Exercise sheet n° 7

Eigenvalues and eigenvectors

1. Compute the eigenvalues and the eigenvectors of the following matrices.

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \quad C = \begin{pmatrix} 4 & 1 & 2 \\ -1 & 1 & -1 \\ -2 & -1 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 2 & -2 \\ 1 & 0 & 0 \end{pmatrix}.$$

Which of the previous matrices is diagonalizable ?

2. Let

$$A = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{pmatrix}.$$

(i) Show that A is diagonalizable and find a matrix P diagonalizing A. Compute A^n for all $n \geq 1$.

(ii) Consider the sequences (u_n), (v_n) and (w_n) defined by the initial values $u_0 = v_0 = 1$, $w_0 = 2$ and the following recursive relations :

$$u_{n+1} = 3u_n - v_n + w_n \qquad v_{n+1} = 2v_n \qquad w_{n+1} = u_n - v_n + 3w_n.$$

Compute u_n, v_n and w_n.

3. Let $a, b \in \mathbb{R}$ and

$$A = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}.$$

Compute A^n for all $n \geq 1$.

4. Let $A \in \mathcal{M}_n(\mathbf{C})$ be a nilpotent matrix, *i.e.* there exists $p \in \mathbf{N}^*$ such that $A^p = 0$. Show that the only eigenvalue of A is 0. Does the converse hold ?

5. Let $A \in \mathcal{M}_n(\mathbf{C})$. Show that the determinant of A is equal to the product of its eigenvalues (counted with multiplicity) and that the trace of A is equal to the sum of its eigenvalues (also counted with multiplicity).

6. Let $A, B \in \mathcal{M}_n(\mathbf{C})$. Show that AB and BA have the same set of eigenvalues, each of them with the same multiplicity.

7. The Italian mathematician Leonardo Fibonacci (c. 1175 – c. 1250) was the first to study the sequence of integers given by 1, 1, 2, 3, 5, 8,. . . and that has his name. It is recursively defined by

$$f_0 = 1,\ f_1 = 1,\ f_{k+1} = f_k + f_{k-1},\ k = 1, 2, ...$$

1. Let $x^{(k)} = (f_{k+1}; f_k)$. Write this relations in matricial form
$$x^{(k+1)} = Ax^{(k)},\ k = 0, 1, ... ,\quad x^{(0)} = (1; 1),$$
, where A is a matrix to be determiend .
2. Compute the eigenvalues and the eigenvectors of A. Is the matrix LA diagonalizable ?
3. Find an explicit formula for the k-th element of the Fibonacci sequence.

8. Compute the eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 4 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Consider now the system of differential equations given by

$$\dot{x}(t) = x(t) + 4z(t), \qquad \dot{y}(t) = y(t) + 4w(t),$$
$$\dot{z}(t) = x(t) + z(t), \qquad \dot{w}(t) = y(t) + w(t).$$

Compute the general solution of the system, and then the particular solution satisfying the conditions $x(0) = y(0) = z(0) = 0, w(0) = 2$.

9. Let $a \in I\!R$. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & a \\ 0 & -1 & 2 & 0 \\ 1 & 0 & 1 & 2 \end{pmatrix}.$$

(i) Compute the characteristic polynomial of A and the corresponding eigenvalues.

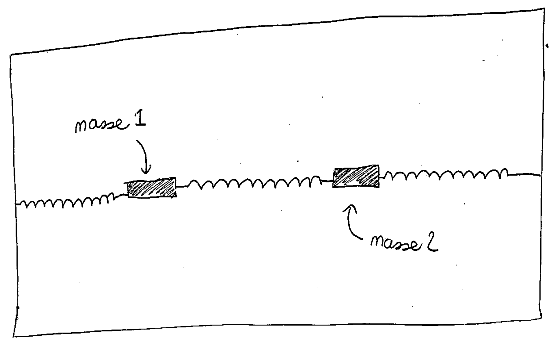
(ii) Compute the eigenspaces of A associated to each eigenvalue. Determine all the values of the parameter a such that the matrix A is diagonalizable. And triangularizable ?

