MAT332 - Series and integration
Fall term — 2022-2023

Exercise sheet 1: Sequences, comparison of sequences and Taylor polynomials

1. *Existence and computations of limits.* Decide whether the following sequences $(u_n)_{n\in\mathbb{N}_0}$ given by

(a) $u_n = 1/(2n+1)$,
(b) $u_n = (n+2)/(2n+3)$,
(c) $u_n = n^2/(n+1)$,
(d) $u_n = (10n^2+1)/(n^3-1)$,
(e) $u_n = 1/(\sqrt{n+1}-\sqrt{n})$,
(f) $u_n = (n+1)^2/((n+1)^3-n^3)$,
(g) $u_n = n^{10}/1.01^n$,

converge or diverge. In the first case, compute the limit.

Solution.

(a) It is easy to see that

$$\lim_{n\to+\infty} \frac{1}{2n+1} = 0,$$

since, given $\epsilon > 0$, we pick $n_0 = \lfloor 1/\epsilon \rfloor + 1$, so

$$\left|\frac{1}{2n+1}\right| = \frac{1}{2n+1} \leq \frac{1}{n} \leq \epsilon,$$

for all integers $n \geq n_0$.

(b) We see that

$$\lim_{n\to+\infty} \frac{n+2}{2n+3} = \lim_{n\to+\infty} \frac{1+2/n}{2+3/n} = \frac{1}{2},$$

since c/n converges to zero as n goes to $+\infty$, for $c \in \mathbb{R}$.

(c) We have that

$$\lim_{n\to+\infty} \frac{n^2}{n+1} = \lim_{n\to+\infty} \frac{n}{1+1/n} = +\infty,$$

since c/n converges to zero as n goes to $+\infty$, for $c \in \mathbb{R}$.

(d) We see that

$$\lim_{n\to+\infty} \frac{10n^2+1}{n^3-1} = \lim_{n\to+\infty} \frac{10+1/n^2}{n(1-1/n^3)} = 0,$$

since c/n^k converges to zero as n goes to $+\infty$, for $c \in \mathbb{R}$ and $k \in \mathbb{N}$.

(e) We have that

$$\lim_{n\to+\infty} \frac{1}{\sqrt{n+1}-\sqrt{n}} = \lim_{n\to+\infty} \frac{\sqrt{n+1}+\sqrt{n}}{(n+1)-n} = \lim_{n\to+\infty} \sqrt{n+1}+\sqrt{n} = +\infty,$$

since $\sqrt{n+c}$ converges to zero as n goes to $+\infty$, for $c \in \mathbb{R}_{>0}$.

(*f*) We see that

$$\lim_{n\to+\infty}\frac{(n+1)^2}{(n+1)^3-n^3}=\lim_{n\to+\infty}\frac{n^2+2n+1}{3n^2+3n+1}=\lim_{n\to+\infty}\frac{1+2/n+1/n^2}{3+3/n+1/n^2}=\frac{1}{3},$$

since c/n^k converges to zero as n goes to $+\infty$, for $c\in\mathbb{R}$ and $k\in\mathbb{N}$.

(*g*) We have that

$$\lim_{n\to+\infty}\frac{n^{10}}{1.01^n}=\lim_{n\to+\infty}\frac{e^{10\ln(n)}}{e^{n\ln(1.01)}}=\lim_{n\to+\infty}e^{n\left(10\frac{\ln(n)}{n}-\ln(1.01)\right)}=0,$$

since $\ln(n)/n$ converges to zero as n goes to $+\infty$, $\ln(1.01)>0$ and e^y goes to zero as y goes to $-\infty$.

2. *Equivalence, domination and negligibility.* For each of the following pair of sequences $(u_n)_{n\in\mathbb{N}_0}$ and $(v_n)_{n\in\mathbb{N}_0}$, verify whether $u_n\sim v_n$, $u_n=\mathrm{O}(v_n)$, $u_n=\mathrm{o}(v_n)$, $v_n=\mathrm{O}(u_n)$, and/or $v_n=\mathrm{o}(u_n)$ when n tends to $+\infty$ hold/s :

(*a*) $u_n=2^{-n}$, $v_n=3^{-n}$;
(*b*) $u_n=1/n$, $v_n=1/\sqrt{n}$;
(*c*) $u_n=n^2$, $v_n=2^n$;
(*d*) $u_n=\cos(1/n)$, $v_n=e^{1/n}$;
(*e*) $u_n=\cos(n)$, $v_n=1$;
(*f*) $u_n=\ln(n)$, $v_n=\sqrt{n}$;
(*g*) $u_n=\sin(1/n)$, $v_n=1/n$.

Solution.

(*a*) Since

$$\lim_{n\to+\infty}\frac{v_n}{u_n}=\lim_{n\to+\infty}\frac{2^n}{3^n}=\lim_{n\to+\infty}\left(\frac{2}{3}\right)^n=0,$$

as c^n goes to zero when n tends to $+\infty$ provided for $c\in\,]\,0,1\,[$, we see that $v_n=\mathrm{o}(u_n)$ when n tends to $+\infty$, and in particular $v_n=\mathrm{O}(u_n)$ when n tends to $+\infty$. The other relations are not verified.

(*b*) Since

$$\lim_{n\to+\infty}\frac{u_n}{v_n}=\lim_{n\to+\infty}\frac{\sqrt{n}}{n}=\lim_{n\to+\infty}\frac{1}{\sqrt{n}}=0,$$

we see that $u_n=\mathrm{o}(v_n)$ when n tends to $+\infty$, and in particular $u_n=\mathrm{O}(v_n)$ when n tends to $+\infty$. The other relations are not verified.

(*c*) Since

$$\lim_{n\to+\infty}\frac{u_n}{v_n}=\lim_{n\to+\infty}\frac{n^2}{2^n}=\lim_{n\to+\infty}\frac{e^{2\ln(n)}}{e^{n\ln(2)}}=\lim_{n\to+\infty}e^{n\left(2\frac{\ln(n)}{n}-\ln(2)\right)}=0,$$

where we used that $\ln(n)/n$ converges to zero as n goes to $+\infty$, $\ln(2)>0$ and e^y goes to zero as y goes to $-\infty$, we see that $u_n=\mathrm{o}(v_n)$ when n tends to $+\infty$, and in particular $u_n=\mathrm{O}(v_n)$ when n tends to $+\infty$. The other relations are not verified.

