MAT332

Fall 2021

Final examination - December 2023

Unjustified answers will be automatically excluded.

The grading is only approximate.

1	
2	
3	
4	

4pt

1. *Questions about the lectures.*

(a) Define the notion of convergence and absolute convergence of a series. Prove that an absolutely convergent series of real numbers is convergent.

(b) State the Leibniz criterion for the convergence of an alternating series.

(c) State the fundamental theorem of calculus for a continuous function.

Solution.

(a) Given a sequence $(a_n)_{n\in\mathbb{N}_0}$ of real numbers, the associated series $\sum_{n=0}^{+\infty} a_n$ is convergent if the sequence $(s_N)_{N\in\mathbb{N}_0}$ converges in $\mathbb{R}$, where $s_N = \sum_{n=0}^{N} a_n$ for all $N \in \mathbb{N}_0$. We say that the series $\sum_{n=0}^{+\infty} a_n$ is absolutely convergent if the series $\sum_{n=0}^{+\infty} |a_n|$ is convergent.

We will now prove that an absolutely convergent series $\sum_{n=0}^{+\infty} a_n$ is convergent. Due to the completeness of $\mathbb{R}$, it suffices to prove that $(s_N)_{N\in\mathbb{N}_0}$ is a Cauchy sequence. Let $(s_N^+)_{N\in\mathbb{N}_0}$ be the sequence given by $s_N^+ = \sum_{n=0}^{N} |a_n|$ for all $N \in \mathbb{N}_0$. Since $\sum_{n=0}^{+\infty} a_n$ is absolutely convergent, then $(s_N^+)_{N\in\mathbb{N}_0}$ is convergent, so in particular it is a Cauchy sequence, *i.e.* given $\epsilon > 0$, there exists $N_0 \in \mathbb{N}_0$ such that $|s_N^+ - s_M^+| \leq \epsilon$ for all integers $N \geq M \geq n_0$. By the triangle inequality for the absolute value we have then

$$|s_N - s_M| = \left| \sum_{n=M+1}^{N} a_n \right| \leq \sum_{n=M+1}^{N} |a_n| = s_N^+ - s_M^+ = |s_N^+ - s_M^+| \leq \epsilon$$

for all integers $N \geq M \geq n_0$. This tells us that $(s_N)_{N\in\mathbb{N}_0}$ is a Cauchy sequence, as claimed.

(b) The Leibniz criterion for the convergence of an alternating series states that, given a sequence $(a_n)_{n\in\mathbb{N}_0}$ of nonnegative real numbers that is decreasing and converges to zero, the series $\sum_{n=0}^{+\infty}(-1)^n a_n$ is convergent.

(c) The fundamental theorem of calculus for a continuous function $f : [a,b] \to \mathbb{R}$, where $a < b$ are real numbers, states that there exists a continuous function $F : [a,b] \to \mathbb{R}$ that is differentiable on $]a,b[$, called a **primitive** of f, and moreover, for any primitive F of f we have that

$$\int_a^b f(x)dx = F(b) - F(a).$$

4pt

2. Determine if the following series are convergent or divergent :

(a) $\sum_{n=1}^{+\infty} \ln\left(1+\frac{1}{n}\right)$,

(b) $\sum_{n=1}^{+\infty} (-1)^n \ln\left(1+\frac{1}{n}\right)$,

(c) $\sum_{n=1}^{+\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{n^\alpha}$, for $\alpha > 0$,

(d) $\sum_{n=1}^{+\infty} 2^{-n^2}$,

(e) $\sum_{n=1}^{+\infty} \frac{n!}{n^n}$.

Solution.

(a) Note first that $\ln(1+1/n) > 0$ for all $n \in \mathbb{N}$. Moreover, notice that

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \ln'(1) = 1,$$

by definition of derivative,which implies that

$$\lim_{n\to+\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = 1. \tag{1}$$

Hence $\ln(1+1/n) \sim 1/n$ as n goes to $+\infty$, and since $\sum_{n=1}^{+\infty} 1/n$ is divergent, $\sum_{n=1}^{+\infty} \ln(1+1/n)$ is also divergent.

