2abac\|(\alpha_{n-1}+\gamma_{n-1}+\alpha_{n-3}\beta_2)$
bac	$abac\|(\alpha_{n-2}\beta+\alpha_{n-2}\gamma)$
$abac$	0

Table 4.2.7: Images of $\tilde{\partial}_n$ for $n \geqslant 3$ and n odd.

and

$x \diagdown y$	$\alpha_{n-1}\gamma$
1	$(a-b)\|\alpha_{n-2}\beta + (b-a)\|\alpha_{n-2}\gamma$
a	$-ab\|\alpha_{n-2}\beta + ba\|\alpha_{n-2}\gamma - (ab+bc+ac)\|(\alpha_{n-1}+\beta_{n-1}+\alpha_{n-3}\beta_2)$
b	$ab\|\alpha_{n-2}\beta - ba\|\alpha_{n-2}\gamma - (ba+ac+bc)\|(\alpha_{n-1}+\beta_{n-1}+\alpha_{n-3}\beta_2)$
c	$(2ac+ba)\|\alpha_{n-2}\beta + (ab+2bc)\|\alpha_{n-2}\gamma$
ab	$(bac-abc)\|(\alpha_{n-1}+\beta_{n-1}+\alpha_{n-3}\beta_2)$
bc	$(abc+bac)\|\alpha_{n-2}\beta + aba\|\alpha_{n-2}\gamma - bac\|(\alpha_{n-1}+\beta_{n-1}+\alpha_{n-3}\beta_2)$
ba	$(abc-bac)\|(\alpha_{n-1}+\beta_{n-1}+\alpha_{n-3}\beta_2)$
ac	$aba\|\alpha_{n-2}\beta + (abc+bac)\|\alpha_{n-2}\gamma - abc\|(\alpha_{n-1}+\beta_{n-1}+\alpha_{n-3}\beta_2)$
aba	$-2abac\|(\alpha_{n-1}+\beta_{n-1}+\alpha_{n-3}\beta_2)$
abc	$abac\|(\alpha_{n-2}\beta+\alpha_{n-2}\gamma)$
bac	$abac\|(\alpha_{n-2}\beta+\alpha_{n-2}\gamma)$
$abac$	0

Table 4.2.8: Images of $\tilde{\partial}_n$ for $n \geqslant 3$ and n odd.

as well as

$x \diagdown y$	$\alpha_{n-2}\beta_2$
1	$(b-c)\|\alpha_{n-2}\beta + (c-b)\|\alpha_{n-2}\gamma$
a	$(ba-ac)\|\alpha_{n-2}\beta - (2ab+bc)\|\alpha_{n-2}\gamma$
b	$(ab-ba)\|(\beta_{n-1}+\gamma_{n-1}+\alpha_{n-3}\beta_2) - bc\|\alpha_{n-2}\beta - (ba+ac)\|\alpha_{n-2}\gamma$
c	$(ab+bc+ac)\|(\beta_{n-1}+\gamma_{n-1}+\alpha_{n-3}\beta_2) + bc\|\alpha_{n-2}\beta + (ba+ac)\|\alpha_{n-2}\gamma$
ab	$-aba\|(\beta_{n-1}+\gamma_{n-1}+\alpha_{n-3}\beta_2) + (aba-abc)\|\alpha_{n-2}\beta + bac\|\alpha_{n-2}\gamma$
bc	$(aba+abc)\|(\beta_{n-1}+\gamma_{n-1}+\alpha_{n-3}\beta_2)$
ba	$aba\|(\beta_{n-1}+\gamma_{n-1}+\alpha_{n-3}\beta_2) - bac\|\alpha_{n-2}\beta + (abc-aba)\|\alpha_{n-2}\gamma$
ac	$abc\|(\beta_{n-1}+\gamma_{n-1}+\alpha_{n-3}\beta_2) + bac\|\alpha_{n-2}\beta + (aba-abc)\|\alpha_{n-2}\gamma$
aba	$-abac\|(\alpha_{n-2}\beta+\alpha_{n-2}\gamma)$
abc	$abac\|(\alpha_{n-2}\beta+\alpha_{n-2}\gamma)$
bac	$2abac\|(\beta_{n-1}+\gamma_{n-1}+\alpha_{n-3}\beta_2)$
$abac$	0

Table 4.2.9: Images of $\tilde{\partial}_n$ for $n \geqslant 3$ and n odd.

Let us now turn to the maps $\tilde{f}_n$. Note first that the $\Bbbk$-linear maps $\tilde{f}_n : A \otimes (A^!_{-n})^* \to A \otimes (A^!_{-(n+3)})^*$ are homogeneous of homological degree 3 and internal degree 6. By degree reasons we see that $\tilde{f}_n(x|y) = 0$ for all $x \in A_m$, $y \in (A^!_{-n})^*$, with $m \in [\![2,4]\!]$ and $n \in \mathbb{N}_0$. A straightforward computation using (4.1.3) tells us that the map $\tilde{f}_0$ is given by

$$\begin{aligned} &\tilde{f}_0(1|\epsilon^!) = 12bac|\alpha_3 + 12abc|\beta_3 - 12aba|\gamma_3 - 6abc|\alpha_2\beta + 6aba|\alpha_2\gamma - 6bac|\alpha\beta_2, \\ &\tilde{f}_0(a|\epsilon^!) = \tilde{f}_0(b|\epsilon^!) = \tilde{f}_0(c|\epsilon^!) = 0. \end{aligned} \tag{4.2.5}$$

Analogously, if $n \in \mathbb{N}$ is odd, then

$$\tilde{f}_n(a|\alpha_n) = \tilde{f}_n(b|\beta_n) = \tilde{f}_n(c|\gamma_n) = -4abac|\alpha_{n+3} - 4abac|\beta_{n+3} - 4abac|\gamma_{n+3} + 6abac|\alpha_{n+1}\beta_2,$$

$$\tilde{f}_n(b|\alpha_{n-1}\beta) = \tilde{f}_n(c|\alpha_{n-1}\gamma) = \tilde{f}_n(a|\alpha_{n-2}\beta_2)$$
$$= -2(n-1)abac|\alpha_{n+3} - 2(n-1)abac|\beta_{n+3} - 2(n-1)abac|\gamma_{n+3},$$

and $\tilde{f}_n(x) = 0$ for

$$x \in \{1|\alpha_n, 1\beta_n, 1|\gamma_n, 1|\alpha_{n-1}\beta, 1|\alpha_{n-1}\gamma, 1|\alpha_{n-2}\beta_2, b|\alpha_n, c|\alpha_n, a|\beta_n, c|\beta_n, a|\gamma_n, b|\gamma_n, a|\alpha_{n-1}\beta, c|\alpha_{n-1}\beta, a|\alpha_{n-1}\gamma, b|\alpha_{n-1}\gamma, b|\alpha_{n-2}\beta_2, c|\alpha_{n-2}\beta_2\}. \tag{4.2.6}$$

Finally, if $n \geqslant 2$ is even, then

$$\tilde{f}_n(1|\alpha_n) = \tilde{f}_n(1|\beta_n) = \tilde{f}_n(1|\gamma_n) = 8bac|\alpha_{n+3} + 8abc|\beta_{n+3} - 8aba|\gamma_{n+3} - 2abc|\alpha_{n+2}\beta + 2aba|\alpha_{n+2}\gamma - 2bac|\alpha_{n+1}\beta_2,$$
$$\tilde{f}_n(1|\alpha_{n-1}\beta) = \tilde{f}_n(1|\alpha_{n-1}\gamma) = 0, \ \tilde{f}_n(1|\alpha_{n-2}\beta_2) = 6(n-2)(bac|\alpha_{n+3} + abc|\beta_{n+3} - aba|\gamma_{n+3}),$$

and $\tilde{f}_n(x) = 0$ for $x \in A_1 \otimes (A^!_{-n})^*$.

From now on, we assume that the characteristic of the field $\Bbbk$ is different from 2 and 3 in Subsection 4.2.1.

4.2.1.3 Computation of the boundaries

In this subsubsection, we will explicitly construct bases $\tilde{\mathfrak{B}}_{n,m}$ and $\mathfrak{B}_{n,m}$ of the $\Bbbk$-vector spaces $\tilde{B}_{n,m} = \mathrm{Im}(\tilde{\partial}_{n+1,m-1})$ and $B_{n,m} = \mathrm{Im}(\partial_{n+1,m-1})$ for $m \in [\![0,4]\!]$ and $n \in \mathbb{N}_0$ respectively, defined before Proposition 4.2.1. This will be done by simply applying the corresponding differential $\tilde{\partial}_{n+1,m-1}$ or $\partial_{n+1,m-1}$ to the usual basis of its domain and extracting a linearly independent generating subset.

Computation of $\tilde{\mathfrak{B}}_{n,m}$ Recall that $\tilde{B}_{n,m} = \mathrm{Im}(\tilde{\partial}_{n+1,m-1})$ and $\tilde{\partial}_{n,m} : \tilde{K}_{n,m} = A_m \otimes (A^!_{-n})^* \to \tilde{K}_{n-1,m+1} = A_{m+1} \otimes (A^!_{-(n-1)})^*$ was defined in Subsubsection 4.2.1.1. Obviously, $\tilde{B}_{n,0} = \mathrm{Im}(\tilde{\partial}_{n+1,-1}) = 0$ for $n \in \mathbb{N}_0$. Then we define $\tilde{\mathfrak{B}}_{n,0} = \emptyset$ for $n \in \mathbb{N}_0$.

Suppose $m = 1$. Table 4.2.1 shows that $\tilde{\partial}_{1,0}(\tilde{K}_{1,0}) = 0$, so $\tilde{B}_{0,1} = \mathrm{Im}(\tilde{\partial}_{1,0}) = 0$. We define $\tilde{\mathfrak{B}}_{0,1} = \emptyset$. For $n \in \mathbb{N}$ with n odd, Tables 4.2.2 - 4.2.5 show that

$$a|\alpha_n = (1/2)\tilde{\partial}_{n+1,0}(1|\alpha_{n+1}), \quad b|\beta_n = (1/2)\tilde{\partial}_{n+1,0}(1|\beta_{n+1}), \quad c|\gamma_n = (1/2)\tilde{\partial}_{n+1,0}(1|\gamma_{n+1}),$$
$$(a+c)|(\beta_n + \alpha_{n-1}\beta) + (b+a)|(\gamma_n + \alpha_{n-1}\gamma) + (c+b)|(\alpha_n + \alpha_{n-2}\beta_2) = \tilde{\partial}_{n+1,0}(1|\alpha_n\beta) = \tilde{\partial}_{n+1,0}(1|\alpha_n\gamma),$$
$$a|\alpha_{n-2}\beta_2 + b|\alpha_{n-1}\beta + c|\alpha_{n-1}\gamma = (1/2)\tilde{\partial}_{n+1,0}(1|\alpha_{n-1}\beta_2).$$

These five elements are linearly independent if none of them vanishes, so they form a $\Bbbk$-basis of $\tilde{B}_{n,1}$. If $n = 1$, we define a basis of $\tilde{B}_{1,1}$ by

$$\tilde{\mathfrak{B}}_{1,1} = \{\underline{a|\alpha}, \underline{b|\beta}, \underline{c|\gamma}, (\underline{a}+c)|\underline{\beta} + (b+a)|\gamma + (c+b)|\alpha\}.$$

If $n \geqslant 3$ is odd, we define a basis of $\tilde{B}_{n,1}$ by

$$\tilde{\mathfrak{B}}_{n,1} = \{\underline{a|\alpha_n}, \underline{b|\beta_n}, \underline{c|\gamma_n}, (a+\underline{c})|(\beta_n + \underline{\alpha_{n-1}\beta}) + (b+a)|(\gamma_n + \alpha_{n-1}\gamma) + (c+b)|(\alpha_n + \alpha_{n-2}\beta_2), \underline{a|\alpha_{n-2}\beta_2} + b|\alpha_{n-1}\beta + c|\alpha_{n-1}\gamma\}.$$

If $n \geqslant 2$ is even, Tables 4.2.6 - 4.2.9 show that

$$0 = \tilde{\partial}_{n+1,0}(1|\alpha_{n+1}) = \tilde{\partial}_{n+1,0}(1|\beta_{n+1}) = \tilde{\partial}_{n+1,0}(1|\gamma_{n+1}),$$
$$(c-a)|(\alpha_{n-1}\beta - \alpha_{n-1}\gamma) = \tilde{\partial}_{n+1,0}(1|\alpha_n\beta), \quad (a-b)|(\alpha_{n-1}\beta - \alpha_{n-1}\gamma) = \tilde{\partial}_{n+1,0}(1|\alpha_n\gamma),$$
$$(b-c)|(\alpha_{n-1}\beta - \alpha_{n-1}\gamma) = \tilde{\partial}_{n+1,0}(1|\alpha_{n-1}\beta_2) = -\tilde{\partial}_{n+1,0}(1|\alpha_n\beta) - \tilde{\partial}_{n+1,0}(1|\alpha_n\gamma).$$

Since the elements $(c-a)|(\alpha_{n-1}\beta-\alpha_{n-1}\gamma)$ and $(a-b)|(\alpha_{n-1}\beta-\alpha_{n-1}\gamma)$ are linearly independent, we define a basis of $\tilde{B}_{n,1}$ by

$$\tilde{\mathfrak{B}}_{n,1} = \{(\underline{c}-a)|(\alpha_{n-1}\beta - \underline{\alpha_{n-1}\gamma}), (a-\underline{b})|(\underline{\alpha_{n-1}\beta} - \alpha_{n-1}\gamma)\}.$$

The dimension of $\tilde{B}_{n,1}$ is then given by

$$\dim \tilde{B}_{n,1} = \begin{cases} 0, & \text{if } n = 0, \\ 4, & \text{if } n = 1, \\ 2, & \text{if } n \geqslant 2 \text{ is even}, \\ 5, & \text{if } n \geqslant 3 \text{ is odd}. \end{cases} \tag{4.2.7}$$

Suppose now $m = 2$. Table 4.2.1 shows that $\tilde{B}_{0,2}$ is spanned by $(ab - ba)|\epsilon^!$, $(ba + ac + bc)|\epsilon^!$ and $(ab + bc + ac)|\epsilon^!$. Since $(ab + ba)|\epsilon^!$ and $(ba + ac + bc)|\epsilon^!$ are linearly independent, and $(ab + bc + ac)|\epsilon^! = (ab - ba)|\epsilon^! + (ba + ac + bc)|\epsilon^!$, we see that

$$\tilde{\mathfrak{B}}_{0,2} = \big\{(ab - \underline{ba})|\underline{\epsilon^!}, (ba + \underline{ac} + bc)|\underline{\epsilon^!}\big\}$$

is a basis of $\tilde{B}_{0,2}$. If $n \in \mathbb{N}$ is odd, let

$$\begin{aligned} \mathcal{E}_{n,2} = \big\{ e_{1,n,2} &= (\underline{ab} + ba)|\underline{\alpha_n} = \tilde{\partial}_{n+1,1}(b|\alpha_{n+1}), \\ e_{2,n,2} &= (\underline{bc} - ba - ac)|\underline{\alpha_n} = -\tilde{\partial}_{n+1,1}(b|\alpha_{n+1}) - \tilde{\partial}_{n+1,1}(c|\alpha_{n+1}), \\ e_{3,n,2} &= (\underline{ab} + ba)|\underline{\beta_n} = \tilde{\partial}_{n+1,1}(a|\beta_{n+1}), \\ e_{4,n,2} &= (\underline{bc} - ba - ac)|\underline{\beta_n} = \tilde{\partial}_{n+1,1}(c|\beta_{n+1}), \\ e_{5,n,2} &= (\underline{ab} + ba)|\underline{\gamma_n} = -\tilde{\partial}_{n+1,1}(a|\gamma_{n+1}) - \tilde{\partial}_{n+1,1}(b|\gamma_{n+1}), \\ e_{6,n,2} &= (\underline{bc} - ba - ac)|\underline{\gamma_n} = \tilde{\partial}_{n+1,1}(b|\gamma_{n+1}), \\ e_{7,n,2} &= bc|(\alpha_n + \alpha_{n-2}\beta_2) - ac|(\beta_n + \alpha_{n-1}\beta) + \underline{ab}|(\gamma_n + \underline{\alpha_{n-1}\gamma}) = \tilde{\partial}_{n+1,1}(b|\alpha_n\beta)\big\}. \end{aligned}$$

Then we define the set $\tilde{\mathfrak{B}}_{1,2} = \mathcal{E}_{1,2}$, and

$$\begin{aligned} \tilde{\mathfrak{B}}_{n,2} = \mathcal{E}_{n,2} \cup \big\{ e_{8,n,2} &= (ab + \underline{ba})|\underline{\alpha_{n-1}\gamma} = \tilde{\partial}_{n+1,1}(b|\alpha_n\beta) + \tilde{\partial}_{n+1,1}(a|\alpha_n\gamma) - e_{5,n,2}, \\ e_{9,n,2} &= (\underline{ab} + ba)|(\underline{\alpha_{n-1}\beta} + \alpha_{n-2}\beta_2) \\ &= \tilde{\partial}_{n+1,1}(a|\alpha_{n-1}\beta_2) + \tilde{\partial}_{n+1,1}(b|\alpha_{n-1}\beta_2) + e_{8,n,2}, \\ e_{10,n,2} &= (bc - ba - \underline{ac})|\underline{\alpha_{n-2}\beta_2} \\ &= \tilde{\partial}_{n+1,1}(c|\alpha_n\beta) - \tilde{\partial}_{n+1,1}(a|\alpha_n\beta) + e_{8,n,2} + e_{5,n,2} - e_{2,n,2}, \\ e_{11,n,2} &= (ab + ba)|\alpha_{n-1}\beta - (\underline{bc} - ba - ac)|\underline{\alpha_{n-1}\gamma} = \tilde{\partial}_{n+1,1}(a|\alpha_{n-1}\beta_2) + e_{8,n,2}, \\ e_{12,n,2} &= (\underline{bc} - ba - ac)|\underline{\alpha_{n-1}\beta} - (ab + ba)|\alpha_{n-2}\beta_2 \\ &= \tilde{\partial}_{n+1,1}(c|\alpha_{n-1}\beta_2) + e_{10,n,2}\big\} \end{aligned}$$

for $n \geqslant 3$ with n odd. We will show that $\tilde{\mathfrak{B}}_{n,2}$ is a basis of $\tilde{B}_{n,2}$ for $n \in \mathbb{N}$ with n odd. As noted before, $\tilde{\mathfrak{B}}_{n,2} \subseteq \tilde{B}_{n,2}$. Since

$$\begin{gathered} \tilde{\partial}_{n+1,1}(a|\alpha_{n+1}) = \tilde{\partial}_{n+1,1}(b|\beta_{n+1}) = \tilde{\partial}_{n+1,1}(c|\gamma_{n+1}) = 0, \\ \tilde{\partial}_{n+1,1}(b|\alpha_{n+1}) = e_{1,n,2},\ \tilde{\partial}_{n+1,1}(c|\alpha_{n+1}) = -e_{1,n,2} - e_{2,n,2},\ \tilde{\partial}_{n+1,1}(a|\beta_{n+1}) = e_{3,n,2}, \\ \tilde{\partial}_{n+1,1}(c|\beta_{n+1}) = e_{4,n,2},\ \tilde{\partial}_{n+1,1}(a|\gamma_{n+1}) = -e_{5,n,2} - e_{6,n,2},\ \tilde{\partial}_{n+1,1}(b|\gamma_{n+1}) = e_{6,n,2}, \\ \tilde{\partial}_{n+1,1}(a|\alpha_n\beta) = -e_{2,n,2} - e_{3,n,2} - e_{4,n,2} + e_{7,n,2} - e_{9,n,2} - e_{10,n,2} - e_{12,n,2}, \\ \tilde{\partial}_{n+1,1}(b|\alpha_n\beta) = e_{7,n,2}, \\ \tilde{\partial}_{n+1,1}(c|\alpha_n\beta) = -e_{3,n,2} - e_{4,n,2} - e_{5,n,2} + e_{7,n,2} - e_{8,n,2} - e_{9,n,2} - e_{12,n,2}, \\ \tilde{\partial}_{n+1,1}(a|\alpha_n\gamma) = e_{5,n,2} - e_{7,n,2} + e_{8,n,2},\ \tilde{\partial}_{n+1,1}(b|\alpha_n\gamma) = -\tilde{\partial}_{n+1,1}(c|\alpha_n\beta) + e_{2,n,2} + e_{10,n,2}, \\ \tilde{\partial}_{n+1,1}(c|\alpha_n\gamma) = -\tilde{\partial}_{n+1,1}(b|\alpha_n\beta) + e_{2,n,2} + e_{10,n,2},\ \tilde{\partial}_{n+1,1}(a|\alpha_{n-1}\beta_2) = -e_{8,n,2} + e_{11,n,2}, \\ \tilde{\partial}_{n+1,1}(b|\alpha_{n-1}\beta_2) = e_{9,n,2} - e_{11,n,2},\ \tilde{\partial}_{n+1,1}(c|\alpha_{n-1}\beta_2) = -e_{10,n,2} + e_{12,n,2}, \end{gathered}$$

the elements in $\tilde{\mathfrak{B}}_{n,2}$ span the space $\tilde{B}_{n,2}$. By Fact 4.1.3, the elements in $\tilde{\mathfrak{B}}_{n,2}$ are linearly independent, so $\tilde{\mathfrak{B}}_{n,2}$ is a basis of $\tilde{B}_{n,2}$, as claimed. If $n \geqslant 2$ is even, let

$$\mathcal{G}_{n,2} = \big\{ g_{1,n,2} = (\underline{ba} - ab)|\underline{\alpha_n} = -\tilde{\partial}_{n+1,1}(b|\alpha_{n+1}),$$

$$
\begin{aligned}
g_{2,n,2} &= (bc+ba+\underline{ac})|\underline{\alpha_n} = \tilde{\partial}_{n+1,1}(c|\alpha_{n+1}) - \tilde{\partial}_{n+1,1}(b|\alpha_{n+1}),\\
g_{3,n,2} &= (\underline{ba}-ab)|\underline{\beta_n} = \tilde{\partial}_{n+1,1}(a|\beta_{n+1}), \quad g_{4,n,2} = (bc+ba+\underline{ac})|\underline{\beta_n} = \tilde{\partial}_{n+1,1}(c|\beta_{n+1}),\\
g_{5,n,2} &= (\underline{ba}-ab)|\underline{\gamma_n} = \tilde{\partial}_{n+1,1}(a|\gamma_{n+1}) - \tilde{\partial}_{n+1,1}(b|\gamma_{n+1}),\\
g_{6,n,2} &= (bc+ba+\underline{ac})|\underline{\gamma_n} = -\tilde{\partial}_{n+1,1}(b|\gamma_{n+1}),\\
g_{9,n,2} &= ab|\alpha_{n-1}\beta - \underline{ba|\alpha_{n-1}\gamma} = (1/3)\big[\tilde{\partial}_{n+1,1}(2a|\alpha_n\beta + 2c|\alpha_n\beta + 3b|\alpha_n\gamma + b|\alpha_{n-1}\beta_2\\
&\quad + c|\alpha_{n-1}\beta_2) - 2g_{1,n,2} + g_{2,n,2} + 2g_{3,n,2} + 2g_{4,n,2} - 3g_{6,n,2}\big],\\
g_{10,n,2} &= \underline{ba|\alpha_{n-1}\beta} - ab|\alpha_{n-1}\gamma = (1/3)\tilde{\partial}_{n+1,1}(a|\alpha_{n-1}\beta_2 - b|\alpha_n\beta),\\
g_{11,n,2} &= \underline{ac|\alpha_{n-1}\beta} + (ab+bc)|\alpha_{n-1}\gamma = -(1/3)\tilde{\partial}_{n+1,1}(b|\alpha_n\beta + 2a|\alpha_{n-1}\beta_2),\\
g_{12,n,2} &= (ab+bc)|\alpha_{n-1}\beta + \underline{ac|\alpha_{n-1}\gamma} = (1/3)\big[\tilde{\partial}_{n+1,1}(c|\alpha_n\beta - 2a|\alpha_n\beta - b|\alpha_{n-1}\beta_2\\
&\quad - c|\alpha_{n-1}\beta_2) + 2g_{1,n,2} - g_{2,n,2} - 2g_{3,n,2} + g_{4,n,2}\big]\big\},
\end{aligned}
$$

Then we define $\tilde{\mathfrak{B}}_{2,2} = \mathcal{G}_{2,2}$, and

$$
\begin{aligned}
\tilde{\mathfrak{B}}_{n,2} = \mathcal{G}_{n,2} \cup \big\{ g_{7,n,2} &= (ba-\underline{ab})|\underline{\alpha_{n-2}\beta_2} = (1/3)[\tilde{\partial}_{n+1,1}(a|\alpha_n\beta + c|\alpha_n\beta - b|\alpha_{n-1}\beta_2\\
&\quad - c|\alpha_{n-1}\beta_2) - g_{1,n,2} - g_{2,n,2} - 2g_{3,n,2} + g_{4,n,2} - 3g_{5,n,2}],\\
g_{8,n,2} &= (\underline{bc}+ba+ac)|\underline{\alpha_{n-2}\beta_2} = (1/3)\big[\tilde{\partial}_{n+1,1}(2a|\alpha_n\beta + 2c|\alpha_n\beta + b|\alpha_{n-1}\beta_2\\
&\quad + c|\alpha_{n-1}\beta_2) - 2g_{1,n,2} - 2g_{2,n,2} + 2g_{3,n,2} - g_{4,n,2} - 3g_{6,n,2}\big]\big\}
\end{aligned}
$$

for $n \geqslant 4$ with n even. We will show that $\tilde{\mathfrak{B}}_{n,2}$ is a basis of $\tilde{B}_{n,2}$ for $n \geqslant 2$ with n even. From the definition, we see that $\tilde{\mathfrak{B}}_{n,2} \subseteq \tilde{B}_{n,2}$. Since

$$
\begin{aligned}
&\tilde{\partial}_{n+1,1}(a|\alpha_{n+1}) = \tilde{\partial}_{n+1,1}(b|\beta_{n+1}) = \tilde{\partial}_{n+1,1}(c|\gamma_{n+1}) = 0,\\
&\tilde{\partial}_{n+1,1}(b|\alpha_{n+1}) = -g_{1,n,2},\ \tilde{\partial}_{n+1,1}(c|\alpha_n) = g_{2,n,2} - g_{1,n,2},\ \tilde{\partial}_{n+1,1}(a|\beta_{n+1}) = g_{3,n,2},\\
&\tilde{\partial}_{n+1,1}(c|\beta_{n+1}) = g_{4,n,2},\ \tilde{\partial}_{n+1,1}(a|\gamma_{n+1}) = g_{5,n,2} - g_{6,n,2},\ \tilde{\partial}_{n+1,1}(b|\gamma_{n+1}) = -g_{6,n,2},\\
&\tilde{\partial}_{n+1,1}(a|\alpha_n\beta) = g_{1,n,2} + g_{5,n,2} + g_{7,n,2} - g_{12,n,2},\ \tilde{\partial}_{n+1,1}(b|\alpha_n\beta) = -2g_{10,n,2} - g_{11,n,2},\\
&\tilde{\partial}_{n+1,1}(c|\alpha_n\beta) = g_{2,n,2} + g_{6,n,2} + g_{8,n,2} + g_{12,n,2},\\
&\tilde{\partial}_{n+1,1}(a|\alpha_n\gamma) = g_{1,n,2} - g_{2,n,2} + g_{3,n,2} - g_{4,n,2} + g_{7,n,2} - g_{8,n,2} - g_{9,n,2},\\
&\tilde{\partial}_{n+1,1}(b|\alpha_n\gamma) = -g_{2,n,2} - g_{4,n,2} - g_{8,n,2} + g_{9,n,2},\ \tilde{\partial}_{n+1,1}(c|\alpha_n\gamma) = g_{10,n,2} + 2g_{11,n,2},\\
&\tilde{\partial}_{n+1,1}(a|\alpha_{n-1}\beta_2) = g_{10,n,2} - g_{11,n,2},\\
&\tilde{\partial}_{n+1,1}(b|\alpha_{n-1}\beta_2) = -g_{3,n,2} - g_{5,n,2} - g_{7,n,2} + g_{9,n,2} - g_{12,n,2},\\
&\tilde{\partial}_{n+1,1}(c|\alpha_{n-1}\beta_2) = -g_{3,n,2} + g_{4,n,2} - g_{5,n,2} + g_{6,n,2} - g_{7,n,2} + g_{8,n,2} - g_{9,n,2} + g_{12,n,2},
\end{aligned}
$$

the elements in $\tilde{\mathfrak{B}}_{n,2}$ span the space $\tilde{B}_{n,2}$. By Fact 4.1.3, the elements in $\tilde{\mathfrak{B}}_{n,2}$ are linearly independent, so $\tilde{\mathfrak{B}}_{n,2}$ is a basis of $\tilde{B}_{n,2}$, as claimed. The dimension of $\tilde{B}_{n,2}$ is thus given by

$$
\dim \tilde{B}_{n,2} = \begin{cases} 2, & \text{if } n = 0,\\ 7, & \text{if } n = 1,\\ 10, & \text{if } n = 2,\\ 12, & \text{if } n \geqslant 3. \end{cases} \tag{4.2.8}
$$

Suppose now $m = 3$. Table 4.2.1 shows that $\tilde{\partial}_{1,2}$ is surjective. We thus define a basis of $\tilde{B}_{0,3}$ by the usual basis of $\tilde{K}_{0,3}$. If $n \in \mathbb{N}$ is odd, let

$$
\begin{aligned}
\mathcal{E}_{n,3} = \big\{ e_{1,n,3} &= \underline{aba|\alpha_n} = \tilde{\partial}_{n+1,2}(ab|\alpha_{n+1}), \quad e_{2,n,3} = \underline{abc|\alpha_n} = -\tilde{\partial}_{n+1,2}(ac|\alpha_{n+1}),\\
e_{3,n,3} &= \underline{aba|\beta_n} = \tilde{\partial}_{n+1,2}(ab|\beta_{n+1}), \quad e_{4,n,3} = \underline{bac|\beta_n} = -\tilde{\partial}_{n+1,2}(bc|\beta_{n+1}),\\
e_{5,n,3} &= \underline{abc|\gamma_n} = -\tilde{\partial}_{n+1,2}(ac|\gamma_{n+1}), \quad e_{6,n,3} = \underline{bac|\gamma_n} = -\tilde{\partial}_{n+1,2}(bc|\gamma_{n+1}),\\
e_{7,n,3} &= \underline{abc}|(\underline{\beta_n} + \alpha_{n-1}\beta) + aba|(\gamma_n + \alpha_{n-1}\gamma) = (1/3)\tilde{\partial}_{n+1,2}((ba-ac)|\alpha_n\beta),\\
e_{8,n,3} &= aba|(\gamma_n + \alpha_{n-1}\gamma) + \underline{bac}|(\underline{\alpha_n} + \alpha_{n-2}\beta_2) = (1/3)\tilde{\partial}_{n+1,2}((2ba+ac)|\alpha_n\beta)\big\}.
\end{aligned}
$$

Then we define $\tilde{\mathfrak{B}}_{1,3} = \mathcal{E}_{1,3}$, and

$$\begin{aligned}
\tilde{\mathfrak{B}}_{n,3} = \mathcal{E}_{n,3} \cup \big\{ e_{9,n,3} &= aba|\alpha_{n-1}\beta + (abc+bac)|\alpha_{n-1}\gamma + aba|\alpha_{n-2}\beta_2 = \tilde{\partial}_{n+1,2}(ba|\alpha_{n-1}\beta_2), \\
e_{10,n,3} &= bac|\alpha_{n-1}\beta + bac|\alpha_{n-1}\gamma + (aba-abc)|\alpha_{n-2}\beta_2 = -\tilde{\partial}_{n+1,2}(bc|\alpha_{n-1}\beta_2), \\
e_{11,n,3} &= (aba+bac)|\alpha_{n-1}\beta + (bac-abc)|\alpha_{n-1}\gamma \\
&= -\tilde{\partial}_{n+1,2}(bc|\alpha_n\beta) - e_{3,n,3} - e_{4,n,3} + e_{5,n,3} - e_{6,n,3}, \\
e_{12,n,3} &= (abc-bac)|\alpha_{n-1}\gamma + (aba+abc)|\alpha_{n-2}\beta_2 \\
&= \tilde{\partial}_{n+1,2}((ab+bc)|\alpha_n\beta) - e_{1,n,3} - e_{2,n,3} - e_{5,n,3} + e_{6,n,3} \big\}
\end{aligned}$$

for $n \geqslant 3$ with n odd. We now show that $\tilde{\mathfrak{B}}_{n,3}$ is a basis of $\tilde{B}_{n,3}$ for $n \in \mathbb{N}$ with n odd. As noted before, $\tilde{\mathfrak{B}}_{n,3} \subseteq \tilde{B}_{n,3}$. Since

$$\begin{aligned}
\tilde{\partial}_{n+1,2}(ab|\alpha_{n+1}) &= \tilde{\partial}_{n+1,2}(ba|\alpha_{n+1}) = e_{1,n,3},\ \tilde{\partial}_{n+1,2}(bc|\alpha_{n+1}) = e_{2,n,3} - e_{1,n,3}, \\
\tilde{\partial}_{n+1,2}(ac|\alpha_{n+1}) &= -e_{2,n,3},\ \tilde{\partial}_{n+1,2}(ab|\beta_{n+1}) = \tilde{\partial}_{n+1,2}(ba|\beta_{n+1}) = e_{3,n,3}, \\
\tilde{\partial}_{n+1,2}(bc|\beta_{n+1}) &= -e_{4,n,3},\ \tilde{\partial}_{n+1,2}(ac|\beta_{n+1}) = e_{4,n,3} - e_{3,n,3}, \\
\tilde{\partial}_{n+1,2}(ab|\gamma_{n+1}) &= \tilde{\partial}_{n+1,2}(ba|\gamma_{n+1}) = e_{5,n,3} + e_{6,n,3},\ \tilde{\partial}_{n+1,2}(bc|\gamma_{n+1}) = -e_{6,n,3}, \\
\tilde{\partial}_{n+1,2}(ac|\gamma_{n+1}) &= -e_{5,n,3}, \\
\tilde{\partial}_{n+1,2}(ab|\alpha_n\beta) &= e_{1,n,3} + e_{2,n,3} + e_{3,n,3} + e_{4,n,3} + e_{11,n,3} + e_{12,n,3}, \\
\tilde{\partial}_{n+1,2}(bc|\alpha_n\beta) &= -e_{3,n,3} - e_{4,n,3} + e_{5,n,3} - e_{6,n,3} - e_{11,n,3}, \\
\tilde{\partial}_{n+1,2}(ba|\alpha_n\beta) &= \tilde{\partial}_{n+1,2}(ab|\alpha_n\gamma) = e_{7,n,3} + e_{8,n,3},\ \tilde{\partial}_{n+1,2}(ac|\alpha_n\beta) = -2e_{7,n,3} + e_{8,n,3}, \\
\tilde{\partial}_{n+1,2}(bc|\alpha_n\gamma) &= e_{7,n,3} - 2e_{8,n,3}, \\
\tilde{\partial}_{n+1,2}(ba|\alpha_n\gamma) &= e_{1,n,3} + e_{2,n,3} + e_{3,n,3} + e_{4,n,3} + e_{11,n,3} + e_{12,n,3}, \\
\tilde{\partial}_{n+1,2}(ac|\alpha_n\gamma) &= -e_{1,n,3} - e_{2,n,3} - e_{5,n,3} + e_{6,n,3} - e_{12,n,3}, \\
\tilde{\partial}_{n+1,2}(ab|\alpha_{n-1}\beta_2) &= \tilde{\partial}_{n+1,2}(ba|\alpha_{n-1}\beta_2) = e_{9,n,3},\ \tilde{\partial}_{n+1,2}(bc|\alpha_{n-1}\beta_2) = -e_{10,n,3}, \\
\tilde{\partial}_{n+1,2}(ac|\alpha_{n-1}\beta_2) &= e_{10,n,3} - e_{9,n,3},
\end{aligned}$$

the elements in $\tilde{\mathfrak{B}}_{n,3}$ span the space $\tilde{B}_{n,3}$. By Fact 4.1.3, the elements $e_{\ell,n,3}$ for $\ell \in [\![1,8]\!]$ are linearly independent. The reader can easily verify that the elements $e_{\ell,n,3}$ for $\ell \in [\![9,12]\!]$ are linearly independent. Since the underlined terms of $e_{\ell,n,3}$ for $\ell \in [\![1,8]\!]$ do not appear in $e_{\ell,n,3}$ for $\ell \in [\![9,12]\!]$, the elements in $\tilde{\mathfrak{B}}_{n,3}$ are linearly independent. So $\tilde{\mathfrak{B}}_{n,3}$ is a basis of $\tilde{B}_{n,3}$, as claimed. If $n \geqslant 2$ is even, let

$$\begin{aligned}
\tilde{\mathfrak{B}}_{n,3} = \big\{ g_{1,n,3} &= \underline{aba|\alpha_n} = \tilde{\partial}_{n+1,2}(ba|\alpha_{n+1}), \quad g_{2,n,3} = \underline{abc|\alpha_n} = \tilde{\partial}_{n+1,2}(ac|\alpha_{n+1}), \\
g_{3,n,3} &= \underline{aba|\beta_n} = \tilde{\partial}_{n+1,2}(ab|\beta_{n+1}), \quad g_{4,n,3} = \underline{bac|\beta_n} = \tilde{\partial}_{n+1,2}(bc|\beta_{n+1}), \\
g_{5,n,3} &= \underline{abc|\gamma_n} = -\tilde{\partial}_{n+1,2}(ac|\gamma_{n+1}), \quad g_{6,n,3} = \underline{bac|\gamma_n} = -\tilde{\partial}_{n+1,2}(bc|\gamma_{n+1}), \\
g_{7,n,3} &= bac|(\alpha_n + \alpha_{n-2}\beta_2) + \underline{aba}|(\underline{\gamma_n} + \alpha_{n-2}\beta_2) \\
&= \tilde{\partial}_{n+1,2}((ab+bc)|\alpha_n\beta) - g_{1,n,3} - g_{6,n,3}, \\
g_{8,n,3} &= \underline{abc}|(\underline{\beta_n} + \alpha_{n-2}\beta_2) + aba|(\gamma_n + \alpha_{n-2}\beta_2) \\
&= \tilde{\partial}_{n+1,2}((ba+ac)|\alpha_{n-1}\beta_2) - g_{3,n,3} - g_{5,n,3}, \\
g_{9,n,3} &= \underline{bac}|(\underline{\alpha_{n-1}\beta} + \alpha_{n-1}\gamma - \alpha_n - \alpha_{n-2}\beta_2) + aba|(\gamma_n + \alpha_{n-2}\beta_2) \\
&= \tilde{\partial}_{n+1,2}(ab|\alpha_n\beta + ac|\alpha_n\gamma) - e_{1,n,3} + e_{2,n,3} - e_{7,n,3} + e_{8,n,3}, \\
g_{10,n,3} &= \underline{aba}|(\alpha_{n-1}\beta + \underline{\alpha_{n-1}\gamma} - 2\gamma_n - 2\alpha_{n-2}\beta_2) \\
&= -\tilde{\partial}_{n+1,2}(ab|\alpha_n\beta + ba|\alpha_{n-1}\beta_2) + g_{1,n,3} + g_{3,n,3}, \\
g_{11,n,3} &= (bac-aba)|\alpha_{n-1}\beta - \underline{abc|\alpha_{n-1}\gamma} + aba|(\gamma_n + \alpha_{n-2}\beta_2) \\
&= \tilde{\partial}_{n+1,2}(ab|\alpha_n\beta) - g_{1,n,3}, \\
g_{12,n,3} &= \underline{abc|\alpha_{n-1}\beta} + (aba-bac)|\alpha_{n-1}\gamma - aba|(\gamma_n + \alpha_{n-2}\beta_2) \\
&= \tilde{\partial}_{n+1,2}(ba|\alpha_n\beta) + g_{1,n,3} \big\}.
\end{aligned}$$

We then show that $\tilde{\mathfrak{B}}_{n,3}$ is a basis of $\tilde{B}_{n,3}$. It follows from the definition that $\tilde{\mathfrak{B}}_{n,3} \subseteq \tilde{B}_{n,3}$. Since

$$\begin{aligned}
&\tilde{\partial}_{n+1,2}(ab|\alpha_{n+1}) = -\tilde{\partial}_{n+1,2}(ba|\alpha_{n+1}) = -g_{1,n,3},\ \tilde{\partial}_{n+1,2}(bc|\alpha_{n+1}) = g_{1,n,3} + g_{2,n,3},\\
&\tilde{\partial}_{n+1,2}(ac|\alpha_{n+1}) = g_{2,n,3},\ \tilde{\partial}_{n+1,2}(ab|\beta_{n+1}) = -\tilde{\partial}_{n+1,2}(ba|\beta_{n+1}) = g_{3,n,3},\\
&\tilde{\partial}_{n+1,2}(bc|\beta_{n+1}) = g_{4,n,3},\ \tilde{\partial}_{n+1,2}(ac|\beta_{n+1}) = g_{3,n,3} + g_{4,n,3},\\
&\tilde{\partial}_{n+1,2}(ab|\gamma_{n+1}) = -\tilde{\partial}_{n+1,2}(ba|\gamma_{n+1}) = g_{6,n,3} - g_{5,n,3},\ \tilde{\partial}_{n+1,2}(bc|\gamma_{n+1}) = -g_{6,n,3},\\
&\tilde{\partial}_{n+1,2}(ac|\gamma_{n+1}) = -g_{5,n,3},\ \tilde{\partial}_{n+1,2}(ab|\alpha_n\beta) = g_{1,n,3} + g_{11,n,3},\\
&\tilde{\partial}_{n+1,2}(bc|\alpha_n\beta) = g_{6,n,3} + g_{7,n,3} - g_{11,n,3},\ \tilde{\partial}_{n+1,2}(ba|\alpha_n\beta) = g_{12,n,3} - g_{1,n,3},\\
&\tilde{\partial}_{n+1,2}(ac|\alpha_n\beta) = g_{1,n,3} + g_{6,n,3} + g_{7,n,3},\ \tilde{\partial}_{n+1,2}(ab|\alpha_n\gamma) = -g_{2,n,3} + g_{4,n,3} + g_{7,n,3} - g_{8,n,3},\\
&\tilde{\partial}_{n+1,2}(bc|\alpha_n\gamma) = -g_{4,n,3} + g_{9,n,3} + g_{12,n,3},\ \tilde{\partial}_{n+1,2}(ba|\alpha_n\gamma) = g_{2,n,3} - g_{4,n,3} - g_{7,n,3} + g_{8,n,3},\\
&\tilde{\partial}_{n+1,2}(ac|\alpha_n\gamma) = -g_{2,n,3} + g_{7,n,3} - g_{8,n,3} + g_{9,n,3} - g_{11,n,3},\\
&\tilde{\partial}_{n+1,2}(ab|\alpha_{n-1}\beta_2) = -g_{3,n,3} + g_{10,n,3} - g_{12,n,3},\ \tilde{\partial}_{n+1,2}(bc|\alpha_{n-1}\beta_2) = g_{3,n,3} + g_{5,n,3} + g_{8,n,3},\\
&\tilde{\partial}_{n+1,2}(ba|\alpha_{n-1}\beta_2) = g_{3,n,3} - g_{10,n,3} - g_{11,n,3},\\
&\tilde{\partial}_{n+1,2}(ac|\alpha_{n-1}\beta_2) = g_{5,n,3} + g_{8,n,3} + g_{10,n,3} + g_{11,n,3},
\end{aligned}$$

the elements in $\tilde{\mathfrak{B}}_{n,3}$ span the space $\tilde{B}_{n,3}$. By Fact 4.1.3, the elements in $\tilde{\mathfrak{B}}_{n,3}$ are linearly independent, so $\tilde{\mathfrak{B}}_{n,3}$ is a basis of $\tilde{B}_{n,3}$, as claimed. Hence, the dimension of $\tilde{B}_{n,3}$ is given by

$$\dim \tilde{B}_{n,3} = \begin{cases} 3, & \text{if } n = 0,\\ 8, & \text{if } n = 1,\\ 12, & \text{if } n \geqslant 2. \end{cases} \tag{4.2.9}$$

Suppose now $m = 4$. Table 4.2.1 tells us that the usual basis of $\tilde{K}_{0,4}$ is a basis of $\tilde{B}_{0,4}$. If $n \in \mathbb{N}$ is odd, Tables 4.2.2 - 4.2.5 show that $\tilde{B}_{n,4}$ is spanned by $\tilde{\partial}_{n+1,3}(aba|\alpha_n\beta)$, $\tilde{\partial}_{n+1,3}(abc|\alpha_n\beta)$ and $\tilde{\partial}_{n+1,3}(bac|\alpha_n\beta)$. Since

$$\tilde{\partial}_{n+1,3}(bac|\alpha_n\beta) = \tilde{\partial}_{n+1,3}(aba|\alpha_n\beta) - \tilde{\partial}_{n+1,3}(abc|\alpha_n\beta),$$

and the elements $\tilde{\partial}_{n+1,3}(aba|\alpha_n\beta)$ and $\tilde{\partial}_{n+1,3}(abc|\alpha_n\beta)$ are linearly independent, we define a basis of $\tilde{B}_{n,4}$ by

$$\begin{aligned}\tilde{\mathfrak{B}}_{n,4} = \big\{&\tilde{\partial}_{n+1,3}(aba|\alpha_n\beta) = \underline{abac}|(\alpha_n + \alpha_{n-2}\beta_2 - \underline{\beta_n} - \alpha_{n-1}\beta),\\ &\tilde{\partial}_{n+1,3}(aba|\alpha_n\beta) = \underline{abac}|(\alpha_n + \alpha_{n-2}\beta_2 - \underline{\gamma_n} - \alpha_{n-1}\gamma)\big\}.\end{aligned}$$

If $n = 2$, by Tables 4.2.6 - 4.2.9, we define a basis of $\tilde{B}_{2,4}$ by

$$\tilde{\mathfrak{B}}_{2,4} = \big\{\underline{abac|\alpha_2}, \underline{abac|\beta_2}, \underline{abac|\gamma_2}, \underline{abac}|(\alpha\beta + \underline{\alpha\gamma})\big\}.$$

If $n \geqslant 4$ is even, we note that $abac|\alpha_{n-2}\beta_2 = (1/2)\tilde{\partial}_{n+1,3}(abc|\alpha_n\beta - bac|\alpha_{n+1} + aba|\gamma_{n+1})$. So we can define a basis of $\tilde{B}_{n,4}$ by

$$\tilde{\mathfrak{B}}_{n,4} = \big\{\underline{abac|\alpha_n}, \underline{abac|\beta_n}, \underline{abac|\gamma_n}, \underline{abac}|(\alpha_{n-1}\beta + \underline{\alpha_{n-1}\gamma}), \underline{abac|\alpha_{n-2}\beta_2}\big\}.$$

In conclusion, the dimension of $\tilde{B}_{n,4}$ is given by

$$\dim \tilde{B}_{n,4} = \begin{cases} 1, & \text{if } n = 0,\\ 2, & \text{if } n \in \mathbb{N} \text{ is odd},\\ 4, & \text{if } n = 2,\\ 5, & \text{if } n \geqslant 4 \text{ is even}. \end{cases} \tag{4.2.10}$$

Computation of $\mathfrak{B}_{n,m}$ Recall that $B_{n,m} = \mathrm{Im}(\partial_{n+1,m-1})$ and $\partial_{n,m} : \tilde{P}_{n,m} \to \tilde{P}_{n-1,m+1}$. Since $\partial_{n,m} = \tilde{\partial}_{n,m}$ for either $m = -1,0,1$ and $n \in \mathbb{N}$, or $m = 2,3$ and $n = 1,2,3$, we get $B_{n,m} = \tilde{B}_{n,m}$

for either $m = 0,1,2$ and $n \in \mathbb{N}_0$, or $m = 3,4$ and $n = 0,1,2$. So we define a basis of $B_{n,m}$ by $\mathfrak{B}_{n,m} = \tilde{\mathfrak{B}}_{n,m}$ for either $m = 0,1,2$ and $n \in \mathbb{N}_0$, or $m = 3,4$ and $n = 0,1,2$.

Suppose $m = 3$. Consider $\partial_{n+1,2} : \tilde{K}_{n+1,2} \oplus \omega_1 \tilde{K}_{n-3,0} \to \tilde{K}_{n,3} \oplus \omega_1 \tilde{K}_{n-4,1}$. If $n = 3$, the element $2bac|\alpha_3 + 2abc|\beta_3 - 2aba|\gamma_3 - abc|\alpha_2\beta + aba|\alpha_2\gamma - bac|\alpha\beta_2 = (1/6)\partial_{4,2}(\omega_1 1|\epsilon^!)$ is not in the space $\tilde{B}_{3,3}$. So we define a basis of $B_{3,3}$ by

$$\mathfrak{B}_{3,3} = \tilde{\mathfrak{B}}_{3,3} \cup \big\{2bac|\alpha_3 + 2abc|\beta_3 - 2aba|\gamma_3 - abc|\alpha_2\beta + aba|\alpha_2\gamma - bac|\alpha\beta_2\big\}.$$

If $n = 5$, we define the set

$$\begin{aligned}\mathfrak{B}_{5,3} = \tilde{\mathfrak{B}}_{5,3} \cup \big\{&4bac|\alpha_5 + 4abc|\beta_5 - 4abc|\gamma_5 - abc|\alpha_4\beta + aba|\alpha_4\gamma - bac|\alpha_3\beta_2 + \underline{\omega_1 a|\alpha}\\ &= (1/2)\partial_{6,2}(\omega_1 1|\alpha_2),\\ &4bac|\alpha_5 + 4abc|\beta_5 - 4abc|\gamma_5 - abc|\alpha_4\beta + aba|\alpha_4\gamma - bac|\alpha_3\beta_2 + \underline{\omega_1 b|\beta}\\ &= (1/2)\partial_{6,2}(\omega_1 1|\beta_2),\\ &4bac|\alpha_5 + 4abc|\beta_5 - 4abc|\gamma_5 - abc|\alpha_4\beta + aba|\alpha_4\gamma - bac|\alpha_3\beta_2 + \underline{\omega_1 c|\gamma}\\ &= (1/2)\partial_{6,2}(\omega_1 1|\gamma_2),\\ &\underline{\omega_1}[(\underline{a}+c)|\underline{\beta} + (b+a)|\gamma + (c+b)|\alpha] = \partial_{6,2}(\omega_1 1|\alpha\beta) = \partial_{6,2}(\omega_1 1|\alpha\gamma)\big\}.\end{aligned}$$

If $n \geqslant 7$ is odd, we define the set

$$\begin{aligned}\mathfrak{B}_{n,3} =&\tilde{\mathfrak{B}}_{n,3}\cup\\ &\big\{4bac|\alpha_n + 4abc|\beta_n - 4abc|\gamma_n - abc|\alpha_{n-1}\beta + aba|\alpha_{n-1}\gamma - bac|\alpha_{n-2}\beta_2 + \underline{\omega_1 a|\alpha_{n-4}}\\ &\quad = (1/2)\partial_{n+1,2}(\omega_1 1|\alpha_{n-3}),\\ &4bac|\alpha_n + 4abc|\beta_n - 4abc|\gamma_n - abc|\alpha_{n-1}\beta + aba|\alpha_{n-1}\gamma - bac|\alpha_{n-2}\beta_2 + \underline{\omega_1 b|\beta_{n-4}}\\ &\quad = (1/2)\partial_{n+1,2}(\omega_1 1|\beta_{n-3}),\\ &4bac|\alpha_n + 4abc|\beta_n - 4abc|\gamma_n - abc|\alpha_{n-1}\beta + aba|\alpha_{n-1}\gamma - bac|\alpha_{n-2}\beta_2 + \underline{\omega_1 c|\gamma_{n-4}}\\ &\quad = (1/2)\partial_{n+1,2}(\omega_1 1|\gamma_{n-3}),\\ &\underline{\omega_1}[(a+\underline{c})|(\beta_{n-4} + \underline{\alpha_{n-5}\beta}) + (b+a)|(\gamma_{n-4} + \alpha_{n-5}\gamma) + (c+b)|(\alpha_{n-4} + \alpha_{n-6}\beta_2)]\\ &\quad = \partial_{n+1,2}(\omega_1 1|\alpha_{n-4}\beta) = \partial_{n+1,2}(\omega_1 1|\alpha_{n-4}\gamma),\\ &3(n-5)(bac|\alpha_n + abc|\beta_n - aba|\gamma_n) + \underline{\omega_1}(\underline{a|\alpha_{n-6}\beta_2} + b|\alpha_{n-5}\beta + c|\alpha_{n-5}\gamma)\\ &\quad = \partial_{n+1,2}(\omega_1 1|\alpha_{n-5}\beta_2)\big\}.\end{aligned}$$

By Fact 4.1.3, the elements in $\mathfrak{B}_{n,3}$ are linearly independent, so $\mathfrak{B}_{n,3}$ is a basis of $B_{n,3}$ for $n \geqslant 5$ with n odd. If $n \geqslant 4$ is even, then $\tilde{f}_{n-3}(\tilde{K}_{n-3,0}) = 0$ since $\tilde{f}$ vanishes on the elements given by (4.2.6). Hence, $B_{n,3} = \tilde{B}_{n,3} \oplus \omega_1 \tilde{B}_{n-4,1}$. We define a basis of $B_{4,3}$ by $\mathfrak{B}_{4,3} = \tilde{\mathfrak{B}}_{4,3}$, and we define a basis of $B_{n,3}$ by

$$\mathfrak{B}_{n,3} = \tilde{\mathfrak{B}}_{n,3} \cup \big\{\underline{\omega_1}(\underline{c} - a)|(\alpha_{n-5}\beta - \underline{\alpha_{n-5}\gamma}), \omega_1(a - \underline{b})|(\underline{\alpha_{n-5}\beta} - \alpha_{n-5}\gamma)\big\}$$

for $n \geqslant 6$ with n even. The dimension of $B_{n,3}$ is then given by

$$\dim B_{n,3} = \begin{cases} 3, & \text{if } n = 0,\\ 8, & \text{if } n = 1,\\ 12, & \text{if } n = 2,4,\\ 13, & \text{if } n = 3,\\ 16, & \text{if } n = 5,\\ 14, & \text{if } n \geqslant 6 \text{ is even},\\ 17, & \text{if } n \geqslant 7 \text{ is odd}.\end{cases} \tag{4.2.11}$$

Suppose $m = 4$. Consider $\partial_{n+1,3} : \tilde{K}_{n+1,3} \oplus \omega_1 \tilde{K}_{n-3,1} \to \tilde{K}_{n,4} \oplus \omega_1 \tilde{K}_{n-4,2} \oplus \omega_2 \tilde{K}_{n-8,0}$. If $n = 3$, then $B_{3,4} = \tilde{B}_{3,4}$ since $\tilde{f}_0(\tilde{K}_{0,1}) = 0$ by the second line of (4.2.5). So, we define $\mathfrak{B}_{3,4} = \tilde{\mathfrak{B}}_{3,4}$. If $n \geqslant 4$ is even, since $\tilde{f}_{n-3}(\tilde{K}_{n-3,1}) \subseteq \tilde{B}_{n,4}$, we have $B_{n,4} = \tilde{B}_{n,4} \oplus \omega_1 B_{n-4,2} = \tilde{B}_{n,4} \oplus \omega_1 \tilde{B}_{n-4,2}$. If $n \geqslant 5$ is odd, since $\tilde{f}_{n-3}(\tilde{K}_{n-3,1}) = 0$ by the last identity of Subsubsection 4.2.1.2, we have

$B_{n,4} = \tilde{B}_{n,4} \oplus \omega_1 \tilde{B}_{n-4,2}$. Hence, for $n \geqslant 4$, we define a basis of $B_{n,4}$ by $\mathfrak{B}_{n,4} = \tilde{\mathfrak{B}}_{n,4} \cup \omega_1 \tilde{\mathfrak{B}}_{n-4,2}$. The dimension of $B_{n,4}$ is then given by

$$\dim B_{n,4} = \begin{cases} 1, & \text{if } n = 0, \\ 2, & \text{if } n = 1,3, \\ 4, & \text{if } n = 2, \\ 7, & \text{if } n = 4, \\ 9, & \text{if } n = 5, \\ 15, & \text{if } n = 6, \\ 14, & \text{if } n \geqslant 7 \text{ is odd}, \\ 17, & \text{if } n \geqslant 8 \text{ is even}. \end{cases} \tag{4.2.12}$$

4.2.1.4 Computation of the cycles

As one can remark rather easily, from the computations in the previous subsubsection we can already deduce the dimensions of the homogeneous components of the spaces of cycles and thus of the Hochschild homology groups. However, since having specific representatives of bases of homology classes is relevant for other computations involving the Hochschild homology groups, we will proceed to do so. More precisely, in this subsubsection, we will explicitly construct bases $\tilde{\mathfrak{D}}_{n,m}$ and $\mathfrak{D}_{n,m}$ of the $\Bbbk$-vector spaces $\tilde{D}_{n,m} = \mathrm{Ker}(\tilde{\partial}_{n,m})$ and $D_{n,m} = \mathrm{Ker}(\partial_{n,m})$ for $m \in [\![0,4]\!]$ and $n \in \mathbb{N}_0$ respectively, defined before Proposition 4.2.1.

Computation of $\tilde{\mathfrak{D}}_{n,m}$ Recall that $\tilde{D}_{n,m} = \mathrm{Ker}(\tilde{\partial}_{n,m})$ and $\tilde{\partial}_{n,m} : \tilde{K}_{n,m} = A_m \otimes (A^!_{-n})^* \to \tilde{K}_{n-1,m+1} = A_{m+1} \otimes (A^!_{-(n-1)})^*$ was defined in Subsubsection 4.2.1.1. Since $\tilde{K}_{n,m}/\tilde{D}_{n,m} \cong \tilde{B}_{n-1,m+1}$, we see that

$$\dim \tilde{D}_{n,m} = \dim \tilde{K}_{n,m} - \dim \tilde{B}_{n-1,m+1}.$$

Hence, from the dimension of $\tilde{B}_{n-1,m+1}$ computed in Subsubsection 4.2.1.3 as well as the dimension of $\tilde{K}_{n,m}$ (see the last paragraph of Subsubsection 4.2.1.1), we deduce the value of the dimension of $\tilde{D}_{n,m}$. We will present them explicitly in the computations below.

For every $(n,m) \in \mathbb{N}_0 \times [\![0,4]\!]$, we are going to provide a set $\tilde{\mathfrak{D}}_{n,m} \subseteq \tilde{D}_{n,m}$ such that $\#\tilde{\mathfrak{D}}_{n,m} = \dim \tilde{D}_{n,m}$ and the elements in $\tilde{\mathfrak{D}}_{n,m}$ are linearly independent. As a consequence, $\tilde{\mathfrak{D}}_{n,m}$ is a basis of $\tilde{D}_{n,m}$. If $\tilde{D}_{n,m} = \tilde{K}_{n,m}$, we pick the usual basis of $\tilde{K}_{n,m}$, defined at the end of Subsubsection 4.2.1.1. We leave to the reader the easy verification in each case that the set $\tilde{\mathfrak{D}}_{n,m}$ satisfies these conditions.

Obviously, $\tilde{D}_{0,m} = \tilde{K}_{0,m}$ for $m \in [\![0,4]\!]$. Then we define the set $\tilde{\mathfrak{D}}_{0,m}$ by the usual basis of $\tilde{K}_{0,m}$.

Suppose $m = 0$. By (4.2.7), the dimension of $\tilde{D}_{n,0}$ is given by

$$\dim \tilde{D}_{n,0} = \begin{cases} 3, & \text{if } n = 1, \\ 1, & \text{if } n \in \mathbb{N}_0 \text{ is even}, \\ 4, & \text{if } n \geqslant 3 \text{ is odd}. \end{cases}$$

If $n = 1$, then $\tilde{D}_{1,0} = \tilde{K}_{1,0}$ since $\dim \tilde{D}_{1,0} = 3 = \dim \tilde{K}_{1,0}$. If $n \geqslant 3$ is odd, we define the set

$$\tilde{\mathfrak{D}}_{n,0} = \big\{\underline{1}|\underline{\alpha_n}, \underline{1}|\underline{\beta_n}, \underline{1}|\underline{\gamma_n}, \underline{1}|(\underline{\alpha_{n-1}\beta} + \alpha_{n-1}\gamma + \alpha_{n-2}\beta_2)\big\}.$$

If $n \geqslant 2$ is even, we define the set

$$\tilde{\mathfrak{D}}_{n,0} = \big\{\underline{1}|(\underline{\alpha_{n-1}\beta} - \alpha_{n-1}\gamma)\big\}.$$

Suppose $m = 1$. By (4.2.8), the dimension of $\tilde{D}_{n,1}$ is given by

$$\dim \tilde{D}_{n,1} = \begin{cases} 3, & \text{if } n = 0, \\ 7, & \text{if } n = 1, \\ 8, & \text{if } n = 2,3, \\ 6, & \text{if } n \geqslant 4. \end{cases}$$

We define the sets

$$\begin{aligned}
\tilde{\mathfrak{D}}_{1,1} &= \{\underline{a|\alpha},\underline{b|\beta},\underline{c|\gamma},\underline{a|\beta}+c|\alpha-c|\beta,\underline{a|\gamma}+c|\alpha,\underline{b|\alpha}-c|\alpha+c|\beta,\underline{b|\gamma}+c|\beta\} \subseteq \tilde{D}_{1,1}, \\
\tilde{\mathfrak{D}}_{2,1} &= \{\underline{a|\alpha_2},\underline{b|\beta_2},\underline{c|\gamma_2},(\underline{c}-a)|(\alpha\beta-\underline{\alpha\gamma}),(a-\underline{b})|(\underline{\alpha\beta}-\alpha\gamma),(\underline{a}+c)|\underline{\beta_2}+a|\alpha\gamma+c|\alpha\beta, \\
&\quad (\underline{a}+b)|\underline{\gamma_2}+a|\alpha\beta+b|\alpha\gamma,(\underline{b}+c)|\underline{\alpha_2}+b|\alpha\gamma+c|\alpha\beta\} \subseteq \tilde{D}_{2,1},
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathfrak{D}}_{3,1} = \{&\underline{a|\alpha_3},\underline{b|\beta_3},\underline{c|\gamma_3},\underline{b|\alpha_2\beta}+c|\alpha_2\gamma+a|\alpha\beta_2,\underline{a}|(\beta_3+\underline{\alpha_2\beta})+b|(\gamma_3+\alpha_2\gamma)+c|(\alpha_3+\alpha\beta_2), \\
&-a|(\beta_3+\gamma_3)+b|(2\alpha_3+\gamma_3-\alpha_2\gamma)+\underline{c}|(\underline{\alpha_2\beta}-\alpha\beta_2-2\alpha_3), \\
&\underline{a}|(\underline{\alpha_2\gamma}-\beta_3)+b|(\alpha_2\gamma-\alpha_3)+2c|(\alpha_3+\beta_3), \\
&2a|(\beta_3+\gamma_3)+\underline{b}|(\underline{\alpha\beta_2}-\gamma_3)+c|(\alpha\beta_2-\beta_3)\} \subseteq \tilde{D}_{3,1}.
\end{aligned}$$

Moreover, if $n \geqslant 4$ is even, we define

$$\begin{aligned}
\tilde{\mathfrak{D}}_{n,1} = \{&\underline{a|\alpha_n},\underline{b|\beta_n},\underline{c|\gamma_n},(\underline{c}-a)|(\alpha_{n-1}\beta-\underline{\alpha_{n-1}\gamma}),(a-\underline{b})|(\underline{\alpha_{n-1}\beta}-\alpha_{n-1}\gamma), \\
&(\underline{a}+b+c)|(\alpha_{n-1}\beta+\alpha_{n-1}\gamma+\alpha_{n-2}\beta_2+\alpha_n+\beta_n+\underline{\gamma_n})\} \subseteq \tilde{D}_{n,1},
\end{aligned}$$

and if $n \geqslant 5$ is odd, we set

$$\begin{aligned}
\tilde{\mathfrak{D}}_{n,1} = \{&\underline{a|\alpha_n},\underline{b|\beta_n},\underline{c|\gamma_n},\underline{b|\alpha_{n-1}\beta}+c|\alpha_{n-1}\gamma+a|\alpha_{n-2}\beta_2, \\
&\underline{a}|(\beta_n+\underline{\alpha_{n-1}\beta})+b|(\gamma_n+\alpha_{n-1}\gamma)+c|(\alpha_n+\alpha_{n-2}\beta_2), \\
&\underline{c}|(\beta_n+\underline{\alpha_{n-1}\beta})+a|(\gamma_n+\alpha_{n-1}\gamma)+b|(\alpha_n+\alpha_{n-2}\beta_2)\} \subseteq \tilde{D}_{n,1}.
\end{aligned}$$

Suppose $m = 2$. By (4.2.9), the dimension of $\tilde{D}_{n,2}$ is given by

$$\dim \tilde{D}_{n,2} = \begin{cases} 4, & \text{if } n = 0, \\ 9, & \text{if } n = 1, \\ 12, & \text{if } n \geqslant 2. \end{cases}$$

We define the sets

$$\begin{aligned}
\tilde{\mathfrak{D}}_{1,2} = \{&(\underline{ab}+ba)|\underline{\gamma},(\underline{bc}-ba-ac)|\underline{\gamma},\underline{ac}|(\underline{\alpha}+\gamma),(\underline{ba}+ac)|(\underline{\beta}+\gamma),\underline{bc|\alpha}-ac|\beta+ab|\gamma, \\
&(\underline{ab}+ba)|\underline{\beta},(\underline{bc}-ba-ac)|\underline{\beta},(bc-\underline{ba}-ac)|\underline{\alpha},(\underline{ab}+ba)|\underline{\alpha}\} \subseteq \tilde{D}_{1,2},
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathfrak{D}}_{2,2} = \{&(ba-\underline{ab})|\underline{\alpha_2},(\underline{bc}+ba+ac)|\underline{\alpha_2},(ba-\underline{ab})|\underline{\beta_2},(\underline{bc}+ba+ac)|\underline{\beta_2},(ba-\underline{ab})|\underline{\gamma_2}, \\
&(\underline{bc}+ba+ac)|\underline{\gamma_2},\underline{ab}|(\underline{\alpha\beta}-2\alpha_2-\beta_2)+bc|(\beta_2-\alpha_2),ab|(\beta_2-\gamma_2)+\underline{bc}|(\underline{\alpha\beta}-\beta_2-2\gamma_2), \\
&\underline{ba|\alpha\beta}-ab|\alpha\gamma,ab|\alpha\beta-\underline{ba|\alpha\gamma},\underline{ac|\alpha\beta}+(ab+bc)|\alpha\gamma,(ab+bc)|\alpha\beta+\underline{ac|\alpha\gamma}\} \subseteq \tilde{D}_{2,2}.
\end{aligned}$$

Moreover, if $n \geqslant 3$ is odd, we define

$$\begin{aligned}
\tilde{\mathfrak{D}}_{n,2} = \{&(\underline{ab}+ba)|\underline{\alpha_n},(\underline{bc}-ba-ac)|\underline{\alpha_n},(\underline{ab}+ba)|\underline{\beta_n},(\underline{bc}-ba-ac)|\underline{\beta_n},(\underline{ab}+ba)|\underline{\gamma_n}, \\
&(\underline{bc}-ba-ac)|\underline{\gamma_n},(\underline{ab}+ba)|(\underline{\alpha_{n-1}\beta}+\alpha_{n-2}\beta_2),(\underline{ab}+ba)|\underline{\alpha_{n-1}\gamma},(\underline{bc}-ba-ac)|\underline{\alpha_{n-2}\beta_2}, \\
&(ab+ba)|\alpha_{n-1}\beta-(\underline{bc}-ba-ac)|\underline{\alpha_{n-1}\gamma}, \\
&bc|(\alpha_n+\alpha_{n-2}\beta_2)-\underline{ac}|(\beta_n+\underline{\alpha_{n-1}\beta})+ab|(\gamma_n+\alpha_{n-1}\gamma), \\
&(\underline{bc}-ba-ac)|\underline{\alpha_{n-1}\beta}-(ab+ba)|\alpha_{n-2}\beta_2\} \subseteq \tilde{D}_{n,2},
\end{aligned}$$

and if $n \geqslant 4$ is even, we set

$$\begin{aligned}
\tilde{\mathfrak{D}}_{n,2} = \{&(ba-\underline{ab})|\underline{\alpha_n},(\underline{bc}+ba+ac)|\underline{\alpha_n},(ba-\underline{ab})|\underline{\beta_n},(\underline{bc}+ba+ac)|\underline{\beta_n},(ba-\underline{ab})|\underline{\gamma_n}, \\
&(\underline{bc}+ba+ac)|\underline{\gamma_n},(ba-\underline{ab})|\underline{\alpha_{n-2}\beta_2},(\underline{bc}+ba+ac)|\underline{\alpha_{n-2}\beta_2},\underline{ba|\alpha_{n-1}\beta}-ab|\alpha_{n-1}\gamma, \\
&\underline{ab|\alpha_{n-1}\beta}-ba|\alpha_{n-1}\gamma,\underline{ac|\alpha_{n-1}\beta}+(ab+bc)|\alpha_{n-1}\gamma,(ab+\underline{bc})|\underline{\alpha_{n-1}\beta}+ac|\alpha_{n-1}\gamma\} \\
\subseteq\ &\tilde{D}_{n,2}.
\end{aligned}$$

Suppose $m = 3$. By (4.2.10), the dimension of $\tilde{D}_{n,3}$ is given by

$$\dim \tilde{D}_{n,3} = \begin{cases} 3, & \text{if } n = 0, \\ 8, & \text{if } n = 1, \\ 13, & \text{if } n = 2 \text{ or } n \geqslant 5 \text{ is odd}, \\ 14, & \text{if } n = 3, \\ 16, & \text{if } n \geqslant 4 \text{ is even.} \end{cases}$$

We define the sets

$$\tilde{\mathfrak{D}}_{1,3} = \big\{\underline{aba|\alpha}, \underline{abc|\alpha}, \underline{aba|\beta}, \underline{bac|\beta}, \underline{abc|\gamma}, \underline{bac|\gamma}, \underline{bac|\alpha} + aba|\gamma, \underline{abc|\beta} + aba|\gamma\big\} \subseteq \tilde{D}_{1,3},$$

and

$$\begin{aligned} \tilde{\mathfrak{D}}_{3,3} = \big\{ & \underline{aba|\alpha_3}, \underline{abc|\alpha_3}, \underline{aba|\beta_3}, \underline{bac|\beta_3}, \underline{abc|\gamma_3}, \underline{bac|\gamma_3}, aba|\alpha_2\beta + \underline{bac|\alpha_2\beta}, aba|\alpha_2\beta + \underline{abc|\alpha_2\gamma}, \\ & aba|\alpha_2\beta + \underline{bac|\alpha_2\gamma}, aba|\alpha_2\beta - \underline{aba|\alpha\beta_2}, aba|\alpha_2\beta + \underline{abc|\alpha\beta_2}, \underline{abc|\alpha_2\beta} - bac|\alpha_3 + aba|\gamma_3, \\ & \underline{aba|\alpha_2\gamma} + bac|\alpha_3 + abc|\beta_3, \underline{bac|\alpha\beta_2} - abc|\beta_3 + aba|\gamma_3\big\} \subseteq \tilde{D}_{3,3}. \end{aligned}$$

Moreover, if $n \geqslant 2$ is even, let

$$\begin{aligned} \mathcal{G}_{n,3} = \big\{ & \underline{aba|\alpha_n}, \underline{abc|\alpha_n}, \underline{bac|\alpha_n}, \underline{aba|\beta_n}, \underline{abc|\beta_n}, \underline{bac|\beta_n}, \underline{aba|\gamma_n}, \underline{abc|\gamma_n}, \underline{bac|\gamma_n}, \\ & \underline{aba|\alpha_{n-1}\beta} + (abc + bac)|\alpha_{n-1}\gamma, \underline{abc}|(\underline{\alpha_{n-1}\beta} + \alpha_{n-1}\gamma), \underline{bac}|(\underline{\alpha_{n-1}\beta} + \alpha_{n-1}\gamma), \\ & \underline{aba}|(\alpha_{n-1}\beta + \underline{\alpha_{n-1}\gamma})\big\}. \end{aligned}$$

Then we define the set $\tilde{\mathfrak{D}}_{2,3} = \mathcal{G}_{2,3}$, and

$$\tilde{\mathfrak{D}}_{n,3} = \mathcal{G}_{n,3} \cup \big\{\underline{aba|\alpha_{n-2}\beta_2}, \underline{abc|\alpha_{n-2}\beta_2}, \underline{bac|\alpha_{n-2}\beta_2}\big\} \subseteq \tilde{D}_{n,3}$$

for $n \geqslant 4$ with n even. If $n \geqslant 5$ is odd, then we define

$$\begin{aligned} \tilde{\mathfrak{D}}_{n,3} = \big\{ & \underline{aba|\alpha_n}, \underline{abc|\alpha_n}, \underline{aba|\beta_n}, \underline{bac|\beta_n}, \underline{abc|\gamma_n}, \underline{bac|\gamma_n}, aba|\alpha_{n-1}\beta + \underline{bac|\alpha_{n-1}\beta}, \\ & aba|\alpha_{n-1}\beta + \underline{abc|\alpha_{n-1}\gamma}, aba|\alpha_{n-1}\beta + \underline{bac|\alpha_{n-1}\gamma}, aba|\alpha_{n-1}\beta - \underline{aba|\alpha_{n-2}\beta_2}, \\ & aba|\alpha_{n-1}\beta + \underline{abc|\alpha_{n-2}\beta_2}, abc|(\beta_n + \alpha_{n-1}\beta) + \underline{aba}|(\gamma_n + \underline{\alpha_{n-1}\gamma}), \\ & \underline{bac}|(\alpha_n + \underline{\alpha_{n-2}\beta_2}) - abc|(\beta_n + \alpha_{n-1}\beta)\big\} \subseteq \tilde{D}_{n,3}. \end{aligned}$$

Finally, if $m = 4$, we immediately see that $\tilde{D}_{n,4} = \tilde{K}_{n,4}$. So we define the set $\tilde{\mathfrak{D}}_{n,4}$ by the usual basis of $\tilde{K}_{n,4}$. The dimension of $\tilde{D}_{n,4}$ is given by

$$\dim \tilde{D}_{n,4} = \begin{cases} 1, & \text{if } n = 0, \\ 3, & \text{if } n = 1, \\ 5, & \text{if } n = 2, \\ 6, & \text{if } n \geqslant 3. \end{cases}$$

Computation of $\mathfrak{D}_{n,m}$ Recall that $D_{n,m} = \operatorname{Ker}(\partial_{n,m})$ and $\partial_{n,m} : \tilde{P}_{n,m} \to \tilde{P}_{n-1,m+1}$. The isomorphism $\tilde{P}_{n,m}/D_{n,m} \cong B_{n-1,m+1}$ tells us that

$$\dim D_{n,m} = \dim \tilde{P}_{n,m} - \dim B_{n-1,m+1}.$$

Hence, from the dimension of $B_{n-1,m+1}$ computed in Subsubsection 4.2.1.3 as well as the dimension of $\tilde{P}_{n,m}$ (see the last paragraph of Subsubsection 4.2.1.1), we deduce the value of the dimension of $D_{n,m}$. We will present them explicitly in the computations below.

For integers $(n,m) \in \mathbb{N}_0 \times [\![0,4]\!]$, we are going to provide a set $\mathfrak{D}_{n,m} \subseteq D_{n,m}$ such that $\#\mathfrak{D}_{n,m} = \dim D_{n,m}$ and the elements in $\mathfrak{D}_{n,m}$ are linearly independent. As a consequence, $\mathfrak{D}_{n,m}$ is a basis of $D_{n,m}$. We leave to the reader the easy verification in each case that the set $\mathfrak{D}_{n,m}$ satisfies these conditions.

For either $m = 0,1$ and $n \in \mathbb{N}_0$, or $m = 2,3,4$ and $n = 0,1,2,3$, note that $\partial_{n,m} = \tilde{\partial}_{n,m}$, then $D_{n,m} = \tilde{D}_{n,m}$. So we define the basis of $D_{n,m}$ by $\mathfrak{D}_{n,m} = \tilde{\mathfrak{D}}_{n,m}$.

Suppose $m = 2$. By (4.2.11), the dimension of $D_{n,2}$ is given by

$$\dim D_{n,2} = \begin{cases} 4, & \text{if } n = 0, \\ 9, & \text{if } n = 1, \\ 12, & \text{if } n = 2,3,4, \\ 15, & \text{if } n = 5, \\ 13, & \text{if } n \geqslant 6 \text{ is even}, \\ 16, & \text{if } n \geqslant 7 \text{ is odd}. \end{cases}$$

If $n = 4$, we define the set $\mathfrak{D}_{4,2} = \tilde{\mathfrak{D}}_{4,2} \subseteq D_{4,2}$. If $n \geqslant 6$ is even, we define

$$\mathfrak{D}_{n,2} = \tilde{\mathfrak{D}}_{n,2} \cup \left\{ \underline{\omega_1 1}|(\underline{\alpha_{n-5}\beta} - \alpha_{n-5}\gamma) \right\} \subseteq D_{n,2}.$$

If $n \geqslant 5$ is odd, we define $\mathfrak{D}_{n,2} = \tilde{\mathfrak{D}}_{n,2} \cup \omega_1 \tilde{\mathfrak{D}}_{n-4,0} \subseteq D_{n,2}$.

Suppose $m = 3$. By (4.2.12), the dimension of $D_{n,3}$ is given by

$$\dim D_{n,3} = \begin{cases} 3, & \text{if } n = 0, \\ 8, & \text{if } n = 1, \\ 13, & \text{if } n = 2, \\ 14, & \text{if } n = 3, \\ 19, & \text{if } n = 4 \text{ or } n \geqslant 9 \text{ is odd}, \\ 20, & \text{if } n = 5, \\ 24, & \text{if } n = 6, \\ 21, & \text{if } n = 7, \\ 22, & \text{if } n \geqslant 8 \text{ is even}. \end{cases}$$

We define the sets

$$\begin{aligned} \mathfrak{D}_{5,3} = \tilde{\mathfrak{D}}_{5,3} \cup \big\{ & 5bac|\alpha_5 + 2abc|\beta_5 - 5aba|\gamma_5 - 3abc|\alpha_4\beta + \underline{\omega_1 a|\alpha}, \\ & 5bac|\alpha_5 + 2abc|\beta_5 - 5aba|\gamma_5 - 3abc|\alpha_4\beta + \underline{\omega_1 b|\beta}, \\ & 5bac|\alpha_5 + 2abc|\beta_5 - 5aba|\gamma_5 - 3abc|\alpha_4\beta + \underline{\omega_1 c|\gamma}, \underline{\omega_1}(\underline{a|\beta} + c|\alpha - c|\beta), \\ & \underline{\omega_1}(\underline{a|\gamma} + c|\alpha), \underline{\omega_1}(\underline{b|\alpha} - c|\alpha + c|\beta), \underline{\omega_1}(\underline{b|\gamma} + c|\beta) \big\} \subseteq D_{5,3}, \end{aligned}$$

and

$$\begin{aligned} \mathfrak{D}_{7,3} = \tilde{\mathfrak{D}}_{7,3} \cup \big\{ & 5bac|\alpha_7 + 2abc|\beta_7 - 5aba|\gamma_7 - 3abc|\alpha_6\beta + \underline{\omega_1 a|\alpha_3}, \\ & 5bac|\alpha_7 + 2abc|\beta_7 - 5aba|\gamma_7 - 3abc|\alpha_6\beta + \underline{\omega_1 b|\beta_3}, \\ & 5bac|\alpha_7 + 2abc|\beta_7 - 5aba|\gamma_7 - 3abc|\alpha_6\beta + \underline{\omega_1 c|\gamma_3}, \\ & 6bac|\alpha_7 + 6abc|\beta_7 - 6aba|\gamma_7 + \underline{\omega_1}(\underline{b|\alpha_2\beta} + c|\alpha_2\gamma + a|\alpha\beta_2), \\ & \underline{\omega_1}[\underline{a}|(\beta_3 + \underline{\alpha_2\beta}) + b|(\gamma_3 + \alpha_2\gamma) + c|(\alpha_3 + \alpha\beta_2)], \\ & \underline{\omega_1}[-a|(\beta_3 + \gamma_3) + b|(2\alpha_3 + \gamma_3 - \alpha_2\gamma) + \underline{c}|(\underline{\alpha_2\beta} - \alpha\beta_2 - 2\alpha_3)], \\ & \underline{\omega_1}[\underline{a}|(\underline{\alpha_2\gamma} - \beta_3) + b|(\alpha_2\gamma - \alpha_3) + 2c|(\alpha_3 + \beta_3)], \\ & \underline{\omega_1}[2a|(\beta_3 + \gamma_3) + \underline{b}|(\underline{\alpha\beta_2} - \gamma_3) + c|(\alpha\beta_2 - \beta_3)] \big\} \subseteq D_{7,3}. \end{aligned}$$

Moreover, if $n \geqslant 4$ is even, we define the set $\mathfrak{D}_{n,3} = \tilde{\mathfrak{D}}_{n,3} \cup \omega_1 \tilde{\mathfrak{D}}_{n-4,1} \subseteq D_{n,3}$, and if $n \geqslant 9$ is odd, we define the set

$$\begin{aligned} \mathfrak{D}_{n,3} = \tilde{\mathfrak{D}}_{n,3} \cup \big\{ & 5bac|\alpha_n + 2abc|\beta_n - 5aba|\gamma_n - 3abc|\alpha_{n-1}\beta + \underline{\omega_1 a|\alpha_{n-4}}, \\ & 5bac|\alpha_n + 2abc|\beta_n - 5aba|\gamma_n - 3abc|\alpha_{n-1}\beta + \underline{\omega_1 b|\beta_{n-4}}, \\ & 5bac|\alpha_n + 2abc|\beta_n - 5aba|\gamma_n - 3abc|\alpha_{n-1}\beta + \underline{\omega_1 c|\gamma_{n-4}}, \\ & 3(n-5)(bac|\alpha_n + abc|\beta_n - aba|\gamma_n) + \underline{\omega_1}(\underline{b|\alpha_{n-5}\beta} + c|\alpha_{n-5}\gamma + a|\alpha_{n-6}\beta_2), \\ & \underline{\omega_1}[\underline{a}|(\beta_{n-4} + \underline{\alpha_{n-5}\beta}) + b|(\gamma_{n-4} + \alpha_{n-5}\gamma) + c|(\alpha_{n-5} + \alpha_{n-6}\beta_2)], \\ & \underline{\omega_1}[\underline{c}|(\beta_{n-4} + \underline{\alpha_{n-5}\beta}) + a|(\gamma_{n-4} + \alpha_{n-5}\gamma) + b|(\alpha_{n-4} + \alpha_{n-6}\beta_2)] \big\} \subseteq D_{n,3}. \end{aligned}$$

Suppose $m = 4$. The space $D_{n,4}$ is given by Proposition 4.2.2. So $\mathfrak{D}_{n,4}$ is given by the usual basis of $\tilde{K}_{n,4}$ and $\omega_1\mathfrak{D}_{n-4,2}$. The dimension of $D_{n,4}$ is then given by

$$\dim D_{n,4} = \begin{cases} 1, & \text{if } n = 0, \\ 3, & \text{if } n = 1, \\ 5, & \text{if } n = 2, \\ 6, & \text{if } n = 3, \\ 10, & \text{if } n = 4, \\ 15, & \text{if } n = 5, \\ 18, & \text{if } n = 6,7,8, \\ 21, & \text{if } n = 9, \\ 19, & \text{if } n \geqslant 10 \text{ is even}, \\ 22, & \text{if } n \geqslant 11 \text{ is odd}. \end{cases}$$

4.2.1.5 Hochschild homology

In this subsubsection, we will explicitly construct a subspace $H_{n,m}$ of $D_{n,m}$ such that $D_{n,m} = H_{n,m} \oplus B_{n,m}$ for $m,n \in \mathbb{N}_0$, and we define $H_{n,m} = 0$ for $(n,m) \in \mathbb{Z}^2 \setminus \mathbb{N}_0^2$. By Proposition 4.2.1, we have the following similar recursive description.

Corollary 4.2.3. *For integers $m \geqslant 3$ and $n \in \mathbb{N}_0$, we have*

$$H_{n,m} \cong \begin{cases} \omega_{\frac{m-3}{2}} H_{n-2m+6,3}, & \text{if } m \text{ is odd}, \\ \omega_{\frac{m}{2}-2} H_{n-2m+8,4}, & \text{if } m \text{ is even}. \end{cases}$$

So it is also sufficient to compute the case $m \in [\![0,4]\!]$. Recall that

$$\dim H_{n,m} = \dim D_{n,m} - \dim B_{n,m} = \dim \tilde{P}_{n,m} - \dim B_{n-1,m+1} - \dim B_{n,m}.$$

Hence, from the dimension of $D_{n,m}$ computed in Subsubsection 4.2.1.4 as well as the dimension of $B_{n,m}$ computed in Subsubsection 4.2.1.3, we deduce the value of the dimension of $H_{n,m}$. We will present them explicitly in the computations below.

For every $(n,m) \in \mathbb{N}_0 \times [\![0,4]\!]$, we are going to provide a set $\mathfrak{H}_{n,m} \subseteq D_{n,m}$ such that $\#\mathfrak{H}_{n,m} = \dim H_{n,m}$ and the elements in $\mathfrak{H}_{n,m} \cup \mathfrak{B}_{n,m}$ are linearly independent. As a consequence, the space $H_{n,m}$ spanned by $\mathfrak{H}_{n,m}$ satisfies $D_{n,m} = H_{n,m} \oplus B_{n,m}$. We leave to the reader the easy verification in each case that the set $\mathfrak{H}_{n,m}$ satisfies these conditions. Note that, unless stated otherwise, the linear independence of the elements in $\mathfrak{H}_{n,m} \cup \mathfrak{B}_{n,m}$ is from Fact 4.1.3, where we put the elements in $\mathfrak{H}_{n,m}$ before the elements in $\mathfrak{B}_{n,m}$.

Suppose $m = 0$. We get immediately $\mathfrak{H}_{n,0} = \mathfrak{D}_{n,0}$ since $B_{n,0} = 0$ for $n \in \mathbb{N}_0$. The dimension of $H_{n,0}$ is given by

$$\dim H_{n,0} = \begin{cases} 1, & \text{if } n \in \mathbb{N}_0 \text{ is even}, \\ 3, & \text{if } n = 1, \\ 4, & \text{if } n \geqslant 3 \text{ is odd}. \end{cases}$$

Suppose $m = 1$. The dimension of $H_{n,1}$ is given by

$$\dim H_{n,1} = \begin{cases} 3, & \text{if } n = 0,1,3, \\ 6, & \text{if } n = 2, \\ 4, & \text{if } n \geqslant 4 \text{ is even}, \\ 1, & \text{if } n \geqslant 5 \text{ is odd}. \end{cases}$$

We define the sets $\mathfrak{H}_{0,1} = \mathfrak{D}_{0,1}$,

$$\mathfrak{H}_{1,1} = \big\{\underline{a|\gamma} + c|\alpha, \underline{b|\alpha} - c|\alpha + c|\beta, \underline{b|\gamma} + c|\beta\big\},$$

$$\begin{aligned} \mathfrak{H}_{2,1} = \mathfrak{D}_{2,1} \backslash \mathfrak{B}_{2,1} = \big\{& \underline{a|\alpha_2}, \underline{b|\beta_2}, \underline{c|\gamma_2}, \underline{a}|(\underline{\beta_2} + \alpha\gamma) + c|(\beta_2 + \alpha\beta), \underline{a}|(\underline{\gamma_2} + \alpha\beta) + b|(\gamma_2 + \alpha\gamma), \\ & \underline{b}|(\underline{\alpha_2} + \alpha\gamma) + c|(\alpha_2 + \alpha\beta)\big\}, \end{aligned}$$

and

$$\begin{aligned}\mathfrak{H}_{3,1} = \big\{&\underline{a}|(\beta_3+\underline{\alpha_2\beta})+b|(\gamma_3+\alpha_2\gamma)+c|(\alpha_3+\alpha\beta_2),\\ &\underline{a}|(\underline{\alpha_2\gamma}-\beta_3)+b|(\alpha_2\gamma-\alpha_3)+2c|(\alpha_3+\beta_3),\\ &2a|(\beta_3+\gamma_3)+\underline{b}|(\underline{\alpha\beta_2}-\gamma_3)+c|(\alpha\beta_2-\beta_3)\big\}.\end{aligned}$$

Moreover, if $n \geqslant 4$ is even, we define the set

$$\mathfrak{H}_{n,1} = \mathfrak{D}_{n,1}\backslash\mathfrak{B}_{n,1} = \big\{\underline{a|\alpha_n}, \underline{b|\beta_n}, \underline{c|\gamma_n}, (\underline{a}+b+c)|(\alpha_{n-1}\beta+\alpha_{n-1}\gamma+\alpha_{n-2}\beta_2+\alpha_n+\beta_n+\underline{\gamma_n})\big\},$$

and if $n \geqslant 5$ is odd, we define

$$\mathfrak{H}_{n,1} = \big\{\underline{a}|(\beta_n+\underline{\alpha_{n-1}\beta})+b|(\gamma_n+\alpha_{n-1}\gamma)+c|(\alpha_n+\alpha_{n-2}\beta_2)\big\}.$$

Suppose $m = 2$. The dimension of $H_{n,2}$ is given by

$$\dim H_{n,2} = \begin{cases} 2, & \text{if } n = 0,1,2,\\ 0, & \text{if } n = 3,4,\\ 3, & \text{if } n = 5,\\ 1, & \text{if } n \geqslant 6 \text{ is even},\\ 4, & \text{if } n \geqslant 7 \text{ is odd}.\end{cases}$$

We define the sets

$$\mathfrak{H}_{0,2} = \big\{\underline{ab|\epsilon^!}, \underline{bc|\epsilon^!}\big\}, \quad \mathfrak{H}_{1,2} = \big\{(\underline{ba}+ac)|(\underline{\beta}+\gamma), \underline{ac}|(\underline{\alpha}+\gamma)\big\},$$

$$\mathfrak{H}_{2,2} = \big\{\underline{ab}|(\underline{\beta_2}-\gamma_2)+bc|(\alpha\beta-\beta_2-2\gamma_2), \underline{ab}|(\alpha\beta-2\underline{\alpha_2}-\beta_2)+bc|(\beta_2-\alpha_2)\big\},$$

and $\mathfrak{H}_{3,2} = \mathfrak{H}_{4,2} = \emptyset$. Moreover, if $n \geqslant 5$ is odd, we define the set $\mathfrak{H}_{n,2} = \omega_1\mathfrak{D}_{n-4,0}$, and if $n \geqslant 6$ is even, we define

$$\mathfrak{H}_{n,2} = \big\{\underline{\omega_1 1}|(\underline{\alpha_{n-5}\beta}-\alpha_{n-5}\gamma)\big\}.$$

Suppose $m = 3$. The dimension of $H_{n,3}$ is given by

$$\dim H_{n,3} = \begin{cases} 0, & \text{if } n = 0,1,\\ 1, & \text{if } n = 2,3,\\ 7, & \text{if } n = 4,\\ 4, & \text{if } n = 5,7,\\ 10, & \text{if } n = 6,\\ 8, & \text{if } n \geqslant 8 \text{ is even},\\ 2, & \text{if } n \geqslant 9 \text{ is odd}.\end{cases}$$

We define the sets $\mathfrak{H}_{0,3} = \mathfrak{H}_{1,3} = \emptyset$,

$$\mathfrak{H}_{2,3} = \big\{\underline{bac|\alpha_2}\big\}, \quad \mathfrak{H}_{3,3} = \big\{aba|\alpha_2\beta+bac|\alpha_2\beta\big\},$$

$$\mathfrak{H}_{4,3} = \big\{\underline{bac|\alpha_4}, \underline{aba|\alpha_2\beta_2}, \underline{abc|\alpha_2\beta_2}, \underline{bac|\alpha_2\beta_2}, \underline{\omega_1 a|\epsilon^!}, \underline{\omega_1 b|\epsilon^!}, \underline{\omega_1 c|\epsilon^!}\big\},$$

$$\mathfrak{H}_{5,3} = \big\{(aba+bac)|\alpha_4\beta, \underline{\omega_1}(\underline{a|\gamma}+c|\alpha), \underline{\omega_1}(b|\alpha-c|\alpha+c|\beta), \underline{\omega_1}(\underline{b|\gamma}+c|\beta)\big\},$$

$$\begin{aligned}\mathfrak{H}_{6,3} = \big\{&\underline{bac|\alpha_6}, \underline{aba|\alpha_4\beta_2}, \underline{abc|\alpha_4\beta_2}, \underline{bac|\alpha_4\beta_2}, \underline{\omega_1 a|\alpha_2}, \underline{\omega_1 b|\beta_2}, \underline{\omega_1 c|\gamma_2},\\ &\underline{\omega_1}[\underline{a}|(\underline{\beta_2}+\alpha\gamma)+c|(\beta_2+\alpha\beta)], \underline{\omega_1}[\underline{a}|(\underline{\gamma_2}+\alpha\beta)+b|(\gamma_2+\alpha\gamma)],\\ &\underline{\omega_1}[\underline{b}|(\underline{\alpha_2}+\alpha\gamma)+c|(\alpha_2+\alpha\beta)]\big\},\end{aligned}$$

and

$$\begin{aligned}\mathfrak{H}_{7,3} = \big\{&(aba+bac)|\alpha_6\beta, \underline{\omega_1}[\underline{a}|(\beta_3+\underline{\alpha_2\beta})+b|(\gamma_3+\alpha_2\gamma)+c|(\alpha_3+\alpha\beta_2)],\\ &\underline{\omega_1}[\underline{a}|(\underline{\alpha_2\gamma}-\beta_3)+b|(\alpha_2\gamma-\alpha_3)+2c|(\alpha_3+\beta_3)],\\ &\underline{\omega_1}[2a|(\beta_3+\gamma_3)+\underline{b}|(\underline{\alpha\beta_2}-\gamma_3)+c|(\alpha\beta_2-\beta_3)]\big\}.\end{aligned}$$

Moreover, if $n \geqslant 8$ is even, we define

$$\begin{aligned}\mathfrak{H}_{n,3} = \big\{&\underline{bac|\alpha_n}, \underline{aba|\alpha_{n-2}\beta_2}, \underline{abc|\alpha_{n-2}\beta_2}, \underline{bac|\alpha_{n-2}\beta_2}, \underline{\omega_1 a|\alpha_{n-4}}, \underline{\omega_1 b|\beta_{n-4}}, \underline{\omega_1 c|\gamma_{n-4}},\\ &\underline{\omega_1}(\underline{a}+b+c)|(\alpha_{n-5}\beta+\alpha_{n-5}\gamma+\alpha_{n-6}\beta_2+\alpha_{n-4}+\beta_{n-4}+\underline{\gamma_{n-4}})\big\},\end{aligned}$$

and if $n \geqslant 9$ is odd, we define

$$\mathfrak{H}_{n,3} = \big\{(aba+bac)|\alpha_{n-1}\beta, \underline{\omega_1}[\underline{a}|(\beta_{n-4}+\underline{\alpha_{n-5}\beta}) + b|(\gamma_{n-4}+\alpha_{n-5}\gamma) + c|(\alpha_{n-4}+\alpha_{n-6}\beta_2)]\big\}.$$

Moreover, the set $\mathfrak{H}_{n,3} \cup \mathfrak{B}_{n,3}$ for $n \geqslant 3$ and n odd is linearly independent. Indeed, Fact 4.1.3 tells us that the elements containing underlined terms do form a linearly independent set. It is then easy to prove that the elements of $\mathfrak{H}_{n,3} \cup \mathfrak{B}_{n,3}$ without any underlining are not a linear combination of the remaining elements, proving the claim.