(*d*) Note first that $u_n=\cos(1/n)\neq 0$ for all $n\in\mathbb{N}$. Since

$$\lim_{n\to+\infty}\frac{u_n}{v_n}=\lim_{n\to+\infty}\frac{\cos(1/n)}{e^{1/n}}=1$$

we see that $u_n \sim v_n$ when n tends to $+\infty$. In consequence, $u_n = \mathrm{O}(v_n)$ when n tends to $+\infty$ and $v_n = \mathrm{O}(u_n)$ when n tends to $+\infty$. The other relations are not verified.

(e) Note first that $u_n = \cos(n) \neq 0$ for all $n \in \mathbb{N}$. Since

$$\left|\frac{u_n}{v_n}\right| = |\cos(n)| \leq 1,$$

for all $n \in \mathbb{N}$, we see that $u_n = \mathrm{O}(v_n)$ when n tends to $+\infty$. Note however that the limit

$$\lim_{n\to+\infty} \frac{u_n}{v_n} = \lim_{n\to+\infty} \cos(n)$$

does not exist, so it is not true that $u_n = \mathrm{o}(v_n)$ when n tends to $+\infty$. The relation $u_n \sim v_n$ when n tends to $+\infty$ is not verified either by the same reason. Moreover, since $\mathbb{Z} + 2\pi\mathbb{Z}$ is dense in $\mathbb{R}$ and cos is continuous, $\{\cos(n) : n \in \mathbb{N}\}$ is dense in $[-1, 1]$, so

$$\left|\frac{v_n}{u_n}\right| = \left|\frac{1}{\cos(n)}\right|$$

is not bounded for $n \in \mathbb{N}$. As a consequence, the other relations are not verified either.

(f) Using that

$$\lim_{x\to 0} \frac{\ln^p(x)}{x^q} = 0$$

for all $p, q > 0$, we see that

$$\lim_{n\to+\infty} \frac{u_n}{v_n} = \lim_{n\to+\infty} \frac{\ln(n)}{\sqrt{n}} = 0,$$

Hence, $u_n = \mathrm{o}(u_n)$ when n tends to $+\infty$, and in particular $u_n = \mathrm{O}(v_n)$ when n tends to $+\infty$. The other relations are not verified.

(g) Using that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \sin'(0) = \cos(0) = 1,$$

we see that

$$\lim_{n\to+\infty} \frac{u_n}{v_n} = \lim_{n\to+\infty} \frac{\sin(1/n)}{1/n} = 1,$$

as $1/n$ goes to zero when n tends to $+\infty$. In consequence, $u_n \sim v_n$ when n tends to $+\infty$. In particular, $u_n = \mathrm{O}(v_n)$ when n tends to $+\infty$ and $v_n = \mathrm{O}(u_n)$ when n tends to $+\infty$. The other relations are not verified.

3. *A few examples.* Give examples of the following situations :

(a) an increasing positive sequence not converging to 0 ;

(b) a bounded sequence which is not convergent ;

(c) a positive sequence which is not bounded and not tending to ∞ ;

(d) a non monotone sequence not converging to 0 ;

(e) two divergent sequences $(u_n)_{n\in\mathbb{N}_0}$ and $(v_n)_{n\in\mathbb{N}_0}$ such that the product sequence $(u_n v_n)_{n\in\mathbb{N}_0}$ is convergent.

Solution.

(a) The sequence $(u_n)_{n\in\mathbb{N}}$ given by $u_n = n$ for $n \in \mathbb{N}$ is positive, since $u_n = n > 0$ for $n \in \mathbb{N}$, and tends to $+\infty$.

(b) The sequence $(u_n)_{n\in\mathbb{N}}$ given by $u_n = (-1)^n$ for $n \in \mathbb{N}$ is bounded, since $|u_n| = |(-1)^n| = 1 \leq 1$ for all $n \in \mathbb{N}$, and it has no limit. Indeed, we note that the subsequences $(u_{2n})_{n\in\mathbb{N}}$ and $(u_{2n+1})_{n\in\mathbb{N}}$ are convergent, with limits 1 and -1 respectively. As a consequence, $(u_n)_{n\in\mathbb{N}}$ is not convergent, since every subsequence of a convergent sequence is also convergent with the same limit.

(c) The sequence $(u_n)_{n\in\mathbb{N}_0}$ given by $u_{2n} = n+1$ and $u_{2n+1} = 1$ for $n \in \mathbb{N}_0$ is positive, is not bounded, as the subsequence $(u_{2n})_{n\in\mathbb{N}}$ tends to $+\infty$ when n tends to $+\infty$, but it is not convergent, since the subsequence $(u_{2n+1})_{n\in\mathbb{N}}$ converges to 1 when n tends to $+\infty$, whereas the subsequence $(u_{2n})_{n\in\mathbb{N}}$ tends to $+\infty$ when n tends to $+\infty$.

(d) The sequence $(u_n)_{n\in\mathbb{N}}$ given by $u_n = (-1)^n$ for $n \in \mathbb{N}$ does not converge to 0, as we saw in the second item, and it is not monotone either, since $u_1 = -1 < 1 = u_2$ but $u_2 = 1 > -1 = u_3$.

(e) The sequences $(u_n)_{n\in\mathbb{N}_0}$ (resp., $(v_n)_{n\in\mathbb{N}_0}$) given by $u_{2n} = n$ and $u_{2n+1} = 0$ (resp., $v_{2n+1} = n$ and $v_{2n} = 0$) for $n \in \mathbb{N}_0$ is not convergent, since the subsequences $(u_{2n})_{n\in\mathbb{N}}$ (resp., $(v_{2n+1})_{n\in\mathbb{N}}$) and $(u_{2n+1})_{n\in\mathbb{N}}$ (resp., $(v_{2n})_{n\in\mathbb{N}}$) tend to $= \infty$ and 0, respectively. On the other hand, the sequence $(u_n v_n)_{n\in\mathbb{N}_0}$ satisfies that $u_n v_n = 0$ for all $n \in \mathbb{N}_0$, so it converges to 0 as n goes to $+\infty$.