(b) Since the logarithm function is increasing and $(1/n)_{n\in\mathbb{N}}$ is a decreasing sequence, $(\ln(1+1/n))_{n\in\mathbb{N}}$ is a decreasing sequence. Fruthermore, since the logarithm is continuous and $(1/n)_{n\in\mathbb{N}}$ converges to zero, $(\ln(1+1/n))_{n\in\mathbb{N}}$ converges to $\ln(1) = 0$. By the Leibniz criterion recalled in the first exercise, the series $\sum_{n=1}^{+\infty}(-1)^n \ln(1+1/n)$ is convergent.

(c) Note first that $\ln(1+1/n)/n^\alpha > 0$ for all $n \in \mathbb{N}$. Using (1), we see that

$$\lim_{n\to+\infty} \frac{\frac{\ln\left(1+\frac{1}{n}\right)}{n^\alpha}}{\frac{1}{n^{1+\alpha}}} = 1,$$

so $\ln(1+1/n)/n^\alpha \sim 1/n^{1+\alpha}$ as n goes to $+\infty$. Since $\sum_{n=1}^{+\infty} 1/n^s$ is convergent if and only if $s > 1$, $\sum_{n=1}^{+\infty} \ln(1+1/n)/n^\alpha$ is convergent for all $\alpha > 0$.

(d) Note that $2^{-n^2} > 0$ for all $n \in \mathbb{N}$. Moreover,

$$\sqrt[n]{2^{-n^2}} = 2^{-\frac{n^2}{n}} = 2^{-n}$$

converges to 0 as n goes to $+\infty$. The root test tells us then that the series $\sum_{n=1}^{+\infty} 2^{-n^2}$ converges.

(e) Note that $n!/n^n > 0$ for all $n \in \mathbb{N}$. Moreover,

$$\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

converges to $e^{-1} < 1$ as n goes to $+\infty$. The ratio test tells us then that the series $\sum_{n=1}^{+\infty} n!/n^n$ converges.

4pt

3. Given $n \in \mathbb{N}_0$, set

$$a_n = \int_0^1 \left(\frac{1+x^2}{2}\right)^n dx.$$

(*a*) Prove that

$$\int_0^1 x\left(\frac{1+x^2}{2}\right)^n dx \le a_n \le \int_0^1 \left(\frac{1+x}{2}\right)^n dx$$

for all $n \in \mathbb{N}_0$.

(*b*) Determine the nature of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} (-1)^n a_n$.

(*c*) Compute the value $\sum_{n=0}^{\infty} (-1)^n a_n$.

Solution.

(*a*) Since $x^2 \le x \le 1$ for $x \in [0,1]$, then

$$x\left(\frac{1+x^2}{2}\right)^n \le \left(\frac{1+x^2}{2}\right)^n \le \left(\frac{1+x}{2}\right)^n$$

for $x \in [0,1]$ and $n \in \mathbb{N}_0$, so the monotonicity of the integral tells that

$$\int_0^1 x\left(\frac{1+x^2}{2}\right)^n dx \le \int_0^1 \left(\frac{1+x^2}{2}\right)^n dx \le \int_0^1 \left(\frac{1+x}{2}\right)^n dx,$$

for all $n \in \mathbb{N}_0$, which gives the desired inequalities.

(*b*) Note that

$$\int_0^1 x\left(\frac{1+x^2}{2}\right)^n dx = \int_{1/2}^1 y^n dy = \left[\frac{y^{n+1}}{n+1}\right]_{1/2}^1 = \frac{1-2^{-n-1}}{n+1}, \tag{2}$$

for all $n \in \mathbb{N}_0$, where we used the change of variables $y = (1+x^2)/2$ for $x \in [0,1]$. Moreover, the series

$$\sum_{n=0}^{+\infty} \frac{1-2^{-n-1}}{n+1}$$

is divergent, since the sequence of partial sums

$$\sum_{n=0}^{N} \frac{1-2^{-n-1}}{n+1} = \sum_{n=0}^{N} \frac{1}{n+1} - \sum_{n=0}^{N} \frac{1}{2^{n+1}(n+1)}$$

is given by the sum of a divergent sequence and a convergent sequence, as the sequence $(\sum_{m=1}^{M} m^{-1})_{M\in\mathbb{N}}$ is divergent and $(\sum_{m=1}^{M} m^{-1}2^{-m})_{M\in\mathbb{N}}$ is convergent. The first inequality of the previous item tells us then that

$$\sum_{n=0}^{N} \frac{1-2^{-n-1}}{n+1} \le \sum_{n=0}^{N} a_n$$

and since the first sum goes to $+\infty$ as N goes to $+\infty$, so does the second sum. In consequence, the series $\sum_{n=0}^{\infty} a_n$ is divergent.