Suppose $m = 4$. The dimension of $H_{n,4}$ is given by

$$\dim H_{n,4} = \begin{cases} 0, & \text{if } n = 0,\\ 1, & \text{if } n = 1,2,8,\\ 4, & \text{if } n = 3,7,\\ 3, & \text{if } n = 4,6,\\ 6, & \text{if } n = 5,\\ 7, & \text{if } n = 9,\\ 2, & \text{if } n \geqslant 10 \text{ is even},\\ 8, & \text{if } n \geqslant 11 \text{ is odd}. \end{cases}$$

We define the sets $\mathfrak{H}_{0,4} = \emptyset$,

$$\mathfrak{H}_{1,4} = \big\{\underline{abac|\alpha}\big\}, \quad \mathfrak{H}_{2,4} = \big\{\underline{abac|\alpha\beta}\big\},$$

$$\mathfrak{H}_{3,4} = \big\{\underline{abac|\alpha_3}, \underline{abac|\alpha_2\beta}, \underline{abac|\alpha_2\gamma}, \underline{abac|\alpha\beta_2}\big\}, \quad \mathfrak{H}_{4,4} = \big\{\underline{abac|\alpha_3\beta}, \underline{\omega_1 ab|\epsilon^!}, \underline{\omega_1 bc|\epsilon^!}\big\},$$

$$\mathfrak{H}_{5,4} = \big\{\underline{abac|\alpha_5}, \underline{abac|\alpha_4\beta}, \underline{abac|\alpha_4\gamma}, \underline{abac|\alpha_3\beta_2}, \underline{\omega_1}(\underline{ba}+ac)|(\underline{\beta}+\gamma), \underline{\omega_1 ac}|(\underline{\alpha}+\gamma)\big\},$$

$$\begin{aligned}\mathfrak{H}_{6,4} = \big\{&\underline{abac|\alpha_5\beta}, \underline{\omega_1}[\underline{ab}|(\underline{\beta_2}-\gamma_2)+bc|(\alpha\beta-\beta_2-2\gamma_2)],\\ &\underline{\omega_1}[\underline{ab}|(\alpha\beta-2\underline{\alpha_2}-\beta_2)+bc|(\beta_2-\alpha_2)]\big\},\end{aligned}$$

$$\mathfrak{H}_{7,4} = \big\{\underline{abac|\alpha_7}, \underline{abac|\alpha_6\beta}, \underline{abac|\alpha_6\gamma}, \underline{abac|\alpha_5\beta_2}\big\},$$

and

$$\mathfrak{H}_{8,4} = \big\{\underline{abac|\alpha_7\beta}\big\}.$$

Moreover, if $n \geqslant 9$ is odd, we define the set

$$\mathfrak{H}_{n,4} = \big\{\underline{abac|\alpha_n}, \underline{abac|\alpha_{n-1}\beta}, \underline{abac|\alpha_{n-1}\gamma}, \underline{abac|\alpha_{n-2}\beta_2}\big\} \cup \omega_2\mathfrak{D}_{n-8,0},$$

and if $n \geqslant 10$ is even, we define

$$\mathfrak{H}_{n,4} = \big\{\underline{abac|\alpha_{n-1}\beta}, \underline{\omega_2 1}|(\underline{\alpha_{n-9}\beta}-\alpha_{n-9}\gamma)\big\}.$$

The previous results can be restated as follows.

Corollary 4.2.4. *Let $m \in [\![0,4]\!]$ and $n \in \mathbb{N}_0$. Then $H_{n,m} = \tilde{H}_{n,m} \oplus \omega_1 H_{n-4,m-2}$ except for $(n,m) = (4,2)$. Moreover, $H_{4,2} = \tilde{H}_{4,2} = 0$. Here, $\tilde{H}_{n,m}$ is the $\Bbbk$-vector space spanned by the set $\tilde{\mathfrak{H}}_{n,m}$. The set $\tilde{\mathfrak{H}}_{n,m}$ is defined as follows. If $m = 0$ or 1, we define the set $\tilde{\mathfrak{H}}_{n,m} = \mathfrak{H}_{n,m}$ for $n \in \mathbb{N}_0$. If $m = 2$, we define the sets*

$$\begin{aligned}&\tilde{\mathfrak{H}}_{0,2} = \big\{ab|\epsilon^!, bc|\epsilon^!\big\}, \quad \tilde{\mathfrak{H}}_{1,2} = \big\{(ba+ac)|(\beta+\gamma), ac|(\alpha+\gamma)\big\},\\ &\tilde{\mathfrak{H}}_{2,2} = \big\{ab|(\alpha\beta-2\alpha_2-\beta_2)+bc|(\beta_2-\alpha_2), ab|(\beta_2-\gamma_2)+bc|(\alpha\beta-\beta_2-2\gamma_2)\big\},\end{aligned}$$

and $\tilde{\mathfrak{H}}_{n,2} = \emptyset$ for $n \geqslant 3$. If $m = 3$, we define the set $\tilde{\mathfrak{H}}_{0,3} = \emptyset$, and

$$\tilde{\mathfrak{H}}_{n,3} = \big\{(aba+bac)|\alpha_{n-1}\beta\big\}$$

for $n \in \mathbb{N}$ with n odd, together with

$$\tilde{\mathfrak{H}}_{n,3} = \{bac|\alpha_n, aba|\alpha_{n-2}\beta_2, abc|\alpha_{n-2}\beta_2, bac|\alpha_{n-2}\beta_2\}$$

for $n \geqslant 2$ with n even. If $m = 4$, we define the set $\tilde{\mathfrak{H}}_{0,4} = \emptyset$, and

$$\tilde{\mathfrak{H}}_{n,4} = \{abac|\alpha_n, abac|\alpha_{n-1}\beta, abac|\alpha_{n-1}\gamma, \alpha_{n-2}\beta_2\}$$

for $n \in \mathbb{N}$ with n odd, together with

$$\tilde{\mathfrak{H}}_{n,4} = \{abac|\alpha_{n-1}\beta\}$$

for $n \geqslant 2$ with n even. Furthermore, if we define $\tilde{H}_{n,m} = 0$ for $(n,m) \in \mathbb{Z}^2 \setminus (\mathbb{N}_0 \times [\![0,4]\!])$, then $H_{n,m} = \tilde{H}_{n,m} \oplus \omega_1 H_{n-4,m-2}$ holds for $(n,m) \in \mathbb{Z}^2 \setminus \{(4,2)\}$ by applying Corollary 4.2.3.

Remark 4.2.5. *The reader can easily check that $\tilde{D}_{n,m} = \tilde{H}_{n,m} \oplus \tilde{B}_{n,m}$ except the case $m = n = 3$.*

Recall that the Hochschild homology is decomposed as $\mathrm{HH}_n(A) = \oplus_{m\in\mathbb{N}_0} H_{n,m}$ for $n \in \mathbb{N}_0$.

Proposition 4.2.6. *Let $n \in \mathbb{N}$. Then*

$$\mathrm{HH}_n(A) = \bigoplus_{\substack{i\in[\![0,\lfloor n/4\rfloor]\!],\\ m\in[\![0,4]\!]}} \omega_i \tilde{H}_{n-4i,m}$$

for $4 \nmid n$, and

$$\mathrm{HH}_n(A) = \Bigg(\bigoplus_{\substack{i\in[\![0,n/4-1]\!],\\ m\in[\![0,4]\!]}} \omega_i \tilde{H}_{n-4i,m}\Bigg) \oplus \Bigg(\bigoplus_{m\in[\![1,4]\!]} \omega_{n/4}\tilde{H}_{0,m}\Bigg)$$

for $4|n$.

Proof. By Corollary 4.2.4, we have

$$\begin{aligned}
&H_{n,2} = \tilde{H}_{n,2} \oplus \omega_1 \tilde{H}_{n-4,0} \text{ for } n \in \mathbb{N}_0 \setminus \{4\}, \quad H_{4,2} = \tilde{H}_{4,2},\\
&H_{n,3} = \tilde{H}_{n,3} \oplus \omega_1 \tilde{H}_{n-4,1} \text{ for } n \in \mathbb{N}_0,\\
&H_{n,4} = \tilde{H}_{n,4} \oplus \omega_1 \tilde{H}_{n-4,2} \oplus \omega_2 \tilde{H}_{n-8,0} \text{ for } n \in \mathbb{N}_0 \setminus \{8\}, \quad H_{8,4} = \tilde{H}_{8,4} \oplus \omega_1 \tilde{H}_{4,2}.
\end{aligned} \tag{4.2.13}$$

If $4 \nmid n$, using Corollary 4.2.3 and (4.2.13), we get

$$\begin{aligned}
\mathrm{HH}_n(A) &= \bigoplus_{m\in[\![0,2\lfloor n/4\rfloor+4]\!]} H_{n,m}\\
&= H_{n,0} \oplus H_{n,1} \oplus H_{n,2} \oplus \Bigg(\bigoplus_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_i H_{n-4i,3}\Bigg) \oplus \Bigg(\bigoplus_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_i H_{n-4i,4}\Bigg)\\
&= \tilde{H}_{n,0} \oplus \tilde{H}_{n,1} \oplus (\tilde{H}_{n,2} \oplus \omega_1 \tilde{H}_{n-4,0}) \oplus \Bigg(\bigoplus_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_i(\tilde{H}_{n-4i,3} \oplus \omega_1 \tilde{H}_{n-4i-4,1})\Bigg)\\
&\quad \oplus \Bigg(\bigoplus_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_i(\tilde{H}_{n-4i,4} \oplus \omega_1 \tilde{H}_{n-4i-4,2} \oplus \omega_2 \tilde{H}_{n-4i-8,0})\Bigg)\\
&= \tilde{H}_{n,0} \oplus \tilde{H}_{n,1} \oplus \tilde{H}_{n,2} \oplus \omega_1 \tilde{H}_{n-4,0} \oplus \Bigg(\bigoplus_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_i \tilde{H}_{n-4i,3}\Bigg)\\
&\quad \oplus \Bigg(\bigoplus_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_{i+1} \tilde{H}_{n-4i-4,1}\Bigg) \oplus \Bigg(\bigoplus_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_i \tilde{H}_{n-4i,4}\Bigg)\\
&\quad \oplus \Bigg(\bigoplus_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_{i+1} \tilde{H}_{n-4i-4,2}\Bigg) \oplus \Bigg(\bigoplus_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_{i+2} \tilde{H}_{n-4i-8,0}\Bigg)\\
&= \tilde{H}_{n,0} \oplus \tilde{H}_{n,1} \oplus \tilde{H}_{n,2} \oplus \omega_1 \tilde{H}_{n-4,0} \oplus \Bigg(\bigoplus_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_i \tilde{H}_{n-4i,3}\Bigg) \oplus \Bigg(\bigoplus_{i\in[\![1,\lfloor n/4\rfloor]\!]} \omega_i \tilde{H}_{n-4i,1}\Bigg)
\end{aligned}$$

$$\oplus\Big(\bigoplus_{i\in[\![0,\lfloor n/4\rfloor]\!]}\omega_i\tilde{H}_{n-4i,4}\Big)\oplus\Big(\bigoplus_{i\in[\![1,\lfloor n/4\rfloor]\!]}\omega_i\tilde{H}_{n-4i,2}\Big)\oplus\Big(\bigoplus_{i\in[\![2,\lfloor n/4\rfloor]\!]}\omega_i\tilde{H}_{n-4i,0}\Big)$$

$$=\bigoplus_{\substack{i\in[\![0,\lfloor n/4\rfloor]\!],\\ m\in[\![0,4]\!]}}\omega_i\tilde{H}_{n-4i,m}.$$

If $4|n$, the proof is similar to above. Note that if $n = 4$, there is no term $\omega_1\tilde{H}_{0,0}$ when decomposing $H_{4,2}$, and if $n \geqslant 8$, there is no term $\omega_{n/4}\tilde{H}_{0,0}$ when decomposing $\omega_{n/4-2}H_{8,4}$. □

Here is a table of the dimensions of $H_{n,m}$ and $\mathrm{HH}_n(A)$ for $n \in [\![0, 19]\!]$ and $m \in [\![0, 12]\!]$.

m \ n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	1	3	1	4	1	4	1	4	1	4	1	4	1	4	1	4	1	4	1	4
1	3	3	6	3	4	1	4	1	4	1	4	1	4	1	4	1	4	1	4	1
2	2	2	2	0	0	3	1	4	1	4	1	4	1	4	1	4	1	4	1	4
3	0	0	1	1	7	4	10	4	8	2	8	2	8	2	8	2	8	2	8	2
4	0	1	1	4	3	6	3	4	1	7	2	8	2	8	2	8	2	8	2	8
5					0	0	1	1	7	4	10	4	8	2	8	2	8	2	8	2
6					0	1	1	4	3	6	3	4	1	7	2	8	2	8	2	8
7									0	0	1	1	7	4	10	4	8	2	8	2
8									0	1	1	4	3	6	3	4	1	7	2	8
9													0	0	1	1	7	4	10	4
10													0	1	1	4	3	6	3	4
11																	0	0	1	1
12																	0	1	1	4
HH_n	6	9	11	12	15	19	21	22	25	29	31	32	35	39	41	42	45	49	51	52

Table 4.2.10: Dimension of $H_{n,m}$ and $\mathrm{HH}_n(A)$.

Proposition 4.2.7. *The dimension of* $\mathrm{HH}_n(A)$ *is given by*

$$\dim\mathrm{HH}_n(A)=\begin{cases}6, & \text{if } n=0,\\ \frac{5}{2}n+5, & \text{if } n=4r \text{ for } r\in\mathbb{N},\\ \frac{5n+13}{2}, & \text{if } n=4r+1 \text{ for } r\in\mathbb{N}_0,\\ \frac{5}{2}n+6, & \text{if } n=4r+2 \text{ for } r\in\mathbb{N}_0,\\ \frac{5n+9}{2}, & \text{if } n=4r+3 \text{ for } r\in\mathbb{N}_0.\end{cases}$$

The Hilbert series of $\mathrm{HH}_n(A)$ is $h_n(t) = \sum_{m\in\mathbb{N}_0}\dim(H_{n,m})t^{m+n}$ for $n \in \mathbb{N}_0$. Note that $m+n$ is the internal degree of $H_{n,m}$.

Corollary 4.2.8. *The Hilbert series* $h_n(t)$ *of* $\mathrm{HH}_n(A)$ *is given as follows. Let* $n \geqslant 6$. *Then*

$$h_n(t)=t^n\Big[1+3\chi_{n+1}+(3\chi_n+1)t+(1+3\chi_{n+1})t^2+\sum_{i=0}^{\mu_n}\big((2+6\chi_n)t^{3+2i}+(2+6\chi_{n+1})t^{4+2i}\big)+p_n(t)\Big],$$

where

$$p_n(t)=\begin{cases}8t^{2\lfloor\frac{n}{4}\rfloor-1}+t^{2\lfloor\frac{n}{4}\rfloor}+7t^{2\lfloor\frac{n}{4}\rfloor+1}+3t^{2\lfloor\frac{n}{4}\rfloor+2}, & \text{if } n\equiv 0 \pmod 4,\\ 2t^{2\lfloor\frac{n}{4}\rfloor-1}+7t^{2\lfloor\frac{n}{4}\rfloor}+4t^{2\lfloor\frac{n}{4}\rfloor+1}+6t^{2\lfloor\frac{n}{4}\rfloor+2}+t^{2\lfloor\frac{n}{4}\rfloor+4}, & \text{if } n\equiv 1 \pmod 4,\\ 10t^{2\lfloor\frac{n}{4}\rfloor+1}+3t^{2\lfloor\frac{n}{4}\rfloor+2}+t^{2\lfloor\frac{n}{4}\rfloor+3}+t^{2\lfloor\frac{n}{4}\rfloor+4}, & \text{if } n\equiv 2 \pmod 4,\\ 4t^{2\lfloor\frac{n}{4}\rfloor+1}+4t^{2\lfloor\frac{n}{4}\rfloor+2}+t^{2\lfloor\frac{n}{4}\rfloor+3}+4t^{2\lfloor\frac{n}{4}\rfloor+4}, & \text{if } n\equiv 3 \pmod 4,\end{cases}$$

and

$$\mu_n=\begin{cases}\lfloor\frac{n}{4}\rfloor-3, & \text{if } n\equiv 0{,}1 \pmod 4,\\ \lfloor\frac{n}{4}\rfloor-2, & \text{if } n\equiv 2{,}3 \pmod 4.\end{cases}$$

Moreover,

$$h_0(t)=1+3t+2t^2, \qquad h_1(t)=3t+3t^2+2t^3+t^5,$$
$$h_2(t)=t^2+6t^3+2t^4+t^5+t^6, \qquad h_3(t)=4t^3+3t^4+t^6+4t^7,$$
$$h_4(t)=t^4+4t^5+7t^7+3t^8, \qquad h_5(t)=4t^5+t^6+3t^7+4t^8+6t^9+t^{11}.$$

Remark 4.2.9. *As we mentioned at the beginning of Subsubsection 4.2.1.4, one can obtain Proposition 4.2.7 and Corollary 4.2.8 directly from the computations in Subsubsection 4.2.1.3 together with Corollary 4.2.3, but a specific choice of cycles for the Hochschild homology can be useful for later computations.*

4.2.1.6 Cyclic homology

In this subsubsection, we assume that the characteristic of the field $\Bbbk$ is zero. Recall that the **reduced Hochschild homology** of A is given by

$$\overline{\mathrm{HH}}_n(A) = \begin{cases} \mathrm{HH}_0(A)/\Bbbk, & \text{if } n = 0, \\ \mathrm{HH}_n(A), & \text{if } n \in \mathbb{N}, \end{cases}$$

and the **reduced cyclic homology** of A is given by

$$\overline{\mathrm{HC}}_n(A) = \begin{cases} \mathrm{HC}_n(A)/\Bbbk, & \text{if } n \in \mathbb{N}_0 \text{ is even}, \\ \mathrm{HC}_n(A), & \text{if } n \in \mathbb{N} \text{ is odd}, \end{cases}$$

where $\mathrm{HC}_n(A)$ for $n \in \mathbb{N}_0$ is the cyclic homology of A (see [16]). As a consequence of Goodwillie's Theorem (see [29], Thm. 9.9.1), we have the isomorphism of graded vector spaces

$$\overline{\mathrm{HC}}_n(A) \cong \begin{cases} \overline{\mathrm{HH}}_0(A), & \text{if } n = 0, \\ \overline{\mathrm{HH}}_n(A)/\overline{\mathrm{HC}}_{n-1}(A), & \text{if } n \in \mathbb{N}. \end{cases} \tag{4.2.14}$$

Corollary 4.2.10. *Assume that the characteristic of $\Bbbk$ is zero. Let $g_n(t)$ be the Hilbert series of $\overline{\mathrm{HC}}_n(A)$ for $n \in \mathbb{N}_0$. Then*

$$g_0(t) = 3t + 2t^2, \quad g_1(t) = t^2 + 2t^3 + t^5, \quad g_2(t) = 4t^3 + 2t^4 + t^6, \quad g_3(t) = t^4 + 4t^7,$$

and for $n \geqslant 4$,

$$g_n(t) = t^{n+1}\Big[1 + 3\chi_n + \sum_{i=0}^{\lfloor \frac{n}{4} \rfloor - 2} \big((1+3\chi_n)t^{2+2i} + (1+3\chi_{n+1})t^{3+2i}\big) + t^{2\lfloor \frac{n}{4} \rfloor} q_n(t)\Big],$$

where

$$q_n(t) = \begin{cases} 3 + 3t, & \text{if } n \equiv 0 \pmod 4, \\ 1 + 6t + t^3, & \text{if } n \equiv 1 \pmod 4, \\ 4 + 3t + t^3, & \text{if } n \equiv 2 \pmod 4, \\ 1 + 4t + 4t^3, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Proof. By (4.2.14), we have

$$g_n(t) = \begin{cases} h_0(t) - 1, & \text{if } n = 0, \\ h_n(t) - g_{n-1}(t), & \text{if } n \in \mathbb{N}. \end{cases}$$

Then we get the result by induction. □

Remark 4.2.11. *The cyclic cohomology of A is isomorphic to the dual space of the cyclic homology of A, so their Hilbert series coincide (see [16]).*

4.2.2 Hochschild cohomology

In this subsection, we will compute the linear structure of the Hochschild cohomology

$$\mathrm{HH}^{\bullet}(A) = \mathrm{Ext}^{\bullet}_{A^e}(A,A)$$

by means of the complex $\mathrm{H}^{\bullet}(\mathrm{Hom}_{A^e}(P^b_{\bullet},A))$. We refer the reader to [30] for further information about Hochschild cohomology.

4.2.2.1 Recursive description of the spaces

Let $K^n = \mathrm{Hom}_{\Bbbk}((A^!_{-n})^*, A)$ for $n \in \mathbb{N}_0$ and $K^n = 0$ for $n \in \mathbb{Z} \setminus \mathbb{N}_0$. We have $\mathrm{Hom}_{A^e}(P^b_n, A) \cong Q^n$ as $\Bbbk$-vector spaces, where $Q^n = \oplus_{i\in[\![0,\lfloor n/4\rfloor]\!]}\omega_i^* K^{n-4i}$ for $n \in \mathbb{N}_0$ and $Q^n = 0$ for $n \in \mathbb{Z}\setminus\mathbb{N}_0$. We will denote by $\partial^n : Q^n \to Q^{n+1}$ the differential

$$(\delta^b_{n+1})^* : \mathrm{Hom}_{A^e}(P^b_n, A) \to \mathrm{Hom}_{A^e}(P^b_{n+1}, A),$$

by $d^n : K^n \to K^{n+1}$ the differential

$$(d^b_{n+1})^* : \mathrm{Hom}_{A^e}(K^b_n, A) \to \mathrm{Hom}_{A^e}(K^b_{n+1}, A),$$

and by $f^n : K^{n+3} \to K^n$ the map

$$(f^b_n)^* : \mathrm{Hom}_{A^e}(K^b_{n+3}, A) \to \mathrm{Hom}_{A^e}(K^b_n, A)$$

for $n \in \mathbb{Z}$. Then the differential ∂^n for $n \in \mathbb{N}_0$ is given by

$$\partial^n\Bigg(\sum_{i\in[\![0,\lfloor n/4\rfloor]\!]}\omega_i^*\xi_{n-4i}\Bigg) = \sum_{i\in[\![0,\lfloor n/4\rfloor]\!]}\big(\omega_i^* d^{n-4i}(\xi_{n-4i}) + \omega_{i+1}^* f^{n-4i-3}(\xi_{n-4i})\big), \tag{4.2.15}$$

where $\xi_j \in K^j$ for $j \in \mathbb{N}_0$. Note that $\partial^n = \tilde{\partial}^n = 0$ for $n \in \mathbb{Z}\setminus\mathbb{N}_0$.

Our aim is to compute the cohomology of $(Q^\bullet, \partial^\bullet)$. Let $K^n_m = \mathrm{Hom}_{\Bbbk}((A^!_{-n})^*, A_m)$ be the subspace of K^n for $(n,m) \in \mathbb{N}_0 \times [\![0,4]\!]$, and $K^n_m = 0$ for $(n,m) \in \mathbb{Z}^2 \setminus (\mathbb{N}_0 \times [\![0,4]\!])$. Let $Q^n_m = \oplus_{i\in[\![0,\lfloor n/4\rfloor]\!]}\omega_i^* K^{n-4i}_{m+2i}$ for $(n,m) \in \mathbb{N}_0 \times \mathbb{Z}_{\leqslant 4}$ and $Q^n_m = 0$ for $(n,m) \in \mathbb{Z}^2 \setminus (\mathbb{N}_0 \times \mathbb{Z}_{\leqslant 4})$, where the symbol ω_i^* has cohomological degree $4i$ and internal degree $-6i$ for $i \in \mathbb{N}_0$, and we usually omit ω_0^* for simplicity. The spaces K^n_m and Q^n_m are concentrated in cohomological degree n and internal degree $m-n$. We have $Q^n = \oplus_{m\leqslant 4} Q^n_m$. Let $\partial^n_m = \partial^n|_{Q^n_m} : Q^n_m \to Q^{n+1}_{m+1}$, and $d^n_m = d^n|_{K^n_m} : K^n_m \to K^{n+1}_{m+1}$. Let $D^n_m = \mathrm{Ker}(\partial^n_m)$, $B^n_m = \mathrm{Im}(\partial^{n-1}_{m-1})$ for $(n,m) \in \mathbb{N}_0 \times \mathbb{Z}_{\leqslant 4}$, and $\tilde{D}^n_m = \mathrm{Ker}(d^n_m)$, $\tilde{B}^n_m = \mathrm{Im}(d^{n-1}_{m-1})$ for $(n,m) \in \mathbb{N}_0 \times [\![0,4]\!]$. Notice that $D^n_m = B^n_m = 0$ for $(n,m) \in \mathbb{Z}^2 \setminus (\mathbb{N}_0 \times \mathbb{Z}_{\leqslant 4})$, and $\tilde{D}^n_m = \tilde{B}^n_m = 0$ for $(n,m) \in \mathbb{Z}^2 \setminus (\mathbb{N}_0 \times [\![0,4]\!])$.

Remark 4.2.12. *We have $Q^n = \oplus_{m\in[\![-2\lfloor n/4\rfloor,4]\!]} Q^n_m$ since the indices in $Q^n_m = \oplus_{i\in[\![0,\lfloor n/4\rfloor]\!]}\omega_i^* K^{n-4i}_{m+2i}$ satisfy $n-4i \in \mathbb{N}_0$ and $m+2i \in [\![0,4]\!]$.*

Proposition 4.2.13. *For integers $m \leqslant 1$ and $n \in \mathbb{N}_0$, we have*

$$B^n_m = \begin{cases} \omega^*_{\frac{1-m}{2}} B_1^{n+2m-2}, & \text{if } m \text{ is odd,} \\ \omega^*_{-\frac{m}{2}} B_0^{n+2m}, & \text{if } m \text{ is even,} \end{cases} \tag{4.2.16}$$

and

$$D^n_m = \begin{cases} \omega^*_{\frac{1-m}{2}} D_1^{n+2m-2}, & \text{if } m \text{ is odd,} \\ \omega^*_{-\frac{m}{2}} D_0^{n+2m}, & \text{if } m \text{ is even,} \end{cases} \tag{4.2.17}$$

where we follow the convention that $\omega_i^*\omega_j^* = \omega_{i+j}^*$ for $i,j \in \mathbb{N}_0$ and $\omega_i^* = 0$ for $i \in \mathbb{Z}\setminus\mathbb{N}_0$.

Proof. Consider $Q^n_m = \oplus_{i\in[\![0,\lfloor n/4\rfloor]\!]}\omega_i^* K^{n-4i}_{m+2i}$ for integers $m \leqslant 4$ and $n \in \mathbb{N}_0$. If m is odd, then $m+2i = 1$ or 3, *i.e.* $i = (1-m)/2$ or $(3-m)/2$. We have

$$Q^n_m = \begin{cases} \omega^*_{\frac{1-m}{2}} K_1^{n+2m-2} \oplus \omega^*_{\frac{3-m}{2}} K_3^{n+2m-6}, & \text{if } n \geqslant 6-2m, \\ \omega^*_{\frac{1-m}{2}} K_1^{n+2m-2}, & \text{if } 2-2m \leqslant n < 6-2m, \\ 0, & \text{if } 0 \leqslant n < 2-2m. \end{cases}$$

If m is even, then $m+2i = 0, 2$ or 4, *i.e.* $i = -m/2, 1-m/2$ or $2-m/2$. We have

$$Q^n_m = \begin{cases} \omega^*_{-\frac{m}{2}} K_0^{n+2m} \oplus \omega^*_{1-\frac{m}{2}} K_2^{n+2m-4} \oplus \omega^*_{2-\frac{m}{2}} K_4^{n+2m-8}, & \text{if } n \geqslant 8-2m, \\ \omega^*_{-\frac{m}{2}} K_0^{n+2m} \oplus \omega^*_{1-\frac{m}{2}} K_2^{n+2m-4}, & \text{if } 4-2m \leqslant n < 8-2m, \\ \omega^*_{-\frac{m}{2}} K_0^{n+2m}, & \text{if } -2m \leqslant n < 4-2m, \\ 0, & \text{if } 0 \leqslant n < -2m. \end{cases} \tag{4.2.18}$$

Hence,

$$Q^n_m = \begin{cases} \omega^*_{\frac{1-m}{2}} Q_1^{n+2m-2}, & \text{if } m \leqslant 1 \text{ is odd,} \\ \omega^*_{-\frac{m}{2}} Q_0^{n+2m}, & \text{if } m \leqslant 0 \text{ is even.} \end{cases} \tag{4.2.19}$$

Since the identities (4.2.16) and (4.2.17) for $m = 1$ are immediate, we suppose $m \leqslant 0$ from now on.

Assume that m is even. Then (4.2.19) tells us that the sequence

$$Q^{n-1}_{m-1} \xrightarrow{\partial^{n-1}_{m-1}} Q^n_m \xrightarrow{\partial^n_m} Q^{n+1}_{m+1}$$

of graded $\Bbbk$-vector spaces is of the form

$$\omega^*_{1-\frac{m}{2}} Q_1^{n+2m-5} \xrightarrow{\partial^{n-1}_{m-1}} \omega^*_{-\frac{m}{2}} Q_0^{n+2m} \xrightarrow{\partial^n_m} \omega^*_{-\frac{m}{2}} Q_1^{n+2m+1}.$$

Since $Q^{n+2m-1}_{-1} = \omega^*_1 Q^{n+2m-5}_1$ by (4.2.19), the above sequence is of the form

$$\omega^*_{-\frac{m}{2}} Q_{-1}^{n+2m-1} \xrightarrow{\partial^{n-1}_{m-1}} \omega^*_{-\frac{m}{2}} Q_0^{n+2m} \xrightarrow{\partial^n_m} \omega^*_{-\frac{m}{2}} Q_1^{n+2m+1}.$$

Note further that $\partial^n_m = \omega^*_{-\frac{m}{2}} \partial^{n+2m}_0$ and $\partial^{n-1}_{m-1} = \omega^*_{1-\frac{m}{2}} \partial^{n+2m-5}_1 = \omega^*_{-\frac{m}{2}} \partial^{n+2m-1}_{-1}$, where the differential $\omega^*_j \partial^{n'}_{m'} : \omega^*_j Q^{n'}_{m'} \to \omega^*_j Q^{n'+1}_{m'+1}$ maps $\omega^*_j x$ to $\omega^*_j \partial^{n'}_{m'}(x)$ for all $x \in Q^{n'}_{m'}$, $j,n' \in \mathbb{N}_0$ and for all integers $m' \leqslant 4$. Hence, $B^n_m = \omega^*_{-\frac{m}{2}} B^{n+2m}_0$ and $D^n_m = \omega^*_{-\frac{m}{2}} D^{n+2m}_0$.

Assume that m is odd (so $m \leqslant -1$). Then (4.2.19) tells us that the sequence

$$Q^{n-1}_{m-1} \xrightarrow{\partial^{n-1}_{m-1}} Q^n_m \xrightarrow{\partial^n_m} Q^{n+1}_{m+1}$$

of graded $\Bbbk$-vector spaces is of the form

$$\omega^*_{\frac{1-m}{2}} Q_0^{n+2m-3} \xrightarrow{\partial^{n-1}_{m-1}} \omega^*_{\frac{1-m}{2}} Q_1^{n+2m-2} \xrightarrow{\partial^n_m} \omega^*_{-\frac{m+1}{2}} Q_0^{n+2m+3}.$$

Note that $\partial^{n-1}_{m-1} = \omega^*_{\frac{1-m}{2}} \partial^{n+2m-3}_0$. Moreover, we also have

$$\omega^*_{-\frac{m+1}{2}} Q_0^{n+2m+3} = \omega^*_{-\frac{m+1}{2}} K_0^{n+2m+3} \oplus \omega^*_{\frac{1-m}{2}} Q_2^{n+2m-1} \tag{4.2.20}$$

by (4.2.18), and the image of $\omega^*_{\frac{1-m}{2}} Q^{n+2m-2}_1$ is contained in $\omega^*_{\frac{1-m}{2}} Q^{n+2m-1}_2$ by the explicit expression of the differential (4.2.15). Furthermore, the composition of ∂^n_m with the canonical projection

$$\omega^*_{-\frac{m+1}{2}} Q_0^{n+2m+3} \longrightarrow \omega^*_{\frac{1-m}{2}} Q_2^{n+2m-1}$$

induced by (4.2.20) is precisely $\omega^*_{\frac{1-m}{2}} \partial^{n+2m-2}_1$. It is thus sufficient to consider the sequence

$$\omega^*_{\frac{1-m}{2}} Q_0^{n+2m-3} \xrightarrow{\omega^*_{\frac{1-m}{2}} \partial_0^{n+2m-3}} \omega^*_{\frac{1-m}{2}} Q_1^{n+2m-2} \xrightarrow{\omega^*_{\frac{1-m}{2}} \partial_1^{n+2m-2}} \omega^*_{\frac{1-m}{2}} Q_2^{n+2m-1}.$$

Hence, $B^n_m = \omega^*_{\frac{1-m}{2}} B^{n+2m-2}_1$ and $D^n_m = \omega^*_{\frac{1-m}{2}} D^{n+2m-2}_1$, as was to be shown. □

Throughout the Subsection 4.2.2 and Section 5.1 we will use the symbol $y|x$, where $x \in \mathscr{B}_m$ and $y \in \mathscr{B}^{!*}_n$, to denote the $\Bbbk$-linear map in $K^n = \mathrm{Hom}_{\Bbbk}((A^!_{-n})^*, A)$, which maps y to x and sends the other usual basis elements of $(A^!_{-n})^*$ to zero. Even though one usually denotes the previous map by $y||x$, we will use $y|x$ for the sake of reducing space in the expressions of the next subsubsection.

In order to compute B^n_m and D^n_m, it is sufficient to compute the case $m \in [\![0,4]\!]$ according to Proposition 4.2.13. First, we will compute the coboundaries, and then we will compute the cocycles. Since this will require handling elements of K^n_m and Q^n_m for $n \in \mathbb{N}_0$ and $m \in [\![0,4]\!]$, we will use the basis $\{y|x \mid x \in \mathscr{B}_m, y \in \mathscr{B}^{!*}_n\}$ of K^n_m and the basis $\{\omega^*_i y|x \mid i \in [\![0, \lfloor n/4 \rfloor]\!], x \in \mathscr{B}_{m+2i}, y \in \mathscr{B}^{!*}_{n-4i}\}$ of Q^n_m, both of which will be called **usual** bases, constructed from the usual bases of the homogeneous components of A and $(A^!)^{\#}$, introduced in Subsection 4.1.1.

4.2.2.2 Explicit description of the differentials

Recall the isomorphism $\mathrm{Hom}_{A^e}(A \otimes (A^!_{-n})^* \otimes A, A) \cong \mathrm{Hom}_{\Bbbk}((A^!_{-n})^*, A)$. We will use it together with Proposition 4.1.15 to explicitly describe d^n and f^n, which were defined at the beginning of Subsubsection 4.2.2.1.

Let $x \in A$. It is then straightforward to see that the differential $d^0 : \mathrm{Hom}_{\Bbbk}((A^!_0)^*, A) \to \mathrm{Hom}_{\Bbbk}((A^!_{-1})^*, A)$ is given by $d^0(\epsilon^!|x) = \alpha|(xa - ax) + \beta|(xb - bx) + \gamma|(xc - cx)$. Analogously, for $n \in \mathbb{N}$, $d^n : \mathrm{Hom}_{\Bbbk}((A^!_{-n})^*, A) \to \mathrm{Hom}_{\Bbbk}((A^!_{-(n+1)})^*, A)$ is given by

$$\begin{aligned}
\alpha_n|x \mapsto{}& \alpha_{n+1}|[(-1)^{n+1}ax + xa] + \alpha_n\beta|(\chi_{n+1}cx - \chi_n bx + xb) \\
&+ \alpha_n\gamma|(\chi_{n+1}bx - \chi_n cx + xc), \\
\beta_n|x \mapsto{}& \beta_{n+1}|[(-1)^{n+1}bx + xb] + \chi_{n+1}\alpha_n\beta|(ax + xc) + \alpha_n\gamma|[(-1)^{n+1}cx + \chi_{n+1}xa + \chi_n xc] \\
&+ \chi_n\alpha_{n-1}\beta_2|(xa - ax), \\
\gamma_n|x \mapsto{}& \gamma_{n+1}|[(-1)^{n+1}cx + xc] + \alpha_n\beta|[(-1)^{n+1}bx + \chi_{n+1}xa + \chi_n xb] + \chi_{n+1}\alpha_n\gamma|(ax + xb) \\
&+ \chi_n\alpha_{n-1}\beta_2|(xa - ax), \\
\alpha_{n-1}\beta|x \mapsto{}& \alpha_n\beta|[(-1)^{n+1}ax + xc] + \alpha_n\gamma|(\chi_{n+1}cx - \chi_n bx + xa) \\
&+ \alpha_{n-1}\beta_2|(\chi_{n+1}bx - \chi_n cx + xb), \\
\alpha_{n-1}\gamma|x \mapsto{}& \alpha_n\beta|(\chi_{n+1}bx - \chi_n cx + xa) + \alpha_n\gamma|[(-1)^{n+1}ax + xb] \\
&+ \alpha_{n-1}\beta_2|(\chi_{n+1}cx - \chi_n bx + xc), \\
\alpha_{n-2}\beta_2|x \mapsto{}& \alpha_n\beta|(\chi_{n+1}cx - \chi_n bx + xb) + \alpha_n\gamma|(\chi_{n+1}bx - \chi_n cx + xc) \\
&+ \alpha_{n-1}\beta_2|[(-1)^{n+1}ax + xa].
\end{aligned}$$

The $\Bbbk$-linear maps $f^n : \mathrm{Hom}_{\Bbbk}((A^!_{-(n+3)})^*, A) \to \mathrm{Hom}_{\Bbbk}((A^!_{-n})^*, A)$ are homogeneous of cohomological degree -3 and internal degree 6. The map f^0 is given by

$$\begin{aligned}
\alpha_3|x &\mapsto \epsilon^!|[2xbac - 2bx(ab + bc) + 2cxba + 2abxc + 2acxb + 2bacx], \\
\beta_3|x &\mapsto \epsilon^!|[2xabc - 2ax(ba + ac) + 2cxab + 2bcxa + 2baxc + 2abcx], \\
\gamma_3|x &\mapsto \epsilon^!|[-2xaba + 2axbc + 2bxac - 2(ab + bc)xb - 2(ba + ac)xa - 2abax], \\
\alpha_2\beta|x &\mapsto \epsilon^!|[-xabc + ax(ba + ac) - cxab - bcxa - baxc - abcx], \\
\alpha_2\gamma|x &\mapsto \epsilon^!|[xaba - axbc - bxac + (ab + bc)xb + (ba + ac)xa + abax], \\
\alpha\beta_2|x &\mapsto \epsilon^!|[-xbac + bx(ab + bc) - cxba - abxc - acxb - bacx].
\end{aligned}$$

For $n \in \mathbb{N}$, f^n is given by

$$\begin{aligned}
\alpha_{n+3}|x \mapsto{}& \alpha_n|[2xbac - \chi_n bx(ab + bc) + \chi_n cxba + \chi_n acxb + \chi_n abxc + (-1)^n 2bacx] \\
&+ \beta_n|[\chi_n xbac - \chi_n bxbc + (-1)^n 2cxba + \chi_n(ba + ac)xb + 2abxc + \chi_n bacx] \\
&+ \gamma_n|[\chi_n xbac + (-1)^{n+1}2bx(ab + bc) + \chi_n cx(ba + ac) + 2acxb - \chi_n bcxc + \chi_n bacx] \\
&+ \chi_{n+1}\alpha_{n-1}\beta|[axab - (n-2)cxba + cxac - baxa + (n-1)abxc + bcxc] \\
&+ \chi_{n+1}\alpha_{n-1}\gamma|[axac + (n-1)bxab + (n-2)bxbc + (n-2)acxb - baxb + (ab + bc)xa] \\
&+ \alpha_{n-2}\beta_2|\{\chi_{n+1}[(n-1)xbac + cx(ab + bc) - bxba + acxc + abxb - (n-1)bacx] \\
&+ \chi_n(n-2)[xbac - bx(ab + bc) + cxba + acxb + abxc + bacx]\}, \\
\beta_{n+3}|x \mapsto{}& \alpha_n|[\chi_n xabc + (-1)^n 2cxab - \chi_n axac + 2baxc + \chi_n(ab + bc)xa + \chi_n abcx] \\
&+ \beta_n|[2xabc + \chi_n cxab - \chi_n ax(ba + ac) + \chi_n baxc + \chi_n bcxa + (-1)^n 2abcx] \\
&+ \gamma_n|[\chi_n xabc + \chi_n cx(ab + bc) + (-1)^{n+1}2ax(ba + ac) - \chi_n acxc + 2bcxa + \chi_n abcx] \\
&+ \chi_{n+1}\alpha_{n-1}\beta|[(n-1)xabc - axab + cx(ba + ac) + baxa + bcxc - (n-1)abcx] \\
&+ \chi_{n+1}\alpha_{n-1}\gamma|[bxbc + (n-1)axba + (n-2)axac + (ba + ac)xb + (n-2)bcxa - abxa] \\
&+ \alpha_{n-2}\beta_2|\{\chi_{n+1}[bxba - (n-2)cxab + cxbc + acxc + (n-1)baxc - abxb] \\
&+ \chi_n(n-2)[xabc + cxab - ax(ba + ac) + baxc + bcxa + abcx]\}, \\
\gamma_{n+3}|x \mapsto{}& \alpha_n|[-\chi_n xaba - \chi_n axab + (-1)^n 2bxac - \chi_n baxa - 2(ab + bc)xb - \chi_n abax] \\
&+ \beta_n|[-\chi_n xaba + (-1)^n 2axbc - \chi_n bxba - 2(ba + ac)xa - \chi_n abxb - \chi_n abax] \\
&+ \gamma_n|[-2xaba + \chi_n axbc + \chi_n bxac - \chi_n(ba + ac)xa - \chi_n(ab + bc)xb + (-1)^{n+1}2abax]
\end{aligned}$$

$$+\chi_{n+1}\alpha_{n-1}\beta|[-axab - cx(ba+ac) - (n-1)axbc - (n-2)baxa - (n-1)acxa - bcxc]$$
$$+\chi_{n+1}\alpha_{n-1}\gamma|[-(n-1)xaba - bxbc - axac - (ba+ac)xb - (ab+bc)xa + (n-1)abax]$$
$$+\alpha_{n-2}\beta_2|\{\chi_{n+1}[-cx(ab+bc) - (n-1)bxac - bxba - acxc - (n-1)bcxb - (n-2)abxb]$$
$$+\chi_n(n-2)[-xaba + axbc + bxac - (ba+ac)xa - (ab+bc)xb - abax]\},$$
$$\alpha_{n+2}\beta|x \mapsto \alpha_n|(-\chi_n cxab - \chi_n baxc) + \beta_n|(-\chi_n xabc - \chi_n abcx) + \gamma_n|[\chi_n ax(ba+ac) - \chi_n bcxa],$$
$$\alpha_{n+2}\gamma|x \mapsto \alpha_n|[-\chi_n bxac + \chi_n(ab+bc)xb] + \beta_n|[-\chi_n axbc + \chi_n(ba+ac)xa] + \gamma_n|(\chi_n xaba + \chi_n abax),$$
$$\alpha_{n+1}\beta_2|x \mapsto \alpha_n|[-xbac + \chi_{n+1}bxac + \chi_{n+1}cxab + \chi_{n+1}(ab+bc)xb - \chi_{n+1}baxc + (-1)^{n+1}bacx]$$
$$+\beta_n|[-\chi_{n+1}xabc + (-1)^{n+1}cxba + \chi_{n+1}axbc - abxc + \chi_{n+1}(ba+ac)xa + \chi_{n+1}abcx]$$
$$+\gamma_n|[\chi_{n+1}xaba - \chi_{n+1}ax(ba+ac) + (-1)^n bx(ab+bc) - \chi_{n+1}bcxa - acxb - \chi_{n+1}abax].$$

For the reader's convenience, we list the images of the differentials d^n and the maps f^n evaluated at elements of the usual $\Bbbk$-basis of the respective domain. In the following tables, $d^n_m(y|x)$ is the entry appearing in the column indexed by x and the row indexed by y, where m is the internal degree of x and n is the internal degree of y.

If $n \in \mathbb{N}$ is odd, the differential d^n is given by

$y \backslash x$	$abac$	aba	abc	bac
α_n	0	$(\alpha_n\gamma - \alpha_n\beta)\|abac$	$(\alpha_n\gamma - \alpha_n\beta)\|abac$	0
β_n	0	$(\alpha_n\beta - \alpha_n\gamma)\|abac$	0	$(\alpha_n\beta - \alpha_n\gamma)\|abac$
γ_n	0	0	$(\alpha_n\beta - \alpha_n\gamma)\|abac$	$(\alpha_n\gamma - \alpha_n\beta)\|abac$
$\alpha_{n-1}\beta$	0	$(\alpha_n\beta - \alpha_n\gamma)\|abac$	0	$(\alpha_n\beta - \alpha_n\gamma)\|abac$
$\alpha_{n-1}\gamma$	0	0	$(\alpha_n\beta - \alpha_n\gamma)\|abac$	$(\alpha_n\gamma - \alpha_n\beta)\|abac$
$\alpha_{n-2}\beta_2$	0	$(\alpha_n\gamma - \alpha_n\beta)\|abac$	$(\alpha_n\gamma - \alpha_n\beta)\|abac$	0

Table 4.2.11: Images of d^n for $n \in \mathbb{N}$ and n odd, where the last three lines are for $n \geqslant 3$ and n odd.

together with

$y \backslash x$	ab	bc
α_n	$\alpha_{n+1}\|aba + \alpha_n\beta\|bac + \alpha_n\gamma\|(aba+abc)$	$\alpha_{n+1}\|(abc-aba) - 2\alpha_n\beta\|bac$
β_n	$\beta_{n+1}\|aba + \alpha_n\beta\|abc + \alpha_n\gamma\|(aba+bac)$	$-\beta_{n+1}\|bac + \alpha_n\beta\|abc - \alpha_n\gamma\|(aba+bac)$
γ_n	$\gamma_{n+1}\|(abc+bac) + 2\alpha_n\beta\|aba$	$-\gamma_{n+1}\|bac - \alpha_n\beta\|aba + \alpha_n\gamma\|(abc-bac)$
$\alpha_{n-1}\beta$	$\alpha_n\beta\|abc + \alpha_n\gamma\|(aba+bac) + \alpha_{n-1}\beta_2\|aba$	$\alpha_n\beta\|abc - \alpha_n\gamma\|(aba+bac) - \alpha_{n-1}\beta_2\|bac$
$\alpha_{n-1}\gamma$	$2\alpha_n\beta\|aba + \alpha_{n-1}\beta_2\|(abc+bac)$	$-\alpha_n\beta\|aba + \alpha_n\gamma\|(abc-bac) - \alpha_{n-1}\beta_2\|bac$
$\alpha_{n-2}\beta_2$	$\alpha_n\beta\|bac + \alpha_n\gamma\|(aba+abc) + \alpha_{n-1}\beta_2\|aba$	$-2\alpha_n\beta\|bac + \alpha_{n-1}\beta_2\|(abc-aba)$

Table 4.2.12: Images of d^n for $n \in \mathbb{N}$ and n odd, where the last three lines are for $n \geqslant 3$ and n odd.

and

$y \backslash x$	ba	ac
α_n	$\alpha_{n+1}\|aba + \alpha_n\beta\|(aba+abc) + \alpha_n\gamma\|bac$	$-\alpha_{n+1}\|abc - \alpha_n\beta\|(aba+abc) + \alpha_n\gamma\|bac$
β_n	$\beta_{n+1}\|aba + \alpha_n\beta\|(aba+bac) + \alpha_n\gamma\|abc$	$\beta_{n+1}\|(bac-aba) - 2\alpha_n\gamma\|abc$
γ_n	$\gamma_{n+1}\|(abc+bac) + 2\alpha_n\gamma\|aba$	$-\gamma_{n+1}\|abc + \alpha_n\beta\|(bac-abc) - \alpha_n\gamma\|aba$
$\alpha_{n-1}\beta$	$\alpha_n\beta\|(aba+bac) + \alpha_n\gamma\|abc + \alpha_{n-1}\beta_2\|aba$	$-2\alpha_n\gamma\|abc + \alpha_{n-1}\beta_2\|(bac-aba)$
$\alpha_{n-1}\gamma$	$2\alpha_n\gamma\|aba + \alpha_{n-1}\beta_2\|(abc+bac)$	$\alpha_n\beta\|(bac-abc) - \alpha_n\gamma\|aba - \alpha_{n-1}\beta_2\|abc$
$\alpha_{n-2}\beta_2$	$\alpha_n\beta\|(aba+abc) + \alpha_n\gamma\|bac + \alpha_{n-1}\beta_2\|aba$	$-\alpha_n\beta\|(aba+abc) + \alpha_n\gamma\|bac - \alpha_{n-1}\beta_2\|abc$

Table 4.2.13: Images of d^n for $n \in \mathbb{N}$ and n odd, where the last three lines are for $n \geqslant 3$ and n odd.

as well as

$y \backslash x$	a	b
α_n	$-\alpha_n\beta\|bc + \alpha_n\gamma\|(ba+ac)$	$\alpha_{n+1}\|(ab+ba) - \alpha_n\beta\|(ba+ac) + \alpha_n\gamma\|bc$
β_n	$\beta_{n+1}\|(ab+ba) + \alpha_n\beta\|ac - \alpha_n\gamma\|(ab+bc)$	$\alpha_n\beta\|(ab+bc) - \alpha_n\gamma\|ac$
γ_n	$\gamma_{n+1}\|(ac-ab-bc) + \alpha_n\beta\|ba + \alpha_n\gamma\|ab$	$\gamma_{n+1}\|(bc-ba-ac) + \alpha_n\beta\|ba + \alpha_n\gamma\|ab$
$\alpha_{n-1}\beta$	$\alpha_n\beta\|ac - \alpha_n\gamma\|(ab+bc) + \alpha_{n-1}\beta_2\|(ab+ba)$	$\alpha_n\beta\|(ab+bc) - \alpha_n\gamma\|ac$
$\alpha_{n-1}\gamma$	$\alpha_n\beta\|ba + \alpha_n\gamma\|ab + \alpha_{n-1}\beta_2\|(ac-ab-bc)$	$\alpha_n\beta\|ba + \alpha_n\gamma\|ab + \alpha_{n-1}\beta_2\|(bc-ba-ac)$
$\alpha_{n-2}\beta_2$	$-\alpha_n\beta\|bc + \alpha_n\gamma\|(ba+ac)$	$-\alpha_n\beta\|(ba+ac) + \alpha_n\gamma\|bc + \alpha_{n-1}\beta_2\|(ab+ba)$

Table 4.2.14: Images of d^n for $n \in \mathbb{N}$ and n odd, where the last three lines are for $n \geqslant 3$ and n odd.

and

$y \backslash x$	c	1
α_n	$\alpha_{n+1}\|(ac-ab-bc)-\alpha_n\beta\|(ba+ac)+\alpha_n\gamma\|bc$	$2\alpha_{n+1}\|a+(\alpha_n\beta+\alpha_n\gamma)\|(b+c)$
β_n	$\beta_{n+1}\|(bc-ba-ac)+\alpha_n\beta\|ac-\alpha_n\gamma\|(ab+bc)$	$2\beta_{n+1}\|b+(\alpha_n\beta+\alpha_n\gamma)\|(a+c)$
γ_n	$-\alpha_n\beta\|ab-\alpha_n\gamma\|ba$	$2\gamma_{n+1}\|c+(\alpha_n\beta+\alpha_n\gamma)\|(a+b)$
$\alpha_{n-1}\beta$	$\alpha_n\beta\|ac-\alpha_n\gamma\|(ab+bc)+\alpha_{n-1}\beta_2\|(bc-ba-ac)$	$(\alpha_n\beta+\alpha_n\gamma)\|(a+c)+2\alpha_{n-1}\beta_2\|b$
$\alpha_{n-1}\gamma$	$-\alpha_n\beta\|ab-\alpha_n\gamma\|ba$	$(\alpha_n\beta+\alpha_n\gamma)\|(a+b)+2\alpha_{n-1}\beta_2\|c$
$\alpha_{n-2}\beta_2$	$-\alpha_n\beta\|(ba+ac)+\alpha_n\gamma\|bc+\alpha_{n-1}\beta_2\|(ac-ab-bc)$	$(\alpha_n\beta+\alpha_n\gamma)\|(b+c)+2\alpha_{n-1}\beta_2\|a$

Table 4.2.15: Images of d^n for $n \in \mathbb{N}$ and n odd, where the last three lines are for $n \geqslant 3$ and n odd.

If $n \geqslant 2$ is even, the differential d^n is given by

$y \backslash x$	$abac$	aba	abc	bac
α_n	0	$2\alpha_n\gamma\|abac$	$-2\alpha_n\beta\|abac$	$-2\alpha_{n+1}\|abac$
β_n	0	$2\alpha_n\gamma\|abac$	$-2\beta_{n+1}\|abac$	$-2\alpha_{n-1}\beta_2\|abac$
γ_n	0	$2\gamma_{n+1}\|abac$	$-2\alpha_n\beta\|abac$	$-2\alpha_{n-1}\beta_2\|abac$
$\alpha_{n-1}\beta$	0	$\alpha_n\beta\|abac+\alpha_{n-1}\beta_2\|abac$	$-\alpha_n\gamma\|abac-\alpha_{n-1}\beta_2\|abac$	$-\alpha_n\beta\|abac-\alpha_n\gamma\|abac$
$\alpha_{n-1}\gamma$	0	$\alpha_n\beta\|abac+\alpha_{n-1}\beta_2\|abac$	$-\alpha_n\gamma\|abac-\alpha_{n-1}\beta_2\|abac$	$-\alpha_n\beta\|abac-\alpha_n\gamma\|abac$
$\alpha_{n-2}\beta_2$	0	$2\alpha_n\gamma\|abac$	$-2\alpha_n\beta\|abac$	$-2\alpha_{n-1}\beta_2\|abac$

Table 4.2.16: Images of d^n for $n \geqslant 2$ and n even, where the last line is for $n \geqslant 4$ and n even.

together with

$y \backslash x$	ab	bc
α_n	$\alpha_{n+1}\|aba-\alpha_n\beta\|aba+\alpha_n\gamma\|(abc-bac)$	$-\alpha_{n+1}\|(aba+abc)-\alpha_n\beta\|bac+\alpha_n\gamma\|bac$
β_n	$-\beta_{n+1}\|aba+\alpha_n\gamma\|(abc-bac)+\alpha_{n-1}\beta_2\|aba$	$-\beta_{n+1}\|bac+\alpha_n\gamma\|bac-\alpha_{n-1}\beta_2\|(aba+abc)$
γ_n	$\gamma_{n+1}\|(abc-bac)-\alpha_n\beta\|aba+\alpha_{n-1}\beta_2\|aba$	$\gamma_{n+1}\|bac-\alpha_n\beta\|bac-\alpha_{n-1}\beta_2\|(aba+abc)$
$\alpha_{n-1}\beta$	$\alpha_n\beta\|abc-\alpha_{n-1}\beta_2\|bac$	$-\alpha_n\beta\|abc-\alpha_n\gamma\|aba$
$\alpha_{n-1}\gamma$	$\alpha_n\beta\|(aba-bac)+\alpha_{n-1}\beta_2\|(abc-aba)$	$\alpha_n\beta\|(bac-aba)-\alpha_n\gamma\|(abc+bac)$
$\alpha_{n-2}\beta_2$	$-\alpha_n\beta\|aba+\alpha_n\gamma\|(abc-bac)+\alpha_{n-1}\beta_2\|aba$	$-\alpha_n\beta\|bac+\alpha_n\gamma\|bac-\alpha_{n-1}\beta_2\|(aba+abc)$

Table 4.2.17: Images of d^n for $n \geqslant 2$ and n even, where the last line is for $n \geqslant 4$ and n even.

and

$y \backslash x$	ba	ac
α_n	$-\alpha_{n+1}\|aba+\alpha_n\beta\|aba+\alpha_n\gamma\|(bac-abc)$	$-\alpha_{n+1}\|abc-\alpha_n\beta\|(aba+bac)+\alpha_n\gamma\|abc$
β_n	$\beta_{n+1}\|aba+\alpha_n\gamma\|(bac-abc)-\alpha_{n-1}\beta_2\|aba$	$-\beta_{n+1}\|(aba+bac)+\alpha_n\gamma\|abc-\alpha_{n-1}\beta_2\|abc$
γ_n	$\gamma_{n+1}\|(bac-abc)+\alpha_n\beta\|aba-\alpha_{n-1}\beta_2\|aba$	$\gamma_{n+1}\|abc-\alpha_n\beta\|(aba+bac)-\alpha_{n-1}\beta_2\|abc$
$\alpha_{n-1}\beta$	$\alpha_n\beta\|(bac-aba)+\alpha_{n-1}\beta_2\|(aba-abc)$	$-\alpha_n\gamma\|(abc+bac)+\alpha_{n-1}\beta_2\|(abc-aba)$
$\alpha_{n-1}\gamma$	$-\alpha_n\beta\|abc+\alpha_{n-1}\beta_2\|bac$	$-\alpha_n\gamma\|aba-\alpha_{n-1}\beta_2\|bac$
$\alpha_{n-2}\beta_2$	$\alpha_n\beta\|aba+\alpha_n\gamma\|(bac-abc)-\alpha_{n-1}\beta_2\|aba$	$-\alpha_n\beta\|(aba+bac)+\alpha_n\gamma\|abc-\alpha_{n-1}\beta_2\|abc$

Table 4.2.18: Images of d^n for $n \geqslant 2$ and n even, where the last line is for $n \geqslant 4$ and n even.

as well as

$y \backslash x$	a	b
α_n	$\alpha_n\beta\|(ab-ba)+\alpha_n\gamma\|(ab+bc+ac)$	$\alpha_{n+1}\|(ba-ab)+\alpha_n\gamma\|(ba+ac+bc)$
β_n	$\beta_{n+1}\|(ab-ba)+\alpha_n\gamma\|(ab+bc+ac)$	$\alpha_n\gamma\|(ba+ac+bc)+\alpha_{n-1}\beta_2\|(ba-ab)$
γ_n	$\gamma_{n+1}\|(ab+bc+ac)+\alpha_n\beta\|(ab-ba)$	$\gamma_{n+1}\|(ba+ac+bc)+\alpha_{n-1}\beta_2\|(ba-ab)$
$\alpha_{n-1}\beta$	$\alpha_n\beta\|ac-\alpha_n\gamma\|ba+\alpha_{n-1}\beta_2\|(2ab+bc)$	$\alpha_n\beta\|(bc-ab)+\alpha_n\gamma\|ba+\alpha_{n-1}\beta_2\|(ba+ac)$
$\alpha_{n-1}\gamma$	$\alpha_n\beta\|(ab+bc)+\alpha_n\gamma\|ab+\alpha_{n-1}\beta_2\|(ac-ba)$	$\alpha_n\beta\|(2ba+ac)-\alpha_n\gamma\|ab+\alpha_{n-1}\beta_2\|bc$
$\alpha_{n-2}\beta_2$	$\alpha_n\beta\|(ab-ba)+\alpha_n\gamma\|(ab+bc+ac)$	$\alpha_n\gamma\|(ba+ac+bc)+\alpha_{n-1}\beta_2\|(ba-ab)$

Table 4.2.19: Images of d^n for $n \geqslant 2$ and n even, where the last line is for $n \geqslant 4$ and n even.

and

$y \backslash x$	c	1
α_n	$-\alpha_{n+1}\|(ab+bc+ac)-\alpha_n\beta\|(bc+ba+ac)$	0
β_n	$-\beta_{n+1}\|(bc+ba+ac)-\alpha_{n-1}\beta_2\|(ab+bc+ac)$	0
γ_n	$-\alpha_n\beta\|(bc+ba+ac)-\alpha_{n-1}\beta_2\|(ab+bc+ac)$	0
$\alpha_{n-1}\beta$	$-\alpha_n\beta\|ac-\alpha_n\gamma\|(ab+2bc)-\alpha_{n-1}\beta_2\|(ba+ac)$	$\alpha_n\beta\|(c-a)+\alpha_n\gamma\|(a-b)+\alpha_{n-1}\beta_2\|(b-c)$
$\alpha_{n-1}\gamma$	$-\alpha_n\beta\|(ab+bc)-\alpha_n\gamma\|(ba+2ac)-\alpha_{n-1}\beta_2\|bc$	$\alpha_n\beta\|(a-c)+\alpha_n\gamma\|(b-a)+\alpha_{n-1}\beta_2\|(c-b)$
$\alpha_{n-2}\beta_2$	$-\alpha_n\beta\|(bc+ba+ac)-\alpha_{n-1}\beta_2\|(ab+bc+ac)$	0

Table 4.2.20: Images of d^n for $n \geqslant 2$ and n even, where the last line is for $n \geqslant 4$ and n even.

Now we turn to the maps f^n. Note that $f^n(u) = 0$ for $u \in K_m^{n+3}$, with $m \in [\![2,4]\!]$ and $n \in \mathbb{N}_0$ by degree reasons. In the following tables, $f^n(y|x)$ is the entry appearing in the column indexed by x and the row indexed by y, where n is the internal degree of y. The map f^0 is given by

$y \setminus x$	1	a	b	c
α_3	$4\epsilon^!\|(bac - aba + abc)$	0	0	0
β_3	$4\epsilon^!\|(bac - aba + abc)$	0	0	0
γ_3	$4\epsilon^!\|(bac - aba + abc)$	0	0	0
$\alpha_2\beta$	$2\epsilon^!\|(aba - abc - bac)$	0	0	0
$\alpha_2\gamma$	$2\epsilon^!\|(aba - abc - bac)$	0	0	0
$\alpha\beta_2$	$2\epsilon^!\|(aba - abc - bac)$	0	0	0

Table 4.2.21: Images of f^0.

If $n \in \mathbb{N}$ is odd, the map f^n is given by

$y \setminus x$	1	a	b	c
α_{n+3}	0	$f^n(\alpha_{n+3}\|a)$	$2(\alpha_{n-1}\gamma - \alpha_{n-2}\beta_2)\|abac$	$2(\alpha_{n-1}\beta - \alpha_{n-2}\beta_2)\|abac$
β_{n+3}	0	$2(\alpha_{n-1}\gamma - \alpha_{n-1}\beta)\|abac$	$f^n(\beta_{n+3}\|b)$	$2(\alpha_{n-2}\beta_2 - \alpha_{n-1}\beta)\|abac$
γ_{n+3}	0	$2(\alpha_{n-1}\beta - \alpha_{n-1}\gamma)\|abac$	$2(\alpha_{n-2}\beta_2 - \alpha_{n-1}\gamma)\|abac$	$f^n(\gamma_{n+3}\|c)$
$\alpha_{n+2}\beta$	0	0	0	0
$\alpha_{n+2}\gamma$	0	0	0	0
$\alpha_{n+1}\beta_2$	0	$-2(\alpha_n + \beta_n + \gamma_n)\|abac$	$-2(\alpha_n + \beta_n + \gamma_n)\|abac$	$-2(\alpha_n + \beta_n + \gamma_n)\|abac$

Table 4.2.22: Images of f^n for $n \in \mathbb{N}$ and n odd.

where

$$\begin{aligned} f^n(\alpha_{n+3}|a) &= [4\alpha_n + 4\beta_n + 4\gamma_n + 2(n-2)\alpha_{n-1}\beta + 2(n-2)\alpha_{n-1}\gamma + 2(n-1)\alpha_{n-2}\beta_2]|abac, \\ f^n(\beta_{n+3}|b) &= [4\alpha_n + 4\beta_n + 4\gamma_n + 2(n-1)\alpha_{n-1}\beta + 2(n-2)\alpha_{n-1}\gamma + 2(n-2)\alpha_{n-2}\beta_2]|abac, \\ f^n(\gamma_{n+3}|c) &= [4\alpha_n + 4\beta_n + 4\gamma_n + 2(n-2)\alpha_{n-1}\beta + 2(n-1)\alpha_{n-1}\gamma + 2(n-2)\alpha_{n-2}\beta_2]|abac. \end{aligned}$$

If $n \geqslant 2$ is even, the map f^n is given by

$$\begin{aligned} f^n(\alpha_{n+3}|1) &= 2\alpha_n|(2bac - aba + abc) + 2\beta_n|(2abc + bac) + 2\gamma_n|(bac - 2aba) \\ &\quad + 2(n-2)\alpha_{n-2}\beta_2|(abc - aba + bac), \\ f^n(\beta_{n+3}|1) &= 2\alpha_n|(2bac + abc) + 2\beta_n|(2abc - aba + bac) + 2\gamma_n|(abc - 2aba) \\ &\quad + 2(n-2)\alpha_{n-2}\beta_2|(abc - aba + bac), \\ f^n(\gamma_{n+3}|1) &= 2\alpha_n|(2bac - aba) + 2\beta_n|(2abc - aba) + 2\gamma_n|(abc - 2aba + bac) \\ &\quad + 2(n-2)\alpha_{n-2}\beta_2|(abc - aba + bac), \\ f^n(\alpha_{n+2}\beta|1) &= -2\alpha_n|bac - 2\beta_n|abc + 2\gamma_n|aba, \\ f^n(\alpha_{n+2}\gamma|1) &= -2\alpha_n|bac - 2\beta_n|abc + 2\gamma_n|aba, \\ f^n(\alpha_{n+1}\beta_2|1) &= -2\alpha_n|bac - 2\beta_n|abc + 2\gamma_n|aba, \end{aligned}$$

and $f^n(x) = 0$ for $x \in K_1^{n+3}$.

From now on, we assume that the characteristic of the field $\mathbb{k}$ is different from 2 and 3 in this subsection.

4.2.2.3 Computation of the coboundaries

In this subsubsection, we will explicitly construct bases $\tilde{\mathfrak{B}}_m^n$ and $\mathfrak{B}_m^n$ of the $\mathbb{k}$-vector spaces $\tilde{B}_m^n = \mathrm{Im}(d_{m-1}^{n-1})$ and $B_m^n = \mathrm{Im}(\partial_{m-1}^{n-1})$ for $m \in [\![0,4]\!]$ and $n \in \mathbb{N}_0$ respectively, defined before Remark 4.2.12. This will be done by simply applying the corresponding differential d_{m-1}^{n-1} or ∂_{m-1}^{n-1} to the usual basis of its domain and extracting a linearly independent generating subset.

Computation of $\tilde{\mathfrak{B}}_m^n$ Recall that $\tilde{B}_m^n = \mathrm{Im}(d_{m-1}^{n-1})$ and

$$d_m^n : K_m^n = \mathrm{Hom}_{\mathbb{k}}((A^!_{-n})^*, A_m) \to K_{m+1}^{n+1} = \mathrm{Hom}_{\mathbb{k}}((A^!_{-(n+1)})^*, A_{m+1})$$

was defined in Subsubsection 4.2.2.1. Obviously, $\tilde{B}^0_m = \mathrm{Im}(d^{-1}_{m-1}) = 0$ for $m \in [\![0,4]\!]$, and $\tilde{B}^n_0 = \mathrm{Im}(d^{n-1}_{-1}) = 0$ for $n \in \mathbb{N}$. Then we define $\tilde{\mathfrak{B}}^0_m = \emptyset$ for $m \in [\![0,4]\!]$, and $\tilde{\mathfrak{B}}^n_0 = \emptyset$ for $n \in \mathbb{N}$.

Suppose $m = 4$. If $n = 1$, since

$$\gamma|abac = (1/2)d^0_3(\epsilon^!|aba), \quad \beta|abac = -(1/2)d^0_3(\epsilon^!|abc), \quad \alpha|abac = -(1/2)d^0_3(\epsilon^!|bac),$$

we have $\tilde{B}^1_4 = K^1_4$. We define a basis of $\tilde{B}^1_4$ by the usual basis of K^1_4. If $n \geqslant 3$ is odd, Table 4.2.16 shows that

$$\begin{aligned} \alpha_n|abac &= -(1/2)d^{n-1}_3(\alpha_{n-1}|bac), & \beta_n|abac &= -(1/2)d^{n-1}_3(\beta_{n-1}|abc), \\ \gamma_n|abac &= (1/2)d^{n-1}_3(\gamma_{n-1}|aba), & \alpha_{n-1}\beta|abac &= -(1/2)d^{n-1}_3(\alpha_{n-1}|abc), \\ \alpha_{n-1}\gamma|abac &= (1/2)d^{n-1}_3(\alpha_{n-1}|aba), & \alpha_{n-2}\beta_2|abac &= -(1/2)d^{n-1}_3(\beta_{n-1}|bac), \end{aligned}$$

so $\tilde{B}^n_4 = K^n_4$. We define a basis of $\tilde{B}^n_4$ by the usual basis of K^n_4. If $n \geqslant 2$ is even, it is easy to see that $\tilde{B}^n_4$ is spanned by the element $(\alpha_{n-1}\beta - \alpha_{n-1}\gamma)|abac$ from Table 4.2.11, so we define a basis of $\tilde{B}^n_4$ by

$$\tilde{\mathfrak{B}}^n_4 = \{(\alpha_{n-1}\beta - \underline{\alpha_{n-1}\gamma})|\underline{abac}\}.$$

The dimension of $\tilde{B}^n_4$ is then given by

$$\dim \tilde{B}^n_4 = \begin{cases} 0, & \text{if } n = 0, \\ 3, & \text{if } n = 1, \\ 1, & \text{if } n \geqslant 2 \text{ is even}, \\ 6, & \text{if } n \geqslant 3 \text{ is odd}. \end{cases} \tag{4.2.21}$$

Suppose $m = 3$. If $n = 1$, since

$$\begin{aligned} (\alpha - \beta)|aba + \gamma|(abc - bac) &= d^0_2(\epsilon^!|ab) = -d^0_2(\epsilon^!|ba), \\ \alpha|(aba + abc) + (\beta - \gamma)|bac &= -d^0_2(\epsilon^!|bc), \\ -\alpha|abc - \beta|(aba + bac) + \gamma|abc &= d^0_2(\epsilon^!|ac) = d^0_2(\epsilon^!|ab) + d^0_2(\epsilon^!|bc), \end{aligned}$$

we define a basis of $\tilde{B}^1_3$ by

$$\tilde{\mathfrak{B}}^1_3 = \{\alpha|(aba + abc) + (\beta - \underline{\gamma})|\underline{bac}, (\alpha - \beta)|aba + \underline{\gamma}|(\underline{abc} - bac)\}.$$

If $n \geqslant 2$ is even, we define the set

$$\begin{aligned} \mathcal{G}^n_3 = \{g^n_{1,3} &= (\underline{\alpha_{n-1}\beta} - \alpha_{n-1}\gamma)|\underline{aba} = (1/2)d^{n-1}_2(\gamma_{n-1}|(ab - ba)), \\ g^n_{2,3} &= (\underline{\alpha_{n-1}\beta} - \alpha_{n-1}\gamma)|\underline{abc} = (1/2)d^{n-1}_2(\beta_{n-1}|(ab + bc + ac)), \\ g^n_{3,3} &= (\underline{\alpha_{n-1}\beta} - \alpha_{n-1}\gamma)|\underline{bac} = -(1/2)d^{n-1}_2(\alpha_{n-1}|(bc + ba + ac)), \\ g^n_{4,3} &= \underline{\alpha_n|aba} + \alpha_{n-1}\beta|bac + \alpha_{n-1}\gamma|(aba + abc) = d^{n-1}_2(\alpha_{n-1}|ab), \\ g^n_{5,3} &= \underline{\alpha_n|abc} - \alpha_{n-1}\beta|bac + \alpha_{n-1}\gamma|(aba + abc) = d^{n-1}_2(\alpha_{n-1}|ab) + d^{n-1}_2(\alpha_{n-1}|bc), \\ g^n_{6,3} &= \underline{\beta_n|aba} + \alpha_{n-1}\beta|abc + \alpha_{n-1}\gamma|(aba + bac) = d^{n-1}_2(\beta_{n-1}|ab), \\ g^n_{7,3} &= \underline{\beta_n|bac} - \alpha_{n-1}\beta|abc + \alpha_{n-1}\gamma|(aba + bac) = -d^{n-1}_2(\beta_{n-1}|bc), \\ g^n_{8,3} &= \underline{\gamma_n|abc} + \alpha_{n-1}\beta|aba + \alpha_{n-1}\gamma|(abc - bac) = d^{n-1}_2(\gamma_{n-1}|ab) + d^{n-1}_2(\gamma_{n-1}|bc), \\ g^n_{9,3} &= \underline{\gamma_n|bac} + \alpha_{n-1}\beta|aba + \alpha_{n-1}\gamma|(bac - abc) = -d^{n-1}_2(\gamma_{n-1}|bc)\}. \end{aligned}$$

Then we define the set $\tilde{\mathfrak{B}}^2_3 = \mathcal{G}^2_3$, and

$$\begin{aligned} \tilde{\mathfrak{B}}^n_3 = \mathcal{G}^n_3 \cup \{g^n_{10,3} &= \underline{\alpha_{n-2}\beta_2|aba} + \alpha_{n-1}\gamma|(aba + abc + bac) = d^{n-1}_2(\alpha_{n-2}\beta|ab) - g^n_{2,3}, \\ g^n_{11,3} &= \underline{\alpha_{n-2}\beta_2|abc} + \alpha_{n-1}\gamma|(aba + abc - bac) = -d^{n-1}_2(\alpha_{n-2}\gamma|ac) - g^n_{2,3} + g^n_{3,3}, \\ g^n_{12,3} &= \underline{\alpha_{n-2}\beta_2|bac} + \alpha_{n-1}\gamma|(aba - abc + bac) = -d^{n-1}_2(\alpha_{n-2}\beta|bc) + g^n_{2,3}\} \end{aligned}$$

for $n \geqslant 4$ with n even. We will show that $\tilde{\mathfrak{B}}^n_3$ is a basis of $\tilde{B}^n_3$ for $n \geqslant 2$ with n even. From the definition, we see that $\tilde{\mathfrak{B}}^n_3 \subseteq \tilde{B}^n_3$. Since

$$d^{n-1}_2(\alpha_{n-1}|ab) = g^n_{4,3}, \; d^{n-1}_2(\alpha_{n-1}|bc) = g^n_{5,3} - g^n_{4,3}, \; d^{n-1}_2(\alpha_{n-1}|ba) = g^n_{1,3} + g^n_{2,3} + g^n_{4,3} - g^n_{3,3},$$

$$d_2^{n-1}(\alpha_{n-1}|ac) = -g_{1,3}^n - g_{2,3}^n - g_{3,3}^n - g_{5,3}^n,\ d_2^{n-1}(\beta_{n-1}|ab) = g_{6,3}^n,\ d_2^{n-1}(\beta_{n-1}|bc) = -g_{7,3}^n,$$
$$d_2^{n-1}(\beta_{n-1}|ba) = g_{1,3}^n - g_{2,3}^n + g_{3,3}^n + g_{6,3}^n,\ d_2^{n-1}(\beta_{n-1}|ac) = 2g_{2,3}^n - g_{6,3}^n + g_{7,3}^n,$$
$$d_2^{n-1}(\gamma_{n-1}|ab) = g_{8,3}^n + g_{9,3}^n,\ d_2^{n-1}(\gamma_{n-1}|bc) = -g_{9,3}^n,\ d_2^{n-1}(\gamma_{n-1}|ba) = g_{8,3}^n + g_{9,3}^n - 2g_{1,3}^n,$$
$$d_2^{n-1}(\gamma_{n-1}|ac) = g_{1,3}^n - g_{2,3}^n + g_{3,3}^n - g_{8,3}^n$$

for $n \geqslant 2$ with n even, and

$$d_2^{n-1}(\alpha_{n-2}\beta|ab) = g_{2,3}^n + g_{10,3}^n,\quad d_2^{n-1}(\alpha_{n-2}\beta|bc) = g_{2,3}^n - g_{12,3}^n,$$
$$d_2^{n-1}(\alpha_{n-2}\beta|ba) = g_{1,3}^n + g_{3,3}^n + g_{10,3}^n,\quad d_2^{n-1}(\alpha_{n-2}\beta|ac) = g_{12,3}^n - g_{10,3}^n,$$
$$d_2^{n-1}(\alpha_{n-2}\gamma|ab) = g_{11,3}^n + g_{12,3}^n + 2g_{1,3}^n,\quad d_2^{n-1}(\alpha_{n-2}\gamma|bc) = -g_{1,3}^n - g_{12,3}^n,$$
$$d_2^{n-1}(\alpha_{n-2}\gamma|ba) = g_{11,3}^n + g_{12,3}^n,\quad d_2^{n-1}(\alpha_{n-2}\gamma|ac) = -g_{2,3}^n + g_{3,3}^n - g_{11,3}^n,$$
$$d_2^{n-1}(\alpha_{n-3}\beta_2|ab) = g_{3,3}^n + g_{10,3}^n,\quad d_2^{n-1}(\alpha_{n-3}\beta_2|bc) = g_{11,3}^n - g_{10,3}^n - 2g_{3,3}^n,$$
$$d_2^{n-1}(\alpha_{n-3}\beta_2|ba) = g_{1,3}^n + g_{2,3}^n + g_{10,3}^n,\quad d_2^{n-1}(\alpha_{n-3}\beta_2|ac) = -g_{1,3}^n - g_{2,3}^n - g_{11,3}^n$$

for $n \geqslant 4$ with n even, the elements in $\tilde{\mathfrak{B}}_3^n$ span the space $\tilde{B}_3^n$. By Fact 4.1.3, the elements in $\tilde{\mathfrak{B}}_3^n$ are linearly independent, so $\tilde{\mathfrak{B}}_3^n$ is a basis of $\tilde{B}_3^n$ for $n \geqslant 2$ with n even. If $n \geqslant 3$ is odd, we define the set

$$\begin{aligned}\mathcal{E}_3^n = \big\{ e_{1,3}^n &= (\underline{\alpha_n} - \alpha_{n-1}\beta)|\underline{aba} + \alpha_{n-1}\gamma|(abc - bac) = d_2^{n-1}(\alpha_{n-1}|ab),\\
e_{2,3}^n &= \underline{\alpha_n}|(aba + \underline{abc}) + (\alpha_{n-1}\beta - \alpha_{n-1}\gamma)|bac = -d_2^{n-1}(\alpha_{n-1}|bc),\\
e_{3,3}^n &= (\alpha_{n-2}\beta_2 - \underline{\beta_n})|\underline{aba} + \alpha_{n-1}\gamma|(abc - bac) = d_2^{n-1}(\beta_{n-1}|ab),\\
e_{4,3}^n &= (\alpha_{n-1}\gamma - \underline{\beta_n})|\underline{bac} - \alpha_{n-2}\beta_2|(aba + abc) = d_2^{n-1}(\beta_{n-1}|bc),\\
e_{5,3}^n &= (\underline{\gamma_n} - \alpha_{n-1}\beta)|\underline{bac} - \alpha_{n-2}\beta_2|(aba + abc) = d_2^{n-1}(\gamma_{n-1}|bc),\\
e_{6,3}^n &= \underline{\gamma_n}|(\underline{abc} - bac) + (\alpha_{n-2}\beta_2 - \alpha_{n-1}\beta)|aba = d_2^{n-1}(\gamma_{n-1}|ab),\\
e_{7,3}^n &= \alpha_{n-1}\beta|abc - \underline{\alpha_{n-2}\beta_2|bac} = d_2^{n-1}(\alpha_{n-2}\beta|ab),\\
e_{8,3}^n &= \alpha_{n-1}\beta|abc + \underline{\alpha_{n-1}\gamma|aba} = -d_2^{n-1}(\alpha_{n-2}\beta|bc),\\
e_{9,3}^n &= \alpha_{n-1}\beta|(bac - aba) + \alpha_{n-2}\beta_2|(aba - abc) = -d_2^{n-1}(\alpha_{n-2}\gamma|ab),\\
e_{10,3}^n &= \alpha_{n-1}\gamma|(abc + bac) + \alpha_{n-2}\beta_2|(aba - abc) = -d_2^{n-1}(\alpha_{n-2}\gamma|(ab + bc))\big\}.\end{aligned}$$

Then we define the set $\tilde{\mathfrak{B}}_3^3 = \mathcal{E}_3^3$, and

$$\begin{aligned}\tilde{\mathfrak{B}}_3^n = \mathcal{E}_3^n \cup \big\{ e_{11,3}^n &= (\alpha_{n-2}\beta_2 - \alpha_{n-1}\beta)|aba + \alpha_{n-1}\gamma|(abc - bac) = d_2^{n-1}(\alpha_{n-3}\beta_2|ab),\\
e_{12,3}^n &= (\alpha_{n-1}\beta - \alpha_{n-1}\gamma)|bac + \alpha_{n-2}\beta_2|(aba + abc) = -d_2^{n-1}(\alpha_{n-3}\beta_2|bc)\big\}\end{aligned}$$

for $n \geqslant 5$ with n odd. We will show that $\tilde{\mathfrak{B}}_3^n$ is a basis of $\tilde{B}_3^n$ for $n \geqslant 3$ with n odd. By definition, $\tilde{\mathfrak{B}}_3^n \subseteq \tilde{B}_3^n$. Since

$$d_2^{n-1}(\alpha_{n-1}|ba) = -e_{1,3}^n,\quad d_2^{n-1}(\alpha_{n-1}|ac) = e_{1,3}^n - e_{2,3}^n,\quad d_2^{n-1}(\beta_{n-1}|ba) = -e_{3,3}^n,$$
$$d_2^{n-1}(\beta_{n-1}|ac) = e_{3,3}^n + e_{4,3}^n,\quad d_2^{n-1}(\gamma_{n-1}|ba) = -e_{6,3}^n,\quad d_2^{n-1}(\gamma_{n-1}|ac) = e_{5,3}^n + e_{6,3}^n,$$
$$d_2^{n-1}(\alpha_{n-2}\beta|ba) = e_{9,3}^n,\quad d_2^{n-1}(\alpha_{n-2}\beta|ac) = -e_{10,3}^n,\quad d_2^{n-1}(\alpha_{n-2}\gamma|bc) = e_{9,3}^n - e_{10,3}^n,$$
$$d_2^{n-1}(\alpha_{n-2}\gamma|ba) = -e_{7,3}^n,\quad d_2^{n-1}(\alpha_{n-2}\gamma|ac) = e_{7,3}^n - e_{8,3}^n$$

for $n \geqslant 3$ with n odd, and

$$d_2^{n-1}(\alpha_{n-3}\beta_2|ba) = -e_{11,3}^n,\quad d_2^{n-1}(\alpha_{n-3}\beta_2|ac) = e_{11,3}^n - e_{12,3}^n$$

for $n \geqslant 5$ with n odd, the elements in $\tilde{\mathfrak{B}}_3^n$ span the space $\tilde{B}_3^n$. By Fact 4.1.3, the elements $e_{\ell,3}^n$ for $\ell \in [\![1,8]\!]$ are linearly independent. The reader can easily verify that the elements $e_{\ell,3}^n$ for $\ell \in [\![9,12]\!]$ are linearly independent. Since the underlined terms of $e_{\ell,3}^n$ for $\ell \in [\![1,8]\!]$ do not appear in $e_{\ell,3}^n$ for $\ell \in [\![9,12]\!]$, the elements in $\tilde{\mathfrak{B}}_3^n$ are linearly independent. So $\tilde{\mathfrak{B}}_3^n$ is a basis of

$\tilde{B}_3^n$. The dimension of $\tilde{B}_3^n$ is thus given by

$$\dim \tilde{B}_3^n = \begin{cases} 0, & \text{if } n = 0, \\ 2, & \text{if } n = 1, \\ 9, & \text{if } n = 2, \\ 10, & \text{if } n = 3, \\ 12, & \text{if } n \geqslant 4. \end{cases} \tag{4.2.22}$$

Suppose $m = 2$. If $n = 1$, since

$$\beta|(ab - ba) + \underline{\gamma}|(\underline{ab} + bc + ac) = d_1^0(\epsilon^!|a), \quad (\alpha + \beta + \underline{\gamma})|(ab - \underline{ba}) = d_1^0(\epsilon^!|(a - b)),$$
$$(\underline{\alpha} + \beta + \gamma)|(ab + \underline{bc} + ac) = d_1^0(\epsilon^!|(a - c)),$$

and these three elements are linearly independent, we define a basis of $\tilde{B}_2^1$ by

$$\tilde{\mathfrak{B}}_2^1 = \{\beta|(ab - ba) + \gamma|(ab + bc + ac), (\alpha + \beta + \gamma)|(ab - ba), (\alpha + \beta + \gamma)|(ab + bc + ac)\}.$$

If $n = 2$, we define the set

$$\begin{aligned} \tilde{\mathfrak{B}}_2^2 = \{ & g_{1,2}^2 = \alpha_2|(ab + ba) - \underline{\alpha\beta}|(\underline{ba} + ac) + \alpha\gamma|bc = d_1^1(\alpha|b), \\ & g_{2,2}^2 = \beta_2|(ab + ba) + \alpha\beta|ac - \underline{\alpha\gamma}|(\underline{ab} + bc) = d_1^1(\beta|a), \\ & g_{3,2}^2 = \underline{\gamma_2}|(\underline{bc} - ba - ac) + \alpha\beta|ba + \alpha\gamma|ab = d_1^1(\gamma|b), \\ & g_{4,2}^2 = \underline{\alpha\beta|ab} + \alpha\gamma|ba = -d_1^1(\gamma|c), \\ & g_{5,2}^2 = \underline{\alpha\beta}|(ab + \underline{bc}) - \alpha\gamma|ac = d_1^1(\beta|b), \\ & g_{6,2}^2 = \underline{\alpha_2}|(2ab + \underline{bc} + ba - ac) = d_1^1(\alpha|(b - c)), \\ & g_{7,2}^2 = \underline{\beta_2}|(ab - \underline{bc} + 2ba + ac) = d_1^1(\beta|(a - c)), \\ & g_{8,2}^2 = \underline{\gamma_2}|(\underline{ab} + 2bc - ba - 2ac) = d_1^1(\gamma|(b - a))\}. \end{aligned}$$

By definition, $\tilde{\mathfrak{B}}_2^2 \subseteq \tilde{B}_2^2$. Since

$$d_1^1(\alpha|a) = g_{4,2}^2 - g_{5,2}^2,$$

the elements in $\tilde{\mathfrak{B}}_2^2$ span the space $\tilde{B}_2^2$. By Fact 4.1.3, the elements in $\tilde{\mathfrak{B}}_2^2$ are linearly independent, so $\tilde{\mathfrak{B}}_2^2$ is a basis of $\tilde{B}_2^2$. If $n \geqslant 3$ is odd, we define the set

$$\begin{aligned} \tilde{\mathfrak{B}}_2^n = \{ & e_{1,2}^n = \alpha_{n-1}\beta|(ab - ba) + \alpha_{n-1}\gamma|(ab + bc + ac) = d_1^{n-1}(\alpha_{n-1}|a), \\ & e_{2,2}^n = (\alpha_{n-1}\beta + \alpha_{n-1}\gamma + \alpha_{n-2}\beta_2)|(ab - ba) = d_1^{n-1}(\alpha_{n-1}|a - \beta_{n-1}|b), \\ & e_{3,2}^n = (\alpha_{n-1}\beta + \alpha_{n-1}\gamma + \alpha_{n-2}\beta_2)|(ab + bc + ac) = d_1^{n-1}(\alpha_{n-1}|a - \gamma_{n-1}|c), \\ & e_{4,2}^n = 2\alpha_{n-1}\beta|bc + \alpha_{n-1}\gamma|(ab + ba) + 2\alpha_{n-2}\beta_2|ac = d_1^{n-1}(\alpha_{n-2}\beta|b + \alpha_{n-2}\gamma|a), \\ & e_{5,2}^n = \alpha_{n-1}\beta|ac - \alpha_{n-1}\gamma|ba + \alpha_{n-2}\beta_2|(2ab + bc) = d_1^{n-1}(\alpha_{n-2}\beta|a), \\ & e_{6,2}^n = \alpha_{n-1}\gamma|(ab + bc - ac) + \alpha_{n-2}\beta_2|(ac - ab - bc) \\ & \quad\;\; = e_{3,2}^n + d_1^{n-1}(\alpha_{n-2}\gamma|c - \alpha_{n-2}\beta|a), \\ & e_{7,2}^n = \underline{\beta_n}|(ab - \underline{ba}) + \alpha_{n-1}\gamma|(ab + bc + ac) = d_1^{n-1}(\beta_{n-1}|a), \\ & e_{8,2}^n = (\underline{\beta_n} + \alpha_{n-1}\gamma + \alpha_{n-2}\beta_2)|(ab + \underline{bc} + ac) = d_1^{n-1}(\beta_{n-1}|(a - c)), \\ & e_{9,2}^n = \underline{\gamma_n}|(ab + \underline{bc} + ac) + \alpha_{n-1}\beta|(ab - ba) = d_1^{n-1}(\gamma_{n-1}|a), \\ & e_{10,2}^n = (\underline{\gamma_n} + \alpha_{n-1}\beta + \alpha_{n-2}\beta_2)|(ab - \underline{ba}) = d_1^{n-1}(\gamma_{n-1}|(a - b)), \\ & e_{11,2}^n = (\underline{\alpha_n} + \beta_n + \alpha_{n-1}\gamma)|(ab - \underline{ba}) = d_1^{n-1}(\beta_{n-1}|a - \alpha_{n-1}|b), \\ & e_{12,2}^n = (\underline{\alpha_n} + \gamma_n + \alpha_{n-1}\beta)|(ab + \underline{bc} + ac) = d_1^{n-1}(\gamma_{n-1}|a - \alpha_{n-1}|c)\}. \end{aligned}$$

By definition, $\tilde{\mathfrak{B}}_2^n \subseteq \tilde{B}_2^n$. Since

$$d_1^{n-1}(\alpha_{n-2}\beta|a) = e_{5,2}^n, \qquad d_1^{n-1}(\alpha_{n-2}\beta|b) = e_{1,2}^n - e_{2,2}^n - e_{3,2}^n + e_{4,2}^n + e_{5,2}^n,$$
$$d_1^{n-1}(\alpha_{n-2}\beta|c) = -e_{1,2}^n + e_{2,2}^n - e_{5,2}^n - e_{6,2}^n, \qquad d_1^{n-1}(\alpha_{n-2}\gamma|a) = -e_{1,2}^n + e_{2,2}^n + e_{3,2}^n - e_{5,2}^n,$$

$$d_1^{n-1}(\alpha_{n-2}\gamma|b) = -2e^n_{1,2} + 2e^n_{3,2} - e^n_{4,2} - e^n_{5,2}, \quad d_1^{n-1}(\alpha_{n-2}\gamma|c) = -e^n_{3,2} + e^n_{5,2} + e^n_{6,2}$$

for $n \geqslant 3$ with n odd, and

$$d_1^{n-1}(\alpha_{n-3}\beta_2|a) = e^n_{1,2}, \quad d_1^{n-1}(\alpha_{n-3}\beta_2|b) = e^n_{1,2} - e^n_{2,2}, \quad d_1^{n-1}(\alpha_{n-3}\beta_2|c) = e^n_{1,2} - e^n_{3,2}$$

for $n \geqslant 5$ with n odd, the elements in $\tilde{\mathfrak{B}}^n_2$ span the space $\tilde{B}^n_2$. The reader can easily verify that the elements $e^n_{\ell,3}$ for $\ell \in [\![1,6]\!]$ are linearly independent. By Fact 4.1.3, the elements $e^n_{\ell,3}$ for $\ell \in [\![7,12]\!]$ are linearly independent. Since the underlined terms of $e^n_{\ell,3}$ for $\ell \in [\![7,12]\!]$ do not appear in $e^n_{\ell,3}$ for $\ell \in [\![1,6]\!]$, the elements in $\tilde{\mathfrak{B}}^n_2$ are linearly independent. So $\tilde{\mathfrak{B}}^n_2$ is a basis of $\tilde{B}^n_2$. If $n \geqslant 4$ is even, we define the set

$$\begin{aligned}
\tilde{\mathfrak{B}}^n_2 = \{ g^n_{1,2} &= \underline{\alpha_n}|(ab + \underline{ba}) = d_1^{n-1}(\alpha_{n-1}|b) + g^n_{11,2} + g^n_{12,2},\\
g^n_{2,2} &= \underline{\alpha_n}|(ab + \underline{bc} - ac) = -d_1^{n-1}(\alpha_{n-1}|c) - g^n_{11,2} - g^n_{12,2}\\
g^n_{3,2} &= \underline{\beta_n}|(ab + \underline{ba}) = d_1^{n-1}(\beta_{n-1}|a) - g^n_{12,2},\\
g^n_{4,2} &= \underline{\beta_n}|(ab + \underline{bc} - ac) = d_1^{n-1}(\beta_{n-1}|(a+c)) - 2g^n_{12,2},\\
g^n_{5,2} &= \underline{\gamma_n}|(ab + \underline{ba}) = -d_1^{n-1}(\gamma_{n-1}|(a+b)) + 2g^n_{11,2},\\
g^n_{6,2} &= \underline{\gamma_n}|(ab + \underline{bc} - ac) = -d_1^{n-1}(\gamma_{n-1}|a) + g^n_{11,2},\\
g^n_{7,2} &= \underline{\alpha_{n-2}\beta_2}|(ab + \underline{ba}) = (1/3)d_1^{n-1}(\alpha_{n-2}\beta|(a-c) + \alpha_{n-3}\beta_2|(b-c)),\\
g^n_{8,2} &= \underline{\alpha_{n-2}\beta_2}|(ab + \underline{bc} - ac) = (1/3)d_1^{n-1}(2\alpha_{n-3}\beta_2|(b-c) - \alpha_{n-2}\beta|(a-c)),\\
g^n_{9,2} &= \alpha_{n-1}\beta|ab + \underline{\alpha_{n-1}\gamma|ba} = -d_1^{n-1}(\gamma_{n-1}|c),\\
g^n_{10,2} &= \alpha_{n-1}\beta|(ab + bc) - \underline{\alpha_{n-1}\gamma|ac} = d_1^{n-1}(\beta_{n-1}|b),\\
g^n_{11,2} &= \underline{\alpha_{n-1}\beta|ba} + \alpha_{n-1}\gamma|ab = d_1^{n-1}(\alpha_{n-2}\gamma|a) + g^n_{8,2},\\
g^n_{12,2} &= \underline{\alpha_{n-1}\beta|ac} - \alpha_{n-1}\gamma|(ab + bc) = d_1^{n-1}(\alpha_{n-2}\beta|a) - g^n_{7,2}\}.
\end{aligned}$$

By definition, $\tilde{\mathfrak{B}}^n_2 \subseteq \tilde{B}^n_2$. Since

$$\begin{aligned}
&d_1^{n-1}(\alpha_{n-1}|a) = d_1^{n-1}(\alpha_{n-3}\beta_2|a) = g^n_{9,2} - g^n_{10,2}, && d_1^{n-1}(\alpha_{n-1}|b) = g^n_{1,2} - g^n_{11,2} - g^n_{12,2},\\
&d_1^{n-1}(\alpha_{n-1}|c) = -g^n_{2,2} - g^n_{11,2} - g^n_{12,2}, && d_1^{n-1}(\beta_{n-1}|a) = g^n_{3,2} + g^n_{12,2},\\
&d_1^{n-1}(\beta_{n-1}|b) = d_1^{n-1}(\alpha_{n-2}\beta|b) = g^n_{10,2}, && d_1^{n-1}(\beta_{n-1}|c) = g^n_{4,2} - g^n_{3,2} + g^n_{12,2},\\
&d_1^{n-1}(\gamma_{n-1}|a) = -g^n_{6,2} + g^n_{11,2}, && d_1^{n-1}(\gamma_{n-1}|b) = -g^n_{5,2} + g^n_{6,2} + g^n_{11,2},\\
&d_1^{n-1}(\gamma_{n-1}|c) = d_1^{n-1}(\alpha_{n-2}\gamma|c) = -g^n_{9,2}, && d_1^{n-1}(\alpha_{n-2}\beta|a) = g^n_{7,2} + g^n_{12,2},\\
&d_1^{n-1}(\alpha_{n-2}\beta|c) = -g^n_{7,2} + g^n_{8,2} + g^n_{12,2}, && d_1^{n-1}(\alpha_{n-2}\gamma|a) = g^n_{11,2} - g^n_{8,2},\\
&d_1^{n-1}(\alpha_{n-2}\gamma|b) = -g^n_{7,2} + g^n_{8,2} + g^n_{11,2}, && d_1^{n-1}(\alpha_{n-3}\beta_2|b) = g^n_{7,2} - g^n_{11,2} - g^n_{12,2},\\
&d_1^{n-1}(\alpha_{n-3}\beta_2|c) = -g^n_{8,2} - g^n_{11,2} - g^n_{12,2},
\end{aligned}$$

the elements in $\tilde{\mathfrak{B}}^n_2$ span the space $\tilde{B}^n_2$. By Fact 4.1.3, the elements in $\tilde{\mathfrak{B}}^n_2$ are linearly independent, so $\tilde{\mathfrak{B}}^n_2$ is a basis of $\tilde{B}^n_2$. Hence, the dimension of $\tilde{B}^n_2$ is given by

$$\dim \tilde{B}^n_2 = \begin{cases} 0, & \text{if } n = 0,\\ 3, & \text{if } n = 1,\\ 8, & \text{if } n = 2,\\ 12, & \text{if } n \geqslant 3. \end{cases} \tag{4.2.23}$$

Suppose finally $m = 1$. If $n = 1$, since $d_0^0(\epsilon^!|1) = 0$, we have $\tilde{B}^1_1 = 0$. We define $\tilde{\mathfrak{B}}^1_1 = \emptyset$. If $n \geqslant 3$ is odd, by Table 4.2.20, the space $\tilde{B}^n_1$ is spanned by the element $\alpha_{n-1}\beta|(c-a) + \alpha_{n-1}\gamma|(a-b) + \alpha_{n-2}\beta_2|(b-c)$. So we define a basis of $\tilde{B}^n_1$ by

$$\tilde{\mathfrak{B}}^n_1 = \{\underline{\alpha_{n-1}\beta}|(c - \underline{a}) + \alpha_{n-1}\gamma|(a-b) + \alpha_{n-2}\beta_2|(b-c)\}.$$

If $n = 2$, by Table 4.2.15, we define a basis of $\tilde{B}^2_1$ by

$$\tilde{\mathfrak{B}}^2_1 = \{2\underline{\alpha_2|a} + (\alpha\beta + \alpha\gamma)|(b+c), 2\underline{\beta_2|b} + (\alpha\beta + \alpha\gamma)|(a+c), 2\underline{\gamma_2|c} + (\alpha\beta + \alpha\gamma)|(a+b)\}.$$

If $n \geqslant 4$ is even, by Table 4.2.15, we define a basis of $\tilde{B}_1^n$ by

$$\tilde{\mathfrak{B}}_1^n = \{2\underline{\alpha_n|a} + (\alpha_{n-1}\beta + \alpha_{n-1}\gamma)|(b+c), 2\underline{\beta_n|b} + (\alpha_{n-1}\beta + \alpha_{n-1}\gamma)|(a+c),$$
$$2\underline{\gamma_n|c} + (\alpha_{n-1}\beta + \alpha_{n-1}\gamma)|(a+b), (\alpha_{n-1}\beta + \alpha_{n-1}\gamma)|(a+c) + 2\underline{\alpha_{n-2}\beta_2|b},$$
$$(\alpha_{n-1}\beta + \alpha_{n-1}\gamma)|(a+b) + 2\underline{\alpha_{n-2}\beta_2|c}, (\alpha_{n-1}\beta + \alpha_{n-1}\gamma)|(b+c) + 2\underline{\alpha_{n-2}\beta_2|a}\}.$$

In conclusion, the dimension of $\tilde{B}_1^n$ is given by

$$\dim \tilde{B}_1^n = \begin{cases} 0, & \text{if } n = 0{,}1, \\ 3, & \text{if } n = 2, \\ 1, & \text{if } n \geqslant 3 \text{ is odd}, \\ 6, & \text{if } n \geqslant 4 \text{ is even}. \end{cases} \tag{4.2.24}$$

Computation of $\mathfrak{B}_m^n$ Recall that $B_m^n = \operatorname{Im}(\partial_{m-1}^{n-1})$ and $\partial_m^n : Q_m^n \to Q_{m+1}^{n+1}$. Since $d_m^n = \partial_m^n$ for either $m = 3$ and $n \geqslant -1$, or $m, n \in [\![-1{,}2]\!]$, we get $B_m^n = \tilde{B}_m^n$ for either $m = 4$ and $n \in \mathbb{N}_0$, or $m, n \in [\![0{,}3]\!]$. So, we define a basis of B_m^n by $\mathfrak{B}_m^n = \tilde{\mathfrak{B}}_m^n$ for either $m = 4$ and $n \in \mathbb{N}_0$, or $m, n \in [\![0{,}3]\!]$.