4. *Limit of a product of sequences.* Let $(u_n)_{n\in\mathbb{N}_0}$ and $(v_n)_{n\in\mathbb{N}_0}$ be complex sequences. Assume that $(u_n)_{n\in\mathbb{N}_0}$ and $(v_n)_{n\in\mathbb{N}_0}$ are convergent. Prove that the product sequence $(u_n v_n)_{n\in\mathbb{N}_0}$ also converges and moreover

$$\lim_{n\to\infty} u_n v_n = \Big(\lim_{n\to\infty} u_n\Big).\Big(\lim_{n\to\infty} v_n\Big).$$

Solution. By assumption, there exist real numbers ℓ_1 and ℓ_2 such that, given $\epsilon > 0$, there exist positive integers $n_1 = n_1(\epsilon)$ and $n_2 = n_2(\epsilon)$ such that

$$\big|u_n - \ell_1\big| \leq \epsilon \text{ and } \big|v_n - \ell_2\big| \leq \epsilon$$

for all $n \geq n_1$ and $n \geq n_2$, respectively. Let

$$n_0 = \max\Big(n_1\Big(\frac{\epsilon}{2(|\ell_2|+1)}\Big), n_1(1), n_2\Big(\frac{\epsilon}{2(|\ell_1|+1)}\Big)\Big).$$

Then, using the reversed triangle inequality given by $|a| - |b| \leq |a-b|$, for $a, b \in \mathbb{R}$, we get that $|u_n - \ell_1| \leq 1$ for $n \geq n_1(1)$, which implies that $|u_n| - |\ell_1| \leq 1$ for $n \geq n_1(1)$, *i.e.* $|u_n| \leq |\ell_1| + 1$ for $n \geq n_1(1)$. Now, we see that

$$\begin{aligned}\big|u_n v_n - \ell_1\ell_2\big| &= \big|u_n v_n - u_n\ell_2 + u_n\ell_2 - \ell_1\ell_2\big| \leq \big|u_n v_n - u_n\ell_2\big| + \big|u_n\ell_2 - \ell_1\ell_2\big| \\ &= \big|u_n\big|\,\big|v_n - \ell_2\big| + \big|u_n - \ell_1\big|\,\big|\ell_2\big|\end{aligned}$$

where we used the triangle inequality given by $|a+b| \leq |a+b|$, for $a, b \in \mathbb{R}$. If $n \geq n_0$, then $|u_n - \ell_1| \leq \epsilon/(2(|\ell_2|+1))$, $|v_n - \ell_2| \leq \epsilon/(2(|\ell_1|+1))$ and $|u_n| \leq |\ell_1| + 1$, which implies that

$$\big|u_n\big|\,\big|v_n - \ell_2\big| + \big|u_n - \ell_1\big|\,\big|\ell_2\big| \leq (|\ell_1|+1)\frac{\epsilon}{2(|\ell_1|+1)} + \frac{\epsilon}{2(|\ell_2|+1)}\big|\ell_2\big| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which in turn implies that $|u_n v_n - \ell_1\ell_2| \leq \epsilon$ for all $n \geq n_0$. This proves the claim of the exercise.

5. *Subsequences.* Let $(u_n)_{n\in\mathbb{N}_0}$ be a sequence of complex numbers.

(a) Show that if $(u_{2n})_{n\in\mathbb{N}_0}$ and $(u_{2n+1})_{n\in\mathbb{N}_0}$ both converge to the same limit, then $(u_n)_{n\in\mathbb{N}_0}$ also converges.

(b) Show that if the sequences $(u_{2n})_{n\in\mathbb{N}_0}$, $(u_{2n+1})_{n\in\mathbb{N}_0}$ and $(u_{3n})_{n\in\mathbb{N}_0}$ are convergent, then $(u_n)_{n\in\mathbb{N}_0}$ also converges.

Solution.

(a) Since $(u_{2n})_{n\in\mathbb{N}_0}$ and $(u_{2n+1})_{n\in\mathbb{N}_0}$ converge to the same limit $\ell \in \mathbb{R}$, then, given $\epsilon > 0$, there exits nonnegative integers $n_1 = n_1(\epsilon)$ and $n_2 = n_2(\epsilon)$, such that

$$|u_{2n} - \ell| \leq \epsilon \text{ and } |u_{2n+1} - \ell| \leq \epsilon$$

for all $n \geq n_1$ and $n \geq n_2$, respectively. Given ϵ, let $N_0 = \max(2n_1(\epsilon), 2n_2(\epsilon)+1)$. We will prove that $|u_N - \ell| \leq \epsilon$ for all $N \geq N_0$. If $N \geq N_0$ is even, we can write $N = 2n$, for $n \in \mathbb{N}_0$. Since $N \geq N_0 \geq 2n_1(\epsilon)$, we conclude that $|u_N - \ell| = |u_{2n} - \ell| \leq \epsilon$. Analogously, if $N \geq N_0$ is odd, we can write $N = 2n+1$, for $n \in \mathbb{N}_0$. Since $N \geq N_0 \geq 2n_2(\epsilon)+1$, we conclude that $|u_N - \ell| = |u_{2n+1} - \ell| \leq \epsilon$. Hence, $(u_N)_{N\in\mathbb{N}_0}$ also converges to ℓ.

(b) Let ℓ_1 (resp., ℓ_2, ℓ_3) be the limit of the sequence $(u_{2n})_{n\in\mathbb{N}_0}$ (resp., $(u_{2n+1})_{n\in\mathbb{N}_0}, (u_{3n})_{n\in\mathbb{N}_0}$). Recall that a subsequence of sequence $(u_n)_{n\in\mathbb{N}_0}$ is defined by a strictly increasing map $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ as $(u_{\varphi(n)})_{n\in\mathbb{N}_0}$. The increasing map $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ sending k to $3k$ for $k \in \mathbb{N}_0$ tells us that $(u_{6n})_{n\in\mathbb{N}_0}$ is a subsequence of $(u_{2n})_{n\in\mathbb{N}_0}$, whereas the increasing map $\psi : \mathbb{N}_0 \to \mathbb{N}_0$ sending j to $2j$ for $j \in \mathbb{N}_0$ tells us that $(u_{6n})_{n\in\mathbb{N}_0}$ is a subsequence of $(u_{3n})_{n\in\mathbb{N}_0}$. Since a subsequence of a convergent sequence is also convergent with the same limit, we conclude that

