A projective resolution for the Fomin-Kirillov algebra $\mathrm{FK}(4)$

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Abstract

In this article we show that, given a quadratic algebra satisfying some assumptions, which we call having a **resolving datum**, one can construct a projective resolution of the trivial module which is obtained as iterated cones of Koszul complexes, and this projective resolution is minimal under some further assumptions. We observe that many examples of quadratic algebras studied so far have a resolving datum, and that the (minimal) projective resolutions constructed for all of them in the literature are an example of our construction. The second main result of the article is that the Fomin-Kirillov algebra $\mathrm{FK}(4)$ of index 4 has a resolving datum. As an application we compute the dimensions of the first cohomology groups of $\mathrm{FK}(4)$.

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1 Introduction

In their study of the Schubert calculus of flag manifolds, S. Fomin and A. Kirillov introduced a family of quadratic algebras over a field $\Bbbk$ of characteristic zero, now called the **Fomin-Kirillov algebras** $\mathrm{FK}(n)$, indexed by the positive integers $n \in \mathbb{N}$ (see [6,9,10]). In case the index n takes the values 3, 4 or 5, the Fomin-Kirillov algebras are also Nichols algebras (see [7,11]), which appear in the classification of finite dimensional pointed Hopf algebras (see [1]). Their (co)homological properties have gained some importance, in particular in relation to the conjecture by P. Etingof and V. Ostrik that claims that the Yoneda algebra of every finite dimensional Hopf algebra is finitely generated.

The explicit homological computations of a graded connected $\Bbbk$-algebra A typically involve the construction of a "small" projective resolution of the trivial module $\Bbbk$. The methods for constructing such projective resolution $P_\bullet$ are typically of "local nature", *i.e.* for $n \in \mathbb{N}$ the projective module P_n is obtained from the previous steps. This is the case for instance of the usual construction of the minimal projective resolution, but also up to some extent of the Anick resolution and variations thereof. The obvious drawback of these procedures is that it is in general rather hard to describe the generic module P_n of $P_\bullet$ or its differential $d_n : P_n \to P_{n-1}$. We also note that for a quadratic algebra A, if there is a choice of monomial order for the generators of A such that the corresponding Anick resolution is minimal then A is Koszul (see [12], Chapter 4, Thm. 3.1), which gives another motivation for a different method of constructing projective resolutions.

In this article, given a quadratic algebra A over $\Bbbk$ satisfying some assumptions, we give a method to construct a projective resolution $P_\bullet$ of the trivial module $\Bbbk$ that is of "global nature" (at least partially). By this we mean that the description of every module P_n of $P_\bullet$ as well as some components of the differential $P_n \to P_{n-1}$ are described in a straightforward manner by means of a quiver associated with A, which we call the **resolving quiver** (see Theorem 2.10). The method is however not completely satisfactory, since the description of the remaining pieces of the differentials is of local nature, and the projective resolution we get is not minimal in general, even though it is minimal in many interesting examples –for which the Anick resolution cannot be minimal, as observed above–. Despite these drawbacks, we remark that many examples of quadratic algebras studied in the literature so far have such a resolving quiver, and that the minimal projective resolutions constructed in the literature for all of them are a specific instance of our construction (see Propositions 2.5 and 2.6, Examples 2.7-2.9 and Remark 2.11), which makes us wonder why the method we present here was not noticed previously. In any case, the main motivation for introducing this machinery is due to the second main result of the article, which states that the Fomin-Kirillov algebra $\mathrm{FK}(4)$ of index 4 has a resolving quiver (see Theorem 3.5), so we can construct an explicit projective resolution of its trivial module. As an explicit application of these results, we have also computed the dimensions of the first cohomological components of $\mathrm{Ext}^{\bullet}_{\mathrm{FK}(4)}(\Bbbk, \Bbbk)$, using the projective resolution given by Theorem 2.10 for the resolving datum of $\mathrm{FK}(4)$ in Theorem 3.5 (see Table 3.1).

The structure of the article is as follows. In the first half of Section 2, Subsection 2.1, we recall the basic definitions and properties of quadratic algebras and quadratic modules, whereas in the second half, Subsection 2.2, we introduce the novel notion of resolving datum. After providing some examples of quadratic algebras having a resolving datum, we prove the first main result of this article, Theorem 2.10. In the first subsection of Section 3 we recall the definition and basic homological properties of the Fomin-Kirillov algebra $\mathrm{FK}(4)$ of index 4. We then present the second main result of the article, Theorem 3.5, which states that $\mathrm{FK}(4)$ has a resolving datum, consisting of quadratic modules $\{\Bbbk = M^0, M^1, M^2, M^3\}$. The proof of the theorem is based on several intermediate results that appear in Subsections 3.3 and 3.4, namely Lemmas 3.11, 3.17, 3.23 and 3.29. We also present in the appendix several auxiliary computations, most of which are obtained using GAP. Namely, in Appendix A.1 we give the GAP code for finding a basis of the Fomin-Kirillov algebra $\mathrm{FK}(4)$ whereas in Appendix A.2 we provide several products in the quadratic dual algebra $\mathrm{FK}(4)^!$, that are used in the sequel. In Appendix A.3 we provide the explicit GAP code to obtain a basis of the quadratic module M^1, whereas in Appendix A.4 we provide the GAP code to compute the homology of the Koszul complex of all the quadratic modules M^i over $\mathrm{FK}(4)$ for $i \in [\![0, 3]\!]$. Finally in Appendix A.5 we provide the GAP code to obtain a basis of $(M^2)^!$, and in Appendix A.6 some products describing the action of $\mathrm{FK}(4)^!$ on $(M^2)^!$. We believe these snippets of code could be useful for the reader interested in dealing with other quadratic algebras.

We will denote by $\mathbb{N}$ (resp., $\mathbb{N}_0$, $\mathbb{Z}$) the set of positive (resp., nonnegative, all) integers. Given $i \in \mathbb{Z}$, we will denote by $\mathbb{Z}_{\leqslant i}$ the set $\{m \in \mathbb{Z} | m \leqslant i\}$. Given $i, j \in \mathbb{Z}$ with $i \leqslant j$, we will denote by $[\![i, j]\!] = \{m \in \mathbb{Z} | i \leqslant m \leqslant j\}$ the integer interval, and we define $\chi_n = 0$ if n is an odd integer and $\chi_n = 1$ if n is an even integer. Moreover, given $r \in \mathbb{R}$, we set $\lfloor r \rfloor = \sup\{n \in \mathbb{Z} | n \leqslant r\}$. Moreover, in the following we will intensively use routines in GAP for many computations, using the GBNP package, so we refer the reader to [5].