Suppose $m = 3$. The differential $\partial_2^{n-1} : K_2^{n-1} \oplus \omega_1^* K_4^{n-5} \to K_3^n$ maps the space $\omega_1^* K_4^{n-5}$ to zero, so $B_3^n = \operatorname{Im}(\partial_2^{n-1}) = \operatorname{Im}(d_2^{n-1}) = \tilde{B}_3^n$. We define a basis of B_3^n by $\mathfrak{B}_3^n = \tilde{\mathfrak{B}}_3^n$.

Suppose $m = 2$. Consider $\partial_1^{n-1} : K_1^{n-1} \oplus \omega_1^* K_3^{n-5} \to K_2^n \oplus \omega_1^* K_4^{n-4}$. If $n \geqslant 4$ is even, we get $B_2^n = \tilde{B}_2^n \oplus \omega_1^* \tilde{B}_4^{n-4}$, since $f^{n-4}(K_1^{n-1}) = 0$ by the last identity of Subsubsection 4.2.2.2 for $n > 4$ and $f^0(K_1^3) = 0$ by the last three columns of Table 4.2.21, as well as $f^{n-8}(u) = 0$ for $u \in K_3^{n-5}$ by degree reasons. If $n \geqslant 5$ is odd, we have $B_2^n = \tilde{B}_2^n \oplus \omega_1^* \tilde{B}_4^{n-4}$, since $\tilde{B}_4^{n-4} = K_4^{n-4}$ as showed in the previous subsubsection. So, we define a basis of B_2^n by $\mathfrak{B}_2^n = \tilde{\mathfrak{B}}_2^n \cup \omega_1^* \tilde{\mathfrak{B}}_4^{n-4}$ for all integers $n \geqslant 4$. The dimension of B_2^n is then given by

$$\dim B_2^n = \begin{cases} 0, & \text{if } n = 0, \\ 3, & \text{if } n = 1, \\ 8, & \text{if } n = 2, \\ 12, & \text{if } n = 3{,}4, \\ 15, & \text{if } n = 5, \\ 13, & \text{if } n \geqslant 6 \text{ is even}, \\ 18, & \text{if } n \geqslant 7 \text{ is odd}. \end{cases} \tag{4.2.25}$$

Suppose $m = 1$. Consider $\partial_0^{n-1} : K_0^{n-1} \oplus \omega_1^* K_2^{n-5} \oplus \omega_2^* K_4^{n-9} \to K_1^n \oplus \omega_1^* K_3^{n-4}$. If $n \geqslant 5$ is odd, we have $B_1^n = \tilde{B}_1^n \oplus \omega_1^* B_3^{n-4} = \tilde{B}_1^n \oplus \omega_1^* \tilde{B}_3^{n-4}$, since $f^{n-4}(K_0^{n-1}) = 0$ by the second column of Table 4.2.22 and $f^{n'}(u) = 0$ for $u \in K_m^{n'+3}$, with $m \in [\![2{,}4]\!]$ and $n' \in \mathbb{N}_0$, by degree reasons. Then we define a basis of B_1^n by $\mathfrak{B}_1^n = \tilde{\mathfrak{B}}_1^n \cup \omega_1^* \tilde{\mathfrak{B}}_3^{n-4}$. If $n = 4$, we define the set

$$\mathfrak{B}_1^4 = \{2\underline{\alpha_4|a} + (\alpha_3\beta + \alpha_3\gamma)|(b+c) + \omega_1^* 4\epsilon^!|(bac - aba + abc) = \partial_0^3(\alpha_3|1),$$
$$2\underline{\beta_4|b} + (\alpha_3\beta + \alpha_3\gamma)|(a+c) + \omega_1^* 4\epsilon^!|(bac - aba + abc) = \partial_0^3(\beta_3|1),$$
$$2\underline{\gamma_4|c} + (\alpha_3\beta + \alpha_3\gamma)|(a+b) + \omega_1^* 4\epsilon^!|(bac - aba + abc) = \partial_0^3(\gamma_3|1),$$
$$(\alpha_3\beta + \alpha_3\gamma)|(a+c) + 2\underline{\alpha_2\beta_2|b} + \omega_1^* 2\epsilon^!|(aba - abc - bac) = \partial_0^3(\alpha_2\beta|1),$$
$$(\alpha_3\beta + \alpha_3\gamma)|(a+b) + 2\underline{\alpha_2\beta_2|c} + \omega_1^* 2\epsilon^!|(aba - abc - bac) = \partial_0^3(\alpha_2\gamma|1),$$
$$(\alpha_3\beta + \alpha_3\gamma)|(b+c) + 2\underline{\alpha_2\beta_2|a} + \omega_1^* 2\epsilon^!|(aba - abc - bac) = \partial_0^3(\alpha\beta_2|1)\}.$$

Since these six elements are linearly independent by Fact 4.1.3, $\mathfrak{B}_1^4$ is a basis of B_1^4. If $n \geqslant 6$ is even, we define the set

$$\mathfrak{B}_1^n = \{2\underline{\alpha_n|a} + (\alpha_{n-1}\beta + \alpha_{n-1}\gamma)|(b+c) + \omega_1^*[2\alpha_{n-4}|(2bac - aba + abc) + 2\beta_{n-4}|(2abc + bac)$$
$$+ 2\gamma_{n-4}|(bac - 2aba) + 2(n-6)\alpha_{n-6}\beta_2|(bac - aba + abc)] = \partial_0^{n-1}(\alpha_{n-1}|1),$$
$$2\underline{\beta_n|b} + (\alpha_{n-1}\beta + \alpha_{n-1}\gamma)|(a+c) + \omega_1^*[2\alpha_{n-4}|(2bac + abc) + 2\beta_{n-4}|(2abc - aba + bac)$$
$$+ 2\gamma_{n-4}|(abc - 2aba) + 2(n-6)\alpha_{n-6}\beta_2|(bac - aba + abc)] = \partial_0^{n-1}(\beta_{n-1}|1),$$

$$
\begin{aligned}
&2\underline{\gamma_n|c} + (\alpha_{n-1}\beta + \alpha_{n-1}\gamma)|(a+b) + \omega_1^*[2\alpha_{n-4}|(2bac - aba) + 2\beta_{n-4}|(2abc - aba)\\
&\ + 2\gamma_{n-4}|(bac - 2aba + abc) + 2(n-6)\alpha_{n-6}\beta_2|(bac - aba + abc)] = \partial_0^{n-1}(\gamma_{n-1}|1),\\
&(\alpha_{n-1}\beta + \alpha_{n-1}\gamma)|(a+c) + 2\underline{\alpha_{n-2}\beta_2|b} + \omega_1^* 2(\gamma_{n-4}|aba - \alpha_{n-4}|bac - \beta_{n-4}|abc)\\
&\ = \partial_0^{n-1}(\alpha_{n-2}\beta|1),\\
&(\alpha_{n-1}\beta + \alpha_{n-1}\gamma)|(a+b) + 2\underline{\alpha_{n-2}\beta_2|c} + \omega_1^* 2(\gamma_{n-4}|aba - \alpha_{n-4}|bac - \beta_{n-4}|abc)\\
&\ = \partial_0^{n-1}(\alpha_{n-2}\gamma|1),\\
&(\alpha_{n-1}\beta + \alpha_{n-1}\gamma)|(b+c) + 2\underline{\alpha_{n-2}\beta_2|a} + \omega_1^* 2(\gamma_{n-4}|aba - \alpha_{n-4}|bac - \beta_{n-4}|abc)\\
&\ = \partial_0^{n-1}(\alpha_{n-3}\beta_2|1)\big\} \cup \omega_1^*\tilde{\mathfrak{B}}_3^{n-4}.
\end{aligned}
$$

Since $f^{n'}(u) = 0$ for $u \in K_m^{n'+3}$, with $m \in [\![2,4]\!]$ and $n' \in \mathbb{N}_0$, by degree reasons, the previous set is a system of generators of B_1^n. By Fact 4.1.3, the elements in $\mathfrak{B}_1^n$ are linearly independent, so $\mathfrak{B}_1^n$ is a basis of B_1^n. The dimension of B_1^n is then given by

$$
\dim B_1^n = \begin{cases} 0, & \text{if } n = 0,1,\\ 3, & \text{if } n = 2,5,\\ 1, & \text{if } n = 3,\\ 6, & \text{if } n = 4,\\ 15, & \text{if } n = 6,\\ 11, & \text{if } n = 7,\\ 18, & \text{if } n \geqslant 8 \text{ is even},\\ 13, & \text{if } n \geqslant 9 \text{ is odd}. \end{cases} \tag{4.2.26}
$$

Suppose finally $m = 0$. Consider $\partial_{-1}^{n-1} : \omega_1^* K_1^{n-5} \oplus \omega_2^* K_3^{n-9} \to K_0^n \oplus \omega_1^* K_2^{n-4} \oplus \omega_2^* K_4^{n-8}$. Note that $B_0^n \subseteq \omega_1^* K_2^{n-4} \oplus \omega_2^* K_4^{n-8}$ and $B_0^n = \operatorname{Im}(\partial_{-1}^{n-1}) = \operatorname{Im}(\omega_1^*\partial_1^{n-5}) = \omega_1^* B_2^{n-4}$. We define a basis of B_0^n by $\mathfrak{B}_0^n = \omega_1^*\mathfrak{B}_2^{n-4}$ for $n \geqslant 4$. The dimension of B_0^n is thus given by

$$
\dim B_0^n = \begin{cases} 0, & \text{if } n \in [\![0,4]\!],\\ 3, & \text{if } n = 5,\\ 8, & \text{if } n = 6,\\ 12, & \text{if } n = 7,8,\\ 15, & \text{if } n = 9,\\ 13, & \text{if } n \geqslant 10 \text{ is even},\\ 18, & \text{if } n \geqslant 11 \text{ is odd}. \end{cases}
$$

4.2.2.4 Computation of the cocycles

As one can remark rather easily, from the computations in the previous subsubsection we can already deduce the dimensions of the homogeneous components of the spaces of cocycles and thus of the Hochschild cohomology groups. However, since we will need specific representatives of the cohomology classes of bases of the Hochschild cohomology $\mathrm{HH}^\bullet(A)$ for computing its algebra structure, we will present them. More precisely, in this subsubsection, we will explicitly construct bases $\tilde{\mathfrak{D}}_m^n$ and $\mathfrak{D}_m^n$ of the $\Bbbk$-vector spaces $\tilde{D}_m^n = \operatorname{Ker}(d_m^n)$ and $D_m^n = \operatorname{Ker}(\partial_m^n)$ for $m \in [\![0,4]\!]$ and $n \in \mathbb{N}_0$, respectively, defined before Remark 4.2.12.

Computation of $\tilde{\mathfrak{D}}_m^n$ Recall that $\tilde{D}_m^n = \operatorname{Ker}(d_m^n)$ and

$$
d_m^n : K_m^n = \operatorname{Hom}_\Bbbk((A^!_{-n})^*, A_m) \to K_{m+1}^{n+1} = \operatorname{Hom}_\Bbbk((A^!_{-(n+1)})^*, A_{m+1})
$$

was defined in Subsubsection 4.2.2.1. Since $K_m^n/\tilde{D}_m^n \cong \tilde{B}_{m+1}^{n+1}$, we see that

$$
\dim \tilde{D}_m^n = \dim K_m^n - \dim \tilde{B}_{m+1}^{n+1}.
$$

Hence, from the dimension of $\tilde{B}_{m+1}^{n+1}$ computed in Subsubsection 4.2.2.3 as well as the dimension of K_m^n (see the last paragraph of Subsubsection 4.2.2.1), we deduce the value of the dimension of $\tilde{D}_m^n$. We will present them explicitly in the computations below.

For every $(n,m) \in \mathbb{N}_0 \times [\![0,4]\!]$, we are going to provide a set $\tilde{\mathfrak{D}}_m^n \subseteq \tilde{D}_m^n$ such that $\#\tilde{\mathfrak{D}}_m^n = \dim \tilde{D}_m^n$ and the elements in $\tilde{\mathfrak{D}}_m^n$ are linearly independent. As a consequence, $\tilde{\mathfrak{D}}_m^n$ is a basis of $\tilde{D}_m^n$. If $\tilde{D}_m^n = K_m^n$, we pick the usual basis of K_m^n, defined at the end of Subsubsection 4.2.2.1. We leave to the reader the easy verification in each case that the set $\tilde{\mathfrak{D}}_m^n$ satisfies these conditions.

Obviously, $\tilde{D}_4^n = K_4^n$ for $n \in \mathbb{N}_0$. Then we define the set $\tilde{\mathfrak{D}}_4^n$ by the usual basis of K_4^n. The dimension of $\tilde{D}_4^n$ is given by

$$\dim \tilde{D}_4^n = \begin{cases} 1, & \text{if } n = 0, \\ 3, & \text{if } n = 1, \\ 5, & \text{if } n = 2, \\ 6, & \text{if } n \geqslant 3. \end{cases}$$

Suppose $m = 3$. By (4.2.21), the dimension of $\tilde{D}_3^n$ is given by

$$\dim \tilde{D}_3^n = \begin{cases} 0, & \text{if } n = 0, \\ 8, & \text{if } n = 1, \\ 9, & \text{if } n = 2, \\ 17, & \text{if } n \geqslant 3 \text{ is odd}, \\ 12, & \text{if } n \geqslant 4 \text{ is even}. \end{cases}$$

We define the sets $\tilde{\mathfrak{D}}_3^0 = \emptyset$,

$$\begin{aligned} \tilde{\mathfrak{D}}_3^1 = \big\{ & \underline{\alpha|bac}, \underline{\beta|abc}, \underline{\gamma|aba}, \underline{\alpha}|(aba - \underline{abc}), (\alpha + \underline{\beta})|\underline{aba}, \alpha|aba + \underline{\beta|bac}, \alpha|aba + \underline{\gamma|abc}, \\ & \alpha|aba - \underline{\gamma|bac} \big\}, \end{aligned}$$

and

$$\begin{aligned} \tilde{\mathfrak{D}}_3^2 = \big\{ & (\underline{\alpha_2} - \beta_2)|\underline{aba}, (\underline{\alpha_2} - \gamma_2)|\underline{abc}, (\underline{\beta_2} - \gamma_2)|\underline{bac}, \alpha_2|abc + \beta_2|bac + 2\underline{\alpha\beta|aba}, \\ & \alpha_2|aba - \beta_2|bac + 2\underline{\alpha\beta|abc}, \alpha_2|(aba - abc) + 2\underline{\alpha\beta|bac}, \alpha_2|abc + \beta_2|bac + 2\underline{\alpha\gamma|aba}, \\ & \alpha_2|aba - \beta_2|bac + 2\underline{\alpha\gamma|abc}, \alpha_2|(aba - abc) + 2\underline{\alpha\gamma|bac} \big\}. \end{aligned}$$

Moreover, if $n \geqslant 3$ is odd, we define

$$\begin{aligned} \tilde{\mathfrak{D}}_3^n = \big\{ & \underline{\alpha_n|bac}, \underline{\beta_n|abc}, \underline{\gamma_n|aba}, \underline{\alpha_{n-1}\beta|abc}, \underline{\alpha_{n-1}\gamma|aba}, \underline{\alpha_{n-2}\beta_2|bac}, \underline{\alpha_n}|(aba - \underline{abc}), (\alpha_n + \underline{\beta_n})|\underline{aba}, \\ & \alpha_n|aba + \underline{\beta_n|bac}, \alpha_n|aba + \underline{\gamma_n|abc}, \alpha_n|aba - \underline{\gamma_n|bac}, (\alpha_n + \underline{\alpha_{n-1}\beta})|\underline{aba}, \\ & \alpha_n|aba + \underline{\alpha_{n-1}\beta|bac}, \alpha_n|aba + \underline{\alpha_{n-1}\gamma|abc}, \alpha_n|aba - \underline{\alpha_{n-1}\gamma|bac}, (\alpha_n - \underline{\alpha_{n-2}\beta_2})|\underline{aba}, \\ & \alpha_n|aba - \underline{\alpha_{n-2}\beta_2|abc} \big\}, \end{aligned}$$

and if $n \geqslant 4$ is even, we set

$$\begin{aligned} \tilde{\mathfrak{D}}_3^n = \big\{ & (\underline{\alpha_n} - \beta_n)|\underline{aba}, (\underline{\alpha_n} - \gamma_n)|\underline{abc}, (\underline{\beta_n} - \gamma_n)|\underline{bac}, \alpha_n|abc + \beta_n|bac + 2\underline{\alpha_{n-1}\beta|aba}, \\ & \alpha_n|aba - \beta_n|bac + 2\underline{\alpha_{n-1}\beta|abc}, \alpha_n|(aba - abc) + 2\underline{\alpha_{n-1}\beta|bac}, \\ & \alpha_n|abc + \beta_n|bac + 2\underline{\alpha_{n-1}\gamma|aba}, \alpha_n|aba - \beta_n|bac + 2\underline{\alpha_{n-1}\gamma|abc}, \\ & \alpha_n|(aba - abc) + 2\underline{\alpha_{n-1}\gamma|bac}, (\underline{\alpha_{n-2}\beta_2} - \alpha_n)|\underline{aba}, (\underline{\alpha_{n-2}\beta_2} - \alpha_n)|\underline{abc}, \\ & (\underline{\alpha_{n-2}\beta_2} - \beta_n)|\underline{bac} \big\}. \end{aligned}$$

Suppose $m = 2$. By (4.2.22), the dimension of $\tilde{D}_2^n$ is given by

$$\dim \tilde{D}_2^n = \begin{cases} 2, & \text{if } n = 0, \\ 3, & \text{if } n = 1, \\ 10, & \text{if } n = 2, \\ 12, & \text{if } n \geqslant 3. \end{cases}$$

We define the sets

$$\tilde{\mathfrak{D}}_2^0 = \big\{ \underline{\epsilon^!}|(ab + \underline{ba}), \underline{\epsilon^!}|(ab + bc - \underline{ac}) \big\},$$

$\tilde{\mathfrak{D}}_2^1 = \big\{\beta|(ab-ba)+\underline{\gamma}|(\underline{ab}+bc+ac), (\alpha+\beta+\underline{\gamma})|(ab-\underline{ba}), \underline{\alpha}|(ab+\underline{bc}+ac)+\beta|(bc+ba+ac)\big\}$,

$$\begin{aligned}\tilde{\mathfrak{D}}_2^2 = \big\{&\underline{\alpha_2}|(ab+\underline{ba}), \underline{\alpha_2}|(ab+\underline{bc}-ac), \underline{\beta_2}|(ab+\underline{ba}), \underline{\beta_2}|(ab+\underline{bc}-ac), \underline{\gamma_2}|(ab+\underline{ba}),\\ &\underline{\gamma_2}|(ab+\underline{bc}-ac), \underline{\alpha\beta|ba}+\alpha\gamma|ab, \underline{\alpha\beta|ab}+\alpha\gamma|ba, \underline{\alpha\beta}|(ba+\underline{ac})-\alpha\gamma|bc,\\ &\underline{\alpha\beta}|(ab+\underline{bc})-\alpha\gamma|ac\big\},\end{aligned}$$

and $\tilde{\mathfrak{D}}_2^n = \tilde{\mathfrak{B}}_2^n$ for $n \geqslant 3$.

Suppose $m = 1$. By (4.2.23), the dimension of $\tilde{D}_1^n$ is given by

$$\dim \tilde{D}_1^n = \begin{cases} 0, & \text{if } n = 0,\\ 1, & \text{if } n = 1,\\ 3, & \text{if } n = 2,\\ 6, & \text{if } n \geqslant 3.\end{cases}$$

We define the sets $\tilde{\mathfrak{D}}_1^0 = \emptyset$, and

$$\tilde{\mathfrak{D}}_1^1 = \big\{\underline{\alpha|a}+\beta|b+\gamma|c\big\}.$$

Moreover, if $n \geqslant 2$ is even, we define $\tilde{\mathfrak{D}}_1^n = \tilde{\mathfrak{B}}_1^n$, and if $n \geqslant 3$ is odd, we define

$$\begin{aligned}\tilde{\mathfrak{D}}_1^n = \big\{&\underline{\alpha_n|a}+\beta_n|b+\gamma_n|c, (\beta_n-\underline{\alpha_{n-1}\beta})|\underline{b}, (\gamma_n-\underline{\alpha_{n-1}\gamma})|\underline{c}, (\alpha_n-\underline{\alpha_{n-2}\beta_2})|\underline{a},\\ &\underline{\alpha_{n-1}\beta|c}+\alpha_{n-1}\gamma|a+\alpha_{n-2}\beta_2|b, \underline{\alpha_{n-1}\beta|a}+\alpha_{n-1}\gamma|b+\alpha_{n-2}\beta_2|c\big\}.\end{aligned}$$

Suppose finally $m = 0$. By (4.2.24), the dimension of $\tilde{D}_0^n$ is given by

$$\dim \tilde{D}_0^n = \begin{cases} 1, & \text{if } n = 0,\\ 0, & \text{if } n \geqslant 1 \text{ is odd},\\ 4, & \text{if } n = 2,\\ 5, & \text{if } n \geqslant 4 \text{ is even}.\end{cases}$$

We define the sets

$$\tilde{\mathfrak{D}}_0^0 = \big\{\underline{\epsilon^!|1}\big\},$$

and

$$\tilde{\mathfrak{D}}_0^2 = \big\{\underline{\alpha_2|1}, \underline{\beta_2|1}, \underline{\gamma_2|1}, (\underline{\alpha\beta}+\alpha\gamma)|\underline{1}\big\}.$$

Moreover, if $n \in \mathbb{N}$ is odd, we define $\tilde{\mathfrak{D}}_0^n = \emptyset$, and if $n \geqslant 4$ is even, we define

$$\tilde{\mathfrak{D}}_0^n = \big\{\underline{\alpha_n|1}, \underline{\beta_n|1}, \underline{\gamma_n|1}, (\underline{\alpha_{n-1}\beta}+\alpha_{n-1}\gamma)|\underline{1}, \underline{\alpha_{n-2}\beta_2|1}\big\}.$$

Computation of $\mathfrak{D}_m^n$ Recall that $D_m^n = \operatorname{Ker}(\partial_m^n)$ and $\partial_m^n : Q_m^n \to Q_{m+1}^{n+1}$. The isomorphism $Q_m^n/D_m^n \cong B_{m+1}^{n+1}$ tells us that

$$\dim D_m^n = \dim Q_m^n - \dim B_{m+1}^{n+1}.$$

Hence, from the dimension of B_{m+1}^{n+1} computed in Subsubsection 4.2.2.3 as well as the dimension of Q_m^n (see the last paragraph of Subsubsection 4.2.2.1), we deduce the value of the dimension of D_m^n. We will present them explicitly in the computations below.

For every $(n,m) \in \mathbb{N}_0 \times [\![0,4]\!]$, we are going to provide a set $\mathfrak{D}_m^n \subseteq D_m^n$ such that $\#\mathfrak{D}_m^n = \dim D_m^n$ and the elements in $\mathfrak{D}_m^n$ are linearly independent. As a consequence, $\mathfrak{D}_m^n$ is a basis of D_m^n. We leave to the reader the easy verification in each case that the set $\mathfrak{D}_m^n$ satisfies these conditions.

For either $m \in [\![3,4]\!]$ and $n \in \mathbb{N}_0$, or $m, n \in [\![0,2]\!]$, note that $\partial_m^n = d_m^n$, then $D_m^n = \tilde{D}_m^n$. So we define the basis of D_m^n by $\mathfrak{D}_m^n = \tilde{\mathfrak{D}}_m^n$.

Suppose $m = 2$. By $B_3^n = \tilde{B}_3^n$ and (4.2.22), the dimension of D_2^n is given by

$$\dim D_2^n = \begin{cases} 2, & \text{if } n = 0, \\ 3, & \text{if } n = 1, \\ 10, & \text{if } n = 2, \\ 12, & \text{if } n = 3, \\ 13, & \text{if } n = 4, \\ 15, & \text{if } n = 5, \\ 17, & \text{if } n = 6, \\ 18, & \text{if } n \geqslant 7. \end{cases}$$

We define the sets $\mathfrak{D}_2^3 = \tilde{\mathfrak{D}}_2^3$ and $\mathfrak{D}_2^n = \tilde{\mathfrak{D}}_2^n \cup \omega_1^* \tilde{\mathfrak{D}}_4^{n-4}$ for $n \geqslant 4$.

Suppose $m = 1$. By (4.2.25), the dimension of D_1^n is given by

$$\dim D_1^n = \begin{cases} 0, & \text{if } n = 0, \\ 1, & \text{if } n = 1, \\ 3, & \text{if } n = 2, \\ 6, & \text{if } n = 3{,}4, \\ 14, & \text{if } n = 5, \\ 15, & \text{if } n = 6, \\ 23, & \text{if } n \geqslant 7 \text{ is odd}, \\ 18, & \text{if } n \geqslant 8 \text{ is even}. \end{cases}$$

We define the set $\mathfrak{D}_1^3 = \tilde{\mathfrak{D}}_1^3$. Moreover, if $n \geqslant 4$ is even, we define $\mathfrak{D}_1^n = \mathfrak{B}_1^n$, and if $n \geqslant 5$ is odd, we define $\mathfrak{D}_1^n = \tilde{\mathfrak{D}}_1^n \cup \omega_1^* \tilde{\mathfrak{D}}_3^{n-4}$.

Suppose finally $m = 0$. By (4.2.26), the dimension of $\mathfrak{D}_0^n$ is given by

$$\dim D_0^n = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n = 1{,}3, \\ 4, & \text{if } n = 2, \\ 7, & \text{if } n = 4, \\ 3, & \text{if } n = 5, \\ 15, & \text{if } n = 6{,}9, \\ 12, & \text{if } n = 7, \\ 18, & \text{if } n = 8 \text{ or } n \geqslant 11 \text{ is odd}, \\ 22, & \text{if } n = 10, \\ 23, & \text{if } n \geqslant 12 \text{ is even}. \end{cases}$$

We define the set $\mathfrak{D}_0^3 = \emptyset$. Moreover, if $n \geqslant 4$ is even, we define the set $\mathfrak{D}_0^n = \tilde{\mathfrak{D}}_0^n \cup \omega_1^* \mathfrak{D}_2^{n-4}$, and if $n \geqslant 5$ is odd, we define $\mathfrak{D}_0^n = \omega_1^* \mathfrak{D}_2^{n-4}$.

4.2.2.5 Hochschild cohomology

In this subsubsection, we will explicitly construct a subspace $H_m^n \subseteq D_m^n$ such that $D_m^n = H_m^n \oplus B_m^n$ for $(n,m) \in \mathbb{N}_0 \times \mathbb{Z}_{\leqslant 4}$, and we define $H_m^n = 0$ for $(n,m) \in \mathbb{Z}^2 \setminus (\mathbb{N}_0 \times \mathbb{Z}_{\leqslant 4})$. By Proposition 4.2.13, we have the following similar recursive description.

Corollary 4.2.14. *For integers $m \leqslant 1$ and $n \in \mathbb{N}_0$, we have*

$$H_m^n \cong \begin{cases} \omega^*_{\frac{1-m}{2}} H_1^{n+2m-2}, & \text{if } m \text{ is odd}, \\ \omega^*_{-\frac{m}{2}} H_0^{n+2m}, & \text{if } m \text{ is even}. \end{cases}$$

So it is also sufficient to compute the case $m \in [\![0,4]\!]$. Recall that

$$\dim H_m^n = \dim D_m^n - \dim B_m^n = \dim Q_m^n - \dim B_{m+1}^{n+1} - \dim B_m^n.$$

Hence, from the dimension of D^n_m computed in Subsubsection 4.2.2.4 as well as the dimension of B^n_m computed in Subsubsection 4.2.2.3, we deduce the value of the dimension of H^n_m. We will present them explicitly in the computations below.

For every $(n,m) \in \mathbb{N}_0 \times [\![0,4]\!]$, we are going to provide a set $\mathfrak{H}^n_m \subseteq D^n_m$ such that $\#\mathfrak{H}^n_m = \dim H^n_m$ and $\mathfrak{H}^n_m \cup \mathfrak{B}^n_m$ is linearly independent. As a consequence, the space H^n_m spanned by $\mathfrak{H}^n_m$ satisfies $D^n_m = H^n_m \oplus B^n_m$. We leave to the reader the easy verification in each case that the set $\mathfrak{H}^n_m$ satisfies these conditions. Note that, unless stated otherwise, the linear independence of the elements in $\mathfrak{H}^n_m \cup \mathfrak{B}^n_m$ follows from Fact 4.1.3, where we put the elements in $\mathfrak{H}^n_m$ before the elements in $\mathfrak{B}^n_m$.

Suppose $m = 4$. The dimension of H^n_4 is given by

$$\dim H^n_4 = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n \in \mathbb{N} \text{ is odd,} \\ 4, & \text{if } n = 2, \\ 5, & \text{if } n \geqslant 4 \text{ is even.} \end{cases}$$

We define the sets

$$\mathfrak{H}^0_4 = \{\underline{\epsilon^!|abac}\},$$

and

$$\mathfrak{H}^2_4 = \{\underline{\alpha_2|abac},\underline{\beta_2|abac},\underline{\gamma_2|abac},\underline{\alpha\beta|abac}\}.$$

Moreover, if $n \in \mathbb{N}$ is odd, we define $\mathfrak{H}^n_4 = \emptyset$, and if $n \geqslant 4$ is even, we define

$$\mathfrak{H}^n_4 = \{\underline{\alpha_n|abac}, \underline{\beta_n|abac}, \underline{\gamma_n|abac}, \underline{\alpha_{n-1}\beta|abac}, \underline{\alpha_{n-2}\beta_2|abac}\}.$$

Suppose $m = 3$. The dimension of H^n_3 is given by

$$\dim H^n_3 = \begin{cases} 0, & \text{if } n \in \mathbb{N}_0 \text{ is even,} \\ 6, & \text{if } n = 1, \\ 7, & \text{if } n = 3, \\ 5, & \text{if } n \geqslant 5 \text{ is odd.} \end{cases}$$

We define the sets

$$\mathfrak{H}^1_3 = \{\underline{\alpha|bac}, \underline{\beta|abc}, \underline{\gamma|aba}, \underline{\alpha}|(aba - \underline{abc}), (\alpha + \underline{\beta})|\underline{aba}, \alpha|aba + \underline{\beta|bac}\},$$

and

$$\mathfrak{H}^3_3 = \{\alpha_3|bac, \beta_3|abc, \gamma_3|aba, \alpha_2\beta|abc, \alpha_3|(aba - abc), (\alpha_3 + \beta_3)|aba, \alpha_3|aba + \beta_3|bac\}.$$

Moreover, if $n \in \mathbb{N}_0$ is even, we define $\mathfrak{H}^n_3 = \emptyset$, and if $n \geqslant 5$ is odd, we define the set

$$\mathfrak{H}^n_3 = \{\alpha_n|bac, \beta_n|abc, \gamma_n|aba, \alpha_{n-1}\beta|abc, \alpha_n|(aba - abc)\}.$$

The reader can easily verify that the set $\mathfrak{H}^n_3 \cup \mathfrak{B}^n_3$ for $n \geqslant 3$ and n odd is linearly independent.

Suppose $m = 2$. The dimension of H^n_2 is given by

$$\dim H^n_2 = \begin{cases} 2, & \text{if } n = 0,2, \\ 0, & \text{if } n \in \mathbb{N} \text{ is odd,} \\ 1, & \text{if } n = 4, \\ 4, & \text{if } n = 6, \\ 5, & \text{if } n \geqslant 8 \text{ is even.} \end{cases}.$$

We define the sets

$$\mathfrak{H}^0_2 = \{\underline{\epsilon^!}|(ab + \underline{ba}),\underline{\epsilon^!}|(ab + bc - \underline{ac})\},$$

and

$$\mathfrak{H}^2_2 = \{\underline{\alpha_2}|(\underline{ab} + ba),\underline{\beta_2}|(\underline{ab} + ba)\}.$$

Moreover, if $n \in \mathbb{N}$ is odd, we define $\mathfrak{H}^n_2 = \emptyset$, and if $n \geqslant 4$ is even, we define $\mathfrak{H}^n_2 = \omega^*_1\mathfrak{H}^{n-4}_4$.

Suppose $m = 1$. The dimension of H_1^n is given by

$$\dim H_1^n = \begin{cases} 0, & \text{if } n \in \mathbb{N}_0 \text{ is even,} \\ 1, & \text{if } n = 1, \\ 5, & \text{if } n = 3, \\ 11, & \text{if } n = 5, \\ 12, & \text{if } n = 7, \\ 10, & \text{if } n \geqslant 9 \text{ is odd.} \end{cases}$$

We define the sets

$$\mathfrak{H}_1^1 = \{\underline{\alpha|a} + \beta|b + \gamma|c\},$$

and

$$\mathfrak{H}_1^3 = \{\underline{\alpha_3|a} + \beta_3|b + \gamma_3|c, (\beta_3 - \underline{\alpha_2\beta})|\underline{b}, (\gamma_3 - \underline{\alpha_2\gamma})|\underline{c}, (\alpha_3 - \underline{\alpha\beta_2})|\underline{a}, \underline{\alpha_2\beta|c} + \alpha_2\gamma|a + \alpha\beta_2|b\}.$$

Moreover, if $n \in \mathbb{N}_0$ is even, we define $\mathfrak{H}_1^n = \emptyset$, and if $n \geqslant 5$ is odd, we define

$$\mathfrak{H}_1^n = \{\underline{\alpha_n|a} + \beta_n|b + \gamma_n|c, (\beta_n - \underline{\alpha_{n-1}\beta})|\underline{b}, (\gamma_n - \underline{\alpha_{n-1}\gamma})|\underline{c}, (\alpha_n - \underline{\alpha_{n-2}\beta_2})|\underline{a},$$
$$\underline{\alpha_{n-1}\beta|c} + \alpha_{n-1}\gamma|a + \alpha_{n-2}\beta_2|b\} \cup \omega_1^*\mathfrak{H}_3^{n-4}.$$

Suppose finally $m = 0$. The dimension of H_0^n is given by

$$\dim H_0^n = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n \in \mathbb{N} \text{ is odd,} \\ 4, & \text{if } n = 2, \\ 7, & \text{if } n = 4{,}6, \\ 6, & \text{if } n = 8, \\ 9, & \text{if } n = 10, \\ 10, & \text{if } n \geqslant 12 \text{ is even.} \end{cases}$$

We define the sets

$$\mathfrak{H}_0^0 = \{\underline{\epsilon^!|1}\},$$

and

$$\mathfrak{H}_0^2 = \{\underline{\alpha_2|1}, \underline{\beta_2|1}, \underline{\gamma_2|1}, (\underline{\alpha\beta} + \alpha\gamma)|\underline{1}\}.$$

Moreover, if $n \in \mathbb{N}$ is odd, we define $\mathfrak{H}_0^n = \emptyset$, and if $n \geqslant 4$ is even, we set

$$\mathfrak{H}_0^n = \{\underline{\alpha_n|1}, \underline{\beta_n|1}, \underline{\gamma_n|1}, (\underline{\alpha_{n-1}\beta} + \alpha_{n-1}\gamma)|\underline{1}, \underline{\alpha_{n-2}\beta_2|1}\} \cup \omega_1^*\mathfrak{H}_2^{n-4}.$$

The previous results can be restated as follows.

Corollary 4.2.15. *Let $m \in [\![0,4]\!]$ and $n \in \mathbb{N}_0$. Then $H_m^n = \tilde{H}_m^n \oplus \omega_1^* H_{m+2}^{n-4}$. Here, $\tilde{H}_m^n$ is the $\mathbb{k}$-vector space spanned by the set $\tilde{\mathfrak{H}}_m^n$, which is defined as follows. If $m \in [\![3,4]\!]$, we define the set $\tilde{\mathfrak{H}}_m^n = \mathfrak{H}_m^n$ for $n \in \mathbb{N}_0$. If $m = 2$, we define the sets*

$$\tilde{\mathfrak{H}}_2^0 = \{\epsilon^!|(ab + ba), \epsilon^!|(ab + bc - ac)\}, \quad \tilde{\mathfrak{H}}_2^2 = \{\alpha_2|(ab + ba), \beta_2|(ab + ba)\},$$

and $\tilde{\mathfrak{H}}_2^n = \emptyset$ for $n = 1$ and $n \geqslant 3$. If $m = 1$, we define the set

$$\tilde{\mathfrak{H}}_1^1 = \{\alpha|a + \beta|b + \gamma|c\},$$

and $\tilde{\mathfrak{H}}_1^n = \emptyset$ for $n \in \mathbb{N}_0$ with n even, together with

$$\tilde{\mathfrak{H}}_1^n = \{\alpha_n|a + \beta_n|b + \gamma_n|c, (\beta_n - \alpha_{n-1}\beta)|b, (\gamma_n - \alpha_{n-1}\gamma)|c, (\alpha_n - \alpha_{n-2}\beta_2)|a,$$
$$\alpha_{n-1}\beta|c + \alpha_{n-1}\gamma|a + \alpha_{n-2}\beta_2|b\}$$

for $n \geqslant 3$ with n odd. If $m = 0$, we define the sets

$$\tilde{\mathfrak{H}}_0^0 = \{\epsilon^!|1\}, \quad \tilde{\mathfrak{H}}_0^2 = \{\alpha_2|1, \beta_2|1, \gamma_2|1, (\alpha\beta + \alpha\gamma)|1\},$$

and $\tilde{\mathfrak{H}}_0^n = \emptyset$ *for* $n \in \mathbb{N}$ *with* n *odd, together with*

$$\tilde{\mathfrak{H}}_0^n = \{\alpha_n|1,\beta_n|1,\gamma_n|1,(\alpha_{n-1}\beta + \alpha_{n-1}\gamma)|1,\alpha_{n-2}\beta_2|1\}$$

for $n \geqslant 4$ *with* n *even. Moreover, if we define* $\tilde{H}_m^n = 0$ *for* $(n,m) \in \mathbb{Z}^2 \setminus (\mathbb{N}_0 \times [\![0,4]\!])$*, then* $H_m^n = \tilde{H}_m^n \oplus \omega_1^* H_{m+2}^{n-4}$ *holds for* $m, n \in \mathbb{Z}$ *by applying Corollary 4.2.14.*

Remark 4.2.16. *The reader can easily check that* $\tilde{D}_m^n = \tilde{H}_m^n \oplus \tilde{B}_m^n$ *for* $m \in [\![0,4]\!]$ *and* $n \in \mathbb{N}_0$.

Recall that the Hochschild cohomology is decomposed as $\mathrm{HH}^n(A) = \oplus_{m\leqslant 4} H_m^n$ for $n \in \mathbb{N}_0$.

Proposition 4.2.17. *For* $n \in \mathbb{N}_0$,

$$\mathrm{HH}^n(A) = \bigoplus_{\substack{i\in[\![0,\lfloor n/4\rfloor]\!],\\ m\in[\![0,4]\!]}} \omega_i^* \tilde{H}_m^{n-4i}.$$

Proof. By Corollary 4.2.15, we have

$$H_2^n = \tilde{H}_2^n \oplus \omega_1^* \tilde{H}_4^{n-4}, \quad H_1^n = \tilde{H}_1^n \oplus \omega_1^* \tilde{H}_3^{n-4}, \quad H_0^n = \tilde{H}_0^n \oplus \omega_1^* \tilde{H}_2^{n-4} \oplus \omega_2^* \tilde{H}_4^{n-8} \tag{4.2.27}$$

for $n \in \mathbb{N}_0$. Using Corollary 4.2.14 and (4.2.27), we get

$$\begin{aligned}
\mathrm{HH}^n(A) &= \bigoplus_{m\in[\![-2\lfloor n/4\rfloor,4]\!]} H_m^n \\
&= H_4^n \oplus H_3^n \oplus H_2^n \oplus \Big(\bigoplus_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_i^* H_1^{n-4i}\Big) \oplus \Big(\bigoplus_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_i^* H_0^{n-4i}\Big) \\
&= \tilde{H}_4^n \oplus \tilde{H}_3^n \oplus (\tilde{H}_2^n \oplus \omega_1^* \tilde{H}_4^{n-4}) \oplus \Big(\bigoplus_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_i^* (\tilde{H}_1^{n-4i} \oplus \omega_1^* \tilde{H}_3^{n-4i-4})\Big) \\
&\quad \oplus \Big(\bigoplus_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_i^* (\tilde{H}_0^{n-4i} \oplus \omega_1^* \tilde{H}_2^{n-4i-4} \oplus \omega_2^* \tilde{H}_4^{n-4i-8})\Big) \\
&= \tilde{H}_4^n \oplus \tilde{H}_3^n \oplus \tilde{H}_2^n \oplus \omega_1^* \tilde{H}_4^{n-4} \oplus \Big(\bigoplus_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_i^* \tilde{H}_1^{n-4i}\Big) \oplus \Big(\bigoplus_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_{i+1}^* \tilde{H}_3^{n-4i-4}\Big) \\
&\quad \oplus \Big(\bigoplus_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_i^* \tilde{H}_0^{n-4i}\Big) \oplus \Big(\bigoplus_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_{i+1}^* \tilde{H}_2^{n-4i-4}\Big) \oplus \Big(\bigoplus_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_{i+2}^* \tilde{H}_4^{n-4i-8}\Big) \\
&= \tilde{H}_4^n \oplus \tilde{H}_3^n \oplus \tilde{H}_2^n \oplus \omega_1^* \tilde{H}_4^{n-4} \oplus \Big(\bigoplus_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_i^* \tilde{H}_1^{n-4i}\Big) \oplus \Big(\bigoplus_{i\in[\![1,\lfloor n/4\rfloor]\!]} \omega_i^* \tilde{H}_3^{n-4i}\Big) \\
&\quad \oplus \Big(\bigoplus_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_i^* \tilde{H}_0^{n-4i}\Big) \oplus \Big(\bigoplus_{i\in[\![1,\lfloor n/4\rfloor]\!]} \omega_i^* \tilde{H}_2^{n-4i}\Big) \oplus \Big(\bigoplus_{i\in[\![2,\lfloor n/4\rfloor]\!]} \omega_i^* \tilde{H}_4^{n-4i}\Big) \\
&= \bigoplus_{\substack{i\in[\![0,\lfloor n/4\rfloor]\!],\\ m\in[\![0,4]\!]}} \omega_i^* \tilde{H}_m^{n-4i}.
\end{aligned}$$

□

Remark 4.2.18. *Let* $\tilde{H}^n = \oplus_{m\in[\![0,4]\!]} \tilde{H}_m^n$ *for* $n \in \mathbb{N}_0$*. Proposition 4.2.17 shows that*

$$\mathrm{HH}^n(A) = \oplus_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_i^* \tilde{H}^{n-4i}.$$

Using Corollary 4.2.15, it is easy to compute that $\dim \tilde{H}^0 = 4$, $\dim \tilde{H}^1 = 7$, $\dim \tilde{H}^2 = 10$, $\dim \tilde{H}^3 = 12$ *and* $\dim \tilde{H}^n = 10$ *for* $n \geqslant 4$.

Using the previous remark, we get the dimension of $\mathrm{HH}^n(A)$.

Proposition 4.2.19. *The dimension of* $\mathrm{HH}^n(A)$ *is given by*

$$\dim \mathrm{HH}^n(A) = \begin{cases} \frac{5}{2}n + 4, & \text{if } n = 4r \text{ for } r \in \mathbb{N}_0, \\ \frac{5}{2}n + 5, & \text{if } n = 4r+2 \text{ for } r \in \mathbb{N}_0, \\ \frac{5n+9}{2}, & \text{if } n = 2r+1 \text{ for } r \in \mathbb{N}_0. \end{cases}$$

The Hilbert series of $\mathrm{HH}^n(A)$ is $h^n(t) = \sum_{m \leqslant 4} \dim(H^n_m) t^{m-n}$ for $n \in \mathbb{N}_0$. Note that $m-n$ is the internal degree of H^n_m.

Corollary 4.2.20. *The Hilbert series $h^n(t)$ of $\mathrm{HH}^n(A)$ is given as follows. Let $n \geqslant 8$. Then*

$$h^n(t) = t^{-n}\big[5\chi_n t^4 + 5\chi_{n+1} t^3 + 5\chi_n t^2 + 10 \sum_{i=0}^{\lfloor \frac{n}{4} \rfloor - 3} t^{\chi_{n+1} - 2i} + t^{-2\lfloor \frac{n}{4} \rfloor} p^n(t)\big],$$

where

$$p^n(t) = \begin{cases} 6t^4 + 7t^2 + 1, & \textit{if } n \equiv 0 \pmod 4, \\ 10t^5 + 11t^3 + t, & \textit{if } n \equiv 1 \pmod 4, \\ 9t^4 + 7t^2 + 4, & \textit{if } n \equiv 2 \pmod 4, \\ 10t^5 + 12t^3 + 5t, & \textit{if } n \equiv 3 \pmod 4. \end{cases}$$

Moreover,

$$\begin{aligned} h^0(t) &= t^4 + 2t^2 + 1, & h^1(t) &= 6t^2 + 1, \\ h^2(t) &= 4t^2 + 2 + 4t^{-2}, & h^3(t) &= 7 + 5t^{-2}, \\ h^4(t) &= 5 + t^{-2} + 7t^{-4} + t^{-6}, & h^5(t) &= 5t^{-2} + 11t^{-4} + t^{-6}, \\ h^6(t) &= 5t^{-2} + 4t^{-4} + 7t^{-6} + 4t^{-8}, & h^7(t) &= 5t^{-4} + 12t^{-6} + 5t^{-8}. \end{aligned}$$

Chapter 5

Algebraic structure and Gerstenhaber structure on Hochschild cohomology of Fomin-Kirillov algebra on 3 generators

In this chapter, we will explicitly determine the algebraic structure and Gerstenhaber structure of the Hochschild cohomology of Fomin-Kirillov algebra on 3 generators. The results of the first section were published in [12], whereas the results of the second section were published in [13].

5.1 Algebraic structure on Hochschild cohomology of $\mathrm{FK}(3)$

In this section, let $\Bbbk$ be a field of characteristic different from 2 and 3, and A the Fomin-Kirillov algebra $\mathrm{FK}(3)$. We will explicitly determine the algebra structure of the Hochschild cohomology of A given by the cup product $\smile$. To do so, we will first find a generating set of the $\Bbbk$-algebra $\mathrm{HH}^\bullet(A) = \oplus_{n\in\mathbb{N}_0} \mathrm{HH}^n(A)$ (see Proposition 5.1.4). Then, after extracting a minimal generating set from the previous set of generators, we will find an explicit presentation of the algebra $\mathrm{HH}^\bullet(A)$ as a quotient of a free algebra F by the ideal generated by an explicit set $\mathscr{R}$ of homogeneous relations. This is done by using a Gröbner basis of $\mathscr{R}$, which allows us to compute the Hilbert series of the quotient $F/(\mathscr{R})$, and then comparing the Hilbert series of the quotient and that of $\mathrm{HH}^\bullet(A)$. We refer the reader to Sections 2.1 and 2.2 of the very nice book [30] for the usual method for computing the cup product. However, to reduce the amount of signs appearing in the computations below we will follow the original definition of cup product by M. Gerstenhaber in [9], Section 7 (*cf.* [30], Definition 1.3.1 and Remark 1.3.3).

We will follow the notation in Subsection 4.2.2.

Remark 5.1.1. *It is easy to see that $f \smile g \in H^{n_1+n_2}_{m_1+m_2}$ for all $f \in H^{n_1}_{m_1}$, $g \in H^{n_2}_{m_2}$. Moreover, it is well-known that the cup product on Hochschild cohomology is graded commutative, i.e. $f \smile g = (-1)^{mn} g \smile f$ for $f \in \mathrm{HH}^m(A)$, $g \in \mathrm{HH}^n(A)$ (see [30], Theorem 1.4.6).*

Lemma 5.1.2. *Let $g = \omega_1^* \epsilon^! | 1 \in \omega_1^* \tilde{H}^0_0 = H^4_{-2}$. Then $f \smile g = \omega_1^* f$ for all $f \in \mathrm{HH}^\bullet(A)$.*

Proof. The map g can be extended to a chain map $g_\bullet : P^b_\bullet \to P^b_\bullet$ with $g_n(\omega_i^* x) = \omega_{i-1}^* x$ for $x \in K^b_{n+4-4i}$ and $i \in [\![0, \lfloor n/4 \rfloor + 1]\!]$. Hence, given $f \in \mathrm{HH}^m(A)$, we get $f \smile g = f g_m = \omega_1^* f$. $\square$

By Lemma 5.1.2 and Proposition 4.2.17, the set

$$\Bigg(\bigcup_{\substack{m \,\in\, [\![0,\,4]\!],\\ n \,\in\, \mathbb{N}_0}} \tilde{\mathfrak{H}}^n_m \cup \{\omega_1^* \epsilon^! | 1\} \Bigg) \setminus \{\epsilon^! | 1\}$$

is a generating set of $\mathrm{HH}^\bullet(A)$ as $\Bbbk$-algebra.

Fact 5.1.3. *Assume $x,y \in A$. Let $n \geqslant 2$ be even. The map $\alpha_n|1$ can be extended to the chain map $g_\bullet : P^b_\bullet \to P^b_\bullet$ satisfying*

$$\begin{aligned}
g_0(x|\alpha_n|y) &= x|\epsilon^!|y,\\
g_0(x|\beta_n|y) &= g_0(x|\gamma_n|y) = g_0(x|\alpha_{n-1}\beta|y) = g_0(x|\alpha_{n-1}\gamma|y) = g_0(x|\alpha_{n-2}\beta_2|y) = 0,\\
g_1(x|\alpha_{n+1}|y) &= x|\alpha|y,\ g_1(x|\beta_{n+1}|y) = g_1(x|\gamma_{n+1}|y) = g_1(x|\alpha_{n-1}\beta_2|y) = 0,\\
g_1(x|\alpha_n\beta|y) &= x|\beta|y,\ g_1(x|\alpha_n\gamma|y) = x|\gamma|y,\\
g_2(x|\alpha_{n+2}|y) &= x|\alpha_2|y,\ g_2(x|\beta_{n+2}|y) = g_2(x|\gamma_{n+2}|y) = 0,\ g_2(x|\alpha_{n+1}\beta|y) = x|\alpha\beta|y,\\
g_2(x|\alpha_{n+1}\gamma|y) &= x|\alpha\gamma|y,\ g_2(x|\alpha_n\beta_2|y) = x|(\beta_2+\gamma_2)|y.
\end{aligned}$$

Moreover, if $n = 2$, the chain map $g_\bullet$ satisfies

$$\begin{aligned}
g_2(\omega_1 x|\epsilon^!|y) = x(&2|\alpha\beta|ac + 1|\alpha\beta|ba + a|\alpha\beta|b - b|\alpha\beta|c + b|\alpha\gamma|a - c|\alpha\gamma|b - b|\alpha_2|b - 3c|\alpha_2|c\\
&+ a|\beta_2|a + a|\gamma_2|a + 2b|\gamma_2|b - 2bc|\alpha\gamma|1 - ab|\alpha\gamma|1)y.
\end{aligned}$$

If $n = 4$, the chain map $g_\bullet$ satisfies

$$\begin{aligned}
g_0(\omega_1 x|\epsilon^!|y) &= 0,\\
g_1(\omega_1 x|\alpha|y) &= 2x(1|\beta|ac + b|\alpha|c + ba|\gamma|1)y,\\
g_1(\omega_1 x|\beta|y) &= -2x(c|\beta|a + a|\beta|c + b|\alpha|a + a|\gamma|a + a|\alpha|b)y,\\
g_1(\omega_1 x|\gamma|y) &= -2x(b|\gamma|a + a|\gamma|b + a|\beta|a + a|\alpha|c + c|\alpha|a)y.
\end{aligned}$$

The map $\beta_n|1$ can be extended to the chain map $g_\bullet : P^b_\bullet \to P^b_\bullet$ satisfying

$$\begin{aligned}
g_0(x|\beta_n|y) &= x|\epsilon^!|y,\\
g_0(x|\alpha_n|y) &= g_0(x|\gamma_n|y) = g_0(x|\alpha_{n-1}\beta|y) = g_0(x|\alpha_{n-1}\gamma|y) = g_0(x|\alpha_{n-2}\beta_2|y) = 0,\\
g_1(x|\beta_{n+1}|y) &= x|\beta|y,\ g_1(x|\alpha_{n+1}|y) = g_1(x|\gamma_{n+1}|y) = g_1(x|\alpha_n\beta|y) = 0,\\
g_1(x|\alpha_n\gamma|y) &= x|\gamma|y,\ g_1(x|\alpha_{n-1}\beta_2|y) = x|\alpha|y,\\
g_2(x|\beta_{n+2}|y) &= x|\beta_2|y,\ g_2(x|\alpha_{n+2}|y) = g_2(x|\gamma_{n+2}|y) = 0,\ g_2(x|\alpha_{n+1}\beta|y) = x|\alpha\beta|y,\\
g_2(x|\alpha_{n+1}\gamma|y) &= x|\alpha\gamma|y,\ g_2(x|\alpha_n\beta_2|y) = x|(\alpha_2+\gamma_2)|y.
\end{aligned}$$

Moreover, if $n = 2$, the chain map $g_\bullet$ satisfies

$$\begin{aligned}
g_2(\omega_1 x|\epsilon^!|y) = x(&1|\alpha\beta|ba - 1|\alpha\beta|ac + b|\alpha\beta|c - c|\alpha\beta|a + c|\alpha\gamma|b - a|\alpha\gamma|c - c|\beta_2|c - 3a|\beta_2|a\\
&+ b|\gamma_2|b + b|\alpha_2|b + 2c|\alpha_2|c + 2ab|\alpha\gamma|1 + bc|\alpha\gamma|1)y.
\end{aligned}$$

If $n = 4$, the chain map $g_\bullet$ satisfies

$$\begin{aligned}
g_0(\omega_1 x|\epsilon^!|y) &= 0,\\
g_1(\omega_1 x|\alpha|y) &= -2x(b|\alpha|c + c|\alpha|b + a|\beta|b + b|\gamma|b + b|\beta|a)y,\\
g_1(\omega_1 x|\beta|y) &= 2x(1|\alpha|bc + a|\beta|c + ab|\gamma|1)y,\\
g_1(\omega_1 x|\gamma|y) &= -2x(a|\gamma|b + b|\gamma|a + b|\alpha|b + b|\beta|c + c|\beta|b)y.
\end{aligned}$$

The map $\gamma_n|1$ can be extended to the chain map $g_\bullet : P^b_\bullet \to P^b_\bullet$ satisfying

$$\begin{aligned}
g_0(x|\gamma_n|y) &= x|\epsilon^!|y,\\
g_0(x|\alpha_n|y) &= g_0(x|\beta_n|y) = g_0(x|\alpha_{n-1}\beta|y) = g_0(x|\alpha_{n-1}\gamma|y) = g_0(x|\alpha_{n-2}\beta_2|y) = 0,\\
g_1(x|\gamma_{n+1}|y) &= x|\gamma|y,\ g_1(x|\alpha_{n+1}|y) = g_1(x|\beta_{n+1}|y) = g_1(x|\alpha_n\gamma|y) = 0,\\
g_1(x|\alpha_n\beta|y) &= x|\beta|y,\ g_1(x|\alpha_{n-1}\beta_2|y) = x|\alpha|y,\\
g_2(x|\gamma_{n+2}|y) &= x|\gamma_2|y,\ g_2(x|\alpha_{n+2}|y) = g_2(x|\beta_{n+2}|y) = 0,\ g_2(x|\alpha_{n+1}\beta|y) = x|\alpha\beta|y,\\
g_2(x|\alpha_{n+1}\gamma|y) &= x|\alpha\gamma|y,\ g_2(x|\alpha_n\beta_2|y) = x|(\alpha_2+\beta_2)|y.
\end{aligned}$$

Moreover, if $n = 2$, the chain map $g_\bullet$ satisfies

$$g_2(\omega_1 x|\epsilon^!|y) = -x(1|\alpha\beta|ac + 2|\alpha\beta|ba + 2a|\alpha\beta|b + b|\alpha\gamma|a + c|\alpha\gamma|b - a|\beta_2|a + 2a|\gamma_2|a + 3b|\gamma_2|b$$

$$+ ba|\alpha\beta|1 + ab|\alpha\gamma|1)y.$$

If $n = 4$, the chain map $g_\bullet$ satisfies

$$g_0(\omega_1 x|\epsilon^!|y) = 0,$$
$$g_1(\omega_1 x|\alpha|y) = 2x(a|\beta|b + b|\gamma|b + b|\beta|a)y,\ g_1(\omega_1 x|\beta|y) = 2x(b|\alpha|a + a|\gamma|a + a|\alpha|b)y,$$
$$g_1(\omega_1 x|\gamma|y) = -2x(1|\alpha|ba + a|\beta|a + ab|\alpha|1)y.$$

The map $(\alpha_{n-1}\beta + \alpha_{n-1}\gamma)|1$ can be extended to the chain map $g_\bullet : P^b_\bullet \to P^b_\bullet$ satisfying

$$g_0(x|\alpha_n|y) = g_0(x|\beta_n|y) = g_0(x|\gamma_n|y) = g_0(x|\alpha_{n-2}\beta_2|y) = 0,$$
$$g_0(x|\alpha_{n-1}\beta|y) = g_0(x|\alpha_{n-1}\gamma|y) = x|\epsilon^!|y,$$
$$g_1(x|\alpha_{n+1}|y) = g_1(x|\beta_{n+1}|y) = g_1(x|\gamma_{n+1}|y) = 0,\ g_1(x|\alpha_n\beta|y) = x|(\alpha+\gamma)|y,$$
$$g_1(x|\alpha_n\gamma|y) = x|(\alpha+\beta)|y,\ g_1(x|\alpha_{n-1}\beta_2|y) = x|(\beta+\gamma)|y,$$
$$g_2(x|\alpha_{n+2}|y) = g_2(x|\beta_{n+2}|y) = g_2(x|\gamma_{n+2}|y) = 0,\ g_2(x|\alpha_{n+1}\beta|y) = x|(\alpha\gamma + \alpha_2 + \beta_2 + \gamma_2)|y,$$
$$g_2(x|\alpha_{n+1}\gamma|y) = x|(\alpha\beta + \alpha_2 + \beta_2 + \gamma_2)|y,\ g_2(x|\alpha_n\beta_2|y) = x|(\alpha\beta + \alpha\gamma)|y.$$

Moreover, if $n = 2$, the chain map $g_\bullet$ satisfies

$$g_2(\omega_1 x|\epsilon^!|y) = x[1|\alpha_2|(ba+ac) - 1|\alpha_2|bc + 1|\beta_2|(ab+bc) - 1|\beta_2|ac - 1|\gamma_2|ab - 1|\gamma_2|ba - b|\alpha_2|c$$
$$- c|\alpha_2|b - a|\beta_2|c - c|\beta_2|a - a|\gamma_2|b - b|\gamma_2|a + (ba+ac)|\alpha_2|1 - bc|\alpha_2|1 - ac|\beta_2|1$$
$$+ (ab+bc)|\beta_2|1 - ab|\gamma_2|1 - ba|\gamma_2|1]y.$$

If $n = 4$, the chain map $g_\bullet$ satisfies

$$g_0(\omega_1 x|\epsilon^!|y) = 0,$$
$$g_1(\omega_1 x|\alpha|y) = g_1(\omega_1 x|\beta|y) = g_1(\omega_1 x|\gamma|y) = 0.$$

Now let $n \geqslant 4$ be even, then the map $\alpha_{n-2}\beta_2|1$ can be extended to the chain map $g_\bullet : P^b_\bullet \to P^b_\bullet$ satisfying

$$g_0(x|\alpha_{n-2}\beta_2|y) = x|\epsilon^!|y,$$
$$g_0(x|\alpha_n|y) = g_0(x|\beta_n|y) = g_0(x|\gamma_n|y) = g_0(x|\alpha_{n-1}\beta|y) = g_0(x|\alpha_{n-1}\gamma|y) = 0,$$
$$g_1(x|\alpha_{n+1}|y) = g_1(x|\beta_{n+1}|y) = g_1(x|\gamma_{n+1}|y) = 0,\ g_1(x|\alpha_n\beta|y) = x|\beta|y,$$
$$g_1(x|\alpha_n\gamma|y) = x|\gamma|y,\ g_1(x|\alpha_{n-1}\beta_2|y) = x|\alpha|y.$$

Moreover, if $n = 4$, the chain map $g_\bullet$ satisfies

$$g_0(\omega_1 x|\epsilon^!|y) = 0.$$

Let $n \in \mathbb{N}$ be odd, then the map $\alpha_n|a + \beta_n|b + \gamma_n|c$ can be extended to the chain map $g_\bullet : P^b_\bullet \to P^b_\bullet$ satisfying

$$g_0(x|\alpha_n|y) = x|\epsilon^!|ay,\ g_0(x|\beta_n|y) = x|\epsilon^!|by,\ g_0(x|\gamma_n|y) = x|\epsilon^!|cy,$$
$$g_0(x|\alpha_{n-1}\beta|y) = g_0(x|\alpha_{n-1}\gamma|y) = g_0(x|\alpha_{n-2}\beta_2|y) = 0,$$
$$g_1(x|\alpha_{n+1}|y) = -x|\alpha|ay,\ g_1(x|\beta_{n+1}|y) = -x|\beta|by,\ g_1(x|\gamma_{n+1}|y) = -x|\gamma|cy,$$
$$g_1(x|\alpha_n\beta|y) = -x|\alpha|by - x|\beta|cy - x|\gamma|ay,\ g_1(x|\alpha_n\gamma|y) = -x|\alpha|cy - x|\beta|ay - x|\gamma|by,$$
$$g_1(x|\alpha_{n-1}\beta_2|y) = 0,$$
$$g_2(x|\alpha_{n+2}|y) = x|\alpha_2|ay,\ g_2(x|\beta_{n+2}|y) = x|\beta_2|by,\ g_2(x|\gamma_{n+2}|y) = x|\gamma_2|cy,$$
$$g_2(x|\alpha_{n+1}\beta|y) = x|\alpha\beta|cy + x|\alpha\gamma|ay + x|(\alpha_2+\gamma_2)|by,$$
$$g_2(x|\alpha_{n+1}\gamma|y) = x|\alpha\beta|ay + x|\alpha\gamma|by + x|(\alpha_2+\beta_2)|cy,$$
$$g_2(x|\alpha_n\beta_2|y) = x|\alpha\beta|by + x|\alpha\gamma|cy + x|(\beta_2+\gamma_2)|ay.$$

Moreover, if $n = 3$, the chain map $g_\bullet$ satisfies

$$g_1(\omega_1 x|\epsilon^!|y) = 2x[1|\alpha|bac + 1|\beta|abc - 1|\gamma|aba + c|\alpha|(ba+ac) - a|\beta|ac - b|\gamma|ba - (ba+ac)|\gamma|a$$

$$
\begin{aligned}
&+ ac|\alpha|b + ba|\beta|c]y,\\
g_2(\omega_1 x|\alpha|y) = {}& 2x[-2|\alpha_2|bac + a|\alpha\gamma|ab - c|\alpha_2|bc + c|\beta_2|ab - b|\gamma_2|ba - bc|\alpha\beta|c + ab|\alpha\gamma|a\\
&- ac|\alpha_2|c - 2ba|\beta_2|c + ab|\gamma_2|b - ba|\gamma_2|c - abc|\alpha\gamma|1 + bac|\alpha_2|1]y,\\
g_2(\omega_1 x|\beta|y) = {}& 2x[-2|\beta_2|abc + b|\alpha\gamma|bc + a|\beta_2|(ab+bc) + a|\gamma_2|bc + c|\alpha_2|(ba+ac)\\
&+ (ab+bc)|\alpha\beta|a + bc|\alpha\gamma|b - ba|\beta_2|a + 2(ba+ac)|\gamma_2|a + bc|\alpha_2|c + (ba+ac)|\alpha_2|a\\
&+ aba|\alpha\gamma|1 + abc|\beta_2|1]y,\\
g_2(\omega_1 x|\gamma|y) = {}& 2x[2|\gamma_2|aba - c|\alpha\gamma|(ab+bc) - b|\gamma_2|ab - b|\alpha_2|(ab+bc) - a|\beta_2|ac - ab|\alpha\beta|b\\
&- (ab+bc)|\alpha\gamma|c + (ba+ac)|\gamma_2|b - 2ac|\alpha_2|b - (ab+bc)|\beta_2|a - ac|\beta_2|b - bac|\alpha\gamma|1\\
&- aba|\gamma_2|1]y.
\end{aligned}
$$

Proposition 5.1.4. *The set*

$$
\mathscr{S} = \Bigg(\bigcup_{\substack{m\,\in\,[\![0,\,4]\!],\\ n\,\in\,[\![0,\,3]\!]}} \tilde{\mathfrak{H}}^n_m \cup \{\omega_1^*\epsilon^!|1\}\Bigg) \setminus \{\epsilon^!|1\}
$$

is a generating set of the $\Bbbk$*-algebra* $\mathrm{HH}^\bullet(A)$*. Hence,* $\mathrm{HH}^\bullet(A)$ *is a finitely generated* $\Bbbk$*-algebra.*

Proof. We will prove the proposition by induction on n. Let $n \geqslant 4$. Assume that $\tilde{\mathfrak{H}}^{n'}_m$ for $m \in [\![0,4]\!]$ and $n' \in [\![0, n-1]\!]$ is generated by the elements of $\mathscr{S}$. We check that $\tilde{\mathfrak{H}}^n_m$ for $m \in [\![0,4]\!]$ is generated by the elements of $\mathscr{S}$. First, we suppose that n is even. Note that $\tilde{\mathfrak{H}}^n_0 = \{\alpha_n|1, \beta_n|1, \gamma_n|1, (\alpha_{n-1}\beta + \alpha_{n-1}\gamma)|1, \alpha_{n-2}\beta_2|1\}$. By Fact 5.1.3, we have

$$
\begin{aligned}
&\alpha_2|1 \smile \alpha_{n-2}|1 \in \alpha_n|1 + \omega_1^* H_2^{n-4}, \quad \beta_2|1 \smile \beta_{n-2}|1 \in \beta_n|1 + \omega_1^* H_2^{n-4},\\
&\gamma_2|1 \smile \gamma_{n-2}|1 \in \gamma_n|1 + \omega_1^* H_2^{n-4}, \quad \gamma_2|1 \smile \alpha_{n-2}|1 \in \alpha_{n-2}\beta_2|1 + \omega_1^* H_2^{n-4},\\
&(\alpha\beta + \alpha\gamma)|1 \smile \alpha_{n-2}|1 \in (\alpha_{n-1}\beta + \alpha_{n-1}\gamma)|1 + \omega_1^* H_2^{n-4}.
\end{aligned}
$$

Hence, the elements in $\tilde{\mathfrak{H}}^n_0$ are generated by the elements in $\mathscr{S}$. Note that $\tilde{\mathfrak{H}}^n_1 = \tilde{\mathfrak{H}}^n_2 = \tilde{\mathfrak{H}}^n_3 = \emptyset$. Finally, we notice that $\tilde{\mathfrak{H}}^n_4 = \{\alpha_n|abac, \beta_n|abac, \gamma_n|abac, \alpha_{n-1}\beta|abac, \alpha_{n-2}\beta_2|abac\}$. Since

$$
\begin{aligned}
&\epsilon^!|abac \smile \alpha_n|1 = \alpha_n|abac, \quad \epsilon^!|abac \smile \beta_n|1 = \beta_n|abac, \quad \epsilon^!|abac \smile \gamma_n|1 = \gamma_n|abac,\\
&\epsilon^!|abac \smile (\alpha_{n-1}\beta + \alpha_{n-1}\gamma)|1 = 2\alpha_{n-1}\beta|abac, \quad \epsilon^!|abac \smile \alpha_{n-2}\beta_2|1 = \alpha_{n-2}\beta_2|abac,
\end{aligned}
$$

the elements in $\tilde{\mathfrak{H}}^n_4$ are also generated by the elements of $\mathscr{S}$.

Next, we suppose that n is odd. Note first that $\tilde{\mathfrak{H}}^n_0 = \tilde{\mathfrak{H}}^n_2 = \tilde{\mathfrak{H}}^n_4 = \emptyset$, and

$$
\begin{aligned}
\tilde{\mathfrak{H}}^n_1 = \{&\alpha_n|a + \beta_n|b + \gamma_n|c, (\beta_n - \alpha_{n-1}\beta)|b, (\gamma_n - \alpha_{n-1}\gamma)|c, (\alpha_n - \alpha_{n-2}\beta_2)|a,\\
&\alpha_{n-1}\beta|c + \alpha_{n-1}\gamma|a + \alpha_{n-2}\beta_2|b\}.
\end{aligned}
$$

By Fact 5.1.3, we see that

$$
\begin{aligned}
&(\alpha|a + \beta|b + \gamma|c) \smile \alpha_{n-1}|1 \in \alpha_n|a + \alpha_{n-1}\beta|b + \alpha_{n-1}\gamma|c + \omega_1^* \tilde{H}_3^{n-4},\\
&(\alpha|a + \beta|b + \gamma|c) \smile \beta_{n-1}|1 \in \beta_n|b + \alpha_{n-1}\gamma|c + \alpha_{n-2}\beta_2|a + \omega_1^* \tilde{H}_3^{n-4},\\
&(\alpha|a + \beta|b + \gamma|c) \smile \gamma_{n-1}|1 \in \gamma_n|c + \alpha_{n-1}\beta|b + \alpha_{n-2}\beta_2|a + \omega_1^* \tilde{H}_3^{n-4},\\
&(\alpha|a + \beta|b + \gamma|c) \smile (\alpha_{n-2}\beta + \alpha_{n-2}\gamma)|1 \in 2(\alpha_{n-1}\beta|c + \alpha_{n-1}\gamma|a + \alpha_{n-2}\beta_2|b) + \omega_1^* \tilde{H}_3^{n-4},\\
&(\alpha|a + \beta|b + \gamma|c) \smile \alpha_{n-3}\beta_2|1 \in \alpha_{n-1}\beta|b + \alpha_{n-1}\gamma|c + \alpha_{n-2}\beta_2|a + \omega_1^* \tilde{H}_3^{n-4}.
\end{aligned}
$$

It is easy to see that the five elements $\alpha_n|a + \alpha_{n-1}\beta|b + \alpha_{n-1}\gamma|c$, $\beta_n|b + \alpha_{n-1}\gamma|c + \alpha_{n-2}\beta_2|a$, $\gamma_n|c + \alpha_{n-1}\beta|b + \alpha_{n-2}\beta_2|a$, $2(\alpha_{n-1}\beta|c + \alpha_{n-1}\gamma|a + \alpha_{n-2}\beta_2|b)$ and $\alpha_{n-1}\beta|b + \alpha_{n-1}\gamma|c + \alpha_{n-2}\beta_2|a$ are linear combinations of elements of $\tilde{\mathfrak{H}}^n_1$. Moreover, they form a basis of $\tilde{H}^n_1$. The elements in $\tilde{H}^n_1$, so *a fortiori* $\tilde{\mathfrak{H}}^n_1$, are thus generated by the elements of $\mathscr{S}$. Note finally that

$$
\tilde{\mathfrak{H}}^n_3 = \{\alpha_n|bac, \beta_n|abc, \gamma_n|aba, \alpha_{n-1}\beta|abc, \alpha_n|(aba - abc)\}.
$$

Since

$$
\begin{aligned}
&\alpha|bac \smile \alpha_{n-1}|1 = \alpha_n|bac, \quad \beta|abc \smile \beta_{n-1}|1 = \beta_n|abc, \quad \gamma|aba \smile \gamma_{n-1}|1 = \gamma_n|aba,\\
&\beta|abc \smile \alpha_{n-1}|1 = \alpha_{n-1}\beta|abc, \quad \alpha|(aba - abc) \smile \alpha_{n-1}|1 = \alpha_n|(aba - abc),
\end{aligned}
$$

the elements in $\tilde{\mathfrak{H}}^n_3$ are generated by the elements of $\mathscr{S}$. □

Proposition 5.1.5. *The set of* 14 *elements given by*

$$\begin{aligned}\mathcal{S} = \{&\epsilon^!|(ab+ba), \epsilon^!|(ab+bc-ac), \epsilon^!|abac, \alpha|a+\beta|b+\gamma|c, \alpha|bac, \beta|abc, \gamma|aba,\\ &\alpha|(aba-abc), \alpha_2|1, \beta_2|1, \gamma_2|1, (\alpha\beta+\alpha\gamma)|1, \alpha_3|a+\beta_3|b+\gamma_3|c, \omega_1^*\epsilon^!|1\} \subseteq \mathrm{HH}^\bullet(A).\end{aligned} \tag{5.1.1}$$

is a minimal generating set of the $\Bbbk$*-algebra* $\mathrm{HH}^\bullet(A)$.

Proof. By Proposition 5.1.4, the 33 element set

$$\begin{aligned}\mathscr{S} = \big\{&\epsilon^!|(ab+ba), \epsilon^!|(ab+bc+ac), \epsilon^!|abac, \alpha|bac, \beta|abc, \gamma|aba, \alpha|(aba-abc), (\alpha+\beta)|aba,\\ &\alpha|aba+\beta|bac, \alpha|a+\beta|b+\gamma|c, \alpha_2|abac, \beta_2|abac, \gamma_2|abac, \alpha\beta|abac, \alpha_2|(ab+ba),\\ &\beta_2|(ab+ba), \alpha_2|1, \beta_2|1, \gamma_2|1, (\alpha\beta+\alpha\gamma)|1, \alpha_3|bac, \beta_3|abc, \gamma_3|aba, \alpha_2\beta|abc, \alpha_3|(aba-abc),\\ &(\alpha_3+\beta_3)|aba, \alpha_3|aba+\beta_3|bac, \alpha_3|a+\beta_3|b+\gamma_3|c, (\beta_3-\alpha_2\beta)|b, (\gamma_3-\alpha_2\gamma)|c,\\ &(\alpha_3-\alpha\beta_2)|a, \alpha_2\beta|c+\alpha_2\gamma|a+\alpha\beta_2|b, \omega_1^*\epsilon^!|1\big\}\end{aligned}$$

is a generating set of $\mathrm{HH}^\bullet(A)$. By Fact 5.1.3 and the computation of coboundaries in Subsubsection 4.2.2.3, we get

$$\begin{aligned}
&\alpha_2|(ab+ba) = \epsilon^!|(ab+ba) \smile \alpha_2|1, \quad \beta_2|(ab+ba) = \epsilon^!|(ab+ba) \smile \beta_2|1,\\
&\alpha_2|abac = \epsilon^!|abac \smile \alpha_2|1, \quad \beta_2|abac = \epsilon^!|abac \smile \beta_2|1, \quad \gamma_2|abac = \epsilon^!|abac \smile \gamma_2|1,\\
&\alpha\beta|abac = (1/2)\epsilon^!|abac \smile (\alpha\beta+\alpha\gamma)|1, \quad \alpha_3|bac = \alpha|bac \smile \alpha_2|1, \quad \beta_3|abc = \beta|abc \smile \beta_2|1,\\
&\gamma_3|aba = \gamma|aba \smile \gamma_2|1, \quad \alpha_2\beta|abc = \alpha|bac \smile \beta_2|1,\\
&(\alpha_3-\alpha\beta_2)|a = (1/2)[(\alpha_3|a+\beta_3|b+\gamma_3|c) + (\alpha|a+\beta|b+\gamma|c) \smile (\alpha_2|1-\beta_2|1-\gamma_2|1)],\\
&(\beta_3-\alpha_2\beta)|b = (1/2)[(\alpha_3|a+\beta_3|b+\gamma_3|c) + (\alpha|a+\beta|b+\gamma|c) \smile (\beta_2|1-\alpha_2|1-\gamma_2|1)],\\
&(\gamma_3-\alpha_2\gamma)|c = (1/2)[(\alpha_3|a+\beta_3|b+\gamma_3|c) + (\alpha|a+\beta|b+\gamma|c) \smile (\gamma_2|1-\alpha_2|1-\beta_2|1)],\\
&\alpha_2\beta|c+\alpha_2\gamma|a+\alpha\beta_2|b = (1/2)(\alpha|a+\beta|b+\gamma|c) \smile (\alpha\beta+\alpha\gamma)|1,\\
&\alpha_3|(aba-abc) = \alpha|(aba-abc) \smile \alpha_2|1, \quad (\alpha_3+\beta_3)|aba = \gamma|aba \smile (\alpha\beta+\alpha\gamma)|1,\\
&\alpha_3|aba+\beta_3|bac = \alpha|(aba-abc) \smile (\beta_2|1+\alpha_2|1) - 2\gamma|aba \smile (\alpha\beta+\alpha\gamma)|1,\\
&(\alpha+\beta)|aba = (1/2)\epsilon^!|(ab+bc-ac) \smile (\alpha|a+\beta|b+\gamma|c) + \alpha|(aba-abc),\\
&\alpha|aba+\beta|bac = (1/2)[\epsilon^!|(ab+ba) \smile (\alpha|a+\beta|b+\gamma|c) - \epsilon^!|(ab+bc-ac) \smile (\alpha|a+\beta|b+\gamma|c)].
\end{aligned} \tag{5.1.2}$$

Hence, the set $\mathcal{S}$ obtained from $\mathscr{S}$ by removing the nineteen elements in (5.1.2) is still a generating set. Similarly, it is easy to check that

$$\begin{aligned}\epsilon^!|(ab+ba) \smile \epsilon^!|(ab+ba) &= \epsilon^!|(ab+bc-ac) \smile \epsilon^!|(ab+bc-ac)\\ &= \epsilon^!|(ab+ba) \smile \epsilon^!|(ab+bc-ac) = 0.\end{aligned} \tag{5.1.3}$$

By Remark 5.1.1, Fact 5.1.3 and (5.1.3), it is easy to check that any one of the fourteen elements of $\mathcal{S}$ can't be generated by the other thirteen elements, so the generating set $\mathcal{S}$ is minimal. □

Let us number the elements of the set $\mathcal{S}$ given in (5.1.1) by $X_1 = \epsilon^!|(ab+ba)$, $X_2 = \epsilon^!|(ab+bc-ac)$, $X_3 = \epsilon^!|abac$, $X_4 = \alpha|bac$, $X_5 = \beta|abc$, $X_6 = \gamma|aba$, $X_7 = \alpha|(aba-abc)$, $X_8 = \alpha|a+\beta|b+\gamma|c$, $X_9 = \alpha_2|1$, $X_{10} = \beta_2|1$, $X_{11} = \gamma_2|1$, $X_{12} = (\alpha\beta+\alpha\gamma)|1$, $X_{13} = \alpha_3|a+\beta_3|b+\gamma_3|c$ and $X_{14} = \omega_1^*\epsilon^!|1$. We define the well-ordered set $\{x_i, i \in [\![1,14]\!]\}$ with $x_i \succ x_j$ for all $i > j$. Let F be the noncommutative associative free $\Bbbk$-algebra generated by x_i for $i \in [\![1,14]\!]$, with length-lexicographic order. We endow the algebra F with the unique grading over $\mathbb{Z}^2$ given by setting the bidegree of x_i to be the same as that of X_i for $i \in [\![1,14]\!]$. Let $\mathscr{R}_1 \subseteq F$ be the set consisting of the following 97 homogeneous elements

$x_1x_2 - x_2x_1, x_1x_3 - x_3x_1, x_1x_4 - x_4x_1, x_1x_5 - x_5x_1, x_1x_6 - x_6x_1, x_1x_7 - x_7x_1, x_1x_8 - x_8x_1,$

$x_1x_9 - x_9x_1, x_1x_{10} - x_{10}x_1, x_1x_{11} - x_{11}x_1, x_1x_{12} - x_{12}x_1, x_1x_{13} - x_{13}x_1, x_1x_{14} - x_{14}x_1,$

$x_2x_3 - x_3x_2, x_2x_4 - x_4x_2, x_2x_5 - x_5x_2, x_2x_6 - x_6x_2, x_2x_7 - x_7x_2, x_2x_8 - x_8x_2, x_2x_9 - x_9x_2,$

$x_2x_{10} - x_{10}x_2, x_2x_{11} - x_{11}x_2, x_2x_{12} - x_{12}x_2, x_2x_{13} - x_{13}x_2, x_2x_{14} - x_{14}x_2, x_3x_4 - x_4x_3,$

$x_3x_5 - x_5x_3, x_3x_6 - x_6x_3, x_3x_7 - x_7x_3, x_3x_8 - x_8x_3, x_3x_9 - x_9x_3, x_3x_{10} - x_{10}x_3, x_3x_{11} - x_{11}x_3,$

$$x_3x_{12} - x_{12}x_3, x_3x_{13} - x_{13}x_3, x_3x_{14} - x_{14}x_3, x_4x_5 + x_5x_4, x_4x_6 + x_6x_4, x_4x_7 + x_7x_4,$$
$$x_4x_8 + x_8x_4, x_4x_9 - x_9x_4, x_4x_{10} - x_{10}x_4, x_4x_{11} - x_{11}x_4, x_4x_{12} - x_{12}x_4, x_4x_{13} + x_{13}x_4,$$
$$x_4x_{14} - x_{14}x_4, x_5x_6 + x_6x_5, x_5x_7 + x_7x_5, x_5x_8 + x_8x_5, x_5x_9 - x_9x_5, x_5x_{10} - x_{10}x_5, \quad (5.1.4)$$
$$x_5x_{11} - x_{11}x_5, x_5x_{12} - x_{12}x_5, x_5x_{13} + x_{13}x_5, x_5x_{14} - x_{14}x_5, x_6x_7 + x_7x_6, x_6x_8 + x_8x_6,$$
$$x_6x_9 - x_9x_6, x_6x_{10} - x_{10}x_6, x_6x_{11} - x_{11}x_6, x_6x_{12} - x_{12}x_6, x_6x_{13} + x_{13}x_6, x_6x_{14} - x_{14}x_6,$$
$$x_7x_8 + x_8x_7, x_7x_9 - x_9x_7, x_7x_{10} - x_{10}x_7, x_7x_{11} - x_{11}x_7, x_7x_{12} - x_{12}x_7, x_7x_{13} + x_{13}x_7,$$
$$x_7x_{14} - x_{14}x_7, x_8x_9 - x_9x_8, x_8x_{10} - x_{10}x_8, x_8x_{11} - x_{11}x_8, x_8x_{12} - x_{12}x_8, x_8x_{13} + x_{13}x_8,$$
$$x_8x_{14} - x_{14}x_8, x_9x_{10} - x_{10}x_9, x_9x_{11} - x_{11}x_9, x_9x_{12} - x_{12}x_9, x_9x_{13} - x_{13}x_9, x_9x_{14} - x_{14}x_9,$$
$$x_{10}x_{11} - x_{11}x_{10}, x_{10}x_{12} - x_{12}x_{10}, x_{10}x_{13} - x_{13}x_{10}, x_{10}x_{14} - x_{14}x_{10}, x_{11}x_{12} - x_{12}x_{11},$$
$$x_{11}x_{13} - x_{13}x_{11}, x_{11}x_{14} - x_{14}x_{11}, x_{12}x_{13} - x_{13}x_{12}, x_{12}x_{14} - x_{14}x_{12}, x_{13}x_{14} - x_{14}x_{13}, x_4^2, x_5^2, x_6^2,$$
$$x_7^2, x_8^2, x_{13}^2.$$

Remark 5.1.6. *Note that the quotient of the free algebra F generated by x_i for $i \in [\![1, 14]\!]$ modulo the (homogeneous) ideal generated by the previous set $\mathscr{R}_1$ is precisely the free graded-commutative (for the homological grading) algebra C generated by the same generators x_i for $i \in [\![1, 14]\!]$.*

Let $\mathscr{R}_2 \subseteq F$ be the set consisting of the following 63 homogeneous elements

$$x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2, x_1x_4, x_1x_5, x_1x_6, x_1x_7, x_2x_4, x_2x_5, x_2x_6, x_2x_7, x_3x_4, x_3x_5, x_3x_6, x_3x_7,$$
$$x_3x_8, x_4x_5, x_4x_6, x_4x_7, x_5x_6, x_5x_7, x_6x_7, x_1x_{11} - 2x_1x_9 - 2x_1x_{10}, x_1x_{12} - x_1x_9 - x_1x_{10},$$
$$x_2x_9 + x_1x_9, x_2x_{10} - 2x_1x_{10}, x_2x_{11} - x_1x_9 - x_1x_{10}, x_2x_{12} - x_1x_{10}, x_3x_9 + x_8x_4, x_3x_{10} + x_8x_5,$$
$$x_3x_{11} - x_8x_6, x_3x_{12} - x_8x_7, x_9x_5 + x_9x_6, x_9x_5 - x_{10}x_4, x_9x_5 + x_{10}x_6, x_9x_5 - x_{11}x_4,$$
$$x_9x_5 - x_{11}x_5, x_{12}x_4 - (1/3)x_9x_7 + (4/3)x_{10}x_7, x_{12}x_5 + (1/3)x_9x_7 - x_{12}x_6 + (5/3)x_{10}x_7,$$
$$x_{10}x_7 - x_{11}x_7, x_{12}x_7 + 2x_9x_5 - (1/3)x_9x_7 - (2/3)x_{10}x_7, x_9x_{10} - x_9x_{11}, x_9x_{10} - x_{10}x_{11},$$
$$x_9x_{12} - x_{12}x_{12} + 2x_9x_{10} - 3x_{14}x_1 + 3x_{14}x_2, x_{10}x_{12} - x_{12}x_{12} + 2x_9x_{10} - 3x_{14}x_2, \quad (5.1.5)$$
$$x_{11}x_{12} - x_{12}x_{12} + 2x_9x_{10} + 3x_{14}x_1, x_1x_{13} - 4x_{12}x_6 + 4x_{10}x_7,$$
$$x_2x_{13} + (4/3)x_9x_7 - 4x_{12}x_6 + (8/3)x_{10}x_7, x_3x_{13}, x_8x_{13} - 6x_{14}x_3, x_{13}x_4 + x_9x_9x_3,$$
$$x_{13}x_5 + x_{10}x_{10}x_3, x_{13}x_6 - x_{11}x_{11}x_3, x_{13}x_7 - x_{12}x_{12}x_3 + 2x_9x_{10}x_3, x_1x_8x_{12} - 2x_{12}x_6 + 2x_{10}x_7,$$
$$x_2x_8x_{12} + (2/3)x_9x_7 - 2x_{12}x_6 + (4/3)x_{10}x_7, x_9x_{13} - x_9x_9x_8 + 6x_{14}x_4,$$
$$x_{10}x_{13} - x_{10}x_{10}x_8 + 6x_{14}x_5, x_{11}x_{13} - x_{11}x_{11}x_8 - 6x_{14}x_6,$$
$$x_{12}x_{13} - x_{11}x_{12}x_8 - 6x_{14}x_7 - 3x_{14}x_2x_8.$$

By abuse of notation, we will also identify $\mathscr{R}_2$ with its image under the canonical projection $F \to F/(\mathscr{R}_1) = C$.

The following theorem is the main result of this article.