$$\ell_1 = \lim_{n\to+\infty} u_{6n} = \ell_3.$$

Analogously, the increasing map $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ sending k to $3k$ for $k \in \mathbb{N}_0$ tells us that $(u_{6n+3})_{n\in\mathbb{N}_0}$ is a subsequence of $(u_{2n+1})_{n\in\mathbb{N}_0}$, whereas the increasing map $\psi : \mathbb{N}_0 \to \mathbb{N}_0$ sending j to $2j+3$ for $j \in \mathbb{N}_0$ tells us that $(u_{6n+3})_{n\in\mathbb{N}_0}$ is a subsequence of $(u_{3n})_{n\in\mathbb{N}_0}$. Since a subsequence of a convergent sequence is also convergent with the same limit, we conclude that

$$\ell_2 = \lim_{n\to+\infty} u_{6n+3} = \ell_3.$$

As a consequence, $\ell_1 = \ell_2$, and by the previous item we conclude that $(u_n)_{n\in\mathbb{N}_0}$ is also convergent with limit $\ell_1 = \ell_2$.

6. *Computation of limits using usual functions.* Compute the limit, if it exists, of the following sequences $(u_n)_{n\in\mathbb{N}}$ given by :

(a) $u_n = n^4(\ln(1-1/n^2)+1/n^2)$,

(b) $u_n = n(e^{2/n}-1)$,

(c) $u_n = n!/n^n$,

(d) $u_n = \tan(1/n)\cos(2n+1)$,

(e) $u_n = (\sqrt{n-3} + i\ln(2n))/\ln(n)$,

(f) $u_n = \ln(n^2+3n-2)/\ln(n^{1/3})$.

Solution.

(a) Consider the function $f : \mathbb{R}_{<1} \to \mathbb{R}$ given by

$$f(x) = \frac{\ln(1-x)+x}{x^2},$$

for $x \in \mathbb{R}_{<1}$. Then, using the Bernoulli-L'Hospital rule we see that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\ln(1-x)+x}{x^2} = \lim_{x\to 0} \frac{-\frac{1}{1-x}+1}{2x} = \lim_{x\to 0} \frac{1}{2(x-1)} = -\frac{1}{2}.$$

Since

$$u_n = n^4\left(\ln\left(1-\frac{1}{n^2}\right)+\frac{1}{n^2}\right) = f(1/n)$$

for $n \in \mathbb{N}$, we conclude that

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} f(1/n) = \lim_{x\to 0} f(x) = -\frac{1}{2}.$$

(b) Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \frac{e^{2x}-1}{x},$$

for $x \in \mathbb{R}$. Then, using the Bernoulli-L'Hospital rule we see that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{e^{2x}-1}{x} = \lim_{x\to 0} \frac{2e^{2x}}{1} = 2.$$

Since

$$u_n = n\left(e^{2/n}-1\right) = f(1/n^2)$$

for $n \in \mathbb{N}$, we conclude that

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} f(1/n^2) = \lim_{x\to 0} f(x) = 2.$$

(c) Note that $0 \leq u_n$ and

$$u_n = \frac{1\cdot 2\cdot\ldots(n-1)\cdot n}{\underbrace{n\cdot n\cdot\ldots n\cdot n}_{n \text{ factors}}} = \frac{1}{n}\frac{2}{n}\ldots\frac{n-1}{n}\frac{n}{n} \leq \frac{1}{n}\cdot\underbrace{1\cdot\cdots\cdot 1\cdot 1}_{n-1 \text{ factors}} = \frac{1}{n},$$

for all $n \in \mathbb{N}$, where we have used that $k/n \leq 1$ for all $k \in [\![2, n]\!]$. In other words, $0 \leq u_n \leq 1/n$ for all $n \in \mathbb{N}$. Since the limit of $1/n$ is zero, we conclude that $(u_n)_{n\in\mathbb{N}}$ converges to zero as n goes to $+\infty$ as well, by the sandwich theorem.

(d) Since the sequence $(v_n)_{n\in\mathbb{N}}$ given by $v_n = \tan(1/n)$ converges to zero as n goes to $+\infty$ and the sequence $(w_n)_{n\in\mathbb{N}}$ given by $w_n = \cos(2n+1)$ is bounded, for $|\cos(2n+1)| \leq 1$, we conclude that the sequence $(u_n)_{n\in\mathbb{N}}$ given by $u_n = v_n w_n$ converges to zero as n goes to $+\infty$ as well.

(e) Recall that a sequence $(u_n)_{n\in\mathbb{N}}$ of complex numbers converges (to $u = a + ib$, with $a, b \in \mathbb{R}$) if and only if the sequences of real numbers given by $(\mathrm{Re}(u_n))_{n\in\mathbb{N}}$ and $(\mathrm{Im}(u_n))_{n\in\mathbb{N}}$ converge (to a and b, respectively). We thus consider the sequences $(\mathrm{Re}(u_n))_{n\in\mathbb{N}}$ and $(\mathrm{Im}(u_n))_{n\in\mathbb{N}}$ given by

$$\mathrm{Re}(u_n) = \frac{\sqrt{n-3}}{\ln(n)} \text{ and } \mathrm{Im}(u_n) = \frac{\ln(2n)}{\ln(n)}.$$

Since

$$\mathrm{Im}(u_n) = \frac{\ln(2n)}{\ln(n)} = \frac{\ln(2)+\ln(n)}{\ln(n)} = 1 + \frac{\ln(2)}{\ln(n)}$$

for $n \in \mathbb{N}$, we conclude that

$$\lim_{n\to+\infty} \mathrm{Im}(u_n) = 1.$$

However, we note that

$$\lim_{n\to+\infty} \mathrm{Re}(u_n) = \lim_{n\to+\infty} \frac{\sqrt{n-3}}{\ln(n)} = +\infty. \tag{1}$$

Indeed, consider the function $f : \mathbb{R}_{>0} \to \mathbb{R}$ given by

$$f(x) = \frac{\sqrt{x-3}}{\ln(x)},$$

for $x \in \mathbb{R}_{>0}$. Then, using the Bernoulli-L'Hospital rule we see that

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\sqrt{x-3}}{\ln(x)} = \lim_{x\to+\infty} \frac{\frac{1}{2\sqrt{x-3}}}{\frac{1}{x}} = \lim_{x\to+\infty} \frac{x}{2\sqrt{x-3}}$$
$$= \lim_{x\to+\infty} \frac{\sqrt{x}}{2\sqrt{1-3/x}} = +\infty.$$

The identity $\mathrm{Re}(u_n) = f(n)$ for $n \in \mathbb{N}$ gives us (1). In consequence, the limit of $(u_n)_{n\in\mathbb{N}}$ does not exist.