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2 Quadratic algebras and modules

In the first subsection we recall the basic definitions of quadratic algebras and modules, as well as their Koszul complexes. In the second subsection we introduce a new property of quadratic algebras that allows to construct projective resolutions.

2.1 Basic definitions

All of the definitions and results in this subsection are classical and can be found in [12].

From now on, let $\Bbbk$ be a field. All vector spaces will be over $\Bbbk$, and all maps between vector spaces will be linear unless otherwise stated. Moreover, we will denote the usual tensor product of vector spaces simply by $\otimes$.

Recall that a unitary associative $\Bbbk$-algebra A is said to be **nonnegatively graded** if $A = \oplus_{n \in \mathbb{N}_0} A_n$ is a direct sum decomposition of vector spaces such that $A_n \cdot A_m \subseteq A_{n+m}$, for all $n, m \in \mathbb{N}_0$, and $1_A \in A_0$. The grading will be called **internal** or **Adams** to emphasize that it does not intervene in the Koszul sign rule. A is said to be **connected** if we have furthermore $A_0 = \Bbbk$, which we assume from now on. Let $A_{>0} = \oplus_{n \in \mathbb{N}} A_n$ and let V be a graded vector subspace of $A_{>0}$ such that the restriction of the canonical projection $A_{>0} \to A_{>0}/(A_{>0} \cdot A_{>0})$ to V is a bijection. We will assume for the rest of this section that V is finite-dimensional. We say in this case that A is a **finitely generated** algebra. Then, the canonical map $\mathbb{T}(V) \to A$ induced by the inclusion $V \subseteq A$ is surjective, where $\mathbb{T}(V) = \oplus_{n\in\mathbb{N}_0} V^{\otimes n}$ denotes the tensor algebra. We will usually write the product of $\mathbb{T}(V)$ by juxtaposition. A nonnegatively graded connected algebra A is said to be **quadratic** if $V = A_1$ and there is a subspace $R \subseteq V^{\otimes 2}$ such that the kernel of $\mathbb{T}(V) \to A$ is the two-sided ideal generated by R. By abuse of terminology, we will identify the quadratic algebra A with its presentation (V, R), where $A = \mathbb{T}(V)/(R)$.

Let V^* be the dual vector space of V and for every integer $n \geqslant 2$ define the pairing $\gamma_n : (V^*)^{\otimes n} \otimes V^{\otimes n} \to \Bbbk$ by $\gamma_n(f_1 \otimes \cdots \otimes f_n, v_1 \otimes \cdots \otimes v_n) = f_1(v_1) \dots f_n(v_n)$, for all $v_1, \dots, v_n \in V$ and $f_1, \dots, f_n \in V^*$. Set $R^\perp \subseteq V^* \otimes V^*$ to be the vector subspace orthogonal to R for γ_2, *i.e.*

$$R^\perp = \Big\{\alpha \in (V^*)^{\otimes 2} \mid \gamma_2(\alpha, r) = 0, \text{for all } r \in R\Big\}.$$

The **quadratic dual** $A^!$ of a quadratic algebra $A = \mathbb{T}(V)/(R)$ is the algebra given by $\mathbb{T}(V^*)/(R^\perp)$. The induced internal grading is denoted by $A^! = \oplus_{n\in\mathbb{N}_0} A^!_{-n}$. Note that $A^!_0 = \Bbbk$ and $A^!_{-1} = V^*$. Moreover, for any integer $n \geqslant 2$, the composition of the isomorphism $(V^*)^{\otimes n} \xrightarrow{\sim} (V^{\otimes n})^*$ induced by the pairing γ_n and the dual of the canonical inclusion $\cap_{i=0}^{n-2} V^{\otimes i} \otimes R \otimes V^{\otimes(n-i-2)} \to V^{\otimes n}$ induces a canonical isomorphism of vector spaces

$$A^!_{-n} \xrightarrow{\sim} \left(\bigcap_{i=0}^{n-2} V^{\otimes i} \otimes R \otimes V^{\otimes(n-i-2)}\right)^*. \tag{2.1}$$

Recall that the graded dual $(A^!)^\# = \oplus_{n\in\mathbb{N}_0} (A^!_{-n})^*$ is a graded bimodule over $A^!$ via $(a \cdot f \cdot b)(c) = f(bca)$, for all $a, b, c \in A^!$ and $f \in (A^!)^\#$. Note in particular that $v \cdot f \in (A^!_{-n})^*$, for all $f \in (A^!_{-n-1})^*$, $v \in V^*$ and $n \in \mathbb{N}_0$. Since $V^* \otimes V \simeq \mathrm{End}_\Bbbk(V)$, there is a unique element $\iota \in V^* \otimes V$ whose image under the previous isomorphism is the identity of V. It is easy to prove that, if $\{v_i\}_{i\in I}$ is a basis of V and $\{f_i\}_{i\in I}$ is the dual basis of V^*, then $\iota = \sum_{i\in I} f_i \otimes v_i$. For $n \in \mathbb{N}_0$ set $K_n(A) = (A^!_{-n})^* \otimes A$, provided with the regular (right) A-module structure, and $d_{n+1} : K_{n+1}(A) \to K_n(A)$ as the multiplication by ι on the left. Furthermore, let $\epsilon : K_0(A) \to \Bbbk$ be the canonical projection from A onto $A_0 = \Bbbk$. To reduce space, we will typically denote the composition $f \circ g$ of maps f and g simply by their juxtaposition fg. It is easy to see that $d_n d_{n+1} = 0$, for all $n \in \mathbb{N}$, and $\epsilon d_1 = 0$. The complex $(K_\bullet(A), d_\bullet)$ is called the **(right) Koszul complex of A**. As usual, we can consider the Koszul complex as a complex indexed by $\mathbb{Z}$, with $K_n(A) = 0$ for all $n \in \mathbb{Z}_{\leqslant -1}$, and $d_n = 0$ for all $n \in \mathbb{Z}_{\leqslant 0}$. Equivalently, if we use the composition of the canonical isomorphism $V^{\otimes n} \xrightarrow{\sim} (V^{\otimes n})^{**}$ and the dual of (2.1) for $A^!_{-n}$, then $d_{n+1} : K_{n+1}(A) \to K_n(A)$ identifies with the restriction of the map $\tilde{d}_{n+1} : V^{\otimes(n+1)} \otimes A \to V^{\otimes n} \otimes A$ determined by

$$(v_1 \otimes \cdots \otimes v_{n+1}) \otimes a \mapsto (v_1 \otimes \cdots \otimes v_n) \otimes v_{n+1}a,$$

for all $v_1, \dots, v_{n+1} \in V$, $a \in A$ and $n \in \mathbb{N}_0$.