Theorem 5.1.7. *Let I be the two-sided ideal of F generated by the set $\mathscr{R} = \mathscr{R}_1 \cup \mathscr{R}_2$ of 160 homogeneous elements and let $D = F/I$. Define the morphism $\varphi : F \to \mathrm{HH}^\bullet(A)$ of bigraded $\Bbbk$-algebras by setting $\varphi(x_i) = X_i$ for $i \in [\![1,14]\!]$. It is easy to check that φ is surjective and $I \subseteq \mathrm{Ker}(\varphi)$, so φ induces the surjective morphism $\bar{\varphi} : D \to \mathrm{HH}^\bullet(A)$. Moreover, $\bar{\varphi}$ is an isomorphism, i.e. $\mathrm{Ker}(\varphi) = I$.*

Before presenting the proof of the previous theorem, let us provide some auxiliary results. We refer the reader to [25] (see also [26]) for the theory of Gröbner bases, as well as the usual terminology we will follow. Using GAP (see [6]) we get a Gröbner basis G of I given by the following 184 elements

$$x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_1x_7, x_1x_{11} - 2x_1x_{10} - 2x_1x_9, x_1x_{12} - x_1x_{10} - x_1x_9,$$
$$x_2x_1 - x_1x_2, x_2^2, x_2x_3, x_2x_4, x_2x_5, x_2x_6, x_2x_7, x_2x_9 + x_1x_9, x_2x_{10} - 2x_1x_{10},$$
$$x_2x_{11} - x_1x_{10} - x_1x_9, x_2x_{12} - x_1x_{10}, x_3x_1 - x_1x_3, x_3x_2 - x_2x_3, x_3^2, x_3x_4, x_3x_5, x_3x_6, x_3x_7, x_3x_8,$$
$$x_3x_{13}, x_4x_1 - x_1x_4, x_4x_2 - x_2x_4, x_4x_3 - x_3x_4, x_4^2, x_4x_5, x_4x_6, x_4x_7, x_4x_8 - x_3x_9, x_4x_{11} - x_4x_{10},$$
$$x_5x_1 - x_1x_5, x_5x_2 - x_2x_5, x_5x_3 - x_3x_5, x_5x_4 + x_4x_5, x_5^2, x_5x_6, x_5x_7, x_5x_8 - x_3x_{10}, x_5x_9 - x_4x_{10},$$
$$x_5x_{11} - x_4x_{10}, x_5x_{12} - x_4x_{12} - (1/2)x_2x_{13} + (1/4)x_1x_{13}, x_6x_1 - x_1x_6, x_6x_2 - x_2x_6,$$
$$x_6x_3 - x_3x_6, x_6x_4 + x_4x_6, x_6x_5 + x_5x_6, x_6^2, x_6x_7, x_6x_8 + x_3x_{11}, x_6x_9 + x_5x_9, x_6x_{10} + x_4x_{10},$$

$x_6x_{12} + (1/2)x_5x_{12} + (1/2)x_4x_{12} - (3/8)x_1x_{13}, x_7x_1 - x_1x_7, x_7x_2 - x_2x_7, x_7x_3 - x_3x_7,$
$x_7x_4 + x_4x_7, x_7x_5 + x_5x_7, x_7x_6 + x_6x_7, x_7^2, x_7x_8 + x_3x_{12}, x_7x_9 - 4x_6x_{12} - 3x_4x_{12} + x_1x_{13},$
$x_7x_{10} - (1/4)x_7x_9 + (3/4)x_4x_{12}, x_7x_{11} - x_7x_{10},$
$x_7x_{12} + x_4x_{12} + 2x_4x_{10} + (1/2)x_2x_{13} - (1/2)x_1x_{13}, x_8x_1 - x_1x_8, x_8x_2 - x_2x_8, x_8x_3 - x_3x_8,$
$x_8x_4 + x_4x_8, x_8x_5 + x_5x_8, x_8x_6 + x_6x_8, x_8x_7 + x_7x_8, x_8^2, x_8x_{13} - 6x_3x_{14}, x_9x_1 - x_1x_9,$
$x_9x_2 - x_2x_9, x_9x_3 - x_3x_9, x_9x_4 - x_4x_9, x_9x_5 - x_5x_9, x_9x_6 - x_6x_9, x_9x_7 - x_7x_9, x_9x_8 - x_8x_9,$
$x_9x_{11} - x_9x_{10}, x_{10}x_1 - x_1x_{10}, x_{10}x_2 - x_2x_{10}, x_{10}x_3 - x_3x_{10}, x_{10}x_4 - x_9x_5, x_{10}x_5 - x_5x_{10},$
$x_{10}x_6 + x_9x_5, x_{10}x_7 - x_7x_{10}, x_{10}x_8 - x_8x_{10}, x_{10}x_9 - x_9x_{10}, x_{10}x_{11} - x_9x_{10},$
$x_{10}x_{12} - x_9x_{12} - 6x_2x_{14} + 3x_1x_{14}, x_{11}x_1 - x_1x_{11}, x_{11}x_2 - x_2x_{11}, x_{11}x_3 - x_3x_{11}, x_{11}x_4 - x_9x_5,$
$x_{11}x_5 - x_9x_5, x_{11}x_6 - x_6x_{11}, x_{11}x_7 - x_{10}x_7, x_{11}x_8 - x_8x_{11}, x_{11}x_9 - x_9x_{11}, x_{11}x_{10} - x_{10}x_{11},$
$x_{11}x_{12} - x_{10}x_{12} + 3x_2x_{14} + 3x_1x_{14}, x_{12}x_1 - x_1x_{12}, x_{12}x_2 - x_2x_{12}, x_{12}x_3 - x_3x_{12},$
$x_{12}x_4 + (4/3)x_{10}x_7 - (1/3)x_9x_7, x_{12}x_5 - x_5x_{12}, x_{12}x_6 - x_6x_{12}, x_{12}x_7 - x_7x_{12}, x_{12}x_8 - x_8x_{12},$
$x_{12}x_9 - x_9x_{12}, x_{12}x_{10} - x_{10}x_{12}, x_{12}x_{11} - x_{11}x_{12}, x_{12}^2 - x_{11}x_{12} - 2x_9x_{10} - 3x_1x_{14},$
$x_{13}x_1 - x_1x_{13}, x_{13}x_2 - x_2x_{13}, x_{13}x_3 - x_3x_{13}, x_{13}x_4 + x_4x_{13}, x_{13}x_5 + x_5x_{13}, x_{13}x_6 + x_6x_{13},$
$x_{13}x_7 + x_7x_{13}, x_{13}x_8 + x_8x_{13}, x_{13}x_9 - x_9x_{13}, x_{13}x_{10} - x_{10}x_{13}, x_{13}x_{11} - x_{11}x_{13}, x_{13}x_{12} - x_{12}x_{13},$
$x_{13}^2, x_{14}x_1 - x_1x_{14}, x_{14}x_2 - x_2x_{14}, x_{14}x_3 - x_3x_{14}, x_{14}x_4 - x_4x_{14}, x_{14}x_5 - x_5x_{14}, x_{14}x_6 - x_6x_{14},$
$x_{14}x_7 - x_7x_{14}, x_{14}x_8 - x_8x_{14}, x_{14}x_9 - x_9x_{14}, x_{14}x_{10} - x_{10}x_{14}, x_{14}x_{11} - x_{11}x_{14}, x_{14}x_{12} - x_{12}x_{14},$
$x_{14}x_{13} - x_{13}x_{14}, x_1x_8x_{12} - 2x_{12}x_6 + 2x_{10}x_7, x_2x_8x_{12} - 2x_{12}x_6 + (4/3)x_{10}x_7 + (2/3)x_9x_7,$
$x_3x_9^2 - x_4x_{13}, x_3x_9x_{12} + x_7x_{13}, x_3x_{10}^2 - x_5x_{13}, x_3x_{11}^2 + x_6x_{13}, x_8x_9^2 - x_9x_{13} - 6x_4x_{14},$
$x_8x_9x_{12} + 6x_2x_8x_{14} - 6x_1x_8x_{14} - x_{12}x_{13} + 6x_7x_{14}, x_8x_{10}^2 - x_{10}x_{13} - 6x_5x_{14},$
$x_8x_{11}^2 - x_{11}x_{13} + 6x_6x_{14}, x_1x_8x_9 + (1/2)x_2x_{13} - (1/2)x_1x_{13}, x_1x_8x_{10} + x_1x_8x_9 - (1/2)x_1x_{13},$
$x_1x_8x_{11} - 2x_1x_8x_{10} - 2x_1x_8x_9, x_1x_9^2, x_1x_9x_{10} + 2x_1x_9^2, x_1x_9x_{12} + x_1x_9^2, x_1x_9x_{13}, x_1x_{10}^2,$
$x_1x_{10}x_{13}, x_2x_8x_9 + x_1x_8x_9, x_2x_8x_{10} + 2x_1x_8x_9 - x_1x_{13}, x_2x_8x_{11} - (1/2)x_1x_{13}, x_3x_9x_{13},$
$x_3x_{10}x_{13}, x_3x_{11}x_{13}, x_3x_{12}x_{13}, x_4x_{10}^2 - x_4x_9x_{10}, x_8x_9x_{13} - 6x_3x_9x_{14}, x_8x_{10}x_{13} - 6x_3x_{10}x_{14},$
$x_8x_{11}x_{13} - 6x_3x_{11}x_{14}, x_8x_{12}x_{13} - 6x_3x_{12}x_{14}, x_9x_{10}^2 - x_9^2x_{10}, x_3x_9x_{10}x_{13},$
$x_8x_9x_{10}x_{13} - 6x_3x_9x_{10}x_{14}.$

We will now compute the standard words with respect to G, *i.e.* the monomials on the letters x_i, $i \in [\![1, 14]\!]$, that are not divisible by the leading terms of the elements of G. This is a direct but tedious computation. We recall that the set of standard words forms a $\Bbbk$-basis S of D. Obviously, $1 \in S$ and $x_i \in S$ for $i \in [\![1, 14]\!]$. The elements in S generated by 2 elements are given by the following 46 elements

$$\begin{aligned}
&x_1x_8, x_1x_9, x_1x_{10}, x_1x_{13}, x_1x_{14},\\
&x_2x_8, x_2x_{13}, x_2x_{14},\\
&x_3x_9, x_3x_{10}, x_3x_{11}, x_3x_{12}, x_3x_{14},\\
&x_4x_9, x_4x_{10}, x_4x_{12}, x_4x_{13}, x_4x_{14},\\
&x_5x_{10}, x_5x_{13}, x_5x_{14},\\
&x_6x_{11}, x_6x_{13}, x_6x_{14},\\
&x_7x_{13}, x_7x_{14},\\
&x_8x_9, x_8x_{10}, x_8x_{11}, x_8x_{12}, x_8x_{14},\\
&x_9^2, x_9x_{10}, x_9x_{12}, x_9x_{13}, x_9x_{14},\\
&x_{10}^2, x_{10}x_{13}, x_{10}x_{14},\\
&x_{11}^2, x_{11}x_{13}, x_{11}x_{14},\\
&x_{12}x_{13}, x_{12}x_{14},\\
&x_{13}x_{14},\\
&x_{14}^2.
\end{aligned} \tag{5.1.6}$$

Analogously, the elements in S generated by 3 elements are given by the following 68 elements

$$
\begin{aligned}
&x_1x_8x_{14}, x_1x_9x_{14}, x_1x_{10}x_{14}, x_1x_{13}x_{14}, x_1x_{14}^2,\\
&x_2x_8x_{14}, x_2x_{13}x_{14}, x_2x_{14}^2,\\
&x_3x_9x_{10}, x_3x_9x_{14}, x_3x_{10}x_{14}, x_3x_{11}x_{14}, x_3x_{12}x_{14}, x_3x_{14}^2,\\
&x_4x_9^2, x_4x_9x_{10}, x_4x_9x_{12}, x_4x_9x_{13}, x_4x_9x_{14}, x_4x_{10}x_{13}, x_4x_{10}x_{14}, x_4x_{12}x_{13}, x_4x_{12}x_{14}, x_4x_{13}x_{14}, x_4x_{14}^2,\\
&x_5x_{10}^2, x_5x_{10}x_{13}, x_5x_{10}x_{14}, x_5x_{13}x_{14}, x_5x_{14}^2,\\
&x_6x_{11}^2, x_6x_{11}x_{13}, x_6x_{11}x_{14}, x_6x_{13}x_{14}, x_6x_{14}^2,\\
&x_7x_{13}x_{14}, x_7x_{14}^2,\\
&x_8x_9x_{10}, x_8x_9x_{14}, x_8x_{10}x_{14}, x_8x_{11}x_{14}, x_8x_{12}x_{14}, x_8x_{14}^2,\\
&x_9^3, x_9^2x_{10}, x_9^2x_{12}, x_9^2x_{13}, x_9^2x_{14}, x_9x_{10}x_{13}, x_9x_{10}x_{14}, x_9x_{12}x_{13}, x_9x_{12}x_{14}, x_9x_{13}x_{14}, x_9x_{14}^2,\\
&x_{10}^3, x_{10}^2x_{13}, x_{10}^2x_{14}, x_{10}x_{13}x_{14}, x_{10}x_{14}^2,\\
&x_{11}^3, x_{11}^2x_{13}, x_{11}^2x_{14}, x_{11}x_{13}x_{14}, x_{11}x_{14}^2,\\
&x_{12}x_{13}x_{14}, x_{12}x_{14}^2,\\
&x_{13}x_{14}^2,\\
&x_{14}^3.
\end{aligned}
\tag{5.1.7}
$$

Finally, the elements in S generated by 4 elements are given by the following 89 elements

$$
\begin{aligned}
&x_1x_8x_{14}^2, x_1x_9x_{14}^2, x_1x_{10}x_{14}^2, x_1x_{13}x_{14}^2, x_1x_{14}^3,\\
&x_2x_8x_{14}^2, x_2x_{13}x_{14}^2, x_2x_{14}^3,\\
&x_3x_9x_{10}x_{14}, x_3x_9x_{14}^2, x_3x_{10}x_{14}^2, x_3x_{11}x_{14}^2, x_3x_{12}x_{14}^2, x_3x_{14}^3,\\
&x_4x_9^3, x_4x_9^2x_{10}, x_4x_9^2x_{12}, x_4x_9^2x_{13}, x_4x_9^2x_{14}, x_4x_9x_{10}x_{13}, x_4x_9x_{10}x_{14}, x_4x_9x_{12}x_{13}, x_4x_9x_{12}x_{14}, x_4x_9x_{13}x_{14},\\
&x_4x_9x_{14}^2, x_4x_{10}x_{13}x_{14}, x_4x_{10}x_{14}^2, x_4x_{12}x_{13}x_{14}, x_4x_{12}x_{14}^2, x_4x_{13}x_{14}^2, x_4x_{14}^3,\\
&x_5x_{10}^3, x_5x_{10}^2x_{13}, x_5x_{10}^2x_{14}, x_5x_{10}x_{13}x_{14}, x_5x_{10}x_{14}^2, x_5x_{13}x_{14}^2, x_5x_{14}^3,\\
&x_6x_{11}^3, x_6x_{11}^2x_{13}, x_6x_{11}^2x_{14}, x_6x_{11}x_{13}x_{14}, x_6x_{11}x_{14}^2, x_6x_{13}x_{14}^2, x_6x_{14}^3,\\
&x_7x_{13}x_{14}^2, x_7x_{14}^3,\\
&x_8x_9x_{10}x_{14}, x_8x_9x_{14}^2, x_8x_{10}x_{14}^2, x_8x_{11}x_{14}^2, x_8x_{12}x_{14}^2, x_8x_{14}^3,\\
&x_9^4, x_9^3x_{10}, x_9^3x_{12}, x_9^3x_{12}, x_9^3x_{13}, x_9^3x_{14}, x_9^2x_{10}x_{13}, x_9^2x_{10}x_{14}, x_9^2x_{12}x_{13}, x_9^2x_{12}x_{14}, x_9^2x_{13}x_{14}, x_9^2x_{14}^2,\\
&x_9x_{10}x_{13}x_{14}, x_9x_{10}x_{14}^2, x_9x_{12}x_{13}x_{14}, x_9x_{12}x_{14}^2, x_9x_{13}x_{14}^2, x_9x_{14}^3,\\
&x_{10}^4, x_{10}^3x_{13}, x_{10}^3x_{14}, x_{10}^2x_{13}x_{14}, x_{10}^2x_{14}^2, x_{10}x_{13}x_{14}^2, x_{10}x_{14}^3,\\
&x_{11}^4, x_{11}^3x_{13}, x_{11}^3x_{14}, x_{11}^2x_{13}x_{14}, x_{11}^2x_{14}^2, x_{11}x_{13}x_{14}^2, x_{11}x_{14}^3,\\
&x_{12}x_{13}x_{14}^2, x_{12}x_{14}^3,\\
&x_{13}x_{14}^3,\\
&x_{14}^4.
\end{aligned}
\tag{5.1.8}
$$

Lemma 5.1.8. *Let x be a word generated by r elements, where $r \geqslant 5$. Then the following statements are equivalent:*

(1) $x \in S$.

(2) $y \in S$ *for any subword* $y \subsetneq x$.

(3) $y \in S$ *for any subword* $y \subsetneq x$ *generated by* $r - 1$ *elements.*

(4) $y \in S$ *for any subword* $y \subsetneq x$ *generated by* 4 *elements.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) follow immediately from the definition of standard word. On the other hand, to prove the implication (4) $\Rightarrow$ (1), it suffices to note that the leading word of any element in G is generated by at most than 4 elements. □

Lemma 5.1.9. *Let x be a word. Then $x \in S$ if and only if $xx_{14} \in S$.*

Proof. To prove the direct implication, suppose that $x \in S$ is generated by r elements. If $r \in [\![0,3]\!]$, we get that $xx_{14} \in S$ directly from (5.1.6) - (5.1.8). If $r \geqslant 4$, write $x = yz$, where y,z are words and z is generated by 3 elements. Obviously, $z \in S$ and $zx_{14} \in S$. By Lemma 5.1.8, $xx_{14} = yzx_{14} \in S$. Finally, note that the converse follows from the definition of standard word. □

The following result is a direct consequence of the previous lemma.

Corollary 5.1.10. *Let $\tilde{S}$ be the elements in S generated by x_i for $i \in [\![1,13]\!]$, and $\tilde{D}$ the subspace of D generated by the elements in $\tilde{S}$. Then*

$$D \cong \tilde{D} \otimes \Bbbk[x_{14}]$$

as graded $\Bbbk$-vector spaces.

We are now ready to prove Theorem 5.1.7.

Proof of Theorem 5.1.7. It is easy to check that the morphism φ vanishes on the set $\mathscr{R}_1$ since the algebra $\mathrm{HH}^{\bullet}(A)$ is graded commutative, and it also vanishes on the set $\mathscr{R}_2$, as the reader can check using Remark 5.1.1, Fact 5.1.3, (5.1.3) and the coboundaries in Subsubsections 4.2.2.3 and 4.2.2.3. Hence, $I \subseteq \mathrm{Ker}(\varphi)$. By Proposition 4.2.17, we have

$$\mathrm{HH}^{\bullet}(A) \cong \Big(\bigoplus_{\substack{m \,\in\, [\![0,\,4]\!], \\ n \,\in\, \mathbb{N}_0}} \tilde{H}^n_m \Big) \otimes \Bbbk[\omega_1^* \epsilon^! | 1]$$

as graded $\Bbbk$-vector spaces. Let S^n_m be the elements in $\tilde{S}$ with cohomological degree $n \in \mathbb{N}_0$ and internal degree $m - n$, where $m \in \mathbb{Z}$. To prove that $\bar{\varphi}$ is an isomorphism, it is sufficient to prove that the cardinality of S^n_m is as same as the dimension of $\tilde{H}^n_m$.

Take $x \in S^n_m$. Since the words $x_i x_j$ for $i > j$ are leading terms of elements in G, we may assume that x is of the form $x_1^{r_1} x_2^{r_2} \cdots x_{13}^{r_{13}}$, where $r_i \in \mathbb{N}_0$ for $i \in [\![1,13]\!]$. Since x_i^2 for $i = 1,2,3,4,5,6,7,8,12,13$ are leading terms of elements in G, we assume $r_i \in [\![0,1]\!]$ for $i = 1,2,3,4,5,6,7,8,12,13$. By degree reasons, we have

$$r_4 + r_5 + r_6 + r_7 + r_8 + 2r_9 + 2r_{10} + 2r_{11} + 2r_{12} + 3r_{13} = n, \tag{5.1.9}$$

and

$$2r_1 + 2r_2 + 4r_3 + 2r_4 + 2r_5 + 2r_6 + 2r_7 - 2r_9 - 2r_{10} - 2r_{11} - 2r_{12} - 2r_{13} = m - n.$$

Adding the two equations together, we get

$$2r_1 + 2r_2 + 4r_3 + 3r_4 + 3r_5 + 3r_6 + 3r_7 + r_8 + r_{13} = m. \tag{5.1.10}$$

Note that the previous identity tells us that $m \in \mathbb{N}_0$.

First, we will first prove that $m \leqslant 4$. If $r_1 = 1$, then $r_i = 0$ for $i = 2,3,4,5,6,7,11,12$, since $x_1 x_i$ is the leading term of an element of the Gröbner basis G for $i = 2,3,4,5,6,7,11,12$. The equation (5.1.10) then shows that $m = 2 + r_8 + r_{13} \leqslant 4$. Assume for the rest of the paragraph that $r_1 = 0$. If $r_2 = 1$, then $r_i = 0$ for $i = 3,4,5,6,7,9,10,11,12$, since $x_2 x_i$ is the leading term of an element of the Gröbner basis G for $i = 3,4,5,6,7,9,10,11,12$. The equation (5.1.10) thus shows that $m = 2 + r_8 + r_{13} \leqslant 4$. Suppose for the rest of the paragraph that $r_2 = 0$. If $r_3 = 1$, then $r_i = 0$ for $i = 4,5,6,7,8,13$, since $x_3 x_i$ is the leading term of an element of the Gröbner basis G for $i = 4,5,6,7,8,13$. The equation (5.1.10) hence shows that $m = 4$. Assume for the rest of the paragraph that $r_3 = 0$. If $r_4 = 1$, then $r_i = 0$ for $i = 5,6,7,8,11$, since $x_4 x_i$ is the leading term of an element of the Gröbner basis G for $i = 5,6,7,8,11$. Then, the equation (5.1.10) shows that $m = 3 + r_{13} \leqslant 4$. Suppose for the rest of the paragraph that $r_4 = 0$. If $r_5 = 1$, then $r_i = 0$ for $i = 6,7,8,9,11,12$, since $x_5 x_i$ is the leading term of an element of the Gröbner basis G for $i = 6,7,8,9,11,12$. The equation (5.1.10) thus shows that $m = 3 + r_{13} \leqslant 4$. Assume for the rest of the paragraph that $r_5 = 0$. If $r_6 = 1$, then $r_i = 0$ for $i = 7,8,9,10,12$, since $x_6 x_i$ is the leading term of an element of the Gröbner basis G for $i = 7,8,9,10,12$. The equation (5.1.10) then shows that $m = 3 + r_{13} \leqslant 4$. Suppose further that $r_6 = 0$. If $r_7 = 1$, then $r_i = 0$ for $i = 8,9,10,11,12$, since

x_7x_i is the leading term of an element of the Gröbner basis G for $i = 8,9,10,11,12$. The equation (5.1.10) shows that $m = 3 + r_{13} \leqslant 4$. Finally, assume also that $r_7 = 0$. Then $m = r_8 + r_{13} \leqslant 2 \leqslant 4$.

Then we suppose $m = 4$. The equation (5.1.10) then becomes

$$2r_1 + 2r_2 + 4r_3 + 3r_4 + 3r_5 + 3r_6 + 3r_7 + r_8 + r_{13} = 4. \tag{5.1.11}$$

If $r_1 = 1$, then $r_i = 0$ for $i = 2,3,4,5,6,7,11,12$, and equation (5.1.11) shows $r_8 + r_{13} = 2$, which gives $r_8 = r_{13} = 1$. This is impossible since x_1x_8 can only be followed by x_{14} by (5.1.7). Assume for the rest of the paragraph that $r_1 = 0$. If $r_2 = 1$, then $r_i = 0$ for $i = 3,4,5,6,7,9,10,11,12$, and equation (5.1.11) shows $r_8 + r_{13} = 2$. In the same way as before, this case is also impossible. Suppose for the rest of the paragraph that $r_2 = 0$. If $r_3 = 1$, then $r_i = 0$ for $i = 4,5,6,7,8,13$. Then $x = x_3$, x_3x_9, x_3x_{10}, x_3x_{11}, x_3x_{12} or $x_3x_9x_{10}$. Suppose for the rest of the paragraph that $r_3 = 0$. If $r_4 = 1$, then $r_i = 0$ for $i = 5,6,7,8,11$, and equation (5.1.11) shows $r_{13} = 1$. By (5.1.9), n is even. Moreover, $x = x_4x_9^{r_9}x_{13}$, $x_4x_9^{r_9}x_{10}x_{13}$ or $x_4x_9^{r_9}x_{12}x_{13}$ for $r_9 \in \mathbb{N}_0$. Suppose for the rest of the paragraph that $r_4 = 0$. If $r_5 = 1$, then $r_i = 0$ for $i = 6,7,8,9,11,12$, and equation (5.1.11) shows $r_{13} = 1$. Then n is even by (5.1.9), and $x = x_5x_{10}^{r_{10}}x_{13}$ for $r_{10} \in \mathbb{N}_0$. Suppose for the rest of the paragraph that $r_5 = 0$. If $r_6 = 1$, then $r_i = 0$ for $i = 7,8,9,10,12$, and equation (5.1.11) shows $r_{13} = 1$. Then n is even by (5.1.9), and $x = x_6x_{11}^{r_{11}}x_{13}$ for $r_{11} \in \mathbb{N}_0$. Suppose for the rest of the paragraph that $r_6 = 0$. If $r_7 = 1$, then $r_i = 0$ for $i = 8,9,10,11,12$, and equation (5.1.11) shows $r_{13} = 1$, which implies that $x = x_7x_{13}$. Finally, assume also that $r_7 = 0$. Then, equation (5.1.11) shows $4 = r_8 + r_{13} \leqslant 2$, which is impossible. To sum up, we have

$$S_4^0 = \{x_3\},\ S_4^2 = \{x_3x_9, x_3x_{10}, x_3x_{11}, x_3x_{12}\},\ S_4^4 = \{x_4x_{13}, x_5x_{13}, x_6x_{13}, x_7x_{13}, x_3x_9x_{10}\},$$
$$S_4^n = \{x_4x_9^{(n-4)/2}x_{13}, x_4x_9^{(n-6)/2}x_{10}x_{13}, x_4x_9^{(n-6)/2}x_{12}x_{13}, x_5x_{10}^{(n-4)/2}x_{13}, x_6x_{11}^{(n-4)/2}x_{13}\}$$

if $n \geqslant 6$ is even, and $S_4^n = \emptyset$ if n is odd.

Suppose $m = 3$. Then (5.1.10) becomes

$$2r_1 + 2r_2 + 4r_3 + 3r_4 + 3r_5 + 3r_6 + 3r_7 + r_8 + r_{13} = 3. \tag{5.1.12}$$

If $r_1 = 1$, then $r_i = 0$ for $i = 2,3,4,5,6,7,11,12$, and equation (5.1.12) shows $r_8 + r_{13} = 1$. Then $r_8 + r_{13}$ is odd. We have thus either $r_8 = 1$ and $r_{13} = 0$, or $r_8 = 0$ and $r_{13} = 1$. Both cases imply that n is odd by (5.1.9). If $r_8 = 1$ and $r_{13} = 0$, then $r_9 = 0$ and $r_{10} = 0$ by (5.1.7), so $x = x_1x_8$. If $r_8 = 0$ and $r_{13} = 1$, then x has the form $x_1x_9^{r_9}x_{10}^{r_{10}}x_{13}$. By (5.1.7), x_1x_9 and x_1x_{10} can only be followed by x_{14}, so $x = x_1x_{13}$. Now assume for the rest of the paragraph that $r_1 = 0$. If $r_2 = 1$, then $r_i = 0$ for $i = 3,4,5,6,7,9,10,11,12$, and equation (5.1.12) shows $r_8 + r_{13} = 1$. We have either $r_8 = 1$ and $r_{13} = 0$, or $r_8 = 0$ and $r_{13} = 1$. Moreover, n is odd. So, $x = x_2x_8$ or x_2x_{13}. Suppose for the rest of the paragraph that $r_2 = 0$. Then $r_3 = 0$ by (5.1.12). If $r_4 = 1$, then $r_i = 0$ for $i = 5,6,7,8,11$, and equation (5.1.12) shows $r_{13} = 0$. Hence, n is odd by (5.1.9), and x has the form $x_4x_9^{r_9}x_{10}^{r_{10}}x_{12}^{r_{12}}$. If $r_9 = 0$, then x can only be x_4, x_4x_{10} or x_4x_{12}. If $r_9 \neq 0$, then $x = x_4x_9^{r_9}$, $x_4x_9^{r_9}x_{10}$ or $x_4x_9^{r_9}x_{12}$. Suppose for the rest of the paragraph that $r_4 = 0$. If $r_5 = 1$, then $r_i = 0$ for $i = 6,7,8,9,11,12$. Then, equation (5.1.12) shows $r_{13} = 0$. Then n is odd by (5.1.9) and $x = x_5x_{10}^{r_{10}}$. Suppose for the rest of the paragraph that $r_5 = 0$. If $r_6 = 1$, then $r_i = 0$ for $i = 7,8,9,10,12$, and equation (5.1.12) shows $r_{13} = 0$. So, n is odd by (5.1.9), and $x = x_6x_{11}^{r_{11}}$. Suppose for the rest of the paragraph that $r_6 = 0$. If $r_7 = 1$, then $r_i = 0$ for $i = 8,9,10,11,12$, and equation (5.1.12) shows $r_{13} = 0$. So, $x = x_7$. Suppose for the rest of the paragraph that $r_7 = 0$. If $r_8 = 1$, then $r_{13} = 0$, and equation (5.1.12) shows $1 = 3$, which is impossible. Finally, assume also that $r_8 = 0$. Then equation (5.1.12) shows $r_{13} = 3$, which is impossible. To sum up, we have

$$S_3^1 = \{x_4, x_5, x_6, x_7, x_1x_8, x_2x_8\},\ S_3^3 = \{x_1x_{13}, x_2x_{13}, x_4x_9, x_4x_{10}, x_4x_{12}, x_5x_{10}, x_6x_{11}\},$$
$$S_3^n = \{x_4x_9^{(n-1)/2}, x_4x_9^{(n-3)/2}x_{10}, x_4x_9^{(n-3)/2}x_{12}, x_5x_{10}^{(n-1)/2}, x_6x_{11}^{(n-1)/2}\}$$

if $n \geqslant 5$ is odd, and $S_3^n = \emptyset$ if n is even.

Suppose $m = 2$. Then (5.1.10) becomes

$$2r_1 + 2r_2 + 4r_3 + 3r_4 + 3r_5 + 3r_6 + 3r_7 + r_8 + r_{13} = 2. \tag{5.1.13}$$

Then $r_i = 0$ for $i = 3,4,5,6,7$. If $r_1 = 1$, then $r_i = 0$ for $i = 2,3,4,5,6,7,11,12$, and equation (5.1.13) shows $r_8 = r_{13} = 0$. Hence, n is even by (5.1.9), and $x = x_1$, x_1x_9 or x_1x_{10}. Assume for the rest of the paragraph that $r_1 = 0$. If $r_2 = 1$, then $r_i = 0$ for $i = 3,4,5,6,7,9,10,11,12$, and equation (5.1.13) shows $r_8 = r_{13} = 0$, so $x = x_2$. Suppose for the rest of the paragraph that $r_2 = 0$. If $r_8 = 1$, (5.1.13) shows $r_{13} = 1$, which is impossible. Finally, if $r_8 = 0$, then $r_{13} = 2$, which is also impossible. We thus have

$$S_2^0 = \{x_1,x_2\}, \quad S_2^2 = \{x_1x_9,x_1x_{10}\},$$

and $S_2^n = \emptyset$ if $n = 1$ and $n \geqslant 3$.

Suppose $m = 1$. Then (5.1.10) becomes

$$2r_1 + 2r_2 + 4r_3 + 3r_4 + 3r_5 + 3r_6 + 3r_7 + r_8 + r_{13} = 1.$$

If $r_8 = 1,r_{13} = 0$, then $x = x_8$, x_8x_9, x_8x_{10}, x_8x_{11}, x_8x_{12} or $x_8x_9x_{10}$. If $r_8 = 0, r_{13} = 1$, then $x = x_{13}$, $x_9^{r_9}x_{13}$, $x_9^{r_9}x_{10}x_{13}$, $x_9^{r_9}x_{12}x_{13}$, $x_{10}^{r_{10}}x_{13}$, $x_{11}^{r_{11}}x_{13}$ or $x_{12}x_{13}$. We then get

$$\begin{aligned} &S_1^1 = \{x_8\},\ S_1^3 = \{x_{13}, x_8x_9,x_8x_{10}, x_8x_{11}, x_8x_{12}\},\\ &S_1^5 = \{x_9x_{13}, x_{10}x_{13}, x_{11}x_{13}, x_{12}x_{13}, x_8x_9x_{10}\},\\ &S_1^n = \{x_9^{(n-3)/2}x_{13}, x_9^{(n-5)/2}x_{10}x_{13}, x_9^{(n-5)/2}x_{12}x_{13}, x_{10}^{(n-3)/2}x_{13}, x_{11}^{(n-3)/2}x_{13}\} \end{aligned}$$

if $n \geqslant 7$ is odd, and $S_1^n = \emptyset$ if n is even.

Suppose $m = 0$. Then (5.1.10) becomes

$$2r_1 + 2r_2 + 4r_3 + 3r_4 + 3r_5 + 3r_6 + 3r_7 + r_8 + r_{13} = 0.$$

Then $r_i = 0$ for $i = 1,2,3,4,5,6,7,8,13$. If $r_i = 0$ for all $i \in [\![1,14]\!]$, then $x = 1$. Otherwise, $x = x_9^{r_9}$, $x_9^{r_9}x_{10}$, $x_9^{r_9}x_{12}$, $x_{10}^{r_{10}}$, $x_{11}^{r_{11}}$ or x_{12}. We thus have

$$S_0^0 = \{1\}, \quad S_0^2 = \{x_9, x_{10}, x_{11}, x_{12}\}, \quad S_0^n = \{x_9^{n/2}, x_9^{(n-2)/2}x_{10}, x_9^{(n-2)/2}x_{12}, x_{10}^{n/2}, x_{11}^{n/2}\}$$

if $n \geqslant 4$ is even, and $S_0^n = \emptyset$ if n is odd.

Finally, we leave to the reader the easy task to check that the cardinality of S_m^n is as same as the dimension of $\tilde{H}_m^n$. □

As a direct consequence of Remark 5.1.6 and Theorem 5.1.7 we get the following result.

Corollary 5.1.11. *Recall that $C = F/(\mathcal{R}_1)$ is precisely the free graded-commutative (for the cohomological degree) algebra generated by the elements x_i for $i \in [\![1, 14]\!]$, where $\mathcal{R}_1$ is the set given in* (5.1.4)*. Let $D' = C/J$, where J is the two-sided ideal of C generated by the elements in $\mathcal{R}_2$ given in* (5.1.5)*. Define the morphism $\varphi' : C \to \mathrm{HH}^\bullet(A)$ of bigraded $\Bbbk$-algebras by setting $\varphi'(x_i) = X_i$ for $i \in [\![1,14]\!]$. It is easy to check that φ' is surjective and $J \subseteq \mathrm{Ker}(\varphi')$, so φ' induces the surjective morphism $\bar{\varphi}' : D' \to \mathrm{HH}^\bullet(A)$. Moreover, $\bar{\varphi}'$ is an isomorphism, i.e. $\mathrm{Ker}(\varphi') = J$.*

5.2 Gerstenhaber structure on Hochschild cohomology of $\mathrm{FK}(3)$

In this section, let $\Bbbk$ be a field of characteristic different from 2 and 3, A the Fomin-Kirillov algebra on 3 generators and $P_\bullet^b$ the projective bimodule resolution constructed in Proposition 4.1.15. We will explicitly determine the Gerstenhaber structure of the Hochschild cohomology of A.

Recall that Corollary 5.1.11 gives the algebra structure of $(\mathrm{HH}^\bullet(A), \smile)$. Given $n \in \mathbb{N}_0$, there is a canonical isomorphism

$$\mathrm{Hom}_{A^e}(P_n^b,A) \cong Q^n \tag{5.2.1}$$

of graded $\Bbbk$-vector spaces, where $Q^n = \oplus_{i\in[\![0,\lfloor n/4\rfloor]\!]}\omega_i^* K^{n-4i}$ and $K^n = \mathrm{Hom}_\Bbbk((A^!_{-n})^*,A)$. Recall that the fourteen generators of $\mathrm{HH}^\bullet(A)$ mentioned in Corollary 5.1.11 are represented in $\mathrm{H}^\bullet(Q^\bullet)$ by the following cocycles: $X_1 = \epsilon^!|(ab + ba)$, $X_2 = \epsilon^!|(ab + bc - ac)$, $X_3 = \epsilon^!|abac$, $X_4 = \alpha|bac$, $X_5 = \beta|abc$, $X_6 = \gamma|aba$, $X_7 = \alpha|(aba - abc)$, $X_8 = \alpha|a + \beta|b + \gamma|c$, $X_9 = \alpha_2|1$, $X_{10} = \beta_2|1$, $X_{11} = \gamma_2|1$, $X_{12} = (\alpha\beta+\alpha\gamma)|1$, $X_{13} = \alpha_3|a+\beta_3|b+\gamma_3|c$ and $X_{14} = \omega_1^*\epsilon^!|1$. Let $Y_i \in \mathrm{Hom}_{A^e}(P_n^b,A)$ be the element associated to X_i via the isomorphism (5.2.1) for $i \in [\![1,14]\!]$. In what follows and to simplify our notation, given a cocycle ϕ, we will use the same symbol ϕ for its cohomology class. To reduce space we will denote the cup product simply by juxtaposition.

Let $i_\bullet : P^b_\bullet \to B_\bullet(A)$ be a morphism of complexes of A-bimodules lifting id_A. It is clear that $i_0 : A \otimes (A^!_0)^* \otimes A \to A \otimes A$ and $i_1 : A \otimes (A^!_{-1})^* \otimes A \to A^{\otimes 3}$ can be chosen as follows

$$i_0(1|\epsilon^!|1) = 1|1, \quad i_1(1|\alpha|1) = -1|a|1, \quad i_1(1|\beta|1) = -1|b|1, \quad i_1(1|\gamma|1) = -1|c|1.$$

5.2.1 Gerstenhaber brackets of $\mathrm{HH}^0(A)$ with $\mathrm{HH}^n(A)$

In this subsection, we are going to use the method introduced in Subsection 1.4.1 to compute the Gerstenhaber bracket of X_i for $i \in [\![1, 14]\!]$ with the elements X_1, X_2, X_3 in $\mathrm{HH}^0(A)$. To wit, for every element X_i with $i \in [\![1, 3]\!]$, we find the associated element ρ in the center $\mathcal{Z}(A)$ such that $\ell_\rho i_0 = X_i$, provide the corresponding self-homotopy $h^\rho_\bullet$ satisfying (1.4.1) and then compute the respective Gerstenhaber brackets by means of Theorem 1.4.1.

We remark first that $[X_i,1] = 0$ for $i \in [\![1, 14]\!]$, since $h^1_\bullet = 0$ gives $[X_i,1] = 0$ for $i \in [\![4, 14]\!]$ and the other follow from Definition 1.3.5. On the other hand, Definition 1.3.5 also tells us that $[X_i,X_j] = 0$ for $i,j \in [\![1,3]\!]$. The proof of the following three results is a lengthy but straightforward computation.

Fact 5.2.1. *Let $\rho = ab + ba \in \mathcal{Z}(A)$. Then, there is a self-contracting homotopy $h^\rho_\bullet$ satisfying (1.4.1) such that*

$$\begin{aligned}
h^\rho_0(1|\epsilon^!|1) &= -b|\alpha|1 - a|\beta|1 - 1|\alpha|b - 1|\beta|a,\\
h^\rho_n(1|\alpha_n|1) &= (-1)^{n+1} b|\alpha_{n+1}|1 - 1|\alpha_{n+1}|b,\\
h^\rho_n(1|\beta_n|1) &= (-1)^{n+1} a|\beta_{n+1}|1 - 1|\beta_{n+1}|a
\end{aligned}$$

for $n \in \mathbb{N}$, and

$$\begin{aligned}
h^\rho_1(1|\gamma|1) &= b|\alpha_2|1 + a|\beta_2|1 + a|\alpha\beta|1 + b|\alpha\gamma|1 - 1|\alpha_2|b - 1|\beta_2|a - 1|\alpha\beta|b - 1|\alpha\gamma|a,\\
h^\rho_2(1|\gamma_2|1) &= a|\gamma_3|1 + b|\gamma_3|1 + c|\alpha_2\beta|1 + c|\alpha\beta_2|1 + 1|\gamma_3|a + 1|\gamma_3|b + 1|\alpha_2\beta|c + 1|\alpha\beta_2|c,\\
h^\rho_2(1|\alpha\beta|1) &= -b|\alpha_3|1 - c|\beta_3|1 - a|\alpha_2\gamma|1 - 1|\alpha_3|c - 1|\beta_3|a - 1|\alpha_2\gamma|b,\\
h^\rho_2(1|\alpha\gamma|1) &= -c|\alpha_3|1 - a|\beta_3|1 - b|\alpha_2\gamma|1 - 1|\alpha_3|b - 1|\beta_3|c - 1|\alpha_2\gamma|a.
\end{aligned}$$

Fact 5.2.2. *Let $\rho = ab + bc - ac \in \mathcal{Z}(A)$. Then, there is a self-contracting homotopy $h^\rho_\bullet$ satisfying (1.4.1) such that*

$$\begin{aligned}
h^\rho_0(1|\epsilon^!|1) &= c|\alpha|1 + a|\gamma|1 + 1|\alpha|c + 1|\gamma|a,\\
h^\rho_n(1|\alpha_n|1) &= (-1)^n c|\alpha_{n+1}|1 + 1|\alpha_{n+1}|c,\\
h^\rho_n(1|\gamma_n|1) &= (-1)^n a|\gamma_{n+1}|1 + 1|\gamma_{n+1}|a
\end{aligned}$$

for $n \in \mathbb{N}$, and

$$\begin{aligned}
h^\rho_1(1|\beta|1) &= -c|\alpha_2|1 - a|\gamma_2|1 - c|\alpha\beta|1 - a|\alpha\gamma|1 + 1|\alpha_2|c + 1|\gamma_2|a + 1|\alpha\beta|a + 1|\alpha\gamma|c,\\
h^\rho_2(1|\beta_2|1) &= -a|\beta_3|1 - c|\beta_3|1 - b|\alpha_2\gamma|1 - b|\alpha\beta_2|1 - 1|\beta_3|a - 1|\beta_3|c - 1|\alpha_2\gamma|b - 1|\alpha\beta_2|b,\\
h^\rho_2(1|\alpha\beta|1) &= b|\alpha_3|1 + a|\gamma_3|1 + c|\alpha_2\beta|1 + 1|\alpha_3|c + 1|\gamma_3|b + 1|\alpha_2\beta|a,\\
h^\rho_2(1|\alpha\gamma|1) &= c|\alpha_3|1 + b|\gamma_3|1 + a|\alpha_2\beta|1 + 1|\alpha_3|b + 1|\gamma_3|a + 1|\alpha_2\beta|c.
\end{aligned}$$

Fact 5.2.3. *Let $\rho = abac \in \mathcal{Z}(A)$. Then, there is a self-contracting homotopy $h^\rho_\bullet$ satisfying (1.4.1) such that*

$$\begin{aligned}
h^\rho_0(1|\epsilon^!|1) &= -aba|\gamma|1 - ab|\alpha|c - a|\beta|ac - 1|\alpha|bac,\\
h^\rho_1(1|\alpha|1) &= aba|\alpha\beta|1 - ab|\alpha_2|b - ba|\beta_2|c + c|\alpha_2|bc + b|\beta_2|ac + b|\alpha\beta|bc - 1|\alpha_2|bac - 1|\alpha\beta|abc,\\
h^\rho_1(1|\beta|1) &= aba|\alpha\gamma|1 - 2ab|\alpha_2|c - ac|\alpha_2|a - ab|\alpha\beta|a + a|\alpha_2|bc - a|\beta_2|ab - a|\beta_2|bc + c|\beta_2|ac\\
&\quad + a|\alpha\gamma|ac - 1|\alpha\gamma|bac,\\
h^\rho_1(1|\gamma|1) &= 2aba|\gamma_2|1 - ba|\alpha\beta|c + b|\alpha_2|bc - a|\gamma_2|ac - c|\alpha\beta|ac - 1|\alpha\gamma|abc,\\
h^\rho_2(1|\alpha_2|1) &= bac|\alpha_3|1 + bc|\beta_3|a - ba|\beta_3|c + ba|\alpha_2\gamma|a - b|\alpha_3|ab - b|\alpha_3|bc + c|\alpha_3|ba + a|\beta_3|ac\\
&\quad + c|\beta_3|bc + a|\alpha_2\gamma|bc + b|\alpha_2\gamma|ba - 2|\alpha_3|bac,
\end{aligned}$$

$$
\begin{aligned}
h_2^\rho(1|\beta_2|1) &= abc|\beta_3|1 - 2ab|\alpha_3|c + ac|\alpha_3|b + ab|\beta_3|a - bc|\beta_3|a + ab|\alpha_2\gamma|b - ba|\alpha_2\gamma|a + b|\alpha_3|ab \\
&\quad + 2b|\alpha_3|bc - c|\alpha_3|ba + c|\alpha_3|ac - a|\beta_3|ac - b|\alpha_2\gamma|ba + b|\alpha_2\gamma|ac - 1|\alpha_3|bac \\
&\quad - 2|\beta_3|abc - 1|\alpha_2\gamma|aba, \\
h_2^\rho(1|\gamma_2|1) &= -3aba|\gamma_3|1 + ba|\beta_3|c - ab|\alpha\beta_2|c - b|\alpha_3|bc - a|\beta_3|ac - c|\beta_3|bc + b|\gamma_3|ac - a|\alpha_2\gamma|bc \\
&\quad + 1|\alpha_3|bac - 1|\alpha_2\beta|abc, \\
h_2^\rho(1|\alpha\beta|1) &= -2aba|\alpha\beta_2|1 - ac|\alpha_3|c - bc|\beta_3|b - 2ba|\beta_3|a - ab|\alpha_2\gamma|c - ba|\alpha_2\gamma|b - a|\alpha_3|bc \\
&\quad + b|\alpha_3|ba + 2b|\alpha_3|ac + b|\beta_3|ab - c|\beta_3|ac + a|\gamma_3|ac - c|\gamma_3|ab + c|\alpha_2\beta|ac - a|\alpha_2\gamma|ac \\
&\quad - a|\alpha\beta_2|bc - 1|\alpha_2\gamma|bac - 1|\alpha\beta_2|abc, \\
h_2^\rho(1|\alpha\gamma|1) &= -abc|\alpha_3|1 - 2aba|\alpha_2\beta|1 - 3ab|\alpha_3|b - ab|\beta_3|c - 2bc|\beta_3|c - 2ba|\alpha_2\gamma|c - ba|\alpha\beta_2|a \\
&\quad + 2a|\alpha_3|ba - c|\alpha_3|bc - b|\beta_3|ac - b|\gamma_3|ab - a|\alpha_2\beta|ab - b|\alpha_2\gamma|bc + c|\alpha_2\gamma|ac \\
&\quad - c|\alpha\beta_2|ab - 1|\alpha_2\beta|bac - 2|\alpha_2\gamma|abc.
\end{aligned}
$$

Using the previous results together with Theorem 1.4.1 we obtain the Gerstenhaber bracket between X_i for $i \in [\![1,14]\!]$ and X_1,X_2,X_3.

Proposition 5.2.4. *The Gerstenhaber bracket on* $\mathrm{HH}^\bullet(A)$ *of* X_i *for* $i \in [\![1,14]\!]$ *with an element* X_j *for* $j \in [\![1,3]\!]$ *is given by*

$$
[X_i,X_1] = \begin{cases} -2X_1, & \text{if } i = 8, \\ -4X_1(X_9 + X_{10}), & \text{if } i = 13, \\ 0, & \text{if } i \in [\![1,14]\!] \setminus \{8,13\}, \end{cases}
$$

$$
[X_i,X_2] = \begin{cases} -2X_2, & \text{if } i = 8, \\ -4X_1X_{10}, & \text{if } i = 13, \\ 0, & \text{if } i \in [\![1,14]\!] \setminus \{8,13\}, \end{cases}
$$

and

$$
[X_i,X_3] = \begin{cases} 0, & \text{if } i \in [\![1,7]\!], \\ -4X_3, & \text{if } i = 8, \\ -2X_{i-5}, & \text{if } i = 9,10, \\ 2X_6, & \text{if } i = 11, \\ 2X_7 - X_1X_8 + X_2X_8, & \text{if } i = 12, \\ -4X_3(X_9 + X_{10} + X_{11}), & \text{if } i = 13, \\ X_{13} - (2/3)X_8(X_9 + X_{10} + X_{11}), & \text{if } i = 14. \end{cases}
$$

Proof. Note that $\ell_{ab+ba}i_0 = Y_1$, $\ell_{ab+bc-ac}i_0 = Y_2$ and $\ell_{abac}i_0 = Y_3$. Applying Theorem 1.4.1 together with Facts 5.2.1, 5.2.2 and 5.2.3, we get the brackets

$$
[X_i,X_1] = \begin{cases} -2X_1, & \text{if } i = 8, \\ -(\alpha_2 + \beta_2 + \gamma_2)|(ab + ba) - \alpha\beta|ba - \alpha\gamma|ab, & \text{if } i = 13, \\ 0, & \text{if } i \in [\![1,12]\!] \setminus \{8\}, \end{cases}
$$

$$
[X_i,X_2] = \begin{cases} -2X_2, & \text{if } i = 8, \\ (\alpha_2 + \beta_2 + \gamma_2)|(ac - ab - bc) + \alpha\beta|ac - \alpha\gamma|(ab + bc), & \text{if } i = 13, \\ 0, & \text{if } i \in [\![1,12]\!] \setminus \{8\}, \end{cases}
$$

and

$$
[X_i,X_3] = \begin{cases} 0, & \text{if } i \in [\![1,7]\!], \\ -4X_3, & \text{if } i = 8, \\ -2X_{i-5}, & \text{if } i = 9,10, \\ 2X_6, & \text{if } i = 11, \\ \alpha|(aba - abc) - \beta|bac - \gamma|bac, & \text{if } i = 12, \\ -4(\alpha_2 + \beta_2 + \gamma_2)|abac, & \text{if } i = 13. \end{cases}
$$

Indeed, this was simply done by computing $[Y_i,Y_1] = Y_i h^{ab+ba}_{\mathfrak{h}(Y_i)-1}$, $[Y_i,Y_2] = Y_i h^{ab+bc-ac}_{\mathfrak{h}(Y_i)-1}$, and $[Y_i,Y_3] = Y_i h^{abac}_{\mathfrak{h}(Y_i)-1}$, where $\mathfrak{h}(Y_i)$ denotes the cohomological degree of Y_i for $i \in [\![1,13]\!]$, and by transport of structures. Note that the vanishing of $[X_i,X_3]$ for $i \in [\![4,7]\!]$ also follows from a simple degree argument using Corollary 1.3.8 together with Corollary 4.2.20. The latter two results also tell us that $[X_{14},X_j] = 0$ (or $[Y_{14},Y_j] = 0$) for $j = 1,2$, by degree reasons. This result also follows from noting that h_3^{ab+ba} is of internal degree 2, so $h_3^{ab+ba}(1|u|1)$ is of internal degree 5 for any $u \in \mathscr{B}_3^{!*}$, which implies that $Y_{14}(h_3^{ab+ba}(1|u|1))$ vanishes, since Y_{14} vanishes on any homogeneous element of internal degree strictly less than 6. Hence, $Y_{14}h_3^{ab+ba} = 0$. We get $Y_{14}h_3^{ab+bc-ac} = 0$ for the same reason.

Next, we compute $\varphi = [Y_{14},Y_3] = Y_{14}h_3^{abac}$. By (1.4.1), the map $h_3^{abac} : P_3^b \to P_4^b$ satisfies $\delta_4^b h_3^{abac} = \eta_3 - h_2^{abac}\delta_3^b$. It is easy to check that

$$\begin{aligned}
(\eta_3 - h_2^{abac}\delta_3^b)(1|\alpha_3|1) &= -bac|\alpha_3|a + abc|\beta_3|a - aba|\beta_3|c + aba|\alpha_2\gamma|a + v_{\alpha_3},\\
(\eta_3 - h_2^{abac}\delta_3^b)(1|\beta_3|1) &= -2aba|\alpha_3|c + bac|\alpha_3|b + aba|\beta_3|a - abc|\beta_3|b + aba|\alpha_2\gamma|b + v_{\beta_3},\\
(\eta_3 - h_2^{abac}\delta_3^b)(1|\gamma_3|1) &= abc|\beta_3|c + 3aba|\gamma_3|c - bac|\alpha\beta_2|c + v_{\gamma_3},\\
(\eta_3 - h_2^{abac}\delta_3^b)(1|\alpha_2\beta|1) &= abc|\alpha_3|a - 4bac|\alpha_3|b - 2aba|\beta_3|a - abc|\beta_3|b + bac|\beta_3|c + 3aba|\gamma_3|b\\
&\quad + 2aba|\alpha_2\beta|a - aba|\alpha_2\gamma|b - 2abc|\alpha_2\gamma|c - abc|\alpha\beta_2|a + aba|\alpha\beta_2|c\\
&\quad + v_{\alpha_2\beta},\\
(\eta_3 - h_2^{abac}\delta_3^b)(1|\alpha_2\gamma|1) &= -4bac|\alpha_3|c + bac|\beta_3|a - 4abc|\beta_3|c + 2aba|\alpha_2\beta|b + bac|\alpha_2\gamma|b\\
&\quad - 3aba|\alpha_2\gamma|c + aba|\alpha\beta_2|a + v_{\alpha_2\gamma},\\
(\eta_3 - h_2^{abac}\delta_3^b)(1|\alpha\beta_2|1) &= -3aba|\alpha_3|b + 2abc|\alpha_3|c - 4abc|\beta_3|a + bac|\beta_3|b + 3aba|\gamma_3|a\\
&\quad + 2aba|\alpha_2\beta|c - aba|\alpha_2\gamma|a - abc|\alpha_2\gamma|b - bac|\alpha_2\gamma|c + 2aba|\alpha\beta_2|b\\
&\quad + v_{\alpha\beta_2},
\end{aligned} \tag{5.2.2}$$

where $v_u \in \oplus_{j\in[\![0,4]\!]\setminus\{3\}}(A_j \otimes (A^!_{-3})^* \otimes A_{4-j})$ for $u \in \mathscr{B}_3^{!*}$. By degree reasons, the element $h_3^{abac}(1|u|1)$ for $u \in \mathscr{B}_3^{!*}$ is of the form

$$h_3^{abac}(1|u|1) = B_u + \omega_1(\lambda_1^u a|\epsilon^!|1 + \lambda_2^u 1|\epsilon^!|a + \lambda_3^u b|\epsilon^!|1 + \lambda_4^u 1|\epsilon^!|b + \lambda_5^u c|\epsilon^!|1 + \lambda_6^u 1|\epsilon^!|c),$$

where $B_u \in K_4^b$ has internal degree 7 and $\lambda_i^u \in \Bbbk$ for $i \in [\![1,6]\!]$. Therefore,

$$\begin{aligned}
(\eta_3 - h_2^{abac}\delta_3^b)(1|u|1) &= \delta_4^b h_3^{abac}(1|u|1)\\
&= d_4^b(B_u) + \lambda_1^u f_0^b(a|\epsilon^!|1) + \lambda_2^u f_0^b(1|\epsilon^!|a) + \lambda_3^u f_0^b(b|\epsilon^!|1) + \lambda_4^u f_0^b(1|\epsilon^!|b)\\
&\quad + \lambda_5^u f_0^b(c|\epsilon^!|1) + \lambda_6^u f_0^b(1|\epsilon^!|c).
\end{aligned} \tag{5.2.3}$$

Using the explicit expression of the differential d_4^b given in Fact 4.1.9, together with an elementary computation we see that, given any homogeneous element $B \in K_4^b$ of internal degree 7, the coefficients of $aba|\gamma_3|a$ and $aba|\alpha_2\gamma|a$ in $d_4^b(B)$ are equal, the coefficients of $aba|\gamma_3|b$ and $aba|\alpha_2\gamma|b$ in $d_4^b(B)$ coincide, and the coefficients of $abc|\beta_3|c$ and $abc|\alpha_2\beta|c$ in $d_4^b(B)$ are also the same. Comparing the coefficients of $aba|\gamma_3|a$ and $aba|\alpha_2\gamma|a$ in both sides of the equation (5.2.3), where the left member is explicitly given by (5.2.2) and the right member is computed using (4.1.3), we obtain

$$\lambda_1^{\alpha_3} + \lambda_2^{\alpha_3} = 1/3, \quad \lambda_1^{\alpha\beta_2} + \lambda_2^{\alpha\beta_2} = -4/3, \quad \lambda_1^u + \lambda_2^u = 0$$

for $u \in \mathscr{B}_3^{!*} \setminus \{\alpha_3,\alpha\beta_2\}$. Similarly, comparing the coefficients of $aba|\gamma_3|b$ and $aba|\alpha_2\gamma|b$ in both sides of the equation (5.2.3), where the left member is explicitly given by (5.2.2) and the right member is computed using (4.1.3), we obtain

$$\lambda_3^{\beta_3} + \lambda_4^{\beta_3} = 1/3, \quad \lambda_3^{\alpha_2\beta} + \lambda_4^{\alpha_2\beta} = -4/3, \quad \lambda_3^u + \lambda_4^u = 0$$

for $u \in \mathscr{B}_3^{!*} \setminus \{\beta_3,\alpha_2\beta\}$. Comparing the coefficients of $abc|\beta_3|c$ and $abc|\alpha_2\beta|c$ in both sides of the equation (5.2.3), where the left member is explicitly given by (5.2.2) and the right member is computed using (4.1.3), we obtain

$$\lambda_5^{\gamma_3} + \lambda_6^{\gamma_3} = 1/3, \quad \lambda_5^{\alpha_2\gamma} + \lambda_6^{\alpha_2\gamma} = -4/3, \quad \lambda_5^u + \lambda_6^u = 0$$

for $u \in \mathscr{B}_3^{!*} \setminus \{\gamma_3,\alpha_2\gamma\}$. Then $\varphi(1|u|1) = Y_{14}h_3^{abac}(1|u|1)$ for $u \in \mathscr{B}_3^{!*}$ is given by

$$\varphi(1|\alpha_3|1) = (1/3)a, \qquad \varphi(1|\beta_3|1) = (1/3)b, \qquad \varphi(1|\gamma_3|1) = (1/3)c,$$
$$\varphi(1|\alpha_2\beta|1) = -(4/3)b, \qquad \varphi(1|\alpha_2\gamma|1) = -(4/3)c, \qquad \varphi(1|\alpha\beta_2|1) = -(4/3)a.$$

Hence, $[X_{14}, X_3] = (1/3)(\alpha_3|a + \beta_3|b + \gamma_3|c) - (4/3)(\alpha_2\beta|b + \alpha_2\gamma|c + \alpha\beta_2|a)$.

We now note the following identities,

$$\begin{aligned} &\alpha_2|(ab+ba) = X_1X_9, \quad \beta_2|(ab+ba) = X_1X_{10}, \quad \alpha|aba + \beta|bac = (1/2)(X_1X_8 - X_2X_8),\\ &\alpha_2|abac = X_3X_9, \qquad \beta_2|abac = X_3X_{10}, \qquad \gamma_2|abac = X_3X_{11}, \end{aligned} \tag{5.2.4}$$

and

$$\begin{aligned} (\alpha_3 - \alpha\beta_2)|a &= (1/2)\{X_{13} + X_8(X_9 - X_{10} - X_{11})\},\\ (\beta_3 - \alpha_2\beta)|b &= (1/2)\{X_{13} + X_8(X_{10} - X_9 - X_{11})\},\\ (\gamma_3 - \alpha_2\gamma)|c &= (1/2)\{X_{13} + X_8(X_{11} - X_9 - X_{10})\}, \end{aligned}$$

given in (5.1.2). Using the previous equalities as well as the coboundaries $g^2_{j,2} \in \tilde{\mathfrak{B}}^2_2$ for $j \in [\![1,8]\!] \setminus \{4,5\}$ and $e^1_{1,3} = \alpha|(aba + abc) + (\beta - \gamma)|bac \in \tilde{\mathfrak{B}}^1_3$ of the sets $\tilde{\mathfrak{B}}^2_2$ and $\tilde{\mathfrak{B}}^1_3$ given in Subsubsection 4.2.2.3, we get

$$\begin{aligned} [X_{13},X_1] &= -(\alpha_2+\beta_2+\gamma_2)|(ab+ba) - \alpha\beta|ba - \alpha\gamma|ab - 3g^2_{1,2} - 3g^2_{2,2} - 2g^2_{3,2} + g^2_{8,2}\\ &= -4(\alpha_2+\beta_2)|(ab+ba) = -4X_1(X_9+X_{10}),\\ [X_{13},X_2] &= (\alpha_2+\beta_2+\gamma_2)|(ac-ab-bc) + \alpha\beta|ac - \alpha\gamma|(ab+bc) - g^2_{1,2} - 2g^2_{2,2} - g^2_{3,2} + g^2_{6,2}\\ &\quad - g^2_{7,2} + g^2_{8,2}\\ &= -4\beta_2|(ab+ba) = -4X_1X_{10},\\ [X_{12},X_3] &= \alpha|(aba - abc) - \beta|bac - \gamma|bac - e^1_{1,3} = 2\alpha|(aba-abc) - 2(\alpha|aba + \beta|bac)\\ &= 2X_7 - X_1X_8 + X_2X_8,\\ [X_{13},X_3] &= -4(\alpha_2+\beta_2+\gamma_2)|abac = -4X_3(X_9+X_{10}+X_{11}),\\ [X_{14},X_3] &= (1/3)(\alpha_3|a+\beta_3|b+\gamma_3|c) - (4/3)(\alpha_2\beta|b+\alpha_2\gamma|c+\alpha\beta_2|a)\\ &= X_{13} - (2/3)X_8(X_9+X_{10}+X_{11}). \end{aligned}$$

The proposition is thus proved. □

5.2.2 Gerstenhaber brackets of $\mathrm{HH}^1(A)$ with $\mathrm{HH}^n(A)$

In this subsection, we are going to use the method recalled in Subsection 1.4.2 to compute the Gerstenhaber bracket of X_i for $i \in [\![4, 8]\!]$ with the elements X_j for $j \in [\![1, 14]\!]$.

Let $\rho : A \to A$ be a derivation of A. By [24], Lemma 1.3, the ρ^e-lifting $\rho_\bullet = \{\rho_n : P^b_n \to P^b_n\}_{n\in\mathbb{N}_0}$ of ρ to $(P^b_\bullet,\delta^b_\bullet)$ exists, and it can be chosen in such a way that

$$\rho_0(x|\epsilon^!|y) = \rho(x)|\epsilon^!|y + x|\epsilon^!|\rho(y) \text{ and } \rho_n(\omega_i x|u|y) = xq_{\omega_i u}y + \omega_i\rho(x)|u|y + \omega_i x|u|\rho(y) \tag{5.2.5}$$

for all $x,y \in A$, $n \in \mathbb{N}$, $i \in [\![0,\lfloor n/4\rfloor]\!]$ and $u \in \mathscr{B}^{!*}_{n-4i}$, where $q_{\omega_i u} \in P^b_n$ satisfies that $\delta^b_n(q_{\omega_i u}) = \rho_{n-1}\delta^b_n(\omega_i 1|u|1)$. To reduce space, we will usually write q_u instead of $q_{\omega_0 u}$. As recalled in Subsection 1.4.2, given $\phi \in \mathrm{HH}^n(A)$, the Gerstenhaber bracket $[G(\rho)i_1,\phi] \in \mathrm{HH}^n(A)$ is given by the cohomology class of $\rho\phi - \phi\rho_n$.

In what follows, we consider a set of derivations of A whose classes give a basis of $\mathrm{HH}^1(A)$ and for each of them we will provide some of the corresponding elements $q_{\omega_i u}$ satisfying (5.2.5). Then, we shall compute the respective Gerstenhaber brackets by means of Theorem 1.4.7.

The proof of the following result follows immediately from the statement.

Proposition 5.2.5. *Let $\rho : A \to A$ be the derivation of A defined by $\rho(x) = \deg(x)x$ for $x \in \mathscr{B}$. Then $\rho_\bullet$ defined by $\rho_n(\omega_i x|u|y) = (\deg(x) + \deg(y) + n + 2i)\omega_i x|u|y$ for $x,y \in \mathscr{B}$, $i \in [\![0,\lfloor n/4\rfloor]\!]$, $u \in \mathscr{B}^{!*}_{n-4i}$ and $n \in \mathbb{N}_0$ is a ρ^e-lifting of ρ. Note that $\deg(x) + \deg(y) + n + 2i$ is the internal degree of*

$\omega_i x|u|y$. *Since* $G(\rho)i_1 = -X_8$, *the Gerstenhaber bracket* $[X_8,\phi] \in \mathrm{HH}^n(A)$ *for* $\phi \in \mathrm{HH}^n(A)$ *is given by the cohomology class* $-\mathfrak{a}(\phi)\phi$, *where* $\mathfrak{a}(\phi)$ *is the internal degree of* ϕ. *Hence,*

$$[X_8,X_j] = \begin{cases} -2X_j, & \text{if } j \in [\![1,7]\!] \setminus \{3\}, \\ -4X_3, & \text{if } j = 3, \\ 0, & \text{if } j = 8, \\ 2X_j, & \text{if } j \in [\![9,13]\!], \\ 6X_{14}, & \text{if } j = 14. \end{cases}$$

The proof of Facts 5.2.6, 5.2.7, 5.2.8 and 5.2.9 below is a lengthy but straightforward computation.

Fact 5.2.6. *Let* $\rho = \rho^4 : A \to A$ *be the derivation of* A *defined by* $\rho^4(a) = bac$ *and* $\rho^4(x) = 0$ *for* $x \in \mathscr{B} \setminus \{a\}$. *Then the elements* $q_{\omega_i u} = q^4_{\omega_i u} \in P^b_n$ *in* (5.2.5) *can be chosen as follows. First,* $q^4_{\beta_n} = q^4_{\gamma_n} = 0$ *for* $n \in \mathbb{N}$. *Moreover,*

$$\begin{aligned}
q^4_{\alpha} &= ba|\gamma|1 + b|\alpha|c + 1|\beta|ac, \\
q^4_{\alpha_2} &= ba|\alpha\beta|1 - b|\alpha_2|b - c|\alpha_2|c - b|\alpha\beta|c + 1|\alpha\beta|ac, \\
q^4_{\alpha\beta} &= ab|\gamma_2|1 - ab|\alpha_2|1 + ba|\alpha\gamma|1 - 2b|\alpha_2|c - c|\alpha_2|a - b|\beta_2|c - a|\alpha\beta|c - b|\alpha\beta|a + 1|\alpha_2|bc \\
&\quad - 1|\beta_2|ab + 1|\alpha\gamma|ac, \\
q^4_{\alpha\gamma} &= ba|\gamma_2|1 + 1|\beta_2|ac, \\
q^4_{\alpha_3} &= bc|\beta_3|1 + ba|\alpha_2\gamma|1 + b|\alpha_3|c + c|\alpha_3|b + b|\beta_3|a - c|\beta_3|c - a|\alpha_2\gamma|c + b|\alpha_2\gamma|b - 1|\beta_3|ac \\
&\quad - 1|\alpha_2\gamma|bc, \\
q^4_{\alpha_2\beta} &= ab|\gamma_3|1 + ba|\alpha\beta_2|1 - a|\alpha_3|b - 2a|\beta_3|c - c|\beta_3|a - b|\gamma_3|c - c|\gamma_3|b - a|\alpha_2\beta|c - c|\alpha\beta_2|c \\
&\quad - b|\alpha_2\gamma|c - a|\alpha_2\gamma|a + 1|\alpha_3|bc + 1|\gamma_3|ba - 1|\alpha_2\gamma|ba - 1|\alpha\beta_2|bc, \\
q^4_{\alpha_2\gamma} &= ba|\alpha_2\beta|1 + ab|\alpha\beta_2|1 - 2ab|\alpha_3|1 - ba|\beta_3|1 + a|\alpha_3|c + 2b|\alpha_3|b + c|\alpha_3|a + a|\beta_3|a + b|\beta_3|c \\
&\quad + c|\beta_3|b + a|\alpha_2\gamma|b + a|\alpha\beta_2|c - 1|\alpha_3|ba - 2|\beta_3|ab + 1|\alpha\beta_2|ac, \\
q^4_{\alpha\beta_2} &= 3ba|\gamma_3|1 - (ab+bc)|\beta_3|1 + (ab+bc)|\alpha_2\beta|1 - 3c|\alpha_3|b - 2a|\beta_3|b - b|\beta_3|a - c|\beta_3|c \\
&\quad - 2a|\gamma_3|c - 2c|\gamma_3|a - a|\alpha_2\beta|b - 2c|\alpha_2\beta|c - 2b|\alpha_2\gamma|b - c|\alpha_2\gamma|a - c|\alpha\beta_2|b + 2|\beta_3|ac \\
&\quad + 1|\gamma_3|ab - 1|\alpha_2\gamma|ab, \\
q^4_{\omega_1\epsilon^!} &= bac|\alpha_4|a + 4abc|\beta_4|b - aba|\gamma_4|c + 4ab|\alpha_3\beta|bc - 4bc|\alpha_3\beta|ab + 2(ba+ac)|\alpha_3\gamma|ac \\
&\quad + 2ba|\alpha_3\gamma|ba + ab|\alpha_2\beta_2|ac + bc|\alpha_2\beta_2|ba - 2ba|\alpha_2\beta_2|(ab+bc) - 4ab|\alpha_4|ba - 2bc|\alpha_4|ba \\
&\quad + (ba+ac)|\alpha_4|bc + 4ab|\alpha_4|ac - 2(ba+ac)|\alpha_4|ab + 8bc|\beta_4|ac - 10ba|\beta_4|ab - 2ac|\beta_4|ab \\
&\quad + 6ab|\beta_4|ac + 4(ab+bc)|\beta_4|ba - 2ab|\gamma_4|ba - 4bc|\gamma_4|ba + 6ab|\gamma_4|ac - 5ba|\gamma_4|ab \\
&\quad + 4ba|\gamma_4|bc - 6ac|\gamma_4|ab + a|\alpha_4|bac + 4b|\beta_4|abc - c|\gamma_4|aba - \omega_1 c|\epsilon^!|c.
\end{aligned}$$

Fact 5.2.7. *Let* $\rho = \rho^5 : A \to A$ *be the derivation of* A *defined by* $\rho^5(b) = abc$ *and* $\rho^5(x) = 0$ *for* $x \in \mathscr{B} \setminus \{b\}$. *Then the elements* $q_{\omega_i u} = q^5_{\omega_i u} \in P^b_n$ *in* (5.2.5) *can be chosen as follows. First,* $q^5_{\alpha_n} = q^5_{\gamma_n} = 0$ *for* $n \in \mathbb{N}$. *Moreover,*

$$\begin{aligned}
q^5_{\beta} &= ab|\gamma|1 + a|\beta|c + 1|\alpha|bc, \\
q^5_{\beta_2} &= ab|\alpha\gamma|1 - a|\beta_2|a - c|\beta_2|c - a|\alpha\gamma|c + 1|\alpha\gamma|bc, \\
q^5_{\alpha\beta} &= ab|\gamma_2|1 + 1|\alpha_2|bc, \\
q^5_{\alpha\gamma} &= ab|\alpha\beta|1 - bc|\alpha\beta|1 - 2ba|\beta_2|1 - a|\alpha_2|c - 2a|\beta_2|c + b|\beta_2|a - c|\beta_2|b + b|\gamma_2|a + b|\alpha\beta|b \\
&\quad - a|\alpha\gamma|b + 1|\beta_2|ac - 1|\alpha_2|ba + 1|\alpha\beta|bc, \\
q^5_{\beta_3} &= (ba+ac)|\alpha_3|1 + ab|\alpha_2\gamma|1 + a|\alpha_3|b + b|\alpha_3|a + 2a|\beta_3|c + c|\beta_3|a + b|\gamma_3|c + c|\gamma_3|b + a|\alpha_2\beta|c \\
&\quad + a|\alpha_2\gamma|a + c|\alpha\beta_2|c - 1|\gamma_3|ba + 1|\alpha\beta_2|bc, \\
q^5_{\alpha_2\beta} &= 2ab|\gamma_3|1 - 2(ba+ac)|\alpha_3|1 - (ab+bc)|\alpha_2\gamma|1 + (ba+ac)|\alpha\beta_2|1 - a|\alpha_3|b - 2b|\alpha_3|a \\
&\quad - 2c|\beta_3|a - b|\gamma_3|c - c|\gamma_3|b - 2a|\alpha_2\gamma|a - c|\alpha\beta_2|c + 2|\alpha_3|bc + 1|\gamma_3|ba - 1|\alpha_2\gamma|ba, \\
q^5_{\alpha_2\gamma} &= ba|\alpha_2\beta|1 - 2ba|\beta_3|1 - ab|\alpha_3|1 + a|\alpha_3|c + b|\alpha_3|b + c|\alpha_3|a + 2a|\beta_3|a + b|\beta_3|c + c|\beta_3|b
\end{aligned}$$

$$
\begin{aligned}
&+ b|\alpha_2\beta|c + b|\alpha_2\gamma|a + b|\alpha\beta_2|b - 2|\alpha_3|ba - 1|\beta_3|ab + 1|\alpha_2\beta|bc, \\
q^5_{\alpha\beta_2} &= ba|\gamma_3|1 + ab|\alpha_2\beta|1 - b|\alpha_3|c - c|\alpha_3|b - b|\beta_3|a + c|\beta_3|c - b|\alpha_2\gamma|b + b|\alpha\beta_2|c + 2|\beta_3|ac \\
&- 1|\alpha_2\gamma|ab + 1|\alpha_2\gamma|bc, \\
q^5_{\omega_1\epsilon^!} &= 4bac|\alpha_4|a + abc|\beta_4|b - aba|\gamma_4|c + ab|\alpha_3\beta|ab + 5ab|\alpha_3\beta|bc - 4bc|\alpha_3\beta|ab \\
&+ 2(ba + ac)|\alpha_3\gamma|ac + 2ba|\alpha_3\gamma|ba - ac|\alpha_3\gamma|ba - ab|\alpha_2\beta_2|ba + 2bc|\alpha_2\beta_2|ba - 2ba|\alpha_2\beta_2|ab \\
&- ba|\alpha_2\beta_2|bc + ac|\alpha_2\beta_2|bc - 11ab|\alpha_4|ba + 2ab|\alpha_4|ac + 6ba|\alpha_4|bc + ac|\alpha_4|ab + 9ac|\alpha_4|bc \\
&+ 3ab|\beta_4|ac + bc|\beta_4|ac - 6ba|\beta_4|ab + ba|\beta_4|bc - 3ac|\beta_4|ab - 5ab|\gamma_4|ba + 4ab|\gamma_4|ac \\
&- 4bc|\gamma_4|ba - 2bc|\gamma_4|ac - 3ba|\gamma_4|ab + 6ba|\gamma_4|bc - 8ac|\gamma_4|ab + 4a|\alpha_4|bac + b|\beta_4|abc \\
&- c|\gamma_4|aba - \omega_1 c|\epsilon^!|c.
\end{aligned}
$$

Fact 5.2.8. *Let $\rho = \rho^6 : A \to A$ be the derivation of A defined by $\rho^6(c) = aba$ and $\rho^6(x) = 0$ for $x \in \mathscr{B} \setminus \{c\}$. Then the elements $q_{\omega_i u} = q^6_{\omega_i u} \in P^b_n$ in (5.2.5) can be chosen as follows. First, $q^6_{\alpha_n} = q^6_{\beta_n} = 0$ for $n \in \mathbb{N}$. Moreover,*

$$
\begin{aligned}
q^6_{\gamma} &= ab|\alpha|1 + a|\beta|a + 1|\alpha|ba, \\
q^6_{\gamma_2} &= ba|\alpha\beta|1 - b|\alpha_2|b + c|\beta_2|c + a|\gamma_2|a + c|\alpha\beta|a + a|\alpha\gamma|c + 1|\alpha\gamma|ab, \\
q^6_{\alpha\beta} &= 2ab|\alpha_2|1 + c|\alpha_2|a + b|\beta_2|c + b|\alpha\beta|a + 1|\beta_2|ab, \\
q^6_{\alpha\gamma} &= ba|\beta_2|1 + a|\alpha_2|c + c|\beta_2|b + a|\alpha\gamma|b + 2|\alpha_2|ba, \\
q^6_{\gamma_3} &= ab|\alpha\beta_2|1 - ba|\beta_3|1 + a|\beta_3|a - a|\gamma_3|b - b|\gamma_3|a - b|\alpha_2\beta|c - c|\alpha_2\beta|b - b|\alpha\beta_2|b - 1|\beta_3|ab \\
&+ 1|\alpha\beta_2|ba, \\
q^6_{\alpha_2\beta} &= ac|\alpha_3|1 + ab|\alpha_2\gamma|1 + 2c|\alpha_3|c + 2a|\beta_3|c + 2c|\beta_3|a + 2a|\alpha_2\gamma|a + b|\alpha_2\gamma|c + c|\alpha_2\gamma|b \\
&+ a|\alpha\beta_2|b + b|\alpha\beta_2|a - 1|\alpha_3|(ab + bc) + 1|\alpha_2\gamma|ba, \\
q^6_{\alpha_2\gamma} &= 3ab|\alpha_3|1 + 2ba|\beta_3|1 - a|\alpha_3|c - c|\alpha_3|a - b|\beta_3|c - c|\beta_3|b + 3|\alpha_3|ba + 2|\beta_3|ab, \\
q^6_{\alpha\beta_2} &= 2bc|\beta_3|1 + 2ba|\alpha_2\gamma|1 + 3b|\alpha_3|c + 3c|\alpha_3|b + 3b|\alpha_2\gamma|b - 2|\beta_3|(ba + ac) + 2|\alpha_2\gamma|ab, \\
q^6_{\omega_1\epsilon^!} &= 2bac|\alpha_2\beta_2|a + aba|\alpha_2\beta_2|c - 5bac|\alpha_4|a - 3abc|\beta_4|b - 2bac|\alpha_3\beta|b + 8ab|\alpha_4|ba - 6ab|\alpha_4|ac \\
&+ 6bc|\alpha_4|ba + 3ba|\alpha_4|bc + 3ac|\alpha_4|bc + 3ab|\beta_4|ac + 3bc|\beta_4|ac + 10ba|\beta_4|ab - 8ba|\beta_4|bc \\
&+ 8ac|\beta_4|ab - 4ab|\gamma_4|ba - 2ab|\alpha_2\beta_2|ba - 5a|\alpha_4|bac - 3b|\beta_4|abc - 2b|\alpha_3\gamma|bac \\
&+ 2a|\alpha_2\beta_2|bac + c|\alpha_2\beta_2|aba - \omega_1 a|\epsilon^!|a.
\end{aligned}
$$

Fact 5.2.9. *Let $\rho = \rho^7 : A \to A$ be the derivation of A defined by $\rho^7(a) = aba - abc$, $\rho^7(ab) = \rho^7(ac) = abac$, $\rho^7(ba) = -abac$ and $\rho^7(x) = 0$ for $x \in \mathscr{B} \setminus \{a,ab,ba,ac\}$. Then the elements $q_{\omega_i u} = q^7_{\omega_i u} \in P^b_n$ in (5.2.5) can be chosen as follows. First, $q^7_{\beta_n} = q^7_{\gamma_n} = 0$ for $n \in \mathbb{N}$. Moreover,*

$$
\begin{aligned}
q^7_{\alpha} &= ab|\alpha|1 - ab|\gamma|1 + a|\beta|a - a|\beta|c + 1|\alpha|ba - 1|\alpha|bc, \\
q^7_{\alpha_2} &= ab|\alpha_2|1 + ac|\alpha_2|1 - a|\alpha\beta|a - 1|\alpha_2|bc + 2|\alpha_2|ba, \\
q^7_{\alpha\beta} &= ba|\beta_2|1 - ba|\gamma_2|1 + (ba + ac)|\alpha\beta|1 + a|\alpha_2|c + c|\beta_2|b + c|\beta_2|c - a|\gamma_2|c + b|\gamma_2|b - c|\gamma_2|b \\
&+ c|\alpha\beta|a - c|\alpha\beta|c + a|\alpha\gamma|b + c|\alpha\gamma|b + 1|\alpha_2|ba - 1|\gamma_2|ba + 1|\alpha\beta|(ba + ac), \\
q^7_{\alpha\gamma} &= ab|\alpha_2|1 - ab|\gamma_2|1 + c|\alpha_2|a - c|\alpha_2|c + b|\beta_2|c + b|\alpha\beta|a - b|\alpha\beta|c + 1|\beta_2|ab + 1|\alpha\beta|ac, \\
q^7_{\alpha_3} &= ab|\alpha_3|1 + ac|\alpha_3|1 - 1|\alpha_3|bc + 2|\alpha_3|ba, \\
q^7_{\alpha_2\beta} &= ab|\alpha_3|1 + 2ba|\beta_3|1 + 2bc|\beta_3|1 - ba|\gamma_3|1 + 2ba|\alpha_2\gamma|1 - ab|\alpha\beta_2|1 - a|\alpha_3|c + 3b|\alpha_3|c \\
&+ c|\alpha_3|b - a|\beta_3|a - a|\beta_3|b + b|\beta_3|a - b|\beta_3|c - 2c|\beta_3|b + c|\beta_3|c - a|\gamma_3|b + a|\gamma_3|c - b|\gamma_3|b \\
&+ 2c|\gamma_3|a + c|\alpha_2\beta|c - a|\alpha_2\gamma|b + a|\alpha_2\gamma|c + b|\alpha_2\gamma|b - a|\alpha\beta_2|c + b|\alpha\beta_2|c - c|\alpha\beta_2|b \\
&+ 2|\alpha_3|ba + 1|\beta_3|ab - 3|\gamma_3|ab - 1|\alpha_2\beta|bc + 1|\alpha_2\beta|ba + 2|\alpha_2\beta|ac, \\
q^7_{\alpha_2\gamma} &= ac|\alpha_3|1 - ba|\gamma_3|1 - (ab + bc)|\alpha_2\beta|1 + ba|\alpha\beta_2|1 - b|\alpha_3|a + 3b|\alpha_3|c + 2c|\alpha_3|b + c|\alpha_3|c \\
&+ a|\beta_3|c + c|\beta_3|a - a|\gamma_3|a + 2a|\gamma_3|c - 2b|\gamma_3|b + 2c|\gamma_3|a + b|\alpha_2\beta|b - c|\alpha_2\beta|a + 2c|\alpha_2\beta|c \\
&+ a|\alpha_2\gamma|a + b|\alpha_2\gamma|b + b|\alpha_2\gamma|c + c|\alpha_2\gamma|a + b|\alpha\beta_2|a - 2|\gamma_3|ab + 2|\alpha_2\beta|ac, \\
q^7_{\alpha\beta_2} &= ab|\alpha_3|1 + ba|\beta_3|1 - 2ab|\gamma_3|1 - a|\alpha_3|c - c|\alpha_3|a + 2c|\alpha_3|c - a|\gamma_3|b + b|\gamma_3|c + c|\alpha_2\gamma|b
\end{aligned}
$$

$$
\begin{aligned}
&+ 1|\alpha_3|ba + 1|\beta_3|ab - 1|\gamma_3|ba,\\
q^7_{\omega_1\epsilon^!} = {}& 2abc|\alpha_2\beta_2|a - 3abc|\alpha_2\beta_2|c - aba|\alpha_2\beta_2|a + 2aba|\alpha_2\beta_2|b - 5bac|\alpha_4|b - 7bac|\alpha_4|c\\
&+ 5abc|\beta_4|c + 3aba|\gamma_4|a - 3aba|\gamma_4|b + 8ab|\alpha_4|ab + 3ab|\alpha_4|bc + bc|\alpha_4|bc - 2ba|\alpha_4|ba\\
&+ 4ac|\alpha_4|ba + 6ac|\alpha_4|ac + 9ab|\beta_4|ab - 4bc|\beta_4|ab - 5bc|\beta_4|bc - 4ba|\beta_4|ba - 7ba|\beta_4|ac\\
&+ 14ab|\gamma_4|ab + 7bc|\gamma_4|ab - ba|\gamma_4|ba - ba|\gamma_4|ac - 8ac|\gamma_4|ba + 7ab|\alpha_3\beta|ac - 3bc|\alpha_3\beta|ba\\
&+ bc|\alpha_3\beta|ac + 2ba|\alpha_3\beta|ab + 3ba|\alpha_3\beta|bc + ba|\alpha_3\beta|ba + 3ba|\alpha_3\beta|ac + 3ac|\alpha_3\beta|bc\\
&- 6ac|\alpha_3\beta|ba - 3ac|\alpha_3\beta|ac - ab|\alpha_3\gamma|ab + 5ab|\alpha_3\gamma|bc - 9ab|\alpha_3\gamma|ba - 5ab|\alpha_3\gamma|ac\\
&+ 3bc|\alpha_3\gamma|ab + 6bc|\alpha_3\gamma|bc - 2bc|\alpha_3\gamma|ba - 2ba|\alpha_3\gamma|ab - 2ba|\alpha_3\gamma|bc + 6ac|\alpha_3\gamma|ab\\
&+ 7ab|\alpha_2\beta_2|ab + 2ab|\alpha_2\beta_2|bc - ba|\alpha_2\beta_2|ba + 2ba|\alpha_2\beta_2|ac + 4ac|\alpha_2\beta_2|ac - 2b|\alpha_4|bac\\
&+ c|\alpha_4|bac + a|\beta_4|abc - 2c|\beta_4|abc + 4a|\gamma_4|aba + 2b|\gamma_4|aba + 2a|\alpha_2\beta_2|abc - 2c|\alpha_2\beta_2|abc\\
&- a|\alpha_2\beta_2|aba - 2b|\alpha_2\beta_2|bac - c|\alpha_2\beta_2|bac + \omega_1 3|\epsilon^!|(ba + ac - bc).
\end{aligned}
$$

We will now apply the previous results to compute the Gerstenhaber brackets of X_i for $i \in [\![4,7]\!]$ with all the other generators of the Hochschild cohomology of A.

Proposition 5.2.10. *The Gerstenhaber bracket $[X_i,X_j] \in \mathrm{HH}^\bullet(A)$ for $i \in [\![4,7]\!]$ and $j \in [\![1,14]\!]$ is given by*

$$
[X_i,X_j] = \begin{cases}
0, & \text{if } (i,j) \in \big([\![4,7]\!] \times [\![1,7]\!]\big) \cup \big([\![4,6]\!] \times [\![9,11]\!]\big),\\
2X_i, & \text{if } i \in [\![4,7]\!] \text{ and } j = 8,\\
4X_1X_9, & \text{if } i = 7 \text{ and } j = 9,\\
X_1X_{10}, & \text{if } i = 7 \text{ and } j = 10,\\
-X_1(X_9 + X_{10}), & \text{if } i = 7 \text{ and } j = 11,\\
2X_1X_{i+5}, & \text{if } i \in [\![4,5]\!] \text{ and } j = 12,\\
2X_1(X_9 + X_{10}), & \text{if } i = 6 \text{ and } j = 12,\\
X_1X_9, & \text{if } i = 7 \text{ and } j = 12,\\
\tau_i 8X_4X_{10}, & \text{if } i \in [\![4,6]\!] \text{ and } j = 13,\\
4X_1X_{13} - 4X_2X_{13} - 8X_4X_{12}, & \text{if } i = 7 \text{ and } j = 13,\\
\tau_i\big((1/3)X_{i+5}^2 - (4/3)X_9X_{10}\big), & \text{if } i \in [\![4,6]\!] \text{ and } j = 14,\\
X_9X_{12}, & \text{if } i = 7 \text{ and } j = 14,
\end{cases}
$$

where $\tau_i = 1$ if $i \in [\![4,5]\!]$ and $\tau_6 = -1$.

Proof. Given $i \in [\![4,7]\!]$, let ρ^i be the derivation of Fact 5.2.6, Fact 5.2.7, Fact 5.2.8, and Fact 5.2.9, respectively. Note that $G(\rho^i)i_1 = -Y_i$. By Theorem 1.4.7, $[-Y_i, Y_j]$ is precisely the cohomology class of $\rho^i Y_j - Y_j \rho^i_n$ for $i \in [\![4,7]\!]$ and $j \in [\![1,13]\!]$, where n is the cohomological degree of Y_j and ρ^i_n is obtained from (5.2.5) together with Fact 5.2.6 for $i = 4$, Fact 5.2.7 for $i = 5$, Fact 5.2.8 for $i = 6$, and Fact 5.2.9 for $i = 7$. It is explicitly given by

$$
[-X_4,X_j] = \begin{cases}
0, & \text{if } j \in [\![1,7]\!] \cup \{9\},\\
-2X_4, & \text{if } j = 8,\\
\alpha\beta|(ab + bc) - \alpha\gamma|ac, & \text{if } j = 10,\\
-\alpha\beta|ab - \alpha\gamma|ba, & \text{if } j = 11,\\
\alpha_2|(bc - ba - ac), & \text{if } j = 12,\\
3\alpha_2\gamma|aba - 5\alpha\beta_2|bac, & \text{if } j = 13,
\end{cases}
$$

and

$$
[-X_5,X_j] = \begin{cases}
0, & \text{if } j \in [\![1,7]\!] \cup \{10\},\\
-2X_5, & \text{if } j = 8,\\
-\alpha\beta|bc + \alpha\gamma|(ba + ac), & \text{if } j = 9,\\
-\alpha\beta|ab - \alpha\gamma|ba, & \text{if } j = 11,\\
-\beta_2|(ab + bc - ac), & \text{if } j = 12,\\
-5\alpha_2\beta|abc + 3\alpha_2\gamma|aba, & \text{if } j = 13,
\end{cases}
$$

as well as

$$[-X_6,X_j] = \begin{cases} 0, & \text{if } j \in [\![1,7]\!] \cup \{11\}, \\ -2X_6, & \text{if } j = 8, \\ \alpha\beta|(bc-ab) - \alpha\gamma|(2ba+ac), & \text{if } j = 9, \\ -\alpha\beta|(ab+bc) + \alpha\gamma|ac, & \text{if } j = 10, \\ \gamma_2|(bc-ba-ac) - \alpha\beta|ba - \alpha\gamma|ab, & \text{if } j = 12, \\ -10\alpha_2\gamma|aba - 2\alpha\beta_2|bac, & \text{if } j = 13, \end{cases}$$

together with

$$[-X_7,X_{13}] = \alpha_3|(abc-2aba) + \alpha_2\beta|(2bac-6aba) - \alpha_2\gamma|(abc+4bac) + 5\alpha\beta_2|(abc-aba),$$

and

$$[-X_7,X_j] = \begin{cases} 0, & \text{if } j \in [\![1,7]\!], \\ -2X_7, & \text{if } j = 8, \\ \alpha_2|(bc-ab-ac-2ba) - \alpha\beta|(ba+ac) + \alpha\gamma|bc, & \text{if } j = 9, \\ \alpha\beta|ac - \alpha\gamma|(ab+bc), & \text{if } j = 10, \\ \alpha\beta|ba + \alpha\gamma|ab, & \text{if } j = 11, \\ (\alpha\beta+\alpha\gamma)|(bc-ba-ac), & \text{if } j = 12. \end{cases}$$

Next, we will compute $\varphi^i = [-Y_i, Y_{14}] = \rho^i Y_{14} - Y_{14}\rho_4^i$ for $i \in [\![4,7]\!]$. Using Fact 5.2.6, it is easy to see that $\varphi^4(1|\beta_4|1) = \varphi^4(1|\gamma_4|1) = \varphi^4(\omega_1 1|\epsilon^!|1) = 0$, whereas Fact 5.2.7 gives us immediately the identities $\varphi^5(1|\alpha_4|1) = \varphi^5(1|\gamma_4|1) = \varphi^5(\omega_1 1|\epsilon^!|1) = 0$, Fact 5.2.8 tells us that $\varphi^6(1|\alpha_4|1) = \varphi^6(1|\beta_4|1) = \varphi^6(\omega_1 1|\epsilon^!|1) = 0$, and Fact 5.2.9 yields that $\varphi^7(1|\beta_4|1) = \varphi^7(1|\gamma_4|1) = 0$ and $\varphi^7(\omega_1 1|\epsilon^!|1) = 3(bc-ba-ac)$. For $i \in \{4,7\}$ and $u \in \mathscr{B}_4^{!*} \setminus \{\beta_4,\gamma_4\}$ (resp., $i = 5$ and $u \in \mathscr{B}_4^{!*} \setminus \{\alpha_4,\gamma_4\}$, $i = 6$ and $u \in \mathscr{B}_4^{!*} \setminus \{\alpha_4,\beta_4\}$), we have that

$$\varphi^i(1|u|1) = (\rho^i Y_{14} - Y_{14}\rho_4^i)(1|u|1) = -Y_{14}(q_u^i) = -\lambda_i^u,$$

where $\lambda_i^u \in \Bbbk$ is the coefficient of $\omega_1 1|\epsilon^!|1$ in q_u^i. It is easy to check that

$$\begin{aligned} \rho_3^4\delta_4^b(1|\alpha_4|1) &= bac|\alpha_3|1 + abc|\beta_3|1 + aba|\alpha_2\gamma|1 + v_{\alpha_4}^4, \\ \rho_3^4\delta_4^b(1|\alpha_3\beta|1) &= -2aba|\alpha_3|1 + 3abc|\gamma_3|1 + bac|\alpha_2\beta|1 + abc|\alpha_2\gamma|1 + 2aba|\alpha\beta_2|1 + v_{\alpha_3\beta}^4, \\ \rho_3^4\delta_4^b(1|\alpha_3\gamma|1) &= -2aba|\beta_3|1 + 2bac|\gamma_3|1 + 2aba|\alpha_2\beta|1 + bac|\alpha_2\gamma|1 + abc|\alpha\beta_2|1 + v_{\alpha_3\gamma}^4, \\ \rho_3^4\delta_4^b(1|\alpha_2\beta_2|1) &= -2bac|\alpha_3|1 - 2abc|\beta_3|1 + 4aba|\gamma_3|1 + 2abc|\alpha_2\beta|1 + 2bac|\alpha\beta_2|1 + v_{\alpha_2\beta_2}^4, \end{aligned}$$

and

$$\begin{aligned} \rho_3^5\delta_4^b(1|\beta_4|1) &= bac|\alpha_3|1 + abc|\beta_3|1 + aba|\alpha_2\gamma|1 + v_{\beta_4}^5, \\ \rho_3^5\delta_4^b(1|\alpha_3\beta|1) &= -2aba|\alpha_3|1 + 2abc|\gamma_3|1 + bac|\alpha_2\beta|1 + aba|\alpha\beta_2|1 + v_{\alpha_3\beta}^5, \\ \rho_3^5\delta_4^b(1|\alpha_3\gamma|1) &= abc|\alpha_3|1 - 2aba|\beta_3|1 + 2bac|\gamma_3|1 + 2aba|\alpha_2\beta|1 + bac|\alpha_2\gamma|1 + abc|\alpha\beta_2|1 + v_{\alpha_3\gamma}^5, \\ \rho_3^5\delta_4^b(1|\alpha_2\beta_2|1) &= -3bac|\alpha_3|1 - 2abc|\beta_3|1 + 3aba|\gamma_3|1 + 2abc|\alpha_2\beta|1 - aba|\alpha_2\gamma|1 + bac|\alpha\beta_2|1 \\ &\quad + v_{\alpha_2\beta_2}^5, \end{aligned}$$

as well as

$$\begin{aligned} \rho_3^6\delta_4^b(1|\gamma_4|1) &= -abc|\beta_3|1 + aba|\gamma_3|1 + bac|\alpha\beta_2|1 + v_{\gamma_4}^6, \\ \rho_3^6\delta_4^b(1|\alpha_3\beta|1) &= 4aba|\alpha_3|1 - 2bac|\beta_3|1 + 2abc|\alpha_2\gamma|1 + 2aba|\alpha\beta_2|1 + v_{\alpha_3\beta}^6, \\ \rho_3^6\delta_4^b(1|\alpha_3\gamma|1) &= -abc|\alpha_3|1 + 2aba|\beta_3|1 + aba|\alpha_2\beta|1 + bac|\alpha_2\gamma|1 + v_{\alpha_3\gamma}^6, \\ \rho_3^6\delta_4^b(1|\alpha_2\beta_2|1) &= 4bac|\alpha_3|1 + 4abc|\beta_3|1 + 4aba|\alpha_2\gamma|1 + v_{\alpha_2\beta_2}^6, \end{aligned}$$

together with

$$\rho_3^7\delta_4^b(1|\alpha_4|1) = (aba - abc)|\alpha_3|1 + v_{\alpha_4}^7,$$

$$\begin{aligned}\rho_3^7\delta_4^b(1|\alpha_3\beta|1) &= -abc|\alpha_3|1 + 3bac|\alpha_3|1 + 3aba|\beta_3|1 + 2abc|\beta_3|1 - aba|\gamma_3|1 - 2bac|\gamma_3|1\\ &\quad - abc|\alpha_2\beta|1 + 2aba|\alpha_2\gamma|1 + v^7_{\alpha_3\beta},\\ \rho_3^7\delta_4^b(1|\alpha_3\gamma|1) &= 2aba|\alpha_3|1 + 2bac|\alpha_3|1 + 2abc|\beta_3|1 - 2bac|\beta_3|1 - 2aba|\gamma_3|1 - 2abc|\gamma_3|1\\ &\quad - abc|\alpha_2\beta|1 + aba|\alpha_2\gamma|1 + abc|\alpha_2\gamma|1 + aba|\alpha\beta_2|1 - bac|\alpha\beta_2|1 + v^7_{\alpha_3\gamma},\\ \rho_3^7\delta_4^b(1|\alpha_2\beta_2|1) &= aba|\alpha_3|1 - abc|\alpha_3|1 + aba|\beta_3|1 - abc|\gamma_3|1 + v^7_{\alpha_2\beta_2},\end{aligned}$$

where $v^i_u \in \oplus_{j\in[\![0,2]\!]}(A_j \otimes (A^!_{-3})^* \otimes A_{3-j})$ if $i \in \{4, 7\}$ and $u \in \mathscr{B}_4^{!*} \setminus \{\beta_4,\gamma_4\}$, or if $i = 5$ and $u \in \mathscr{B}_4^{!*} \setminus \{\alpha_4,\gamma_4\}$, or if $i = 6$ and $u \in \mathscr{B}_4^{!*} \setminus \{\alpha_4,\beta_4\}$.