We will show that the series $\sum_{n=0}^{\infty}(-1)^n a_n$ is convergent. To prove this, it suffices to show that $(a_n)_{n\in\mathbb{N}_0}$ is a nonnegative decreasing sequence converging to zero, since the Leibniz criterion tells us then that $\sum_{n=0}^{\infty}(-1)^n a_n$ is convergent. It is clear that $a_0 \geq 0$ for all $n \in \mathbb{N}_0$, since a_n is given as the integral of a nonnegative continuous function over a finite bounded interval. Moreover, since $(1+x^2)/2 \leq 1$ for $x \in [0,1]$, we have that

$$\left(\frac{1+x^2}{2}\right)^{n+1} \leq \left(\frac{1+x^2}{2}\right)^{n}$$

for $x \in [0,1]$ and $n \in \mathbb{N}_0$. The monotonicity of the integral then implies that

$$a_{n+1} = \int_0^1 \left(\frac{1+x^2}{2}\right)^{n+1} dx \leq \int_0^1 \left(\frac{1+x^2}{2}\right)^{n} dx = a_n$$

for all $n \in \mathbb{N}_0$, so the sequence $(a_n)_{n\in\mathbb{N}_0}$ is decreasing. Finally, the fact that $a_n \geq 0$ for all $n \in \mathbb{N}_0$ and the second inequality of the first item tells us that

$$0 \leq a_n \leq \int_0^1 \left(\frac{1+x}{2}\right)^{n} dx = \frac{2-2^{-n}}{n+1},$$

for all $n \in \mathbb{N}_0$, so

$$0 \leq \lim_{n\to+\infty} a_n \leq \lim_{n\to+\infty} \int_0^1 \left(\frac{1+x}{2}\right)^{n} dx = \lim_{n\to+\infty} \frac{1-2^{-n-1}}{n+1} = 0,$$

which says that $(a_n)_{n\in\mathbb{N}_0}$ converges to zero, as was to be shown.

(c) It is clear that

$$\begin{aligned}\sum_{n=0}^{N}(-1)^n a_n &= \int_0^1 \sum_{n=0}^{N}\left(-\frac{1+x^2}{2}\right)^{n} dx = \int_0^1 \frac{1-\left(-\frac{1+x^2}{2}\right)^{N+1}}{1-\left(-\frac{1+x^2}{2}\right)} dx \\ &= \int_0^1 \frac{1}{1-\left(-\frac{1+x^2}{2}\right)} dx - \int_0^1 \frac{\left(-\frac{1+x^2}{2}\right)^{N+1}}{1-\left(-\frac{1+x^2}{2}\right)} dx \\ &= \int_0^1 \frac{2}{3+x^2} dx + \frac{(-1)^N}{2^N} \int_0^1 \frac{(1+x^2)^{N+1}}{3+x^2} dx,\end{aligned}$$

for all $N \in \mathbb{N}_0$, where we used the usual identity $\sum_{n=0}^{N} q^n = (1-q^{N+1})/(1-q)$ for all $q \in \mathbb{R} \setminus \{1\}$. Using the change of variables $y = x/\sqrt{3}$ we get that

$$\begin{aligned}\int_0^1 \frac{2}{3+x^2} dx = \frac{2}{\sqrt{3}} \int_0^{1/\sqrt{3}} \frac{1}{1+y^2} dy &= \frac{2}{\sqrt{3}} \Big[\arctan(y) \Big]_0^{1/\sqrt{3}} \\ &= \frac{2}{\sqrt{3}} \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{3\sqrt{3}}.\end{aligned}$$