The following result is immediate.

Fact 2.1. *Let A be a quadratic algebra. Then,* $\mathrm{Ker}(\epsilon) = \mathrm{Im}(d_1)$ *and* $\mathrm{Ker}(d_1) = \mathrm{Im}(d_2)$, *and in fact* $(K_\bullet(A), d_\bullet)$ *coincides with the minimal projective resolution of the trivial right A-module $\Bbbk$ in the category of bounded below graded right A-modules, up to homological degree* 2.

Recall that a quadratic algebra A is said to be **Koszul** if its Koszul complex is exact in positive homological degrees.

Let $M = \oplus_{n\in\mathbb{Z}} M_n$ be a graded right module over a quadratic algebra A such that $\dim(M_n)$ is finite for all $n \in \mathbb{Z}$. Given $j \in \mathbb{Z}$, we denote by $M(j)$ the same underlying module with shifted (internal) grading given by $M(j)_i = M_{j+i}$ for $i \in \mathbb{Z}$. We remark that a **morphism of graded right A-modules** $f : M \to N$ is a homogeneous A-linear map of degree zero. Moreover, for a nonzero graded module M over A, if there exist integers $s \leqslant t$ such that $\dim(M_n) = 0$ for all $n \in \mathbb{Z} \setminus [\![s, t]\!]$ and $\dim(M_s) \cdot \dim(M_t) \neq 0$, then we say that the **dimension vector** of M is $(\dim(M_s), \dots, \dim(M_t))$.

Let $M = \oplus_{n\in\mathbb{N}_0} M_n$ be a graded right module over the quadratic algebra $A = \mathbb{T}(V)/(R)$. We say that M is **quadratic** if the canonical map $\rho|_{M_0\otimes A} : M_0 \otimes A \to M$ given by the restriction of the action $\rho : M \otimes A \to A$ is surjective and the kernel of $\rho|_{M_0\otimes A}$ is the A-submodule generated by a vector subspace $R_M \subseteq M_0 \otimes V$, called the **space of relations** of M. We will typically denote M_0 by V_M, and call it the **space of generators** of M. We will assume for the rest of this section that V_M is finite-dimensional. Note that, by definition, a quadratic module is generated by a space of generators concentrated in degree zero. By abuse of notation we will usually denote a quadratic module by its presentation $(V_M, R_M)_A$, which we will also be denoted simply by (V_M, R_M) if A is clear from the context. Note that, if A is a quadratic algebra, then the trivial module $\Bbbk$ is quadratic with $V_{\Bbbk} = \Bbbk$ and $R_{\Bbbk} = V_{\Bbbk} \otimes V$.

Given a quadratic right module M with presentation $(V_M, R_M)_A$, we define its **quadratic dual** $M^{!_m}$ as the quotient of the graded right $A^!$-module $V_M^* \otimes A^!$ by the graded submodule generated by $R_M^\perp \subseteq V_M^* \otimes V^*$, where

$$R_M^\perp = \{\beta \in V_M^* \otimes V^* \mid \gamma_{2,M}(\beta, r) = 0 \text{ for all } r \in R_M\},$$

where $\gamma_{2,M} : V_M^* \otimes V^* \otimes V_M \otimes V \to \Bbbk$ is the map given by $\gamma_{2,M}(f \otimes g \otimes u \otimes v) = f(u)g(v)$ for $f \in V_M^*$, $g \in V^*$, $u \in V_M$ and $v \in V$. Note that $M^{!_m} = \oplus_{n \in \mathbb{N}_0} M^{!_m}_{-n}$ is a graded right module over $A^!$, where $M^{!_m}_{-n}$ sits in degree $-n$. As a consequence, the graded dual $(M^{!_m})^\# = \oplus_{n\in\mathbb{N}_0}(M^{!_m}_{-n})^*$ is a graded left module over $A^!$. On the other hand, it is easy to see that $\Bbbk^{!_m} = A^!$.

Let M be a quadratic right module M over the quadratic algebra A. For $n \in \mathbb{N}_0$, set $\mathrm{K}_n^{\mathrm{mod}}(M) = (M^{!_m}_{-n})^* \otimes A$, provided with the regular right A-module structure, and d_{n+1} : $\mathrm{K}_{n+1}^{\mathrm{mod}}(M) \to \mathrm{K}_n^{\mathrm{mod}}(M)$ as the multiplication by ι on the left. Furthermore, let $\epsilon : \mathrm{K}_0^{\mathrm{mod}}(M) \to M$ be the canonical morphism $\rho_{V_M \otimes A} : V_M \otimes A \to M$ onto M. It is easy to see that $d_n d_{n+1} = 0$, for all $n \in \mathbb{N}$, and $\rho_{V_M\otimes A} d_1 = 0$. The complex $(\mathrm{K}_\bullet^{\mathrm{mod}}(M), d_\bullet)$ is called the **(right) Koszul complex of M**. As usual, we can consider the Koszul complex as a complex indexed by $\mathbb{Z}$, with $\mathrm{K}_n^{\mathrm{mod}}(M) = 0$ for all $n \in \mathbb{Z}_{\leqslant -1}$, and $d_n = 0$ for all $n \in \mathbb{Z}_{\leqslant 0}$.

The following immediate result is the analogous of Fact 2.1 for right modules.

Fact 2.2. *Let M be a quadratic right module over a quadratic algebra A. Then,* $\mathrm{Ker}(\rho_{V_M\otimes A}) = \mathrm{Im}(d_1)$, *and in fact $(\mathrm{K}_\bullet^{\mathrm{mod}}(M), d_\bullet)$ coincides with the minimal projective resolution of M in the category of bounded below graded right A-modules, up to homological degree 1.*

Let M and N be two quadratic right modules over the quadratic algebra $A = \mathbb{T}(V)/(R)$, with presentations (V_M, R_M) and (V_N, R_N), respectively. Let us denote by $\hom_A(M, N)$ the vector space formed by all homogeneous morphisms $f : M \to N$ of right A-modules of degree zero, and by $\mathrm{Hom}((V_M, R_M), (V_N, R_N))$ the vector space formed by all linear morphisms g : $V_M \to V_N$ satisfying that $(g \otimes \mathrm{id}_V)(R_M) \subseteq R_N$. Then, it is clear that the map

$$\hom_A(M, N) \to \mathrm{Hom}\big((V_M, R_M), (V_N, R_N)\big)$$

sending f to its restriction $f|_{V_M} : V_M \to V_N$ is an isomorphism. This tells us that $f : M \to N$ is a monomorphism (resp., epimorphism) in the category of quadratic right A-modules with homogeneous morphisms of A-modules of degree zero if and only if $f|_{M_0} : M_0 \to N_0$ is injective (resp., surjective). In particular, a morphism of the category of quadratic right A-modules with homogeneous morphisms of A-modules of degree zero is an epimorphism if and only if it is a surjection.