Since, q^i_u is of the form $q^i_u = B^i_u + \lambda^u_i \omega_1 1|\epsilon^!|1$ by degree reasons, where $B^i_u \in K^b_4$, and we have by definition that $\delta_4^b(q^i_u) = \rho_3^i\delta_4^b(1|u|1)$, we see that

$$d_4^b(B^i_u) = \delta_4^b(q^i_u) - \lambda^u_i f_0^b(1|\epsilon^!|1) = \rho_3^i\delta_4^b(1|u|1) - \lambda^u_i f_0^b(1|\epsilon^!|1) \tag{5.2.6}$$

for $i \in [\![4, 7]\!]$. Using the explicit expression of the differential d_4^b given in Fact 4.1.9, it is clear that the coefficients of $aba|\gamma_3|1$ and $aba|\alpha_2\gamma|1$ in $d_4^b(B)$ coincide for all $B \in K_4^b$. Comparing the coefficients of $aba|\gamma_3|1$ and $aba|\alpha_2\gamma|1$ in both sides of the equation (5.2.6), together with the expression of $f_0^b(1|\epsilon^!|1)$ given in (4.1.3), we get

$$\lambda_4^{\alpha_4} = \lambda_5^{\beta_4} = -\lambda_6^{\gamma_4} = 1/3,\quad \lambda_i^{\alpha_3\beta} = \lambda_i^{\alpha_3\gamma} = 0,\quad \lambda_i^{\alpha_2\beta_2} = -\tau_i 4/3,$$
$$\lambda_7^{\alpha_4} = \lambda_7^{\alpha_2\beta_2} = 0,\quad \text{and}\quad \lambda_7^{\alpha_3\beta} = \lambda_7^{\alpha_3\gamma} = 1,$$

for $i \in [\![4, 6]\!]$, where $\tau_i = 1$ if $i \in [\![4, 5]\!]$ and $\tau_6 = -1$. Hence, we obtain that

$$\begin{aligned}&\varphi^4(1|\alpha_4|1) = -1/3,\ \ \varphi^4(1|\alpha_3\beta|1) = \varphi^4(1|\alpha_3\gamma|1) = 0 \ \text{ and }\ \varphi^4(1|\alpha_2\beta_2|1) = 4/3,\\ &\varphi^5(1|\beta_4|1) = -1/3,\ \ \varphi^5(1|\alpha_3\beta|1) = \varphi^5(1|\alpha_3\gamma|1) = 0 \ \text{ and }\ \varphi^5(1|\alpha_2\beta_2|1) = 4/3,\\ &\varphi^6(1|\gamma_4|1) = 1/3,\quad \varphi^6(1|\alpha_3\beta|1) = \varphi^6(1|\alpha_3\gamma|1) = 0 \ \text{ and }\ \varphi^6(1|\alpha_2\beta_2|1) = -4/3,\\ &\varphi^7(1|\alpha_4|1) = \varphi^7(1|\alpha_2\beta_2|1) = 0 \quad \text{and} \quad \varphi^7(1|\alpha_3\beta|1) = \varphi^7(1|\alpha_3\gamma|1) = -1.\end{aligned}$$

In consequence, we get

$$\begin{aligned}&[-X_4, X_{14}] = (4/3)\alpha_2\beta_2|1 - (1/3)\alpha_4|1,\\ &[-X_5, X_{14}] = (4/3)\alpha_2\beta_2|1 - (1/3)\beta_4|1,\\ &[-X_6, X_{14}] = (1/3)\gamma_4|1 - (4/3)\alpha_2\beta_2|1,\\ &[-X_7, X_{14}] = -(\alpha_3\beta + \alpha_3\gamma)|1 + 3\omega_1^*\epsilon^!|(bc - ba - ac).\end{aligned}$$

Using the coboundaries $g^2_{j,2} \in \tilde{\mathfrak{B}}^2_2$ for $j \in [\![4,6]\!]$ and $e^3_{k,3} \in \tilde{\mathfrak{B}}^3_3$ for $k \in [\![7,8]\!]$ given in Subsubsection 4.2.2.3, (5.2.4) as well as the identities

$$\alpha_2\beta|abc = X_4X_{10},\quad \alpha_4|1 = X_9^2,\quad \text{and}\quad \alpha_2\beta_2|1 = X_9X_{10}, \tag{5.2.7}$$

which follow from Fact 5.1.3 and (5.1.2), we can rewrite several brackets as

$$\begin{aligned}&[-X_4, X_{10}] = \alpha\beta|(ab + bc) - \alpha\gamma|ac - g^2_{5,2} = 0,\\ &[-X_4,X_{11}] = -\alpha\beta|ab - \alpha\gamma|ba + g^2_{4,2} = 0,\\ &[-X_4,X_{12}] = \alpha_2|(bc - ba - ac) - g^2_{6,2} = -2\alpha_2|(ab + ba) = -2X_1X_9,\\ &[-X_4,X_{13}] = 3\alpha_2\gamma|aba - 5\alpha\beta_2|bac - 5e^3_{7,3} - 3e^3_{8,3} = -8\alpha_2\beta|abc = -8X_4X_{10},\\ &[-X_4,X_{14}] = (4/3)\alpha_2\beta_2|1 - (1/3)\alpha_4|1 = (4/3)X_9X_{10} - (1/3)X_9^2.\end{aligned}$$

Analogously, using the coboundaries $g^2_{j,2} \in \tilde{\mathfrak{B}}^2_2$ for $j \in \{4,5,7\}$ and $e^3_{8,3} \in \tilde{\mathfrak{B}}^3_3$ given in Subsubsection 4.2.2.3, (5.2.4), (5.2.7) and the identity $\beta_4|1 = X_{10}^2$ given in Fact 5.1.3, we get that

$$\begin{aligned}&[-X_5, X_9] = -\alpha\beta|bc + \alpha\gamma|(ba + ac) - g^2_{4,2} + g^2_{5,2} = 0,\\ &[-X_5,X_{11}] = -\alpha\beta|ab - \alpha\gamma|ba + g^2_{4,2} = 0,\\ &[-X_5,X_{12}] = -\beta_2|(ab + bc - ac) - g^2_{7,2} = -2\beta_2|(ab + ba) = -2X_1X_{10},\end{aligned}$$

$$[-X_5,X_{13}] = -5\alpha_2\beta|abc + 3\alpha_2\gamma|aba - 3e^3_{8,3} = -8\alpha_2\beta|abc = -8X_4X_{10},$$
$$[-X_5,X_{14}] = (4/3)\alpha_2\beta_2|1 - (1/3)\beta_4|1 = (4/3)X_9X_{10} - (1/3)X^2_{10}.$$

Moreover, using the coboundaries $g^2_{j,2} \in \tilde{\mathfrak{B}}^2_2$ for $j \in [\![1,5]\!]$ and $e^3_{k,3} \in \tilde{\mathfrak{B}}^3_3$ for $k \in [\![7,8]\!]$ given in Subsubsection 4.2.2.3, (5.2.4), (5.2.7) and the identity $\gamma_4|1 = X^2_{11}$ given in Fact 5.1.3, we obtain

$$[-X_6, X_9] = \alpha\beta|(bc - ab) - \alpha\gamma|(2ba + ac) + 2g^2_{4,2} - g^2_{5,2} = 0,$$
$$[-X_6,X_{10}] = -\alpha\beta|(ab + bc) + \alpha\gamma|ac + g^2_{5,2} = 0,$$
$$[-X_6,X_{12}] = \gamma_2|(bc - ba - ac) - \alpha\beta|ba - \alpha\gamma|ab - 2g^2_{1,2} - 2g^2_{2,2} - g^2_{3,2} = -2(\alpha_2 + \beta_2)|(ab + ba)$$
$$= -2X_1(X_9 + X_{10}),$$
$$[-X_6,X_{13}] = -10\alpha_2\gamma|aba - 2\alpha\beta_2|bac - 2e^3_{7,3} + 10e^3_{8,3} = 8\alpha_2\beta|abc = 8X_4X_{10},$$
$$[-X_6,X_{14}] = (1/3)\gamma_4|1 - (4/3)\alpha_2\beta_2|1 = (1/3)X^2_{11} - (4/3)X_9X_{10}.$$

Finally, using the coboundaries $g^2_{j,2} \in \tilde{\mathfrak{B}}^2_2$ for $j \in [\![1,6]\!] \setminus \{3\}$ and $e^3_{k,3} \in \tilde{\mathfrak{B}}^3_3$ for $k \in [\![1,4]\!] \cup [\![9,10]\!]$ given in Subsubsection 4.2.2.3, (5.2.4) and

$$\alpha_3|(aba - abc) = X_7X_9, \quad \alpha_3|aba + \beta_3|bac = X_7(X_9 + X_{10}) - 2X_6X_{12},$$
$$(\alpha_3 + \beta_3)|aba = X_6X_{12}, \quad (\alpha_3\beta + \alpha_3\gamma)|1 + 3\omega_1\epsilon^!|(ba - bc + ac) = X_9X_{12},$$

given in Fact 5.1.3, or in (5.1.2), together with the second element in the fifth and the eighth line, the first element in the ninth line of (5.1.5), we have that

$$[-X_7, X_9] = \alpha_2|(bc - ab - ac - 2ba) - \alpha\beta|(ba + ac) + \alpha\gamma|bc - g^2_{1,2} - g^2_{6,2} = -4\alpha_2|(ab + ba)$$
$$= -4X_1X_9,$$
$$[-X_7,X_{10}] = \alpha\beta|ac - \alpha\gamma|(ab + bc) - g^2_{2,2} = -\beta_2|(ab + ba) = -X_1X_{10},$$
$$[-X_7,X_{11}] = \alpha\beta|ba + \alpha\gamma|ab + g^2_{1,2} + g^2_{2,2} = (\alpha_2 + \beta_2)|(ab + ba) = X_1(X_9 + X_{10}),$$
$$[-X_7,X_{12}] = (\alpha\beta + \alpha\gamma)|(bc - ba - ac) - g^2_{1,2} + g^2_{4,2} - g^2_{5,2} = -\alpha_2|(ab + ba) = -X_1X_9,$$
$$[-X_7,X_{13}] = \alpha_3|(abc - 2aba) + \alpha_2\beta|(2bac - 6aba) - \alpha_2\gamma|(abc + 4bac) + 5\alpha\beta_2|(abc - aba)$$
$$- (1/3)(23e^3_{1,3} + 11e^3_{2,3} - 32e^3_{3,3} - 16e^3_{4,3} - 5e^3_{9,3} + 6e^3_{10,3})$$
$$= (8/3)\alpha_3|(aba - abc) - (32/3)(\alpha_3 + \beta_3)|aba - (16/3)(\alpha_3|aba + \beta_3|bac)$$
$$= -(8/3)X_7(X_9 + 2X_{10}) = -4X_1X_{13} + 4X_2X_{13} + 8X_4X_{12},$$
$$[-X_7,X_{14}] = -(\alpha_3\beta + \alpha_3\gamma)|1 + 3\omega^*_1\epsilon^!|(bc - ba - ac) = -X_9X_{12}.$$

The proposition is thus proved. □

Remark 5.2.11. *Note that vanishing of $[X_i,X_j]$ for $i \in [\![4,7]\!]$ and $j \in [\![3,7]\!]$ in Proposition 5.2.10 also follows from a simple degree argument based on Corollary 1.3.8 and the Hilbert series of the Hochschild cohomology given in Corollary 4.2.20.*

5.2.3 Gerstenhaber brackets

We will finally compute the remaining Gerstenhaber brackets. We start with the following result, which is a sort of descending argument.

Lemma 5.2.12. *Let $H = \oplus_{n\in\mathbb{N}_0} H^n$ be a Gerstenhaber algebra with bracket $[\,,]$. Let $x \in H^{n+1}$, $y \in H^n$, $a_x \in H^0$, $a_y \in H^1$ and $z \in H^m$ satisfy that $a_x x = a_y y$, and there is a vector subspace $M \subseteq H^{n+m-1}$ such that $[y,z] \in M$ and the map $\mu_{a_y} : M \to H^{n+m}$ sending $v \in M$ to $a_y v$ is injective. Then, $[y,z]$ is the unique element $v \in M$ satisfying that $a_y v$ coincides with*

$$(-1)^{m-1}\big(a_x[x,z] + [a_x,z]x - [a_y,z]y\big). \tag{5.2.8}$$

Proof. By (1.3.4) we get that

$$[a_x x,z] = [a_x,z]x + a_x[x,z] \text{ and } [a_y y,z] = [a_y,z]y + (-1)^{m-1}a_y[y,z].$$

These identities together with $a_x x = a_y y$ imply

$$a_y[y,z] = (-1)^{m-1}\big(a_x[x,z] + [a_x,z]x - [a_y,z]y\big).$$

Hence, the right member is in the image of the injective map μ_{a_y}, and the result follows. □

Remark 5.2.13. *We will apply the previous lemma to the case when H is the Hochschild cohomology of a graded algebra, so H is endowed with an extra grading, called internal (see Corollary 1.3.8), the elements x,y,z,a_x,a_y are homogeneous for both gradings and $M \subseteq H^{n+m-1}$ is the subspace of internal degree equal to the sum of those of y and z. In this case, the methods given in Subsections 1.4.1 and 1.4.2 allow to compute the last two brackets of (5.2.8), whereas the first one will usually vanish by degree reasons.*

Proposition 5.2.14. *Let $A = \mathrm{FK}(3)$ be the Fomin-Kirillov algebra on 3 generators. Then, we have the Gerstenhaber brackets $[X_i,X_j] = 0$ for $i,j \in [\![9,14]\!] \setminus \{13\}$ and*

$$[X_{13},X_j] = \begin{cases} 2X_j^2, & \text{if } j \in [\![9,11]\!], \\ -6X_1X_{14} + 6X_2X_{14} + 2X_9X_{12}, & \text{if } j = 12, \\ 0, & \text{if } j = 13, \\ 4(X_9 + X_{10} + X_{11})X_{14}, & \text{if } j = 14. \end{cases}$$

Proof. Recall that, by Corollary 1.3.8, the Gerstenhaber bracket satisfies that $[\,,] : H^{n_1}_{m_1} \times H^{n_2}_{m_2} \to H^{n_1+n_2-1}_{m_1+m_2-1}$, where $H^{n_i}_{m_i}$ has internal degree $m_i - n_i$ for $i = 1,2$. Using this degree argument together with the Hilbert series of the Hochschild cohomology computed in Corollary 4.2.20, we easily see that $[X_i,X_j] = 0$ for $i,j \in [\![9,14]\!] \setminus \{13\}$. Moreover, $[X_{13},X_{13}] = 0$ by (1.3.3).

It remains to compute $[X_{13},X_j]$ for all $j \in [\![9,14]\!] \setminus \{13\}$. Note first the identities

$$\begin{aligned} [X_8,X_9]X_{13} - 6[X_3,X_9]X_{14} &= 2X_9X_{13} + 12X_4X_{14} = 2X_8X_9^2, \\ [X_8,X_{10}]X_{13} - 6[X_3,X_{10}]X_{14} &= 2X_{10}X_{13} + 12X_5X_{14} = 2X_8X_{10}^2, \\ [X_8,X_{11}]X_{13} - 6[X_3,X_{11}]X_{14} &= 2X_{11}X_{13} - 12X_6X_{14} = 2X_8X_{11}^2, \\ [X_8,X_{12}]X_{13} - 6[X_3,X_{12}]X_{14} &= 2X_{12}X_{13} - 12X_7X_{14} + 6X_1X_8X_{14} - 6X_2X_8X_{14} \\ &= 2X_8X_{11}X_{12} + 6X_1X_8X_{14} = 2X_8X_{12}^2 - 4X_8X_9X_{10} \\ &= 2X_8X_9X_{12} - 6X_1X_8X_{14} + 6X_2X_8X_{14}, \\ [X_8,X_{14}]X_{13} - 6[X_3,X_{14}]X_{14} &= 4X_8(X_9 + X_{10} + X_{11})X_{14}, \end{aligned} \tag{5.2.9}$$

where the first equality of the first fourth lines as well as that of the last line follows from Propositions 5.2.4 and 5.2.5, and we used the first element of the seventh and the eighth line of (5.1.5), as well as its last four elements, for the remaining equalities. The penultimate element of the ninth line of (5.1.5), also tells us that $6X_3X_{14} = X_8X_{13} \in \mathrm{HH}^\bullet(A)$.

Notice now that, by degree reasons, $[X_{13},X_j] \in H^4_0$ for $j \in [\![9,12]\!]$ and H^4_0 is precisely the subspace of $\mathrm{HH}^4(A)$ spanned by the elements X_9^2, X_{10}^2, X_{11}^2, $X_9X_{12} - 3X_1X_{14} + 3X_2X_{14}$, X_9X_{10}, X_1X_{14} and X_2X_{14}. On the other hand, $[X_{13},X_{14}] \in H^6_{-2} = \omega_1^* H^2_0$, by degree reasons, and $\omega_1^* H^2_0$ is the subspace of $\mathrm{HH}^4(A)$ spanned by X_9X_{14}, $X_{10}X_{14}$, $X_{11}X_{14}$ and $X_{12}X_{14}$. Let us denote by ${}^jM \subseteq \mathrm{HH}^4(A)$ the subspace given by H^4_0 if $j \in [\![9,12]\!]$ and by H^6_{-2} if $j = 14$. Since the elements $X_8X_9^2$, $X_8X_{10}^2$, $X_8X_{11}^2$, $X_8X_9X_{12} - 3X_1X_8X_{14} + 3X_2X_8X_{14}$, $X_8X_9X_{10}$, $X_1X_8X_{14}$, and $X_2X_8X_{14}$ are linearly independent, by the second equalities of the first four lines of (5.2.9) together with (5.1.6) and (5.1.7), the map ${}^jM \to \mathrm{HH}^5(A)$ given by left multiplication by X_8 is injective for $j \in [\![9,12]\!]$. Similarly, the elements $X_8X_9X_{14}$, $X_8X_{10}X_{14}$, $X_8X_{11}X_{14}$ and $X_8X_{12}X_{14}$ are linearly independent, by (5.1.7), so the map ${}^{14}M \to \mathrm{HH}^7(A)$ given by left multiplication by X_8 is also injective.

Finally, applying Lemma 5.2.12 to $x = X_{14}$, $y = X_{13}$, $z = X_j$, $a_x = 6X_3$, $a_y = X_8$ and $M = {}^jM$ for $j \in [\![9,14]\!] \setminus \{13\}$, together with the fact remarked at the beginning of the proof that $[X_{14},X_j] = 0$ and (5.2.9), the result follows. □

We can summarize the calculations of the Gerstenhaber brackets on $\mathrm{HH}^\bullet(A)$ done in Propositions 5.2.4, 5.2.5, 5.2.10 and 5.2.14 in the following table, where the brackets strictly below the diagonal are not displayed since they can be obtained using Lemma 1.3.6.

$\rho \backslash \phi$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}
X_1	0	0	0	0	0	0	0	$2X_1$	0	0	0	0	$4X_1(X_9+X_{10})$	0
X_2		0	0	0	0	0	0	$2X_2$	0	0	0	0	$4X_1X_{10}$	0
X_3			0	0	0	0	0	$4X_3$	$-2X_4$	$-2X_5$	$2X_6$	$2X_7 - X_1X_8 + X_2X_8$	$4X_3(X_9+X_{10}+X_{11})$	$X_{13} - (2/3)X_8(X_9+X_{10}+X_{11})$
X_4				0	0	0	0	$2X_4$	0	0	0	$2X_1X_9$	$8X_4X_{10}$	$(1/3)X_9^2 - (4/3)X_9X_{10}$
X_5					0	0	0	$2X_5$	0	0	0	$2X_1X_{10}$	$8X_4X_{10}$	$(1/3)X_{10}^2 - (4/3)X_9X_{10}$
X_6						0	0	$2X_6$	0	0	0	$2X_1(X_9+X_{10})$	$-8X_4X_{10}$	$(4/3)X_9X_{10} - (1/3)X_{11}^2$
X_7							0	$2X_7$	$4X_1X_9$	X_1X_{10}	$-X_1(X_9+X_{10})$	X_1X_9	$4X_1X_{13} - 4X_2X_{13} - 8X_4X_{12}$	X_9X_{12}
X_8								0	$2X_9$	$2X_{10}$	$2X_{11}$	$2X_{12}$	$2X_{13}$	$6X_{14}$
X_9									0	0	0	0	$-2X_9^2$	0
X_{10}										0	0	0	$-2X_{10}^2$	0
X_{11}											0	0	$-2X_{11}^2$	0
X_{12}												0	$6X_1X_{14} - 6X_2X_{14} - 2X_9X_{12}$	0
X_{13}													0	$4(X_9+X_{10}+X_{11})X_{14}$
X_{14}														0

Table 5.2.1: Gerstenhaber brackets $[\rho,\phi]$.

Proposition 5.2.15. *There is no generator of the Gerstenhaber bracket on the Hochschild cohomology* $\mathrm{HH}^\bullet(A)$ *of* $A = \mathrm{FK}(3)$, *i.e. there is no map* $\Delta : \mathrm{HH}^\bullet(A) \to \mathrm{HH}^\bullet(A)$ *of degree* -1 *such that*

$$[x,y] = (-1)^{|x|}\big(\Delta(xy) - \Delta(x)y - (-1)^{|x|}x\Delta(y)\big) \tag{5.2.10}$$

for all homogeneous elements $x,y \in \mathrm{HH}^\bullet(A)$, *where* $|x|$ *is the cohomological degree of* x. *In particular, there is no Batalin-Vilkovisky structure on* $\mathrm{HH}^\bullet(A)$ *inducing the Gerstenhaber bracket.*

Proof. Assume that (5.2.10) holds. Obviously, $\Delta(\mathrm{HH}^0(A)) = 0$. Applying the results in Table 5.2.1 and (5.2.10), we get $-4X_3 = [X_8,X_3] = \Delta(X_8)X_3$, and $0 = [X_i,X_j] = \Delta(X_i)X_j$ for $i \in [\![4,7]\!]$ and $j \in [\![1,3]\!]$, since $X_8X_3 = X_iX_j = 0$ in that case (see the first two lines of (5.1.5)). Hence, $\Delta(X_8) \in -4 + \mathrm{span}_{\Bbbk}\{X_1,X_2,X_3\}$ and $\Delta(X_i) \in \mathrm{span}_{\Bbbk}\{X_1,X_2,X_3\}$ for $i \in [\![4,7]\!]$, where $\mathrm{span}_{\Bbbk}\{X_1,X_2,X_3\}$ is the $\Bbbk$-subspace spanned by $\{X_1,X_2,X_3\}$. Moreover,

$$\begin{aligned} -2X_4 &= [X_3,X_9] = \Delta(X_3X_9) - X_3\Delta(X_9) = \Delta(X_3X_9), \\ 2X_4 &= [X_4,X_8] = -\Delta(X_4X_8) + \Delta(X_4)X_8 - X_4\Delta(X_8) = -\Delta(X_4X_8) + \Delta(X_4)X_8 + 4X_4, \end{aligned} \tag{5.2.11}$$

where we used that $X_4X_i = X_3X_k = 0$ for $i \in [\![1,3]\!]$ and $k \in [\![4,8]\!]$, by the first two lines of (5.1.5). Since $X_3X_9 = X_4X_8 \in \mathrm{HH}^2(A)$ (see the penultimate element of the third line of (5.1.5)), adding the equations (5.2.11), we obtain $\Delta(X_4)X_8 + 4X_4 = 0$. The identity $\Delta(X_4) = k_1X_1 + k_2X_2 + k_3X_3$ for $k_1,k_2,k_3 \in \Bbbk$, which we proved before, implies that $k_1X_1X_8 + k_2X_2X_8 + 4X_4 = 0$. This is impossible since the elements X_1X_8, X_2X_8 and X_4 are linearly independent in $\mathrm{HH}^1(A)$ (see (5.1.6)). The proposition thus follows. ☐

Chapter 6

Fomin-Kirillov algebra of index 4

6.1 Resolving datum on $\mathrm{FK}(4)$

We will compute a connected resolving datum on the Fomin-Kirillov algebra $\mathrm{FK}(4)$ of index 4 (see Theorem 6.1.5), then we obtain immediately a projective resolution of the trivial module in the category of bounded-below graded right $\mathrm{FK}(4)$-modules by Theorem 1.2.5.

In this section, $\Bbbk$ is a field of characteristic different from 2 and 3, and we will denote the Fomin-Kirillov algebra $\mathrm{FK}(4)$ of index 4 simply by A. For a set S, we denote by $\Bbbk S$ the $\Bbbk$-vector space spanned by all elements of S. Let $\mathscr{I}$ be the set $\{(i,j) \in [\![1,4]\!]^2 \mid i < j\}$, $\mathscr{I}_1$ the set $\{(1,2),(1,3),(2,3)\}$ and $\mathscr{J}$ the set $\{(i,j) \in [\![1,4]\!] \mid i \neq j\}$.

6.1.1 Generalities

We recall that the Fomin-Kirillov algebra A of index 4 is the quadratic $\Bbbk$-algebra generated by the $\Bbbk$-vector space V spanned by $X = \{x_{i,j} \mid (i,j) \in \mathscr{I}\}$, modulo the ideal generated by the vector space $R \subseteq V^{\otimes 2}$ spanned by the following 17 elements

$$x_{1,2}^2, x_{1,3}^2, x_{2,3}^2, x_{1,4}^2, x_{2,4}^2, x_{3,4}^2, x_{1,2}x_{2,3} - x_{2,3}x_{1,3} - x_{1,3}x_{1,2}, x_{2,3}x_{1,2} - x_{1,2}x_{1,3} - x_{1,3}x_{2,3},$$
$$x_{1,2}x_{2,4} - x_{2,4}x_{1,4} - x_{1,4}x_{1,2}, x_{2,4}x_{1,2} - x_{1,2}x_{1,4} - x_{1,4}x_{2,4}, x_{1,3}x_{3,4} - x_{3,4}x_{1,4} - x_{1,4}x_{1,3},$$
$$x_{3,4}x_{1,3} - x_{1,3}x_{1,4} - x_{1,4}x_{3,4}, x_{2,3}x_{3,4} - x_{3,4}x_{2,4} - x_{2,4}x_{2,3}, x_{3,4}x_{2,3} - x_{2,3}x_{2,4} - x_{2,4}x_{3,4},$$
$$x_{1,2}x_{3,4} - x_{3,4}x_{1,2}, x_{1,3}x_{2,4} - x_{2,4}x_{1,3}, x_{1,4}x_{2,3} - x_{2,3}x_{1,4}.$$

Recall that the dimension of A is 576 and the Hilbert series of A is

$$[2]^2[3]^2[4]^2 = 1 + 6t + 19t^2 + 42t^3 + 71t^4 + 96t^5 + 106t^6 + 96t^7 + 71t^8 + 42t^9 + 19t^{10} + 6t^{11} + t^{12},$$

where $[n] = \sum_{i=0}^{n-1} t^i$, for $n \in \mathbb{N}$. Note that $A = \oplus_{m \in [\![0,12]\!]} A_m$, where A_m is the subspace of A concentrated in internal degree m.

If the free monoid generated by X is equipped with the homogeneous lexicographic order induced by the well order $x_{1,2} \prec x_{1,3} \prec x_{2,3} \prec x_{1,4} \prec x_{2,4} \prec x_{3,4}$ on X, then a Gröbner basis G_A of the ideal (R) in the algebra $\mathbb{T}(V)$ is given by the following 30 elements

$$x_{1,2}^2, x_{1,3}^2, x_{2,3}x_{1,2} - x_{1,3}x_{2,3} - x_{1,2}x_{1,3}, x_{2,3}x_{1,3} + x_{1,3}x_{1,2} - x_{1,2}x_{2,3}, x_{2,3}^2, x_{1,4}x_{2,3} - x_{2,3}x_{1,4},$$
$$x_{1,4}^2, x_{2,4}x_{1,2} - x_{1,4}x_{2,4} - x_{1,2}x_{1,4}, x_{2,4}x_{1,3} - x_{1,3}x_{2,4}, x_{2,4}x_{1,4} + x_{1,4}x_{1,2} - x_{1,2}x_{2,4}, x_{2,4}^2,$$
$$x_{3,4}x_{1,2} - x_{1,2}x_{3,4}, x_{3,4}x_{1,3} - x_{1,4}x_{3,4} - x_{1,3}x_{1,4}, x_{3,4}x_{2,3} - x_{2,4}x_{3,4} - x_{2,3}x_{2,4},$$
$$x_{3,4}x_{1,4} + x_{1,4}x_{1,3} - x_{1,3}x_{3,4}, x_{3,4}x_{2,4} + x_{2,4}x_{2,3} - x_{2,3}x_{3,4}, x_{3,4}^2, x_{1,3}x_{1,2}x_{1,3} + x_{1,2}x_{1,3}x_{1,2},$$
$$x_{1,4}x_{1,2}x_{1,4} + x_{1,2}x_{1,4}x_{1,2}, x_{1,4}x_{1,3}x_{1,2} - x_{1,4}x_{1,2}x_{2,3} + x_{2,3}x_{1,4}x_{1,3},$$
$$x_{1,4}x_{1,3}x_{2,3} + x_{1,4}x_{1,2}x_{1,3} - x_{2,3}x_{1,4}x_{1,2}, x_{1,4}x_{1,3}x_{1,4} + x_{1,3}x_{1,4}x_{1,3},$$
$$x_{2,4}x_{2,3}x_{1,4} + x_{1,4}x_{1,2}x_{2,3} - x_{1,2}x_{2,4}x_{2,3}, x_{2,4}x_{2,3}x_{2,4} + x_{2,3}x_{2,4}x_{2,3},$$
$$x_{1,4}x_{1,2}x_{1,3}x_{2,3} - x_{2,3}x_{1,4}x_{1,2}x_{2,3}, x_{1,4}x_{1,2}x_{1,3}x_{1,4} + x_{1,3}x_{1,4}x_{1,2}x_{1,3} + x_{1,2}x_{1,3}x_{1,4}x_{1,2},$$
$$x_{1,4}x_{1,2}x_{2,3}x_{1,4} + x_{1,2}x_{1,4}x_{1,2}x_{2,3}, x_{1,4}x_{1,2}x_{1,3}x_{1,2}x_{2,3} + x_{2,3}x_{1,4}x_{1,2}x_{1,3}x_{1,2},$$
$$x_{1,4}x_{1,2}x_{1,3}x_{1,2}x_{1,4}x_{1,2} - x_{1,3}x_{1,4}x_{1,2}x_{1,3}x_{1,2}x_{1,4},$$

$x_{1,4}x_{1,2}x_{1,3}x_{1,2}x_{1,4}x_{1,3} - x_{1,2}x_{1,4}x_{1,2}x_{1,3}x_{1,2}x_{1,4}$,

which are obtained using the GAP code in Appendix A.4. The standard words with respect to G_A form a $\Bbbk$-basis of A. The classes in A of the standard words of $\mathbb{T}(V)$ with respect to G_A thus form a homogeneous $\Bbbk$-basis $\mathscr{B}$ of A. We set $\mathscr{B}_m = \mathscr{B} \cap A_m$ for $m \in [\![0, 12]\!]$.

We denote by $\{y_{i,j} = x^*_{i,j} \mid (i,j) \in \mathscr{I}\}$ the basis of V^* dual to the basis X of V. Then, the quadratic dual algebra of A is given by $A^! = \mathbb{T}(V^*)/(R^\perp) = \oplus_{n\in\mathbb{N}_0} A^!_{-n}$, where the space $R^\perp \subseteq (V^*)^{\otimes 2}$ is spanned by the following 19 elements

$$\begin{gathered}
y_{1,2}y_{2,3} + y_{2,3}y_{1,3}, y_{1,3}y_{2,3} + y_{2,3}y_{1,2}, y_{1,2}y_{2,3} + y_{1,3}y_{1,2}, y_{1,2}y_{1,3} + y_{2,3}y_{1,2},\\
y_{1,2}y_{2,4} + y_{2,4}y_{1,4}, y_{1,4}y_{2,4} + y_{2,4}y_{1,2}, y_{1,2}y_{2,4} + y_{1,4}y_{1,2}, y_{1,2}y_{1,4} + y_{2,4}y_{1,2},\\
y_{1,3}y_{3,4} + y_{3,4}y_{1,4}, y_{1,4}y_{3,4} + y_{3,4}y_{1,3}, y_{1,3}y_{3,4} + y_{1,4}y_{1,3}, y_{1,3}y_{1,4} + y_{3,4}y_{1,3},\\
y_{2,3}y_{3,4} + y_{3,4}y_{2,4}, y_{2,4}y_{3,4} + y_{3,4}y_{2,3}, y_{2,3}y_{3,4} + y_{2,4}y_{2,3}, y_{2,3}y_{2,4} + y_{3,4}y_{2,3},\\
y_{1,2}y_{3,4} + y_{3,4}y_{1,2}, y_{1,3}y_{2,4} + y_{2,4}y_{1,3}, y_{2,3}y_{1,4} + y_{1,4}y_{2,3}.
\end{gathered}$$

Using the GAP code in Appendix A.4, we get a Gröbner basis G_B of the ideal $(R^\perp)$ in $\mathbb{T}(V^*)$ given by the following 31 elements

$$\begin{gathered}
y_{1,3}y_{1,2} + y_{1,2}y_{2,3},\ y_{1,3}y_{2,3} - y_{1,2}y_{1,3},\ y_{2,3}y_{1,2} + y_{1,3}y_{2,3},\ y_{2,3}y_{1,3} + y_{1,2}y_{2,3},\ y_{1,4}y_{1,2} + y_{1,2}y_{2,4},\\
y_{1,4}y_{1,3} + y_{1,3}y_{3,4},\ y_{1,4}y_{2,3} + y_{2,3}y_{1,4},\ y_{1,4}y_{2,4} - y_{1,2}y_{1,4},\ y_{1,4}y_{3,4} - y_{1,3}y_{1,4},\ y_{2,4}y_{1,2} + y_{1,4}y_{2,4},\\
y_{2,4}y_{1,3} + y_{1,3}y_{2,4},\ y_{2,4}y_{2,3} + y_{2,3}y_{3,4},\ y_{2,4}y_{1,4} + y_{1,2}y_{2,4},\ y_{2,4}y_{3,4} - y_{2,3}y_{2,4},\ y_{3,4}y_{1,2} + y_{1,2}y_{3,4},\\
y_{3,4}y_{1,3} + y_{1,4}y_{3,4},\ y_{3,4}y_{2,3} + y_{2,4}y_{3,4},\ y_{3,4}y_{1,4} + y_{1,3}y_{3,4},\ y_{3,4}y_{2,4} + y_{2,3}y_{3,4},\ y_{1,2}y_{2,3}^2 - y_{1,2}y_{1,3}^2,\\
y_{1,2}y_{2,4}^2 - y_{1,2}y_{1,4}^2,\ y_{1,3}y_{3,4}^2 - y_{1,3}y_{1,4}^2,\ y_{2,3}y_{3,4}^2 - y_{2,3}y_{2,4}^2,\ y_{1,2}y_{1,3}^3 - y_{1,2}^3y_{1,3},\\
y_{1,2}y_{1,3}y_{2,4}^2 - y_{1,2}y_{1,3}y_{1,4}^2,\ y_{1,2}y_{2,3}y_{2,4}^2 - y_{1,2}y_{2,3}y_{1,4}^2,\ y_{1,2}y_{1,4}^3 - y_{1,2}^3y_{1,4},\ y_{1,3}y_{1,4}^3 - y_{1,3}^3y_{1,4},\\
y_{2,3}y_{2,4}^3 - y_{2,3}^3y_{2,4},\ y_{1,2}y_{1,3}^2y_{2,4}^2 - y_{1,2}y_{1,3}^2y_{1,4}^2,\ y_{1,2}y_{2,3}y_{1,4}^3 - y_{1,2}^3y_{2,3}y_{1,4}.
\end{gathered} \tag{6.1.1}$$

Let $\mathscr{B}^!_0 = \{1\} \subseteq \Bbbk$, let $\mathscr{B}^!_1 = \{y_{i,j} \mid (i,j) \in \mathscr{I}\} \subseteq V^*$, let $\mathscr{B}^!_2 \subseteq A^!_{-2}$ be the set formed by the following 17 elements

$$\begin{gathered}
y_{1,2}^2, y_{1,2}y_{1,3}, y_{1,2}y_{2,3}, y_{1,2}y_{1,4}, y_{1,2}y_{2,4}, y_{1,2}y_{3,4}, y_{1,3}^2, y_{1,3}y_{1,4}, y_{1,3}y_{2,4}, y_{1,3}y_{3,4}, y_{2,3}^2, y_{2,3}y_{1,4},\\
y_{2,3}y_{2,4}, y_{2,3}y_{3,4}, y_{1,4}^2, y_{2,4}^2, y_{3,4}^2,
\end{gathered}$$

let $\mathscr{B}^!_3 \subseteq A^!_{-3}$ be the set formed by the following 30 elements

$$\begin{gathered}
y_{1,2}^3, y_{1,2}^2y_{1,3}, y_{1,2}^2y_{2,3}, y_{1,2}^2y_{1,4}, y_{1,2}^2y_{2,4}, y_{1,2}^2y_{3,4}, y_{1,2}y_{1,3}^2, y_{1,2}y_{1,3}y_{1,4}, y_{1,2}y_{1,3}y_{2,4},\\
y_{1,2}y_{1,3}y_{3,4}, y_{1,2}y_{2,3}y_{1,4}, y_{1,2}y_{2,3}y_{2,4}, y_{1,2}y_{2,3}y_{3,4}, y_{1,2}y_{1,4}^2, y_{1,2}y_{3,4}^2, y_{1,3}^3, y_{1,3}^2y_{1,4}, y_{1,3}^2y_{2,4},\\
y_{1,3}^2y_{3,4}, y_{1,3}y_{1,4}^2, y_{1,3}y_{2,4}^2, y_{2,3}^3, y_{2,3}^2y_{1,4}, y_{2,3}^2y_{2,4}, y_{2,3}^2y_{3,4}, y_{2,3}y_{1,4}^2, y_{2,3}y_{2,4}^2, y_{1,4}^3, y_{2,4}^3, y_{3,4}^3,
\end{gathered}$$

and let $\mathscr{B}^!_4 \subseteq A^!_{-4}$ be the set formed by the following 38 elements

$$\begin{gathered}
y_{1,2}^4, y_{1,2}^3y_{1,3}, y_{1,2}^3y_{2,3}, y_{1,2}^3y_{1,4}, y_{1,2}^3y_{2,4}, y_{1,2}^3y_{3,4}, y_{1,2}^2y_{1,3}^2, y_{1,2}^2y_{1,3}y_{1,4}, y_{1,2}^2y_{1,3}y_{2,4},\\
y_{1,2}^2y_{1,3}y_{3,4}, y_{1,2}^2y_{2,3}y_{1,4}, y_{1,2}^2y_{2,3}y_{2,4}, y_{1,2}^2y_{2,3}y_{3,4}, y_{1,2}^2y_{1,4}^2, y_{1,2}^2y_{3,4}^2, y_{1,2}y_{1,3}^2y_{1,4}, y_{1,2}y_{1,3}^2y_{2,4},\\
y_{1,2}y_{1,3}^2y_{3,4}, y_{1,2}y_{1,3}y_{1,4}^2, y_{1,2}y_{2,3}y_{1,4}^2, y_{1,2}y_{3,4}^3, y_{1,3}^4, y_{1,3}^3y_{1,4}, y_{1,3}^3y_{2,4}, y_{1,3}^3y_{3,4}, y_{1,3}^2y_{1,4}^2,\\
y_{1,3}^2y_{2,4}^2, y_{1,3}y_{2,4}^3, y_{2,3}^4, y_{2,3}^3y_{1,4}, y_{2,3}^3y_{2,4}, y_{2,3}^3y_{3,4}, y_{2,3}^2y_{1,4}^2, y_{2,3}^2y_{2,4}^2, y_{2,3}y_{1,4}^3, y_{1,4}^4, y_{2,4}^4, y_{3,4}^4.
\end{gathered}$$

Moreover, for every integer $n \geqslant 5$, define $\mathscr{B}^!_n = \mathscr{U}^!_n \cup \mathscr{C}^!_n$, where the set $\mathscr{U}^!_n \subseteq A^!_{-n}$ consists of the following 24 elements

$$\begin{gathered}
y_{1,2}^{n-1}y_{1,3}, y_{1,2}^{n-1}y_{2,3}, y_{1,2}^{n-1}y_{1,4}, y_{1,2}^{n-1}y_{2,4}, y_{1,2}^{n-2}y_{1,3}^2, y_{1,2}^{n-2}y_{1,3}y_{1,4}, y_{1,2}^{n-2}y_{1,3}y_{2,4}, y_{1,2}^{n-2}y_{1,3}y_{3,4},\\
y_{1,2}^{n-2}y_{2,3}y_{1,4}, y_{1,2}^{n-2}y_{2,3}y_{2,4}, y_{1,2}^{n-2}y_{2,3}y_{3,4}, y_{1,2}^{n-2}y_{1,4}^2, y_{1,2}^{n-3}y_{1,3}^2y_{1,4}, y_{1,2}^{n-3}y_{1,3}^2y_{2,4}, y_{1,2}^{n-3}y_{1,3}^2y_{3,4},\\
y_{1,2}^{n-3}y_{1,3}y_{1,4}^2, y_{1,2}^{n-3}y_{2,3}y_{1,4}^2, y_{1,2}^{n-4}y_{1,3}^2y_{1,4}^2, y_{1,3}^{n-1}y_{1,4}, y_{1,3}^{n-1}y_{3,4}, y_{1,3}^{n-2}y_{1,4}^2, y_{2,3}^{n-1}y_{2,4}, y_{2,3}^{n-1}y_{3,4}, y_{2,3}^{n-2}y_{2,4}^2,
\end{gathered}$$

and $\mathscr{C}^!_n \subseteq A^!_{-n}$ is the set of $3(n+1)$ elements given by

$$\mathscr{C}^!_n = \{y_{1,2}^{n-r}y_{3,4}^r,\ y_{1,3}^{n-r}y_{2,4}^r,\ y_{2,3}^{n-r}y_{1,4}^r \mid r \in [\![0,n]\!]\}.$$

The following result is proved directly from the explicit description of the Gröbner basis G_B given in (6.1.1) for the ideal $(R^\perp) \subseteq \mathbb{T}(V^*)$.

Fact 6.1.1. *The set $\mathscr{B}^!_n$ is a basis of $A^!_{-n}$ for $n \in \mathbb{N}_0$, consisting of standard words with respect to the Gröbner basis G_B. In consequence, $\#(\mathscr{B}^!_n) = 3n + 27$ for $n \geqslant 5$, and the Hilbert series $h(t)$ of $A^!$ is given by*

$$h(t) = 1 + 6t + 17t^2 + 30t^3 + 38t^4 + \sum_{n=5}^{\infty}(3n+27)t^n = \frac{1 + 4t + 6t^2 + 2t^3 - 5t^4 - 4t^5 - t^6}{(t-1)^2}.$$

The following result describes several identities expressing products of the generators of the quadratic dual algebra $A^!$ in terms of the basis $\mathscr{B}^! = \cup_{n\in\mathbb{N}_0}\mathscr{B}^!_n$. The proof is a straightforward but rather lengthy verification.

Fact 6.1.2. *We have the following identities*

$$y_{i,j}^{n-r} y_{k,l}^r y_{i,j} = (-1)^r y_{i,j}^{n-r+1} y_{k,l}^r, \quad y_{i,j}^{n-r} y_{k,l}^r y_{k,l} = y_{i,j}^{n-r} y_{k,l}^{r+1} \tag{6.1.2}$$

and

$$y_{i,j} y_{i,j}^{n-r} y_{k,l}^r = y_{i,j}^{n-r+1} y_{k,l}^r, \quad y_{k,l} y_{i,j}^{n-r} y_{k,l}^r = (-1)^{n-r} y_{i,j}^{n-r} y_{k,l}^{r+1}$$

in $A^!$, for all integers $n \geqslant 2$, $r \in [\![1,n-1]\!]$, $(i,j) \in \mathscr{I}_1$, $(k,l) \in \mathscr{I}$ with $\#\{i,j,k,l\} = 4$. Moreover, we also have the identities

$$\begin{aligned}
y_{1,2}^{n-r} y_{3,4}^r y_{1,3} &= \chi_r y_{1,2}^{n-2} y_{1,3} y_{1,4}^2 - \chi_{r+1} y_{1,2}^{n-1} y_{1,3} y_{1,4},\\
y_{1,2}^{n-r} y_{3,4}^r y_{2,3} &= \chi_r y_{1,2}^{n-2} y_{2,3} y_{1,4}^2 - \chi_{r+1} y_{1,2}^{n-1} y_{2,3} y_{2,4},\\
y_{1,2}^{n-r} y_{3,4}^r y_{1,4} &= \chi_r y_{1,2}^{n-2} y_{1,3}^2 y_{1,4} - \chi_{r+1} y_{1,2}^{n-1} y_{1,3} y_{3,4},\\
y_{1,2}^{n-r} y_{3,4}^r y_{2,4} &= \chi_r y_{1,2}^{n-2} y_{1,3}^2 y_{2,4} - \chi_{r+1} y_{1,2}^{n-1} y_{2,3} y_{3,4},\\
y_{1,3}^{n-r} y_{2,4}^r y_{1,2} &= \chi_n \chi_r y_{1,2}^{n-3} y_{1,3}^2 y_{1,4}^2 - \chi_{n+1} \chi_r y_{1,2}^{n-2} y_{2,3} y_{1,4}^2 + \chi_n \chi_{r+1} y_{1,2}^{n-1} y_{2,3} y_{1,4}\\
&\quad - \chi_{n+1} \chi_{r+1} y_{1,2}^{n-2} y_{1,3}^2 y_{1,4},\\
y_{1,3}^{n-r} y_{2,4}^r y_{2,3} &= \chi_n \chi_r y_{1,2}^{n-2} y_{2,3} y_{1,4}^2 + \chi_{n+1} \chi_r y_{1,2}^{n-2} y_{1,3} y_{1,4}^2 - \chi_n \chi_{r+1} y_{1,2}^{n-1} y_{1,3} y_{3,4}\\
&\quad - \chi_{n+1} \chi_{r+1} y_{1,2}^{n-1} y_{2,3} y_{3,4},\\
y_{1,3}^{n-r} y_{2,4}^r y_{1,4} &= \chi_n \chi_r y_{1,2}^{n-2} y_{1,3}^2 y_{1,4} + \chi_{n+1} \chi_r y_{1,2}^{n-1} y_{1,3} y_{1,4} + \chi_n \chi_{r+1} y_{1,2}^{n-1} y_{2,3} y_{2,4}\\
&\quad - \chi_{n+1} \chi_{r+1} y_{1,2}^{n-2} y_{1,3}^2 y_{2,4},\\
y_{1,3}^{n-r} y_{2,4}^r y_{3,4} &= \chi_n \chi_r y_{1,2}^{n-2} y_{1,3}^2 y_{3,4} + \chi_{n+1} \chi_r y_{1,2}^{n-1} y_{1,3} y_{3,4} + \chi_n \chi_{r+1} y_{1,2}^{n-1} y_{1,3} y_{2,4}\\
&\quad + \chi_{n+1} \chi_{r+1} y_{1,2}^{n-1} y_{2,3} y_{2,4},\\
y_{2,3}^{n-r} y_{1,4}^r y_{1,2} &= \chi_n \chi_r y_{1,2}^{n-3} y_{1,3}^2 y_{1,4}^2 - \chi_{n+1} \chi_r y_{1,2}^{n-2} y_{1,3} y_{1,4}^2 + \chi_n \chi_{r+1} y_{1,2}^{n-1} y_{1,3} y_{2,4}\\
&\quad - \chi_{n+1} \chi_{r+1} y_{1,2}^{n-2} y_{1,3}^2 y_{2,4},\\
y_{2,3}^{n-r} y_{1,4}^r y_{1,3} &= \chi_n \chi_r y_{1,2}^{n-2} y_{1,3} y_{1,4}^2 - \chi_{n+1} \chi_r y_{1,2}^{n-2} y_{2,3} y_{1,4}^2 + \chi_n \chi_{r+1} y_{1,2}^{n-1} y_{2,3} y_{3,4}\\
&\quad - \chi_{n+1} \chi_{r+1} y_{1,2}^{n-1} y_{1,3} y_{3,4},\\
y_{2,3}^{n-r} y_{1,4}^r y_{2,4} &= \chi_n \chi_r y_{1,2}^{n-2} y_{1,3}^2 y_{2,4} + \chi_{n+1} \chi_r y_{1,2}^{n-1} y_{2,3} y_{2,4} - \chi_n \chi_{r+1} y_{1,2}^{n-1} y_{1,3} y_{1,4}\\
&\quad + \chi_{n+1} \chi_{r+1} y_{1,2}^{n-2} y_{1,3}^2 y_{1,4},\\
y_{2,3}^{n-r} y_{1,4}^r y_{3,4} &= \chi_n \chi_r y_{1,2}^{n-2} y_{1,3}^2 y_{3,4} + \chi_{n+1} \chi_r y_{1,2}^{n-1} y_{2,3} y_{3,4} - \chi_n \chi_{r+1} y_{1,2}^{n-1} y_{2,3} y_{1,4}\\
&\quad + \chi_{n+1} \chi_{r+1} y_{1,2}^{n-1} y_{1,3} y_{1,4},
\end{aligned} \tag{6.1.3}$$

and

$$\begin{aligned}
y_{1,3} y_{1,2}^{n-r} y_{3,4}^r &= \chi_n \chi_r y_{1,2}^{n-2} y_{1,3} y_{1,4}^2 - \chi_n \chi_{r+1} y_{1,2}^{n-1} y_{2,3} y_{3,4} - \chi_{n+1} \chi_r y_{1,2}^{n-2} y_{2,3} y_{1,4}^2\\
&\quad + \chi_{n+1} \chi_{r+1} y_{1,2}^{n-1} y_{1,3} y_{3,4},\\
y_{2,4} y_{1,2}^{n-r} y_{3,4}^r &= \chi_n \chi_r y_{1,2}^{n-2} y_{1,3}^2 y_{2,4} - \chi_n \chi_{r+1} y_{1,2}^{n-1} y_{1,3} y_{1,4} - \chi_{n+1} \chi_r y_{1,2}^{n-2} y_{1,3}^2 y_{1,4}\\
&\quad + \chi_{n+1} \chi_{r+1} y_{1,2}^{n-1} y_{2,3} y_{2,4},\\
y_{2,3} y_{1,2}^{n-r} y_{3,4}^r &= \chi_n \chi_r y_{1,2}^{n-2} y_{2,3} y_{1,4}^2 - \chi_n \chi_{r+1} y_{1,2}^{n-1} y_{1,3} y_{3,4} - \chi_{n+1} \chi_r y_{1,2}^{n-2} y_{1,3} y_{1,4}^2
\end{aligned}$$

$$
\begin{aligned}
&+ \chi_{n+1}\chi_{r+1} y_{1,2}^{n-1} y_{2,3} y_{3,4},\\
y_{1,4} y_{1,2}^{n-r} y_{3,4}^{r} &= \chi_n\chi_r y_{1,2}^{n-2} y_{1,3}^{2} y_{1,4} - \chi_n\chi_{r+1} y_{1,2}^{n-1} y_{2,3} y_{2,4} - \chi_{n+1}\chi_r y_{1,2}^{n-2} y_{1,3}^{2} y_{2,4}\\
&+ \chi_{n+1}\chi_{r+1} y_{1,2}^{n-1} y_{1,3} y_{1,4},\\
y_{1,2} y_{1,3}^{n-r} y_{2,4}^{r} &= \chi_n\chi_r y_{1,2}^{n-3} y_{1,3}^{2} y_{1,4}^{2} + \chi_n\chi_{r+1} y_{1,2}^{n-1} y_{1,3} y_{2,4} + \chi_{n+1}\chi_r y_{1,2}^{n-2} y_{1,3} y_{1,4}^{2}\\
&+ \chi_{n+1}\chi_{r+1} y_{1,2}^{n-2} y_{1,3}^{2} y_{2,4}, \qquad (6.1.4)\\
y_{3,4} y_{1,3}^{n-r} y_{2,4}^{r} &= \chi_n\chi_r y_{1,2}^{n-2} y_{1,3}^{2} y_{3,4} + \chi_n\chi_{r+1} y_{1,2}^{n-1} y_{2,3} y_{1,4} - \chi_{n+1}\chi_r y_{1,2}^{n-1} y_{1,3} y_{1,4}\\
&- \chi_{n+1}\chi_{r+1} y_{1,2}^{n-1} y_{2,3} y_{3,4},\\
y_{2,3} y_{1,3}^{n-r} y_{2,4}^{r} &= (-1)^n \chi_r y_{1,2}^{n-2} y_{2,3} y_{1,4}^{2} + (-1)^{n+1} \chi_{r+1} y_{1,2}^{n-1} y_{2,3} y_{2,4},\\
y_{1,4} y_{1,3}^{n-r} y_{2,4}^{r} &= \chi_{n-r} y_{1,2}^{n-2} y_{1,3}^{2} y_{1,4} + (-1)^n \chi_{n-r+1} y_{1,2}^{n-1} y_{1,3} y_{3,4},\\
y_{1,2} y_{2,3}^{n-r} y_{1,4}^{r} &= \chi_n\chi_r y_{1,2}^{n-3} y_{1,3}^{2} y_{1,4}^{2} + \chi_n\chi_{r+1} y_{1,2}^{n-1} y_{2,3} y_{1,4} + \chi_{n+1}\chi_r y_{1,2}^{n-2} y_{2,3} y_{1,4}^{2}\\
&+ \chi_{n+1}\chi_{r+1} y_{1,2}^{n-2} y_{1,3}^{2} y_{1,4},\\
y_{3,4} y_{2,3}^{n-r} y_{1,4}^{r} &= \chi_n\chi_r y_{1,2}^{n-2} y_{1,3}^{2} y_{3,4} - \chi_n\chi_{r+1} y_{1,2}^{n-1} y_{1,3} y_{2,4} - \chi_{n+1}\chi_r y_{1,2}^{n-1} y_{2,3} y_{2,4}\\
&- \chi_{n+1}\chi_{r+1} y_{1,2}^{n-1} y_{1,3} y_{3,4},\\
y_{1,3} y_{2,3}^{n-r} y_{1,4}^{r} &= \chi_r y_{1,2}^{n-2} y_{1,3} y_{1,4}^{2} + \chi_{r+1} y_{1,2}^{n-1} y_{1,3} y_{1,4},\\
y_{2,4} y_{2,3}^{n-r} y_{1,4}^{r} &= (-1)^n \chi_{n-r} y_{1,2}^{n-2} y_{1,3}^{2} y_{2,4} - \chi_{n-r+1} y_{1,2}^{n-1} y_{2,3} y_{3,4},
\end{aligned}
$$

together with

$$
\begin{aligned}
y_{1,3} y_{1,2}^{n} &= \chi_n y_{1,2}^{n} y_{1,3} - \chi_{n+1} y_{1,2}^{n} y_{2,3}, & y_{1,3} y_{3,4}^{n} &= \chi_n y_{1,3}^{n-1} y_{1,4}^{2} + \chi_{n+1} y_{1,3}^{n} y_{3,4},\\
y_{2,4} y_{1,2}^{n} &= \chi_n y_{1,2}^{n} y_{2,4} - \chi_{n+1} y_{1,2}^{n} y_{1,4}, & y_{2,4} y_{3,4}^{n} &= y_{2,3}^{n} y_{2,4},\\
y_{2,3} y_{1,2}^{n} &= \chi_n y_{1,2}^{n} y_{2,3} - \chi_{n+1} y_{1,2}^{n} y_{1,3}, & y_{2,3} y_{3,4}^{n} &= \chi_n y_{2,3}^{n-1} y_{2,4}^{2} + \chi_{n+1} y_{2,3}^{n} y_{3,4},\\
y_{1,4} y_{1,2}^{n} &= \chi_n y_{1,2}^{n} y_{1,4} - \chi_{n+1} y_{1,2}^{n} y_{2,4}, & y_{1,4} y_{3,4}^{n} &= y_{1,3}^{n} y_{1,4},\\
y_{1,2} y_{1,3}^{n} &= \chi_n y_{1,2}^{n-1} y_{1,3}^{2} + \chi_{n+1} y_{1,2}^{n} y_{1,3}, & y_{1,2} y_{2,4}^{n} &= \chi_n y_{1,2}^{n-1} y_{1,4}^{2} + \chi_{n+1} y_{1,2}^{n} y_{2,4},\\
y_{3,4} y_{1,3}^{n} &= \chi_n y_{1,3}^{n} y_{3,4} - \chi_{n+1} y_{1,3}^{n} y_{1,4}, & y_{3,4} y_{2,4}^{n} &= (-1)^n y_{2,3}^{n} y_{3,4}, \qquad (6.1.5)\\
y_{2,3} y_{1,3}^{n} &= (-1)^n y_{1,2}^{n} y_{2,3}, & y_{2,3} y_{2,4}^{n} &= \chi_n y_{2,3}^{n-1} y_{2,4}^{2} + \chi_{n+1} y_{2,3}^{n} y_{2,4},\\
y_{1,4} y_{1,3}^{n} &= \chi_n y_{1,3}^{n} y_{1,4} - \chi_{n+1} y_{1,3}^{n} y_{3,4}, & y_{1,4} y_{2,4}^{n} &= y_{1,2}^{n} y_{1,4},\\
y_{1,2} y_{2,3}^{n} &= \chi_n y_{1,2}^{n-1} y_{1,3}^{2} + \chi_{n+1} y_{1,2}^{n} y_{2,3}, & y_{1,2} y_{1,4}^{n} &= \chi_n y_{1,2}^{n-1} y_{1,4}^{2} + \chi_{n+1} y_{1,2}^{n} y_{1,4},\\
y_{3,4} y_{2,3}^{n} &= \chi_n y_{2,3}^{n} y_{3,4} - \chi_{n+1} y_{2,3}^{n} y_{2,4}, & y_{3,4} y_{1,4}^{n} &= (-1)^n y_{1,3}^{n} y_{3,4},\\
y_{1,3} y_{2,3}^{n} &= y_{1,2}^{n} y_{1,3}, & y_{1,3} y_{1,4}^{n} &= \chi_n y_{1,3}^{n-1} y_{1,4}^{2} + \chi_{n+1} y_{1,3}^{n} y_{1,4},\\
y_{2,4} y_{2,3}^{n} &= \chi_n y_{2,3}^{n} y_{2,4} - \chi_{n+1} y_{2,3}^{n} y_{3,4}, & y_{2,4} y_{1,4}^{n} &= (-1)^n y_{1,2}^{n} y_{2,4},
\end{aligned}
$$

in $A^!$, for all integers $n \geqslant 2$ and $r \in [\![1,n-1]\!]$.

Recall that the graded dual $(A^!)^{\#} = \oplus_{n\in\mathbb{N}_0}(A^!_{-n})^*$ is a graded bimodule over $A^!$ via the identity $(ufv)(w) = f(vwu)$ for $u,v,w \in A^!$ and $f \in (A^!)^{\#}$. Let $\mathscr{B}_n^{!*}$ be the dual basis to the basis $\mathscr{B}_n^!$ for $n \in \mathbb{N}_0$. We write $\mathscr{B}_0^{!*} = \{\epsilon^!\}$ and $z_{n_1}^{i_1,j_1} \dots z_{n_r}^{i_r,j_r} = (y_{i_1,j_1}^{n_1} \dots y_{i_r,j_r}^{n_r})^* \in \mathscr{B}_n^{!*}$ for $y_{i_1,j_1}^{n_1} \dots y_{i_r,j_r}^{n_r} \in \mathscr{B}_n^!$, where $n = n_1 + \dots + n_r$, $n,r,n_1,\dots,n_r \in \mathbb{N}$ and $(i_1,j_1),\dots,(i_r,j_r) \in \mathscr{I}$. We will omit the index n_j for $j \in [\![1,r]\!]$ if $n_j = 1$ in the element $z_{n_1}^{i_1,j_1} \dots z_{n_r}^{i_r,j_r}$ or $y_{i_1,j_1}^{n_1} \dots y_{i_r,j_r}^{n_r}$. Obviously, $y_{i,j}z^{i,j} = \epsilon^!$ for $(i,j) \in \mathscr{I}$ and the other actions of $\mathscr{B}_1^!$ on $\mathscr{B}_1^{!*}$ vanish.

Let $(K_\bullet,d_\bullet)$ be the Koszul complex of $\Bbbk$ in the category of bounded below graded right A-modules and $\epsilon : K_0 \to \Bbbk$ the canonical projection. The differential $d_n : K_n \to K_{n-1}$ for $n \in \mathbb{N}$ is given by the multiplication of $\sum_{(i,j)\in\mathscr{I}} y_{i,j} \otimes x_{i,j}$ on the left. As usual, we can consider the Koszul complex as a complex indexed by $\mathbb{Z}$, with $K_n = 0$ for all $n \in \mathbb{Z}_{\leqslant -1}$, and $d_n = 0$ for all $n \in \mathbb{Z}_{\leqslant 0}$. To reduce space, we will typically use vertical bars instead of the tensor product symbols $\otimes$.

The differential $d_\bullet$ of the Koszul complex of A can be explicitly described in the following result. Its proof is a straightforward but lengthy verification, using the identities listed in Fact 6.1.2 and in Appendix A.1.

Fact 6.1.3. *Let $d_n : K_n \to K_{n-1}$ be the differential of the Koszul complex of A for $n \in \mathbb{N}$. It can be explicitly described as follows. First, $d_1(z^{i,j}|1) = \epsilon^{!}|x_{i,j}$ for $(i,j) \in \mathcal{J}$, and*

$$d_n(z^{i,j}_{n-r}z^{k,l}_{r}|1) = (-1)^r z^{i,j}_{n-r-1}z^{k,l}_{r}|x_{i,j} + z^{i,j}_{n-r}z^{k,l}_{r-1}|x_{k,l}, \tag{6.1.6}$$

for $n \geqslant 2$, $r \in [\![0,n]\!]$, $(i,j) \in \mathcal{J}_1$, $(k,l) \in \mathcal{J}$ with $\#\{i,j,k,l\} = 4$, where we follow the convention that $z^{i,j}_{n}z^{k,l}_{0} = z^{i,j}_{n}$, $z^{i,j}_{0}z^{k,l}_{n} = z^{k,l}_{n}$, $z^{i,j}_{n}z^{k,l}_{-1} = 0$ and $z^{i,j}_{-1}z^{k,l}_{n} = 0$ for $n \in \mathbb{N}$. Moreover, for $n \geqslant 5$, the differential d_{n+1} is given by (6.1.6) *and*

$$\begin{aligned}
z^{1,2}_{n}z^{1,3}|1 \mapsto{}& -(z^{1,2}_{n-1}z^{2,3} + \chi_{n+1}z^{2,3}_{n})|x_{1,2} + (z^{1,2}_{n} + z^{1,2}_{n-2}z^{1,3}_{2} + \chi_{n}z^{2,3}_{n})|x_{1,3} \\
&+ (z^{1,2}_{n-1}z^{1,3} + \chi_{n+1}z^{1,3}_{n})|x_{2,3}, \\
z^{1,2}_{n}z^{2,3}|1 \mapsto{}& -\{z^{1,2}_{n-1}z^{1,3} + \chi_{n+1}z^{1,3}_{n}\}|x_{1,2} - \{z^{1,2}_{n-1}z^{2,3} + \chi_{n+1}z^{2,3}_{n}\}|x_{1,3} \\
&+ \{z^{1,2}_{n} + z^{1,2}_{n-2}z^{1,3}_{2} + \chi_{n}z^{1,3}_{n}\}|x_{2,3}, \\
z^{1,2}_{n}z^{1,4}|1 \mapsto{}& -\{z^{1,2}_{n-1}z^{2,4} + \chi_{n+1}z^{2,4}_{n}\}|x_{1,2} + \{z^{1,2}_{n} + z^{1,2}_{n-2}z^{1,4}_{2} + \chi_{n}z^{2,4}_{n}\}|x_{1,4} \\
&+ \{z^{1,2}_{n-1}z^{1,4} + \chi_{n+1}z^{1,4}_{n}\}|x_{2,4}, \\
z^{1,2}_{n}z^{2,4}|1 \mapsto{}& -\{z^{1,2}_{n-1}z^{1,4} + \chi_{n+1}z^{1,4}_{n}\}|x_{1,2} - \{z^{1,2}_{n-1}z^{2,4} + \chi_{n+1}z^{2,4}_{n}\}|x_{1,4} \\
&+ \{z^{1,2}_{n} + z^{1,2}_{n-2}z^{1,4}_{2} + \chi_{n}z^{1,4}_{n}\}|x_{2,4}, \\
z^{1,2}_{n-1}z^{1,3}_{2}|1 \mapsto{}& \{z^{1,2}_{n-2}z^{1,3}_{2} + \chi_{n}(z^{1,3}_{n} + z^{2,3}_{n})\}|x_{1,2} + z^{1,2}_{n-1}z^{1,3}|x_{1,3} + z^{1,2}_{n-1}z^{2,3}|x_{2,3}, \\
z^{1,2}_{n-1}z^{1,3}z^{1,4}|1 \mapsto{}& \{z^{1,2}_{n-2}z^{2,3}z^{2,4} + \chi_{n}z^{2,3}_{n-1}z^{2,4}\}|x_{1,2} \\
&- \Big\{z^{1,2}_{n-3}z^{1,3}_{2}z^{3,4} + \chi_{n+1}z^{2,3}_{n-1}z^{3,4} + \sum_{s=1}^{\lfloor \frac{n}{2} \rfloor} z^{1,2}_{n-2s+1}z^{3,4}_{2s-1}\Big\}|x_{1,3} \\
&- \{z^{1,2}_{n-2}z^{1,3}z^{1,4} + \chi_{n}z^{1,3}_{n-1}z^{1,4}\}|x_{2,3} \\
&+ \Big\{z^{1,2}_{n-1}z^{1,3} + z^{1,2}_{n-3}z^{1,3}z^{1,4}_{2} + \chi_{n+1}\sum_{s=1}^{\frac{n-1}{2}} z^{1,3}_{n-2s}z^{2,4}_{2s}\Big\}|x_{1,4} \\
&- \Big\{z^{1,2}_{n-2}z^{2,3}z^{1,4} + \chi_{n}\sum_{s=1}^{\frac{n}{2}} z^{2,3}_{n-2s+1}z^{1,4}_{2s-1}\Big\}|x_{2,4} \\
&+ \Big\{z^{1,2}_{n-1}z^{1,4} + z^{1,2}_{n-3}z^{1,3}_{2}z^{1,4} + \chi_{n+1}\sum_{s=1}^{\frac{n-1}{2}} z^{2,3}_{n-2s+1}z^{1,4}_{2s-1}\Big\}|x_{3,4}, \\
z^{1,2}_{n-1}z^{1,3}z^{2,4}|1 \mapsto{}& \Big\{z^{1,2}_{n-2}z^{2,3}z^{1,4} + \chi_{n}\sum_{s=1}^{\frac{n}{2}} z^{2,3}_{n-2s+1}z^{1,4}_{2s-1}\Big\}|x_{1,2} \\
&- \{z^{1,2}_{n-1}z^{2,4} + z^{1,2}_{n-3}z^{1,3}_{2}z^{2,4} + \chi_{n+1}z^{2,3}_{n-1}z^{2,4}\}|x_{1,3} - \{z^{1,2}_{n-2}z^{1,3}z^{3,4} + \chi_{n}z^{1,3}_{n-1}z^{3,4}\}|x_{2,3} \\
&+ \{z^{1,2}_{n-2}z^{2,3}z^{2,4} + \chi_{n}z^{2,3}_{n-1}z^{2,4}\}|x_{1,4} + \{z^{1,2}_{n-1}z^{1,3} + z^{1,2}_{n-3}z^{1,3}z^{1,4}_{2} + \chi_{n+1}z^{1,3}_{n-2}z^{1,4}_{2}\}|x_{2,4} \\
&+ \Big\{z^{1,2}_{n-2}z^{1,3}z^{2,4} + \chi_{n}\sum_{s=1}^{\frac{n}{2}} z^{1,3}_{n-2s+1}z^{2,4}_{2s-1}\Big\}|x_{3,4}, \\
z^{1,2}_{n-1}z^{1,3}z^{3,4}|1 \mapsto{}& \{z^{1,2}_{n-2}z^{2,3}z^{3,4} + \chi_{n}z^{2,3}_{n-1}z^{3,4}\}|x_{1,2} \\
&- \Big\{z^{1,2}_{n-1}z^{1,4} + z^{1,2}_{n-3}z^{1,3}_{2}z^{1,4} + \chi_{n+1}\sum_{s=1}^{\frac{n-1}{2}} z^{2,3}_{n-2s+1}z^{1,4}_{2s-1}\Big\}|x_{1,3} \\
&- \Big\{z^{1,2}_{n-2}z^{1,3}z^{2,4} + \chi_{n}\sum_{s=1}^{\frac{n}{2}} z^{1,3}_{n-2s+1}z^{2,4}_{2s-1}\Big\}|x_{2,3} \\
&- \Big\{z^{1,2}_{n-3}z^{1,3}_{2}z^{3,4} + \chi_{n+1}z^{2,3}_{n-1}z^{3,4} + \sum_{s=1}^{\lfloor \frac{n}{2} \rfloor} z^{1,2}_{n-2s+1}z^{3,4}_{2s-1}\Big\}|x_{1,4} \\
&- \{z^{1,2}_{n-2}z^{1,3}z^{3,4} + \chi_{n}z^{1,3}_{n-1}z^{3,4}\}|x_{2,4} \\
&+ \Big\{z^{1,2}_{n-1}z^{1,3} + z^{1,2}_{n-3}z^{1,3}z^{1,4}_{2} + \chi_{n+1}\sum_{s=1}^{\frac{n-1}{2}} z^{1,3}_{n-2s}z^{2,4}_{2s}\Big\}|x_{3,4}, \\
z^{1,2}_{n-1}z^{2,3}z^{1,4}|1 \mapsto{}& \Big\{z^{1,2}_{n-2}z^{1,3}z^{2,4} + \chi_{n}\sum_{s=1}^{\frac{n}{2}} z^{1,3}_{n-2s+1}z^{2,4}_{2s-1}\Big\}|x_{1,2} + \{z^{1,2}_{n-2}z^{2,3}z^{3,4} + \chi_{n}z^{2,3}_{n-1}z^{3,4}\}|x_{1,3}
\end{aligned}$$

$$
\begin{aligned}
&- \{z_{n-1}^{1,2} z^{1,4} + z_{n-3}^{1,2} z_2^{1,3} z^{1,4} + \chi_{n+1} z_{n-1}^{1,3} z^{1,4}\}|x_{2,3} \\
&+ \{z_{n-1}^{1,2} z^{2,3} + z_{n-3}^{1,2} z^{2,3} z_2^{1,4} + \chi_{n+1} z_{n-2}^{2,3} z_2^{2,4}\}|x_{1,4} - \{z_{n-2}^{1,2} z^{1,3} z^{1,4} + \chi_n z_{n-1}^{1,3} z^{1,4}\}|x_{2,4} \\
&- \left\{z_{n-2}^{1,2} z^{2,3} z^{1,4} + \chi_n \sum_{s=1}^{\frac{n}{2}} z_{n-2s+1}^{2,3} z_{2s-1}^{1,4}\right\}|x_{3,4},
\end{aligned}
$$

$$
\begin{aligned}
z_{n-1}^{1,2} z^{2,3} z^{2,4}|1 \mapsto &\{z_{n-2}^{1,2} z^{1,3} z^{1,4} + \chi_n z_{n-1}^{1,3} z^{1,4}\}|x_{1,2} + \{z_{n-2}^{1,2} z^{2,3} z^{2,4} + \chi_n z_{n-1}^{2,3} z^{2,4}\}|x_{1,3} \\
&- \left\{z_{n-3}^{1,2} z_2^{1,3} z^{3,4} + \chi_{n+1} z_{n-1}^{1,3} z^{3,4} + \sum_{s=1}^{\lfloor \frac{n}{2} \rfloor} z_{n-2s+1}^{1,2} z_{2s-1}^{3,4}\right\}|x_{2,3} \\
&+ \left\{z_{n-2}^{1,2} z^{1,3} z^{2,4} + \chi_n \sum_{s=1}^{\frac{n}{2}} z_{n-2s+1}^{1,3} z_{2s-1}^{2,4}\right\}|x_{1,4} \\
&+ \left\{z_{n-1}^{1,2} z^{2,3} + z_{n-3}^{1,2} z^{2,3} z_2^{1,4} + \chi_{n+1} \sum_{s=1}^{\frac{n-1}{2}} z_{n-2s}^{2,3} z_{2s}^{1,4}\right\}|x_{2,4} \\
&+ \left\{z_{n-1}^{1,2} z^{2,4} + z_{n-3}^{1,2} z_2^{1,3} z^{2,4} + \chi_{n+1} \sum_{s=1}^{\frac{n-1}{2}} z_{n-2s+1}^{1,3} z_{2s-1}^{2,4}\right\}|x_{3,4},
\end{aligned}
$$

$$
\begin{aligned}
z_{n-1}^{1,2} z^{2,3} z^{3,4}|1 \mapsto &\{z_{n-2}^{1,2} z^{1,3} z^{3,4} + \chi_n z_{n-1}^{1,3} z^{3,4}\}|x_{1,2} + \left\{z_{n-2}^{1,2} z^{2,3} z^{1,4} + \chi_n \sum_{s=1}^{\frac{n}{2}} z_{n-2s+1}^{2,3} z_{2s-1}^{1,4}\right\}|x_{1,3} \\
&- \left\{z_{n-1}^{1,2} z^{2,4} + z_{n-3}^{1,2} z_2^{1,3} z^{2,4} + \chi_{n+1} \sum_{s=1}^{\frac{n-1}{2}} z_{n-2s+1}^{1,3} z_{2s-1}^{2,4}\right\}|x_{2,3} \\
&+ \{z_{n-2}^{1,2} z^{2,3} z^{3,4} + \chi_n z_{n-1}^{2,3} z^{3,4}\}|x_{1,4} \\
&- \left\{z_{n-3}^{1,2} z_2^{1,3} z^{3,4} + \chi_{n+1} z_{n-1}^{1,3} z^{3,4} + \sum_{s=1}^{\lfloor \frac{n}{2} \rfloor} z_{n-2s+1}^{1,2} z_{2s-1}^{3,4}\right\}|x_{2,4} \\
&+ \left\{z_{n-1}^{1,2} z^{2,3} + z_{n-3}^{1,2} z^{2,3} z_2^{1,4} + \chi_{n+1} \sum_{s=1}^{\frac{n-1}{2}} z_{n-2s}^{2,3} z_{2s}^{1,4}\right\}|x_{3,4},
\end{aligned}
$$

$$
z_{n-1}^{1,2} z_2^{1,4}|1 \mapsto \{z_{n-2}^{1,2} z_2^{1,4} + \chi_n (z_n^{1,4} + z_n^{2,4})\}|x_{1,2} + z_{n-1}^{1,2} z^{1,4}|x_{1,4} + z_{n-1}^{1,2} z^{2,4}|x_{2,4}, \tag{6.1.7}
$$

$$
\begin{aligned}
z_{n-2}^{1,2} z_2^{1,3} z^{1,4}|1 \mapsto &-\left\{z_{n-3}^{1,2} z_2^{1,3} z^{2,4} + \chi_{n+1}\left(z_{n-1}^{2,3} z^{2,4} + \sum_{s=1}^{\frac{n-1}{2}} z_{n-2s+1}^{1,3} z_{2s-1}^{2,4}\right)\right\}|x_{1,2} - z_{n-2}^{1,2} z^{1,3} z^{3,4}|x_{1,3} \\
&- z_{n-2}^{1,2} z^{2,3} z^{1,4}|x_{2,3} \\
&+ \left\{z_{n-2}^{1,2} z_2^{1,3} + z_{n-4}^{1,2} z_2^{1,3} z_2^{1,4} + \chi_n \left(z_{n-2}^{2,3} z_2^{2,4} + \sum_{s=1}^{\frac{n-2}{2}} z_{n-2s}^{1,3} z_{2s}^{2,4}\right) + \sum_{s=1}^{\lfloor \frac{n-1}{2} \rfloor} z_{n-2s}^{1,2} z_{2s}^{3,4}\right\}|x_{1,4} \\
&+ \left\{z_{n-3}^{1,2} z_2^{1,3} z^{1,4} + \chi_{n+1}\left(z_{n-1}^{1,3} z^{1,4} + \sum_{s=1}^{\frac{n-1}{2}} z_{n-2s+1}^{2,3} z_{2s-1}^{1,4}\right)\right\}|x_{2,4} + z_{n-2}^{1,2} z^{1,3} z^{1,4}|x_{3,4},
\end{aligned}
$$

$$
\begin{aligned}
z_{n-2}^{1,2} z_2^{1,3} z^{2,4}|1 \mapsto &-\left\{z_{n-3}^{1,2} z_2^{1,3} z^{1,4} + \chi_{n+1}\left(z_{n-1}^{1,3} z^{1,4} + \sum_{s=1}^{\frac{n-1}{2}} z_{n-2s+1}^{2,3} z_{2s-1}^{1,4}\right)\right\}|x_{1,2} - z_{n-2}^{1,2} z^{1,3} z^{2,4}|x_{1,3} \\
&- z_{n-2}^{1,2} z^{2,3} z^{3,4}|x_{2,3} - \left\{z_{n-3}^{1,2} z_2^{1,3} z^{2,4} + \chi_{n+1}\left(z_{n-1}^{2,3} z^{2,4} + \sum_{s=1}^{\frac{n-1}{2}} z_{n-2s+1}^{1,3} z_{2s-1}^{2,4}\right)\right\}|x_{1,4} \\
&+ \left\{z_{n-2}^{1,2} z_2^{1,3} + z_{n-4}^{1,2} z_2^{1,3} z_2^{1,4} + \chi_n \left(z_{n-2}^{1,3} z_2^{1,4} + \sum_{s=1}^{\frac{n-2}{2}} z_{n-2s}^{2,3} z_{2s}^{1,4}\right) + \sum_{s=1}^{\lfloor \frac{n-1}{2} \rfloor} z_{n-2s}^{1,2} z_{2s}^{3,4}\right\}|x_{2,4} \\
&+ z_{n-2}^{1,2} z^{2,3} z^{2,4}|x_{3,4},
\end{aligned}
$$

$$
\begin{aligned}
z_{n-2}^{1,2} z_2^{1,3} z^{3,4}|1 \mapsto &-\{z_{n-3}^{1,2} z_2^{1,3} z^{3,4} + \chi_{n+1}(z_{n-1}^{1,3} z^{3,4} + z_{n-1}^{2,3} z^{3,4})\}|x_{1,2} - z_{n-2}^{1,2} z^{1,3} z^{1,4}|x_{1,3} \\
&- z_{n-2}^{1,2} z^{2,3} z^{2,4}|x_{2,3} - z_{n-2}^{1,2} z^{1,3} z^{3,4}|x_{1,4} - z_{n-2}^{1,2} z^{2,3} z^{3,4}|x_{2,4} \\
&+ \left\{z_{n-2}^{1,2} z_2^{1,3} + z_{n-2}^{1,2} z_2^{1,4} + z_{n-4}^{1,2} z_2^{1,3} z_2^{1,4} + \chi_n \sum_{s=1}^{\frac{n-2}{2}} (z_{n-2s}^{1,3} z_{2s}^{2,4} + z_{n-2s}^{2,3} z_{2s}^{1,4})\right\}|x_{3,4},
\end{aligned}
$$

$$
\begin{aligned}
z^{1,2}_{n-2}z^{1,3}z^{1,4}_{2}|1 \mapsto &-\Big\{z^{1,2}_{n-3}z^{2,3}z^{1,4}_{2} + \chi_{n+1}\Big(z^{2,3}_{n-2}z^{2,4}_{2} + \sum_{s=1}^{\frac{n-1}{2}} z^{2,3}_{n-2s}z^{1,4}_{2s}\Big)\Big\}|x_{1,2} \\
&+ \Big\{z^{1,2}_{n-2}z^{1,4}_{2} + z^{1,2}_{n-4}z^{1,3}_{2}z^{1,4}_{2} + \chi_{n}\Big(z^{2,3}_{n-2}z^{2,4}_{2} + \sum_{s=1}^{\frac{n-2}{2}} z^{2,3}_{n-2s}z^{1,4}_{2s}\Big) + \sum_{s=1}^{\lfloor\frac{n-1}{2}\rfloor} z^{1,2}_{n-2s}z^{3,4}_{2s}\Big\}|x_{1,3} \\
&+ \Big\{z^{1,2}_{n-3}z^{1,3}z^{1,4}_{2} + \chi_{n+1}\Big(z^{1,3}_{n-2}z^{1,4}_{2} + \sum_{s=1}^{\frac{n-1}{2}} z^{1,3}_{n-2s}z^{2,4}_{2s}\Big)\Big\}|x_{2,3} \\
&+ z^{1,2}_{n-2}z^{1,3}z^{1,4}|x_{1,4} + z^{1,2}_{n-2}z^{1,3}z^{2,4}|x_{2,4} + z^{1,2}_{n-2}z^{1,3}z^{3,4}|x_{3,4}, \\
z^{1,2}_{n-2}z^{2,3}z^{1,4}_{2}|1 \mapsto &-\Big\{z^{1,2}_{n-3}z^{1,3}z^{1,4}_{2} + \chi_{n+1}\Big(z^{1,3}_{n-2}z^{1,4}_{2} + \sum_{s=1}^{\frac{n-1}{2}} z^{1,3}_{n-2s}z^{2,4}_{2s}\Big)\Big\}|x_{1,2} \\
&- \Big\{z^{1,2}_{n-3}z^{2,3}z^{1,4}_{2} + \chi_{n+1}\Big(z^{2,3}_{n-2}z^{2,4}_{2} + \sum_{s=1}^{\frac{n-1}{2}} z^{2,3}_{n-2s}z^{1,4}_{2s}\Big)\Big\}|x_{1,3} \\
&+ \Big\{z^{1,2}_{n-2}z^{1,4}_{2} + z^{1,2}_{n-4}z^{1,3}_{2}z^{1,4}_{2} + \chi_{n}\Big(z^{1,3}_{n-2}z^{1,4}_{2} + \sum_{s=1}^{\frac{n-2}{2}} z^{1,3}_{n-2s}z^{2,4}_{2s}\Big) + \sum_{s=1}^{\lfloor\frac{n-1}{2}\rfloor} z^{1,2}_{n-2s}z^{3,4}_{2s}\Big\}|x_{2,3} \\
&+ z^{1,2}_{n-2}z^{2,3}z^{1,4}|x_{1,4} + z^{1,2}_{n-2}z^{2,3}z^{2,4}|x_{2,4} + z^{1,2}_{n-2}z^{2,3}z^{3,4}|x_{3,4}, \\
z^{1,2}_{n-3}z^{1,3}_{2}z^{1,4}_{2}|1 \mapsto &\Big\{z^{1,2}_{n-4}z^{1,3}_{2}z^{1,4}_{2} + \chi_{n}\Big(z^{1,3}_{n-2}z^{1,4}_{2} + z^{2,3}_{n-2}z^{2,4}_{2} + \sum_{s=1}^{\frac{n-2}{2}} (z^{1,3}_{n-2s}z^{2,4}_{2s} + z^{2,3}_{n-2s}z^{1,4}_{2s})\Big)\Big\}|x_{1,2} \\
&+ z^{1,2}_{n-3}z^{1,3}z^{1,4}_{2}|x_{1,3} + z^{1,2}_{n-3}z^{2,3}z^{1,4}_{2}|x_{2,3} + z^{1,2}_{n-3}z^{1,3}_{2}z^{1,4}|x_{1,4} + z^{1,2}_{n-3}z^{1,3}_{2}z^{2,4}|x_{2,4} \\
&+ z^{1,2}_{n-3}z^{1,3}_{2}z^{3,4}|x_{3,4}, \\
z^{1,3}_{n}z^{1,4}|1 \mapsto &-\{z^{1,3}_{n-1}z^{3,4} + \chi_{n+1}z^{3,4}_{n}\}|x_{1,3} + \{z^{1,3}_{n} + z^{1,3}_{n-2}z^{1,4}_{2} + \chi_{n}z^{3,4}_{n}\}|x_{1,4} \\
&+ \{z^{1,3}_{n-1}z^{1,4} + \chi_{n+1}z^{1,4}_{n}\}|x_{3,4}, \\
z^{1,3}_{n}z^{3,4}|1 \mapsto &-\{z^{1,3}_{n-1}z^{1,4} + \chi_{n+1}z^{1,4}_{n}\}|x_{1,3} - \{z^{1,3}_{n-1}z^{3,4} + \chi_{n+1}z^{3,4}_{n}\}|x_{1,4} \\
&+ \{z^{1,3}_{n} + z^{1,3}_{n-2}z^{1,4}_{2} + \chi_{n}z^{1,4}_{n}\}|x_{3,4}, \\
z^{1,3}_{n-1}z^{1,4}_{2}|1 \mapsto &\{z^{1,3}_{n-2}z^{1,4}_{2} + \chi_{n}(z^{1,4}_{n} + z^{3,4}_{n})\}|x_{1,3} + z^{1,3}_{n-1}z^{1,4}|x_{1,4} + z^{1,3}_{n-1}z^{3,4}|x_{3,4}, \\
z^{2,3}_{n}z^{2,4}|1 \mapsto &-\{z^{2,3}_{n-1}z^{3,4} + \chi_{n+1}z^{3,4}_{n}\}|x_{2,3} + \{z^{2,3}_{n} + z^{2,3}_{n-2}z^{2,4}_{2} + \chi_{n}z^{3,4}_{n}\}|x_{2,4} \\
&+ \{z^{2,3}_{n-1}z^{2,4} + \chi_{n+1}z^{2,4}_{n}\}|x_{3,4}, \\
z^{2,3}_{n}z^{3,4}|1 \mapsto &-\{z^{2,3}_{n-1}z^{2,4} + \chi_{n+1}z^{2,4}_{n}\}|x_{2,3} - \{z^{2,3}_{n-1}z^{3,4} + \chi_{n+1}z^{3,4}_{n}\}|x_{2,4} \\
&+ \{z^{2,3}_{n} + z^{2,3}_{n-2}z^{2,4}_{2} + \chi_{n}z^{2,4}_{n}\}|x_{3,4}, \\
z^{2,3}_{n-1}z^{2,4}_{2}|1 \mapsto &\{z^{2,3}_{n-2}z^{2,4}_{2} + \chi_{n}(z^{2,4}_{n} + z^{3,4}_{n})\}|x_{2,3} + z^{2,3}_{n-1}z^{2,4}|x_{2,4} + z^{2,3}_{n-1}z^{3,4}|x_{3,4}.
\end{aligned}
$$

For a quadratic right A-module M, let $(\mathrm{K}_\bullet(M),d_\bullet(M))$ be the Koszul complex of M in the category of bounded-below graded right A-modules, and $\epsilon' : \mathrm{K}_0(M) \to M$ the canonical projection. Recall that $\mathrm{K}_n(M) = (M^{!}_{-n})^* \otimes A$ for $n \in \mathbb{N}_0$, the map ϵ' is the surjective map $W \otimes A \to M$, and the differential $d_n(M) : \mathrm{K}_n(M) \to \mathrm{K}_{n-1}(M)$ is given by $d_n(M)(u|v) = \sum_{(i,j)\in\mathscr{J}} y_{i,j}u|x_{i,j}v$ for $u \in (M^{!}_{-n})^*$, $v \in A$ and $n \in \mathbb{N}$. Recall also that the complex

$$
\mathrm{K}_1(M) \xrightarrow{d_1(M)} \mathrm{K}_0(M) \xrightarrow{\epsilon'} M \longrightarrow 0
$$

is always exact for a quadratic module M. As usual, we can consider the Koszul complex as a complex indexed by $\mathbb{Z}$, with $\mathrm{K}_n(M) = 0$ for all $n \in \mathbb{Z}_{\leqslant -1}$, and $d_n(M) = 0$ for all $n \in \mathbb{Z}_{\leqslant 0}$. Notice that the notation $(K_\bullet,d_\bullet)$ is exactly $(\mathrm{K}_\bullet(\Bbbk),d_\bullet(\Bbbk))$.

Let $M = \oplus_{n\in\mathbb{Z}} M_n$ be a graded right A-module such that $\dim(M_n)$ is finite for all $n \in \mathbb{Z}$. Given $j \in \mathbb{Z}$, we denote by $M(j)$ the same underlying module with shifted (internal) grading given by $M(j)_i = M_{j+i}$ for $i \in \mathbb{Z}$. We remark that a **morphism of graded right A-modules** $f : M \to N$ is a homogeneous A-linear map of degree zero. Moreover, for a nonzero graded module M over A, if there exist integers $s \leqslant t$ such that $\dim(M_n) = 0$ for all $n \in \mathbb{Z} \setminus [\![s,t]\!]$ and $\dim(M_s)\cdot\dim(M_t) \neq 0$, then we say that the **dimension vector** of M is $(\dim(M_s),\ldots,\dim(M_t))$.

6.1.2 Resolving datum

We will now define some quadratic A-modules M^i for $i \in [\![1,3]\!]$. Let M^1 be the A-module generated by two homogeneous elements a_1,a_2 of degree zero, subject to the following 6 relations

$$a_1x_{1,2} + a_2x_{1,2}, a_1x_{1,3}, a_2x_{2,3}, a_2x_{1,4}, a_1x_{2,4}, a_1x_{3,4} + a_2x_{3,4}. \tag{6.1.8}$$

Let M^2 be the A-module generated by the set $\{h_i \mid i \in [\![1,7]\!]\}$ of seven homogeneous elements of degree zero, subject to the following 24 relations

$$\begin{aligned}&h_1x_{1,2}, h_1x_{1,3}, h_1x_{2,3}, h_2x_{1,2}, h_2x_{1,4}, h_2x_{2,4}, h_3x_{1,3}, h_3x_{1,4}, h_3x_{3,4}, h_4x_{2,3}, h_4x_{2,4}, h_4x_{3,4},\\ &h_1x_{2,4} - h_3x_{2,4} - h_5x_{1,3}, h_2x_{1,3} - h_4x_{1,3} + h_5x_{2,4}, h_5x_{3,4} - h_6x_{1,2}, h_1x_{1,4} - h_4x_{1,4} + h_6x_{2,3},\\ &h_2x_{2,3} - h_3x_{2,3} - h_6x_{1,4}, h_5x_{1,2} + h_6x_{3,4}, h_1x_{3,4} + h_2x_{3,4} + h_7x_{1,2}, h_6x_{2,4} + h_7x_{1,3},\\ &h_5x_{1,4} + h_7x_{2,3}, h_5x_{2,3} - h_7x_{1,4}, h_6x_{1,3} - h_7x_{2,4}, h_3x_{1,2} + h_4x_{1,2} + h_7x_{3,4}.\end{aligned} \tag{6.1.9}$$

Finally, let M^3 be the A-module generated by the set $\{e_i \mid i \in [\![1,8]\!]\}$ of eight homogeneous elements of degree zero, subject to the following 24 relations

$$\begin{aligned}&e_1x_{1,2} + e_2x_{3,4}, e_1x_{3,4} - e_2x_{1,2}, e_3x_{1,2} - e_4x_{3,4}, e_3x_{3,4} + e_4x_{1,2}, e_4x_{1,3} + e_2x_{2,4}, e_4x_{2,4} - e_2x_{1,3},\\ &e_3x_{1,3} + e_1x_{2,4}, e_3x_{2,4} - e_1x_{1,3}, e_1x_{2,3} - e_4x_{1,4}, e_1x_{1,4} + e_4x_{2,3}, e_3x_{2,3} - e_2x_{1,4}, e_3x_{1,4} + e_2x_{2,3},\\ &e_5x_{1,2}, e_5x_{1,3}, e_5x_{2,3}, e_6x_{1,2}, e_6x_{1,4}, e_6x_{2,4}, e_7x_{1,3}, e_7x_{1,4}, e_7x_{3,4}, e_8x_{2,3}, e_8x_{2,4}, e_8x_{3,4}.\end{aligned} \tag{6.1.10}$$

Since the previous modules are finite dimensional, we use GAP to obtain a homogeneous $\Bbbk$-basis of M^i, and in particular, the Hilbert series of M^i, for $i \in [\![1,3]\!]$. See Appendix A.2 for a basis of M^1.

Fact 6.1.4. *Given $i \in [\![1,3]\!]$, the Hilbert series $h_{M^i}(t)$ of the quadratic A-module M^i introduced in the previous paragraph is given by*

$$\begin{aligned}h_{M^1}(t) &= 2 + 6t + 11t^2 + 12t^3 + 11t^4 + 6t^5 + 2t^6,\\ h_{M^2}(t) &= 7 + 18t + 32t^2 + 42t^3 + 40t^4 + 30t^5 + 16t^6 + 6t^7 + 1t^8,\\ h_{M^3}(t) &= 8 + 24t + 48t^2 + 72t^3 + 80t^4 + 72t^5 + 48t^6 + 24t^7 + 8t^8.\end{aligned}$$

Theorem 6.1.5. *Let $\mathscr{M} = \{M^0 = \Bbbk, M^1, M^2, M^3\}$ be the family of quadratic A-modules introduced in the first paragraph of this subsection, and let $\hbar : [\![0, N]\!]^2 \times \mathbb{N}^2 \to \mathbb{N}_0^2$ be the map given by*

$$\begin{aligned}\hbar(0,2,3,6) &= \hbar(0,0,3,6) = \hbar(1,2,1,4) = (1,0),\\ \hbar(0,0,3,8) &= \hbar(0,1,4,8) = \hbar(0,0,5,16) = \hbar(1,0,1,6) = \hbar(1,0,1,8) = \hbar(2,0,1,4)\\ &= \hbar(2,0,1,6) = \hbar(2,1,2,6) = \hbar(2,3,3,6) = \hbar(3,3,3,6) = (0,1),\end{aligned}$$

and $\hbar(i,j,k,\ell)$ vanishes on other (i,j,k,ℓ). Then this gives a connected resolving datum on A, whose associated resolving quiver is given in Figure 6.1, where we denote by ${}_j\alpha_i^{d',d''}$ the unique arrow from M^i to M^j having bidegree (d',d''). In this case, the strict partial order on the arrows is given by ${}_0\alpha_0^{4,8} < {}_0\alpha_0^{4,6}, {}_2\alpha_0^{4,6}$, and ${}_0\alpha_1^{2,6}, {}_0\alpha_1^{2,8} < {}_2\alpha_1^{2,4}$. The arrows ${}_1\alpha_0^{5,8}$ and ${}_1\alpha_2^{3,6}$ of odd difference degrees appear in red.

Proof. Lemma 6.1.6, 6.1.7, 6.1.8 and 6.1.9, Corollary 6.1.11, Proposition 6.1.17, and Corollary 6.1.24 and 6.1.29 show that the homology of Koszul complex of $\Bbbk$ and M^i for $i \in [\![1,3]\!]$ is given by

$$\mathrm{H}_n(\Bbbk) \cong \begin{cases} M^1(-8), & \text{if } n = 4,\\ \Bbbk(-16), & \text{if } n = 5,\\ 0, & \text{if } n \in \mathbb{N} \setminus \{3, 4, 5\},\end{cases}$$

and

$$\mathrm{H}_n(M^1) = 0, \text{ if } n \geqslant 2,$$

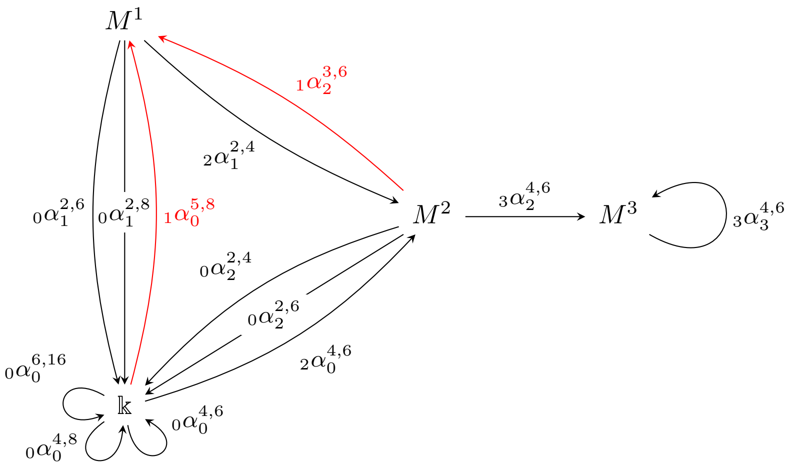


Figure 6.1: Resolving quiver of FK(4).

and

$$\mathrm{H}_n(M^2) \cong \begin{cases} \Bbbk(-4) \oplus \Bbbk(-6), & \text{if } n = 1, \\ M^1(-6), & \text{if } n = 2, \\ M^3(-6), & \text{if } n = 3, \\ 0, & \text{if } n \geqslant 4, \end{cases}$$

together with

$$\mathrm{H}_n(M^3) \cong \begin{cases} M^3(-6), & \text{if } n = 3, \\ 0, & \text{if } n \in \mathbb{N} \setminus \{3\}, \end{cases}$$

as well as the non-split short exact sequences of graded right A-modules

$$0 \to M^2(-6) \oplus \Bbbk(-6) \to \mathrm{H}_3(\Bbbk) \to \Bbbk(-8) \to 0, \tag{6.1.11}$$

and

$$0 \to M^2(-4) \to \mathrm{H}_1(M^1) \to \Bbbk(-6) \oplus \Bbbk(-8) \to 0. \tag{6.1.12}$$

The theorem is thus proved. □

For the quiver in Figure 6.1 we have an explicit description of the paths starting at $\Bbbk$. Here we write a path $(a_1, \dots, a_n)$ by $a_n \cdots a_1$.

