1
2
3
4
5

MAT332
Fall 2022

Midterm examination

Unjustified answers will be automatically excluded.

The grading is only approximative.

3pt

1. Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers.

(a) Recall the definition of convergence for the series $\sum_{n=1}^{+\infty} z_n$.

(b) Recall the definition of absolute convergence for the series $\sum_{n=1}^{+\infty} z_n$.

(c) Prove that $\sum_{n=1}^{+\infty} z_n$ is convergent if it is absolutely convergent.

Solution.

(a) Let $S_N = \sum_{n=1}^{N} z_n$ for $N \in \mathbb{N}$. If $\lim_{N\to+\infty} S_N = S$ for some $S \in \mathbb{C}$, we say that the series $\sum_{n=1}^{+\infty} z_n$ converges.

More specifically, the series $\sum_{n=1}^{+\infty} z_n$ converges if there exists a complex number S satisfying that for any $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $|\sum_{n=1}^{N} z_n - S| < \epsilon$ for all $N \geqslant N_0$.

(b) We say that the series $\sum_{n=1}^{+\infty} z_n$ is absolutely convergent if the series $\sum_{n=1}^{+\infty} |z_n|$ converges.

More specifically, the series $\sum_{n=1}^{+\infty} z_n$ is absolutely convergent if there exists a real number S satisfying that for any $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $|\sum_{n=1}^{N} |z_n| - S| < \epsilon$ for all $N \geqslant N_0$.

(c) Consider the sequence $\{s_N\}_{N\in\mathbb{N}}$ of partial sums $s_N = \sum_{n=1}^{N} z_n$ for $N \in \mathbb{N}$.

Since $\sum_{n\in\mathbb{N}} z_n$ converges absolutely, the sequence $\{S_N\}_{N\in\mathbb{N}}$ of partial sums given by $S_N = \sum_{n=1}^{N} |z_n|$ for $N \in \mathbb{N}$ is a Cauchy sequence.

Given $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{M} |z_n| = |S_M - S_N| < \epsilon$$

for all $M > N \geqslant N_0$.

Using the triangle inequality we see that

$$|s_M - s_N| = \left| \sum_{n=N+1}^{M} z_n \right| \leqslant \sum_{n=N+1}^{M} |z_n| = |S_M - S_N| < \epsilon$$

for all $M > N \geqslant N_0$.

This tells us that $\{s_N\}_{N\in\mathbb{N}}$ is a Cauchy sequence of complex numbers.

Since any Cauchy sequence of complex numbers is convergent, then $\{s_N\}_{N\in\mathbb{N}}$ is convergent, *i.e.* $\sum_{n\in\mathbb{N}} z_n$ converges.

1,5pt

2. Prove that the series

$$\sum_{n=2}^{+\infty} \frac{n+1}{n!}$$

converges and compute the sum of the series.

Solution. Let $u_n = \frac{n+1}{n!}$ for $n \geqslant 2$. Then $u_n > 0$ and

$$\frac{u_{n+1}}{u_n} = \frac{\frac{n+2}{(n+1)!}}{\frac{n+1}{n!}} = \frac{n+2}{(n+1)^2}$$

for $n \geqslant 2$. Since

$$\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to+\infty} \frac{n+2}{n^2+2n+1} = \lim_{n\to+\infty} \frac{1+\frac{2}{n}}{n+2+\frac{1}{n}} = 0,$$

we obtain that the series $\sum_{n=2}^{+\infty} u_n$ converges by D'Alembert rule.

Note that $e = \sum_{n=0}^{+\infty} \frac{1}{n!}$.