Remark 2.3. *Assume the space of generators of the quadratic algebra A has nonzero dimension. Then, the category of quadratic right A-modules with homogeneous morphisms of A-modules of degree zero is not abelian, since the canonical projection $A \to \Bbbk$ is a monomorphism and an epimorphism but it is not an isomorphism. In particular, the example shows that monomorphisms of the category of quadratic right modules are not necessarily injective. For a less trivial example, consider $\Bbbk$ of characteristic different from 2, $A = \Bbbk\langle x, y\rangle/(xy - yx) = \Bbbk[x, y]$, $M = e.A$, $M' = (e_1.A \oplus e_2.A)/(e_1.x + e_2.x, e_1.y - e_2.y)$ and the morphism $f : M \to M'$ of right A-modules sending e to e_1. Then f is a non-injective monomorphism of quadratic right modules, since*

$$f(e.xy + e.yx) = (e_1.x + e_2.x).y + (e_1.y - e_2.y).x$$

vanishes, but $f|_{M_0}$ and $f|_{M_1}$ are injective.

Given $f \in \hom_A(M, N)$, define the homogeneous morphism $f^{!_m} : N^{!_m} \to M^{!_m}$ of right $A^!$-modules of degree zero whose restriction to V_N^* is precisely the dual $(f|_{V_M})^*$ of $f|_{V_M} : V_M \to V_N$. Since $((f|_{V_M})^* \otimes \mathrm{id}_{V^*})(R_N^\perp) \subseteq R_M^\perp$, the map $f^{!_m}$ is well defined. By taking the graded dual

$(f^{!_m})^{\#} : (M^{!_m})^{\#} \to (N^{!_m})^{\#}$ we obtain a homogeneous morphism of left $A^!$-modules of degree zero. We finally define the morphism

$$\mathrm{K}^{\mathrm{mod}}_\bullet(f) : \mathrm{K}^{\mathrm{mod}}_\bullet(M) \to \mathrm{K}^{\mathrm{mod}}_\bullet(N)$$

of complexes of right A-modules by $\mathrm{K}^{\mathrm{mod}}_\bullet(f) = (f^{!_m})^{\#} \otimes \mathrm{id}_A$. It is clear that $\mathrm{K}^{\mathrm{mod}}_\bullet(fg) = \mathrm{K}^{\mathrm{mod}}_\bullet(f) \circ \mathrm{K}^{\mathrm{mod}}_\bullet(g)$ and $\mathrm{K}^{\mathrm{mod}}_\bullet(\mathrm{id}_M) = \mathrm{id}_{\mathrm{K}^{\mathrm{mod}}_\bullet(M)}$, for $f \in \hom_A(M, N)$, $g \in \hom_A(N', M)$ and N' a quadratic right A-module.

Remark 2.4. *If f is injective, then $f|_{V_M}$ is also injective, which implies that its dual $(f|_{V_M})^*$ is surjective, so $f^{!_m}$ is surjective as well, which in turn implies that $(f^{!_m})^{\#}$ and $\mathrm{K}^{\mathrm{mod}}_\bullet(f)$ are injective.*

From now on, by (resp., graded, quadratic) module over a (resp., graded, quadratic) algebra A we will refer to (resp., graded, quadratic) right A-module, unless otherwise stated.

2.2 Resolving data on quadratic algebras

We introduce the following definition. A **resolving datum** on a quadratic algebra A is a finite set $\mathscr{M} = \{M^0, \dots, M^N\}$ of pairwise non-isomorphic quadratic (right) A-modules with $N \in \mathbb{N}_0$ such that $M^0 = \Bbbk$ is the trivial module and a map

$$\hbar : [\![0, N]\!]^2 \times \mathbb{N}^2 \to \mathbb{N}_0^2$$

such that

(A.1) $\hbar$ has finite support (*i.e.* there exists a finite set $S \subseteq [\![0, N]\!]^2 \times \mathbb{N}^2$ such that $\hbar(i, j, k, \ell) = (0, 0)$ for all $(i, j, k, \ell) \in ([\![0, N]\!]^2 \times \mathbb{N}^2) \setminus S$),

(A.2) there are short exact sequences of right A-modules

$$0 \to \bigoplus_{j=0}^{N} \bigoplus_{\ell \in \mathbb{N}} \Big(M^j(-\ell)\Big)^{\pi_1(\hbar(i,j,k,\ell))} \to \mathrm{H}_k(\mathrm{K}^{\mathrm{mod}}_\bullet(M^i)) \to \bigoplus_{j=0}^{N} \bigoplus_{\ell \in \mathbb{N}} \Big(M^j(-\ell)\Big)^{\pi_2(\hbar(i,j,k,\ell))} \to 0 \qquad (2.2)$$

with homogeneous morphisms of degree zero for all $(i, k) \in [\![0, N]\!] \times \mathbb{N}$, where $\pi_i : \mathbb{N}_0^2 \to \mathbb{N}_0$ is the canonical projection on the i-th component for $i \in \{1, 2\}$,

(A.3) If (2.2) splits for some $i_0 \in [\![0, N]\!]$ and $k_0 \in \mathbb{N}$, then $\pi_1(\hbar(i_0, j, k_0, \ell)) = 0$ for all $j \in [\![0, N]\!]$ and $\ell \in \mathbb{N}$.

Recall that a **quiver** is the datum of a set Q_0, called **set of vertices**, and a set Q_1, called **set of arrows**, together with maps $s, t : Q_1 \to Q_0$ called the **source** and **target** maps of the quiver. We say the quiver is **bigraded** if we further have a map $\mathrm{bideg} : Q_1 \to \mathbb{Z}^2$. We will denote the bidegree of an arrow α of Q_1 by $\mathrm{bideg}(\alpha) = (\mathrm{bideg}_1(\alpha), \mathrm{bideg}_2(\alpha)) \in \mathbb{Z}^2$. The **difference degree** of an arrow α is defined as $\mathrm{dfdeg}(\alpha) = \mathrm{bideg}_2(\alpha) - \mathrm{bideg}_1(\alpha) \in \mathbb{Z}$.