(P.1) the set of paths ending at $\Bbbk$ form the free monoid $M_\Bbbk$ generated by the following 11 types of generators

$$\Big\{ {}_0\alpha_0^{4,6}, {}_0\alpha_0^{4,8}, {}_0\alpha_2^{2,4}({}_2\alpha_1^{2,4}\,{}_1\alpha_2^{3,6})^p\,{}_2\alpha_0^{4,6}, {}_0\alpha_2^{2,6}({}_2\alpha_1^{2,4}\,{}_1\alpha_2^{3,6})^p\,{}_2\alpha_0^{4,6}, {}_0\alpha_1^{2,6}\,{}_1\alpha_2^{3,6}({}_2\alpha_1^{2,4}\,{}_1\alpha_2^{3,6})^p\,{}_2\alpha_0^{4,6}, \\ {}_0\alpha_1^{2,8}\,{}_1\alpha_2^{3,6}({}_2\alpha_1^{2,4}\,{}_1\alpha_2^{3,6})^p\,{}_2\alpha_0^{4,6}, {}_0\alpha_1^{2,6}({}_1\alpha_2^{3,6}\,{}_2\alpha_1^{2,4})^p\,{}_1\alpha_0^{5,8}, {}_0\alpha_1^{2,8}({}_1\alpha_2^{3,6}\,{}_2\alpha_1^{2,4})^p\,{}_1\alpha_0^{5,8}, \\ {}_0\alpha_2^{2,4}\,{}_2\alpha_1^{2,4}({}_1\alpha_2^{3,6}\,{}_2\alpha_1^{2,4})^p\,{}_1\alpha_0^{5,8}, {}_0\alpha_2^{2,6}\,{}_2\alpha_1^{2,4}({}_1\alpha_2^{3,6}\,{}_2\alpha_1^{2,4})^p\,{}_1\alpha_0^{5,8}, {}_0\alpha_0^{6,16} \mid p \in \mathbb{N}_0 \Big\};$$

(P.2) the set of paths from $\Bbbk$ to M^1 are given by

$$\Big\{ ({}_1\alpha_2^{3,6}\,{}_2\alpha_1^{2,4})^p\,{}_1\alpha_0^{5,8}\omega, {}_1\alpha_2^{3,6}({}_2\alpha_1^{2,4}\,{}_1\alpha_2^{3,6})^p\,{}_2\alpha_0^{4,6}\omega \mid \omega \in M_\Bbbk, p \in \mathbb{N}_0 \Big\};$$

(P.3) the set of paths from $\Bbbk$ to M^2 are given by

$$\Big\{ {}_2\alpha_1^{2,4}({}_1\alpha_2^{3,6}\,{}_2\alpha_1^{2,4})^p\,{}_1\alpha_0^{5,8}\omega, ({}_2\alpha_1^{2,4}\,{}_1\alpha_2^{3,6})^p\,{}_2\alpha_0^{4,6}\omega \mid \omega \in M_\Bbbk, p \in \mathbb{N}_0 \Big\};$$

(P.4) the set of paths from $\Bbbk$ to M^3 are given by

$$\Big\{({}_3\alpha_3^{4,6})^q\,{}_3\alpha_2^{4,6}\,{}_2\alpha_1^{2,4}({}_1\alpha_2^{3,6}\,{}_2\alpha_1^{2,4})^p\,{}_1\alpha_0^{5,8}\omega,\ ({}_3\alpha_3^{4,6})^q\,{}_3\alpha_2^{4,6}({}_2\alpha_1^{2,4}\,{}_1\alpha_2^{3,6})^p\,{}_2\alpha_0^{4,6}\omega \mid \omega\in M_{\Bbbk}, p,q\in\mathbb{N}_0\Big\}.$$

Note also that ${}_0\alpha_0^{4,6}, {}_2\alpha_0^{4,6} < {}_1\alpha_0^{5,8}$, and ${}_0\alpha_2^{2,4} < {}_1\alpha_2^{3,6}$, as well as

$$\begin{aligned}\mathrm{dfdeg}({}_0\alpha_0^{4,6}) = \mathrm{dfdeg}({}_2\alpha_0^{4,6}) = 2 = \mathrm{dfdeg}({}_1\alpha_0^{5,8}) - 1,\\ \mathrm{dfdeg}({}_0\alpha_2^{2,4}) = 2 = \mathrm{dfdeg}({}_1\alpha_2^{3,6}) - 1.\end{aligned}$$

6.1.2.1 Computation of the Koszul complex of some quadratic modules using GAP

Recall that $(K_\bullet, d_\bullet)$ is the Koszul complex of the trivial module $\Bbbk$ in the category of bounded below graded right A-modules. Let $K_{n,m} = (A^!_{-n})^* \otimes A_m$, $d_{n,m} = d_n|_{K_{n,m}} : K_{n,m} \to K_{n-1,m+1}$, $B_{n,m} = \mathrm{Im}(d_{n+1,m-1})$, $D_{n,m} = \mathrm{Ker}(d_{n,m})$, $H_{n,m} = D_{n,m}/B_{n,m}$ for $n \in \mathbb{N}_0$ and $m \in [\![0,12]\!]$. Let $H_n = \oplus_{m\in[\![0,12]\!]} H_{n,m}$ for $n \in \mathbb{N}_0$. For a quadratic A-module M, recall that $(\mathrm{K}_\bullet(M), d_\bullet(M))$ is the Koszul complex of M. Let

$$B^M_{n,m} = \mathrm{Im}(d_{n+1}(M)|_{(M^!_{-(n+1)})^*\otimes A_{m-1}}),\quad D^M_{n,m} = \mathrm{Ker}(d_n(M)|_{(M^!_{-n})^*\otimes A_m})$$

and $\mathrm{H}_{n,m}(M) = D^M_{n,m}/B^M_{n,m}$ for $n \in \mathbb{N}_0$ and $m \in [\![0,12]\!]$. Let $\mathrm{H}_n(M) = \oplus_{m\in[\![0,12]\!]} \mathrm{H}_{n,m}(M)$ for $n \in \mathbb{N}_0$. Using GAP, we can compute the dimension of the $B^{M^i}_{n,m}$ for n less than some arbitrary positive integer, $m \in [\![1,12]\!]$ and $i \in [\![0,3]\!]$ by using the code in Appendix A.4 together with the following simple routine.

```
for i in [0..3] do
    for j1 in [0..9] do
        for j2 in [1..12] do
            Print(i, " ", j1, " ", j2, " ", RankMat(FF(i,j1,j2)), "\n");
        od;
    od;
od;
```

For the rest of the section, we will only indicate the extra code added to the one in Appendix A.4 for every computation, and, for the reader's convenience, we will often indicate the output of many of the intermediate commands in the corresponding successive line and preceded by a pound sign #.

By above GAP code, the dimension of $B_{n,m}$ for $n \in [\![0,9]\!]$ and $m \in [\![0,12]\!]$ is given in Table 6.1.1.

n \ m	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	6	19	42	71	96	106	96	71	42	19	6	1
1	0	17	72	181	330	470	540	505	384	233	108	35	6
2	0	30	142	384	737	1092	1297	1248	974	606	288	96	17
3	0	38	186	515	1020	1550	1890	1866	1494	956	468	162	30
4	0	42	207	576	1146	1752	2151	2142	1731	1122	558	198	38
5	0	45	222	618	1230	1881	2310	2301	1860	1206	600	213	42
6	0	48	237	660	1314	2010	2469	2460	1989	1290	642	228	45
7	0	51	252	702	1398	2139	2628	2619	2118	1374	684	243	48
8	0	54	267	744	1482	2268	2787	2778	2247	1458	726	258	51
9	0	57	282	786	1566	2397	2946	2937	2376	1542	768	273	54

Table 6.1.1: Dimension of $B_{n,m}$.

By $\dim D_{n,m} = \dim K_{n,m} - \dim B_{n-1,m+1}$ and Table 6.1.1, we get the dimension of $D_{n,m}$ for $n \in [\![0,5]\!]$ and $m \in [\![0,12]\!]$ in Table 6.1.2.

n \ m	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	6	19	42	71	96	106	96	71	42	19	6	1
1	0	17	72	181	330	470	540	505	384	233	108	35	6
2	0	30	142	384	737	1092	1297	1248	974	606	288	96	17
3	0	38	186	523	1038	1583	1932	1906	1524	972	474	163	30
4	0	42	207	576	1148	1758	2162	2154	1742	1128	560	198	38
5	0	45	222	618	1230	1881	2310	2301	1860	1206	600	214	42

Table 6.1.2: Dimension of $D_{n,m}$.

By $\dim H_{n,m} = \dim D_{n,m} - \dim B_{n,m}$, we get the dimension of $H_{n,m}$ for $n \in [\![0,5]\!]$ and $m \in [\![0,12]\!]$ in Table 6.1.3. The dimensions that are not listed in the following table are zeros.

n \ m	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1												
3				8	18	33	42	40	30	16	6	1	
4					2	6	11	12	11	6	2		
5												1	

Table 6.1.3: Dimension of $H_{n,m}$.

By above GAP code, we obtain the dimension of $B^{M^1}_{n,m}$ for $n \in [\![0,9]\!]$ and $m \in [\![1,12]\!]$. The dimension of homology of the Koszul complex of M^1 for $n \in [\![1,9]\!]$ and $m \in [\![0,12]\!]$ is given in Table 6.1.4. The dimensions that are not listed in the following table are zeros.

n \ m	0	1	2	3	4	5	6	7	8	9	10	11	12
1				7	18	33	42	41	30	16	6	1	

Table 6.1.4: Dimension of $\mathrm{H}_{n,m}(M^1)$.

By above GAP code, we obtain the dimension of $B^{M^2}_{n,m}$ for $n \in [\![0,9]\!]$ and $m \in [\![1,12]\!]$. Then the dimension of $\mathrm{H}_{n,m}(M^2)$ for $n \in [\![1,9]\!]$ and $m \in [\![0,12]\!]$ is given by Table 6.1.5. The dimensions that are not listed in the following table are zeros.

n \ m	0	1	2	3	4	5	6	7	8	9	10	11	12
1				1		1							
2					2	6	11	12	11	6	2		
3				8	24	48	72	80	72	48	24	8	

Table 6.1.5: Dimension of $\mathrm{H}_{n,m}(M^2)$.

By above GAP code, we obtain the dimension of $B^{M^3}_{n,m}$ for $n \in [\![0,9]\!]$ and $m \in [\![1,12]\!]$. Then the dimension of $\mathrm{H}_{n,m}(M^3)$ is given by Table 6.1.6 for $n \in [\![1,9]\!]$ and $m \in [\![0,12]\!]$. The dimensions that are not listed in the following table are zeros.

n \ m	0	1	2	3	4	5	6	7	8	9	10	11	12
3				8	24	48	72	80	72	48	24	8	

Table 6.1.6: Dimension of $\mathrm{H}_{n,m}(M^3)$.

Lemma 6.1.6. *We have* $\mathrm{H}_4(\Bbbk) \cong M^1(-8)$, $\mathrm{H}_5(\Bbbk) \cong \Bbbk(-16)$ *and the non-split short exact sequence* (6.1.11) *of graded A-modules.*

Proof. The isomorphism $\mathrm{H}_5(\Bbbk) \cong \Bbbk(-16)$ follows immediately from Table 6.1.3. Recall that we write H_n instead of $\mathrm{H}_n(\Bbbk)$ for $n \in \mathbb{N}$ to simplify the notation.

Let us prove the isomorphism $\mathrm{H}_4(\Bbbk) \cong M^1(-8)$. The following GAP code shows that the dimension vector of the submodule of H_4 generated by two basis elements a'_1, a'_2 of $H_{4,4}$ is (2,6,11,12,11,6,2). So, Table 6.1.3 tells us that H_4 is generated by a'_1, a'_2 as an A-module.

```
Imm:=Im(0,4,4);;
RankMat(Imm);
# 1146
gene:=geneMH(0,4,4);;
Append(Imm,gene);
```

```
RankMat(Imm);
# 1148
Uh:=UU(gene,4);; Vh:=VV(gene,4);; Wh:=WW(gene,4);;
for r in [5..10] do
   hxr:=HXR(0,Uh,Vh,Wh,4,4,r-4);
   Im4r:=Im(0,4,r);
   Append(Im4r, hxr);
   Print(r, " ", RankMat(Im4r)-RankMat(Im(0,4,r)), "\n");
od;
# 5 6
# 6 11
# 7 12
# 8 11
# 9 6
# 10 2
```

On the other hand, it is direct to check that the generators a'_1,a'_2 of H_4 satisfy the quadratic relations (6.1.8) defining M^1. Indeed, the following code shows that the dimension of the subspace generated by $B_{4,5}$ together with the elements of the form (6.1.8) with a'_i instead of a_i coincides with the dimension of $B_{4,5}$.

```
gene:=geneMH(0,4,4);;
Uh:=UU(gene,4);; Vh:=VV(gene,4);; Wh:=WW(gene,4);;
hx:=HXR(0,Uh,Vh,Wh,4,4,1);;
cc:=0*[1..6];;
cc[1]:=hx[1]+hx[7];; cc[2]:=hx[2];; cc[3]:=hx[5];; cc[4]:=hx[6]+hx[12];;
cc[5]:=hx[9];; cc[6]:=hx[10];;
Imm:=Im(0,4,5);;
RankMat(Imm);
# 1752
Append(Imm,cc);
RankMat(Imm);
# 1752
```

Hence, there is a surjective morphism $M^1(-8) \to H_4$ of graded A-modules. Since the dimension vector of M^1 is (2,6,11,12,11,6,2) by Fact 6.1.4, we have $H_4 \cong M^1(-8)$ as graded A-modules, as claimed.

Let us now prove the existence of the short exact sequence (6.1.11). The following GAP code shows that the dimension vector of the submodule of H_3 generated by the basis elements $c'_i, i \in [\![1,8]\!]$ of $H_{3,3}$ is $(8, 18, 32, 42, 40, 30, 16, 6, 1)$.

```
Imm:=Im(0,3,3);;
RankMat(Imm);
# 515
gene:=geneMH(0,3,3);;
Append(Imm,gene);
RankMat(Imm);
# 523
Uh:=UU(gene,3);; Vh:=VV(gene,3);; Wh:=WW(gene,3);;
for r in [4..11] do
   hxr:=HXR(0,Uh,Vh,Wh,3,3,r-3);
   Im3r:=Im(0,3,r);
   Append(Im3r, hxr);
   Print(r, " ", RankMat(Im3r)-RankMat(Im(0,3,r)), "\n");
od;
# 4 18
# 5 32
# 6 42
# 7 40
# 8 30
# 9 16
# 10 6
# 11 1
```

Let M^4 be the quadratic module generated by the set $\{c_i \mid i \in [\![1,8]\!]\}$ of eight homogeneous elements of degree zero, subject to the following 30 relations

$$\begin{aligned}&c_1x_{1,2}, c_1x_{1,3}, c_1x_{2,3}, c_2x_{1,2}, c_2x_{1,4}, c_2x_{2,4}, c_3x_{1,3}, c_3x_{1,4}, c_3x_{3,4}, c_4x_{2,3}, c_4x_{2,4}, c_4x_{3,4},\\&c_5x_{1,3} - c_1x_{2,4} + c_3x_{2,4}, c_5x_{2,4} + c_2x_{1,3} - c_4x_{1,3}, c_6x_{2,3} + c_1x_{1,4} - c_4x_{1,4},\\&c_6x_{1,4} - c_2x_{2,3} + c_3x_{2,3}, c_7x_{1,2} + c_1x_{3,4} + c_2x_{3,4}, c_7x_{3,4} + c_3x_{1,2} + c_4x_{1,2}, c_5x_{1,2} + c_6x_{3,4},\\&c_5x_{3,4} - c_6x_{1,2}, c_6x_{1,3} - c_7x_{2,4}, c_6x_{2,4} + c_7x_{1,3}, c_5x_{1,4} + c_7x_{2,3}, c_5x_{2,3} - c_7x_{1,4},\\&c_8x_{1,2}, c_8x_{1,3}, c_8x_{2,3}, c_8x_{1,4}, c_8x_{2,4}, c_8x_{3,4}.\end{aligned} \tag{6.1.13}$$

Using GAP we get that the dimension vector of M^4 is $(8, 18, 32, 42, 40, 30, 16, 6, 1)$. It is direct to check that the elements $c'_i, i \in [\![1,8]\!]$ of H_3 satisfy the quadratic relations (6.1.13). Indeed, the following code shows that the dimension of the subspace generated by $B_{3,4}$ together with the elements of the form (6.1.13) with c'_i instead of c_i coincides with the dimension of $B_{3,4}$.

```
gene:=geneMH(0,3,3);;
Uh:=UU(gene,3);; Vh:=VV(gene,3);; Wh:=WW(gene,3);;
hx:=HXR(0,Uh,Vh,Wh,3,3,1);;
cc:=0*[1..30];;
cc[1]:=hx[1];; cc[2]:=hx[2];; cc[3]:=hx[3];; cc[4]:=hx[7];; cc[5]:=hx[10];;
cc[6]:=hx[11];; cc[7]:=hx[14];; cc[8]:=hx[16];; cc[9]:=hx[18];; cc[10]:=hx[21];;
cc[11]:=hx[23];; cc[12]:=hx[24];; cc[13]:=hx[5]-hx[17]-hx[26];;
cc[14]:=hx[8]-hx[20]+hx[29];; cc[15]:=hx[30]-hx[31];; cc[16]:=hx[4]-hx[22]+hx[33];;
cc[17]:=hx[9]-hx[15]-hx[34];; cc[18]:=hx[25]+hx[36];; cc[19]:=hx[6]+hx[12]+hx[37];;
cc[20]:=hx[35]+hx[38];; cc[21]:=hx[28]+hx[39];; cc[22]:=hx[27]-hx[40];;
cc[23]:=hx[32]-hx[41];; cc[24]:=hx[13]+hx[19]+hx[42];; cc[25]:=hx[43];;
cc[26]:=hx[44];; cc[27]:=hx[45];; cc[28]:=hx[46];; cc[29]:=hx[47];; cc[30]:=hx[48];;
Imm:=Im(0,3,4);;
RankMat(Imm);
# 1020
Append(Imm,cc);
RankMat(Imm);
# 1020
```

Hence, there is a morphism $M^4(-6) \to H_3$ of graded A-modules whose image is the submodule of H_3 generated by $c'_i, i \in [\![1,8]\!]$. Since the dimension vectors of M^4 and the submodule of H_3 generated by $c'_i, i \in [\![1,8]\!]$ are the same, the previous morphism is injective. Moreover, the submodule of M^4 generated by $c_i, i \in [\![1,7]\!]$ is isomorphic to M^2 via the map given by $c_i \mapsto h_i$ for $i \in [\![1,7]\!]$, and the submodule of M^4 generated by c_8 is isomorphic to the trivial A-module $\Bbbk$. It is direct to check that these submodules have trivial intersection, by degree reasons. By comparing the Hilbert series of M^4, M^2 and $\Bbbk$ we obtain the isomorphism $M^4 \cong M^2 \oplus \Bbbk$ of graded A-modules. In consequence, there is an injective morphism $M^2(-6) \oplus \Bbbk(-6) \to H_3$. By a direct dimension and grading argument using Table 6.1.3, its cokernel is exactly $\Bbbk(-8)$.

Finally, we prove that the short exact sequence (6.1.11) is non-split. Let c_i for $i \in [\![1,33]\!]$ be the basis elements of space $H_{3,5}$ and $p : \mathrm{H}_3(\Bbbk) \to \Bbbk(-8)$ the surjection in (6.1.11), satisfying that $p(\mathsf{c}_i) = 0$ for $i \in [\![1,32]\!]$, and $p(\mathsf{c}_{33}) = \mathsf{e}_1$, where e_1 is the identity element of $\Bbbk(-8)$. The short exact sequence (6.1.11) is split if and only if there exists a morphism $s : \Bbbk(-8) \to \mathrm{H}_3(\Bbbk)$ of graded A-modules such that the composition ps is the identity map. Assume that there exists such a map s. Let $m = s(\mathsf{e}_1) \in H_{3,5}$. Then m is of the form $\sum_{i=1}^{32} \lambda_i \mathsf{c}_i + \mathsf{c}_{33}$ for $\lambda_i \in \Bbbk$, and $m.x = s(\mathsf{e}_1).x = s(\mathsf{e}_1.x) = s(0) = 0$ for all $x \in A_+$. In particular, $\sum_{i=1}^{32} \lambda_i \mathsf{c}_i x_{1,2} + \mathsf{c}_{33} x_{1,2} = 0$ for some $\lambda_i \in \Bbbk$, *i.e.* $\mathsf{c}_{33}x_{1,2}$ is a linear combination of $\mathsf{c}_i x_{1,2}$ for $i \in [\![1,32]\!]$. Using GAP, we choose suitable representative elements $\mathsf{c}'_i \in D_{3,5}$ of c_i for $i \in [\![1,33]\!]$, and get that the dimension of the space spanned by $\mathsf{c}'_i x_{1,2}$ for $i \in [\![1,33]\!]$ and elements in $B_{3,6}$, is strictly larger than the dimension of the space spanned by $\mathsf{c}'_i x_{1,2}$ for $i \in [\![1,32]\!]$ and elements in $B_{3,6}$, as the following code shows.

```
gene:=geneMH(0,3,3);;
Uh:=UU(gene,3);; Vh:=VV(gene,3);; Wh:=WW(gene,3);;
hx:=HXR(0,Uh,Vh,Wh,3,3,2);;
hxx:=0*[1..33];;
hxx[1]:=hx[14];; hxx[2]:=hx[15];; hxx[3]:=hx[16];; hxx[4]:=hx[17];; hxx[5]:=hx[18];;
hxx[6]:=hx[19];; hxx[7]:=hx[25];; hxx[8]:=hx[26];; hxx[9]:=hx[27];; hxx[10]:=hx[29];;
hxx[11]:=hx[31];; hxx[12]:=hx[32];; hxx[13]:=hx[39];; hxx[14]:=hx[40];;
hxx[15]:=hx[41];; hxx[16]:=hx[42];; hxx[17]:=hx[50];; hxx[18]:=hx[51];;
hxx[19]:=hx[58];; hxx[20]:=hx[59];; hxx[21]:=hx[60];; hxx[22]:=hx[61];;
hxx[23]:=hx[65];; hxx[24]:=hx[67];; hxx[25]:=hx[77];; hxx[26]:=hx[78];;
```

```
hxx[27]:=hx[79];; hxx[28]:=hx[80];; hxx[29]:=hx[88];; hxx[30]:=hx[89];;
hxx[31]:=hx[91];; hxx[32]:=hx[93];; hxx[33]:=Ker(0,3,5)[79];;
Imm:=Im(0,3,5);;
RankMat(Imm);
# 1550
Append(Imm, hxx);
RankMat(Imm);
# 1583
gene:=hxx;;
Uh:=UU(gene,5);; Vh:=VV(gene,5);; Wh:=WW(gene,5);;
cc:=HXR(0,Uh,Vh,Wh,3,5,1);;
cc12:=0*[1..32];;
for i in [1..32] do
    cc12[i]:=cc[6*i-5];
od;
Imm:=Im(0,3,6);;
RankMat(Imm);
# 1890
Append(Imm,cc12);
RankMat(Imm);
# 1910
Append(Imm,[cc[6*33-5]]);
RankMat(Imm);
# 1911
```

This shows that $\mathsf{c}_{33}x_{1,2} \neq 0$, and it is not a linear combination of $\mathsf{c}_i x_{1,2}$ for $i \in [\![1,32]\!]$, which is a contradiction. So, (6.1.11) is non-split. □

Lemma 6.1.7. *We have the non-split short exact sequence* (6.1.12) *of graded A-modules.*

Proof. The following GAP code shows that the dimension vector of the A-submodule of $\mathrm{H}_1(M^1)$ generated by basis elements $h'_i, i \in [\![1,7]\!]$ of $\mathrm{H}_{1,3}(M^1)$ is $(7,18,32,42,40,30,16,6,1)$.

```
Imm:=Im(1,1,3);;
RankMat(Imm);
# 114
gene:=geneMH(1,1,3);;
Append(Imm,gene);
RankMat(Imm);
# 121
Uh:=UU(gene,3);; Vh:=VV(gene,3);; Wh:=WW(gene,3);;
for r in [4..11] do
   hxr:=HXR(1,Uh,Vh,Wh,1,3,r-3);
   Im1r:=Im(1,1,r);
   Append(Im1r, hxr);
   Print(r, " ", RankMat(Im1r)-RankMat(Im(1,1,r)), "\n");
od;
# 4 18
# 5 32
# 6 42
# 7 40
# 8 30
# 9 16
# 10 6
# 11 1
```

Moreover, it is direct to check that the elements $h'_i, i \in [\![1,7]\!]$ of $\mathrm{H}_1(M^1)$ satisfy the quadratic relations (6.1.9) defining M^2. Indeed, the following code shows that the dimension of the subspace generated by $B^{M^1}_{1,4}$ together with the elements of the form (6.1.9) with h'_i instead of h_i coincides with the dimension of $B^{M^1}_{1,4}$.

```
gene:=geneMH(1,1,3);;
Uh:=UU(gene,3);; Vh:=VV(gene,3);; Wh:=WW(gene,3);;
hx:=HXR(1,Uh,Vh,Wh,1,3,1);;
cc:=0*[1..24];;
cc[1]:=hx[1];; cc[2]:=hx[2];; cc[3]:=hx[3];; cc[4]:=hx[7];; cc[5]:=hx[10];;
cc[6]:=hx[11];; cc[7]:=hx[14];; cc[8]:=hx[16];; cc[9]:=hx[18];; cc[10]:=hx[21];;
cc[11]:=hx[23];; cc[12]:=hx[24];; cc[13]:=hx[5]-hx[17]-hx[26];;
cc[14]:=hx[8]-hx[20]+hx[29];; cc[15]:=hx[30]-hx[31];; cc[16]:=hx[4]-hx[22]+hx[33];;
cc[17]:=hx[9]-hx[15]-hx[34];; cc[18]:=hx[25]+hx[36];; cc[19]:=hx[6]+hx[12]+hx[37];;
cc[20]:=hx[35]+hx[38];; cc[21]:=hx[28]+hx[39];; cc[22]:=hx[27]-hx[40];;
```

```
cc[23]:=hx[32]-hx[41];; cc[24]:=hx[13]+hx[19]+hx[42];;
Imm:=Im(1,1,4);;
RankMat(Imm);
# 222
Append(Imm,cc);
RankMat(Imm);
# 222
```

Hence, there is a surjective morphism from $M^2(-4)$ to the submodule of $\mathrm{H}_1(M^1)$ generated by $h'_i, i \in [\![1,7]\!]$, which is an isomorphism of graded A-modules since the dimension vector of M^2 is also $(7, 18, 32, 42, 40, 30, 16, 6, 1)$. Namely, there is an injective morphism $M^2(-4) \to \mathrm{H}_1(M^1)$ of graded modules. A simple argument using dimensions and grading together with Table 6.1.4 tells us that the cokernel of this injective morphism is exactly the graded A-module $\Bbbk(-6) \oplus \Bbbk(-8)$, as was to be shown.

We finally show that (6.1.12) is non-split. Let c_i for $i \in [\![1,33]\!]$ be the basis elements of space $\mathrm{H}_{1,5}(M^1)$ and $p : \mathrm{H}_1(M^1) \to \Bbbk(-6) \oplus \Bbbk(-8)$ the surjection in (6.1.12), satisfying that $p(\mathsf{c}_i) = 0$ for $i \in [\![1,32]\!]$, and $p(\mathsf{c}_{33}) = \mathsf{e}_1$, where e_1 is the identity element of $\Bbbk(-6)$. The short exact sequence (6.1.12) is split if and only if there exists a morphism $s : \Bbbk(-6) \oplus \Bbbk(-8) \to \mathrm{H}_1(M^1)$ of graded A-modules such that the composition ps is the identity map. Assume there is such a map s. Let $m = s(\mathsf{e}_1) \in \mathrm{H}_{1,5}(M^1)$. Then m is of the form $\sum_{i=1}^{32} \lambda_i \mathsf{c}_i + \mathsf{c}_{33}$ for $\lambda_i \in \Bbbk$, and $m.x = s(\mathsf{e}_1).x = s(\mathsf{e}_1.x) = s(0) = 0$ for all $x \in A_+$. In particular, $\sum_{i=1}^{32} \lambda_i \mathsf{c}_i x_{1,2} + \mathsf{c}_{33} x_{1,2} = 0$ for some $\lambda_i \in \Bbbk$, *i.e.* $\mathsf{c}_{33} x_{1,2}$ is a linear combination of $\mathsf{c}_i x_{1,2}$ for $i \in [\![1,32]\!]$. Using GAP, we choose suitable representative elements $\mathsf{c}'_i \in D_{1,5}^{M^1}$ of c_i for $i \in [\![1,33]\!]$, and get that the dimension of the space spanned by $\mathsf{c}'_i x_{1,2}$ for $i \in [\![1,33]\!]$ and elements in $B_{1,6}^{M^1}$, is strictly larger than the dimension of the space spanned by $\mathsf{c}'_i x_{1,2}$ for $i \in [\![1,32]\!]$ and elements in $B_{1,6}^{M^1}$, as the following code shows.

```
gene:=geneMH(1,1,3);;
Uh:=UU(gene,3);; Vh:=VV(gene,3);; Wh:=WW(gene,3);;
hx:=HXR(1,Uh,Vh,Wh,1,3,2);;
hxx:=0*[1..33];;
hxx[1]:=hx[14];; hxx[2]:=hx[15];; hxx[3]:=hx[16];; hxx[4]:=hx[17];; hxx[5]:=hx[18];;
hxx[6]:=hx[19];; hxx[7]:=hx[25];; hxx[8]:=hx[26];; hxx[9]:=hx[27];; hxx[10]:=hx[29];;
hxx[11]:=hx[31];; hxx[12]:=hx[32];; hxx[13]:=hx[39];; hxx[14]:=hx[40];;
hxx[15]:=hx[41];; hxx[16]:=hx[42];; hxx[17]:=hx[50];; hxx[18]:=hx[51];;
hxx[19]:=hx[58];; hxx[20]:=hx[59];; hxx[21]:=hx[60];; hxx[22]:=hx[61];;
hxx[23]:=hx[65];; hxx[24]:=hx[67];; hxx[25]:=hx[77];; hxx[26]:=hx[78];;
hxx[27]:=hx[79];; hxx[28]:=hx[80];; hxx[29]:=hx[88];; hxx[30]:=hx[89];;
hxx[31]:=hx[91];; hxx[32]:=hx[93];; hxx[33]:=Ker(1,1,5)[76];;
Imm:=Im(1,1,5);;
RankMat(Imm);
# 333
Append(Imm, hxx);
RankMat(Imm);
# 366
gene:=hxx;;
Uh:=UU(gene,5);; Vh:=VV(gene,5);; Wh:=WW(gene,5);;
cc:=HXR(1,Uh,Vh,Wh,1,5,1);;
cc12:=0*[1..32];;
for i in [1..32] do
    cc12[i]:=cc[6*i-5];
od;
Imm:=Im(1,1,6);;
RankMat(Imm);
# 402
Append(Imm,cc12);
RankMat(Imm);
# 422
Append(Imm,[cc[6*33-5]]);
RankMat(Imm);
# 423
```

This shows that $\mathsf{c}_{33} x_{1,2} \neq 0$, and it is not a linear combination of $\mathsf{c}_i x_{1,2}$ for $i \in [\![1,32]\!]$, which is a contradiction. So, (6.1.12) is non-split. □

Lemma 6.1.8. *We have the isomorphisms* $\mathrm{H}_1(M^2) \cong \Bbbk(-4) \oplus \Bbbk(-6)$, $\mathrm{H}_2(M^2) \cong M^1(-6)$ *and* $\mathrm{H}_3(M^2) \cong M^3(-6)$ *of graded A-modules.*

Proof. A simple argument using dimension and grading together with Table 6.1.5 gives the isomoprhism $\mathrm{H}_1(M^2) \cong \Bbbk(-4) \oplus \Bbbk(-6)$.

We prove that the space $\mathrm{H}_2(M^2)$ is a quadratic module, which is isomorphic to $M^1(-6)$. The following GAP code shows that the dimension vector of the submodule of $\mathrm{H}_2(M^2)$ generated by two basis elements a_1'',a_2'' of $\mathrm{H}_{2,4}(M^2)$ is (2,6,11,12,11,6,2). So, $\mathrm{H}_2(M^2)$ is generated by the two elements as an A-module.

```
Imm:=Im(2,2,4);;
RankMat(Imm);
# 1474
gene:=geneMH(2,2,4);;
Append(Imm,gene);
RankMat(Imm);
# 1476
Uh:=UU(gene,4);; Vh:=VV(gene,4);; Wh:=WW(gene,4);;
for r in [5..10] do
   hxr:=HXR(2,Uh,Vh,Wh,2,4,r-4);
   Im2r:=Im(2,2,r);
   Append(Im2r, hxr);
   Print(r, " ", RankMat(Im2r)-RankMat(Im(2,2,r)), "\n");
od;
# 5 6
# 6 11
# 7 12
# 8 11
# 9 6
# 10 2
```

Furthermore, it is direct to check that the generators a_1'',a_2'' of $\mathrm{H}_2(M^2)$ satisfy the quadratic relations (6.1.8) defining M^1. Indeed, the following code shows that the dimension of the subspace generated by $B_{2,5}^{M^2}$ together with the elements of the form (6.1.8) with a_i'' instead of a_i coincides with the dimension of $B_{2,5}^{M^2}$.

```
gene:=geneMH(2,2,4);;
Uh:=UU(gene,4);; Vh:=VV(gene,4);; Wh:=WW(gene,4);;
hx:=HXR(2,Uh,Vh,Wh,2,4,1);;
cc:=0*[1..6];;
cc[1]:=hx[1]+hx[7];; cc[2]:=hx[2];; cc[3]:=hx[5];; cc[4]:=hx[6]+hx[12];;
cc[5]:=hx[9];; cc[6]:=hx[10];;
Imm:=Im(2,2,5);;
RankMat(Imm);
# 2244
Append(Imm,cc);
RankMat(Imm);
# 2244
```

Hence, there is a surjective morphism $M^1(-6) \to \mathrm{H}_2(M^2)$ of graded A-modules. Since the dimension vector of M^1 is (2,6,11,12,11,6,2), we have $\mathrm{H}_2(M^2) \cong M^1(-6)$ as graded A-modules, as claimed.

Next, we prove that the space $\mathrm{H}_3(M^2)$ is also a quadratic module, which is isomorphic to $M^3(-6)$. The following code shows that the dimension vector of the submodule of $\mathrm{H}_3(M^2)$ generated by basis elements $e_i',i \in [\![1,8]\!]$ of $\mathrm{H}_{3,3}(M^2)$ is (8,24,48,72,80,72,48,24,8). So, $\mathrm{H}_3(M^2)$ is generated by the eight elements $e_i',i \in [\![1,8]\!]$ as an A-module.

```
Imm:=Im(2,3,3);;
RankMat(Imm);
# 786
gene:=geneMH(2,3,3);;
Append(Imm,gene);
RankMat(Imm);
# 794
Uh:=UU(gene,3);; Vh:=VV(gene,3);; Wh:=WW(gene,3);;
for r in [4..11] do
   hxr:=HXR(2,Uh,Vh,Wh,3,3,r-3);
   Im3r:=Im(2,3,r);
   Append(Im3r, hxr);
   Print(r, " ", RankMat(Im3r)-RankMat(Im(2,3,r)), "\n");
od;
```

```
# 4 24
# 5 48
# 6 72
# 7 80
# 8 72
# 9 48
# 10 24
# 11 8
```

Moreover, it is direct to check that the generators $e'_i, i \in [\![1,8]\!]$ of $\mathrm{H}_3(M^2)$ satisfy the quadratic relations (6.1.10). Indeed, the following code shows that the dimension of the subspace generated by $B_{3,4}^{M^2}$ together with the elements of the form (6.1.10) with e'_i instead of e_i coincides with the dimension of $B_{3,4}^{M^2}$.

```
gene:=geneMH(2,3,3);;
Uh:=UU(gene,3);; Vh:=VV(gene,3);; Wh:=WW(gene,3);;
hx:=HXR(2,Uh,Vh,Wh,3,3,1);;
cc:=0*[1..24];;
cc[1]:=hx[1]+hx[12];; cc[2]:=hx[6]-hx[7];; cc[3]:=hx[13]-hx[24];;
cc[4]:=hx[18]+hx[19];; cc[5]:=hx[20]+hx[11];; cc[6]:=hx[23]-hx[8];;
cc[7]:=hx[14]+hx[5];; cc[8]:=hx[17]-hx[2];; cc[9]:=hx[3]-hx[22];;
cc[10]:=hx[4]+hx[21];; cc[11]:=hx[15]-hx[10];; cc[12]:=hx[16]+hx[9];;
cc[13]:=hx[25];; cc[14]:=hx[26];; cc[15]:=hx[27];; cc[16]:=hx[31];; cc[17]:=hx[34];;
cc[18]:=hx[35];; cc[19]:=hx[38];; cc[20]:=hx[40];; cc[21]:=hx[42];; cc[22]:=hx[45];;
cc[23]:=hx[47];; cc[24]:=hx[48];;
Imm:=Im(2,3,4);;
RankMat(Imm);
# 1566
Append(Imm,cc);
RankMat(Imm);
# 1566
```

Hence, there is a surjective morphism $M^3(-6) \to \mathrm{H}_3(M^2)$ of graded A-modules. Since the dimension vector of M^3 is (8,24,48,72,80,72,48,24,8), we have $\mathrm{H}_3(M^2) \cong M^3(-6)$ as graded A-modules, as claimed. □

Lemma 6.1.9. *We have* $\mathrm{H}_3(M^3) \cong M^3(-6)$ *of graded A-modules.*

Proof. The following GAP code shows that the dimension vector of the submodule of $\mathrm{H}_3(M^3)$ generated by basis elements $e''_i, i \in [\![1,8]\!]$ of $\mathrm{H}_{3,3}(M^3)$ is (8,24,48,72,80,72,48,24,8). So, $\mathrm{H}_3(M^3)$ is generated by the eight elements as an A-module.

```
Imm:=Im(3,3,3);;
RankMat(Imm);
# 672
gene:=geneMH(3,3,3);;
Append(Imm,gene);
RankMat(Imm);
# 680
Uh:=UU(gene,3);; Vh:=VV(gene,3);; Wh:=WW(gene,3);;
for r in [4..11] do
   hxr:=HXR(3,Uh,Vh,Wh,3,3,r-3);
   Im3r:=Im(3,3,r);
   Append(Im3r, hxr);
   Print(r, " ", RankMat(Im3r)-RankMat(Im(3,3,r)), "\n");
od;
# 4 24
# 5 48
# 6 72
# 7 80
# 8 72
# 9 48
# 10 24
# 11 8
```

Furthermore, it is direct to check that the generators $e''_i, i \in [\![1,8]\!]$ of $\mathrm{H}_3(M^3)$ satisfy the quadratic relations (6.1.10). Indeed, the following code shows that the dimension of the subspace generated by $B_{3,4}^{M^3}$ together with the elements of the form (6.1.10) with e''_i instead of e_i coincides with the dimension of $B_{3,4}^{M^3}$.

```
gene:=geneMH(3,3,3);;
Uh:=UU(gene,3);; Vh:=VV(gene,3);; Wh:=WW(gene,3);;
hx:=HXR(3,Uh,Vh,Wh,3,3,1);;
cc:=0*[1..24];;
cc[1]:=hx[1]+hx[12];; cc[2]:=hx[6]-hx[7];; cc[3]:=hx[13]-hx[24];;
cc[4]:=hx[18]+hx[19];; cc[5]:=hx[20]+hx[11];; cc[6]:=hx[23]-hx[8];;
cc[7]:=hx[14]+hx[5];; cc[8]:=hx[17]-hx[2];; cc[9]:=hx[3]-hx[22];;
cc[10]:=hx[4]+hx[21];; cc[11]:=hx[15]-hx[10];; cc[12]:=hx[16]+hx[9];;
cc[13]:=hx[25];; cc[14]:=hx[26];; cc[15]:=hx[27];; cc[16]:=hx[31];; cc[17]:=hx[34];;
cc[18]:=hx[35];; cc[19]:=hx[38];; cc[20]:=hx[40];; cc[21]:=hx[42];; cc[22]:=hx[45];;
cc[23]:=hx[47];; cc[24]:=hx[48];;
Imm:=Im(3,3,4);;
RankMat(Imm);
# 1344
Append(Imm,cc);
RankMat(Imm);
# 1344
```

Hence, we see that there is a surjective morphism $M^3(-6) \to \mathrm{H}_3(M^3)$ of graded A-modules. Since the dimension vector of M^3 is (8,24,48,72,80,72,48,24,8), we have $\mathrm{H}_3(M^3) \cong M^3(-6)$ as graded A-modules, as claimed. □

6.1.2.2 Homology of the Koszul complex of the trivial module

In this subsubsection, we will compute the homology of the Koszul complex of the trivial module.

Proposition 6.1.10. *For $n \geqslant 5$, the dimension of $B_{n,m}$ is given by*

$$\dim B_{n,m} = \begin{cases} 0, & \text{if } m = 0, \\ 3n+30, & \text{if } m = 1, \\ 15n+147, & \text{if } m = 2, \\ 42n+408, & \text{if } m = 3, \\ 84n+810, & \text{if } m = 4, \\ 129n+1236, & \text{if } m = 5, \\ 159n+1515, & \text{if } m = 6, \\ 159n+1506, & \text{if } m = 7, \\ 129n+1215, & \text{if } m = 8, \\ 84n+786, & \text{if } m = 9, \\ 42n+390, & \text{if } m = 10, \\ 15n+138, & \text{if } m = 11, \\ 3n+27, & \text{if } m = 12. \end{cases} \tag{6.1.14}$$

Corollary 6.1.11. *We have $\dim H_n = 0$ for $n \in \mathbb{N} \setminus \{3,4,5\}$. Moreover, the dimension of $H_{n,m}$ for $n = 0,3,4,5$ and $m \in [\![0,12]\!]$ is given in Table 6.1.3.*

Proof. By $\dim D_{n,m} = \dim K_{n,m} - \dim B_{n-1,m+1}$, Proposition 6.1.10, together with Table 6.1.1 and 6.1.2, we have $\dim D_{n,m} = \dim B_{n,m}$ for $n \in \mathbb{N} \setminus \{3,4,5\}$ and $m \in [\![0,12]\!]$. Then the corollary holds. □

In order to prove Proposition 6.1.10, we need some preparatory results. Let $\mathscr{C}_n = \cup_{(i,j)\in\mathscr{I}_1} \mathscr{C}_n^{i,j}$, where

$$\mathscr{C}_n^{i,j} = \{z_{n-r}^{i,j} z_r^{k,l} \mid (k,l) \in \mathscr{I} \text{ such that } \#\{i,j,k,l\} = 4, r \in [\![0,n]\!]\} \subseteq \mathscr{B}_n^{!*}$$

for $(i,j) \in \mathscr{I}_1$ and $n \in \mathbb{N}$, and let $\mathscr{U}_n = \mathscr{B}_n^{!*} \setminus \mathscr{C}_n$ for $n \in \mathbb{N}$. Note that the pair $(k,l) \in \mathscr{I}$ is uniquely determined in the definition of $\mathscr{C}_n^{i,j}$. Given $m,n \in \mathbb{N}$, let $C_{n,m}$ be the subspace of $\Bbbk\mathscr{C}_n \otimes A_m$ spanned by $\{d_{n+1}(z|x) \mid z \in \mathscr{C}_{n+1}, x \in A_{m-1}\}$, $C_{n,m}^{i,j}$ be the subspace of $C_{n,m}$ spanned by $\{d_{n+1}(z|x) \mid z \in \mathscr{C}_{n+1}^{i,j}, x \in A_{m-1}\}$ for $(i,j) \in \mathscr{I}_1$, and $U_{n,m}$ be the subspace of $B_{n,m}$ spanned by $\{d_{n+1}(z|x) \mid z \in \mathscr{U}_{n+1}, x \in A_{m-1}\}$.

Fixing the order $x_{1,2} \prec x_{3,4} \prec x_{1,3} \prec x_{2,3} \prec x_{1,4} \prec x_{2,4}$ (resp., $x_{1,3} \prec x_{2,4} \prec x_{1,2} \prec x_{2,3} \prec x_{1,4} \prec x_{3,4}$, $x_{2,3} \prec x_{1,4} \prec x_{1,2} \prec x_{1,3} \prec x_{2,4} \prec x_{3,4}$), the corresponding basis of A consisting of standard words will be denoted by $W^{1,2}$ (resp., $W^{1,3}$, $W^{2,3}$). It can be explicitly computed using GAP (see Appendix A.3 for $W^{1,2}$). For $(i,j) \in \mathcal{I}_1$, let $(k,l) \in \mathcal{I}$ such that $\#\{i,j,k,l\} = 4$, set $W^{i,j}_m = W^{i,j} \cap A_m$. Set $E^{i,j}_m$ as the subset of $W^{i,j}_m$ containing elements whose first element is not $x_{i,j}$, and set $\tilde{E}^{i,j}_m$ as the subset of $W^{i,j}_m$ containing elements whose first element is neither $x_{i,j}$ nor $x_{k,l}$. Let $\mathfrak{a}^{i,j}_m = \#E^{i,j}_m$ and $\mathfrak{b}^{i,j}_m = \#\tilde{E}^{i,j}_m$ for $m \in [\![0,11]\!]$. The integers $\mathfrak{a}^{i,j}_m$ and $\mathfrak{b}^{i,j}_m$ are easily computed from the explicit description of the bases $W^{i,j}_m$, they are independent of (i,j), so they will be denoted simply by $\mathfrak{a}_m$ and $\mathfrak{b}_m$, respectively, and are given in Table 6.1.7.

m	0	1	2	3	4	5	6	7	8	9	10	11
$\mathfrak{a}_m$	1	5	14	28	43	53	53	43	28	14	5	1
$\mathfrak{b}_m$	1	4	10	18	25	28	25	18	10	4	1	0

Table 6.1.7: Values of $\mathfrak{a}_m$ and $\mathfrak{b}_m$.

Lemma 6.1.12. *We have* $C_{n,m} = \bigoplus_{(i,j)\in\mathcal{I}_1} C^{i,j}_{n,m}$ *and the dimension of* $C^{i,j}_{n,m}$ *is given by*

$$\dim C^{i,j}_{n,m} = \begin{cases} n+2, & \text{if } m = 1, \\ 5n+9, & \text{if } m = 2, \\ 14n+24, & \text{if } m = 3, \\ 28n+46, & \text{if } m = 4, \\ 43n+68, & \text{if } m = 5, \\ 53n+81, & \text{if } m = 6, \\ 53n+78, & \text{if } m = 7, \\ 43n+61, & \text{if } m = 8, \\ 28n+38, & \text{if } m = 9, \\ 14n+18, & \text{if } m = 10, \\ 5n+6, & \text{if } m = 11, \\ n+1, & \text{if } m = 12, \end{cases}$$

for all $(i,j) \in \mathcal{I}_1$ *and* $n \in \mathbb{N}$. *Else* $\dim C^{i,j}_{n,m} = 0$.

Proof. Given $(i,j) \in \mathcal{I}_1$, fix $(k,l) \in \mathcal{I}$ such that $\#\{i,j,k,l\} = 4$. Then, the maps $\Bbbk E^{i,j}_{m-1} \to A_m$ and $\Bbbk \tilde{E}^{i,j}_{m-1} \to A_m$ given by left multiplication by $x_{i,j}$ and by left multiplication by $x_{k,l}$, respectively, are injective for $m \in [\![1,12]\!]$. Hence, using (6.1.6), we see that the set formed by the elements $(-1)^r z^{i,j}_{n-r} z^{k,l}_r | x_{i,j} x + z^{i,j}_{n-r+1} z^{k,l}_{r-1} | x_{k,l} x$, for $x \in E^{i,j}_{m-1}$ and $r \in [\![0,n]\!]$, together with the elements $z^{k,l}_n | x_{k,l} y$ for $y \in \tilde{E}^{i,j}_{m-1}$ gives a basis of $C^{i,j}_{n,m}$. Then, $\dim C^{i,j}_{n,m} = \mathfrak{a}_{m-1}(n+1) + \mathfrak{b}_{m-1}$, which together with Table 6.1.7 proves the claim. ☐

Lemma 6.1.13. *We have* $\dim U_{n,m} = \dim U_{n+2,m}$ *and* $\dim(U_{n,m} \cap C_{n,m}) = \dim(U_{n+2,m} \cap C_{n+2,m})$ *for* $n \geqslant 5$ *and* $m \in [\![1,12]\!]$.

Proof. For $n \geqslant 5$, set

$$u^{i,j}_n = \sum_{\substack{r \in [\![1, n-1]\!], \\ r \text{ odd}}} z^{i,j}_{n-r} z^{k,l}_r \quad \text{and} \quad v^{i,j}_n = \sum_{\substack{r \in [\![1, n-1]\!], \\ r \text{ even}}} z^{i,j}_{n-r} z^{k,l}_r,$$

for $(i,j) \in \mathcal{I}_1$, $(k,l) \in \mathcal{I}$, $\#\{i,j,k,l\} = 4$, and

$$\mathbb{Q}_n = \mathcal{U}_n \cup \{ z^{i,j}_n \mid (i,j) \in \mathcal{I} \} \cup \{ u^{i,j}_n, v^{i,j}_n \mid (i,j) \in \mathcal{I}_1 \} \subseteq (A^!_{-n})^*.$$

Let $\mathcal{U}^{i,j}_n$ be the subset of $\mathcal{U}_n$ consisting of elements whose first element is $z^{i,j}$ for $(i,j) \in \mathcal{I}_1$. There is an isomorphism $\mathfrak{f}_n : \Bbbk\mathbb{Q}_n \to \Bbbk\mathbb{Q}_{n+2}$ of vector spaces defined by $\mathfrak{f}_n(z) = z^{i,j}_2 z$ for $z \in \mathcal{U}^{i,j}_n$ and $(i,j) \in \mathcal{I}_1$, $\mathfrak{f}_n(u^{i,j}_n) = u^{i,j}_{n+2}$, $\mathfrak{f}_n(v^{i,j}_n) = v^{i,j}_{n+2}$, and $\mathfrak{f}_n(z^{i,j}_n) = z^{i,j}_{n+2}$ for $(i,j) \in \mathcal{I}$. Then, the map $\mathfrak{g}_n = \mathfrak{f}_n \otimes \mathrm{id}_A : \Bbbk\mathbb{Q}_n \otimes A \to \Bbbk\mathbb{Q}_{n+2} \otimes A$ is a linear isomorphism. By (6.1.7),

$U_{n,m} \subseteq \Bbbk\mathbb{Q}_n \otimes A_m$ and $\mathfrak{g}_n(U_{n,m}) = U_{n+2,m}$, giving an isomorphism $U_{n,m} \cong U_{n+2,m}$ of vector spaces for $n \geqslant 5$ and $m \in [\![1,12]\!]$. This proves the first part of the lemma.

Set $F_{n,m} = (\Bbbk\mathbb{Q}_n \otimes A_m) \cap C_{n,m}$ and define $L^{i,j}_{n,m} = \Bbbk\{z^{i,j}_n, z^{k,l}_n, u^{i,j}_n, v^{i,j}_n\} \otimes A_m$ as the subspace of $\Bbbk\mathscr{C}^{i,j}_n \otimes A_m$, where $(i,j) \in \mathscr{I}_1$, $(k,l) \in \mathscr{I}$ with $\#\{i,j,k,l\} = 4$. It is clear that $F_{n,m} = \oplus_{(i,j)\in\mathscr{I}_1}(L^{i,j}_{n,m} \cap C^{i,j}_{n,m})$. Fix $(i,j) \in \mathscr{I}_1$, $(k,l) \in \mathscr{I}$ with $\#\{i,j,k,l\} = 4$. Let $\xi^{i,j} \in C^{i,j}_{n,m}$. Then $\xi^{i,j}$ is of the form

$$\xi^{i,j} = \sum_{\substack{r \in [\![0,n]\!],\\ x \in E^{i,j}_{m-1}}} \lambda_{r,x}\{(-1)^r z^{i,j}_{n-r} z^{k,l}_r | x_{i,j}x + z^{i,j}_{n-r+1} z^{k,l}_{r-1} | x_{k,l}x\} + \sum_{y \in \tilde{E}^{i,j}_{m-1}} \mu_y z^{k,l}_n | x_{k,l}y \tag{6.1.15}$$

for $\lambda_{r,x}, \mu_y \in \Bbbk$. If $\xi^{i,j} \in L^{i,j}_{n,m}$, then $\xi^{i,j}$ is of the form

$$\xi^{i,j} = \sum_{w \in W^{i,j}_m} (\alpha_w z^{i,j}_n | w + \beta_w z^{k,l}_n | w + \gamma_w u^{i,j}_n | w + \eta_w v^{i,j}_n | w) \tag{6.1.16}$$

for $\alpha_w, \beta_w, \gamma_w, \eta_w \in \Bbbk$. Comparing the coefficients in (6.1.15) and (6.1.16), we obtain

$$\begin{aligned}
&\alpha_{x_{i,j}x} = \lambda_{0,x}, \quad \alpha_{x_{k,l}y} = \lambda_{1,y}, \quad \beta_{x_{i,j}x} = (-1)^n \lambda_{n,x}, \quad \beta_{x_{k,l}y} = \mu_y,\\
&\gamma_{x_{i,j}x} = -\lambda_{p,x} \text{ for } p \in [\![1,n-1]\!] \text{ with } p \text{ odd},\\
&\gamma_{x_{k,l}y} = \lambda_{q,y} \text{ for } q \in [\![2,n]\!] \text{ with } q \text{ even},\\
&\eta_{x_{i,j}x} = \lambda_{q,x} \text{ for } q \in [\![2,n-1]\!] \text{ with } q \text{ even},\\
&\eta_{x_{k,l}y} = \lambda_{p,y} \text{ for } p \in [\![3,n]\!] \text{ with } p \text{ odd},
\end{aligned}$$

where $x \in E^{i,j}_{m-1}$ and $y \in \tilde{E}^{i,j}_{m-1}$. Hence, if n is even, the space $L^{i,j}_{n,m} \cap C^{i,j}_{n,m}$ is spanned by $z^{i,j}_n | x_{i,j}x$, $(v^{i,j}_n | x_{k,l} - u^{i,j}_n | x_{i,j} + z^{i,j}_n | x_{k,l})x$ for $x \in E^{i,j}_{m-1}$, $z^{k,l}_n | x_{k,l}y$, $(u^{i,j}_n | x_{k,l} + v^{i,j}_n | x_{i,j} + z^{k,l}_n | x_{i,j})y$ for $y \in \tilde{E}^{i,j}_{m-1}$, $v^{i,j}_n | x_{i,j}w$ and $z^{k,l}_n | x_{i,j}w$ for $w \in E^{i,j}_{m-1} \setminus \tilde{E}^{i,j}_{m-1}$. If n is odd, the space $L^{i,j}_{n,m} \cap C^{i,j}_{n,m}$ is spanned by $z^{i,j}_n | x_{i,j}x$, $(u^{i,j}_n | x_{k,l} + v^{i,j}_n | x_{i,j})x$ for $x \in E^{i,j}_{m-1}$, $z^{k,l}_n | x_{k,l}y$, $(v^{i,j}_n | x_{k,l} - u^{i,j}_n | x_{i,j} + z^{i,j}_n | x_{k,l} - z^{k,l}_n | x_{i,j})y$ for $y \in \tilde{E}^{i,j}_{m-1}$, $u^{i,j}_n | x_{i,j}w$ and $z^{k,l}_n | x_{i,j}w$ for $w \in E^{i,j}_{m-1} \setminus \tilde{E}^{i,j}_{m-1}$. We finally note that $U_{n,m} \cap C_{n,m} = U_{n,m} \cap F_{n,m}$ and $\mathfrak{g}_n(F_{n,m}) = F_{n+2,m}$. Hence, $U_{n,m} \cap C_{n,m} \cong U_{n+2,m} \cap C_{n+2,m}$ as vector spaces. This proves the second part of the lemma. □

Proof of Proposition 6.1.10. By Table 6.1.1, we obtain that (6.1.14) holds for $(n,m) \in [\![5,6]\!] \times [\![0,12]\!]$. On the other hand, by Lemma 6.1.13, we get that $\dim B_{n+2,m} - \dim B_{n,m} = \dim C_{n+2,m} - \dim C_{n,m}$ for $n \geqslant 5$ and $m \in [\![1,12]\!]$. The statement then follows. □

Moreover, using GAP, the dimension of $U_{n,m}$ for $n \geqslant 3$ and $m \in [\![1,12]\!]$ is given by Table 6.1.8.

n \ m	1	2	3	4	5	6	7	8	9	10	11	12
3	23	138	422	896	1428	1800	1815	1468	947	466	162	30
$n \geqslant 4$ with n even	24	136	408	850	1344	1690	1716	1406	924	466	168	34
$n \geqslant 5$ with n odd	24	144	434	912	1452	1836	1872	1536	1008	504	180	36

Table 6.1.8: Dimension of $U_{n,m}$.

Remark 6.1.14. *By Lemma 6.1.12, the subcomplex $\Bbbk\mathscr{C}_n \otimes A$ of K_n is exact for $n \geqslant 2$.*

6.1.2.3 Homology of the Koszul complex of M^1

In this subsubsection, we compute $\mathrm{H}_n(M^1)$ for all $n \in \mathbb{N}_0$.

Recall that $M^1 = (W \otimes A)/(I)$, where W is the 2-dimensional vector space spanned by a_1,a_2, and I is the subspace of $W \otimes V$ spanned by (6.1.8). The quadratic dual $(M^1)^! = \oplus_{n\in\mathbb{N}_0}(M^1)^!_{-n} = (U \otimes A^!)/(J)$ of M^1 is an $A^!$-module, where U is the 2-dimensional vector space spanned by b_1,b_2 (for $\{b_1,b_2\}$ the dual basis to $\{a_1,a_2\}$), and J is the subspace of $U \otimes V^*$ spanned by

$$\{b_1y_{1,2} - b_2y_{1,2}, b_2y_{1,3}, b_1y_{2,3}, b_1y_{1,4}, b_2y_{2,4}, b_1y_{3,4} - b_2y_{3,4}\}. \tag{6.1.17}$$

Lemma 6.1.15. *Recall that $\mathscr{B}^! = \cup_{n\in\mathbb{N}_0}\mathscr{B}^!_n$ is the basis of $A^!$. Let $u,v \in \mathscr{B}^!$ and*

$$Y_{1,2} = \{\pm y_{1,2}^{r_1} y_{3,4}^{r_2} | r_1,r_2 \in \mathbb{N}_0\}, Y_{1,3} = \{\pm y_{1,3}^{r_1} y_{2,4}^{r_2} | r_1,r_2 \in \mathbb{N}_0\}, Y_{2,3} = \{\pm y_{2,3}^{r_1} y_{1,4}^{r_2} | r_1,r_2 \in \mathbb{N}_0\}. \tag{6.1.18}$$

If $uv \in Y_{i,j}$ for $(i,j) \in \mathscr{I}_1$, then $u,v \in Y_{i,j}$.

Proof. We will prove the lemma by induction on the degree of v. Let $u \in \mathscr{B}^!_m$ and $v \in \mathscr{B}^!_n$ for $m,n \in \mathbb{N}_0$. Obviously, the lemma holds for $n = 0$ and $m \in \mathbb{N}_0$. Assume that $v = v'y$ for $y \in \{y_{s,t} | (s,t) \in \mathscr{I}\}$ and $v' \in \mathscr{B}^!_{n-1}$. Note that $uv' = \pm c$, where $c \in \mathscr{B}^!_{m+n-1}$. By Tables A.1.1 - A.1.7 together with (6.1.2) and (6.1.3), $cy \in Y_{i,j}$ implies that $c,y \in Y_{i,j}$. Then, by induction hypothesis we get that $u, v' \in Y_{i,j}$. In consequence, $v = v'y \in Y_{i,j}$, as was to be shown. □

Lemma 6.1.16. *Set $T_n = \{b_1 y_{1,2}^k y_{3,4}^{n-k}, b_1 y_{1,3}^k y_{2,4}^{n-k}, b_2 y_{2,3}^k y_{1,4}^{n-k} \mid k \in [\![0,n]\!]\} \subseteq (M^1)^!_{-n}$ for $n \in \mathbb{N}_0$. Note that T_n has cardinal $3(n+1)$ for $n \in \mathbb{N}$, and cardinal 2 for $n = 0$, since $T_0 = \{b_1,b_2\}$. Then, T_n is a basis of the space $(M^1)^!_{-n}$ for $n \in \mathbb{N}_0$.*

Proof. Note that the space $(M^1)^!_{-n}$ is spanned by $\{b_1 y, b_2 y \mid y \in \mathscr{B}^!_n\}$ for $n \in \mathbb{N}_0$. It is easy to check that

$$\begin{aligned}
b_j y_{1,2}^m y_{1,3} &= \chi_m b_1 y_{2,3}^m y_{1,3} - \chi_{m+1} b_1 y_{2,3}^m y_{1,2} = 0,\\
b_j y_{1,2}^m y_{2,3} &= \chi_m b_2 y_{1,3}^m y_{2,3} - \chi_{m+1} b_2 y_{1,3}^m y_{1,2} = 0,\\
b_j y_{1,2}^m y_{1,4} &= \chi_m b_2 y_{2,4}^m y_{1,4} - \chi_{m+1} b_2 y_{2,4}^m y_{1,2} = 0,\\
b_j y_{1,2}^m y_{2,4} &= \chi_m b_1 y_{1,4}^m y_{2,4} - \chi_{m+1} b_1 y_{1,4}^m y_{1,2} = 0,\\
b_1 y_{1,3}^m y_{1,4} &= \chi_m b_1 y_{3,4}^m y_{1,4} + \chi_{m+1} b_1 y_{1,4}^m y_{3,4} = \chi_m b_2 y_{3,4}^m y_{1,4} + \chi_{m+1} b_1 y_{1,4}^m y_{3,4}\\
&= \chi_m b_2 y_{1,3}^m y_{1,4} + \chi_{m+1} b_1 y_{1,4}^m y_{3,4} = 0,\\
b_1 y_{1,3}^m y_{3,4} &= \chi_m b_1 y_{1,4}^m y_{3,4} - \chi_{m+1} b_1 y_{1,4}^m y_{1,3} = 0,\\
b_2 y_{2,3}^m y_{2,4} &= \chi_m b_2 y_{3,4}^m y_{2,4} + \chi_{m+1} b_2 y_{2,4}^m y_{3,4} = \chi_m b_1 y_{3,4}^m y_{2,4} + \chi_{m+1} b_2 y_{2,4}^m y_{3,4}\\
&= \chi_m b_1 y_{2,3}^m y_{2,4} + \chi_{m+1} b_2 y_{2,4}^m y_{3,4} = 0,\\
b_2 y_{2,3}^m y_{3,4} &= \chi_m b_2 y_{2,4}^m y_{3,4} - \chi_{m+1} b_2 y_{2,4}^m y_{2,3} = 0,
\end{aligned}$$

for $j \in [\![1,2]\!]$ and $m \in \mathbb{N}$. Together with (6.1.17), we get that the space $(M^1)^!_{-n}$ is spanned by T_n for $n \in \mathbb{N}_0$.

It is clear that T_0 is linearly independent. Next, we prove that the elements in T_n are linearly independent for $n \in \mathbb{N}$. Suppose that

$$\sum_{k\in[\![0,n]\!]} \alpha_k b_1 y_{1,2}^k y_{3,4}^{n-k} + \sum_{k\in[\![0,n]\!]} \beta_k b_1 y_{1,3}^k y_{2,4}^{n-k} + \sum_{k\in[\![0,n]\!]} \gamma_k b_2 y_{2,3}^k y_{1,4}^{n-k} = 0$$

in $(M^1)^!_{-n}$, where $\alpha_k,\beta_k,\gamma_k \in \Bbbk$ for $k \in [\![0,n]\!]$. Then

$$\begin{aligned}
&\sum_{k\in[\![0,n]\!]} \alpha_k b_1 y_{1,2}^k y_{3,4}^{n-k} + \sum_{k\in[\![0,n]\!]} \beta_k b_1 y_{1,3}^k y_{2,4}^{n-k} + \sum_{k\in[\![0,n]\!]} \gamma_k b_2 y_{2,3}^k y_{1,4}^{n-k}\\
&= \sum_{u\in\mathscr{B}^!_{n-1}} \lambda_{1,u}(b_1 y_{1,2} - b_2 y_{1,2})u + \sum_{u\in\mathscr{B}^!_{n-1}} \lambda_{2,u} b_2 y_{1,3} u + \sum_{u\in\mathscr{B}^!_{n-1}} \lambda_{3,u} b_1 y_{2,3} u\\
&\quad + \sum_{u\in\mathscr{B}^!_{n-1}} \lambda_{4,u} b_1 y_{1,4} u + \sum_{u\in\mathscr{B}^!_{n-1}} \lambda_{5,u} b_2 y_{2,4} u + \sum_{u\in\mathscr{B}^!_{n-1}} \lambda_{6,u}(b_1 y_{3,4} - b_2 y_{3,4})u \in U \otimes A^!,
\end{aligned}$$

where $\lambda_{i,u} \in \Bbbk$ for $i \in [\![1,6]\!]$ and $u \in \mathscr{B}^!_{n-1}$. So,

$$\begin{aligned}
\sum_{k\in[\![0,n]\!]} \gamma_k b_2 y_{2,3}^k y_{1,4}^{n-k} = &- \sum_{u\in\mathscr{B}^!_{n-1}} \lambda_{1,u} b_2 y_{1,2} u + \sum_{u\in\mathscr{B}^!_{n-1}} \lambda_{2,u} b_2 y_{1,3} u + \sum_{u\in\mathscr{B}^!_{n-1}} \lambda_{5,u} b_2 y_{2,4} u\\
&- \sum_{u\in\mathscr{B}^!_{n-1}} \lambda_{6,u} b_2 y_{3,4} u \in \Bbbk\{b_2\} \otimes A^! \cong A^!
\end{aligned} \tag{6.1.19}$$

and

$$\begin{aligned}&\sum_{k\in[\![0,n]\!]}\alpha_k b_1 y_{1,2}^k y_{3,4}^{n-k}+\sum_{k\in[\![0,n]\!]}\beta_k b_1 y_{1,3}^k y_{2,4}^{n-k}\\ &=\sum_{u\in\mathscr{B}^!_{n-1}}\lambda_{1,u}b_1y_{1,2}u+\sum_{u\in\mathscr{B}^!_{n-1}}\lambda_{3,u}b_1y_{2,3}u+\sum_{u\in\mathscr{B}^!_{n-1}}\lambda_{4,u}b_1y_{1,4}u+\sum_{u\in\mathscr{B}^!_{n-1}}\lambda_{6,u}b_1y_{3,4}u \\ &\in \Bbbk\{b_1\}\otimes A^!\cong A^!.\end{aligned} \tag{6.1.20}$$

Lemma 6.1.15 and (6.1.19) imply that

$$\sum_{k\in[\![0,n]\!]}\gamma_k y_{2,3}^k y_{1,4}^{n-k}=\sum_{u\in\mathscr{B}^!_{n-1}\cap Y_{1,2}}\lambda_{1,u}y_{1,2}u+\sum_{u\in\mathscr{B}^!_{n-1}\cap Y_{1,2}}\lambda_{6,u}y_{3,4}u=0$$

in $A^!$, whereas Lemma 6.1.15 and (6.1.20) imply that

$$\sum_{k\in[\![0,n]\!]}\alpha_k y_{1,2}^k y_{3,4}^{n-k}=\sum_{u\in\mathscr{B}^!_{n-1}\cap Y_{1,2}}\lambda_{1,u}y_{1,2}u+\sum_{u\in\mathscr{B}^!_{n-1}\cap Y_{1,2}}\lambda_{6,u}y_{3,4}u,\quad \sum_{k\in[\![0,n]\!]}\beta_k y_{1,3}^k y_{2,4}^{n-k}=0$$

in $A^!$. Hence, $\alpha_k=\beta_k=\gamma_k=0$ for $k\in[\![0,n]\!]$. The lemma is thus proved. □

Given $n\in\mathbb{N}$, we will denote by $T_n^*=\{x^*\mid x\in T_n\}$ the dual basis of T_n. Note that the differential $d_1(M^1):\mathrm{K}_1(M^1)\to\mathrm{K}_0(M^1)$ is given by

$$\begin{aligned}&(b_1y_{1,2})^*|1\mapsto b_1^*|x_{1,2}+b_2^*|x_{1,2},(b_1y_{1,3})^*|1\mapsto b_1^*|x_{1,3},(b_1y_{2,4})^*|1\mapsto b_1^*|x_{2,4},\\&(b_1y_{3,4})^*|1\mapsto b_1^*|x_{3,4}+b_2^*|x_{3,4},(b_2y_{2,3})^*|1\mapsto b_2^*|x_{2,3},(b_2y_{1,4})^*|1\mapsto b_2^*|x_{1,4},\end{aligned}$$

where $b_sy_{i,j}\in T_1$ and $(b_sy_{i,j})^*\in T_1^*$ is the dual element of $b_sy_{i,j}$. The differential $d_n(M^1)$: $\mathrm{K}_n(M^1)\to\mathrm{K}_{n-1}(M^1)$ for $n\geqslant 2$ is given by

$$(b_sy_{i,j}^{n-r}y_{k,l}^r)^*|1\mapsto(-1)^r(b_sy_{i,j}^{n-1-r}y_{k,l}^r)^*|x_{i,j}+(b_sy_{i,j}^{n-r}y_{k,l}^{r-1})^*|x_{k,l},$$

where $s\in[\![1,2]\!]$, $r\in[\![0,n]\!]$, $(i,j)\in\mathscr{I}_1$, $(k,l)\in\mathscr{I}$ with $\#\{i,j,k,l\}=4$, $b_sy_{i,j}^{n-r}y_{k,l}^r\in T_n$ and $x^*\in((M^1)^!_{-n})^*\in T_n^*$ is the dual element of $x\in T_n\subseteq(M^1)^!_{-n}$.

Proposition 6.1.17. *We have* $\dim\mathrm{H}_n(M^1)=0$ *for integers* $n\geqslant 2$.

Proof. It is clear that there is an isomorphism $((M^1)^!_{-n})^*\otimes A\to\Bbbk\mathscr{C}_n\otimes A$ of chain complex of graded A-modules given by $(b_sy_{i,j}^{n-r}y_{k,l}^r)^*|x\mapsto z_{n-r}^{i,j}z_r^{k,l}|x$, where $x\in A$ and $n\in\mathbb{N}$. So, $\dim B_{n,m}^{M^1}=\dim C_{n,m}$ for $m\in[\![0,12]\!]$ and $n\in\mathbb{N}$, where $\dim C_{n,m}$ is given by Lemma 6.1.12. The result now follows from the fact that the Koszul complex $\mathrm{K}_\bullet(M^1)$ is isomorphic to the complex $\Bbbk\mathscr{C}_\bullet\otimes A$ for $\bullet\in\mathbb{N}$ and Remark 6.1.14. □

Corollary 6.1.18. *The dimension of* $B_{n,m}^{M^1}$ *for* $n\in\mathbb{N}_0$ *and* $m\in[\![0,12]\!]$ *is given by*

$n \backslash m$	0	1	2	3	4	5	6	7	8	9	10	11	12
$n=0$	0	6	27	72	131	186	210	192	142	84	38	12	2
$n\in\mathbb{N}$	0	$3n+6$	$15n+27$	$42n+72$	$84n+138$	$129n+204$	$159n+243$	$159n+234$	$129n+183$	$84n+114$	$42n+54$	$15n+18$	$3n+3$

Table 6.1.9: Dimension of $B_{n,m}^{M^1}$.

Proof. The last row of Table 6.1.9 follows from Lemma 6.1.12, since $\dim B_{n,m}^{M^1}=\dim C_{n,m}$ for $n\in\mathbb{N}$ and $m\in[\![0,12]\!]$, as explained in the proof of Proposition 6.1.17. For the remaining case, note that $\dim B_{0,m}^{M^1}=\dim D_{0,m}^{M^1}=\dim(((M^1)^!_0)^*\otimes A_m)-\dim H_{4,m+4}=2\dim A_m-\dim H_{4,m+4}$ for $m\in[\![0,12]\!]$. The result now follows. □

Corollary 6.1.19. *The dimension of* $D_{1,m}^{M^1}$ *for* $m\in[\![0,12]\!]$ *is given by*

m	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim D_{1,m}^{M^1}$	0	9	42	121	240	366	444	434	342	214	102	34	6

Table 6.1.10: Dimension of $D_{1,m}^{M^1}$.

Hence, the dimension of $\mathrm{H}_{1,m}(M^1)$ *for* $m\in[\![0,12]\!]$ *is exactly given in Table 6.1.4, by* $\dim\mathrm{H}_{1,m}(M^1)=\dim D_{1,m}^{M^1}-\dim B_{1,m}^{M^1}$. *In particular,* $\dim\mathrm{H}_1(M^1)=194$.

Proof. The result follows directly from $\dim D^{M^1}_{1,m} = \dim(((M^1)^!_{-1})^* \otimes A_m) - \dim B^{M^1}_{0,m+1} = 6\dim A_m - \dim B^{M^1}_{0,m+1}$ for $m \in [\![0,12]\!]$, together with Corollary 6.1.18. $\square$

6.1.2.4 Homology of the Koszul complex of M^2

In this subsubsection, we show that $\mathrm{H}_n(M^2) = 0$ for $n \geqslant 4$.

Recall the definition of the quadratic module M^2 given in Subsection 6.1.2. Let $\{g_i \mid i \in [\![1,7]\!]\}$ be the dual basis to the basis $\{h_i \mid i \in [\![1,7]\!]\}$ of the space of generators of M^2. Then, it is easy to see that the $A^!$-module $(M^2)^!$ is generated by $g_i, i \in [\![1,7]\!]$, subject to the following 18 relations

$$\begin{gathered} g_1y_{3,4} - g_2y_{3,4}, g_3y_{1,2} - g_4y_{1,2}, g_5y_{1,2} - g_6y_{3,4}, g_5y_{3,4} + g_6y_{1,2}, g_1y_{3,4} - g_7y_{1,2}, g_3y_{1,2} - g_7y_{3,4}, \\ g_1y_{2,4} + g_3y_{2,4}, g_2y_{1,3} + g_4y_{1,3}, g_6y_{1,3} + g_7y_{2,4}, g_6y_{2,4} - g_7y_{1,3}, g_1y_{2,4} + g_5y_{1,3}, g_2y_{1,3} - g_5y_{2,4}, \\ g_1y_{1,4} + g_4y_{1,4}, g_2y_{2,3} + g_3y_{2,3}, g_5y_{2,3} + g_7y_{1,4}, g_5y_{1,4} - g_7y_{2,3}, g_1y_{1,4} - g_6y_{2,3}, g_2y_{2,3} + g_6y_{1,4}. \end{gathered} \tag{6.1.21}$$

Using GAP we get the basis of $(M^2)^!_{-n}$ for $n \in [\![0,3]\!]$ given in Appendix A.5. Let $\mathcal{U}^{!,M^2}_n$ be the subset of $(M^2)^!_{-n}$ consisting of the following 24 elements

$$\begin{gathered} g_1y_{1,2}^{n-1}y_{1,3}, g_1y_{1,2}^{n-1}y_{2,3}, g_1y_{1,2}^{n-1}y_{1,4}, g_1y_{1,2}^{n-1}y_{2,4}, g_1y_{1,2}^{n-2}y_{1,3}^2, g_1y_{1,2}^{n-2}y_{1,3}y_{1,4}, g_1y_{1,2}^{n-2}y_{1,3}y_{2,4}, \\ g_1y_{1,2}^{n-2}y_{1,3}y_{3,4}, g_1y_{1,2}^{n-2}y_{2,3}y_{1,4}, g_1y_{1,2}^{n-2}y_{2,3}y_{2,4}, g_1y_{1,2}^{n-2}y_{2,3}y_{3,4}, g_1y_{1,2}^{n-2}y_{1,4}^2, g_1y_{1,2}^{n-3}y_{1,3}^2y_{3,4}, \\ g_1y_{1,2}^{n-3}y_{1,3}y_{1,4}^2, g_1y_{1,2}^{n-3}y_{2,3}y_{1,4}^2, g_2y_{1,2}^{n-1}y_{1,4}, g_2y_{1,2}^{n-1}y_{2,4}, g_2y_{1,2}^{n-2}y_{1,4}^2, g_3y_{1,3}^{n-1}y_{1,4}, g_3y_{1,3}^{n-1}y_{3,4}, \\ g_3y_{1,3}^{n-2}y_{1,4}^2, g_4y_{2,3}^{n-1}y_{2,4}, g_4y_{2,3}^{n-1}y_{3,4}, g_4y_{2,3}^{n-2}y_{2,4}^2, \end{gathered} \tag{6.1.22}$$

and $\mathscr{C}^{!,M^2}_n$ the subset of $(M^2)^!_{-n}$ consisting of the following $3n + 21$ elements

$$\begin{gathered} g_1y_{1,2}^n, g_1y_{1,2}^{n-r}y_{3,4}^r, g_1y_{3,4}^n, g_2y_{1,2}^n, g_3y_{1,2}y_{3,4}^{n-1}, g_3y_{3,4}^n, g_4y_{3,4}^n, g_5y_{1,2}^n, g_5y_{1,2}^{n-1}y_{3,4}, \\ g_1y_{1,3}^n, g_1y_{1,3}^{n-r}y_{2,4}^r, g_1y_{2,4}^n, g_2y_{1,3}y_{2,4}^{n-1}, g_2y_{2,4}^n, g_3y_{1,3}^n, g_4y_{2,4}^n, g_6y_{1,3}^n, g_6y_{1,3}^{n-1}y_{2,4}, \\ g_1y_{2,3}^n, g_1y_{2,3}^{n-r}y_{1,4}^r, g_1y_{1,4}^n, g_2y_{2,3}y_{1,4}^{n-1}, g_2y_{1,4}^n, g_3y_{1,4}^n, g_4y_{2,3}^n, g_5y_{2,3}^n, g_5y_{2,3}^{n-1}y_{1,4}, \end{gathered} \tag{6.1.23}$$

where $r \in [\![1,n-1]\!]$ and $n \geqslant 4$.

Lemma 6.1.20. *The set $T^{M^2}_n = \mathcal{U}^{!,M^2}_n \cup \mathscr{C}^{!,M^2}_n$ is a basis of $(M^2)^!_{-n}$ for $n \geqslant 4$. Moreover, $\dim(M^2)^!_0 = 7$, $\dim(M^2)^!_{-1} = 24$, $\dim(M^2)^!_{-2} = 43$ and $\dim(M^2)^!_{-n} = 3n + 45$ for $n \geqslant 3$.*

Proof. We will prove that the set $T^{M^2}_n$ is a basis of $(M^2)^!_{-n}$ for $n \geqslant 4$. Firstly, using GAP, $T^{M^2}_n$ is a basis of $(M^2)^!_{-n}$ for $n \in [\![4,7]\!]$. Note that the space $(M^2)^!_{-n}$ is spanned by $\{g_iy \mid i \in [\![1,7]\!], y \in \mathscr{B}^!_n\}$ for $n \in \mathbb{N}_0$. Moreover, the following identities are straightforward to verify and are left to the reader:

$$\begin{aligned} g_1y_{1,2}^{n-3}y_{1,3}^2y_{1,4} &= g_1y_{1,2}^{n-3}y_{1,4}y_{1,3}^2 = -\chi_n g_1y_{2,4}^{n-3}y_{1,2}y_{1,3}^2 + \chi_{n+1}g_1y_{2,4}^{n-3}y_{1,4}y_{1,3}^2 \\ &= \chi_n g_5y_{1,3}y_{2,4}^{n-4}y_{1,2}y_{1,3}^2 - \chi_{n+1}g_5y_{1,3}y_{2,4}^{n-4}y_{1,4}y_{1,3}^2 = -g_5y_{1,2}^3y_{2,3}y_{2,4}^{n-4} \\ &= -g_6y_{3,4}y_{1,2}^2y_{2,3}y_{2,4}^{n-4} = g_6y_{2,3}y_{1,2}^2y_{2,4}^{n-3} = g_1y_{1,4}y_{1,2}^2y_{2,4}^{n-3} = g_1y_{1,2}^{n-1}y_{1,4}, \\ g_1y_{1,2}^{n-3}y_{1,3}^2y_{2,4} &= g_1y_{1,2}^{n-3}y_{2,4}y_{1,3}^2 = -\chi_n g_1y_{1,4}^{n-3}y_{1,2}y_{1,3}^2 + \chi_{n+1}g_1y_{1,4}^{n-3}y_{2,4}y_{1,3}^2 \\ &= -\chi_n g_6y_{2,3}y_{1,4}^{n-4}y_{1,2}y_{1,3}^2 + \chi_{n+1}g_6y_{2,3}y_{1,4}^{n-4}y_{2,4}y_{1,3}^2 \\ &= \chi_n g_6y_{1,2}^3y_{1,3}y_{1,4}^{n-4} - \chi_{n+1}g_6y_{1,2}^{n-2}y_{1,3}y_{1,4} \\ &= -\chi_n g_5y_{3,4}y_{1,2}^2y_{1,3}y_{1,4}^{n-4} + \chi_{n+1}g_5y_{3,4}y_{1,2}^{n-3}y_{1,3}y_{1,4} \\ &= \chi_n g_5y_{1,3}y_{1,2}^2y_{1,4}^{n-3} - \chi_{n+1}g_5y_{1,3}y_{1,4}y_{1,2}^{n-3}y_{1,4} \\ &= -\chi_n g_1y_{2,4}y_{1,2}^2y_{1,4}^{n-3} + \chi_{n+1}g_1y_{2,4}y_{1,4}y_{1,2}^{n-3}y_{1,4} = g_1y_{1,2}^{n-1}y_{2,4}, \\ g_1y_{1,3}^{n-1}y_{1,4} &= -\chi_n g_1y_{3,4}^{n-1}y_{1,3} + \chi_{n+1}g_1y_{3,4}^{n-1}y_{1,4} = -\chi_n g_7y_{1,2}y_{3,4}^{n-2}y_{1,3} + \chi_{n+1}g_7y_{1,2}y_{3,4}^{n-2}y_{1,4} \\ &= -\chi_n g_7y_{1,4}^2y_{1,2}^{n-3}y_{1,3} + \chi_{n+1}g_7y_{2,3}y_{1,2}y_{3,4}y_{1,2}^{n-3} \end{aligned}$$

$$
\begin{aligned}
&= \chi_n g_5 y_{2,3} y_{1,4} y_{1,2}^{n-3} y_{1,3} + \chi_{n+1} g_5 y_{1,4} y_{1,2} y_{3,4} y_{1,2}^{n-3} \\
&= -\chi_n g_5 y_{1,3}^2 y_{1,2} y_{2,4} y_{1,2}^{n-4} - \chi_{n+1} g_5 y_{1,3} y_{1,4} y_{1,2}^{n-2} \\
&= \chi_n g_1 y_{2,4} y_{1,3} y_{1,2} y_{2,4} y_{1,2}^{n-4} + \chi_{n+1} g_1 y_{2,4} y_{1,4} y_{1,2}^{n-2} \\
&= \chi_n g_1 y_{1,2}^{n-2} y_{1,3} y_{1,4} + \chi_{n+1} g_1 y_{1,2}^{n-1} y_{1,4}, \\
g_1 y_{1,3}^{n-1} y_{3,4} &= -\chi_n g_1 y_{3,4} y_{1,4} y_{1,3}^{n-2} + \chi_{n+1} g_1 y_{3,4} y_{1,3}^{n-1} = -\chi_n g_7 y_{1,2} y_{1,4} y_{1,3}^{n-2} + \chi_{n+1} g_7 y_{1,2} y_{1,3}^{n-1} \\
&= \chi_n g_7 y_{2,4} y_{1,2} y_{1,3}^{n-2} + \chi_{n+1} g_7 y_{1,3}^2 y_{1,2}^{n-2} \\
&= -\chi_n g_6 y_{1,3} y_{1,2} y_{1,3}^{n-2} + \chi_{n+1} g_6 y_{2,4} y_{1,3} y_{1,2}^{n-2} \\
&= -\chi_n g_6 y_{2,3}^3 y_{1,3}^{n-3} + \chi_{n+1} g_6 y_{2,3} y_{1,3} y_{1,4} y_{1,2}^{n-3} \\
&= -\chi_n g_1 y_{1,4} y_{2,3}^2 y_{1,3}^{n-3} + \chi_{n+1} g_1 y_{1,4} y_{1,3} y_{1,4} y_{1,2}^{n-3} \\
&= \chi_n g_1 y_{1,2}^{n-2} y_{1,3} y_{3,4} + \chi_{n+1} g_1 y_{1,2}^{n-3} y_{1,3}^2 y_{3,4}, \\
g_1 y_{2,3}^{n-1} y_{2,4} &= g_1 y_{2,4} y_{3,4}^{n-1} = -g_5 y_{1,3} y_{3,4}^{n-1} = \chi_n g_5 y_{1,4} y_{1,3}^{n-1} - \chi_{n+1} g_5 y_{1,4}^2 y_{1,3}^{n-2} \\
&= \chi_n g_7 y_{2,3} y_{1,3}^{n-1} - \chi_{n+1} g_7 y_{2,3} y_{1,4} y_{1,3}^{n-2} = -\chi_n g_7 y_{1,2}^{n-1} y_{2,3} - \chi_{n+1} g_7 y_{1,2}^{n-2} y_{2,3} y_{3,4} \\
&= -\chi_n g_1 y_{3,4} y_{1,2}^{n-2} y_{2,3} - \chi_{n+1} g_1 y_{3,4} y_{1,2}^{n-3} y_{2,3} y_{3,4} \\
&= \chi_n g_1 y_{1,2}^{n-2} y_{2,3} y_{2,4} + \chi_{n+1} g_1 y_{1,2}^{n-3} y_{1,3}^2 y_{2,4} = \chi_n g_1 y_{1,2}^{n-2} y_{2,3} y_{2,4} + \chi_{n+1} g_1 y_{1,2}^{n-1} y_{2,4}, \\
g_1 y_{2,3}^{n-1} y_{3,4} &= (-1)^{n+1} g_1 y_{3,4} y_{2,4}^{n-1} = (-1)^{n+1} g_7 y_{1,2} y_{2,4}^{n-1} \\
&= -\chi_n g_7 y_{2,4}^2 y_{1,2} y_{2,4}^{n-3} + \chi_{n+1} g_7 y_{2,4}^{n-1} y_{1,2} \\
&= \chi_n g_6 y_{1,3} y_{2,4} y_{1,2} y_{2,4}^{n-3} - \chi_{n+1} g_6 y_{1,3} y_{2,4}^{n-2} y_{1,2} \\
&= -\chi_n g_6 y_{2,3} y_{1,3} y_{1,4} y_{2,4}^{n-3} + \chi_{n+1} g_6 y_{2,3} y_{1,3}^{n-2} y_{1,4} \\
&= -\chi_n g_1 y_{1,4} y_{1,3} y_{1,4} y_{2,4}^{n-3} + \chi_{n+1} g_1 y_{1,4} y_{1,3}^{n-2} y_{1,4} \\
&= \chi_n g_1 y_{1,2}^{n-2} y_{2,3} y_{3,4} + \chi_{n+1} g_1 y_{1,3}^{n-1} y_{3,4} = \chi_n g_1 y_{1,2}^{n-2} y_{2,3} y_{3,4} + \chi_{n+1} g_1 y_{1,2}^{n-3} y_{1,3}^2 y_{3,4}, \\
g_2 y_{1,3}^2 &= g_5 y_{2,4} y_{1,3} = -g_5 y_{1,3} y_{2,4} = g_1 y_{2,4}^2, \ g_2 y_{2,3}^2 = -g_6 y_{1,4} y_{2,3} = g_6 y_{2,3} y_{1,4} = g_1 y_{1,4}^2, \\
g_2 y_{1,2}^{n-1} y_{1,3} &= -\chi_n g_2 y_{2,3}^3 y_{1,2}^{n-3} + \chi_{n+1} g_2 y_{2,3}^2 y_{1,3} y_{1,2}^{n-3} \\
&= -\chi_n g_1 y_{1,4}^2 y_{2,3} y_{1,2}^{n-3} + \chi_{n+1} g_1 y_{1,4}^2 y_{1,3} y_{1,2}^{n-3} = g_1 y_{1,2}^{n-3} y_{1,3} y_{1,4}^2, \\
g_2 y_{1,2}^{n-1} y_{2,3} &= -\chi_n g_2 y_{1,3}^3 y_{1,2}^{n-3} + \chi_{n+1} g_2 y_{1,3}^2 y_{2,3} y_{1,2}^{n-3} \\
&= -\chi_n g_1 y_{2,4}^2 y_{1,3} y_{1,2}^{n-3} + \chi_{n+1} g_1 y_{2,4}^2 y_{2,3} y_{1,2}^{n-3} = g_1 y_{1,2}^{n-3} y_{2,3} y_{1,4}^2, \\
g_3 y_{1,2}^2 &= g_7 y_{3,4} y_{1,2} = -g_7 y_{1,2} y_{3,4} = -g_1 y_{3,4}^2, \\
g_5 y_{3,4}^2 &= -g_6 y_{1,2} y_{3,4} = g_6 y_{3,4} y_{1,2} = g_5 y_{1,2}^2, \ g_5 y_{1,4}^2 = g_7 y_{2,3} y_{1,4} = -g_7 y_{1,4} y_{2,3} = g_5 y_{2,3}^2, \\
g_5 y_{1,2}^{n-1} y_{1,3} &= \chi_n g_5 y_{1,3} y_{2,3} y_{1,2}^{n-2} + \chi_{n+1} g_5 y_{1,3} y_{1,2}^{n-1} = -\chi_n g_1 y_{2,4} y_{2,3} y_{1,2}^{n-2} - \chi_{n+1} g_1 y_{2,4} y_{1,2}^{n-1} \\
&= \chi_n g_1 y_{1,2}^{n-2} y_{2,3} y_{3,4} - \chi_{n+1} g_1 y_{1,2}^{n-1} y_{2,4}, \\
g_5 y_{1,2}^{n-1} y_{2,3} &= -\chi_n g_5 y_{1,3} y_{1,2}^{n-1} + \chi_{n+1} g_5 y_{1,3}^2 y_{2,3} y_{1,2}^{n-3} \\
&= \chi_n g_1 y_{2,4} y_{1,2}^{n-1} - \chi_{n+1} g_1 y_{2,4} y_{1,3} y_{2,3} y_{1,2}^{n-3} \\
&= -\chi_n g_1 y_{1,2}^{n-1} y_{1,4} - \chi_{n+1} g_1 y_{1,2}^{n-2} y_{1,3} y_{3,4}, \\
g_5 y_{1,2}^{n-1} y_{1,4} &= \chi_n g_5 y_{3,4}^2 y_{1,2}^{n-3} y_{1,4} = \chi_n g_5 y_{1,3}^2 y_{1,4} y_{2,4} y_{1,2}^{n-4} + \chi_{n+1} g_5 y_{1,3}^2 y_{1,4} y_{1,2}^{n-3} \\
&= -\chi_n g_1 y_{2,4} y_{1,3} y_{1,4} y_{2,4} y_{1,2}^{n-4} - \chi_{n+1} g_1 y_{2,4} y_{1,3} y_{1,4} y_{1,2}^{n-3} \\
&= \chi_n g_1 y_{1,2}^{n-3} y_{2,3} y_{1,4}^2 + \chi_{n+1} g_1 y_{1,2}^{n-2} y_{2,3} y_{2,4}, \\
g_5 y_{1,2}^{n-1} y_{2,4} &= \chi_n g_5 y_{3,4}^2 y_{1,2}^{n-3} y_{2,4} = -\chi_n g_5 y_{1,3}^2 y_{1,4} y_{1,2}^{n-3} - \chi_{n+1} g_5 y_{1,3}^2 y_{1,4} y_{2,4} y_{1,2}^{n-4} \\
&= \chi_n g_1 y_{2,4} y_{1,3} y_{1,4} y_{1,2}^{n-3} + \chi_{n+1} g_1 y_{2,4} y_{1,3} y_{1,4} y_{2,4} y_{1,2}^{n-4} \\
&= -\chi_n g_1 y_{1,2}^{n-2} y_{1,3} y_{1,4} + \chi_{n+1} g_1 y_{1,2}^{n-3} y_{1,3} y_{1,4}^2, \\
g_5 y_{2,3}^{n-1} y_{2,4} &= g_5 y_{1,4}^2 y_{2,3}^{n-3} y_{2,4} = \chi_n g_5 y_{1,3}^2 y_{2,3} y_{2,4} y_{2,3}^{n-4} - \chi_{n+1} g_5 y_{1,3}^2 y_{2,3} y_{3,4} y_{2,3}^{n-4} \\
&= -\chi_n g_1 y_{2,4} y_{1,3} y_{2,3} y_{2,4} y_{2,3}^{n-4} + \chi_{n+1} g_1 y_{2,4} y_{1,3} y_{2,3} y_{3,4} y_{2,3}^{n-4} \\
&= \chi_n g_1 y_{1,2}^{n-2} y_{1,3} y_{3,4} + \chi_{n+1} g_1 y_{1,2}^{n-3} y_{1,3} y_{1,4}^2, \\
g_5 y_{2,3}^{n-1} y_{3,4} &= g_5 y_{1,4}^2 y_{2,3}^{n-3} y_{3,4} = \chi_n g_5 y_{1,3}^2 y_{2,3} y_{3,4} y_{2,3}^{n-4} + \chi_{n+1} g_5 y_{1,3}^2 y_{3,4} y_{2,3}^{n-3}
\end{aligned}
$$

$$= -\chi_n g_1 y_{2,4} y_{1,3} y_{2,3} y_{3,4} y_{2,3}^{n-4} - \chi_{n+1} g_1 y_{2,4} y_{1,3} y_{3,4} y_{2,3}^{n-3}$$
$$= -\chi_n g_1 y_{1,2}^{n-3} y_{1,3} y_{1,4}^2 + \chi_{n+1} g_1 y_{1,2}^{n-2} y_{1,3} y_{2,4},$$
$$g_6 y_{2,4}^2 = g_7 y_{1,2} y_{2,4} = -g_7 y_{2,4} y_{1,3} = g_6 y_{1,3}^2,$$
$$g_6 y_{1,3}^{n-1} y_{1,4} = -\chi_n g_6 y_{3,4} y_{1,3}^{n-1} + \chi_{n+1} g_6 y_{1,4} y_{1,3}^{n-1} = -\chi_n g_5 y_{1,2} y_{1,3}^{n-1} - \chi_{n+1} g_2 y_{2,3} y_{1,3}^{n-1}$$
$$= -\chi_n g_1 y_{1,2}^{n-2} y_{2,3} y_{3,4} - \chi_{n+1} g_1 y_{1,2}^{n-3} y_{2,3} y_{1,4}^2,$$
$$g_6 y_{1,3}^{n-1} y_{3,4} = -\chi_n g_6 y_{1,4} y_{1,3}^{n-1} + \chi_{n+1} g_6 y_{3,4} y_{1,3}^{n-1} = \chi_n g_2 y_{2,3} y_{1,3}^{n-1} + \chi_{n+1} g_5 y_{1,2} y_{1,3}^{n-1}$$
$$= -\chi_n g_1 y_{1,2}^{n-3} y_{2,3} y_{1,4}^2 + \chi_{n+1} g_1 y_{1,2}^{n-2} y_{2,3} y_{1,4},$$

for $n \geqslant 5$. Using the previous identities together with (6.1.21) we see that the space $(M^2)^!_{-n}$ is spanned by $T_n^{M^2}$ for $n \geqslant 8$.