On the other hand, note that

$$\begin{aligned} 0 &\leq \left| \frac{(-1)^N}{2^N} \int_0^1 \frac{(1+x^2)^{N+1}}{3+x^2} dx \right| = \frac{1}{2^N} \int_0^1 \frac{(1+x^2)^{N+1}}{3+x^2} dx \\ &\leq \int_0^1 \frac{(1+x^2)^{N+1}}{2^{N+1}} dx \leq \int_0^1 \frac{(1+x)^{N+1}}{2^{N+1}} dx = \frac{2-2^{-N-1}}{N+1} \end{aligned}$$

for all $N \in \mathbb{N}_0$, where we used that $3+x^2 \geq 2$ and the last inequality of the first item. Hence

$$\lim_{N\to+\infty} \frac{(-1)^N}{2^N} \int_0^1 \frac{(1+x^2)^{N+1}}{3+x^2} dx = 0$$

and

$$\sum_{n=0}^{+\infty} (-1)^n a_n = \lim_{N\to+\infty} \sum_{n=0}^{N} (-1)^n a_n = \int_0^1 \frac{2}{3+x^2} dx = \frac{\pi}{3\sqrt{3}}.$$

3pt

4. Consider the function $f : \mathbb{R} \setminus \{0,-1\} \to \mathbb{R}$ given by

$$f(x) = \frac{1}{x(x+1)}$$

for $x \in \mathbb{R} \setminus \{0,-1\}$.

(a) Find A and B in $\mathbb{R}$ such that

$$f(x) = \frac{A}{x} + \frac{B}{x+1}$$

for $x \in \mathbb{R} \setminus \{0,-1\}$.

(b) Compute $\int_1^2 f(x) dx$.

(c) Compute

$$\int_1^2 \frac{\ln(1+x)}{x^2} dx.$$

Solution.

(a) It is clear that

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

for $x \in \mathbb{R} \setminus \{0,-1\}$, *i.e.* $A = -B = 1$.

(*b*) We have that

$$\int_1^2 f(x)dx = \int_1^2 \frac{1}{x}dx - \int_1^2 \frac{1}{x+1}dx = \Big[\ln\big(|x|\big)\Big]_1^2 - \Big[\ln\big(|x+1|\big)\Big]_1^2$$
$$= \ln(2) - \ln(3) + \ln(2) = \ln\left(\frac{4}{3}\right).$$

(*c*) By integrating by parts with $u = \ln(x+1)$ and $v = -1/x$ (so $v' = 1/x^2$) we see that

$$\int_1^2 \frac{\ln(1+x)}{x^2}dx = \left[-\frac{\ln(1+x)}{x}\right]_1^2 + \int_1^2 \frac{1}{(x+1)x}dx$$
$$= -\frac{\ln(3)}{2} + \ln(2) + \ln\left(\frac{4}{3}\right) = \ln\left(\frac{8}{3\sqrt{3}}\right),$$

where we used the value computed in the previous item.

2pt

5. Compute the value of the following integrals :

(*a*) $\int_0^\pi \sin^2(x)\cos^2(x)dx$, (*b*) $\int_0^{\pi/2} \frac{\sin(x)}{\cos^2(x)+2\cos(x)+2}dx$.

Solution.

(*a*) Recall that, by using integration by parts twice, we have that

$$\int \sin^n(x)dx = -\frac{\cos(x)\sin^{n-1}(x)}{n} + \frac{n-1}{n}\int \sin^{n-2}(x)dx$$

for all integers $n \geq 2$. In particular, this identity implies that

$$\int_0^\pi \sin^2(x) = \left[-\frac{\cos(x)\sin(x)}{2}\right]_0^\pi + \frac{1}{2}\int_0^\pi dx = \frac{\pi}{2}$$

and

$$\int_0^\pi \sin^4(x) = \left[-\frac{\cos(x)\sin^3(x)}{4}\right]_0^\pi + \frac{3}{4}\int_0^\pi \sin^2(x)dx = \frac{3\pi}{8},$$

where we used the previous identity. Using these equalities together with the Pithagorean identity $\cos^2(x) = 1 - \sin^2(x)$, we see that

$$\int_0^\pi \sin^2(x)\cos^2(x)dx = \int_0^\pi \big(\sin^2(x) - \sin^4(x)\big)dx = \frac{\pi}{2} - \frac{3\pi}{8} = \frac{\pi}{8}.$$

(b) Note first that

$$\int \frac{\sin(x)}{\cos^2(x)+2\cos(x)+2}dx = -\int \frac{1}{y^2+2y+2}dy = -\int \frac{1}{1+(1+y)^2}dy$$
$$= -\arctan(1+y)+C$$
$$= -\arctan\big(1+\cos(x)\big)+C,$$

where we used the change of variable $y=\cos(x)$. As a consequence,

$$\int_0^{\pi/2} \frac{\sin(x)}{\cos^2(x)+2\cos(x)+2}dx = \Big[-\arctan\big(1+\cos(x)\big)\Big]_0^{\pi/2} = -\frac{\pi}{4}+\arctan(2).$$

5pt

6. Consider the function $f:\mathbb{R}\to\mathbb{R}$ given by

$$f(x) = \frac{1}{1+x^4\sin^2(x)}$$

for $x\in\mathbb{R}$.