We also recall that, given a quiver with set of vertices Q_0 and set of arrows Q_1, a **path** of length $n \in \mathbb{N}_0$ is a vertex if $n = 0$, and a tuple $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$ in Q_1^n for $n \in \mathbb{N}$ such that $t(\alpha_i) = s(\alpha_{i+1})$ for all $i \in [\![1, n-1]\!]$. As usual, we define $s(e) = t(e) = e$ for any vertex e, $s(\alpha_1, \dots, \alpha_n) = s(\alpha_1)$ and $t(\alpha_1, \dots, \alpha_n) = t(\alpha_n)$ for every path $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$ of length $n \in \mathbb{N}$. Furthermore, if the quiver is bigraded, given a path $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$ of length $n \in \mathbb{N}$, we define its bidegree $\mathrm{bideg}(\bar{\alpha}) = (\mathrm{bideg}_1(\bar{\alpha}), \mathrm{bideg}_2(\bar{\alpha})) \in \mathbb{Z}^2$ by $(\sum_{i=1}^n \mathrm{bideg}_1(\alpha_i), \sum_{i=1}^n \mathrm{bideg}_2(\alpha_i))$. The bidegree of a path of length zero given by a vertex e is defined as $\mathrm{bideg}(e) = (\mathrm{bideg}_1(e), \mathrm{bideg}_2(e)) = (0, 0)$. The **difference degree** of a path $\bar{\alpha}$ is defined as $\mathrm{dfdeg}(\bar{\alpha}) = \mathrm{bideg}_2(\bar{\alpha}) - \mathrm{bideg}_1(\bar{\alpha}) \in \mathbb{Z}$.

Given a quadratic algebra together with a resolving datum as in the first paragraph of this subsection, we define the associated **resolving quiver** $\mathscr{R}\mathbb{Q}_A$ as the unique bigraded quiver with set of vertices $\{M^0, \dots, M^N\}$, and whose set of arrows of degree (d', d'') from M^i to M^j has cardinality $\pi_1(\hbar(i, j, d' - 1, d'')) + \pi_2(\hbar(i, j, d' - 1, d''))$. To be able to manipulate these arrows, assume we have chosen a fixed set $\mathscr{A}r'_{i,j,d',d''}$ of arrows of degree (d', d'') from M^i to M^j of cardinality $\pi_1(\hbar(i, j, d' - 1, d''))$ and another fixed set $\mathscr{A}r''_{i,j,d',d''}$ of arrows of degree (d', d'') from M^i to M^j of cardinality $\pi_2(\hbar(i, j, d' - 1, d''))$, such that $\mathscr{A}r'_{i,j,d',d''}$ and $\mathscr{A}r''_{i,j,d',d''}$ are disjoint. For every $i \in [\![0, N]\!]$ and $d' \in \mathbb{N}$, we also set a strict partial order on the set of all arrows α of $\mathscr{R}\mathbb{Q}_A$ such that $s(\alpha) = M^i$ and $\mathrm{bideg}_1(\alpha) = d'$ by setting precisely that every arrow of $\mathscr{A}r''_{i,j,d',d''}$ is strictly less than every arrow of $\mathscr{A}r'_{i,j',d',d'''}$ for all $j, j' \in [\![0, N]\!]$ and $d'', d''' \in \mathbb{N}$.

Note that this quiver is finite by (A.1). We will say that the resolving datum is **connected** if the associated resolving quiver is connected.

As we will see in Theorem 2.10, the resolving quiver $\mathscr{R}\mathbb{Q}_A$ contains some homological information of the algebra A. The first clues in this direction are given by the following results, the first of which is trivial.

Proposition 2.5. *A quadratic algebra A is Koszul if and only if the resolving quiver associated to a (equivalently, to every) connected resolving datum on A has no arrows.*

Proposition 2.6. *Let $p, q \geqslant 2$ be integers. A quadratic algebra A is (p, q)-Koszul (in the sense introduced by S. Brenner, M. Butler and A. King in [2]) if and only if it is finite dimensional with $\dim(A_p) \neq 0$ and $\dim(A_{p+1}) = 0$, the Koszul complex of A has finite length q and the resolving quiver associated to a (equivalently, to every) connected resolving datum on A has only one vertex and $\dim(A_p) \cdot \dim(A^!_q)$ arrows of bidegree $(q + 1, q + p)$.*

Proof. This is precisely Prop. 3.9 of [2]. □

We also have the following three (families of) examples of resolving quivers. They show that the notion of resolving quiver pervades many of the examples of quadratic algebras considered so far (see also Remark 2.11).

Example 2.7. *Let $m \geqslant 5$ be an integer. Set $V(m)$ as the vector space of dimension $3m$ generated by the set $\cup_{i=1}^{m+1} S_i$, where $S_1 = \{n\}$, $S_2 = \{p, q, r\}$, $S_3 = \{s, t, u\}$, $S_4 = \{v, w, x_1, y_1, z_1\}$, $S_i = \{x_{i-3}, y_{i-3}, z_{i-3}\}$ for $i \in [\![5, m-1]\!]$, $S_m = \{x_{m-3}, y_{m-3}\}$ and $S_{m+1} = \{x_{m-2}\}$. Let $R(m)$ be the vector subspace of dimension $3m + 4$ of $V(m)^{\otimes 2}$ generated by*

$$\begin{aligned}&\big\{np - nq, np - nr, ps - pt, qt - qu, rs - ru, sv - sw, tw - tx_1, uv - ux_1, vx_2, wx_2, sv - sy_1,\\ &tw - ty_1, ux_1 - uy_1, sz_1, tz_1, uz_1\big\} \cup \big\{x_i x_{i+1}, y_{i-1}x_i + z_{i-1}y_i \mid i \in [\![1, m-3]\!]\big\}\\ &\cup \big\{z_i z_{i+1} \mid i \in [\![1, m-5]\!]\big\},\end{aligned}$$

where we denote the tensor product $\otimes$ by simple juxtaposition. Then, let C_m be the quadratic algebra defined as $\mathbb{T}V(m)/(R(m))$. This algebra was defined in Section 2 of [3]. Let $\mathscr{M} = \{\Bbbk, M^1\}$ where M^1 is the standard right module C_m, and let $\hbar : \{0,1\}^2 \times \mathbb{N}^2 \to \mathbb{N}_0^2$ be the map given by $\hbar(0, 1, m-1, m+1) = (0, 1)$ and $\hbar(i, j, k, \ell) = (0, 0)$ if $(i, j, k, \ell) \neq (0, 1, m-1, m+1)$. Then, [3], Thm. 2.7, tells us that this gives us a connected resolving datum on C_m, whose associated resolving quiver is

$$\Bbbk \longrightarrow M^1$$

such that its arrow has bidegree $(m, m + 1)$.