We will next prove that the elements in $T_n^{M^2}$ for $n \geqslant 8$ are linearly independent. Suppose that we have the identity

$$\sum_{i \in [\![1,24]\!]} \alpha_i t_i + \sum_{i \in [\![1,n+7]\!]} \alpha_i^{1,2} t_i^{1,2} + \sum_{i \in [\![1,n+7]\!]} \alpha_i^{1,3} t_i^{1,3} + \sum_{i \in [\![1,n+7]\!]} \alpha_i^{2,3} t_i^{2,3} = \sum_{\substack{i \in [\![1,18]\!], \\ u \in \mathscr{B}^!_{n-1}}} \lambda_u^i r_i u, \tag{6.1.24}$$

in $\Bbbk\{g_i | i \in [\![1,7]\!]\} \otimes A^!$, where t_i is the i-th element in (6.1.22) for $i \in [\![1,24]\!]$, $t_i^{1,2}$ is the i-th element in the first line of (6.1.23), $t_i^{1,3}$ is the i-th element in the second line of (6.1.23), and $t_i^{2,3}$ is the i-th element in the last line of (6.1.23) for $i \in [\![1, n+7]\!]$, r_i is the i-th element in (6.1.21), and $\alpha_i, \alpha_i^{1,2}, \alpha_i^{1,3}, \alpha_i^{2,3}, \lambda_u^i \in \Bbbk$. We need to prove that the coefficients α_i vanish for all $i \in [\![1,24]\!]$, as well as that $\alpha_i^{1,2}$, $\alpha_i^{1,3}$ and $\alpha_i^{2,3}$ vanish for all $i \in [\![1, n+7]\!]$. By Lemma 6.1.15, (6.1.24) implies that

$$\sum_{i \in [\![1,n+7]\!]} \alpha_i^{1,2} t_i^{1,2} = \sum_{\substack{i \in [\![1,6]\!], \\ u \in \mathscr{B}^!_{n-1} \cap Y_{1,2}}} \lambda_u^i r_i u, \tag{6.1.25}$$

$$\sum_{i \in [\![1,n+7]\!]} \alpha_i^{1,3} t_i^{1,3} = \sum_{\substack{i \in [\![7,12]\!], \\ u \in \mathscr{B}^!_{n-1} \cap Y_{1,3}}} \lambda_u^i r_i u, \tag{6.1.26}$$

$$\sum_{i \in [\![1,n+7]\!]} \alpha_i^{2,3} t_i^{2,3} = \sum_{\substack{i \in [\![13,18]\!], \\ u \in \mathscr{B}^!_{n-1} \cap Y_{2,3}}} \lambda_u^i r_i u, \tag{6.1.27}$$

and

$$\sum_{i \in [\![1,24]\!]} \alpha_i t_i = \sum_{\substack{i \in [\![1,6]\!], \\ u \in \mathscr{B}^!_{n-1} \setminus Y_{1,2}}} \lambda_u^i r_i u + \sum_{\substack{i \in [\![7,12]\!], \\ u \in \mathscr{B}^!_{n-1} \setminus Y_{1,3}}} \lambda_u^i r_i u + \sum_{\substack{i \in [\![13,18]\!], \\ u \in \mathscr{B}^!_{n-1} \setminus Y_{2,3}}} \lambda_u^i r_i u, \tag{6.1.28}$$

in $\Bbbk\{g_i | i \in [\![1,7]\!]\} \otimes A^!$. By (6.1.25), we get $\alpha_i^{1,2} = 0$ for $i \in [\![1,n+7]\!]$. Indeed, since there is no $g_2 y_{1,2}^n$, $g_3 y_{3,4}^n$, $g_4 y_{3,4}^n$ on the right side of (6.1.25), we get that $\alpha_{n+2}^{1,2} = \alpha_{n+4}^{1,2} = \alpha_{n+5}^{1,2} = 0$. Furthermore, as there is no $g_4 y_{1,2}^{n-r} y_{3,4}^r$ for $n - r \in \mathbb{N}$ on the left side of (6.1.25), we see that $\lambda_u^2 = 0$ for $u \in \mathscr{B}^!_{n-1} \cap Y_{1,2}$. Moreover, since there is no $g_7 y_{3,4}^n$ on the left side of (6.1.25), we obtain that $\lambda^6_{y_{3,4}^{n-1}} = 0$. This implies that $\alpha_{n+3}^{1,2} = 0$ and $\lambda_u^6 = 0$ for $u \in \mathscr{B}^!_{n-1} \cap Y_{1,2}$. Finally, since there is no $g_2 u$ and $g_7 u$ for $u \in \mathscr{B}^!_n$ on the left side of (6.1.25), we have that $\lambda_u^1 = \lambda_u^5 = 0$ for $u \in \mathscr{B}^!_{n-1} \cap Y_{1,2}$. In consequence, we get $\alpha_i^{1,2} = 0$ for $i \in [\![1, n+1]\!]$. Now, we have that

$$\alpha_{n+6}^{1,2} g_5 y_{1,2}^n + \alpha_{n+7}^{1,2} g_5 y_{1,2}^{n-1} y_{3,4} = \sum_{r \in [\![0,n-1]\!]} \mathscr{a}_r (g_5 y_{1,2} - g_6 y_{3,4}) u + \sum_{r \in [\![0,n-1]\!]} \mathscr{b}_r (g_5 y_{3,4} + g_6 y_{1,2}) u$$

in $\Bbbk\{g_i | i \in [\![1,7]\!]\} \otimes A^!$, where $\mathscr{a}_r = \lambda^3_{y_{1,2}^{n-1-r} y_{3,4}^r}$ and $\mathscr{b}_r = \lambda^4_{y_{1,2}^{n-1-r} y_{3,4}^r}$ for $r \in [\![0, n-1]\!]$. Hence,

$$\alpha_{n+6}^{1,2} g_5 y_{1,2}^n + \alpha_{n+7}^{1,2} g_5 y_{1,2}^{n-1} y_{3,4}$$
$$= \mathscr{a}_0 g_5 y_{1,2}^n + \sum_{r \in [\![1,n-1]\!]} \Big(\mathscr{a}_r + \big((-1)^r \chi_n + (-1)^{r-1} \chi_{n+1}\big) \mathscr{b}_{r-1} \Big) g_5 y_{1,2}^{n-r} y_{3,4}^r + \mathscr{b}_{n-1} g_5 y_{3,4}^n$$
$$+ \mathscr{b}_0 g_6 y_{1,2}^n + \sum_{r \in [\![1,n-1]\!]} \Big(\mathscr{b}_r + \big((-1)^{r-1} \chi_n + (-1)^r \chi_{n+1}\big) \mathscr{a}_{r-1} \Big) g_6 y_{1,2}^{n-r} y_{3,4}^r - \mathscr{a}_{n-1} g_6 y_{3,4}^n$$

in $\Bbbk\{g_i | i \in [\![1,7]\!]\} \otimes A^!$. Comparing the coefficients, it is easy to see that $\alpha^{1,2}_{n+6} = \alpha^{1,2}_{n+7} = 0$ and $a_r = b_r = 0$ for $r \in [\![0,n-1]\!]$. Similarly, (6.1.26) implies $\alpha^{1,3}_i = 0$ for $i \in [\![1,n+7]\!]$, and (6.1.27) implies $\alpha^{2,3}_i = 0$ for $i \in [\![1,n+7]\!]$. By regarding the coefficients of g_i in (6.1.28) for $i \in [\![1,7]\!]$, we get that (6.1.28) is tantamount to

$$\begin{aligned}
g_1\big(y_{3,4}\Delta^1 + y_{3,4}\Delta^5 + y_{2,4}\Delta^7 + y_{2,4}\Delta^{11} + y_{1,4}\Delta^{13} + y_{1,4}\Delta^{17}\big) &= \sum_{i\in[\![1,15]\!]} \alpha_i t_i,\\
g_2\big(-y_{3,4}\Delta^1 + y_{1,3}\Delta^8 + y_{1,3}\Delta^{12} + y_{2,3}\Delta^{14} + y_{2,3}\Delta^{18}\big) &= \sum_{i\in[\![16,18]\!]} \alpha_i t_i,\\
g_3\big(y_{1,2}\Delta^2 + y_{1,2}\Delta^6 + y_{2,4}\Delta^7 + y_{2,3}\Delta^{14}\big) &= \sum_{i\in[\![19,21]\!]} \alpha_i t_i,\\
g_4\big(-y_{1,2}\Delta^2 + y_{1,3}\Delta^8 + y_{1,4}\Delta^{13}\big) &= \sum_{i\in[\![22,24]\!]} \alpha_i t_i,\\
g_5\big(y_{1,2}\Delta^3 + y_{3,4}\Delta^4 + y_{1,3}\Delta^{11} - y_{2,4}\Delta^{12} + y_{2,3}\Delta^{15} + y_{1,4}\Delta^{16}\big) &= 0,\\
g_6\big(-y_{3,4}\Delta^3 + y_{1,2}\Delta^4 + y_{1,3}\Delta^9 + y_{2,4}\Delta^{10} - y_{2,3}\Delta^{11} + y_{1,4}\Delta^{12}\big) &= 0,\\
g_7\big(-y_{1,2}\Delta^5 - y_{3,4}\Delta^6 + y_{2,4}\Delta^9 - y_{1,3}\Delta^{10} + y_{1,4}\Delta^{15} - y_{2,3}\Delta^{16}\big) &= 0,
\end{aligned} \tag{6.1.29}$$

in $\Bbbk\{g_i\} \otimes A^!$ for $i \in [\![1,7]\!]$ respectively, where $\Delta^j = \sum_{u\in\mathscr{B}^!_{n-1}\setminus Y_{1,2}} \lambda^j_u u$ for $j \in [\![1,6]\!]$, $\Delta^j = \sum_{u\in\mathscr{B}^!_{n-1}\setminus Y_{1,3}} \lambda^j_u u$ for $j \in [\![7,12]\!]$, $\Delta^j = \sum_{u\in\mathscr{B}^!_{n-1}\setminus Y_{2,3}} \lambda^j_u u$ for $j \in [\![13,18]\!]$ and $Y_{i,j}$ is defined in (6.1.18). In consequence, we see that the elements in $T^{M^2}_n$ are linearly independent if and only if equation (6.1.29) implies that $\alpha_i = 0$ for $i \in [\![1,24]\!]$.

Let

$$\begin{aligned}
a^i_0 &= \lambda^i_{y^{n-1}_{1,2}}, a'^i_0 = \lambda^i_{y^{n-1}_{3,4}}, a^i_1 = \sum_{\substack{r\,\in\,[\![1,\,n\,-\,2]\!],\\ r\ odd}} \lambda^i_{y^{n-1-r}_{1,2}y^r_{3,4}}, a^i_2 = \sum_{\substack{r\,\in\,[\![1,\,n\,-\,2]\!],\\ r\ even}} \lambda^i_{y^{n-1-r}_{1,2}y^r_{3,4}},\\
b^j_0 &= \lambda^j_{y^{n-1}_{1,3}}, b'^j_0 = \lambda^j_{y^{n-1}_{2,4}}, b^j_1 = \sum_{\substack{r\,\in\,[\![1,\,n\,-\,2]\!],\\ r\ odd}} \lambda^j_{y^{n-1-r}_{1,3}y^r_{2,4}}, b^j_2 = \sum_{\substack{r\,\in\,[\![1,\,n\,-\,2]\!],\\ r\ even}} \lambda^j_{y^{n-1-r}_{1,3}y^r_{2,4}},\\
c^k_0 &= \lambda^k_{y^{n-1}_{2,3}}, c'^k_0 = \lambda^k_{y^{n-1}_{1,4}}, c^k_1 = \sum_{\substack{r\,\in\,[\![1,\,n\,-\,2]\!],\\ r\ odd}} \lambda^k_{y^{n-1-r}_{2,3}y^r_{1,4}}, c^k_2 = \sum_{\substack{r\,\in\,[\![1,\,n\,-\,2]\!],\\ r\ even}} \lambda^k_{y^{n-1-r}_{2,3}y^r_{1,4}},
\end{aligned}$$

for $i \in [\![7,18]\!]$, $j \in [\![1,6]\!] \cup [\![13,18]\!]$ and $k \in [\![1,12]\!]$. From (6.1.29) as well as the products (6.1.4) and (6.1.5) in $A^!$, we get a system E_n of linear equations in the field $\Bbbk$, which contains $24 \times 7 = 168$ linear equations and $24 + 24 \times 18 + 4 \times 12 \times 3 = 600$ variables α_i, λ^j_u for $u \in \mathscr{U}^!_{n-1}$, a^i_0, a'^i_0, a^i_1, a^i_2, b^j_0, b'^j_0, b^j_1, b^j_2, c^k_0, c'^k_0, c^k_1, c^k_2. Moreover, the linear independence of $T^{M^2}_n$ (or, equivalently, the fact that (6.1.29) implies that $\alpha_i = 0$ for $i \in [\![1,24]\!]$) is equivalent to the fact that the linear system E_n implies that $\alpha_i = 0$ for $i \in [\![1,24]\!]$. Note that E_n has the same form when n increases by 2. Using GAP, the elements in $T^{M^2}_n$ are linearly independent for $n \in \{8,9\}$, so the lemma holds for all integers $n \geqslant 8$. □

Let $\mathscr{U}^{M^2}_n$ be the dual basis to $\mathscr{U}^{!,M^2}_n$, $\mathscr{C}^{M^2}_n$ the dual basis to $\mathscr{C}^{!,M^2}_n$, and $\mathscr{C}^{M^2}_n = \cup_{(i,j)\in\mathscr{I}_1} \mathscr{C}^{i,j,M^2}_n$, where $\mathscr{C}^{i,j,M^2}_n$ is the subset of $\mathscr{C}^{M^2}_n$ consisting of elements of the form $(g_s y^{n-\bullet}_{i,j} y^{\bullet}_{k,l})^*$ for $(i,j) \in \mathscr{I}_1$, $(k,l) \in \mathscr{I}$ with $\#\{i,j,k,l\} = 4$. Given $n, m \in \mathbb{N}$, let $C^{M^2}_{n,m}$ be the subspace of $\Bbbk\mathscr{C}^{M^2}_n \otimes A_m$ spanned by

$$\{d_{n+1}(M^2)(z|x) \mid z \in \mathscr{C}^{M^2}_{n+1}, x \in A_{m-1}\},$$

$C^{i,j,M^2}_{n,m}$ the subspace of $C^{M^2}_{n,m}$ spanned by

$$\{d_{n+1}(M^2)(z|x) \mid z \in \mathscr{C}^{i,j,M^2}_{n+1}, x \in A_{m-1}\}$$

for $(i,j) \in \mathscr{I}_1$, and $U^{M^2}_{n,m}$ the subspace of $B^{M^2}_{n,m}$ spanned by

$$\{d_{n+1}(M^2)(z|x) \mid z \in \mathscr{U}^{M^2}_{n+1}, x \in A_{m-1}\}.$$

Using the actions listed in Appendix A.6, it is direct but lengthy to check that the differential in the subcomplex $\Bbbk\mathscr{C}^{1,2,M^2}_{n+1} \otimes A$ of the Koszul complex is given by

$$
\begin{aligned}
(g_1 y_{1,2}^{n+1})^*|1 &\mapsto (g_1 y_{1,2}^{n})^*|x_{1,2},\\
(g_1 y_{1,2}^{n} y_{3,4})^*|1 &\mapsto -(g_1 y_{1,2}^{n-1} y_{3,4})^*|x_{1,2} + (g_1 y_{1,2}^{n})^*|x_{3,4} + (g_2 y_{1,2}^{n})^*|x_{3,4},\\
(g_1 y_{1,2}^{n+1-r} y_{3,4}^{r})^*|1 &\mapsto (-1)^r (g_1 y_{1,2}^{n-r} y_{3,4}^{r})^*|x_{1,2} + (g_1 y_{1,2}^{n+1-r} y_{3,4}^{r-1})^*|x_{3,4} \text{ for } r \in [\![2, n-1]\!],\\
(g_1 y_{1,2} y_{3,4}^{n})^*|1 &\mapsto (-1)^n (g_1 y_{3,4}^{n})^*|x_{1,2} + (g_1 y_{1,2} y_{3,4}^{n-1})^*|x_{3,4},\\
(g_1 y_{3,4}^{n+1})^*|1 &\mapsto (g_1 y_{3,4}^{n})^*|x_{3,4} + (-1)^n (g_3 y_{1,2} y_{3,4}^{n-1})^*|x_{1,2},\\
(g_2 y_{1,2}^{n+1})^*|1 &\mapsto (g_2 y_{1,2}^{n})^*|x_{1,2},\\
(g_3 y_{1,2} y_{3,4}^{n})^*|1 &\mapsto (-1)^n (g_3 y_{3,4}^{n})^*|x_{1,2} + (-1)^n (g_4 y_{3,4}^{n})^*|x_{1,2} + (g_3 y_{1,2} y_{3,4}^{n-1})^*|x_{3,4},\\
(g_3 y_{3,4}^{n+1})^*|1 &\mapsto (g_3 y_{3,4}^{n})^*|x_{3,4},\\
(g_4 y_{3,4}^{n+1})^*|1 &\mapsto (g_4 y_{3,4}^{n})^*|x_{3,4},\\
(g_5 y_{1,2}^{n+1})^*|1 &\mapsto (g_5 y_{1,2}^{n})^*|x_{1,2} + (g_5 y_{1,2}^{n-1} y_{3,4})^*|x_{3,4},\\
(g_5 y_{1,2}^{n} y_{3,4})^*|1 &\mapsto -(g_5 y_{1,2}^{n-1} y_{3,4})^*|x_{1,2} + (g_5 y_{1,2}^{n})^*|x_{3,4},
\end{aligned}
$$

for $n \geqslant 4$. Similarly, the differential in $\Bbbk\mathscr{C}^{1,3,M^2}_{n+1} \otimes A$ is given by

$$
\begin{aligned}
(g_1 y_{1,3}^{n+1})^*|1 &\mapsto (g_1 y_{1,3}^{n})^*|x_{1,3},\\
(g_1 y_{1,3}^{n} y_{2,4})^*|1 &\mapsto -(g_1 y_{1,3}^{n-1} y_{2,4})^*|x_{1,3} + (g_1 y_{1,3}^{n})^*|x_{2,4} - (g_3 y_{1,3}^{n})^*|x_{2,4},\\
(g_1 y_{1,3}^{n+1-r} y_{2,4}^{r})^*|1 &\mapsto (-1)^r (g_1 y_{1,3}^{n-r} y_{2,4}^{r})^*|x_{1,3} + (g_1 y_{1,3}^{n+1-r} y_{2,4}^{r-1})^*|x_{2,4} \text{ for } r \in [\![2, n-1]\!],\\
(g_1 y_{1,3} y_{2,4}^{n})^*|1 &\mapsto (-1)^n (g_1 y_{2,4}^{n})^*|x_{1,3} + (g_1 y_{1,3} y_{2,4}^{n-1})^*|x_{2,4},\\
(g_1 y_{2,4}^{n+1})^*|1 &\mapsto (g_1 y_{2,4}^{n})^*|x_{2,4} - (-1)^n (g_2 y_{1,3} y_{2,4}^{n-1})^*|x_{1,3},\\
(g_2 y_{1,3} y_{2,4}^{n})^*|1 &\mapsto (-1)^n (g_2 y_{2,4}^{n})^*|x_{1,3} - (-1)^n (g_4 y_{2,4}^{n})^*|x_{1,3} + (g_2 y_{1,3} y_{2,4}^{n-1})^*|x_{2,4},\\
(g_2 y_{2,4}^{n+1})^*|1 &\mapsto (g_2 y_{2,4}^{n})^*|x_{2,4},\\
(g_3 y_{1,3}^{n+1})^*|1 &\mapsto (g_3 y_{1,3}^{n})^*|x_{1,3},\\
(g_4 y_{2,4}^{n+1})^*|1 &\mapsto (g_4 y_{2,4}^{n})^*|x_{2,4},\\
(g_6 y_{1,3}^{n+1})^*|1 &\mapsto (g_6 y_{1,3}^{n})^*|x_{1,3} + (g_6 y_{1,3}^{n-1} y_{2,4})^*|x_{2,4},\\
(g_6 y_{1,3}^{n} y_{2,4})^*|1 &\mapsto -(g_6 y_{1,3}^{n-1} y_{2,4})^*|x_{1,3} + (g_6 y_{1,3}^{n})|x_{2,4},
\end{aligned}
$$

for $n \geqslant 4$, whereas the differential in $\Bbbk\mathscr{C}^{2,3,M^2}_{n+1} \otimes A$ is given by

$$
\begin{aligned}
(g_1 y_{2,3}^{n+1})^*|1 &\mapsto (g_1 y_{2,3}^{n})^*|x_{2,3},\\
(g_1 y_{2,3}^{n} y_{1,4})^*|1 &\mapsto -(g_1 y_{2,3}^{n-1} y_{1,4})^*|x_{2,3} + (g_1 y_{2,3}^{n})^*|x_{1,4} - (g_4 y_{2,3}^{n})^*|x_{1,4},\\
(g_1 y_{2,3}^{n+1-r} y_{1,4}^{r})^*|1 &\mapsto (-1)^r (g_1 y_{2,3}^{n-r} y_{1,4}^{r})^*|x_{2,3} + (g_1 y_{2,3}^{n+1-r} y_{1,4}^{r-1})^*|x_{1,4} \text{ for } r \in [\![2, n-1]\!],\\
(g_1 y_{2,3} y_{1,4}^{n})^*|1 &\mapsto (-1)^n (g_1 y_{1,4}^{n})^*|x_{2,3} + (g_1 y_{2,3} y_{1,4}^{n-1})^*|x_{1,4},\\
(g_1 y_{1,4}^{n+1})^*|1 &\mapsto (g_1 y_{1,4}^{n})^*|x_{1,4} - (-1)^n (g_2 y_{2,3} y_{1,4}^{n-1})^*|x_{2,3},\\
(g_2 y_{2,3} y_{1,4}^{n})^*|1 &\mapsto (-1)^n (g_2 y_{1,4}^{n})^*|x_{2,3} - (-1)^n (g_3 y_{1,4}^{n})^*|x_{2,3} + (g_2 y_{2,3} y_{1,4}^{n-1})^*|x_{1,4},\\
(g_2 y_{1,4}^{n+1})^*|1 &\mapsto (g_2 y_{1,4}^{n})^*|x_{1,4},\\
(g_3 y_{1,4}^{n+1})^*|1 &\mapsto (g_3 y_{1,4}^{n})^*|x_{1,4},\\
(g_4 y_{2,3}^{n+1})^*|1 &\mapsto (g_4 y_{2,3}^{n})^*|x_{2,3},\\
(g_5 y_{2,3}^{n+1})^*|1 &\mapsto (g_5 y_{2,3}^{n})^*|x_{2,3} + (g_5 y_{2,3}^{n-1} y_{1,4})^*|x_{1,4},\\
(g_5 y_{2,3}^{n} y_{1,4})^*|1 &\mapsto -(g_5 y_{2,3}^{n-1} y_{1,4})^*|x_{2,3} + (g_5 y_{2,3}^{n})^*|x_{1,4},
\end{aligned}
$$

for $n \geqslant 4$.

Recall that the sets $W^{i,j}_m$, $E^{i,j}_m$ and $\tilde{E}^{i,j}_m$ for $(i,j) \in \mathscr{J}_1$ are defined in the paragraph before Table 6.1.7. For $(i,j) \in \mathscr{J}_1$, $(k,l) \in \mathscr{J}$ with $\#\{i,j,k,l\} = 4$, let $\hat{E}^{i,j}_m$ be the subset of $W^{i,j}_m$ containing elements whose first element is $x_{i,j}$ and second element is not $x_{k,l}$. Let $E'^{i,j}_m$ be the subset of $W^{i,j}_m$ containing elements whose first element is $x_{i,j}$ and the second element is $x_{k,l}$. The left multiplication of $x_{k,l}$ from $\Bbbk\hat{E}^{i,j}_{m-1}$ to $\Bbbk E'^{i,j}_m$ is isomorphic. It is easy to check that $\#(\hat{E}^{i,j}_m \cup \tilde{E}^{i,j}_m) = \mathfrak{a}_m$, where $\mathfrak{a}_m$ is given in Table 6.1.7. A basis of $C^{1,2,M^2}_{n,m}$ is given by $d_{n+1}(M^2)(t_i|x)$ for $i \in [\![1,n+1]\!] \cup \{n+3, n+7, n+8\}$ and $x \in E^{1,2}_{m-1}$, $d_{n+1}(M^2)(t_i|x)$ for $i \in [\![n+4, n+6]\!]$ and $x \in \hat{E}^{1,2}_{m-1} \cup \tilde{E}^{1,2}_{m-1}$, $d_{n+1}(M^2)((g_1 y^{n+1}_{3,4})^*|x)$ for $x \in \tilde{E}^{1,2}_{m-1}$, where $t_i \in \mathscr{C}^{1,2,M^2}_{n+1}$ is the i-th element in the following sequence

$$(g_1 y^{n+1}_{1,2})^*, (g_1 y^{n+1-r}_{1,2} y^r_{3,4})^* \text{ for } r \in [\![1,n]\!], (g_1 y^{n+1}_{3,4})^*, (g_2 y^{n+1}_{1,2})^*, (g_3 y_{1,2} y^n_{3,4})^*, (g_3 y^{n+1}_{3,4})^*, (g_4 y^{n+1}_{3,4})^*, (g_5 y^{n+1}_{1,2})^*, (g_5 y^n_{1,2} y_{3,4})^*. \tag{6.1.30}$$

A basis of $C^{1,3,M^2}_{n,m}$ is given by $d_{n+1}(M^2)(t_i|x)$ for $i \in [\![1,n+1]\!] \cup \{n+5, n+7, n+8\}$ and $x \in E^{1,3}_{m-1}$, $d_{n+1}(M^2)(t_i|x)$ for $i \in \{n+3, n+4, n+6\}$ and $x \in \hat{E}^{1,3}_{m-1} \cup \tilde{E}^{1,3}_{m-1}$, $d_{n+1}(M^2)((g_1 y^{n+1}_{2,4})^*|x)$ for $x \in \tilde{E}^{1,3}_{m-1}$, where $t_i \in \mathscr{C}^{1,3,M^2}_{n+1}$ is the i-th element in the following sequence

$$(g_1 y^{n+1}_{1,3})^*, (g_1 y^{n+1-r}_{1,3} y^r_{2,4})^* \text{ for } r \in [\![1,n]\!], (g_1 y^{n+1}_{2,4})^*, (g_2 y_{1,3} y^n_{2,4})^*, (g_2 y^{n+1}_{2,4})^*, (g_3 y^{n+1}_{1,3})^*, (g_4 y^{n+1}_{2,4})^*, (g_6 y^{n+1}_{1,3})^*, (g_6 y^n_{1,3} y_{2,4})^*. \tag{6.1.31}$$

A basis of $C^{2,3,M^2}_{n,m}$ is given by $d_{n+1}(M^2)(t_i|x)$ for $i \in [\![1,n+1]\!] \cup [\![n+6, n+8]\!]$ and $x \in E^{2,3}_{m-1}$, $d_{n+1}(M^2)(t_i|x)$ for $i \in [\![n+3, n+5]\!]$ and $x \in \hat{E}^{2,3}_{m-1} \cup \tilde{E}^{2,3}_{m-1}$, $d_{n+1}(M^2)((g_1 y^{n+1}_{1,4})^*|x)$ for $x \in \tilde{E}^{2,3}_{m-1}$, where $t_i \in \mathscr{C}^{2,3,M^2}_{n+1}$ is the i-th element in the following sequence

$$(g_1 y^{n+1}_{2,3})^*, (g_1 y^{n+1-r}_{2,3} y^r_{1,4})^* \text{ for } r \in [\![1,n]\!], (g_1 y_{2,3} y^n_{1,4})^*, (g_1 y^{n+1}_{1,4})^*, (g_2 y^{n+1}_{1,4})^*, (g_3 y^{n+1}_{1,4})^*, (g_4 y^{n+1}_{2,3})^*, (g_5 y^{n+1}_{2,3})^*, (g_5 y^n_{2,3} y_{1,4})^*. \tag{6.1.32}$$

So, $\dim C^{i,j,M^2}_{n,m} = \mathfrak{a}_{m-1}(n+7) + \mathfrak{b}_{m-1}$, where $\mathfrak{a}_m$ and $\mathfrak{b}_m$ are given in Table 6.1.7.

Lemma 6.1.21. *We have* $C^{M^2}_{n,m} = \bigoplus_{(i,j) \in \mathscr{I}_1} C^{i,j,M^2}_{n,m}$ *and the dimension of* $C^{i,j,M^2}_{n,m}$ *is given by*

$$\dim C^{i,j,M^2}_{n,m} = \begin{cases} n+8, & \text{if } m = 1, \\ 5n+39, & \text{if } m = 2, \\ 14n+108, & \text{if } m = 3, \\ 28n+214, & \text{if } m = 4, \\ 43n+326, & \text{if } m = 5, \\ 53n+399, & \text{if } m = 6, \\ 53n+396, & \text{if } m = 7, \\ 43n+319, & \text{if } m = 8, \\ 28n+206, & \text{if } m = 9, \\ 14n+102, & \text{if } m = 10, \\ 5n+36, & \text{if } m = 11, \\ n+7, & \text{if } m = 12, \end{cases}$$

for all $(i,j) \in \mathscr{I}_1$ *and* $n \geqslant 4$*. Moreover, if* $(i,j) \in \mathscr{I}_1$, $n \geqslant 4$ *and* $m \geqslant 13$, $\dim C^{i,j,M^2}_{n,m} = 0$.

Lemma 6.1.22. *We have* $\dim U^{M^2}_{n,m} = \dim U^{M^2}_{n+2,m}$ *and* $\dim(U^{M^2}_{n,m} \cap C^{M^2}_{n,m}) = \dim(U^{M^2}_{n+2,m} \cap C^{M^2}_{n+2,m})$ *for* $n \geqslant 4$ *and* $m \in [\![1,12]\!]$.

Proof. Let

$$u^{i,j}_n = \sum_{\substack{r \in [\![1,\, n-1]\!], \\ r \text{ odd}}} (g_1 y^{n-r}_{i,j} y^r_{k,l})^*, \quad v^{i,j}_n = \sum_{\substack{r \in [\![1,\, n-1]\!], \\ r \text{ even}}} (g_1 y^{n-r}_{i,j} y^r_{k,l})^*,$$

for $(i,j) \in \mathscr{I}_1$, $(k,l) \in \mathscr{I}$ with $\#\{i,j,k,l\} = 4$. Let

$$\mathbb{Q}^{i,j}_n = \left(\mathscr{C}^{i,j,M^2}_n \setminus \{t^{i,j,n}_r | r \in [\![2,n]\!]\}\right) \cup \{u^{i,j}_n, v^{i,j}_n\}$$

for $(i,j) \in \mathcal{I}_1$, where $t_r^{i,j,n} = (g_1 y_{i,j}^{n-r+1} y_{k,l}^{r-1})^* \in \mathscr{C}_n^{i,j,M^2}$ for $r \in [\![2,n]\!]$. Let

$$\mathbb{Q}_n = \mathscr{U}_n^{M^2} \cup (\cup_{(i,j)\in\mathcal{I}_1} \mathbb{Q}_n^{i,j}).$$

It is clear that there is an isomorphism $\mathfrak{f}_n : \Bbbk\mathbb{Q}_n \to \Bbbk\mathbb{Q}_{n+2}$ of vector spaces. Consider thus the linear isomorphism $\mathfrak{g}_n = \mathfrak{f}_n \otimes \mathrm{id}_A : \Bbbk\mathbb{Q}_n \otimes A \to \Bbbk\mathbb{Q}_{n+2} \otimes A$. Then $U_{n,m}^{M^2} \subseteq \Bbbk\mathbb{Q}_n \otimes A_m$ and $\mathfrak{g}_n(U_{n,m}^{M^2}) = U_{n+2,m}^{M^2}$ for $n \geqslant 4$. Hence, $U_{n,m}^{M^2} \cong U_{n+2,m}^{M^2}$ as vector spaces for $n \geqslant 4$.

Let $F_{n,m}^{i,j} = (\Bbbk\mathbb{Q}_n^{i,j} \otimes A_m) \cap C_{n,m}^{i,j,M^2}$ for $(i,j) \in \mathcal{I}_1$. Then

$$U_{n,m}^{M^2} \cap C_{n,m}^{M^2} = U_{n,m}^{M^2} \cap (\Bbbk\mathbb{Q}_n \otimes A_m) \cap C_{n,m}^{M^2} = U_{n,m}^{M^2} \cap \Bigg(\bigoplus_{(i,j)\in\mathcal{I}_1} F_{n,m}^{i,j} \Bigg).$$

To prove that $\dim(U_{n,m}^{M^2} \cap C_{n,m}^{M^2}) = \dim(U_{n+2,m}^{M^2} \cap C_{n+2,m}^{M^2})$, it is sufficient to show that $\mathfrak{g}_n(F_{n,m}^{i,j}) = F_{n+2,m}^{i,j}$ for $(i,j) \in \mathcal{I}_1$. This follows directly from the next simple facts, whose proof is left to the reader. If n is even, $F_{n,m}^{i,j}$ is spanned by the elements

(E.1) $(g_1 y_{i,j}^n)^* | x_{i,j} x$, $(v_n^{i,j} | x_{k,l} - u_n^{i,j} | x_{i,j} + (g_1 y_{i,j}^n)^* | x_{k,l} + \xi^{i,j} | x_{k,l}) x$ for $x \in E_{m-1}^{i,j}$,

(E.2) $((g_1 y_{k,l}^n)^* | x_{k,l} + \eta^{i,j} | x_{i,j}) y$, $(v_n^{i,j} | x_{i,j} + u_n^{i,j} | x_{k,l} + (g_1 y_{k,l}^n)^* | x_{i,j}) y$ for $y \in \tilde{E}_{m-1}^{i,j}$,

(E.3) $v_n^{i,j} | x_{i,j} w$, $(g_1 y_{k,l}^n)^* | x_{i,j} w$ for $w \in E_{m-1}^{i,j} \setminus \tilde{E}_{m-1}^{i,j}$,

(E.4) $d_{n+1}(M^2)(t_r^{i,j} | x)$ for $(r,x) \in \Delta^{i,j}$,

whereas, if n is odd, $F_{n,m}^{i,j}$ is spanned by the elements

(O.1) $(g_1 y_{i,j}^n)^* | x_{i,j} x$, $(u_n^{i,j} | x_{k,l} + v_n^{i,j} | x_{i,j}) x$ for $x \in E_{m-1}^{i,j}$,

(O.2) $((g_1 y_{k,l}^n)^* | x_{k,l} + \eta^{i,j} | x_{i,j}) y$, $(v_n^{i,j} | x_{k,l} - u_n^{i,j} | x_{i,j} + (g_1 y_{i,j}^n)^* | x_{k,l} + \xi^{i,j} | x_{k,l} - (g_1 y_{k,l}^n)^* | x_{i,j}) y$ for $y \in \tilde{E}_{m-1}^{i,j}$,

(O.3) $u_n^{i,j} | x_{i,j} w$, $(g_1 y_{k,l}^n)^* | x_{i,j} w$ for $w \in E_{m-1}^{i,j} \setminus \tilde{E}_{m-1}^{i,j}$,

(O.4) $d_{n+1}(M^2)(t_r^{i,j} | x)$ for $(r,x) \in \Delta^{i,j}$.

Here, $t_r^{1,2}$ (resp., $t_r^{1,3}$, $t_r^{2,3}$) is the r-th element in (6.1.30) (resp., (6.1.31), (6.1.32)), $\xi^{1,2} = (g_2 y_{1,2}^n)^*$, $\xi^{1,3} = -(g_3 y_{1,3}^n)^*$, $\xi^{2,3} = -(g_4 y_{2,3}^n)^*$, $\eta^{1,2} = (-1)^n (g_3 y_{1,2} y_{3,4}^{n-1})^*$, $\eta^{1,3} = (-1)^{n+1} (g_2 y_{1,3} y_{2,4}^{n-1})^*$, $\eta^{2,3} = (-1)^{n+1} (g_2 y_{2,3} y_{1,4}^{n-1})^*$, $\Delta^{1,2} = (\{n+3,n+7,n+8\} \times E_{m-1}^{1,2}) \cup ([\![n+4,n+6]\!] \times (\hat{E}_{m-1}^{1,2} \cup \tilde{E}_{m-1}^{1,2}))$, $\Delta^{1,3} = (\{n+5,n+7,n+8\} \times E_{m-1}^{1,3}) \cup (\{n+3, n+4, n+6\} \times (\hat{E}_{m-1}^{1,3} \cup \tilde{E}_{m-1}^{1,3}))$, and $\Delta^{2,3} = ([\![n+6,n+8]\!] \times E_{m-1}^{2,3}) \cup ([\![n+3,n+5]\!] \times (\hat{E}_{m-1}^{2,3} \cup \tilde{E}_{m-1}^{2,3}))$. □

Recall that $B_{n,m}^{M^2}$ (resp., $D_{n,m}^{M^2}$) is the image (resp., kernel) concentrated in homological degree n and internal degree $m+n$ of the Koszul complex of M^2.

Proposition 6.1.23. *The dimension of* $B^{M^2}_{n,m}$ *is given by*

$$\dim B^{M^2}_{n,m} = \begin{cases} 0, & \text{if } m = 0, \\ 3n + 48, & \text{if } m = 1, \\ 15n + 237, & \text{if } m = 2, \\ 42n + 660, & \text{if } m = 3, \\ 84n + 1314, & \text{if } m = 4, \\ 129n + 2010, & \text{if } m = 5, \\ 159n + 2469, & \text{if } m = 6, \\ 159n + 2460, & \text{if } m = 7, \\ 129n + 1989, & \text{if } m = 8, \\ 84n + 1290, & \text{if } m = 9, \\ 42n + 642, & \text{if } m = 10, \\ 15n + 228, & \text{if } m = 11, \\ 3n + 45, & \text{if } m = 12, \end{cases}$$

for $n \geqslant 3$.

Proof. By Lemma 6.1.22, we have $\dim B^{M^2}_{n+2,m} - \dim B^{M^2}_{n,m} = \dim C^{M^2}_{n+2,m} - \dim C^{M^2}_{n,m}$ for $n \geqslant 4$ and $m \in [\![1,12]\!]$. Using GAP we get the value of $\dim B^{M^2}_{n,m}$ for $n \in [\![3,5]\!]$ and $m \in [\![1,12]\!]$. □

Corollary 6.1.24. *We have* $\mathrm{H}_n(M^2) = 0$ *for* $n \geqslant 4$.

Proof. The result follows from $\dim D^{M^2}_{n,m} = (3n + 45) \dim A_m - \dim B^{M^2}_{n-1,m+1}$ for $n \geqslant 4$ and $m \in [\![0,12]\!]$, and $\dim \mathrm{H}_{n,m}(M^2) = \dim D^{M^2}_{n,m} - \dim B^{M^2}_{n,m}$. □

6.1.2.5 Homology of the Koszul complex of M^3

In this subsubsection, we show that $\mathrm{H}_n(M^3) = 0$ for $n \in \mathbb{N} \setminus \{3\}$.

Note first that $M^3 \cong N \oplus (\oplus_{k \in [\![1,4]\!]} S_k)$ as graded A-modules, where N is the submodule of M^3 generated by $e_i, i \in [\![1,4]\!]$, and S_k is the submodule generated by e_{k+4} for $k \in [\![1,4]\!]$. Let $\{f_i \mid i \in [\![1,8]\!]\}$ be the dual basis to $\{e_i \mid i \in [\![1,8]\!]\}$. It is easy to see that the $A^!$-module $(M^3)^!$ is generated by f_i for $i \in [\![1,8]\!]$, subject to the following 24 relations

$$\begin{gathered} f_1y_{1,2} - f_2y_{3,4}, f_1y_{3,4} + f_2y_{1,2}, f_3y_{1,2} + f_4y_{3,4}, f_3y_{3,4} - f_4y_{1,2}, f_4y_{1,3} - f_2y_{2,4}, f_4y_{2,4} + f_2y_{1,3}, \\ f_3y_{1,3} - f_1y_{2,4}, f_3y_{2,4} + f_1y_{1,3}, f_1y_{2,3} + f_4y_{1,4}, f_1y_{1,4} - f_4y_{2,3}, f_3y_{2,3} + f_2y_{1,4}, f_3y_{1,4} - f_2y_{2,3}, \\ f_5y_{1,4}, f_5y_{2,4}, f_5y_{3,4}, f_6y_{1,3}, f_6y_{2,3}, f_6y_{3,4}, f_7y_{1,2}, f_7y_{2,3}, f_7y_{2,4}, f_8y_{1,2}, f_8y_{1,3}, f_8y_{1,4}. \end{gathered}$$

Using GAP, a basis of $(M^3)^!_{-1}$ is given by the 24 elements

$$\begin{gathered} f_1y_{1,2}, f_1y_{1,3}, f_1y_{2,3}, f_1y_{1,4}, f_1y_{2,4}, f_1y_{3,4}, f_2y_{1,3}, f_2y_{2,3}, f_2y_{1,4}, f_2y_{2,4}, f_3y_{1,2}, f_3y_{3,4}, f_5y_{1,2}, f_5y_{1,3}, \\ f_5y_{2,3}, f_6y_{1,2}, f_6y_{1,4}, f_6y_{2,4}, f_7y_{1,3}, f_7y_{1,4}, f_7y_{3,4}, f_8y_{2,3}, f_8y_{2,4}, f_8y_{3,4}, \end{gathered}$$

and a basis of $(M^3)^!_{-2}$ is given by the 40 elements

$$\begin{gathered} f_1y_{1,2}^2, f_1y_{1,2}y_{1,3}, f_1y_{1,2}y_{2,3}, f_1y_{1,2}y_{1,4}, f_1y_{1,2}y_{2,4}, f_1y_{1,2}y_{3,4}, f_1y_{1,3}^2, f_1y_{1,3}y_{1,4}, f_1y_{1,3}y_{2,4}, \\ f_1y_{1,3}y_{3,4}, f_1y_{2,3}^2, f_1y_{2,3}y_{1,4}, f_1y_{2,3}y_{2,4}, f_1y_{2,3}y_{3,4}, f_2y_{1,3}^2, f_2y_{1,3}y_{2,4}, f_2y_{2,3}^2, f_2y_{2,3}y_{1,4}, f_3y_{1,2}^2, \\ f_3y_{1,2}y_{3,4}, f_5y_{1,2}^2, f_5y_{1,2}y_{1,3}, f_5y_{1,2}y_{2,3}, f_5y_{1,3}^2, f_5y_{2,3}^2, f_6y_{1,2}^2, f_6y_{1,2}y_{1,4}, f_6y_{1,2}y_{2,4}, f_6y_{1,4}^2, f_6y_{2,4}^2, \\ f_7y_{1,3}^2, f_7y_{1,3}y_{1,4}, f_7y_{1,3}y_{3,4}, f_7y_{1,4}^2, f_7y_{3,4}^2, f_8y_{2,3}^2, f_8y_{2,3}y_{2,4}, f_8y_{2,3}y_{3,4}, f_8y_{2,4}^2, f_8y_{3,4}^2. \end{gathered}$$

Remark 6.1.25. *Let* $k \in [\![1,4]\!]$ *and* $f_{k+4}y_{i_1,j_1} \dots y_{i_s,j_s}$ *be a monomial in* $(S_k)^!$, *where* $s \in \mathbb{N}$ *and* $(i_1,j_1), \dots, (i_s,j_s) \in \mathcal{J}$. *If* $5 - k \in \{i_1,j_1,\dots,i_s,j_s\}$, *then* $f_{k+4}y_{i_1,j_1} \dots y_{i_s,j_s} = 0 \in (S_k)^!$.

Remark 6.1.26. *Let* $k \in [\![1,4]\!]$, $y_{i_1,j_1} \dots y_{i_s,j_s} = y_{i'_1,j'_1} \dots y_{i'_s,j'_s}$ *in* $A^!$ *for* $(i_1,j_1),\dots,(i_s,j_s) \in \mathcal{J}$, $(i'_1,j'_1),\dots,(i'_s,j'_s) \in \mathcal{J}$ *and* $s \in \mathbb{N}$. *If* $k \in \{i_1,j_1,\dots,i_s,j_s\}$, *then* $k \in \{i'_1,j'_1,\dots,i'_s,j'_s\}$.

Lemma 6.1.27. *Let*

$$\begin{aligned}
T_n^{S_1} &= \{f_5y_{1,2}^n, f_5y_{1,2}^{n-1}y_{1,3}, f_5y_{1,2}^{n-1}y_{2,3}, f_5y_{1,2}^{n-2}y_{1,3}^2, f_5y_{1,3}^n, f_5y_{2,3}^n\},\\
T_n^{S_2} &= \{f_6y_{1,2}^n, f_6y_{1,2}^{n-1}y_{1,4}, f_6y_{1,2}^{n-1}y_{2,4}, f_6y_{1,2}^{n-2}y_{1,4}^2, f_6y_{1,4}^n, f_6y_{2,4}^n\},\\
T_n^{S_3} &= \{f_7y_{1,3}^n, f_7y_{1,3}^{n-1}y_{1,4}, f_7y_{1,3}^{n-1}y_{3,4}, f_7y_{1,3}^{n-2}y_{1,4}^2, f_7y_{1,4}^n, f_7y_{3,4}^n\},\\
T_n^{S_4} &= \{f_8y_{2,3}^n, f_8y_{2,3}^{n-1}y_{2,4}, f_8y_{2,3}^{n-1}y_{3,4}, f_8y_{2,3}^{n-2}y_{2,4}^2, f_8y_{2,4}^n, f_8y_{3,4}^n\},
\end{aligned}$$

for $n \geqslant 3$. *Then* $T_n^{S_k}$ *is a basis of* $(S_k)^!_{-n}$ *for* $k \in [\![1,4]\!]$ *and* $n \geqslant 3$. *Note that* $T_n^{S_k}$ *has cardinal* 6 *for* $k \in [\![1,4]\!]$ *and* $n \geqslant 3$.

Proof. The space $(S_k)^!_{-n}$ is spanned by $\{f_{k+4}y \mid y \in \mathscr{B}^!_n\}$. By Remark 6.1.25, the space $(S_k)^!_{-n}$ is spanned by $T_n^{S_k}$ for $n \geqslant 3$. By Remark 6.1.26, the elements of $T_n^{S_k}$ are linearly independent. Indeed, let $\sum_{i\in[\![1,6]\!]} \alpha_i q_i = \sum_{u\in\mathscr{B}^!_{n-1}} \lambda^1_u f_5y_{1,4}u + \sum_{u\in\mathscr{B}^!_{n-1}} \lambda^2_u f_5y_{2,4}u + \sum_{u\in\mathscr{B}^!_{n-1}} \lambda^3_u f_5y_{3,4}u$ in $\Bbbk\{f_5\} \otimes A^!$, where $\alpha_i,\lambda^j_u \in \Bbbk$, and q_i is the i-th element of $T_n^{S_1}$. By Remark 6.1.26, the right side of the equation is a linear combination of elements of form $f_5y_{i_1,j_1}\dots y_{i_n,j_n} \in f_5\mathscr{B}^!_n$ for $4 \in \{i_1,j_1,\dots,i_n,j_n\}$. This implies $\alpha_i = 0$ for $i \in [\![1,6]\!]$. Hence, $T_n^{S_1}$ are linearly independent. The other cases are similar. □

Lemma 6.1.28. *The set* T_n^N *consisting of the following* 24 *elements*

$$\begin{aligned}
&f_1y_{1,2}^n, f_1y_{1,2}^{n-1}y_{3,4}, f_1y_{1,3}^n, f_1y_{1,3}^{n-1}y_{2,4}, f_1y_{2,3}^n, f_1y_{2,3}^{n-1}y_{1,4}, f_2y_{1,3}^n, f_2y_{1,3}^{n-1}y_{2,4}, f_2y_{2,3}^n,\\
&f_2y_{2,3}^{n-1}y_{1,4}, f_3y_{1,2}^n, f_3y_{1,2}^{n-1}y_{3,4}, f_1y_{1,2}^{n-1}y_{1,3}, f_1y_{1,2}^{n-1}y_{2,3}, f_1y_{1,2}^{n-1}y_{1,4}, f_1y_{1,2}^{n-1}y_{2,4},\\
&f_1y_{1,2}^{n-2}y_{1,3}^2, f_1y_{1,2}^{n-2}y_{1,3}y_{1,4}, f_1y_{1,2}^{n-2}y_{1,3}y_{2,4}, f_1y_{1,2}^{n-2}y_{1,3}y_{3,4}, f_1y_{1,2}^{n-2}y_{2,3}y_{1,4},\\
&f_1y_{1,2}^{n-2}y_{2,3}y_{2,4}, f_1y_{1,2}^{n-2}y_{2,3}y_{3,4}, f_1y_{1,2}^{n-3}y_{1,3}^2y_{3,4}
\end{aligned} \tag{6.1.33}$$

is a basis of $N^!_{-n}$ *for* $n \geqslant 3$.

Proof. Firstly, using GAP, T_n^N is a basis of $N^!_{-n}$ for $n \in [\![3,5]\!]$. Note that the space $N^!_{-n}$ is spanned by $\{f_iy \mid i \in [\![1,4]\!], y \in \mathscr{B}^!_n\}$ for $n \in \mathbb{N}_0$. By the dual relations, it is easy to see that $N^!_{-n}$ is spanned by

$$\begin{aligned}
&f_1y_{1,2}^n, f_1y_{1,2}^{n-1}y_{3,4}, f_1y_{1,3}^n, f_1y_{1,3}^{n-1}y_{2,4}, f_1y_{2,3}^n, f_1y_{2,3}^{n-1}y_{1,4}, f_2y_{1,3}^n, f_2y_{1,3}^{n-1}y_{2,4}, f_2y_{2,3}^n, f_2y_{2,3}^{n-1}y_{1,4},\\
&f_3y_{1,2}^n, f_3y_{1,2}^{n-1}y_{3,4}, f_iy,
\end{aligned}$$

for $i \in [\![1,3]\!]$, $y \in \mathscr{U}^!_n$ and $n \geqslant 2$. Note that $y_{i,j}^2$ is central in $A^!$ and $f_sy_{i,j}^2 = f_sy_{k,l}^2$ for $s \in [\![1,4]\!]$ and $(i,j),(k,l) \in \mathcal{J}$ with $\#\{i,j,k,l\} = 4$. For $n \geqslant 5$ and $i \in [\![1,3]\!]$,

$$\begin{aligned}
f_iy_{1,2}^{n-2}y_{1,4}^2 &= f_iy_{1,2}^{n-2}y_{2,3}^2 = f_iy_{1,2}^{n-2}y_{1,3}^2,\ f_iy_{1,2}^{n-3}y_{1,3}^2y_{1,4} = f_iy_{1,2}^{n-3}y_{2,4}^2y_{1,4} = f_iy_{1,2}^{n-1}y_{1,4},\\
f_iy_{1,2}^{n-3}y_{1,3}^2y_{2,4} &= f_iy_{1,2}^{n-3}y_{2,4}^3 = f_iy_{1,2}^{n-1}y_{2,4},\ f_iy_{1,2}^{n-3}y_{1,3}y_{1,4}^2 = f_iy_{1,2}^{n-3}y_{1,3}y_{2,3}^2 = f_iy_{1,2}^{n-1}y_{1,3},\\
f_iy_{1,2}^{n-3}y_{2,3}y_{1,4}^2 &= f_iy_{1,2}^{n-3}y_{2,3}^3 = f_iy_{1,2}^{n-1}y_{2,3},\ f_iy_{1,2}^{n-4}y_{1,3}^2y_{1,4}^2 = f_iy_{1,2}^{n-4}y_{1,3}^2y_{2,3}^2 = f_iy_{1,2}^{n-2}y_{1,3}^2,\\
f_iy_{1,3}^{n-1}y_{1,4} &= \chi_nf_iy_{1,3}y_{2,4}^{n-2}y_{1,4} + \chi_{n+1}f_iy_{2,4}^{n-1}y_{1,4} = \chi_nf_iy_{1,3}y_{1,2}^{n-2}y_{1,4} + \chi_{n+1}f_iy_{1,2}^{n-1}y_{1,4}\\
&= \chi_nf_iy_{1,2}^{n-2}y_{1,3}y_{1,4} + \chi_{n+1}f_iy_{1,2}^{n-1}y_{1,4},\\
f_iy_{1,3}^{n-1}y_{3,4} &= f_iy_{1,3}^{n-3}y_{2,4}^2y_{3,4} = f_iy_{1,3}^{n-3}y_{2,3}^2y_{3,4} = \chi_nf_iy_{1,2}^{n-3}y_{1,3}y_{2,3}y_{3,4} + \chi_{n+1}f_iy_{1,2}^{n-3}y_{2,3}^2y_{3,4}\\
&= \chi_nf_iy_{1,2}^{n-2}y_{1,3}y_{3,4} + \chi_{n+1}f_iy_{1,2}^{n-3}y_{1,3}^2y_{3,4},\\
f_iy_{2,3}^{n-1}y_{2,4} &= \chi_nf_iy_{2,3}y_{1,4}^{n-2}y_{2,4} + \chi_{n+1}f_iy_{1,4}^{n-1}y_{2,4} = \chi_nf_iy_{2,3}y_{1,2}^{n-2}y_{2,4} + \chi_{n+1}f_iy_{1,2}^{n-1}y_{2,4}\\
&= \chi_nf_iy_{1,2}^{n-2}y_{2,3}y_{2,4} + \chi_{n+1}f_iy_{1,2}^{n-1}y_{2,4},\\
f_iy_{2,3}^{n-1}y_{3,4} &= f_iy_{2,3}^{n-3}y_{1,4}^2y_{3,4} = f_iy_{2,3}^{n-3}y_{1,3}^2y_{3,4} = -\chi_nf_iy_{1,2}^{n-3}y_{2,3}y_{1,3}y_{3,4} + \chi_{n+1}f_iy_{1,2}^{n-3}y_{1,3}^2y_{3,4}\\
&= \chi_nf_iy_{1,2}^{n-2}y_{2,3}y_{3,4} + \chi_{n+1}f_iy_{1,2}^{n-3}y_{1,3}^2y_{3,4}.
\end{aligned}$$

Moreover, by the dual relation $f_2y_{1,2} = -f_1y_{3,4}$, and

$$f_3y_{1,2}^{n-1}y_{1,3} = \chi_nf_3y_{1,2}y_{1,3}y_{1,2}^{n-2} + \chi_{n+1}f_3y_{1,3}y_{1,2}^{n-1} = -\chi_nf_3y_{2,3}y_{1,2}y_{1,2}^{n-2} + \chi_{n+1}f_3y_{1,3}y_{1,2}^{n-1}$$

$$
\begin{aligned}
&= \chi_n f_2 y_{1,4} y_{1,2}^{n-1} + \chi_{n+1} f_1 y_{2,4} y_{1,2}^{n-1}, \\
f_3 y_{1,2}^{n-1} y_{2,3} &= \chi_n f_3 y_{1,2} y_{2,3} y_{1,2}^{n-2} + \chi_{n+1} f_3 y_{2,3} y_{1,2}^{n-1} = -\chi_n f_3 y_{2,3} y_{1,3} y_{1,2}^{n-2} + \chi_{n+1} f_3 y_{2,3} y_{1,2}^{n-1} \\
&= \chi_n f_2 y_{1,4} y_{1,3} y_{1,2}^{n-2} - \chi_{n+1} f_2 y_{1,4} y_{1,2}^{n-1}, \\
f_3 y_{1,2}^{n-1} y_{1,4} &= \chi_n f_3 y_{1,2} y_{1,4} y_{1,2}^{n-2} + \chi_{n+1} f_3 y_{1,4} y_{1,2}^{n-1} = -\chi_n f_3 y_{2,4} y_{1,2} y_{1,2}^{n-2} + \chi_{n+1} f_3 y_{1,4} y_{1,2}^{n-1} \\
&= \chi_n f_1 y_{1,3} y_{1,2}^{n-1} + \chi_{n+1} f_2 y_{2,3} y_{1,2}^{n-1}, \\
f_3 y_{1,2}^{n-1} y_{2,4} &= \chi_n f_3 y_{1,2} y_{2,4} y_{1,2}^{n-2} + \chi_{n+1} f_3 y_{2,4} y_{1,2}^{n-1} = -\chi_n f_3 y_{1,4} y_{1,2} y_{1,2}^{n-2} + \chi_{n+1} f_3 y_{2,4} y_{1,2}^{n-1} \\
&= -\chi_n f_2 y_{2,3} y_{1,2} y_{1,2}^{n-2} - \chi_{n+1} f_1 y_{1,3} y_{1,2}^{n-1},
\end{aligned}
$$

for $n \geqslant 3$, the space $N^!_{-n}$ is spanned by T_n^N for $n \geqslant 5$.

Next, we prove that the elements in T_n^N for $n \geqslant 6$ are linearly independent. Suppose that we have the identity

$$
\begin{aligned}
\sum_{i \in [\![1,24]\!]} \alpha_i t_i = &\sum_{u \in \mathscr{B}^!_{n-1}} \lambda_u^1 (f_1 y_{1,2} - f_2 y_{3,4}) u + \sum_{u \in \mathscr{B}^!_{n-1}} \lambda_u^2 (f_1 y_{3,4} + f_2 y_{1,2}) u \\
&+ \sum_{u \in \mathscr{B}^!_{n-1}} \lambda_u^3 (f_3 y_{1,2} + f_4 y_{3,4}) u + \sum_{u \in \mathscr{B}^!_{n-1}} \lambda_u^4 (f_3 y_{3,4} - f_4 y_{1,2}) u \\
&+ \sum_{u \in \mathscr{B}^!_{n-1}} \lambda_u^5 (f_4 y_{1,3} - f_2 y_{2,4}) u + \sum_{u \in \mathscr{B}^!_{n-1}} \lambda_u^6 (f_4 y_{2,4} + f_2 y_{1,3}) u \\
&+ \sum_{u \in \mathscr{B}^!_{n-1}} \lambda_u^7 (f_3 y_{1,3} - f_1 y_{2,4}) u + \sum_{u \in \mathscr{B}^!_{n-1}} \lambda_u^8 (f_3 y_{2,4} + f_1 y_{1,3}) u \\
&+ \sum_{u \in \mathscr{B}^!_{n-1}} \lambda_u^9 (f_1 y_{2,3} + f_4 y_{1,4}) u + \sum_{u \in \mathscr{B}^!_{n-1}} \lambda_u^{10} (f_1 y_{1,4} - f_4 y_{2,3}) u \\
&+ \sum_{u \in \mathscr{B}^!_{n-1}} \lambda_u^{11} (f_3 y_{2,3} + f_2 y_{1,4}) u + \sum_{u \in \mathscr{B}^!_{n-1}} \lambda_u^{12} (f_3 y_{1,4} - f_2 y_{2,3}) u,
\end{aligned}
\tag{6.1.34}
$$

in $\Bbbk\{f_1,f_2,f_3,f_4\} \otimes A^!$, where $\alpha_i, \lambda_u^j \in \Bbbk$, and t_i is the i-th element in (6.1.33). We need to show that $\alpha_i = 0$ for all $i \in [\![1,24]\!]$. By inspecting the coefficients of the term $f_s y_{i,j}^{n-r} y_{k,l}^r$ for $\#\{i,j,k,l\} = 4$, it is easy to see that $\alpha_i = 0$ for $i \in [\![1,12]\!]$. Then (6.1.34) is equivalent to

$$
\begin{aligned}
f_1\big(y_{1,2}\Delta^1 + y_{3,4}\Delta^2 + y_{1,3}\Delta^8 - y_{2,4}\Delta^7 + y_{2,3}\Delta^9 + y_{1,4}\Delta^{10}\big) &= \sum_{i \in [\![13,24]\!]} \alpha_i t_i, \\
f_2\big(y_{1,2}\Delta^2 - y_{3,4}\Delta^1 + y_{1,3}\Delta^6 - y_{2,4}\Delta^5 - y_{2,3}\Delta^{12} + y_{1,4}\Delta^{11}\big) &= 0, \\
f_3\big(y_{1,2}\Delta^3 + y_{3,4}\Delta^4 + y_{1,3}\Delta^7 + y_{2,4}\Delta^8 + y_{2,3}\Delta^{11} + y_{1,4}\Delta^{12}\big) &= 0, \\
f_4\big(-y_{1,2}\Delta^4 + y_{3,4}\Delta^3 + y_{1,3}\Delta_5 + y_{2,4}\Delta^6 - y_{2,3}\Delta^{10} + y_{1,4}\Delta^9\big) &= 0,
\end{aligned}
\tag{6.1.35}
$$

in $\Bbbk\{f_i\} \otimes A^!$ for $i \in [\![1,4]\!]$ respectively, where $\Delta^j = \sum_{u \in \mathscr{B}^!_{n-1} \setminus Y_{1,2}} \lambda_u^j u$ for $j \in [\![1,4]\!]$, $\Delta^j = \sum_{u \in \mathscr{B}^!_{n-1} \setminus Y_{1,3}} \lambda_u^j u$ for $j \in [\![5,8]\!]$, $\Delta^j = \sum_{u \in \mathscr{B}^!_{n-1} \setminus Y_{2,3}} \lambda_u^j u$ for $j \in [\![9,12]\!]$ and $Y_{i,j}$ is defined in (6.1.18). In particular, we see that the elements in T_n^N are linearly independent if and only if equation (6.1.35) implies that $\alpha_i = 0$ for all $i \in [\![13,24]\!]$.

Let

$$
a_0^j = \lambda^j_{y_{1,2}^{n-1}},\ a_0'^j = \lambda^j_{y_{3,4}^{n-1}},\ a_1^j = \sum_{\substack{r \in [\![1,\, n-2]\!], \\ r \text{ odd}}} \lambda^j_{y_{1,2}^{n-1-r} y_{3,4}^r},\ a_2^j = \sum_{\substack{r \in [\![1,\, n-2]\!], \\ r \text{ even}}} \lambda^j_{y_{1,2}^{n-1-r} y_{3,4}^r} \text{ for } j \in [\![5,12]\!],
$$

$$
b_0^j = \lambda^j_{y_{1,3}^{n-1}},\ b_0'^j = \lambda^j_{y_{2,4}^{n-1}},\ b_1^j = \sum_{\substack{r \in [\![1,\, n-2]\!], \\ r \text{ odd}}} \lambda^j_{y_{1,3}^{n-1-r} y_{2,4}^r},\ b_2^j = \sum_{\substack{r \in [\![1,\, n-2]\!], \\ r \text{ even}}} \lambda^j_{y_{1,3}^{n-1-r} y_{2,4}^r} \text{ for } j \in [\![1,4]\!] \cup [\![9,12]\!],
$$

$$
c_0^j = \lambda^j_{y_{2,3}^{n-1}},\ c_0'^j = \lambda^j_{y_{1,4}^{n-1}},\ c_1^j = \sum_{\substack{r \in [\![1,\, n-2]\!], \\ r \text{ odd}}} \lambda^j_{y_{2,3}^{n-1-r} y_{1,4}^r},\ c_2^j = \sum_{\substack{r \in [\![1,\, n-2]\!], \\ r \text{ even}}} \lambda^j_{y_{2,3}^{n-1-r} y_{1,4}^r} \text{ for } j \in [\![1,8]\!].
$$

Using (6.1.35) together with the products (6.1.4) and (6.1.5), in $A^!$, we get a system of linear equations E_n, which contains $24 \times 4 = 96$ linear equations and $12 + 24 \times 12 + 4 \times 8 \times 3 = 396$

variables α_i, λ_u^j for $u \in \mathcal{U}_{n-1}^!$, a_0^j, $a_0'^j$, a_1^j, a_2^j, b_0^j, $b_0'^j$, b_1^j, b_2^j, c_0^j, $c_0'^j$, c_1^j, c_2^j. Hence, the linear independence of T_n^N (or, equivalently, the fact that equation (6.1.35) implies that $\alpha_i = 0$ for all $i \in [\![13, 24]\!]$) is tantamount to the fact that the linear system E_n implies that $\alpha_i = 0$ for all $i \in [\![13, 24]\!]$. Furthermore, it is easy to see that E_n has the same form as E_{n+2}. We then use GAP to check that the elements in T_n^N are linearly independent for $n \in [\![6, 7]\!]$, and conclude that the lemma holds for all integers $n \geqslant 6$. □

Corollary 6.1.29. *We have* $\mathrm{H}_n(M^3) = 0$ *for* $n \in \mathbb{N} \setminus \{3\}$.

Proof. By Tables A.1.6 and A.1.7, and the reductions in the proof of Lemma 6.1.28, the differential at homological degree n in the Koszul complex N or S_k has the same form when $n \geqslant 4$ increases by 2. Then $\mathrm{H}_{n+2}(M^3) = \mathrm{H}_n(M^3)$ for $n \geqslant 4$. Using GAP, $\mathrm{H}_n(M^3) = 0$ for $n \in [\![1,5]\!] \setminus \{3\}$. By induction on n, $\mathrm{H}_n(M^3) = 0$ for $n \in \mathbb{N} \setminus \{3\}$. □

Appendix A

Some computations

In this Appendix, we list some computations about the Fomin-Kirillov algebra $\mathrm{FK}(4)$ of index 4. We will denote $\mathrm{FK}(4)$ simply by A.

A.1 Products in $\mathrm{FK}(4)$

It is easy to check the products in $A^!$, listed in Table A.1.1-A.1.4, by using GAP or by computing them directly, and to check the products listed in A.1.6-A.1.9 by induction on integers $n \geqslant 5$. In Tables A.1.1-A.1.4, A.1.6 and A.1.7, the entry appearing in the row indexed by y and the column indexed by y' is the product yy'. In Tables A.1.8 and A.1.9, the entry appearing in the column indexed by y' and the row indexed by y is the product $y'y$. To reduce space, in Table A.1.3, we write the product yy' by $\pm m$, where $m \in [\![55,92]\!]$ is the integer appearing in the first column of Table A.1.4, and indicating the element in the second column of Table A.1.4 that is in the same row as it. In Table A.1.4, we write the product yy' by $\pm m$, where $m \in [\![93,134]\!]$ is the integer appearing in the first column of Table A.1.5, and indicating the element in the second column of Table A.1.5 that is in the same row as it. In Table A.1.8 and A.1.9, we write the product yy' by $\pm m$, where $m \in [\![1,24]\!]$ is the integer appearing in the first column of Table A.1.8 (or A.1.9), and indicating the element a_m^{n+1}, where

$$
\begin{array}{llll}
\mathsf{a}_1^n = y_{1,2}^{n-1}y_{1,3}, & \mathsf{a}_2^n = y_{1,2}^{n-2}y_{1,3}^2, & \mathsf{a}_3^n = y_{1,2}^{n-1}y_{2,3}, & \mathsf{a}_4^n = y_{1,2}^{n-1}y_{1,4}, \\
\mathsf{a}_5^n = y_{1,2}^{n-2}y_{1,3}y_{1,4}, & \mathsf{a}_6^n = y_{1,2}^{n-3}y_{1,3}^2y_{1,4}, & \mathsf{a}_7^n = y_{1,3}^{n-1}y_{1,4}, & \mathsf{a}_8^n = y_{1,2}^{n-2}y_{2,3}y_{1,4}, \\
\mathsf{a}_9^n = y_{1,2}^{n-2}y_{1,4}^2, & \mathsf{a}_{10}^n = y_{1,2}^{n-3}y_{1,3}y_{1,4}^2, & \mathsf{a}_{11}^n = y_{1,2}^{n-4}y_{1,3}^2y_{1,4}^2, & \mathsf{a}_{12}^n = y_{1,3}^{n-2}y_{1,4}^2, \\
\mathsf{a}_{13}^n = y_{1,2}^{n-3}y_{2,3}y_{1,4}^2, & \mathsf{a}_{14}^n = y_{1,2}^{n-1}y_{2,4}, & \mathsf{a}_{15}^n = y_{1,2}^{n-2}y_{1,3}y_{2,4}, & \mathsf{a}_{16}^n = y_{1,2}^{n-3}y_{1,3}^2y_{2,4}, \\
\mathsf{a}_{17}^n = y_{1,2}^{n-2}y_{2,3}y_{2,4}, & \mathsf{a}_{18}^n = y_{2,3}^{n-1}y_{2,4}, & \mathsf{a}_{19}^n = y_{2,3}^{n-2}y_{2,4}^2, & \mathsf{a}_{20}^n = y_{1,2}^{n-2}y_{1,3}y_{3,4}, \\
\mathsf{a}_{21}^n = y_{1,2}^{n-3}y_{1,3}^2y_{3,4}, & \mathsf{a}_{22}^n = y_{1,3}^{n-1}y_{3,4}, & \mathsf{a}_{23}^n = y_{1,2}^{n-2}y_{2,3}y_{3,4}, & \mathsf{a}_{24}^n = y_{2,3}^{n-1}y_{3,4},
\end{array}
$$

for $n \geqslant 5$ and $m \in [\![1,24]\!]$.

1	$y \backslash y'$	$y_{1,2}$	$y_{1,3}$	$y_{2,3}$	$y_{1,4}$	$y_{2,4}$	$y_{3,4}$
2	$y_{1,2}$	$y_{1,2}^2$	$y_{1,2}y_{1,3}$	$y_{1,2}y_{2,3}$	$y_{1,2}y_{1,4}$	$y_{1,2}y_{2,4}$	$y_{1,2}y_{3,4}$
3	$y_{1,3}$	$-y_{1,2}y_{2,3}$	$y_{1,3}^2$	$y_{1,2}y_{1,3}$	$y_{1,3}y_{1,4}$	$y_{1,3}y_{2,4}$	$y_{1,3}y_{3,4}$
4	$y_{2,3}$	$-y_{1,2}y_{1,3}$	$-y_{1,2}y_{2,3}$	$y_{2,3}^2$	$y_{2,3}y_{1,4}$	$y_{2,3}y_{2,4}$	$y_{2,3}y_{3,4}$
5	$y_{1,4}$	$-y_{1,2}y_{2,4}$	$-y_{1,3}y_{3,4}$	$-y_{2,3}y_{1,4}$	$y_{1,4}^2$	$y_{1,2}y_{1,4}$	$y_{1,3}y_{1,4}$
6	$y_{2,4}$	$-y_{1,2}y_{1,4}$	$-y_{1,3}y_{2,4}$	$-y_{2,3}y_{3,4}$	$-y_{1,2}y_{2,4}$	$y_{2,4}^2$	$y_{2,3}y_{2,4}$
7	$y_{3,4}$	$-y_{1,2}y_{3,4}$	$-y_{1,3}y_{1,4}$	$-y_{2,3}y_{2,4}$	$-y_{1,3}y_{3,4}$	$-y_{2,3}y_{3,4}$	$y_{3,4}^2$

Table A.1.1: Products yy'.

1	y \ y'	$y_{1,2}$	$y_{1,3}$	$y_{2,3}$	$y_{1,4}$	$y_{2,4}$	$y_{3,4}$
8	$y_{1,2}^2$	$y_{1,2}^3$	$y_{1,2}^2 y_{1,3}$	$y_{1,2}^2 y_{2,3}$	$y_{1,2}^2 y_{1,4}$	$y_{1,2}^2 y_{2,4}$	$y_{1,2}^2 y_{3,4}$
9	$y_{1,2} y_{1,3}$	$-y_{1,2}^2 y_{2,3}$	$y_{1,2} y_{1,3}^2$	$y_{1,2}^2 y_{1,3}$	$y_{1,2} y_{1,3} y_{1,4}$	$y_{1,2} y_{1,3} y_{2,4}$	$y_{1,2} y_{1,3} y_{3,4}$
10	$y_{1,2} y_{2,3}$	$-y_{1,2}^2 y_{1,3}$	$-y_{1,2}^2 y_{2,3}$	$y_{1,2} y_{1,3}^2$	$y_{1,2} y_{2,3} y_{1,4}$	$y_{1,2} y_{2,3} y_{2,4}$	$y_{1,2} y_{2,3} y_{3,4}$
11	$y_{1,2} y_{1,4}$	$-y_{1,2}^2 y_{2,4}$	$-y_{1,2} y_{1,3} y_{3,4}$	$-y_{1,2} y_{2,3} y_{1,4}$	$y_{1,2} y_{1,4}^2$	$y_{1,2}^2 y_{1,4}$	$y_{1,2} y_{1,3} y_{1,4}$
12	$y_{1,2} y_{2,4}$	$-y_{1,2}^2 y_{1,4}$	$-y_{1,2} y_{1,3} y_{2,4}$	$-y_{1,2} y_{2,3} y_{3,4}$	$-y_{1,2}^2 y_{2,4}$	$y_{1,2} y_{1,4}^2$	$y_{1,2} y_{2,3} y_{2,4}$
13	$y_{1,2} y_{3,4}$	$-y_{1,2}^2 y_{3,4}$	$-y_{1,2} y_{1,3} y_{1,4}$	$-y_{1,2} y_{2,3} y_{2,4}$	$-y_{1,2} y_{1,3} y_{3,4}$	$-y_{1,2} y_{2,3} y_{3,4}$	$y_{1,2} y_{3,4}^2$
14	$y_{1,3}^2$	$y_{1,2} y_{1,3}^2$	$y_{1,3}^3$	$y_{1,2}^2 y_{2,3}$	$y_{1,3}^2 y_{1,4}$	$y_{1,3}^2 y_{2,4}$	$y_{1,3}^2 y_{3,4}$
15	$y_{1,3} y_{1,4}$	$y_{1,2} y_{2,3} y_{2,4}$	$-y_{1,3}^2 y_{3,4}$	$-y_{1,2} y_{1,3} y_{1,4}$	$y_{1,3} y_{1,4}^2$	$-y_{1,2} y_{2,3} y_{1,4}$	$y_{1,3}^2 y_{1,4}$
16	$y_{1,3} y_{2,4}$	$y_{1,2} y_{2,3} y_{1,4}$	$-y_{1,3}^2 y_{2,4}$	$-y_{1,2} y_{1,3} y_{3,4}$	$y_{1,2} y_{2,3} y_{2,4}$	$y_{1,3} y_{2,4}^2$	$y_{1,2} y_{1,3} y_{2,4}$
17	$y_{1,3} y_{3,4}$	$y_{1,2} y_{2,3} y_{3,4}$	$-y_{1,3}^2 y_{1,4}$	$-y_{1,2} y_{1,3} y_{2,4}$	$-y_{1,3}^2 y_{3,4}$	$-y_{1,2} y_{1,3} y_{3,4}$	$y_{1,3} y_{1,4}^2$
18	$y_{2,3}^2$	$y_{1,2} y_{1,3}^2$	$y_{1,2}^2 y_{1,3}$	$y_{2,3}^3$	$y_{2,3}^2 y_{1,4}$	$y_{2,3}^2 y_{2,4}$	$y_{2,3}^2 y_{3,4}$
19	$y_{2,3} y_{1,4}$	$y_{1,2} y_{1,3} y_{2,4}$	$y_{1,2} y_{2,3} y_{3,4}$	$-y_{2,3}^2 y_{1,4}$	$y_{2,3} y_{1,4}^2$	$-y_{1,2} y_{1,3} y_{1,4}$	$-y_{1,2} y_{2,3} y_{1,4}$
20	$y_{2,3} y_{2,4}$	$y_{1,2} y_{1,3} y_{1,4}$	$y_{1,2} y_{2,3} y_{2,4}$	$-y_{2,3}^2 y_{3,4}$	$y_{1,2} y_{1,3} y_{2,4}$	$y_{2,3} y_{2,4}^2$	$y_{2,3}^2 y_{2,4}$
21	$y_{2,3} y_{3,4}$	$y_{1,2} y_{1,3} y_{3,4}$	$y_{1,2} y_{2,3} y_{1,4}$	$-y_{2,3}^2 y_{2,4}$	$y_{1,2} y_{2,3} y_{3,4}$	$-y_{2,3}^2 y_{3,4}$	$y_{2,3} y_{2,4}^2$
22	$y_{1,4}^2$	$y_{1,2} y_{1,4}^2$	$y_{1,3} y_{1,4}^2$	$y_{2,3} y_{1,4}^2$	$y_{1,4}^3$	$y_{1,2}^2 y_{2,4}$	$y_{1,3}^2 y_{3,4}$
23	$y_{2,4}^2$	$y_{1,2} y_{1,4}^2$	$y_{1,3} y_{2,4}^2$	$y_{2,3} y_{2,4}^2$	$y_{1,2}^2 y_{1,4}$	$y_{2,4}^3$	$y_{2,3}^2 y_{3,4}$
24	$y_{3,4}^2$	$y_{1,2} y_{3,4}^2$	$y_{1,3} y_{1,4}^2$	$y_{2,3} y_{2,4}^2$	$y_{1,3}^2 y_{1,4}$	$y_{2,3}^2 y_{2,4}$	$y_{3,4}^3$

Table A.1.2: Products yy'.

1	y \ y'	$y_{1,2}$	$y_{1,3}$	$y_{2,3}$	$y_{1,4}$	$y_{2,4}$	$y_{3,4}$
25	$y_{1,2}^3$	55	56	57	58	59	60
26	$y_{1,2}^2 y_{1,3}$	−57	61	56	62	63	64
27	$y_{1,2}^2 y_{2,3}$	−56	−57	61	65	66	67
28	$y_{1,2}^2 y_{1,4}$	−59	−64	−65	68	58	62
29	$y_{1,2}^2 y_{2,4}$	−58	−63	−67	−59	68	66
30	$y_{1,2}^2 y_{3,4}$	−60	−62	−66	−64	−67	69
31	$y_{1,2} y_{1,3}^2$	61	56	57	70	71	72
32	$y_{1,2} y_{1,3} y_{1,4}$	66	−72	−62	73	−65	70
33	$y_{1,2} y_{1,3} y_{2,4}$	65	−71	−64	66	73	63
34	$y_{1,2} y_{1,3} y_{3,4}$	67	−70	−63	−72	−64	73
35	$y_{1,2} y_{2,3} y_{1,4}$	63	67	−70	74	−62	−65
36	$y_{1,2} y_{2,3} y_{2,4}$	62	66	−72	63	74	71
37	$y_{1,2} y_{2,3} y_{3,4}$	64	65	−71	67	−72	74
38	$y_{1,2} y_{1,4}^2$	68	73	74	58	59	72
39	$y_{1,2} y_{3,4}^2$	69	73	74	70	71	75
40	$y_{1,3}^3$	−57	76	56	77	78	79
41	$y_{1,3}^2 y_{1,4}$	−71	−79	−65	80	70	77
42	$y_{1,3}^2 y_{2,4}$	−70	−78	−67	−71	81	66
43	$y_{1,3}^2 y_{3,4}$	−72	−77	−66	−79	−67	80
44	$y_{1,3} y_{1,4}^2$	−74	80	73	77	63	79
45	$y_{1,3} y_{2,4}^2$	−74	81	73	62	82	64
46	$y_{2,3}^3$	−56	−57	83	84	85	86
47	$y_{2,3}^2 y_{1,4}$	−71	−64	−84	87	70	62
48	$y_{2,3}^2 y_{2,4}$	−70	−63	−86	−71	88	85
49	$y_{2,3}^2 y_{3,4}$	−72	−62	−85	−64	−86	88
50	$y_{2,3} y_{1,4}^2$	−73	−74	87	89	66	67
51	$y_{2,3} y_{2,4}^2$	−73	−74	88	65	85	86
52	$y_{1,4}^3$	−59	−79	−89	90	58	77
53	$y_{2,4}^3$	−58	−82	−86	−59	91	85
54	$y_{3,4}^3$	−75	−77	−85	−79	−86	92

Table A.1.3: Products yy'.

1	$y \backslash y'$	$y_{1,2}$	$y_{1,3}$	$y_{2,3}$	$y_{1,4}$	$y_{2,4}$	$y_{3,4}$
55	$y_{1,2}^4$	93	94	95	96	97	98
56	$y_{1,2}^3 y_{1,3}$	−95	99	94	100	101	102
57	$y_{1,2}^3 y_{2,3}$	−94	−95	99	103	104	105
58	$y_{1,2}^3 y_{1,4}$	−97	−102	−103	106	96	100
59	$y_{1,2}^3 y_{2,4}$	−96	−101	−105	−97	106	104
60	$y_{1,2}^3 y_{3,4}$	−98	−100	−104	−102	−105	107
61	$y_{1,2}^2 y_{1,3}^2$	99	94	95	108	109	110
62	$y_{1,2}^2 y_{1,3} y_{1,4}$	104	−110	−100	111	−103	108
63	$y_{1,2}^2 y_{1,3} y_{2,4}$	103	−109	−102	104	111	101
64	$y_{1,2}^2 y_{1,3} y_{3,4}$	105	−108	−101	−110	−102	111
65	$y_{1,2}^2 y_{2,3} y_{1,4}$	101	105	−108	112	−100	−103
66	$y_{1,2}^2 y_{2,3} y_{2,4}$	100	104	−110	101	112	109
67	$y_{1,2}^2 y_{2,3} y_{3,4}$	102	103	−109	105	−110	112
68	$y_{1,2}^2 y_{1,4}^2$	106	111	112	96	97	110
69	$y_{1,2}^2 y_{3,4}^2$	107	111	112	108	109	113
70	$y_{1,2} y_{1,3}^2 y_{1,4}$	−109	−102	−103	114	108	100
71	$y_{1,2} y_{1,3}^2 y_{2,4}$	−108	−101	−105	−109	114	104
72	$y_{1,2} y_{1,3}^2 y_{3,4}$	−110	−100	−104	−102	−105	114
73	$y_{1,2} y_{1,3} y_{1,4}^2$	−112	114	111	100	101	102
74	$y_{1,2} y_{2,3} y_{1,4}^2$	−111	−112	114	103	104	105
75	$y_{1,2} y_{3,4}^3$	−113	−100	−104	−102	−105	115
76	$y_{1,3}^4$	99	116	95	117	118	119
77	$y_{1,3}^3 y_{1,4}$	104	−119	−100	120	−103	117
78	$y_{1,3}^3 y_{2,4}$	103	−118	−102	104	121	101
79	$y_{1,3}^3 y_{3,4}$	105	−117	−101	−119	−102	120
80	$y_{1,3}^2 y_{1,4}^2$	114	120	112	117	109	119
81	$y_{1,3}^2 y_{2,4}^2$	114	121	112	108	122	110
82	$y_{1,3} y_{2,4}^3$	103	−122	−102	104	123	101
83	$y_{2,3}^4$	99	94	124	125	126	127
84	$y_{2,3}^3 y_{1,4}$	101	105	−125	128	−100	−103
85	$y_{2,3}^3 y_{2,4}$	100	104	−127	101	129	126
86	$y_{2,3}^3 y_{3,4}$	102	103	−126	105	−127	129
87	$y_{2,3}^2 y_{1,4}^2$	114	111	128	130	109	110
88	$y_{2,3}^2 y_{2,4}^2$	114	111	129	108	126	127
89	$y_{2,3} y_{1,4}^3$	101	105	−130	131	−100	−103
90	$y_{1,4}^4$	106	120	131	132	97	119
91	$y_{2,4}^4$	106	123	129	96	133	127
92	$y_{3,4}^4$	115	120	129	117	126	134

Table A.1.4: Products yy'.

1	
93	$y_{1,2}^5$
94	$y_{1,2}^4 y_{1,3}$
95	$y_{1,2}^4 y_{2,3}$
96	$y_{1,2}^4 y_{1,4}$
97	$y_{1,2}^4 y_{2,4}$
98	$y_{1,2}^4 y_{3,4}$
99	$y_{1,2}^3 y_{1,3}^2$
100	$y_{1,2}^3 y_{1,3} y_{1,4}$
101	$y_{1,2}^3 y_{1,3} y_{2,4}$
102	$y_{1,2}^3 y_{1,3} y_{3,4}$
103	$y_{1,2}^3 y_{2,3} y_{1,4}$
104	$y_{1,2}^3 y_{2,3} y_{2,4}$
105	$y_{1,2}^3 y_{2,3} y_{3,4}$
106	$y_{1,2}^3 y_{1,4}^2$
107	$y_{1,2}^3 y_{3,4}^2$
108	$y_{1,2}^2 y_{1,3}^2 y_{1,4}$
109	$y_{1,2}^2 y_{1,3}^2 y_{2,4}$
110	$y_{1,2}^2 y_{1,3}^2 y_{3,4}$
111	$y_{1,2}^2 y_{1,3} y_{1,4}^2$
112	$y_{1,2}^2 y_{2,3} y_{1,4}^2$
113	$y_{1,2}^2 y_{3,4}^3$
114	$y_{1,2} y_{1,3}^2 y_{1,4}^2$
115	$y_{1,2} y_{3,4}^4$
116	$y_{1,3}^5$
117	$y_{1,3}^4 y_{1,4}$
118	$y_{1,3}^4 y_{2,4}$
119	$y_{1,3}^4 y_{3,4}$
120	$y_{1,3}^3 y_{1,4}^2$
121	$y_{1,3}^3 y_{2,4}^2$
122	$y_{1,3}^2 y_{2,4}^3$
123	$y_{1,3} y_{2,4}^4$
124	$y_{2,3}^5$
125	$y_{2,3}^4 y_{1,4}$
126	$y_{2,3}^4 y_{2,4}$
127	$y_{2,3}^4 y_{3,4}$
128	$y_{2,3}^3 y_{1,4}^2$
129	$y_{2,3}^3 y_{2,4}^2$
130	$y_{2,3}^2 y_{1,4}^3$
131	$y_{2,3} y_{1,4}^4$
132	$y_{1,4}^5$
133	$y_{2,4}^5$
134	$y_{3,4}^5$

Table A.1.5: Elements in $\mathscr{B}_5^!$.

y \ y'	$y_{1,2}$	$y_{1,3}$	$y_{2,3}$
$y_{1,2}^{n}$	$y_{1,2}^{n+1}$	$y_{1,2}^{n}y_{1,3}$	$y_{1,2}^{n}y_{2,3}$
$y_{1,2}^{n-1}y_{1,3}$	$-y_{1,2}^{n}y_{2,3}$	$y_{1,2}^{n-1}y_{1,3}^{2}$	$y_{1,2}^{n}y_{1,3}$
$y_{1,2}^{n-1}y_{2,3}$	$-y_{1,2}^{n}y_{1,3}$	$-y_{1,2}^{n}y_{2,3}$	$y_{1,2}^{n-1}y_{1,3}^{2}$
$y_{1,2}^{n-1}y_{1,4}$	$-y_{1,2}^{n}y_{2,4}$	$-y_{1,2}^{n-1}y_{1,3}y_{3,4}$	$-y_{1,2}^{n-1}y_{2,3}y_{1,4}$
$y_{1,2}^{n-1}y_{2,4}$	$-y_{1,2}^{n}y_{1,4}$	$-y_{1,2}^{n-1}y_{1,3}y_{2,4}$	$-y_{1,2}^{n-1}y_{2,3}y_{3,4}$
$y_{1,2}^{n-2}y_{1,3}^{2}$	$y_{1,2}^{n-1}y_{1,3}^{2}$	$y_{1,2}^{n}y_{1,3}$	$y_{1,2}^{n}y_{2,3}$
$y_{1,2}^{n-2}y_{1,3}y_{1,4}$	$y_{1,2}^{n-1}y_{2,3}y_{2,4}$	$-y_{1,2}^{n-2}y_{1,3}^{2}y_{3,4}$	$-y_{1,2}^{n-1}y_{1,3}y_{1,4}$
$y_{1,2}^{n-2}y_{1,3}y_{2,4}$	$y_{1,2}^{n-1}y_{2,3}y_{1,4}$	$-y_{1,2}^{n-2}y_{1,3}^{2}y_{2,4}$	$-y_{1,2}^{n-1}y_{1,3}y_{3,4}$
$y_{1,2}^{n-2}y_{1,3}y_{3,4}$	$y_{1,2}^{n-1}y_{2,3}y_{3,4}$	$-y_{1,2}^{n-2}y_{1,3}^{2}y_{1,4}$	$-y_{1,2}^{n-1}y_{1,3}y_{2,4}$
$y_{1,2}^{n-2}y_{2,3}y_{1,4}$	$y_{1,2}^{n-1}y_{1,3}y_{2,4}$	$y_{1,2}^{n-1}y_{2,3}y_{3,4}$	$-y_{1,2}^{n-2}y_{1,3}^{2}y_{1,4}$
$y_{1,2}^{n-2}y_{2,3}y_{2,4}$	$y_{1,2}^{n-1}y_{1,3}y_{1,4}$	$y_{1,2}^{n-1}y_{2,3}y_{2,4}$	$-y_{1,2}^{n-2}y_{1,3}^{2}y_{3,4}$
$y_{1,2}^{n-2}y_{2,3}y_{3,4}$	$y_{1,2}^{n-1}y_{1,3}y_{3,4}$	$y_{1,2}^{n-1}y_{2,3}y_{1,4}$	$-y_{1,2}^{n-2}y_{1,3}^{2}y_{2,4}$
$y_{1,2}^{n-2}y_{1,4}^{2}$	$y_{1,2}^{n-1}y_{1,4}^{2}$	$y_{1,2}^{n-2}y_{1,3}y_{1,4}^{2}$	$y_{1,2}^{n-2}y_{2,3}y_{1,4}^{2}$
$y_{1,2}^{n-3}y_{1,3}^{2}y_{1,4}$	$-y_{1,2}^{n-2}y_{1,3}^{2}y_{2,4}$	$-y_{1,2}^{n-1}y_{1,3}y_{3,4}$	$-y_{1,2}^{n-1}y_{2,3}y_{1,4}$
$y_{1,2}^{n-3}y_{1,3}^{2}y_{2,4}$	$-y_{1,2}^{n-2}y_{1,3}^{2}y_{1,4}$	$-y_{1,2}^{n-1}y_{1,3}y_{2,4}$	$-y_{1,2}^{n-1}y_{2,3}y_{3,4}$
$y_{1,2}^{n-3}y_{1,3}^{2}y_{3,4}$	$-y_{1,2}^{n-2}y_{1,3}^{2}y_{3,4}$	$-y_{1,2}^{n-1}y_{1,3}y_{1,4}$	$-y_{1,2}^{n-1}y_{2,3}y_{2,4}$
$y_{1,2}^{n-3}y_{1,3}y_{1,4}^{2}$	$-y_{1,2}^{n-2}y_{2,3}y_{1,4}^{2}$	$y_{1,2}^{n-3}y_{1,3}^{2}y_{1,4}^{2}$	$y_{1,2}^{n-2}y_{1,3}y_{1,4}^{2}$
$y_{1,2}^{n-3}y_{2,3}y_{1,4}^{2}$	$-y_{1,2}^{n-2}y_{1,3}y_{1,4}^{2}$	$-y_{1,2}^{n-2}y_{2,3}y_{1,4}^{2}$	$y_{1,2}^{n-3}y_{1,3}^{2}y_{1,4}^{2}$
$y_{1,2}^{n-4}y_{1,3}^{2}y_{1,4}^{2}$	$y_{1,2}^{n-3}y_{1,3}^{2}y_{1,4}^{2}$	$y_{1,2}^{n-2}y_{1,3}y_{1,4}^{2}$	$y_{1,2}^{n-2}y_{2,3}y_{1,4}^{2}$
$y_{1,3}^{n}$	$\chi_n y_{1,2}^{n-1}y_{1,3}^{2} - \chi_{n+1}y_{1,2}^{n}y_{2,3}$	$y_{1,3}^{n+1}$	$\chi_n y_{1,2}^{n}y_{2,3} + \chi_{n+1}y_{1,2}^{n}y_{1,3}$
$y_{1,3}^{n-1}y_{1,4}$	$\chi_n y_{1,2}^{n-1}y_{2,3}y_{2,4} - \chi_{n+1}y_{1,2}^{n-2}y_{1,3}^{2}y_{2,4}$	$-y_{1,3}^{n}y_{3,4}$	$-\chi_n y_{1,2}^{n-1}y_{1,3}y_{1,4} - \chi_{n+1}y_{1,2}^{n-1}y_{2,3}y_{1,4}$
$y_{1,3}^{n-1}y_{3,4}$	$\chi_n y_{1,2}^{n-1}y_{2,3}y_{3,4} - \chi_{n+1}y_{1,2}^{n-2}y_{1,3}^{2}y_{3,4}$	$-y_{1,3}^{n}y_{1,4}$	$-\chi_n y_{1,2}^{n-1}y_{1,3}y_{2,4} - \chi_{n+1}y_{1,2}^{n-1}y_{2,3}y_{2,4}$
$y_{1,3}^{n-2}y_{1,4}^{2}$	$\chi_n y_{1,2}^{n-3}y_{1,3}^{2}y_{1,4}^{2} - \chi_{n+1}y_{1,2}^{n-2}y_{2,3}y_{1,4}^{2}$	$y_{1,3}^{n-1}y_{1,4}^{2}$	$\chi_n y_{1,2}^{n-2}y_{2,3}y_{1,4}^{2} + \chi_{n+1}y_{1,2}^{n-2}y_{1,3}y_{1,4}^{2}$
$y_{2,3}^{n}$	$\chi_n y_{1,2}^{n-1}y_{1,3}^{2} - \chi_{n+1}y_{1,2}^{n}y_{1,3}$	$\chi_n y_{1,2}^{n}y_{1,3} - \chi_{n+1}y_{1,2}^{n}y_{2,3}$	$y_{2,3}^{n+1}$
$y_{2,3}^{n-1}y_{2,4}$	$\chi_n y_{1,2}^{n-1}y_{1,3}y_{1,4} - \chi_{n+1}y_{1,2}^{n-2}y_{1,3}^{2}y_{1,4}$	$\chi_n y_{1,2}^{n-1}y_{2,3}y_{2,4} - \chi_{n+1}y_{1,2}^{n-1}y_{1,3}y_{2,4}$	$-y_{2,3}^{n}y_{3,4}$
$y_{2,3}^{n-1}y_{3,4}$	$\chi_n y_{1,2}^{n-1}y_{1,3}y_{3,4} - \chi_{n+1}y_{1,2}^{n-2}y_{1,3}^{2}y_{3,4}$	$\chi_n y_{1,2}^{n-1}y_{2,3}y_{1,4} - \chi_{n+1}y_{1,2}^{n-1}y_{1,3}y_{1,4}$	$-y_{2,3}^{n}y_{2,4}$
$y_{2,3}^{n-2}y_{2,4}^{2}$	$\chi_n y_{1,2}^{n-3}y_{1,3}^{2}y_{1,4}^{2} - \chi_{n+1}y_{1,2}^{n-2}y_{1,3}y_{1,4}^{2}$	$\chi_n y_{1,2}^{n-2}y_{1,3}y_{1,4}^{2} - \chi_{n+1}y_{1,2}^{n-2}y_{2,3}y_{1,4}^{2}$	$y_{2,3}^{n-1}y_{2,4}^{2}$
$y_{1,4}^{n}$	$\chi_n y_{1,2}^{n-1}y_{1,4}^{2} - \chi_{n+1}y_{1,2}^{n}y_{2,4}$	$\chi_n y_{1,3}^{n-1}y_{1,4}^{2} - \chi_{n+1}y_{1,3}^{n}y_{3,4}$	$(-1)^n y_{2,3}y_{1,4}^{n}$
$y_{2,4}^{n}$	$\chi_n y_{1,2}^{n-1}y_{1,4}^{2} - \chi_{n+1}y_{1,2}^{n}y_{1,4}$	$(-1)^n y_{1,3}y_{2,4}^{n}$	$\chi_n y_{2,3}^{n-1}y_{2,4}^{2} - \chi_{n+1}y_{2,3}^{n}y_{3,4}$
$y_{3,4}^{n}$	$(-1)^n y_{1,2}y_{3,4}^{n}$	$\chi_n y_{1,3}^{n-1}y_{1,4}^{2} - \chi_{n+1}y_{1,3}^{n}y_{1,4}$	$\chi_n y_{2,3}^{n-1}y_{2,4}^{2} - \chi_{n+1}y_{2,3}^{n}y_{2,4}$

Table A.1.6: Products yy' and $n \geqslant 5$.

y \ y'	$y_{1,4}$	$y_{2,4}$	$y_{3,4}$
$y_{1,2}^{n}$	$y_{1,2}^{n}y_{1,4}$	$y_{1,2}^{n}y_{2,4}$	$y_{1,2}^{n}y_{3,4}$
$y_{1,2}^{n-1}y_{1,3}$	$y_{1,2}^{n-1}y_{1,3}y_{1,4}$	$y_{1,2}^{n-1}y_{1,3}y_{2,4}$	$y_{1,2}^{n-1}y_{1,3}y_{3,4}$
$y_{1,2}^{n-1}y_{2,3}$	$y_{1,2}^{n-1}y_{2,3}y_{1,4}$	$y_{1,2}^{n-1}y_{2,3}y_{2,4}$	$y_{1,2}^{n-1}y_{2,3}y_{3,4}$
$y_{1,2}^{n-1}y_{1,4}$	$y_{1,2}^{n-1}y_{1,4}^{2}$	$y_{1,2}^{n}y_{1,4}$	$y_{1,2}^{n-1}y_{1,3}y_{1,4}$
$y_{1,2}^{n-1}y_{2,4}$	$-y_{1,2}^{n}y_{2,4}$	$y_{1,2}^{n-1}y_{1,4}^{2}$	$y_{1,2}^{n-1}y_{2,3}y_{2,4}$
$y_{1,2}^{n-2}y_{1,3}^{2}$	$y_{1,2}^{n-2}y_{1,3}^{2}y_{1,4}$	$y_{1,2}^{n-2}y_{1,3}^{2}y_{2,4}$	$y_{1,2}^{n-2}y_{1,3}^{2}y_{3,4}$
$y_{1,2}^{n-2}y_{1,3}y_{1,4}$	$y_{1,2}^{n-2}y_{1,3}y_{1,4}^{2}$	$-y_{1,2}^{n-1}y_{2,3}y_{1,4}$	$y_{1,2}^{n-2}y_{1,3}^{2}y_{1,4}$
$y_{1,2}^{n-2}y_{1,3}y_{2,4}$	$y_{1,2}^{n-1}y_{2,3}y_{2,4}$	$y_{1,2}^{n-2}y_{1,3}y_{1,4}^{2}$	$y_{1,2}^{n-1}y_{1,3}y_{2,4}$
$y_{1,2}^{n-2}y_{1,3}y_{3,4}$	$-y_{1,2}^{n-2}y_{1,3}^{2}y_{3,4}$	$-y_{1,2}^{n-1}y_{1,3}y_{3,4}$	$y_{1,2}^{n-2}y_{1,3}y_{1,4}^{2}$
$y_{1,2}^{n-2}y_{2,3}y_{1,4}$	$y_{1,2}^{n-2}y_{2,3}y_{1,4}^{2}$	$-y_{1,2}^{n-1}y_{1,3}y_{1,4}$	$-y_{1,2}^{n-1}y_{2,3}y_{1,4}$
$y_{1,2}^{n-2}y_{2,3}y_{2,4}$	$y_{1,2}^{n-1}y_{1,3}y_{2,4}$	$y_{1,2}^{n-2}y_{2,3}y_{1,4}^{2}$	$y_{1,2}^{n-2}y_{1,3}^{2}y_{2,4}$
$y_{1,2}^{n-2}y_{2,3}y_{3,4}$	$y_{1,2}^{n-1}y_{2,3}y_{3,4}$	$-y_{1,2}^{n-2}y_{1,3}^{2}y_{3,4}$	$y_{1,2}^{n-2}y_{2,3}y_{1,4}^{2}$
$y_{1,2}^{n-2}y_{1,4}^{2}$	$y_{1,2}^{n}y_{1,4}$	$y_{1,2}^{n}y_{2,4}$	$y_{1,2}^{n-2}y_{1,3}^{2}y_{3,4}$
$y_{1,2}^{n-3}y_{1,3}^{2}y_{1,4}$	$y_{1,2}^{n-3}y_{1,3}^{2}y_{1,4}^{2}$	$y_{1,2}^{n-2}y_{1,3}^{2}y_{1,4}$	$y_{1,2}^{n-1}y_{1,3}y_{1,4}$
$y_{1,2}^{n-3}y_{1,3}^{2}y_{2,4}$	$-y_{1,2}^{n-2}y_{1,3}^{2}y_{2,4}$	$y_{1,2}^{n-3}y_{1,3}^{2}y_{1,4}^{2}$	$y_{1,2}^{n-1}y_{2,3}y_{2,4}$
$y_{1,2}^{n-3}y_{1,3}^{2}y_{3,4}$	$-y_{1,2}^{n-1}y_{1,3}y_{3,4}$	$-y_{1,2}^{n-1}y_{2,3}y_{3,4}$	$y_{1,2}^{n-3}y_{1,3}^{2}y_{1,4}^{2}$
$y_{1,2}^{n-3}y_{1,3}y_{1,4}^{2}$	$y_{1,2}^{n-1}y_{1,3}y_{1,4}$	$y_{1,2}^{n-1}y_{1,3}y_{2,4}$	$y_{1,2}^{n-1}y_{1,3}y_{3,4}$
$y_{1,2}^{n-3}y_{2,3}y_{1,4}^{2}$	$y_{1,2}^{n-1}y_{2,3}y_{1,4}$	$y_{1,2}^{n-1}y_{2,3}y_{2,4}$	$y_{1,2}^{n-1}y_{2,3}y_{3,4}$
$y_{1,2}^{n-4}y_{1,3}^{2}y_{1,4}^{2}$	$y_{1,2}^{n-2}y_{1,3}^{2}y_{1,4}$	$y_{1,2}^{n-2}y_{1,3}^{2}y_{2,4}$	$y_{1,2}^{n-2}y_{1,3}^{2}y_{3,4}$
$y_{1,3}^{n}$	$y_{1,3}^{n}y_{1,4}$	$y_{1,3}^{n}y_{2,4}$	$y_{1,3}^{n}y_{3,4}$
$y_{1,3}^{n-1}y_{1,4}$	$y_{1,3}^{n-1}y_{1,4}^{2}$	$\chi_{n+1}y_{1,2}^{n-2}y_{1,3}^{2}y_{1,4} - \chi_n y_{1,2}^{n-1}y_{2,3}y_{1,4}$	$y_{1,3}^{n}y_{1,4}$
$y_{1,3}^{n-1}y_{3,4}$	$-y_{1,3}^{n}y_{3,4}$	$-\chi_n y_{1,2}^{n-1}y_{1,3}y_{3,4} - \chi_{n+1}y_{1,2}^{n-1}y_{2,3}y_{3,4}$	$y_{1,3}^{n-1}y_{1,4}^{2}$
$y_{1,3}^{n-2}y_{1,4}^{2}$	$y_{1,3}^{n}y_{1,4}$	$\chi_n y_{1,2}^{n-2}y_{1,3}^{2}y_{2,4} + \chi_{n+1}y_{1,2}^{n-1}y_{1,3}y_{2,4}$	$y_{1,3}^{n}y_{3,4}$
$y_{2,3}^{n}$	$y_{2,3}^{n}y_{1,4}$	$y_{2,3}^{n}y_{2,4}$	$y_{2,3}^{n}y_{3,4}$
$y_{2,3}^{n-1}y_{2,4}$	$\chi_n y_{1,2}^{n-1}y_{1,3}y_{2,4} - \chi_{n+1}y_{1,2}^{n-2}y_{1,3}^{2}y_{2,4}$	$y_{2,3}^{n-1}y_{2,4}^{2}$	$y_{2,3}^{n}y_{2,4}$
$y_{2,3}^{n-1}y_{3,4}$	$\chi_n y_{1,2}^{n-1}y_{2,3}y_{3,4} - \chi_{n+1}y_{1,2}^{n-1}y_{1,3}y_{3,4}$	$-y_{2,3}^{n}y_{3,4}$	$y_{2,3}^{n-1}y_{2,4}^{2}$
$y_{2,3}^{n-2}y_{2,4}^{2}$	$\chi_n y_{1,2}^{n-2}y_{1,3}^{2}y_{1,4} + \chi_{n+1}y_{1,2}^{n-2}y_{2,3}y_{1,4}$	$y_{2,3}^{n}y_{2,4}$	$y_{2,3}^{n}y_{3,4}$
$y_{1,4}^{n}$	$y_{1,4}^{n+1}$	$\chi_n y_{1,2}^{n}y_{2,4} + \chi_{n+1}y_{1,2}^{n}y_{1,4}$	$\chi_n y_{1,3}^{n}y_{3,4} + \chi_{n+1}y_{1,3}^{n}y_{1,4}$
$y_{2,4}^{n}$	$\chi_n y_{1,2}^{n}y_{1,4} - \chi_{n+1}y_{1,2}^{n}y_{2,4}$	$y_{2,4}^{n+1}$	$\chi_n y_{2,3}^{n}y_{3,4} + \chi_{n+1}y_{2,3}^{n}y_{2,4}$
$y_{3,4}^{n}$	$\chi_n y_{1,3}^{n}y_{1,4} - \chi_{n+1}y_{1,3}^{n}y_{3,4}$	$\chi_n y_{2,3}^{n}y_{2,4} - \chi_{n+1}y_{2,3}^{n}y_{3,4}$	$y_{3,4}^{n+1}$

Table A.1.7: Products yy' and $n \geqslant 5$.