(a) Show that $\int\limits_{-\infty}^{+\infty} f(x)dx$ converges if and only if $\int\limits_{0}^{+\infty} f(x)dx$ converges.

(b) Given $n\in\mathbb{N}_0$, set

$$I_n = \int_{n\pi}^{(n+1)\pi} f(x)dx.$$

Show that $\int\limits_{0}^{+\infty} f(x)dx$ converges if and only if $\sum\limits_{n=0}^{\infty} I_n$ converges.

(c) Prove that

$$I_n \le \int_0^{\pi} \frac{dx}{1+n^4\pi^4\sin^2(x)} = 2\int_0^{\pi/2} \frac{dx}{1+n^4\pi^4\sin^2(x)}$$

for all $n\in\mathbb{N}_0$.

(d) Prove that $\sin(x)\ge 2x/\pi$ for all $x\in[0,\pi/2]$, and deduce that $I_n\le 1/n^2$ for all $n\in\mathbb{N}$. Conclude that $\int\limits_{-\infty}^{+\infty} f(x)dx$ converges.

Solution.

(a) Define $I_A^+ = \int_0^A \frac{1}{1+x^4\sin^2(x)}dx$ and $I_{-A,B} == \int_{-A}^B \frac{1}{1+x^4\sin^2(x)}dx$ for all $A,B\in\mathbb{R}_{>0}$. Note that, by doing the change of variables $y=-x$, we get that

$$I_{A,0} = \int_{-A}^{0} f(x)dx = \int_{-A}^{0} \frac{1}{1+x^4\sin^2(x)}dx = \int_0^A \frac{1}{1+y^4\sin^2(y)}dy = I_A^+.$$

As a consequence,

$$I_{A,B} = I_A^+ + I_B^+ \tag{3}$$

for all $A, B \in \mathbb{R}_{>0}$. Recall that $\int_0^{+\infty} f(x)dx$ converges if and only if I_A^+ converges to a real value as A goes to $+\infty$, and $\int_{-\infty}^{+\infty} f(x)dx$ converges if and only if $I_{A,B}$ converges to a real value as A and B go to $+\infty$. It is then clear, by (3), that, if $\int_0^{+\infty} f(x)dx$ converges, then $\int_{-\infty}^{+\infty} f(x)dx$ also converges. Conversely, if $\int_{-\infty}^{+\infty} f(x)dx$ converges, then, since the parameters A and B are independent, the convergence of $I_{A,B}$ implies in particular that of $I_{A,A} = 2I_A^+$ converges as A goes to $+\infty$, so $\int_0^{+\infty} f(x)dx$ converges.

(*b*) Note first that

$$I_A^+ = I_{\lfloor A \rfloor}^+ + \int_{\lfloor A/\pi \rfloor \pi}^{A} f(x)dx \tag{4}$$

for all $A \in \mathbb{R}_{>0}$, where $\lfloor B \rfloor$ denotes the integer part of $B > 0$. Since $f(x) > 0$ and $f(x)$ converges to zero as x goes to $+\infty$, we see that, given $\epsilon > 0$, there exists $C > 0$ such that $0 < f(x) \leq \epsilon$ for all $x > C$. Then,

$$0 \leq \int_{\lfloor A/\pi \rfloor \pi}^{A} f(x)dx \leq \int_{\lfloor A/\pi \rfloor \pi}^{A} \frac{\epsilon}{\pi} dx \leq \epsilon,$$

for all $A > C + \pi$, since $A - \lfloor A/\pi \rfloor \pi \leq \pi$. As a consequence,

$$\lim_{A \to +\infty} \int_{\lfloor A/\pi \rfloor \pi}^{A} f(x)dx = 0,$$

which tells us that the convergence of I_A as A goes to $+\infty$, *i.e.* the convergence of $\int_0^{+\infty} f(x)dx$, is equivalente to the convergence of $I_{N\pi}^+ = \sum_{n=0}^{N-1} I_n$ as N goes to $+\infty$, *i.e.* the convergence of the series $\sum_{n=0}^{+\infty} I_n$.