Then

$$\sum_{n=2}^{+\infty} \frac{n}{n!} = \sum_{n=2}^{+\infty} \frac{1}{(n-1)!} = \sum_{n=1}^{+\infty} \frac{1}{n!}.$$

So,

$$\begin{aligned}\sum_{n=2}^{+\infty} \frac{n+1}{n!} &= \sum_{n=2}^{+\infty} \frac{n}{n!} + \sum_{n=2}^{+\infty} \frac{1}{n!} = \sum_{n=1}^{+\infty} \frac{1}{n!} + \sum_{n=2}^{+\infty} \frac{1}{n!} \\ &= \sum_{n=0}^{+\infty} \frac{1}{n!} - 1 + \sum_{n=0}^{+\infty} \frac{1}{n!} - 1 - 1 \\ &= 2e - 3.\end{aligned}$$

9pt

3. Consider the sequences $(u_n)_{n\in\mathbb{N}}$ given by

(i) $u_n = \frac{2-\cos(n)}{\sqrt{n}}$,

(ii) $u_n = \sin\left(1+e^{-n}\right)$,

(iii) $u_n = \left(1-\frac{1}{\sqrt{n}}\right)^{n^{3/2}}$,

(iv) $u_n = \frac{4^{n+1}(n+1)!}{(2n-1)!}$,

(v) $u_n = \frac{(-1)^n}{5+\sqrt{\ln(n)}}$,

(vi) $u_n = \frac{(-1)^n}{n+2\sin(n^3)}$,

for $n \in \mathbb{N}$, respectively. Determine if the series $\sum_{n=1}^{+\infty} u_n$ is absolutely convergent, convergent or none of them.

Solution. Note that absolutely convergence implies convergence, and divergence implies absolutely divergence.

(i) The series $\sum_{n=1}^{+\infty} u_n$ is divergent.

Since $\cos(n) \in [-1,1]$, we have $2-\cos(n) \in [1,3]$ for $n \in \mathbb{N}$. So, $u_n > 0$ and

$$u_n \geqslant \frac{1}{\sqrt{n}}$$

for $n \in \mathbb{N}$. The Riemann series $\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n}}$ is divergent. We get that the series $\sum_{n=1}^{+\infty} u_n$ is divergent by comparison criteria.

(ii) The series $\sum_{n=1}^{+\infty} u_n$ is divergent.

We have that e^{-n} tends to 0 when n tends to infinity. So, the general term u_n tends to $\sin(1) \neq 0$ when n tends to infinity. Since the sequence of general terms of a convergent series converges to 0, we get that the series $\sum_{n=1}^{+\infty} u_n$ is divergent.

(iii) The series $\sum_{n=1}^{+\infty} u_n$ is absolutely convergent.

Let $f(x) = (1-x)^{\frac{1}{x}}$ for $x \in \mathbb{R}$. Then

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} e^{\frac{1}{x}\ln(1-x)}.$$

By L'Hospital rule,

$$\lim_{x\to 0+} \frac{\ln(1-x)}{x} = \lim_{x\to 0+} \frac{\frac{-1}{1-x}}{1} = -1.$$

Then, $\lim_{x\to 0+} f(x) = e^{-1}$. Take $x = \frac{1}{\sqrt{n}}$, we obtain

$$\lim_{n\to+\infty} (1-\frac{1}{\sqrt{n}})^{\sqrt{n}} = e^{-1}.$$

Note that $(1-\frac{1}{\sqrt{n}})^{\sqrt{n}}$ is increasing. Then $(1-\frac{1}{\sqrt{n}})^{\sqrt{n}} \leqslant e^{-1}$ for $n \in \mathbb{N}$. Hence,

$$0 \leqslant u_n = (1-\frac{1}{\sqrt{n}})^{n^{3/2}} = \left((1-\frac{1}{\sqrt{n}})^{\sqrt{n}}\right)^n \leqslant (\frac{1}{e})^n$$

for $n \in \mathbb{N}$. Since the geometric series $\sum_{n\in\mathbb{N}} e^{-n}$ converges, the series $\sum_{n=1}^{+\infty} u_n$ is absolutely convergent by comparison criteria.

(iv) The series $\sum_{n=1}^{+\infty} u_n$ is absolutely convergent.

We have $u_n > 0$ and

$$\frac{u_{n+1}}{u_n} = \frac{\frac{4^{n+2}(n+2)!}{(2n+1)!}}{\frac{4^{n+1}(n+1)!}{(2n-1)!}} = \frac{4^{n+2}(n+2)!}{(2n+1)!}\,\frac{(2n-1)!}{4^{n+1}(n+1)!} = \frac{2n+4}{2n^2+n}$$

for $n \in \mathbb{N}$. Then $\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = 0$. By D'Alembert rule, the series $\sum_{n=1}^{+\infty} u_n$ is absolutely convergent.