(f) Note that

$$u_n = \frac{\ln(n^2+3n-2)}{\ln(n^{1/3})} = \frac{\ln\left(n^2(1+3/n-2/n^2)\right)}{\ln(n)/3} = 3\frac{\ln(n^2)+\ln(1+3/n-2/n^2)}{\ln(n)}$$
$$= 3\frac{2\ln(n)+\ln(1+3/n-2/n^2)}{\ln(n)} = 6\frac{\ln(n)}{\ln(n)} + 3\frac{\ln(1+3/n-2/n^2)}{\ln(n)}$$
$$= 6 + 3\frac{\ln(1+3/n-2/n^2)}{\ln(n)}$$

for all $n \in \mathbb{N}$. Since the numerator of the last summand goes to zero as n goes to $+\infty$ and the denominator goes to $+\infty$ as n goes to $+\infty$, we conclude that

$$\lim_{n\to+\infty} u_n = 6.$$

7. *Adjacent sequences.*

(a) Prove that each of the following pair of sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are adjacent.

(i) $u_n = \sum_{k=1}^{n} 1/k^2$ and $v_n = u_n + 1/n$.

(ii) $u_n = \sum_{k=1}^{n} 1/k^3$ and $v_n = u_n + 1/n^2$.

(iii) $u_0 = a > 0$, $v_0 = b > a$, $v_{n+1} = (u_n + v_n)/2$ and $u_{n+1} = \sqrt{u_n v_n}$.

(*b*) Define the real sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ by

$$u_n = \sum_{k=0}^{n} \frac{1}{k!} \text{ and } v_n = u_n + \frac{1}{n!n}.$$

(i) Show that these sequences are adjacent, with a common limit e (it's a possible definition of e).

(ii) Show that e is not rational.
Hint : Suppose that $e = p/q$ and note that for $n \in \mathbb{N}$ we have the inequalities $n!u_n < n!p/q < n!v_n$. Then choose n such that $n!p/q$ is an integer.

Solution.

(*a*) Recall that two sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are said to be **adjacent** if

$$\lim_{n\to+\infty}(u_n - v_n) = 0.$$

(i) Since $v_n - u_n = 1/n$ for all $n \in \mathbb{N}$, we conclude that $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are adjacent. Note moreover that $u_n \leq v_n$ for all $n \in \mathbb{N}$.

(ii) Since $v_n - u_n = 1/n^2$ for all $n \in \mathbb{N}$, we conclude that $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are adjacent. Note moreover that $u_n \leq v_n$ for all $n \in \mathbb{N}$.

(iii) We will first prove the identity

$$\sqrt{xy} \leq \frac{x+y}{2} \tag{2}$$

for all $x, y \in \mathbb{R}_{\geq 0}$. It is clear that (2) holds for $x = 0$ or $y = 0$. Assume that $x, y > 0$. Then, by dividing (2) by y, we see that (2) for $x, y > 0$ is tantamount to

$$\sqrt{\frac{x}{y}} \leq \frac{x/y+1}{2} \tag{3}$$

for all $x, y > 0$. By setting $t = x/y$, we see thus that (3) for $x, y > 0$ is equivalent to

$$\sqrt{t} \leq \frac{t+1}{2} \tag{4}$$

for all $t > 0$. Note however that, by multiplying by 2 and taking the square, (4) is tantamount to $4t \leq (t+1)^2$, *i.e.* $4t \leq t^2 + 2t + 1$, which is equivalent to $0 \leq t^2 - 2t + 1 = (t-1)^2$, which is a tautology. We have thus proved (2).

We now claim that

$$0 \leq u_n \leq v_n \tag{5}$$

for all $n \in \mathbb{N}_0$. Indeed, the case for $n = 0$ follows from the assumptions. Assume that $0 \leq u_n \leq v_n$ holds for $n \in \mathbb{N}_0$. Then,

$$u_{n+1} = \sqrt{u_n v_n} \geq 0 \text{ and } u_{n+1} = \sqrt{u_n v_n} \leq \frac{u_n + v_n}{2} = v_{n+1},$$

where we used (2). This proves (5). Moreover, (5) tells us that

$$u_n \leq u_{n+1} \text{ and } v_{n+1} \leq v_n \tag{6}$$

for all $n \in \mathbb{N}_0$. Indeed, using (5) we get

$$u_{n+1} = \sqrt{u_n v_n} \geq \sqrt{u_n u_n} = u_n$$

and

$$v_{n+1} = \frac{u_n + v_n}{2} \leq \frac{v_n + v_n}{2} = v_n$$

for all $n \in \mathbb{N}_0$. Hence, $(u_n)_{n\in\mathbb{N}}$ is an increasing sequence with upper bound v_1 and $(v_n)_{n\in\mathbb{N}}$ is a decreasing sequence with lower bound u_1. In consequence, $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are convergent. Let c be the limit of $(u_n)_{n\in\mathbb{N}}$ and d be the limit of $(v_n)_{n\in\mathbb{N}}$. Moreover,

$$d = \lim_{n\to+\infty} v_{n+1} = \lim_{n\to+\infty} \frac{u_n + v_n}{2} = \frac{c+d}{2}$$

tells us that $c = d$. As a consequence,

$$\lim_{n\to+\infty}(u_n - v_n) = \lim_{n\to+\infty} u_n - \lim_{n\to+\infty} v_n = c - d = 0,$$

i.e. the sequences are adjacent.