(iii) Assume that $a = 0$ and consider the system of differential equations given by

$$\dot{x}(t) = x(t) + y(t) - z(t), \qquad \dot{y}(t) = y(t),$$
$$\dot{z}(t) = -y(t) + 2z(t), \qquad \dot{w}(t) = x(t) + z(t) + 2w(t).$$

Compute the general solution of the system, and then the particular solution satisfying the conditions $x(0) = y(0) = w(0) = 0, z(0) = 1$.

10. Consider the system formed by two masses and three springs with the same force constant $k > 0$ as indicated in the following diagram :



Assume that the two are longitudinally separated from the equilibrium position (using an external force) at time $t = 0$ and the systems stars moving freely. Let $x_1(t)$ (resp. $x_2(t)$) the (signed) longitudinal distance of the first (resp. second) mass from its equilibrium position at t. We assume that the two springs obey Hooke's law, so

$$\begin{aligned} mx_1'' &= -2kx_1 + kx_2, \\ mx_2'' &= kx_1 - 2kx_2. \end{aligned}$$

1. Rewrite these equations in matrix form :

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}'' = M \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

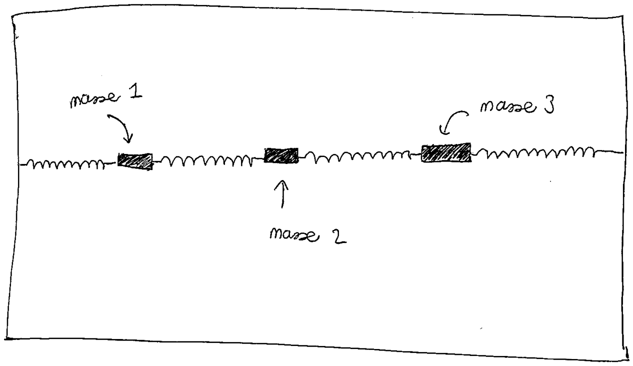
   where M s a matrix to be determined.
2. Diagonalize M and compute the general solution of this equation.
3. What are normal modes of oscillation of the system ?

We consider now the analogous situation with 3 masses and 4 springs.

1. Write down the equations of motion of this system.
2. Rewrite them in matrix form :

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}'' = M \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

   where M s a matrix to be determined.
3. Diagonalize M and compute the general solution of this equation.
4. What is the normal mode of oscillation of the system corresponding to the eigenvalue $2k/m$.



Eigenvalues of symmetric matrices and singular value decomposition

11. Show that the following matrices are orthogonal and compute their eigenvalues.

$$(a)\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad (b)\ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad (c)\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

12. Show that the following matrices are symmetric and diagonalize them by means of an orhtogonal matrix.

$$(a)\begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix} \quad (b)\begin{bmatrix} 2 & 36 \\ 36 & 23 \end{bmatrix} \quad (c)\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (d)\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

13. Compute the singular value decomposition of the following matrices.

$$(a)\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad (b)\begin{bmatrix} -2 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \quad (c)\begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \end{bmatrix} \quad (d)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

14. Let $A = U\Sigma V^t$ be the singular value decomposition of a matrix $A \in I\!R^{m\times n}$ of rank $p < \min(m, n)$. We denote by $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_p$ the nonzero singular values of A, and by $\bar{U} = [u_1, ..., u_p]$ and $\bar{V} = [v_1, ..., v_p]$ the left and right associated orthogonal matrices, respectively. We have thus

$$A = \bar{U}\,\mathrm{diag}(\sigma_1, ..., \sigma_p)\,\bar{V}^t$$

(minimal decomposition).

Let A^+ be the matrix of size $n \times m$ given by

$$A^+ = \bar{V}\,\mathrm{diag}(\sigma_1^{-1}, ..., \sigma_p^{-1})\,\bar{U}^t.$$

It is called the *pseudo-inverse matrix of A*.

1. What is the size of A^+? Express A^+A and AA^+ in terms of the singular value decomposition of A. Verify that $A\,A^+A = A$ and $A^+A\,A^t = A^t\,AA^+ = A^t$. Explain their meaning.

2. Let A be a matrix of size $m \times n$ and rank n, and let b be a vector column of size m. Assume that the system $Ax = b$ has a solution x. Show that the solution $x \in \mathbb{R}^n$ satisfies $x = A^+b$.