	$y \backslash y'$	$y_{1,2}$	$y_{3,4}$	$y_{1,3}$	$y_{2,4}$	$y_{2,3}$	$y_{1,4}$
1	$y_{1,2}^{n-1}y_{1,3}$	1	5	3	20	-2	15
2	$y_{1,2}^{n-2}y_{1,3}^2$	2	21	1	16	3	6
3	$y_{1,2}^{n-1}y_{2,3}$	3	17	-2	8	-1	23
4	$y_{1,2}^{n-1}y_{1,4}$	4	20	-8	-9	-5	14
5	$y_{1,2}^{n-2}y_{1,3}y_{1,4}$	5	-10	6	-17	-8	21
6	$y_{1,2}^{n-3}y_{1,3}^2y_{1,4}$	6	20	-8	-11	-5	16
7	$y_{1,3}^{n-1}y_{1,4}$	5	-12	7	-17	-8	22
8	$y_{1,2}^{n-2}y_{2,3}y_{1,4}$	8	-15	5	-23	6	-13
9	$y_{1,2}^{n-2}y_{1,4}^2$	9	21	10	14	13	4
10	$y_{1,2}^{n-3}y_{1,3}y_{1,4}^2$	10	5	13	20	-11	15
11	$y_{1,2}^{n-4}y_{1,3}^2y_{1,4}^2$	11	21	10	16	13	6
12	$y_{1,3}^{n-2}y_{1,4}^2$	11	22	12	16	13	7
13	$y_{1,2}^{n-3}y_{2,3}y_{1,4}^2$	13	17	-11	8	-10	23
14	$y_{1,2}^{n-1}y_{2,4}$	14	23	-17	-4	-15	-9
15	$y_{1,2}^{n-2}y_{1,3}y_{2,4}$	15	8	16	-10	-17	20
16	$y_{1,2}^{n-3}y_{1,3}^2y_{2,4}$	16	23	-17	-6	-15	-11
17	$y_{1,2}^{n-2}y_{2,3}y_{2,4}$	17	-13	15	21	16	5
18	$y_{2,3}^{n-1}y_{2,4}$	17	-19	15	24	18	5
19	$y_{2,3}^{n-2}y_{2,4}^2$	11	24	10	18	19	6
20	$y_{1,2}^{n-2}y_{1,3}y_{3,4}$	20	-6	21	-15	-23	-10
21	$y_{1,2}^{n-3}y_{1,3}^2y_{3,4}$	21	-11	-23	-5	-20	-17
22	$y_{1,3}^{n-1}y_{3,4}$	20	-7	22	-15	-23	-12
23	$y_{1,2}^{n-2}y_{2,3}y_{3,4}$	23	-16	20	-13	21	8
24	$y_{2,3}^{n-1}y_{3,4}$	23	-18	20	-19	24	8

Table A.1.8: Products $y'y$ and $n \geqslant 6$ even.

	$y \backslash y'$	$y_{1,2}$	$y_{3,4}$	$y_{1,3}$	$y_{2,4}$	$y_{2,3}$	$y_{1,4}$
1	$y_{1,2}^{n-1}y_{1,3}$	1	-5	2	-15	-3	-20
2	$y_{1,2}^{n-2}y_{1,3}^2$	2	-21	-3	-6	-1	-16
3	$y_{1,2}^{n-1}y_{2,3}$	3	-17	1	-23	2	-8
4	$y_{1,2}^{n-1}y_{1,4}$	4	-20	5	-14	8	9
5	$y_{1,2}^{n-2}y_{1,3}y_{1,4}$	5	10	8	-21	-6	17
6	$y_{1,2}^{n-3}y_{1,3}^2y_{1,4}$	6	-20	5	-16	8	11
7	$y_{1,3}^{n-1}y_{1,4}$	6	-22	7	-16	8	12
8	$y_{1,2}^{n-2}y_{2,3}y_{1,4}$	8	15	-6	13	-5	23
9	$y_{1,2}^{n-2}y_{1,4}^2$	9	-21	-13	-4	-10	-14
10	$y_{1,2}^{n-3}y_{1,3}y_{1,4}^2$	10	-5	11	-15	-13	-20
11	$y_{1,2}^{n-4}y_{1,3}^2y_{1,4}^2$	11	-21	-13	-6	-10	-16
12	$y_{1,3}^{n-2}y_{1,4}^2$	10	-7	12	-15	-13	-22
13	$y_{1,2}^{n-3}y_{2,3}y_{1,4}^2$	13	-17	10	-23	11	-8
14	$y_{1,2}^{n-1}y_{2,4}$	14	-23	15	9	17	4
15	$y_{1,2}^{n-2}y_{1,3}y_{2,4}$	15	-8	17	-20	-16	10
16	$y_{1,2}^{n-3}y_{1,3}^2y_{2,4}$	16	-23	15	11	17	6
17	$y_{1,2}^{n-2}y_{2,3}y_{2,4}$	17	13	-16	-5	-15	-21
18	$y_{2,3}^{n-1}y_{2,4}$	16	-24	15	19	18	6
19	$y_{2,3}^{n-2}y_{2,4}^2$	13	-18	10	-24	19	-8
20	$y_{1,2}^{n-2}y_{1,3}y_{3,4}$	20	6	23	10	-21	15
21	$y_{1,2}^{n-3}y_{1,3}^2y_{3,4}$	21	11	20	17	23	5
22	$y_{1,3}^{n-1}y_{3,4}$	21	12	22	17	23	7
23	$y_{1,2}^{n-2}y_{2,3}y_{3,4}$	23	16	-21	-8	-20	13
24	$y_{2,3}^{n-1}y_{3,4}$	21	19	20	18	24	5

Table A.1.9: Products $y'y$ and $n \geqslant 5$ odd.

A.2 A basis of M^1

We present here the GAP code for computing a basis of the quadratic module M^1, defined at the beginning of Subsection 6.1.2. The code was provided by J.W. Knopper.

```
# Knopper's code
LoadPackage("GBNP");
A:=FreeAssociativeAlgebraWithOne(Rationals,"x12","x13","x23","x14","x24","x34");;
x12:=A.x12;; x13:=A.x13;; x23:=A.x23;; x14:=A.x14;; x24:=A.x24;; x34:=A.x34;;
oA:=One(A);;
relationsA:=[x12^2, x13^2, x23^2, x14^2, x24^2, x34^2, x12*x23-x23*x13-x13*x12,
 x23*x12-x12*x13-x13*x23, x12*x24-x24*x14-x14*x12, x24*x12-x12*x14-x14*x24,
 x13*x34-x34*x14-x14*x13, x34*x13-x13*x14-x14*x34, x23*x34-x34*x24-x24*x23,
 x34*x23-x23*x24-x24*x34, x12*x34-x34*x12, x13*x24-x24*x13, x14*x23-x23*x14];;
relsANP:=GP2NPList(relationsA);;
GBNP.ConfigPrint(A);
GA:=Grobner(relsANP);;

MD:=A^2;;
ab:=GeneratorsOfLeftModule(MD);
c2:=ab[1]; c1:=ab[2];
modrels:=[c1*x13, c1*x24, c2*x23, c2*x14, c1*x12+c2*x12, c1*x34+c2*x34];;
modrelsNP:=GP2NPList(modrels);
PrintNPList(modrelsNP);

GBNP.CheckHom:=function(G,wtv)
    local i,j,k,l,mon,h1,h2,ans;
    mon:=LMonsNP(G);
    ans:=GBNP.WeightedDegreeList(mon,wtv);
    for i in [1..Length(G)] do
        h1:=ans[i];
        l:=Length(G[i][1]);
        for j in [2..l] do
            mon:=G[i][1][j];
            h2:=0;
            for k in [1..Length(mon)] do
                if mon[k]>0 then
                # Don't count module generators, which have a negative index.
                # Only count two-sided generators with index 1 or more
                       h2:=h2+wtv[mon[k]];
                fi;
            od;
            if h2<>h1 then return(false); fi;
        od;
    od;
    Info(InfoGBNP,1,"Input is homogeneous");
    return(ans);
end;

GBNP.WeightedDegreeMon:=function(mon,lst)
    local i,ans;
    ans:=0;
    for i in mon do
        # Don't count module generators, which have a negative index.
        # Only count two-sided generators with index 1 or more
        if i>0 then
            ans:=ans+lst[i];
        fi;
    od;
    return(ans);
end;;

SetInfoLevel(InfoGBNP,1);
SetInfoLevel(InfoGBNPTime,1);
combinedrelsNP:=Concatenation(GA,modrelsNP);
GAT:=SGrobnerTrunc(combinedrelsNP, 9, [1,1,1,1,1,1]);
PrintNPList(GAT);

splitGAT:=function(GAT)
    local p, ts, rel, lm;
    # p: list of module or prefix relations, ts: list of two-sided relations,
```

```
    # rel: current relation, lm: leading monomial of current relation rel
    p:=[];
    ts:=[];
    for rel in GAT do
        # get leading monomial
        lm := rel[1,1];
        if Length(lm)>1 and lm[1]<0 then
            # module relations start with a negative generator.
            # if 1 is part of the GB then it does not have a generator,
            # furthermore it is two-sided.
            Add(p, rel);
        else
            Add(ts, rel);
        fi;
    od;
    return rec(p:=p, ts:=ts);
end;;

split:=splitGAT(GAT);
GBR:=rec(p:=split.p, pg:=2, ts:=split.ts);
BQM:=BaseQM(GBR,6,2,0);;
PrintNPList(BQM);

[ 0, 1 ]
[ 1 , 0]
[ 0, x12 ]
[ 0, x23 ]
[ 0, x14 ]
[ 0, x34 ]
[ x13 , 0]
[ x24 , 0]
[ 0, x12x13 ]
[ 0, x12x23 ]
[ 0, x12x14 ]
[ 0, x12x24 ]
[ 0, x12x34 ]
[ 0, x23x14 ]
[ 0, x23x24 ]
[ 0, x23x34 ]
[ 0, x14x13 ]
[ 0, x14x34 ]
[ x13x24 , 0]
[ 0, x12x13x14 ]
[ 0, x12x13x24 ]
[ 0, x12x13x34 ]
[ 0, x12x23x14 ]
[ 0, x12x23x24 ]
[ 0, x12x23x34 ]
[ 0, x12x14x13 ]
[ 0, x12x14x34 ]
[ 0, x12x24x23 ]
[ 0, x12x24x34 ]
[ 0, x23x14x12 ]
[ 0, x23x14x34 ]
[ 0, x12x13x14x12 ]
[ 0, x12x13x14x13 ]
[ 0, x12x13x14x24 ]
[ 0, x12x13x14x34 ]
[ 0, x12x13x24x23 ]
[ 0, x12x13x24x34 ]
[ 0, x12x23x14x13 ]
[ 0, x12x23x14x24 ]
[ 0, x12x23x24x34 ]
[ 0, x12x14x13x34 ]
[ 0, x12x24x23x34 ]
[ 0, x12x13x14x12x23 ]
[ 0, x12x13x14x12x24 ]
[ 0, x12x13x14x12x34 ]
[ 0, x12x13x14x13x34 ]
[ 0, x12x13x14x24x23 ]
[ 0, x12x13x24x23x34 ]
[ 0, x12x13x14x12x23x34 ]
[ 0, x12x13x14x12x24x23 ]
```

A.3 A basis of $\mathrm{FK}(4)$

We present here the GAP code as well the result to compute the basis $W^{1,2}$ (consisting of standard words) of A under the order $x_{1,2} \prec x_{3,4} \prec x_{1,3} \prec x_{2,3} \prec x_{1,4} \prec x_{2,4}$.

```
LoadPackage("GBNP");
A:=FreeAssociativeAlgebraWithOne(Rationals,"x12","x34","x13","x23","x14","x24");;
x12:=A.x12;; x13:=A.x13;; x23:=A.x23;; x14:=A.x14;; x24:=A.x24;; x34:=A.x34;;
oA:=One(A);;
relationsA:=[x12^2, x13^2, x23^2, x14^2, x24^2, x34^2, x12*x23-x23*x13-x13*x12,
 x23*x12-x12*x13-x13*x23, x12*x24-x24*x14-x14*x12, x24*x12-x12*x14-x14*x24,
 x13*x34-x34*x14-x14*x13, x34*x13-x13*x14-x14*x34, x23*x34-x34*x24-x24*x23,
 x34*x23-x23*x24-x24*x34, x12*x34-x34*x12, x13*x24-x24*x13, x14*x23-x23*x14];;
relsANP:=GP2NPList(relationsA);;
GA:=Grobner(relsANP);;
GBNP.ConfigPrint(A);

 x12^2
 x34x12 - x12x34
 x34^2
 x13^2
 x23x12 - x13x23 - x12x13
 x23x13 + x13x12 - x12x23
 x23^2
 x14x34 + x13x14 - x34x13
 x14x13 - x13x34 + x34x14
 x14x23 - x23x14
 x14^2
 x24x12 - x14x24 - x12x14
 x24x34 + x23x24 - x34x23
 x24x13 - x13x24
 x24x23 - x23x34 + x34x24
 x24x14 + x14x12 - x12x24
 x24^2
 x13x12x13 + x12x13x12
 x13x34x13 - x34x13x34
 x23x34x23 - x34x23x34
 x14x12x34 + x13x14x12 - x34x13x12
 x14x12x13 - x23x14x12 + x13x34x23 - x34x23x14
 x14x12x23 + x23x34x14 + x34x14x12 - x12x23x34
 x14x12x14 + x12x14x12
 x23x34x13x12 - x13x34x23x34 + x34x23x34x13 - x12x13x34x23
 x23x34x13x34 + x13x12x34x13 - x12x23x34x13
 x23x34x13x23 + x13x23x34x13 - x34x13x23x34
 x13x12x34x13x12 + x34x13x12x34x13
 x13x12x34x13x34 + x12x13x12x34x13

W:=BaseQA(GA,6,0);;
PrintNPList(W);
```

The basis $W^{1,2}$ is given by the following 576 elements

$1, x_{1,2}, x_{3,4}, x_{1,3}, x_{2,3}, x_{1,4}, x_{2,4}, x_{1,2}x_{3,4}, x_{1,2}x_{1,3}, x_{1,2}x_{2,3}, x_{1,2}x_{1,4}, x_{1,2}x_{2,4}, x_{3,4}x_{1,3}, x_{3,4}x_{2,3},$
$x_{3,4}x_{1,4}, x_{3,4}x_{2,4}, x_{1,3}x_{1,2}, x_{1,3}x_{3,4}, x_{1,3}x_{2,3}, x_{1,3}x_{1,4}, x_{1,3}x_{2,4}, x_{2,3}x_{3,4}, x_{2,3}x_{1,4}, x_{2,3}x_{2,4}, x_{1,4}x_{1,2},$
$x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}, x_{1,2}x_{3,4}x_{2,3}, x_{1,2}x_{3,4}x_{1,4}, x_{1,2}x_{3,4}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}, x_{1,2}x_{1,3}x_{3,4}, x_{1,2}x_{1,3}x_{2,3},$
$x_{1,2}x_{1,3}x_{1,4}, x_{1,2}x_{1,3}x_{2,4}, x_{1,2}x_{2,3}x_{3,4}, x_{1,2}x_{2,3}x_{1,4}, x_{1,2}x_{2,3}x_{2,4}, x_{1,2}x_{1,4}x_{1,2}, x_{1,2}x_{1,4}x_{2,4},$
$x_{3,4}x_{1,3}x_{1,2}, x_{3,4}x_{1,3}x_{3,4}, x_{3,4}x_{1,3}x_{2,3}, x_{3,4}x_{1,3}x_{1,4}, x_{3,4}x_{1,3}x_{2,4}, x_{3,4}x_{2,3}x_{3,4}, x_{3,4}x_{2,3}x_{1,4},$
$x_{3,4}x_{2,3}x_{2,4}, x_{3,4}x_{1,4}x_{1,2}, x_{3,4}x_{1,4}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}, x_{1,3}x_{1,2}x_{2,3}, x_{1,3}x_{1,2}x_{1,4}, x_{1,3}x_{1,2}x_{2,4},$
$x_{1,3}x_{3,4}x_{2,3}, x_{1,3}x_{3,4}x_{1,4}, x_{1,3}x_{3,4}x_{2,4}, x_{1,3}x_{2,3}x_{3,4}, x_{1,3}x_{2,3}x_{1,4}, x_{1,3}x_{2,3}x_{2,4}, x_{1,3}x_{1,4}x_{1,2},$
$x_{1,3}x_{1,4}x_{2,4}, x_{2,3}x_{3,4}x_{1,3}, x_{2,3}x_{3,4}x_{1,4}, x_{2,3}x_{3,4}x_{2,4}, x_{2,3}x_{1,4}x_{1,2}, x_{2,3}x_{1,4}x_{2,4}, x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}, x_{1,2}x_{3,4}x_{1,3}x_{2,3}, x_{1,2}x_{3,4}x_{1,3}x_{1,4}, x_{1,2}x_{3,4}x_{1,3}x_{2,4},$
$x_{1,2}x_{3,4}x_{2,3}x_{3,4}, x_{1,2}x_{3,4}x_{2,3}x_{1,4}, x_{1,2}x_{3,4}x_{2,3}x_{2,4}, x_{1,2}x_{3,4}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{1,4}x_{2,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{3,4}, x_{1,2}x_{1,3}x_{1,2}x_{2,3}, x_{1,2}x_{1,3}x_{1,2}x_{1,4}, x_{1,2}x_{1,3}x_{1,2}x_{2,4}, x_{1,2}x_{1,3}x_{3,4}x_{2,3},$
$x_{1,2}x_{1,3}x_{3,4}x_{1,4}, x_{1,2}x_{1,3}x_{3,4}x_{2,4}, x_{1,2}x_{1,3}x_{2,3}x_{3,4}, x_{1,2}x_{1,3}x_{2,3}x_{1,4}, x_{1,2}x_{1,3}x_{2,3}x_{2,4},$
$x_{1,2}x_{1,3}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{1,4}x_{2,4}, x_{1,2}x_{2,3}x_{3,4}x_{1,3}, x_{1,2}x_{2,3}x_{3,4}x_{1,4}, x_{1,2}x_{2,3}x_{3,4}x_{2,4},$

$x_{1,2}x_{2,3}x_{1,4}x_{1,2}, x_{1,2}x_{2,3}x_{1,4}x_{2,4}, x_{1,2}x_{1,4}x_{1,2}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}, x_{3,4}x_{1,3}x_{1,2}x_{2,3},$

$x_{3,4}x_{1,3}x_{1,2}x_{1,4}, x_{3,4}x_{1,3}x_{1,2}x_{2,4}, x_{3,4}x_{1,3}x_{3,4}x_{2,3}, x_{3,4}x_{1,3}x_{3,4}x_{1,4}, x_{3,4}x_{1,3}x_{3,4}x_{2,4},$

$x_{3,4}x_{1,3}x_{2,3}x_{3,4}, x_{3,4}x_{1,3}x_{2,3}x_{1,4}, x_{3,4}x_{1,3}x_{2,3}x_{2,4}, x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{3,4}x_{1,3}x_{1,4}x_{2,4},$

$x_{3,4}x_{2,3}x_{3,4}x_{1,3}, x_{3,4}x_{2,3}x_{3,4}x_{1,4}, x_{3,4}x_{2,3}x_{3,4}x_{2,4}, x_{3,4}x_{2,3}x_{1,4}x_{1,2}, x_{3,4}x_{2,3}x_{1,4}x_{2,4},$

$x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}, x_{1,3}x_{1,2}x_{3,4}x_{2,3}, x_{1,3}x_{1,2}x_{3,4}x_{1,4}, x_{1,3}x_{1,2}x_{3,4}x_{2,4},$

$x_{1,3}x_{1,2}x_{2,3}x_{3,4}, x_{1,3}x_{1,2}x_{2,3}x_{1,4}, x_{1,3}x_{1,2}x_{2,3}x_{2,4}, x_{1,3}x_{1,2}x_{1,4}x_{1,2}, x_{1,3}x_{1,2}x_{1,4}x_{2,4},$

$x_{1,3}x_{3,4}x_{2,3}x_{3,4}, x_{1,3}x_{3,4}x_{2,3}x_{1,4}, x_{1,3}x_{3,4}x_{2,3}x_{2,4}, x_{1,3}x_{3,4}x_{1,4}x_{1,2}, x_{1,3}x_{3,4}x_{1,4}x_{2,4},$

$x_{1,3}x_{2,3}x_{3,4}x_{1,3}, x_{1,3}x_{2,3}x_{3,4}x_{1,4}, x_{1,3}x_{2,3}x_{3,4}x_{2,4}, x_{1,3}x_{2,3}x_{1,4}x_{1,2}, x_{1,3}x_{2,3}x_{1,4}x_{2,4},$

$x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{2,3}x_{3,4}x_{1,3}x_{2,4}, x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{2,3}x_{3,4}x_{1,4}x_{2,4},$

$x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{1,4},$

$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{1,4}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,4},$

$x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}, x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}, x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{1,2},$

$x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}, x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}, x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{2,4},$

$x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3},$

$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{3,4},$

$x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{1,4}, x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{1,2}x_{1,4}x_{2,4},$

$x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{3,4}, x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{1,4}, x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{2,4}, x_{1,2}x_{1,3}x_{3,4}x_{1,4}x_{1,2},$

$x_{1,2}x_{1,3}x_{3,4}x_{1,4}x_{2,4}, x_{1,2}x_{1,3}x_{2,3}x_{3,4}x_{1,3}, x_{1,2}x_{1,3}x_{2,3}x_{3,4}x_{1,4}, x_{1,2}x_{1,3}x_{2,3}x_{3,4}x_{2,4},$

$x_{1,2}x_{1,3}x_{2,3}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{2,3}x_{1,4}x_{2,4}, x_{1,2}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4},$

$x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{2,4}, x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{1,2}x_{2,3}x_{1,4}x_{1,2}x_{2,4},$

$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,4},$

$x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}, x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{1,4}, x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{1,4}x_{1,2},$

$x_{3,4}x_{1,3}x_{1,2}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}, x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{1,4}, x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{2,4},$

$x_{3,4}x_{1,3}x_{3,4}x_{1,4}x_{1,2}, x_{3,4}x_{1,3}x_{3,4}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}, x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4},$

$x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{2,4}, x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{1,2}, x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$

$x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{2,4}, x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{2,4},$

$x_{3,4}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,4},$

$x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}, x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}, x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}x_{1,4}x_{1,2},$

$x_{1,3}x_{1,2}x_{3,4}x_{1,4}x_{2,4}, x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}, x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4}, x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{2,4},$

$x_{1,3}x_{1,2}x_{2,3}x_{1,4}x_{1,2}, x_{1,3}x_{1,2}x_{2,3}x_{1,4}x_{2,4}, x_{1,3}x_{1,2}x_{1,4}x_{1,2}x_{2,4}, x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3},$

$x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}, x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{2,4}, x_{1,3}x_{3,4}x_{2,3}x_{1,4}x_{1,2}, x_{1,3}x_{3,4}x_{2,3}x_{1,4}x_{2,4},$

$x_{1,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{2,4}, x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2},$

$x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{1,3}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4},$

$x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,4},$

$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{1,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{2,4},$

$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{1,4},$

$x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3},$

$x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}, x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{2,4},$

$x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{2,4}, x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2},$

$x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4},$

$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{2,4},$

$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,4}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}, x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4},$

$x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{1,4}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{1,4}x_{1,2}x_{2,4},$

$x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}, x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}, x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{2,4}, x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{1,4}x_{1,2},$

$x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{1,4}x_{2,4}, x_{1,2}x_{1,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{1,2}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{2,4},$

$x_{1,2}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{1,2}x_{1,3}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2},$

$x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4},$

$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{2,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,4}x_{1,2}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}, x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{1,4}x_{1,2}, x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{1,4}x_{1,2}x_{2,4},$
$x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}, x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}, x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{2,4}, x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{1,4}x_{1,2},$
$x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{2,4},$
$x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2},$
$x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4},$
$x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3},$
$x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}, x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{1,2}, x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{2,4},$
$x_{1,3}x_{1,2}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{2,4}, x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{1,2},$
$x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{1,3}x_{1,2}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{2,4},$
$x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{1,3}x_{3,4}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2},$
$x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{1,4}x_{1,2},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3},$
$x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{1,4}x_{1,2},$
$x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4},$
$x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{2,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{2,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{1,2},$
$x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4},$
$x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{2,4}, x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{2,4},$
$x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4},$
$x_{1,2}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{1,2},$
$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{1,2}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{2,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{2,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{2,4}, x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2},$
$x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2},$
$x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{2,4},$
$x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{1,2}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2},$
$x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2},$
$x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2},$
$x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3},$

$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{1,2},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{2,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{2,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{1,2}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{2,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{2,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4},$
$x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4},$
$x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{2,4},$
$x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{1,2},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{2,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{2,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2},$
$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{2,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{2,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2},$
$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4},$
$x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4},$

$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}.$

A.4 Koszul complex of M^i for $i \in \{0,1,2,3\}$

We present here the GAP code for computing the differential of the Koszul complex of the quadratic modules $M^0 = \Bbbk$ and M^i for $i \in [\![1,3]\!]$ defined in Subsection 6.1.2. We also present a basis of $\mathrm{H}_{n,m}(M^i)$ for some pairs (n,m). In the following code, the matrix FF(i,n,m) represents the linear map $d_{n+1,m-1}(M^i) : \mathrm{K}_{n+1,m-1}(M^i) \to \mathrm{K}_{n,m}(M^i)$, Im(i,n,m) is a basis of the space $B^{M^i}_{n,m}$ and Ker(i,n,m) is a basis of the space $D^{M^i}_{n,m}$. Moreover, geneMH(i,n,m) are some elements in $D^{M^i}_{n,m}$, and we can show that it represents a basis of $\mathrm{H}_{n,m}(M^i)$ since the dimension of the space spanned by $B^{M^i}_{n,m}$ and geneMH(i,n,m) coincides with the dimension of $D^{M^i}_{n,m}$.

```
LoadPackage("GBNP");
A:=FreeAssociativeAlgebraWithOne(Rationals,"x12","x13","x23","x14","x24","x34");;
x12:=A.x12;; x13:=A.x13;; x23:=A.x23;; x14:=A.x14;; x24:=A.x24;; x34:=A.x34;;
oA:=One(A);;
relationsA:=[x12^2, x13^2, x23^2, x14^2, x24^2, x34^2, x12*x23-x23*x13-x13*x12,
 x23*x12-x12*x13-x13*x23, x12*x24-x24*x14-x14*x12, x24*x12-x12*x14-x14*x24,
 x13*x34-x34*x14-x14*x13, x34*x13-x13*x14-x14*x34, x23*x34-x34*x24-x24*x23,
 x34*x23-x23*x24-x24*x34, x12*x34-x34*x12, x13*x24-x24*x13, x14*x23-x23*x14];;
# A/relationsA is the Fomin-Kirillov algbera on 4 generators.
relsANP:=GP2NPList(relationsA);;
GBNP.ConfigPrint(A);
GA:=Grobner(relsANP);; # GA is a Gröbner basis of the ideal in A.
# PrintNPList(GA);
C:=BaseQA(GA,6,0);;  # C is the set of standard words with respect to GA.
# PrintNPList(C);

f:=function(n)
    if n=0 then return 1;
    elif n=1 then return 6;
    elif n=2 then return 19;
    elif n=3 then return 42;
    elif n=4 then return 71;
    elif n=5 then return 96;
    elif n=6 then return 106;
    elif n=7 then return 96;
    elif n=8 then return 71;
    elif n=9 then return 42;
    elif n=10 then return 19;
    elif n=11 then return 6;
    elif n=12 then return 1;
    fi;
end;
# f(n) is the dimension of $A_n$.

g:=function(n)
    if n=-1 then return 0;
    elif n=0 then return f(0);
    elif n>0 then return Sum(List([0..n], s->f(s)));
    fi;
end;
# g(n)-g(n-1)=f(n).

B:=FreeAssociativeAlgebraWithOne(Rationals,"y12","y13","y23","y14","y24","y34");;
```

```
y12:=B.y12;; y13:=B.y13;; y23:=B.y23;; y14:=B.y14;; y24:=B.y24;; y34:=B.y34;;
oB:=One(B);;
relationsB:=[y12*y23+y23*y13, y13*y23+y23*y12, y12*y23+y13*y12, y12*y13+y23*y12,
 y12*y24+y24*y14, y14*y24+y24*y12, y12*y24+y14*y12, y12*y14+y24*y12, y13*y34+y34*y14,
 y14*y34+y34*y13, y13*y34+y14*y13, y13*y14+y34*y13, y23*y34+y34*y24, y24*y34+y34*y23,
 y23*y34+y24*y23, y23*y24+y34*y23, y12*y34+y34*y12, y13*y24+y24*y13,
 y23*y14+y14*y23];;
# B/relationsB is the quadratic dual of the Fomin-Kirillov algebra on 4 generators.
relsBNP:=GP2NPList(relationsB);;
wtv:= [1,1,1,1,1,1];;
GBNP.ConfigPrint(B);;
GB:=Grobner(relsBNP);; # GB is a Gröbner basis of the ideal in B.
# PrintNPList(GB);
D:= BaseQATrunc(GB,15,wtv);;
for degpart in D do for mon in degpart do PrintNP([[mon],[1]]); od; od;
DT:=[];
for degpart in D do for mon in degpart do Append(DT,[[[mon],[1]]]); od; od;

S:=B^8;;
ab:=GeneratorsOfLeftModule(S);;
g8:=ab[1];; g7:=ab[2];; g6:=ab[3];; g5:=ab[4];;
g4:=ab[5];; g3:=ab[6];; g2:=ab[7];; g1:=ab[8];;
modrels:=[g1*y12-g2*y34, g1*y34+g2*y12, g3*y12+g4*y34, g3*y34-g4*y12, g4*y13-g2*y24,
 g4*y24+g2*y13, g3*y13-g1*y24, g3*y24+g1*y13, g1*y23+g4*y14, g1*y14-g4*y23,
 g3*y23+g2*y14, g3*y14-g2*y23, g5*y14, g5*y24, g5*y34, g6*y13, g6*y23, g6*y34, g7*y12,
 g7*y23, g7*y24, g8*y12, g8*y13, g8*y14 ];;
modrelsNP:=GP2NPList(modrels);;
# PrintNPList(modrelsNP);;

GBNP.CheckHom:=function(G,wtv)
    local i,j,k,l,mon,h1,h2,ans;
    mon:=LMonsNP(G);
    ans:=GBNP.WeightedDegreeList(mon,wtv);
    for i in [1..Length(G)] do
        h1:=ans[i];
        l:=Length(G[i][1]);
        for j in [2..l] do
            mon:=G[i][1][j];
            h2:=0;
            for k in [1..Length(mon)] do
                if mon[k]>0 then
                    h2:=h2+wtv[mon[k]];
                fi;
            od;
            if h2<>h1 then return(false); fi;
        od;
    od;
    Info(InfoGBNP,1,"Input is homogeneous");
    return(ans);
end;

GBNP.WeightedDegreeMon:=function(mon,lst)
    local i,ans;
    ans:=0;
    for i in mon do
        if i>0 then
            ans:=ans+lst[i];
        fi;
    od;
    return(ans);
end;;

SetInfoLevel(InfoGBNP,1);;
SetInfoLevel(InfoGBNPTime,1);;
combinedrelsNP:=Concatenation(GB,modrelsNP);;
GBT:=SGrobnerTrunc(combinedrelsNP, 15, [1,1,1,1,1,1]);;
# PrintNPList(GBT);

splitGBT:=function(GBT)
    local p, ts, rel, lm;
    p:=[];
    ts:=[];
    for rel in GBT do
```

```
        lm := rel[1,1];
        if Length(lm)>1 and lm[1]<0 then
            Add(p, rel);
        else
            Add(ts, rel);
        fi;
    od;
    return rec(p:=p, ts:=ts);
end;;

split:=splitGBT(GBT);;
GBRM3:=rec(p:=split.p, pg:=8, ts:=split.ts);;
BQMM3:=BaseQM(GBRM3,6,8,650);;
# PrintNPList(BQMM3);

S:=B^7;;
ab:=GeneratorsOfLeftModule(S);;
g7:=ab[1];; g6:=ab[2];; g5:=ab[3];; g4:=ab[4];; g3:=ab[5];; g2:=ab[6];; g1:=ab[7];;
modrels:=[g1*y14+g4*y14, g1*y24+g3*y24, g1*y34-g2*y34, g2*y13+g4*y13, g2*y23+g3*y23,
 g3*y12-g4*y12, g5*y12-g6*y34, g1*y24+g5*y13, g5*y23+g7*y14, g5*y14-g7*y23,
 g2*y13-g5*y24, g5*y34+g6*y12, g6*y13+g7*y24, g1*y14-g6*y23, g2*y23+g6*y14,
 g6*y24-g7*y13, g1*y34-g7*y12, g3*y12-g7*y34];;
modrelsNP:=GP2NPList(modrels);;
# PrintNPList(modrelsNP);

SetInfoLevel(InfoGBNP,1);;
SetInfoLevel(InfoGBNPTime,1);;
combinedrelsNP:=Concatenation(GB,modrelsNP);;
GBT:=SGrobnerTrunc(combinedrelsNP, 15, [1,1,1,1,1,1]);;
# PrintNPList(GBT);

split:=splitGBT(GBT);;
GBRM2:=rec(p:=split.p, pg:=7, ts:=split.ts);;
BQMM2:=BaseQM(GBRM2,6,7,1000);;
# PrintNPList(BQMM2);

MD:=B^2;;
ab:=GeneratorsOfLeftModule(MD);;
g2:=ab[1];; g1:=ab[2];;
modrels:=[g1*y12-g2*y12, g2*y13, g1*y23, g1*y14, g2*y24, g1*y34-g2*y34];;
modrelsNP:=GP2NPList(modrels);;
# PrintNPList(modrelsNP);

SetInfoLevel(InfoGBNP,1);;
SetInfoLevel(InfoGBNPTime,1);;
combinedrelsNP:=Concatenation(GB,modrelsNP);;
GBT:=SGrobnerTrunc(combinedrelsNP, 15, [1,1,1,1,1,1]);;
# PrintNPList(GBT);

split:=splitGBT(GBT);;
GBRM1:=rec(p:=split.p, pg:=2, ts:=split.ts);;
BQMM1:=BaseQM(GBRM1,6,2,400);;
# PrintNPList(BQMM1);

ff:=function(i,n)
    if i=0 and n=-1 then return 0;
    elif i=0 and n=0 then return 1;
    elif i=0 and n=1 then return 7;
    elif i=0 and n=2 then return 24;
    elif i=0 and n=3 then return 54;
    elif i=0 and n=4 then return 92;
    elif i=0 and n>4 then return (3*n+69)*(n-4)/2+92;

    elif i=1 and n=-1 then return 0;
    elif i=1 and n=0 then return 2;
    elif i=1 and n>0 then return (3*n+9)*n/2+2;

    elif i=2 and n=-1 then return 0;
    elif i=2 and n=0 then return 7;
    elif i=2 and n=1 then return 31;
    elif i=2 and n=2 then return 74;
    elif i=2 and n>2 then return (3*n+99)*(n-2)/2+74;
```

```
    elif i=3 and n=-1 then return 0;
    elif i=3 and n=0 then return 8;
    elif i=3 and n=1 then return 32;
    elif i=3 and n=2 then return 72;
    elif i=3 and n>2 then return 48*n-24;
    fi;
end;

FFM0:=function(j,i)
    local F,RDF,H,L,RFA,DFA,rra,dda,s,LAs,k,t;
    RDF:=List([ff(0,j-1)+1..ff(0,j+1)], p -> DT[p]);
    H:=List([1..6], s -> TransposedMat(MatrixQA(s,RDF,GB)));;
    L:=List([1..6], s -> List([ff(0,j)-ff(0,j-1)+1..ff(0,j+1)-ff(0,j-1)],
     q -> List([1..ff(0,j)-ff(0,j-1)], p -> H[s][q][p])));;
    RFA:=List([g(i-1)+1..g(i)], p -> C[p]);
    DFA:=List([g(i-2)+1..g(i-1)], p -> C[p]);
    rra:=Length(RFA);
    dda:=Length(DFA);
    F:=[];
    for s in [1..6] do
        LAs:=0*[1..dda];
        for k in [1..dda] do
            LAs[k]:=0*[1..rra];
            for t in [1..Length(MulQA(C[s+1], DFA[k], GA)[1])] do
                LAs[k][Position(RFA,[[MulQA(C[s+1], DFA[k], GA)[1][t]],[1]])]:=
                 MulQA(C[s+1], DFA[k], GA)[2][t];
            od;
        od;
        F:=F+KroneckerProduct(L[s],LAs);
    od;
    return F;
end;;

FFM1:=function(j,i)
    local FF,RF,DF,rr,dd,RFA,DFA,rra,dda,s,LLs,LAs,k,t;
    RF:=List([ff(1,j-1)+1..ff(1,j)], p -> BQMM1[p]);
    DF:=List([ff(1,j)+1..ff(1,j+1)], p -> BQMM1[p]);
    rr:=Length(RF);
    dd:=Length(DF);
    RFA:=List([g(i-1)+1..g(i)], p -> C[p]);
    DFA:=List([g(i-2)+1..g(i-1)], p -> C[p]);
    rra:=Length(RFA);
    dda:=Length(DFA);
    FF:=[];
    for s in [1..6] do
        LLs:=0*[1..rr];
        LAs:=0*[1..dda];
        for k in [1..rr] do
            LLs[k]:=0*[1..dd];
            for t in [1..Length(MulQM(RF[k], DT[s+1], GBRM1)[1])] do
                LLs[k][Position(DF,[[MulQM(RF[k], DT[s+1], GBRM1)[1][t]],[1]])]:=
                 MulQM(RF[k], DT[s+1], GBRM1)[2][t];
            od;
        od;
        for k in [1..dda] do
            LAs[k]:=0*[1..rra];
            for t in [1..Length(MulQA(C[s+1], DFA[k], GA)[1])] do
                LAs[k][Position(RFA,[[MulQA(C[s+1], DFA[k], GA)[1][t]],[1]])]:=
                 MulQA(C[s+1], DFA[k], GA)[2][t];
            od;
        od;
        FF:=FF+KroneckerProduct(TransposedMat(LLs),LAs);
    od;
    return FF;
end;

FFM2:=function(j,i)
    local FF,RF,DF,rr,dd,RFA,DFA,rra,dda,s,LLs,LAs,k,t;
    RF:=List([ff(2,j-1)+1..ff(2,j)], p -> BQMM2[p]);
    DF:=List([ff(2,j)+1..ff(2,j+1)], p -> BQMM2[p]);
    rr:=Length(RF);
    dd:=Length(DF);
    RFA:=List([g(i-1)+1..g(i)], p -> C[p]);
```

```
    DFA:=List([g(i-2)+1..g(i-1)], p -> C[p]);
    rra:=Length(RFA);
    dda:=Length(DFA);
    FF:=[];
    for s in [1..6] do
        LLs:=0*[1..rr];
        LAs:=0*[1..dda];
        for k in [1..rr] do
            LLs[k]:=0*[1..dd];
            for t in [1..Length(MulQM(RF[k], DT[s+1], GBRM2)[1])] do
                LLs[k][Position(DF,[[MulQM(RF[k], DT[s+1], GBRM2)[1][t]],[1]])]:=
                 MulQM(RF[k], DT[s+1], GBRM2)[2][t];
            od;
        od;
        for k in [1..dda] do
            LAs[k]:=0*[1..rra];
            for t in [1..Length(MulQA(C[s+1], DFA[k], GA)[1])] do
                LAs[k][Position(RFA,[[MulQA(C[s+1], DFA[k], GA)[1][t]],[1]])]:=
                 MulQA(C[s+1], DFA[k], GA)[2][t];
            od;
        od;
        FF:=FF+KroneckerProduct(TransposedMat(LLs),LAs);
    od;
    return FF;
end;

FFM3:=function(j,i)
    local FF,RF,DF,rr,dd,RFA,DFA,rra,dda,s,LLs,LAs,k,t;
    RF:=List([ff(3,j-1)+1..ff(3,j)], p -> BQMM3[p]);
    DF:=List([ff(3,j)+1..ff(3,j+1)], p -> BQMM3[p]);
    rr:=Length(RF);
    dd:=Length(DF);
    RFA:=List([g(i-1)+1..g(i)], p -> C[p]);
    DFA:=List([g(i-2)+1..g(i-1)], p -> C[p]);
    rra:=Length(RFA);
    dda:=Length(DFA);
    FF:=[];
    for s in [1..6] do
        LLs:=0*[1..rr];
        LAs:=0*[1..dda];
        for k in [1..rr] do
            LLs[k]:=0*[1..dd];
            for t in [1..Length(MulQM(RF[k], DT[s+1], GBRM3)[1])] do
                LLs[k][Position(DF,[[MulQM(RF[k], DT[s+1], GBRM3)[1][t]],[1]])]:=
                 MulQM(RF[k], DT[s+1], GBRM3)[2][t];
            od;
        od;
        for k in [1..dda] do
            LAs[k]:=0*[1..rra];
            for t in [1..Length(MulQA(C[s+1], DFA[k], GA)[1])] do
                LAs[k][Position(RFA,[[MulQA(C[s+1], DFA[k], GA)[1][t]],[1]])]:=
                 MulQA(C[s+1], DFA[k], GA)[2][t];
            od;
        od;
        FF:=FF+KroneckerProduct(TransposedMat(LLs),LAs);
    od;
    return FF;
end;

FF:=function(ii,j,i)
    if ii=0 then return FFM0(j,i);
    elif ii=1 then return FFM1(j,i);
    elif ii=2 then return FFM2(j,i);
    elif ii=3 then return FFM3(j,i);
    fi;
end;

Im:=function(ii,j,i)
    local Imm;
    Imm:=TriangulizedMat(BaseMatDestructive(FF(ii,j,i)));
    return Imm;
end;
```

```
Ker:=function(ii,j,i)
    local Kerr;
    Kerr:=TriangulizedNullspaceMatDestructive(FF(ii,j-1,i+1));
    return Kerr;
end;

HXR:=function(ii,Uh,Vh,Wh,n,m,r)
    local hxr,Vhxr,CC,s,t,le,yy,VP,j,i,k;
    VP:=[];;
    hxr:=0*[1..Length(Uh)*f(r)];;
    Vhxr:=0*[1..Length(Uh)*f(r)];;
    CC:=List([g(m+r-1)+1..g(m+r)], p -> C[p]);;
    for s in [1..Length(Uh)] do
        for t in [1..f(r)] do
            le:=Length(Uh[s]);;
            yy:=C[g(r-1)+t];;
            hxr[(s-1)*f(r)+t]:=0*[1..(ff(ii,n)-ff(ii,n-1))*f(m+r)];;
            VP:=0*[1..le];;
            Vhxr[(s-1)*f(r)+t]:=0*[1..le];;
            for j in [1..le] do
                VP[j]:=[ [  ], [  ] ];
                for i in [1..Length(Vh[s][j])] do
                    VP[j]:=AddNP(VP[j],MulQA(C[g(m-1)+Vh[s][j][i]],yy,GA),1,
                     Wh[s][j][i]);
                od;
                Vhxr[(s-1)*f(r)+t][j]:=List([1..Length(VP[j][1])], k ->
                 Position(CC, [ [VP[j][1][k]], [ 1 ] ] ));
                for k in [1..Length(VP[j][1])] do
                    hxr[(s-1)*f(r)+t][f(m+r)*(Uh[s][j]-1)+Vhxr[(s-1)*f(r)+t][j][k]]:=
                     VP[j][2][k];
                od;
            od;
        od;
    od;
    return hxr;
end;

UU:=function(gene,ii)
    local Rest,Uh,Vh,Wh,Post,k,aa,Quo,Res,Sig,Qu,Re,Sg,i,j;
    Rest:=function(n)
        if n mod f(ii) > 0 then return n mod f(ii);
        else return f(ii);
        fi;
    end;
    Uh:=0*[1..Length(gene)];;
    Vh:=0*[1..Length(gene)];;
    Wh:=0*[1..Length(gene)];;
    Post:=[];;
    for k in [1..Length(gene)] do
        Uh[k]:=[]; Vh[k]:=[]; Wh[k]:=[];
        aa:=gene[k];
        Post:=[1..Length(aa)];
        SubtractSet(Post, Positions(aa,0));
        Quo:=List([1..Length(Post)], s->(Post[s]-Rest(Post[s]))/f(ii)+1);
        Res:=List([1..Length(Post)], s->Rest(Post[s]));
        Sig:=List([1..Length(Post)], s->gene[k][Post[s]]);
        Qu:=Set(Quo);
        Re:=0*[1..Length(Qu)];
        Sg:=0*[1..Length(Qu)];
        for i in [1..Length(Qu)] do
            Re[i]:=[];
            Sg[i]:=[];
            for j in [1..Length(Positions(Quo,Qu[i]))] do
                Re[i][j]:=Res[Position(Quo,Qu[i])+j-1];
                Sg[i][j]:=Sig[Position(Quo,Qu[i])+j-1];
            od;
        od;
        Uh[k]:=Qu;
        Vh[k]:=Re;
        Wh[k]:=Sg;
    od;
    return Uh;
end;
```

```
VV:=function(gene,ii)
    local Rest,Uh,Vh,Wh,Post,k,aa,Quo,Res,Sig,Qu,Re,Sg,i,j;
    Rest:=function(n)
        if n mod f(ii) > 0 then return n mod f(ii);
        else return f(ii);
        fi;
    end;
    Uh:=0*[1..Length(gene)];;
    Vh:=0*[1..Length(gene)];;
    Wh:=0*[1..Length(gene)];;
    Post:=[];;
    for k in [1..Length(gene)] do
        Uh[k]:=[]; Vh[k]:=[]; Wh[k]:=[];
        aa:=gene[k];
        Post:=[1..Length(aa)];
        SubtractSet(Post, Positions(aa,0));
        Quo:=List([1..Length(Post)], s->(Post[s]-Rest(Post[s]))/f(ii)+1);
        Res:=List([1..Length(Post)], s->Rest(Post[s]));
        Sig:=List([1..Length(Post)], s->gene[k][Post[s]]);
        Qu:=Set(Quo);
        Re:=0*[1..Length(Qu)];
        Sg:=0*[1..Length(Qu)];
        for i in [1..Length(Qu)] do
            Re[i]:=[];
            Sg[i]:=[];
            for j in [1..Length(Positions(Quo,Qu[i]))] do
                Re[i][j]:=Res[Position(Quo,Qu[i])+j-1];
                Sg[i][j]:=Sig[Position(Quo,Qu[i])+j-1];
            od;
        od;
        Uh[k]:=Qu;
        Vh[k]:=Re;
        Wh[k]:=Sg;
    od;
    return Vh;
end;

WW:=function(gene,ii)
    local Rest,Uh,Vh,Wh,Post,k,aa,Quo,Res,Sig,Qu,Re,Sg,i,j;
    Rest:=function(n)
        if n mod f(ii) > 0 then return n mod f(ii);
        else return f(ii);
        fi;
    end;
    Uh:=0*[1..Length(gene)];;
    Vh:=0*[1..Length(gene)];;
    Wh:=0*[1..Length(gene)];;
    Post:=[];;
    for k in [1..Length(gene)] do
        Uh[k]:=[]; Vh[k]:=[]; Wh[k]:=[];
        aa:=gene[k];
        Post:=[1..Length(aa)];
        SubtractSet(Post, Positions(aa,0));
        Quo:=List([1..Length(Post)], s->(Post[s]-Rest(Post[s]))/f(ii)+1);
        Res:=List([1..Length(Post)], s->Rest(Post[s]));
        Sig:=List([1..Length(Post)], s->gene[k][Post[s]]);
        Qu:=Set(Quo);
        Re:=0*[1..Length(Qu)];
        Sg:=0*[1..Length(Qu)];
        for i in [1..Length(Qu)] do
            Re[i]:=[];
            Sg[i]:=[];
            for j in [1..Length(Positions(Quo,Qu[i]))] do
                Re[i][j]:=Res[Position(Quo,Qu[i])+j-1];
                Sg[i][j]:=Sig[Position(Quo,Qu[i])+j-1];
            od;
        od;
        Uh[k]:=Qu;
        Vh[k]:=Re;
        Wh[k]:=Sg;
    od;
    return Wh;
```

```
end;

geneMH:=function(i,n,m)
    if i=0 and n=3 and m=3 then return
    [Ker(0,3,3)[99], Ker(0,3,3)[378], Ker(0,3,3)[164], -Ker(0,3,3)[467],
     Ker(0,3,3)[219], -Ker(0,3,3)[40], Ker(0,3,3)[301],
     Ker(0,3,3)[206]-Ker(0,3,3)[99]-Ker(0,3,3)[378]+Ker(0,3,3)[164]-Ker(0,3,3)[467]];
    elif i=0 and n=3 and m=5 then return [Ker(0,3,5)[79]];
    elif i=0 and n=4 and m=4 then return
    [Ker(0,4,4)[550], Ker(0,4,4)[450]-Ker(0,4,4)[550]];
    elif i=0 and n=5 and m=11 then return [Ker(0,5,11)[90]];
    elif i=1 and n=1 and m=3 then return
    [Ker(1,1,3)[15], Ker(1,1,3)[27], Ker(1,1,3)[53], -Ker(1,1,3)[67],
     Ker(1,1,3)[19], -Ker(1,1,3)[16], Ker(1,1,3)[22]];
    elif i=1 and n=1 and m=5 then return [Ker(1,1,5)[76]];
    elif i=1 and n=1 and m=7 then return [Ker(1,1,7)[64]];
    elif i=2 and n=1 and m=3 then return [Ker(2,1,3)[257]];
    elif i=2 and n=1 and m=5 then return [Ker(2,1,5)[908]];
    elif i=2 and n=2 and m=4 then return
    [Ker(2,2,4)[783]-Ker(2,2,4)[784], Ker(2,2,4)[784]];
    elif i=2 and n=3 and m=3 then return
    [Ker(2,3,3)[36]-Ker(2,3,3)[193]-Ker(2,3,3)[470]-Ker(2,3,3)[570]-Ker(2,3,3)[658],
     Ker(2,3,3)[200], Ker(2,3,3)[197], -Ker(2,3,3)[16],
     Ker(2,3,3)[193], Ker(2,3,3)[470], Ker(2,3,3)[570], Ker(2,3,3)[658]];
    elif i=3 and n=3 and m=3 then return
    [Ker(3,3,3)[179], -Ker(3,3,3)[185], -Ker(3,3,3)[174], Ker(3,3,3)[176],
     Ker(3,3,3)[355], Ker(3,3,3)[452], Ker(3,3,3)[540], Ker(3,3,3)[628]];
    else return [];
    fi;
end;
```

A.5 A basis of $(M^2)^!$

We present here the GAP code to compute a basis of $(M^2)^!_{-n}$ for n less than some positive integer, where the quadratic module M^2 is defined at the beginning of Subsection 6.1.2. We also list the basis of $(M^2)^!_{-n}$ for $n \in [\![0,3]\!]$.

```
LoadPackage("GBNP");
B:=FreeAssociativeAlgebraWithOne(Rationals,"y12","y13","y23","y14","y24","y34");;
y12:=B.y12;; y13:=B.y13;; y23:=B.y23;; y14:=B.y14;; y24:=B.y24;; y34:=B.y34;;
oB:=One(B);;
relationsB:=[y12*y23+y23*y13, y13*y23+y23*y12, y12*y23+y13*y12, y12*y13+y23*y12,
 y12*y24+y24*y14, y14*y24+y24*y12, y12*y24+y14*y12, y12*y14+y24*y12, y13*y34+y34*y14,
 y14*y34+y34*y13, y13*y34+y14*y13, y13*y14+y34*y13, y23*y34+y34*y24, y24*y34+y34*y23,
 y23*y34+y24*y23, y23*y24+y34*y23, y12*y34+y34*y12, y13*y24+y24*y13,
 y23*y14+y14*y23];;
relsBNP:=GP2NPList(relationsB);;
wtv:= [1,1,1,1,1,1];;
GB:=Grobner(relsBNP);;
GBNP.ConfigPrint(B);;
PrintNPList(GB);

D:= BaseQATrunc(GB,12,wtv);;
for degpart in D do
    for mon in degpart do
        PrintNP([[mon],[1]]);
    od;
od;
DT:=[];
for degpart in D do
    for mon in degpart do
        Append(DT,[[[mon],[1]]]);
    od;
od;

S:=B^7;
ab:=GeneratorsOfLeftModule(S);
g7:=ab[1]; g6:=ab[2]; g5:=ab[3]; g4:=ab[4]; g3:=ab[5]; g2:=ab[6]; g1:=ab[7];
modrels:=[g1*y14+g4*y14, g1*y24+g3*y24, g1*y34-g2*y34, g2*y13+g4*y13, g2*y23+g3*y23,
```

```
 g3*y12-g4*y12, g5*y12-g6*y34, g1*y24+g5*y13, g5*y23+g7*y14, g5*y14-g7*y23,
 g2*y13-g5*y24, g5*y34+g6*y12, g6*y13+g7*y24, g1*y14-g6*y23, g2*y23+g6*y14,
 g6*y24-g7*y13, g1*y34-g7*y12, g3*y12-g7*y34];
modrelsNP:=GP2NPList(modrels);
PrintNPList(modrelsNP);

GBNP.CheckHom:=function(G,wtv)
    local i,j,k,l,mon,h1,h2,ans;
    mon:=LMonsNP(G);
    ans:=GBNP.WeightedDegreeList(mon,wtv);
    for i in [1..Length(G)] do
        h1:=ans[i];
        l:=Length(G[i][1]);
        for j in [2..l] do
            mon:=G[i][1][j];
            h2:=0;
            for k in [1..Length(mon)] do
                if mon[k]>0 then
                    h2:=h2+wtv[mon[k]];
                fi;
            od;
            if h2<>h1 then return(false); fi;
        od;
    od;
    Info(InfoGBNP,1,"Input is homogeneous");
    return(ans);
end;

GBNP.WeightedDegreeMon:=function(mon,lst)
    local i,ans;
    ans:=0;
    for i in mon do
        if i>0 then
            ans:=ans+lst[i];
        fi;
    od;
    return(ans);
end;;

SetInfoLevel(InfoGBNP,1);
SetInfoLevel(InfoGBNPTime,1);
combinedrelsNP:=Concatenation(GB,modrelsNP);
GBT:=SGrobnerTrunc(combinedrelsNP, 15, [1,1,1,1,1,1]);
PrintNPList(GBT);

splitGBT:=function(GBT)
    local p, ts, rel, lm;
    p:=[];
    ts:=[];
    for rel in GBT do
        lm := rel[1,1];
        if Length(lm)>1 and lm[1]<0 then
            Add(p, rel);
        else
            Add(ts, rel);
        fi;
    od;
    return rec(p:=p, ts:=ts);
end;;

split:=splitGBT(GBT);
GBR:=rec(p:=split.p, pg:=7, ts:=split.ts);
BQM:=BaseQM(GBR,6,7,500);;
PrintNPList(BQM);

[ 0, 0, 0, 0, 0, 0, 1 ]
[ 0, 0, 0, 0, 0, 1 , 0]
[ 0, 0, 0, 0, 1 , 0, 0]
[ 0, 0, 0, 1 , 0, 0, 0]
[ 0, 0, 1 , 0, 0, 0, 0]
[ 0, 1 , 0, 0, 0, 0, 0]
[ 1 , 0, 0, 0, 0, 0, 0]
[ 0, 0, 0, 0, 0, 0, y12 ]
```

```
[ 0, 0, 0, 0, 0, 0, y13 ]
[ 0, 0, 0, 0, 0, 0, y23 ]
[ 0, 0, 0, 0, 0, 0, y14 ]
[ 0, 0, 0, 0, 0, 0, y24 ]
[ 0, 0, 0, 0, 0, 0, y34 ]
[ 0, 0, 0, 0, 0, y12 , 0]
[ 0, 0, 0, 0, 0, y13 , 0]
[ 0, 0, 0, 0, 0, y23 , 0]
[ 0, 0, 0, 0, 0, y14 , 0]
[ 0, 0, 0, 0, 0, y24 , 0]
[ 0, 0, 0, 0, y12 , 0, 0]
[ 0, 0, 0, 0, y13 , 0, 0]
[ 0, 0, 0, 0, y14 , 0, 0]
[ 0, 0, 0, 0, y34 , 0, 0]
[ 0, 0, 0, y23 , 0, 0, 0]
[ 0, 0, 0, y24 , 0, 0, 0]
[ 0, 0, 0, y34 , 0, 0, 0]
[ 0, 0, y12 , 0, 0, 0, 0]
[ 0, 0, y23 , 0, 0, 0, 0]
[ 0, 0, y14 , 0, 0, 0, 0]
[ 0, 0, y34 , 0, 0, 0, 0]
[ 0, y13 , 0, 0, 0, 0, 0]
[ 0, y24 , 0, 0, 0, 0, 0]
[ 0, 0, 0, 0, 0, 0, y12^2 ]
[ 0, 0, 0, 0, 0, 0, y12y13 ]
[ 0, 0, 0, 0, 0, 0, y12y23 ]
[ 0, 0, 0, 0, 0, 0, y12y14 ]
[ 0, 0, 0, 0, 0, 0, y12y24 ]
[ 0, 0, 0, 0, 0, 0, y12y34 ]
[ 0, 0, 0, 0, 0, 0, y13^2 ]
[ 0, 0, 0, 0, 0, 0, y13y14 ]
[ 0, 0, 0, 0, 0, 0, y13y24 ]
[ 0, 0, 0, 0, 0, 0, y13y34 ]
[ 0, 0, 0, 0, 0, 0, y23^2 ]
[ 0, 0, 0, 0, 0, 0, y23y14 ]
[ 0, 0, 0, 0, 0, 0, y23y24 ]
[ 0, 0, 0, 0, 0, 0, y23y34 ]
[ 0, 0, 0, 0, 0, 0, y14^2 ]
[ 0, 0, 0, 0, 0, 0, y24^2 ]
[ 0, 0, 0, 0, 0, 0, y34^2 ]
[ 0, 0, 0, 0, 0, y12^2 , 0]
[ 0, 0, 0, 0, 0, y12y13 , 0]
[ 0, 0, 0, 0, 0, y12y23 , 0]
[ 0, 0, 0, 0, 0, y12y14 , 0]
[ 0, 0, 0, 0, 0, y12y24 , 0]
[ 0, 0, 0, 0, 0, y13y24 , 0]
[ 0, 0, 0, 0, 0, y23y14 , 0]
[ 0, 0, 0, 0, 0, y14^2 , 0]
[ 0, 0, 0, 0, 0, y24^2 , 0]
[ 0, 0, 0, 0, y12y34 , 0, 0]
[ 0, 0, 0, 0, y13^2 , 0, 0]
[ 0, 0, 0, 0, y13y14 , 0, 0]
[ 0, 0, 0, 0, y13y34 , 0, 0]
[ 0, 0, 0, 0, y14^2 , 0, 0]
[ 0, 0, 0, 0, y34^2 , 0, 0]
[ 0, 0, 0, y23^2 , 0, 0, 0]
[ 0, 0, 0, y23y24 , 0, 0, 0]
[ 0, 0, 0, y23y34 , 0, 0, 0]
[ 0, 0, 0, y24^2 , 0, 0, 0]
[ 0, 0, 0, y34^2 , 0, 0, 0]
[ 0, 0, y12^2 , 0, 0, 0, 0]
[ 0, 0, y12y34 , 0, 0, 0, 0]
[ 0, 0, y23^2 , 0, 0, 0, 0]
[ 0, 0, y23y14 , 0, 0, 0, 0]
[ 0, y13^2 , 0, 0, 0, 0, 0]
[ 0, y13y24 , 0, 0, 0, 0, 0]
[ 0, 0, 0, 0, 0, 0, y12^3 ]
[ 0, 0, 0, 0, 0, 0, y12^2y13 ]
[ 0, 0, 0, 0, 0, 0, y12^2y23 ]
[ 0, 0, 0, 0, 0, 0, y12^2y14 ]
[ 0, 0, 0, 0, 0, 0, y12^2y24 ]
[ 0, 0, 0, 0, 0, 0, y12^2y34 ]
[ 0, 0, 0, 0, 0, 0, y12y13^2 ]
```

```
[ 0, 0, 0, 0, 0, 0, y12y13y14 ]
[ 0, 0, 0, 0, 0, 0, y12y13y24 ]
[ 0, 0, 0, 0, 0, 0, y12y13y34 ]
[ 0, 0, 0, 0, 0, 0, y12y23y14 ]
[ 0, 0, 0, 0, 0, 0, y12y23y24 ]
[ 0, 0, 0, 0, 0, 0, y12y23y34 ]
[ 0, 0, 0, 0, 0, 0, y12y14^2 ]
[ 0, 0, 0, 0, 0, 0, y12y34^2 ]
[ 0, 0, 0, 0, 0, 0, y13^3 ]
[ 0, 0, 0, 0, 0, 0, y13^2y24 ]
[ 0, 0, 0, 0, 0, 0, y13^2y34 ]
[ 0, 0, 0, 0, 0, 0, y13y14^2 ]
[ 0, 0, 0, 0, 0, 0, y13y24^2 ]
[ 0, 0, 0, 0, 0, 0, y23^3 ]
[ 0, 0, 0, 0, 0, 0, y23^2y14 ]
[ 0, 0, 0, 0, 0, 0, y23y14^2 ]
[ 0, 0, 0, 0, 0, 0, y23y24^2 ]
[ 0, 0, 0, 0, 0, 0, y14^3 ]
[ 0, 0, 0, 0, 0, 0, y24^3 ]
[ 0, 0, 0, 0, 0, 0, y34^3 ]
[ 0, 0, 0, 0, 0, y12^3 , 0]
[ 0, 0, 0, 0, 0, y12^2y14 , 0]
[ 0, 0, 0, 0, 0, y12^2y24 , 0]
[ 0, 0, 0, 0, 0, y12y14^2 , 0]
[ 0, 0, 0, 0, 0, y13y24^2 , 0]
[ 0, 0, 0, 0, 0, y23y14^2 , 0]
[ 0, 0, 0, 0, 0, y14^3 , 0]
[ 0, 0, 0, 0, 0, y24^3 , 0]
[ 0, 0, 0, 0, y12y34^2 , 0, 0]
[ 0, 0, 0, 0, y13^3 , 0, 0]
[ 0, 0, 0, 0, y13^2y14 , 0, 0]
[ 0, 0, 0, 0, y13^2y34 , 0, 0]
[ 0, 0, 0, 0, y13y14^2 , 0, 0]
[ 0, 0, 0, 0, y14^3 , 0, 0]
[ 0, 0, 0, 0, y34^3 , 0, 0]
[ 0, 0, 0, y23^3 , 0, 0, 0]
[ 0, 0, 0, y23^2y24 , 0, 0, 0]
[ 0, 0, 0, y23^2y34 , 0, 0, 0]
[ 0, 0, 0, y23y24^2 , 0, 0, 0]
[ 0, 0, 0, y24^3 , 0, 0, 0]
[ 0, 0, 0, y34^3 , 0, 0, 0]
[ 0, 0, y12^3 , 0, 0, 0, 0]
[ 0, 0, y12^2y34 , 0, 0, 0, 0]
[ 0, 0, y23^3 , 0, 0, 0, 0]
[ 0, 0, y23^2y14 , 0, 0, 0, 0]
[ 0, y13^3 , 0, 0, 0, 0, 0]
[ 0, y13^2y24 , 0, 0, 0, 0, 0]
```

A.6 Right action of $\mathrm{FK}(4)^!$ on $(M^2)^!$

We list below the right action of some elements of $A^!$ on $(M^2)^!$, where M^2 is the quadratic right A-module defined at the beginning of Subsection 6.1.2. In Tables A.6.1-A.6.4, the entry appearing in the row indexed by y and the column indexed by y' is the product yy'. To reduce space, the integer $m \in [\![1{,}24]\!]$, appearing in the third to fifth columns of Tables A.6.1-A.6.4 indicates the element b_m^{n+1}, where b_m^n is the m-th element in (6.1.22) for $n \geqslant 4$ and $m \in [\![1, 24]\!]$.

	y \ y'	$y_{1,2}$	$y_{1,3}$	$y_{2,3}$
1	$g_1y_{1,2}^{n-1}y_{1,3}$	-2	5	1
2	$g_1y_{1,2}^{n-1}y_{2,3}$	-1	-2	5
3	$g_1y_{1,2}^{n-1}y_{1,4}$	-4	-8	-9
4	$g_1y_{1,2}^{n-1}y_{2,4}$	-3	-7	-11
5	$g_1y_{1,2}^{n-2}y_{1,3}^2$	5	1	2
6	$g_1y_{1,2}^{n-2}y_{1,3}y_{1,4}$	10	-13	-6
7	$g_1y_{1,2}^{n-2}y_{1,3}y_{2,4}$	9	-4	-8
8	$g_1y_{1,2}^{n-2}y_{1,3}y_{3,4}$	11	-3	-7
9	$g_1y_{1,2}^{n-2}y_{2,3}y_{1,4}$	7	11	-3
10	$g_1y_{1,2}^{n-2}y_{2,3}y_{2,4}$	6	10	-13
11	$g_1y_{1,2}^{n-2}y_{2,3}y_{3,4}$	8	9	-4
12	$g_1y_{1,2}^{n-2}y_{1,4}^2$	12	14	15
13	$g_1y_{1,2}^{n-3}y_{1,3}^2y_{3,4}$	-13	-6	-10
14	$g_1y_{1,2}^{n-3}y_{1,3}y_{1,4}^2$	-15	12	14
15	$g_1y_{1,2}^{n-3}y_{2,3}y_{1,4}^2$	-14	-15	12
16	$g_2y_{1,2}^{n-1}y_{1,4}$	-17	-8	-9
17	$g_2y_{1,2}^{n-1}y_{2,4}$	-16	-7	-11
18	$g_2y_{1,2}^{n-2}y_{1,4}^2$	18	14	15
19	$g_3y_{1,3}^{n-1}y_{1,4}$	-10	-20	6
20	$g_3y_{1,3}^{n-1}y_{3,4}$	-11	-19	7
21	$g_3y_{1,3}^{n-2}y_{1,4}^2$	-12	21	-15
22	$g_4y_{2,3}^{n-1}y_{2,4}$	-6	-10	-23
23	$g_4y_{2,3}^{n-1}y_{3,4}$	-8	-9	-22
24	$g_4y_{2,3}^{n-2}y_{2,4}^2$	-12	-14	24
	$g_1y_{1,2}^n$	$g_1y_{1,2}^{n+1}$	1	2
	$g_1y_{1,2}^{n-r}y_{3,4}^r$	$(-1)^rg_1y_{1,2}^{n-r+1}y_{3,4}^r$	$\chi_r14 - \chi_{r+1}6$	$\chi_r15 - \chi_{r+1}10$
	$g_1y_{3,4}^n$	$g_1y_{1,2}y_{3,4}^n$	14	15
	$g_2y_{1,2}^n$	$g_2y_{1,2}^{n+1}$	14	15
	$g_3y_{1,2}y_{3,4}^{n-1}$	$g_1y_{3,4}^{n+1}$	6	10
	$g_3y_{3,4}^n$	$g_3y_{1,2}y_{3,4}^n$	21	-15
	$g_4y_{3,4}^n$	$g_3y_{1,2}y_{3,4}^n$	-14	24
	$g_5y_{1,2}^n$	$g_5y_{1,2}^{n+1}$	-4	-8
	$g_5y_{1,2}^{n-1}y_{3,4}$	$-g_5y_{1,2}^ny_{3,4}$	-11	3
	$g_1y_{1,3}^n$	5	$g_1y_{1,3}^{n+1}$	2
	$g_1y_{1,3}^{n-r}y_{2,4}^r$	$\chi_r12 + \chi_{r+1}9$	$(-1)^rg_1y_{1,3}^{n-r+1}y_{2,4}^r$	$\chi_r15 - \chi_{r+1}8$
	$g_1y_{2,4}^n$	12	$g_1y_{1,3}y_{2,4}^n$	15
	$g_2y_{1,3}y_{2,4}^{n-1}$	9	$-g_1y_{2,4}^{n+1}$	-8
	$g_2y_{2,4}^n$	18	$g_2y_{1,3}y_{2,4}^n$	15
	$g_3y_{1,3}^n$	-12	$g_3y_{1,3}^{n+1}$	-15
	$g_4y_{2,4}^n$	-12	$-g_2y_{1,3}y_{2,4}^n$	24
	$g_6y_{1,3}^n$	-7	$g_6y_{1,3}^{n+1}$	3
	$g_6y_{1,3}^{n-1}y_{2,4}$	-13	$-g_6y_{1,3}^ny_{2,4}$	-10
	$g_1y_{2,3}^n$	5	1	$g_1y_{2,3}^{n+1}$
	$g_1y_{2,3}^{n-r}y_{1,4}^r$	$\chi_r12 + \chi_{r+1}7$	$\chi_r14 + \chi_{r+1}11$	$(-1)^rg_1y_{2,3}^{n-r+1}y_{1,4}^r$
	$g_1y_{1,4}^n$	12	14	$g_1y_{2,3}y_{1,4}^n$
	$g_2y_{2,3}y_{1,4}^{n-1}$	7	11	$-g_1y_{1,4}^{n+1}$
	$g_2y_{1,4}^n$	18	14	$g_2y_{2,3}y_{1,4}^n$
	$g_3y_{1,4}^n$	-12	21	$-g_2y_{2,3}y_{1,4}^n$
	$g_4y_{2,3}^n$	-12	-14	$g_4y_{2,3}^{n+1}$
	$g_5y_{2,3}^n$	9	-4	$g_5y_{2,3}^{n+1}$
	$g_5y_{2,3}^{n-1}y_{1,4}$	-13	-6	$-g_5y_{2,3}^ny_{1,4}$

Table A.6.1: Products yy' for $n \geqslant 4$ even.

	y \ y'	$y_{1,4}$	$y_{2,4}$	$y_{3,4}$
1	$g_1 y_{1,2}^{n-1} y_{1,3}$	6	7	8
2	$g_1 y_{1,2}^{n-1} y_{2,3}$	9	10	11
3	$g_1 y_{1,2}^{n-1} y_{1,4}$	12	3	6
4	$g_1 y_{1,2}^{n-1} y_{2,4}$	-4	12	10
5	$g_1 y_{1,2}^{n-2} y_{1,3}^2$	3	4	13
6	$g_1 y_{1,2}^{n-2} y_{1,3} y_{1,4}$	14	-9	3
7	$g_1 y_{1,2}^{n-2} y_{1,3} y_{2,4}$	10	14	7
8	$g_1 y_{1,2}^{n-2} y_{1,3} y_{3,4}$	-13	-8	14
9	$g_1 y_{1,2}^{n-2} y_{2,3} y_{1,4}$	15	-6	-9
10	$g_1 y_{1,2}^{n-2} y_{2,3} y_{2,4}$	7	15	4
11	$g_1 y_{1,2}^{n-2} y_{2,3} y_{3,4}$	11	-13	15
12	$g_1 y_{1,2}^{n-2} y_{1,4}^2$	3	4	13
13	$g_1 y_{1,2}^{n-3} y_{1,3}^2 y_{3,4}$	-8	-11	12
14	$g_1 y_{1,2}^{n-3} y_{1,3} y_{1,4}^2$	6	7	8
15	$g_1 y_{1,2}^{n-3} y_{2,3} y_{1,4}^2$	9	10	11
16	$g_2 y_{1,2}^{n-1} y_{1,4}$	18	16	6
17	$g_2 y_{1,2}^{n-1} y_{2,4}$	-17	18	10
18	$g_2 y_{1,2}^{n-2} y_{1,4}^2$	16	17	13
19	$g_3 y_{1,3}^{n-1} y_{1,4}$	21	9	19
20	$g_3 y_{1,3}^{n-1} y_{3,4}$	-20	8	21
21	$g_3 y_{1,3}^{n-2} y_{1,4}^2$	19	-4	20
22	$g_4 y_{2,3}^{n-1} y_{2,4}$	-7	24	22
23	$g_4 y_{2,3}^{n-1} y_{3,4}$	-11	-23	24
24	$g_4 y_{2,3}^{n-2} y_{2,4}^2$	-3	22	23
	$g_1 y_{1,2}^n$	3	4	$g_1 y_{1,2}^n y_{3,4}$
	$g_1 y_{1,2}^{n-r} y_{3,4}^r$	$\chi_r 3 - \chi_{r+1} 8$	$\chi_r 4 - \chi_{r+1} 11$	$g_1 y_{1,2}^{n-r} y_{3,4}^{r+1}$
	$g_1 y_{3,4}^n$	3	4	$g_1 y_{3,4}^{n+1}$
	$g_2 y_{1,2}^n$	16	17	$g_1 y_{1,2}^n y_{3,4}$
	$g_3 y_{1,2} y_{3,4}^{n-1}$	8	11	$g_3 y_{1,2} y_{3,4}^n$
	$g_3 y_{3,4}^n$	19	-4	$g_3 y_{3,4}^{n+1}$
	$g_4 y_{3,4}^n$	-3	22	$g_4 y_{3,4}^{n+1}$
	$g_5 y_{1,2}^n$	10	14	$g_5 y_{1,2}^n y_{3,4}$
	$g_5 y_{1,2}^{n-1} y_{3,4}$	-15	6	$g_5 y_{1,2}^{n+1}$
	$g_1 y_{1,3}^n$	3	$g_1 y_{1,3}^n y_{2,4}$	13
	$g_1 y_{1,3}^{n-r} y_{2,4}^r$	$\chi_r 3 + \chi_{r+1} 10$	$g_1 y_{1,3}^{n-r} y_{2,4}^{r+1}$	$\chi_r 13 + \chi_{r+1} 7$
	$g_1 y_{2,4}^n$	3	$g_1 y_{2,4}^{n+1}$	13
	$g_2 y_{1,3} y_{2,4}^{n-1}$	10	$g_2 y_{1,3} y_{2,4}^n$	7
	$g_2 y_{2,4}^n$	16	$g_2 y_{2,4}^{n+1}$	13
	$g_3 y_{1,3}^n$	19	$-g_1 y_{1,3}^n y_{2,4}$	20
	$g_4 y_{2,4}^n$	-3	$g_4 y_{2,4}^{n+1}$	23
	$g_6 y_{1,3}^n$	-15	$g_6 y_{1,3}^n y_{2,4}$	9
	$g_6 y_{1,3}^{n-1} y_{2,4}$	-8	$g_6 y_{1,3}^{n+1}$	12
	$g_1 y_{2,3}^n$	$g_1 y_{2,3}^n y_{1,4}$	4	13
	$g_1 y_{2,3}^{n-r} y_{1,4}^r$	$g_1 y_{2,3}^{n-r} y_{1,4}^{r+1}$	$\chi_r 4 - \chi_{r+1} 6$	$\chi_r 13 - \chi_{r+1} 9$
	$g_1 y_{1,4}^n$	$g_1 y_{1,4}^{n+1}$	4	13
	$g_2 y_{2,3} y_{1,4}^{n-1}$	$g_2 y_{2,3} y_{1,4}^n$	-6	-9
	$g_2 y_{1,4}^n$	$g_2 y_{1,4}^{n+1}$	17	13
	$g_3 y_{1,4}^n$	$g_3 y_{1,4}^{n+1}$	-4	20
	$g_4 y_{2,3}^n$	$-g_1 y_{2,3}^n y_{1,4}$	22	23
	$g_5 y_{2,3}^n$	$g_5 y_{2,3}^n y_{1,4}$	14	7
	$g_5 y_{2,3}^{n-1} y_{1,4}$	$g_5 y_{2,3}^{n+1}$	-11	12

Table A.6.2: Products yy' for $n \geqslant 4$ even.

	$y \backslash y'$	$y_{1,2}$	$y_{1,3}$	$y_{2,3}$
1	$g_1 y_{1,2}^{n-1} y_{1,3}$	-2	5	1
2	$g_1 y_{1,2}^{n-1} y_{2,3}$	-1	-2	5
3	$g_1 y_{1,2}^{n-1} y_{1,4}$	-4	-8	-9
4	$g_1 y_{1,2}^{n-1} y_{2,4}$	-3	-7	-11
5	$g_1 y_{1,2}^{n-2} y_{1,3}^2$	5	1	2
6	$g_1 y_{1,2}^{n-2} y_{1,3} y_{1,4}$	10	-13	-6
7	$g_1 y_{1,2}^{n-2} y_{1,3} y_{2,4}$	9	-4	-8
8	$g_1 y_{1,2}^{n-2} y_{1,3} y_{3,4}$	11	-3	-7
9	$g_1 y_{1,2}^{n-2} y_{2,3} y_{1,4}$	7	11	-3
10	$g_1 y_{1,2}^{n-2} y_{2,3} y_{2,4}$	6	10	-13
11	$g_1 y_{1,2}^{n-2} y_{2,3} y_{3,4}$	8	9	-4
12	$g_1 y_{1,2}^{n-2} y_{1,4}^2$	12	14	15
13	$g_1 y_{1,2}^{n-3} y_{1,3}^2 y_{3,4}$	-13	-6	-10
14	$g_1 y_{1,2}^{n-3} y_{1,3} y_{1,4}^2$	-15	12	14
15	$g_1 y_{1,2}^{n-3} y_{2,3} y_{1,4}^2$	-14	-15	12
16	$g_2 y_{1,2}^{n-1} y_{1,4}$	-17	-8	-9
17	$g_2 y_{1,2}^{n-1} y_{2,4}$	-16	-7	-11
18	$g_2 y_{1,2}^{n-2} y_{1,4}^2$	18	14	15
19	$g_3 y_{1,3}^{n-1} y_{1,4}$	4	-20	9
20	$g_3 y_{1,3}^{n-1} y_{3,4}$	13	-19	10
21	$g_3 y_{1,3}^{n-2} y_{1,4}^2$	15	21	-14
22	$g_4 y_{2,3}^{n-1} y_{2,4}$	3	7	-23
23	$g_4 y_{2,3}^{n-1} y_{3,4}$	13	6	-22
24	$g_4 y_{2,3}^{n-2} y_{2,4}^2$	14	15	24
	$g_1 y_{1,2}^n$	$g_1 y_{1,2}^{n+1}$	1	2
	$g_1 y_{1,2}^{n-r} y_{3,4}^r$	$(-1)^r g_1 y_{1,2}^{n-r+1} y_{3,4}^r$	$\chi_r 14 - \chi_{r+1} 6$	$\chi_r 15 - \chi_{r+1} 10$
	$g_1 y_{3,4}^n$	$-g_1 y_{1,2} y_{3,4}^n$	-6	-10
	$g_2 y_{1,2}^n$	$g_2 y_{1,2}^{n+1}$	14	15
	$g_3 y_{1,2} y_{3,4}^{n-1}$	$-g_1 y_{3,4}^{n+1}$	-14	-15
	$g_3 y_{3,4}^n$	$-g_3 y_{1,2} y_{3,4}^n$	-19	10
	$g_4 y_{3,4}^n$	$-g_3 y_{1,2} y_{3,4}^n$	6	-22
	$g_5 y_{1,2}^n$	$g_5 y_{1,2}^{n+1}$	11	-3
	$g_5 y_{1,2}^{n-1} y_{3,4}$	$-g_5 y_{1,2}^n y_{3,4}$	-4	-8
	$g_1 y_{1,3}^n$	-2	$g_1 y_{1,3}^{n+1}$	1
	$g_1 y_{1,3}^{n-r} y_{2,4}^r$	$-\chi_r 15 - \chi_{r+1} 3$	$(-1)^r g_1 y_{1,3}^{n-r+1} y_{2,4}^r$	$\chi_r 14 - \chi_{r+1} 11$
	$g_1 y_{2,4}^n$	-3	$-g_1 y_{1,3} y_{2,4}^n$	-11
	$g_2 y_{1,3} y_{2,4}^{n-1}$	-15	$g_1 y_{2,4}^{n+1}$	14
	$g_2 y_{2,4}^n$	-16	$-g_2 y_{1,3} y_{2,4}^n$	-11
	$g_3 y_{1,3}^n$	15	$g_3 y_{1,3}^{n+1}$	-14
	$g_4 y_{2,4}^n$	3	$g_2 y_{1,3} y_{2,4}^n$	-23
	$g_6 y_{1,3}^n$	-8	$g_6 y_{1,3}^{n+1}$	4
	$g_6 y_{1,3}^{n-1} y_{2,4}$	10	$-g_6 y_{1,3}^n y_{2,4}$	-6
	$g_1 y_{2,3}^n$	-1	-2	$g_1 y_{2,3}^{n+1}$
	$g_1 y_{2,3}^{n-r} y_{1,4}^r$	$-\chi_r 14 - \chi_{r+1} 4$	$-\chi_r 15 - \chi_{r+1} 8$	$(-1)^r g_1 y_{2,3}^{n-r+1} y_{1,4}^r$
	$g_1 y_{1,4}^n$	-4	-8	$-g_1 y_{2,3} y_{1,4}^n$
	$g_2 y_{2,3} y_{1,4}^{n-1}$	-14	-15	$g_1 y_{1,4}^{n+1}$
	$g_2 y_{1,4}^n$	-17	-8	$-g_2 y_{2,3} y_{1,4}^n$
	$g_3 y_{1,4}^n$	4	-20	$g_2 y_{2,3} y_{1,4}^n$
	$g_4 y_{2,3}^n$	14	15	$g_4 y_{2,3}^{n+1}$
	$g_5 y_{2,3}^n$	-11	3	$g_5 y_{2,3}^{n+1}$
	$g_5 y_{2,3}^{n-1} y_{1,4}$	6	10	$-g_5 y_{2,3}^n y_{1,4}$

Table A.6.3: Products yy' for $n \geqslant 5$ odd.

	$y \backslash y'$	$y_{1,4}$	$y_{2,4}$	$y_{3,4}$
1	$g_1y_{1,2}^{n-1}y_{1,3}$	6	7	8
2	$g_1y_{1,2}^{n-1}y_{2,3}$	9	10	11
3	$g_1y_{1,2}^{n-1}y_{1,4}$	12	3	6
4	$g_1y_{1,2}^{n-1}y_{2,4}$	-4	12	10
5	$g_1y_{1,2}^{n-2}y_{1,3}^2$	3	4	13
6	$g_1y_{1,2}^{n-2}y_{1,3}y_{1,4}$	14	-9	3
7	$g_1y_{1,2}^{n-2}y_{1,3}y_{2,4}$	10	14	7
8	$g_1y_{1,2}^{n-2}y_{1,3}y_{3,4}$	-13	-8	14
9	$g_1y_{1,2}^{n-2}y_{2,3}y_{1,4}$	15	-6	-9
10	$g_1y_{1,2}^{n-2}y_{2,3}y_{2,4}$	7	15	4
11	$g_1y_{1,2}^{n-2}y_{2,3}y_{3,4}$	11	-13	15
12	$g_1y_{1,2}^{n-2}y_{1,4}^2$	3	4	13
13	$g_1y_{1,2}^{n-3}y_{1,3}^2y_{3,4}$	-8	-11	12
14	$g_1y_{1,2}^{n-3}y_{1,3}y_{1,4}^2$	6	7	8
15	$g_1y_{1,2}^{n-3}y_{2,3}y_{1,4}^2$	9	10	11
16	$g_2y_{1,2}^{n-1}y_{1,4}$	18	16	6
17	$g_2y_{1,2}^{n-1}y_{2,4}$	-17	18	10
18	$g_2y_{1,2}^{n-2}y_{1,4}^2$	16	17	13
19	$g_3y_{1,3}^{n-1}y_{1,4}$	21	-3	19
20	$g_3y_{1,3}^{n-1}y_{3,4}$	-20	11	21
21	$g_3y_{1,3}^{n-2}y_{1,4}^2$	19	-7	20
22	$g_4y_{2,3}^{n-1}y_{2,4}$	4	24	22
23	$g_4y_{2,3}^{n-1}y_{3,4}$	8	-23	24
24	$g_4y_{2,3}^{n-2}y_{2,4}^2$	-9	22	23
	$g_1y_{1,2}^n$	3	4	$g_1y_{1,2}^ny_{3,4}$
	$g_1y_{1,2}^{n-r}y_{3,4}^r$	$\chi_r3-\chi_{r+1}8$	$\chi_r4-\chi_{r+1}11$	$g_1y_{1,2}^{n-r}y_{3,4}^{r+1}$
	$g_1y_{3,4}^n$	-8	-11	$g_1y_{3,4}^{n+1}$
	$g_2y_{1,2}^n$	16	17	$g_1y_{1,2}^ny_{3,4}$
	$g_3y_{1,2}y_{3,4}^{n-1}$	-3	-4	$g_3y_{1,2}y_{3,4}^n$
	$g_3y_{3,4}^n$	-20	11	$g_3y_{3,4}^{n+1}$
	$g_4y_{3,4}^n$	8	-23	$g_4y_{3,4}^{n+1}$
	$g_5y_{1,2}^n$	15	-6	$g_5y_{1,2}^ny_{3,4}$
	$g_5y_{1,2}^{n-1}y_{3,4}$	10	14	$g_5y_{1,2}^{n+1}$
	$g_1y_{1,3}^n$	6	$g_1y_{1,3}^ny_{2,4}$	8
	$g_1y_{1,3}^{n-r}y_{2,4}^r$	$\chi_r6-\chi_{r+1}4$	$g_1y_{1,3}^{n-r}y_{2,4}^{r+1}$	$\chi_r8+\chi_{r+1}10$
	$g_1y_{2,4}^n$	-4	$g_1y_{2,4}^{n+1}$	10
	$g_2y_{1,3}y_{2,4}^{n-1}$	6	$g_2y_{1,3}y_{2,4}^n$	8
	$g_2y_{2,4}^n$	-17	$g_2y_{2,4}^{n+1}$	10
	$g_3y_{1,3}^n$	19	$-g_1y_{1,3}^ny_{2,4}$	20
	$g_4y_{2,4}^n$	4	$g_4y_{2,4}^{n+1}$	22
	$g_6y_{1,3}^n$	-11	$g_6y_{1,3}^ny_{2,4}$	-15
	$g_6y_{1,3}^{n-1}y_{2,4}$	14	$g_6y_{1,3}^{n+1}$	3
	$g_1y_{2,3}^n$	$g_1y_{2,3}^ny_{1,4}$	10	11
	$g_1y_{2,3}^{n-r}y_{1,4}^r$	$g_1y_{2,3}^{n-r}y_{1,4}^{r+1}$	$\chi_r10+\chi_{r+1}3$	$\chi_r11+\chi_{r+1}6$
	$g_1y_{1,4}^n$	$g_1y_{1,4}^{n+1}$	3	6
	$g_2y_{2,3}y_{1,4}^{n-1}$	$g_2y_{2,3}y_{1,4}^n$	10	11
	$g_2y_{1,4}^n$	$g_2y_{1,4}^{n+1}$	16	6
	$g_3y_{1,4}^n$	$g_3y_{1,4}^{n+1}$	-3	19
	$g_4y_{2,3}^n$	$-g_1y_{2,3}^ny_{1,4}$	22	23
	$g_5y_{2,3}^n$	$g_5y_{2,3}^ny_{1,4}$	8	-14
	$g_5y_{2,3}^{n-1}y_{1,4}$	$g_5y_{2,3}^{n+1}$	15	4

Table A.6.4: Products yy' for $n \geqslant 5$ odd.

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