(*b*) (i) Since

$$\lim_{n\to+\infty}(u_n - v_n) = \lim_{n\to+\infty} \frac{1}{n!n} = 0$$

the sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are adjacent. Moreover, the sequence $(u_n)_{n\in\mathbb{N}}$ is clearly strictly increasing. Analogously, the sequence $(v_n)_{n\in\mathbb{N}}$ is strictly decreasing, since

$$\begin{aligned} v_{n+1} - v_n &= u_{n+1} - u_n + \frac{1}{(n+1)!(n+1)} - \frac{1}{n!n} \\ &= \frac{1}{(n+1)!} + \frac{1}{(n+1)!(n+1)} - \frac{1}{n!n} = \frac{n+2}{(n+1)!(n+1)} - \frac{1}{n!n} \\ &= \frac{1}{n!}\left[\frac{n+2}{(n+1)^2} - \frac{1}{n}\right] = \frac{1}{n!}\,\frac{-1}{n(n+1)^2} < 0, \end{aligned}$$

for all $n \in \mathbb{N}$.

(ii) Since $(u_n)_{n\in\mathbb{N}}$ is increasing, $(v_n)_{n\in\mathbb{N}}$ is decreasing, $u_n \leq v_n$ for all $n \in \mathbb{N}$, and they are adjacent, we conclude that

$$u_n < \lim_{n\to+\infty} u_n = e = \lim_{n\to+\infty} v_n < v_n \tag{7}$$

for all $n \in \mathbb{N}$. Suppose that $e = p/q$ for some $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Choose $n \in \mathbb{N}$ such that q divides $n!$ (*e.g.* $n = q$). Hence, $n!p/q$ is an integer and multiplying (7) by $n!$ we get

$$\sum_{k=0}^{n} \frac{n!}{k!} = n!u_n < n!\frac{p}{q} = n!e < n!v_n = n!u_n + \frac{1}{n}.$$

Note that $n!u_n \in \mathbb{N}$ and since $n > 1$, $0 < n!(e - u_n) < n!(v_n - u_n) < 1/n \leq 1$. Since $n!e - u_n$ is an integer, but there are no integers strictly larger than 0 and strictly less than 1, we conclude that e cannot be a rational number.

8. *Sequences defined recursively.*

(*a*) Let $f:[0,1]\to[0,1]$ be a continuous function such that $f(0)=0$, $f(1)=1$ and $f(x)<x$ for all $x\in\,]0,1[$. Define recursively a sequence $(u_n)_{n\in\mathbb{N}_0}$ by

$$\begin{cases} u_0\in[0,1], \\ u_{n+1}=f(u_n), \text{ for all } n\in\mathbb{N}_0. \end{cases}$$

Show that the sequence $(u_n)_{n\in\mathbb{N}_0}$ converges and compute its limit.

(*b*) Define recursively a sequence $(v_n)_{n\in\mathbb{N}_0}$ by

$$\begin{cases} v_0=\frac{1}{2}, \\ v_{n+1}=\frac{v_n}{2-\sqrt{v_n}}, \text{ for all } n\in\mathbb{N}_0. \end{cases}$$

Show that the sequence $(v_n)_{n\in\mathbb{N}_0}$ converges and compute its limit.

Solution.

(*a*) Assume that $u_0=1$, then $u_n=1$ for all $n\in\mathbb{N}_0$, since $f(1)=1$. In this case, $(u_n)_{n\in\mathbb{N}_0}$ converges to 1. Analogously, if $u_0=0$, then $u_n=0$ for all $n\in\mathbb{N}_0$, since $f(0)=0$. In this case, $(u_n)_{n\in\mathbb{N}_0}$ converges to 0.

Finally, assume that $u_0\in\,]0,1[$. We claim that $u_{n+1}\leq u_n$ for all $n\in\mathbb{N}_0$. Indeed, note that the statement holds for $n=0$, since $u_1=f(u_0)<u_0$. If the previous statement holds for $n\in\mathbb{N}_0$, then $u_{n+1}\leq u_0<1$ is either zero or lies in $]0,1[$. If it vanishes, then $u_{n+2}=f(u_{n+1})=0\leq 0=u_{n+1}$. If $u_{n+1}\in\,]0,1[$, then $u_{n+2}=f(u_{n+1})<u_{n+1}$. We conclude that the sequence $(u_n)_{n\in\mathbb{N}_0}$ is decreasing and bounded below (by 0), so it is convergent. Let c be its limit, which is strictly less than 1, since $c\leq u_0<1$. Since f is continuous, then

$$c=\lim_{n\to+\infty}u_{n+1}=\lim_{n\to+\infty}f(u_n)=f\Big(\lim_{n\to+\infty}u_n\Big)=f(c),$$

so c is a fixed point of f. Since the only fixed point of f in $[0,1[$ is 0, we conclude that $c=0$.

(*b*) ejitems It is clear that the function $f:[0,1]\to[0,1]$

$$f(x)=\frac{x}{2-\sqrt{x}}$$

is continuous and satisfies that $f(0)=0$, $f(1)=1$ and $f(x)<x$ for all $x\in\,]0,1[$, since the latter is tantamount to $1<2-\sqrt{x}$ for all $x\in\,]0,1[$, *i.e.* $\sqrt{x}<1$ for all $x\in\,]0,1[$. Since $u_0\in\,]0,1[$, we conclude that the sequence $(v_n)_{n\in\mathbb{N}_0}$ converges and its limit is zero.

9. *Cesàro average.* Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers. Define

$$S_n=\frac{u_1+\ldots+u_n}{n}$$

for all $n\in\mathbb{N}$.

(*a*) Show that if $(u_n)_{n\in\mathbb{N}}$ converges in $\mathbb{C}$, then $(S_n)_{n\in\mathbb{N}}$ converges to the same limit.

(*b*) Give an example of a divergent sequence $(u_n)_{n\in\mathbb{N}}$ such that $(S_n)_{n\in\mathbb{N}}$ converges.

(*c*) Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of (strictly) positive real numbers such that u_{n+1}/u_n converges. Show that $(u_n^{1/n})_{n\in\mathbb{N}}$ converges to the same limit.

Solution.