(*c*) Note first that

$$I_n = \int_{n\pi}^{(n+1)\pi} \frac{dy}{1 + y^4 \sin^2(y)} \leq \int_{n\pi}^{(n+1)\pi} \frac{dy}{1 + n^4\pi^4 \sin^2(y)}$$

for all $n \in \mathbb{N}_0$, since $y \geq n\pi$ for $y \in [n\pi, (n+1)\pi]$. Moreover, using the change of variables $x = y + n\pi$ with $x \in [0, \pi]$, and the fact that $\sin(y + n\pi) = \sin(y)$ for $n \in \mathbb{N}_0$, we get that

$$\int_{n\pi}^{(n+1)\pi} \frac{dy}{1 + n^4\pi^4 \sin^2(y)} = \int_0^{\pi} \frac{dx}{1 + n^4\pi^4 \sin^2(x)}$$

for all $n \in \mathbb{N}_0$. Finally, note that

$$\begin{aligned}\int_0^{\pi} \frac{dx}{1+n^4\pi^4\sin^2(x)} &= \int_0^{\pi/2} \frac{dx}{1+n^4\pi^4\sin^2(x)} + \int_{\pi/2}^{\pi} \frac{dx}{1+n^4\pi^4\sin^2(x)} \\ &= \int_0^{\pi/2} \frac{dx}{1+n^4\pi^4\sin^2(x)} + \int_0^{\pi/2} \frac{dz}{1+n^4\pi^4\sin^2(z)} \\ &= 2\int_0^{\pi/2} \frac{dx}{1+n^4\pi^4\sin^2(x)}\end{aligned}$$

for all $n \in \mathbb{N}_0$, where we used in the second equality the change of variables $z = \pi - x$ for the second integral. We have thus proved the required inequalities.

(d) To prove the inequality $\sin(x) \geq 2x/\pi$ for $x \in [0, \pi/2]$, recall that the sine function is concave on the interval $[0, \pi]$, since its double derivative $\sin'' = -\sin$ is positive on $]0, \pi[$. By definition of concavity, we have that $\sin(t\pi/2) = \sin((1-t)0 + t\pi/2) \geq (1-t)\sin(0) + t\sin(\pi/2) = t$ for all $t \in [0, 1]$, which is tantamount to $\sin(x) \geq 2x/\pi$ for $x \in [0, \pi/2]$, by setting $x = t\pi/2$. Using this inequality and the monotonicity of the integral, we get that

$$I_n \leq 2\int_0^{\pi/2} \frac{dx}{1+n^4\pi^4\sin^2(x)} \leq 2\int_0^{\pi/2} \frac{dx}{1+n^4\pi^4\left(\frac{2x}{\pi}\right)^2} = 2\int_0^{\pi/2} \frac{dx}{1+4n^4\pi^2x^2}$$

for all $n \in \mathbb{N}_0$. Using the change of variables $u = 2n^2\pi x$, we get that

$$\begin{aligned}2\int_0^{\pi/2} \frac{dx}{1+4n^4\pi^2x^2} &= \frac{1}{n^2\pi}\int_0^{n^2\pi^2} \frac{du}{1+u^2} \leq \frac{1}{n^2\pi}\int_0^{+\infty} \frac{du}{1+u^2} \\ &= \frac{1}{n^2\pi}\Big[\arctan(u)\Big]_0^{+\infty} = \frac{1}{2n^2} \leq \frac{1}{n^2}\end{aligned}$$

for all $n \in \mathbb{N}_0$. We also remark that, since f is positive, $I_n \geq 0$ for all $n \in \mathbb{N}_0$. Hence, $0 \leq I_n \leq n^{-2}$ for all $n \in \mathbb{N}_0$, which implies that $\sum_{n=0}^{+\infty} I_n$ converges, and by the first two items, the integral $\int\limits_{-\infty}^{+\infty} f(x)dx$ converges.