Example 2.8. *Given $g \in \Bbbk$, let V be the vector space generated by the set $\{x, y, z\}$ and let $R(g)$ be the vector subspace of $V^{\otimes 2}$ generated by the set*

$$\{xy - yx, z^2, xz - zx - y^2 - gx^2\},$$

where we denote the tensor product $\otimes$ by simple juxtaposition. Define $A(g)$ as the quadratic algebra given by $\mathbb{T}V/(R(g))$. This is precisely the algebra $T(g, 0)$ of Lemma 5.2 of [4], where we interchanged the role of x and y, and we wrote z instead of w. Let $\mathscr{M} = \{\Bbbk, M^1\}$ where M^1 is the quadratic right module $v.A(g)/(v.z)$, and let $\hbar : \{0,1\}^2 \times \mathbb{N}^2 \to \mathbb{N}_0^2$ be the map given by $\hbar(0, 1, 3, 4) = (0, 1)$ and $\hbar(i, j, k, \ell) = (0, 0)$ if $(i, j, k, \ell) \neq (0, 1, 3, 4)$. Then, [4], Thm. 5.6, tells us that this gives us a connected resolving datum on $A(g)$, whose associated resolving quiver is

$$\Bbbk \longrightarrow M^1$$

such that its arrow has bidegree $(3, 4)$.

Example 2.9. *Consider the Fomin-Kirillov algebra $\mathrm{FK}(3)$ on three generators (see [8], Section 2.3). Let $\mathscr{M} = \{\Bbbk\}$ and let $\hbar : \{0\}^2 \times \mathbb{N}^2 \to \mathbb{N}_0^2$ be the map given by $\hbar(0, 0, 3, 6) = (0, 1)$ and $\hbar(i, j, k, \ell) = (0, 0)$ if $(i, j, k, \ell) \neq (0, 0, 3, 6)$. Then, [8], Prop. 3.1, tells us that this gives a resolving datum on $\mathrm{FK}(3)$ whose associated resolving quiver is*

$$\Bbbk \circlearrowleft$$

such that its arrow has bidegree $(4, 6)$.

From the resolving quiver associated to a resolving datum of the form given in the first paragraph of this subsection we can define the **set of paths** $\mathscr{P}\!a_{M^i}$ given as the set formed by all paths $\bar{\alpha}$ of the quiver $\mathscr{R}\mathbb{Q}_A$ such that $s(\bar{\alpha}) = M^i$. Moreover, we will define the following strict partial order on $\mathscr{P}\!a_{M^i}$ for every $i \in [\![0, N]\!]$ as follows. First, we set the vertex at M^i to be strictly greater than any other path of $\mathscr{P}\!a_{M^i}$. Given $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\bar{\beta} = (\beta_1, \dots, \beta_m)$ in $\mathscr{P}\!a_{M^i}$ with $n, m \in \mathbb{N}$, we say that $\bar{\alpha} < \bar{\beta}$ if $\alpha_j = \beta_j$ for all $j \in [\![1, j_0]\!]$ for some $j_0 \in [\![0, \min(n, m)]\!]$, and one of the following possibilities holds:

(O.1) $n, m > j_0$, $\mathrm{bideg}_1(\alpha_{j_0+1}) = \mathrm{bideg}_1(\beta_{j_0+1})$ and $\alpha_{j_0+1} < \beta_{j_0+1}$;

(O.2) $n, m > j_0$, $\mathrm{bideg}_1(\alpha_{j_0+1}) < \mathrm{bideg}_1(\beta_{j_0+1})$;

(O.3) $j_0 = m < n$.

It is clear that this defines a strict partial order on $\mathscr{P}\!a_{M^i}$.

We now give the first main result of this article, which gives a description of a projective resolution of every quadratic module M^i in a connected resolving datum $\{M^0, \dots, M^N\}$ of a quadratic algebra A.

Theorem 2.10. *Assume we have a connected resolving datum on a quadratic algebra A with set of quadratic modules $\mathscr{M} = \{M^0, \dots, M^N\}$ and whose resolving quiver is denoted by $\mathscr{R}\mathbb{Q}_A$. Then, there exists a projective resolution $P_\bullet^{M^i}$ of M^i in the category of bounded below graded right A-modules such that*

$$P_n^{M^i} = \bigoplus_{\substack{\bar{\alpha} \in \mathscr{P}\!a_{M^i},\\ \mathrm{bideg}_1(\bar{\alpha}) \leqslant n}} \bar{\alpha}.\mathrm{K}^{\mathrm{mod}}_{n-\mathrm{bideg}_1(\bar{\alpha})}\big(t(\bar{\alpha})\big)\big(-\mathrm{bideg}_2(\bar{\alpha})\big) \tag{2.3}$$

for all $n \in \mathbb{N}_0$ and $i \in [\![0, N]\!]$, where the symbol $\bar{\alpha}$ multiplying the Koszul complex on the left is only a formal symbol used as a simple bookkeeping device. Moreover, if

$$\mathrm{dfdeg}(\bar{\alpha}) \neq \mathrm{dfdeg}(\bar{\beta}) - 1 \tag{2.4}$$

for all $\bar{\alpha}, \bar{\beta} \in \mathscr{P}\!a_{M^i}$ such that $\bar{\alpha} < \bar{\beta}$ (e.g. if $\mathrm{dfdeg}(\alpha)$ is even for all arrows α of $\mathscr{R}\mathbb{Q}_A$), then the previous projective resolution is minimal.

Proof. We are going to use the following notation. Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a short exact sequence of right A-modules and let $P'_\bullet \to M'$ and $P''_\bullet \to M''$ be two projective resolutions (resp., up to homological degree $m \in \mathbb{N}$) with differentials $d'_\bullet$ and $d''_\bullet$, respectively. Then, we will note by $P_\bullet = P'_\bullet \hat{\oplus} P''_\bullet \to M$ a fixed projective resolution (resp., up to homological degree $m \in \mathbb{N}$) given by the Horseshoe lemma (see [14], Lemma 2.2.8). We recall that $P_n = P'_n \oplus P''_n$ for all $n \in \mathbb{N}_0$ (resp., for all $n \in [\![0, m]\!]$) with differential $d_\bullet$ satisfying that $d_\bullet|_{P'_\bullet} = d'_\bullet$ and $d_\bullet|_{P''_\bullet} = d''_\bullet + f_\bullet$ for some family $\{f_n : P''_n \to P'_{n-1} \mid n \in \mathbb{N}\}$ (resp., $\{f_n : P''_n \to P'_{n-1} \mid n \in [\![1, m]\!]\}$) of morphisms of A-modules.