(*a*) We note first that

$$\left|S_n-\ell\right|=\left|\frac{\sum_{k=1}^n u_k}{n}-\ell\right|=\left|\frac{\left(\sum_{k=1}^n u_k\right)-n\ell}{n}\right|=\frac{\left|\sum_{k=1}^n (u_k-\ell)\right|}{n}\leq\frac{\sum_{k=1}^n |u_k-\ell|}{n},$$

for all $n\in\mathbb{N}$. Assume that $(u_n)_{n\in\mathbb{N}}$ converges to $\ell\in\mathbb{C}$. This implies that, given $\epsilon>0$, there exists $n_0\in\mathbb{N}$ such that $|u_n-\ell|\leq\epsilon/2$ for all $n\geq n_0$. Then,

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}|u_k-\ell|&=\frac{1}{n}\sum_{k=1}^{n_0}|u_k-\ell|+\frac{1}{n}\sum_{k=n_0+1}^{n}|u_k-\ell|\leq\frac{1}{n}\sum_{k=1}^{n_0}|u_k-\ell|+\frac{\epsilon(n-n_0)}{2n}\\&\leq\frac{1}{n}\sum_{k=1}^{n_0}|u_k-\ell|+\frac{\epsilon}{2},\end{aligned}$$

for all $n\geq n_0$. Now, since $C=\sum_{k=1}^{n_0}|u_k-\ell|$ is a finite value, let $n_1\in\mathbb{N}$ be such that $C/n\leq\epsilon/2$ (take for instance $n_1=\lfloor 2C/\epsilon\rfloor+1$). Set $N_0=\max(n_0,n_1)$. Then,

$$\frac{1}{n}\sum_{k=1}^{n_0}|u_k-\ell|+\frac{\epsilon}{2}\leq\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$$

for all $n\geq N_0$, which implies that $|S_n-\ell|\leq\epsilon$, for all $n\geq N_0$. This proves that $(S_n)_{n\in\mathbb{N}}$ converges to ℓ as n goes to $+\infty$.

(*b*) Let $(u_n)_{n\in\mathbb{N}}$ be given by $u_n=(1+(-1)^n)/2$ for $n\in\mathbb{N}$, *i.e.*

$$u_n=\begin{cases}0, & \text{if } n \text{ is odd},\\ 1, & \text{if } n \text{ is even}.\end{cases}$$

It is easy to see that $(u_n)_{n\in\mathbb{N}}$ is not convergent, since $(u_{2n})_{n\in\mathbb{N}}$ converges to 1 whereas $(u_{2n+1})_{n\in\mathbb{N}_0}$ converges to 0. Note in this case that

$$S_n=\begin{cases}\frac{m}{2m+1}, & \text{if } n=2m+1 \text{ is odd with } m\in\mathbb{N}_0,\\ \frac{1}{2}, & \text{if } n=2m \text{ is even with } m\in\mathbb{N}.\end{cases}$$

Since both subsequence $(S_{2m})_{m\in\mathbb{N}}$ and $(S_{2m+1})_{m\in\mathbb{N}_0}$ converge to $1/2$, we see that $(S_n)_{n\in\mathbb{N}}$ converges.

(*c*) Assume that u_{n+1}/u_n converges to ℓ as n goes to $+\infty$. Note that $\ell\geq 0$. Then, given $\epsilon>0$ such that $\epsilon<\ell$, there exists n_0 such that

$$\left|\frac{u_{n+1}}{u_n}-\ell\right|\leq\frac{\epsilon}{2}$$

for all $n\geq n_0$. To reduce some expressions, we will write $\epsilon'=\epsilon/2$. This implies that $(\ell-\epsilon')\leq u_{n+1}/u_n\leq(\ell+\epsilon')$ for all $n\geq n_0$, *i.e.* $(\ell-\epsilon')u_n\leq u_{n+1}\leq(\ell+\epsilon')u_n$ for all $n\geq n_0$. Note that $0<\ell-\epsilon'<\ell+\epsilon'$, since $\epsilon<\ell$. By a recursive argument we conclude that

$$(\ell-\epsilon')^k u_n\leq u_{n+k}\leq(\ell+\epsilon')^k u_n, \tag{8}$$

for all $n\geq n_0$ and $k\in\mathbb{N}_0$. Indeed, this is trivially verified if $k=0$ and any $n\geq n_0$. Assuming that it holds for k and a fixed $n\geq n_0$ tells us that

$$(\ell-\epsilon')^{k+1}u_n\leq(\ell-\epsilon')u_{n+k}\leq u_{n+k+1}\leq(\ell+\epsilon')u_{n+k}\leq(\ell+\epsilon')^{k+1}u_n,$$

as we wanted to show. In particular, (8) tells us that

$$(\ell-\epsilon')^{n-n_0}u_{n_0} \leq u_n \leq (\ell+\epsilon')^{n-n_0}u_{n_0}, \tag{9}$$

for all $n \geq n_0$, which yields

$$(\ell-\epsilon')\frac{u_{n_0}^{1/n}}{(\ell-\epsilon')^{n_0/n}} = (\ell-\epsilon')^{1-n_0/n}u_{n_0}^{1/n} \leq u_n^{1/n} \leq (\ell+\epsilon')^{1-n_0/n}u_{n_0}^{1/n} = (\ell+\epsilon')\frac{u_{n_0}^{1/n}}{(\ell+\epsilon')^{n_0/n}}, \tag{10}$$

for all $n \geq n_0$. Pick $n_1 \geq n_0$ such that

$$\frac{u_{n_0}^{1/n}}{(\ell+\epsilon')^{n_0/n}} \leq \frac{\ell+\epsilon}{\ell+\epsilon'} \text{ and } \frac{u_{n_0}^{1/n}}{(\ell-\epsilon')^{n_0/n}} \geq \frac{\ell-\epsilon}{\ell-\epsilon'}$$

for all $n \geq n_1$. This is possible since

$$\lim_{n\to+\infty}\frac{u_{n_0}^{1/n}}{(\ell+\epsilon')^{n_0/n}} = \lim_{n\to+\infty}\frac{u_{n_0}^{1/n}}{(\ell-\epsilon')^{n_0/n}} = 1,$$

but

$$0 < \frac{\ell-\epsilon}{\ell-\epsilon'} < 1 < \frac{\ell+\epsilon}{\ell+\epsilon'}.$$

Hence,

$$(\ell+\epsilon')\frac{u_{n_0}^{1/n}}{(\ell+\epsilon')^{n_0/n}} \leq \ell+\epsilon$$

and

$$(\ell-\epsilon')\frac{u_{n_0}^{1/n}}{(\ell-\epsilon')^{n_0/n}} \geq \ell-\epsilon$$

for all $n \geq n_1$. Using the previous inequalities together with (10), we obtain that

$$\ell-\epsilon \leq u_n^{1/n} \leq \ell+\epsilon$$

for all $n \geq n_1$. As a consequence, the sequence $(u_n^{1/n})_{n\in\mathbb{N}}$ converges to ℓ as n goes to $+\infty$.