(v) The series $\sum_{n=1}^{+\infty} u_n$ is convergent, but absolutely divergent.

The series $\sum_{n=1}^{+\infty} u_n$ is an alternate series. Since $\frac{1}{5+\sqrt{\ln(n)}} > 0$ is decreasing and convergent to 0, the series $\sum_{n=1}^{+\infty} u_n$ is convergent by Leibniz criteria.

Note that $\ln(n) < n$ for $n \in \mathbb{N}$. Then

$$|u_n| = \frac{1}{5+\sqrt{\ln(n)}} > \frac{1}{5+\sqrt{n}}$$

for $n \in \mathbb{N}$. Since $\frac{1}{5+\sqrt{n}} \sim \frac{1}{\sqrt{n}}$ when n tends to infinity, and the Riemann series $\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n}}$ is divergent, we get that the series $\sum_{n=1}^{+\infty} \frac{1}{5+\sqrt{n}}$ is divergent. Hence, the series $\sum_{n=1}^{+\infty} u_n$ is absolutely divergent by comparison criteria.

(vi) The series $\sum_{n=1}^{+\infty} u_n$ is convergent, but absolutely divergent.

Note that $\sin(x) \in [-1,1]$ for $x \in \mathbb{R}$. For $n \geqslant 3$, we have

$$|u_n| = \frac{1}{n+2\sin(n^3)} \geqslant \frac{1}{n+2}.$$

The series $\sum_{n\geqslant 3} \frac{1}{n+2}$ is divergent. By comparison criteria, the series $\sum_{n\geqslant 3} |u_n|$ is divergent, *i.e.* the series $\sum_{n\in\mathbb{N}} u_n$ is absolutely divergent.

Let $v_n = u_{2n-1} + u_n$ for $n \in \mathbb{N}$. Note that $\lim_{n\to+\infty} u_n = 0$. By the technique of block summation, the series $\sum_{n\in\mathbb{N}} u_n$ and $\sum_{n\in\mathbb{N}} v_n$ simultaneously converge or diverge. We have

$$\begin{aligned} v_n = u_{2n-1} + u_{2n} &= \frac{-1}{2n-1+2\sin\left((2n-1)^3\right)} + \frac{1}{2n+2\sin\left((2n)^3\right)} \\ &= \frac{-1-2\sin\left((2n)^3\right)+2\sin\left((2n-1)^3\right)}{\left(2n-1+2\sin\left((2n-1)^3\right)\right)\left(2n+2\sin\left((2n)^3\right)\right)}. \end{aligned}$$

Then for $n \geqslant 2$,

$$\begin{aligned} |v_n| &\leqslant \frac{|-1| + |2\sin\left((2n)^3\right)| + \left|2\sin\left((2n-1)^3\right)\right|}{\left(|2n-1| - |2\sin\left((2n-1)^3\right)|\right)\left(|2n| - |2\sin\left((2n)^3\right)|\right)} \\ &\leqslant \frac{5}{(2n-3)(2n-2)}. \end{aligned}$$

Since $\frac{5}{(2n-3)(2n-2)} \sim \frac{5}{4n^2}$ when n tends to infinity, the series $\sum_{n\geqslant 2} \frac{5}{(2n-3)(2n-2)}$ converges. Then the series $\sum_{n\geqslant 2} |v_n|$ converges by comparison criteria, *i.e.* the series $\sum_{n\in\mathbb{N}} v_n$ is absolutely convergent. This implies that the series $\sum_{n\in\mathbb{N}} v_n$ is convergent. Hence, the series $\sum_{n\in\mathbb{N}} u_n$ converges.

2pt

4. Consider the sequence $(u_n)_{n\in\mathbb{N}}$ given by

$$u_n = \frac{(-1)^{n+1}}{\sqrt{n}}$$

for $n \in \mathbb{N}$.