Given $i \in [\![0, N]\!]$, let $m_i \in \mathbb{N}$ be the largest positive integer such that $\mathrm{H}_{m_i}(\mathrm{K}^{\mathrm{mod}}_\bullet(M^i)) \neq 0$ and $\mathrm{H}_k(\mathrm{K}^{\mathrm{mod}}_\bullet(M^i)) = 0$ for all integers $k > m_i$. If $\mathrm{H}_k(\mathrm{K}^{\mathrm{mod}}_\bullet(M^i)) = 0$ for all $k \in \mathbb{N}$, then we set $m_i = 0$ in this case.

We will denote by $d^i_{k+1} : \mathrm{K}^{\mathrm{mod}}_{k+1}(M^i) \to \mathrm{K}^{\mathrm{mod}}_k(M^i)$ the differential of the Koszul complex of M^i for $k \in \mathbb{N}_0$ and $i \in [\![0, N]\!]$. For every $i \in [\![0, N]\!]$, we will construct a projective resolution $P^i_\bullet$ of M^i. By Fact 2.2 we will assume that $P^i_n = \mathrm{K}^{\mathrm{mod}}_n(M^i)$ for $i \in [\![0, N]\!]$ and $n \in \{0, 1\}$. In particular, P^i_n coincides with (2.3) for all $i \in [\![0, N]\!]$ and $n \in \{0, 1\}$. We will in fact prove that P^i_n coincides with (2.3) for all $i \in [\![0, N]\!]$ and $n \in \mathbb{N}_0$ by induction on the homological degree n. If $m_i = 0$, we set $P^i_\bullet = \mathrm{K}^{\mathrm{mod}}_\bullet(M^i)$ for all $\bullet \in \mathbb{N}_0$. It is straightforward to see that the resolutions $P^i_\bullet$ and (2.3) coincide.

We will now construct $P^i_\bullet$ for all $\bullet \in \mathbb{N}_0$ for $i \in [\![0, N]\!]$ such that $m_i > 0$. Let $m \in \mathbb{N}$. Assume that we have defined P^i_n for all $i \in [\![0, N]\!]$ such that $m_i > 0$ and $n \in [\![0, m]\!]$ such that P^i_n coincides with (2.3) for all $n \in [\![0, m]\!]$. Using the Horseshoe lemma for (2.2), we get a projective resolution of $\mathrm{H}_k(\mathrm{K}^{\mathrm{mod}}_\bullet(M^i))$ of the form

$${}^mQ^{i,k}_\bullet = \Big(\bigoplus_{j=0}^{N} \bigoplus_{\ell \in \mathbb{N}} \Big(P^j_\bullet(-\ell)\Big)^{\pi_1(\hbar(i,j,k,\ell))}\Big) \hat{\oplus} \Big(\bigoplus_{j=0}^{N} \bigoplus_{\ell \in \mathbb{N}} \Big(P^j_\bullet(-\ell)\Big)^{\pi_2(\hbar(i,j,k,\ell))}\Big)$$

defined for homological degrees $\bullet \in [\![0, m]\!]$, $i \in [\![1, N]\!]$ and $k \in [\![1, m_i]\!]$. We will construct by induction on the index $k \in [\![0, m_i]\!]$ a family of complexes of right A-modules ${}^m R^{i,k}_\bullet$ for $\bullet \in [\![0, m+1]\!]$ such that ${}^m R^{i,k}_\bullet$ is a projective resolution of $\mathrm{Im}(d^i_{m_i-k+1})$ up to homological degree $m+1$. For $k = 0$, we set ${}^m R^{i,0}_\bullet$ as the complex of right A-modules given by $(\mathrm{K}^{\mathrm{mod}}_{\bullet+m_i+1}(M^i), d^i_{\bullet+m_i+1})_{\bullet \in \mathbb{N}_0}$. Note that ${}^m R^{i,0}_\bullet$ is a projective resolution of $\mathrm{Im}(d^i_{m_i+1})$ for $i \in [\![0, N]\!]$ such that $m_i > 0$, and it is independent of m. Assume now we have defined a complex of right A-modules ${}^m R^{i,k-1}_\bullet$ for some $k \in [\![1, m_i]\!]$ and $\bullet \in [\![0, m+1]\!]$ such that ${}^m R^{i,k-1}_\bullet$ is a projective resolution of $\mathrm{Im}(d^i_{m_i-k+2})$ up to homological degree $m+1$. Then, we define the complex of right A-modules ${}^m R^{i,k}_\bullet$ by

$$ {}^m R^{i,k}_0 = \mathrm{K}^{\mathrm{mod}}_{m_i-k+1}(M^i) \text{ and } {}^m R^{i,k}_\bullet = {}^m R^{i,k-1}_{\bullet-1} \hat{\oplus} {}^m Q^{i,m_i-k+1}_{\bullet-1} $$

for $\bullet \in [\![1, m+1]\!]$, the differential $d^{i,k}_\bullet$ for $\bullet \geqslant 2$ is induced by that of ${}^m R^{i,k-1}_{\bullet-1} \hat{\oplus} {}^m Q^{i,m_i-k+1}_{\bullet-1}$ and $d^{i,k}_1 : {}^m R^{i,k}_1 \to {}^m R^{i,k}_0$ is given as the composition of the augmentation ${}^m R^{i,k-1}_\bullet \hat{\oplus} {}^m Q^{i,m_i-k+1}_\bullet \to \mathrm{Ker}(d^i_{m_i-k+1})$ and the inclusion $\mathrm{Ker}(d_{m_i-k+1}) \hookrightarrow \mathrm{K}^{\mathrm{mod}}_{m_i-k+1}(M^i)$. Using the Horseshoe lemma for

$$ 0 \longrightarrow \mathrm{Im}(d^i_{m_i-k+2}) \longrightarrow \mathrm{Ker}(d^i_{m_i-k+1}) \longrightarrow \mathrm{H}_{m_i-k+1}\left(\mathrm{K}^{\mathrm{mod}}_\bullet(M^i)\right) \longrightarrow 0 $$