10. *Lim sup and lim inf.* Let $(u_n)_{n\in\mathbb{N}_0}$ be a bounded sequence of real numbers. Define sequences $(i_n)_{n\in\mathbb{N}_0}$ and $(s_n)_{n\in\mathbb{N}_0}$ by

$$i_n = \inf\{u_k : k \geq n\} \text{ and } s_n = \sup\{u_k : k \geq n\}$$

for all $n \in \mathbb{N}_0$.

(*a*) Show that both $(i_n)_{n\in\mathbb{N}_0}$ and $(s_n)_{n\in\mathbb{N}_0}$ converge. The limit of $(i_n)_{n\in\mathbb{N}_0}$ is called **limit inferior** or **lower limit** of the sequence $(u_n)_{n\in\mathbb{N}_0}$, and is denoted by

$$\liminf_{n\to\infty} u_n.$$

The limit of $(s_n)_{n\in\mathbb{N}_0}$ is called **limit superior** or **upper limit** of the sequence

$(u_n)_{n\in\mathbb{N}_0}$, and is written

$$\limsup_{n\to\infty} u_n.$$

(*b*) Show that there exists a subsequence of $(u_n)_{n\in\mathbb{N}_0}$ converging to the limit inferior of $(u_n)_{n\in\mathbb{N}_0}$ and another subsequence of $(u_n)_{n\in\mathbb{N}_0}$ converging to the limit superior of $(u_n)_{n\in\mathbb{N}_0}$.

(*c*) Prove that $(u_n)_{n\in\mathbb{N}_0}$ converges if and only if $(i_n)_{n\in\mathbb{N}_0}$ and $(s_n)_{n\in\mathbb{N}_0}$ converge to the same limit in $\mathbb{R}$.

Solution.

(*a*) Let $a, b \in \mathbb{R}$ satisfy that $a \leq u_n \leq b$ for all $n \in \mathbb{N}_0$. Then $\{u_k : k \geq n\} \subseteq [a, b]$ for all $n \in \mathbb{N}_0$, which implies that

$$i_n = \inf\{u_k : k \geq n\} \in [a, b] \text{ and } s_n = \sup\{u_k : k \geq n\} \in [a, b].$$

In consequence, the sequences $(i_n)_{n\in\mathbb{N}_0}$ and $(s_n)_{n\in\mathbb{N}_0}$ are bounded. Moreover, $(i_n)_{n\in\mathbb{N}_0}$ is an increasing sequence and $(s_n)_{n\in\mathbb{N}_0}$ is a decreasing sequence, since the inclusion $\{u_k : k \geq n+1\} \subseteq \{u_k : k \geq n\}$ implies that

$$i_{n+1} = \inf\{u_k : k \geq n+1\} \leq \inf\{u_k : k \geq n\} = i_n$$

and

$$s_{n+1} = \sup\{u_k : k \geq n+1\} \geq \inf\{u_k : k \geq n\} = s_n$$

for all $n \in \mathbb{N}_0$. Since bounded monotone sequences are convergent, we conclude that $(i_n)_{n\in\mathbb{N}_0}$ and $(s_n)_{n\in\mathbb{N}_0}$ converge.

(*b*) We prove the case for $(i_n)_{n\in\mathbb{N}_0}$, since the one for $(s_n)_{n\in\mathbb{N}_0}$ is analogous. We construct a strictly increasing map $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ satisfying that $u_{\varphi(n)} \leq i_{\varphi(n-1)} + 1/2^n$ by recursion. Assume we have constructed $\varphi(0), \dots, \varphi(n-1)$ as before, for some $n \in \mathbb{N}_0$. Since $i_{\varphi(n-1)+1} = \inf\{u_k : k \geq \varphi(n-1)+1\}$, then there exists $\varphi(n) > \varphi(n-1)$ such that $u_{\varphi(n)} \leq i_{\varphi(n-1)} + 1/2^n$. Note also that $i_{\varphi(n)} \leq u_{\varphi(n)}$, by definition of the sequence $(i_n)_{n\in\mathbb{N}_0}$. Since the sequence $(i_n)_{n\in\mathbb{N}_0}$ is convergent, its subsequence $(i_{\varphi(n)})_{n\in\mathbb{N}_0}$ is also convergent with the same limit, and the inequalities $i_{\varphi(n)} \leq u_{\varphi(n)} \leq i_{\varphi(n-1)} + 1/2^n$ for all $n \in \mathbb{N}_0$ then tell us that the sequence $(u_{\varphi(n)})_{n\in\mathbb{N}_0}$ also converges to the limit of $(i_{\varphi(n)})_{n\in\mathbb{N}_0}$, *i.e.* $(u_{\varphi(n)})_{n\in\mathbb{N}_0}$ converges to the lower limit of $(u_n)_{n\in\mathbb{N}_0}$.

(*c*) Assume that $(u_n)_{n\in\mathbb{N}_0}$ converges to $\ell \in \mathbb{R}$. The previous item tells us that there exists subsequences $(u_{\varphi(n)})_{n\in\mathbb{N}_0}$ and $(u_{\psi(n)})_{n\in\mathbb{N}_0}$ of $(u_n)_{n\in\mathbb{N}_0}$ converging to the lower limit inferior and the upper limit of $(u_n)_{n\in\mathbb{N}_0}$, respectively. Since $(u_n)_{n\in\mathbb{N}_0}$ is convergent, the limits of the subsequences $(u_{\varphi(n)})_{n\in\mathbb{N}_0}$ and $(u_{\psi(n)})_{n\in\mathbb{N}_0}$ should coincide with ℓ. Conversely, assume that $(i_n)_{n\in\mathbb{N}_0}$ and $(s_n)_{n\in\mathbb{N}_0}$ converge to the same limit ℓ in $\mathbb{R}$. Since

$$i_n = \inf\{u_k : k \geq n\} \leq u_n \leq \sup\{u_k : k \geq n\} = s_n$$

for all $n \in \mathbb{N}_0$, the Sandwich Theorem tells that $(u_n)_{n\in\mathbb{N}_0}$ converges to ℓ.