(a) Define $v_k = u_{2k-1} + u_{2k}$ for $k \in \mathbb{N}$. Determine if the series

$$\sum_{k=1}^{+\infty} v_k$$

is convergent or divergent.

(b) Using the previous item, determine if the series

$$\sum_{n=1}^{+\infty} u_n$$

is convergent or divergent.

Solution.

(a) The series $\sum_{k=1}^{+\infty} v_k$ is convergent.

We have

$$v_k = u_{2k-1} + u_{2k} = \frac{1}{\sqrt{2k-1}} - \frac{1}{\sqrt{2k}} = \frac{\sqrt{2k} - \sqrt{2k-1}}{\sqrt{2k(2k-1)}}$$
$$= \frac{(\sqrt{2k} - \sqrt{2k-1})(\sqrt{2k} + \sqrt{2k-1})}{\sqrt{2k(2k-1)}(\sqrt{2k} + \sqrt{2k-1})} = \frac{1}{\sqrt{2k(2k-1)}(\sqrt{2k} + \sqrt{2k-1})}$$

for $k \in \mathbb{N}$.

Then $v_k > 0$ and

$$\lim_{k\to+\infty} \frac{v_k}{\frac{1}{4\sqrt{2}k^{3/2}}} = \lim_{k\to+\infty} \frac{4\sqrt{2}k^{3/2}}{\sqrt{2k(2k-1)}(\sqrt{2k} + \sqrt{2k-1})}$$
$$= \lim_{k\to+\infty} \frac{4\sqrt{2}}{\sqrt{4-\frac{2}{k}}\left(\sqrt{2} + \sqrt{2-\frac{1}{k}}\right)} = 1.$$

This implies that the general terms of two positive series $\sum_{k=1}^{+\infty} v_k$ and $\frac{1}{4\sqrt{2}} \sum_{k=1}^{+\infty} \frac{1}{k^{3/2}}$ are equivalent. Since the Riemann series $\sum_{k=1}^{+\infty} \frac{1}{k^{3/2}}$ converges, we obtain that the series $\sum_{k=1}^{+\infty} v_k$ is convergent.

(b) The series $\sum_{n=1}^{+\infty} u_n$ is convergent, since a summation by blocks of it gives the series $\sum_{k=1}^{+\infty} v_k$, which is convergent by the previous item.

4,5pt

5. Consider the sequence $(u_n)_{n\in\mathbb{N}}$ given by

$$u_n = \frac{e^{1/n}}{n^2}$$

for $n \in \mathbb{N}$.

(*a*) Show that the map $f:\mathbb{R}_{\geq 1}\to\mathbb{R}$ given by

$$f(x)=\frac{e^{1/x}}{x^2}$$

for $x\in\mathbb{R}_{\geq 1}$ is decreasing and compute

$$\lim_{x\to+\infty} f(x).$$

(*b*) Determine if the series $\sum_{n=1}^{+\infty} u_n$ is convergent or divergent.

(*c*) Show that

$$\sum_{n=p+1}^{q}\frac{e^{1/n}}{n^2}\leq\int_p^q\frac{e^{1/x}}{x^2}dx\leq\sum_{n=p}^{q-1}\frac{e^{1/n}}{n^2}$$

for all $p,q\in\mathbb{N}$ such that $p<q$.

(*d*) Given $N\in\mathbb{N}$, set $R_N=\sum_{n=N+1}^{+\infty} u_n$. Using the previous item, show that

$$e^{1/(N+1)}-1\leq R_N\leq e^{1/N}-1.$$

and from this prove that $R_N\sim e^{1/N}-1$ as $N\to+\infty$.

Solution.

(*a*) It is clear that f is differentiable, since it is obtained as the quotient with nonzero denominator and composition of differentiable functions. Moreover, we also have

$$f'(x)=-\frac{2x+1}{x^4}e^{1/x}<0$$

for all $x\in\mathbb{R}_{>1}$, which in turn implies that f is strictly decreasing. Finally, since $1\leq e^{1/x}\leq e$ for all $x\in\mathbb{R}_{\geq 1}$, we see immediately that

$$\lim_{x\to+\infty} f(x)=\lim_{x\to+\infty}\frac{e^{1/x}}{x^2}=0.$$

(*b*) Note that $u_n=f(n)$ for all $n\in\mathbb{N}$. It is clear that f is a nonnegative function and we have showed that f is decreasing with limit equal to zero at $+\infty$. Moreover,

$$\int_a^{+\infty} f(x)dx=\lim_{M\to+\infty}\Big[-e^{1/x}\Big]_a^M=e^{1/a}-\lim_{M\to+\infty}e^{1/M}=e^{1/a}-1, \tag{1}$$

for all $a\in\mathbb{R}_{\geq 1}$. Using the integral test and (1) for $a=1$, we conclude that $\sum_{n=1}^{+\infty} u_n$ converges.

(*c*) Since f is decreasing, we have that $f(n+1)\leq f(x)\leq f(n)$ for all $x\in[n,n+1]$ and $n\in\mathbb{N}$. In consequence,

$$\frac{e^{1/(n+1)}}{(n+1)^2}=\int_n^{n+1}\frac{e^{1/(n+1)}}{(n+1)^2}dx\leq\int_n^{n+1}\frac{e^{1/x}}{x^2}dx\leq\int_n^{n+1}\frac{e^{1/n}}{n^2}dx=\frac{e^{1/n}}{n^2},$$

for all $n \in \mathbb{N}$, which in turn implies that

$$\sum_{n=p+1}^{q} \frac{e^{1/n}}{n^2} = \sum_{n=p}^{q-1} \frac{e^{1/(n+1)}}{(n+1)^2} \leq \underbrace{\sum_{n=p}^{q-1} \int_n^{n+1} \frac{e^{1/x}}{x^2} dx}_{=\int_p^q \frac{e^{1/x}}{x^2} dx} \leq \sum_{n=p}^{q-1} \frac{e^{1/n}}{n^2}$$

for all $p, q \in \mathbb{N}$ such that $p < q$.

(d) Fixing $p = N$ and letting q go to $+\infty$ in the previous item, we have that

$$R_N = \sum_{n=N+1}^{+\infty} \frac{e^{1/(n+1)}}{(n+1)^2} \leq \underbrace{\int_N^{+\infty} \frac{e^{1/x}}{x^2} dx}_{=e^{1/N}-1} \leq \sum_{n=N}^{+\infty} \frac{e^{1/(n+1)}}{(n+1)^2} = R_{N-1},$$

for all $N \in \mathbb{N}$, where we used (1). The first inequality tells us that

$$R_N \leq e^{1/N} - 1,$$

for all $N \in \mathbb{N}$, whereas the second can be rewritten as

$$R_{N-1} \geq e^{1/N} - 1,$$

for all $N \in \mathbb{N}$. Replacing N by $N+1$, we have thus

$$R_N \geq e^{1/(N+1)} - 1,$$

for all $N \in \mathbb{N}$, as was to be shown. Putting together these inequalities we get

$$\frac{e^{1/(N+1)} - 1}{e^{1/N} - 1} \leq \frac{R_N}{e^{1/N} - 1} \leq 1$$

for all $N \in \mathbb{N}$, which implies that

$$\lim_{N \to +\infty} \frac{R_N}{e^{1/N} - 1} = 1,$$

since

$$\lim_{N \to +\infty} \frac{e^{1/(N+1)} - 1}{e^{1/N} - 1} = \lim_{x \to +\infty} \frac{e^{1/(x+1)} - 1}{e^{1/x} - 1} = \lim_{x \to +\infty} \frac{-\frac{e^{1/(x+1)}}{(x+1)^2}}{-\frac{e^{1/x}}{x^2}}$$
$$= \lim_{x \to +\infty} \frac{e^{1/(x+1)} x^2}{e^{1/x}(x+1)^2} = 1,$$

where we have used the Bernoulli-L'Hospital in the second equality. In particular, we conclude that $R_N \sim e^{1/N} - 1$ as $N \to +\infty$.