together with the projective resolutions ${}^m R^{i,k-1}_\bullet$ and ${}^m Q^{i,m_i-k+1}_\bullet$ for $\bullet \in [\![0, m]\!]$, we obtain that the complex ${}^m R^{i,k-1}_\bullet \hat{\oplus} {}^m Q^{i,m_i-k+1}_\bullet$ for $\bullet \in [\![0, m]\!]$ is a projective resolution of $\mathrm{Ker}(d^i_{m_i-k+1})$ up to homological degree m, and thus ${}^m R^{i,k}_\bullet$ for for $\bullet \in [\![0, m+1]\!]$ is a projective resolution of $\mathrm{Im}(d^i_{m_i-k+1})$ up to homological degree $m+1$, as was to be shown. In particular, ${}^m R^{i,m_i}_\bullet$ for $\bullet \in [\![0, m+1]\!]$ is a projective resolution of $\mathrm{Im}(d^i_1)$ up to homological degree $m+1$. Let ${}^m R^i_\bullet = \mathrm{K}^{\mathrm{mod}}_0(M^i)$ and ${}^m R^i_\bullet = {}^m R^{i,m_i}_{\bullet-1}$ for $\bullet \in [\![1, m+2]\!]$. Then ${}^m R^i_\bullet$ for $\bullet \in [\![0, m+2]\!]$ is a projective resolution of M^i up to homological degree $m+2$. A long but straightforward computation shows that ${}^m R^i_\bullet$ coincides with (2.3) for $\bullet \in [\![0, m+2]\!]$, and that we can take the complexes ${}^m R^i_\bullet$ and ${}^{m-1} R^i_\bullet$ to coincide up to homological degree $m+1$. Hence, if $i \in [\![0, N]\!]$ such that $m_i > 0$, we define the complex $P^i_\bullet$ to be equal to ${}^m R^i_\bullet$ up to homological degree $m+2$. Since this holds for every $m \in \mathbb{N}$, the first part of the theorem is proved.

To prove the last one, let us denote by $P^i_{n,\bar{\alpha}}$ the direct summand in (2.3) indexed by $\bar{\alpha} \in \mathcal{P}\alpha_{M^i}$. The construction of the projective resolution $P^i_\bullet$ given in the first part of the proof tells us that, given $\bar{\alpha}, \bar{\beta} \in \mathcal{P}\alpha_{M^i}$, if the component

$$ d^{\bar{\alpha},\bar{\beta}}_{n+1} \otimes_A \mathrm{id}_{\Bbbk} : P^i_{n+1,\bar{\alpha}} \otimes_A \Bbbk \to P^i_{n,\bar{\beta}} \otimes_A \Bbbk $$

of the differential of $P^i_\bullet \otimes_A \Bbbk$ is nonzero, then $\bar{\alpha} < \bar{\beta}$ and $\mathrm{dfdeg}(\bar{\alpha}) = \mathrm{dfdeg}(\bar{\beta}) - 1$. The minimality result then follows. ☐

Remark 2.11. *It is easy to check that conditions (2.4) are verified in the case of Proposition 2.6, as well as in Examples 2.7, 2.8 and 2.9, so the corresponding projective resolution (2.3) is minimal, coinciding with the resolutions constructed in those references. We also remark that the Anick resolution for the three previous examples cannot be minimal, regardless of the choice of the order on the generators of the algebras, as it is the case for any quadratic algebra that is not Koszul (see [12], Chapter 4, Thm. 3.1). This in particular implies that our resolution can be minimal even when Anick's resolution is not.*

Remark 2.12. *All the definitions and results of this section were done in the context of quadratic algebras. More generally, these definitions and results extend directly (by the same underlying arguments) to the case of N-homogeneous algebras for any integer $N \geq 2$. The main reason for restricting to the quadratic case (i.e. $N = 2$) is simply because we are not aware of any interesting example of N-homogeneous algebra with $N > 3$ that is not generalized Koszul.*

3 Resolving datum on $\mathrm{FK}(4)$

We will prove in this section the second main result of this article, namely that the Fomin-Kirillov algebra $\mathrm{FK}(4)$ of index 4 has a connected resolving datum (see Theorem 3.5). In consequence, combining this with Theorem 2.10 we obtain immediately a projective resolution of the trivial module in the category of bounded-below graded right $\mathrm{FK}(4)$-modules.

3.1 Generalities on the Fomin-Kirillov algebra $\mathrm{FK}(4)$ and its quadratic dual

From now on we assume that the field $\Bbbk$ has characteristic different from 2 and 3. For a set S, we denote by $\Bbbk S$ the $\Bbbk$-vector space spanned by all elements of S.

Let $\mathscr{I}$ be the set $\{(i,j) \in [\![1,4]\!]^2 \mid i < j\}$, $\mathscr{I}_1$ the set $\{(1,2),(1,3),(2,3)\}$ and $\mathscr{J}$ the set $\{(i,j) \in [\![1,4]\!]^2 \mid i \neq j\}$. We recall that the **Fomin-Kirillov algebra $\mathrm{FK}(4)$ of index 4** is the quadratic $\Bbbk$-algebra generated by the $\Bbbk$-vector space V spanned by $X = \{x_{i,j} \mid (i,j) \in \mathscr{I}\}$, modulo the ideal generated by the vector space $R \subseteq V^{\otimes 2}$ spanned by the following 17 elements

$$x_{1,2}^2, x_{1,3}^2, x_{2,3}^2, x_{1,4}^2, x_{2,4}^2, x_{3,4}^2, x_{1,2}x_{2,3} - x_{2,3}x_{1,3} - x_{1,3}x_{1,2}, x_{2,3}x_{1,2} - x_{1,2}x_{1,3} - x_{1,3}x_{2,3},$$
$$x_{1,2}x_{2,4} - x_{2,4}x_{1,4} - x_{1,4}x_{1,2}, x_{2,4}x_{1,2} - x_{1,2}x_{1,4} - x_{1,4}x_{2,4}, x_{1,3}x_{3,4} - x_{3,4}x_{1,4} - x_{1,4}x_{1,3},$$
$$x_{3,4}x_{1,3} - x_{1,3}x_{1,4} - x_{1,4}x_{3,4}, x_{2,3}x_{3,4} - x_{3,4}x_{2,4} - x_{2,4}x_{2,3}, x_{3,4}x_{2,3} - x_{2,3}x_{2,4} - x_{2,4}x_{3,4},$$
$$x_{1,2}x_{3,4} - x_{3,4}x_{1,2}, x_{1,3}x_{2,4} - x_{2,4}x_{1,3}, x_{1,4}x_{2,3} - x_{2,3}x_{1,4}.$$

To simplify, we will denote the Fomin-Kirillov algebra FK(4) of index 4 simply by A. Recall that the dimension of A is 576 and the Hilbert series of A is

$$[2]^2[3]^2[4]^2 = 1 + 6t + 19t^2 + 42t^3 + 71t^4 + 96t^5 + 106t^6 + 96t^7 + 71t^8 + 42t^9 + 19t^{10} + 6t^{11} + t^{12},$$

where $[n] = \sum_{i=0}^{n-1} t^i$, for $n \in \mathbb{N}$. Note that $A = \oplus_{m\in[\![0,12]\!]} A_m$, where A_m is the subspace of A concentrated in internal degree m. We refer the reader to [6, 11] for more information on Fomin-Kirillov algebras.

The previous Hilbert series can be reobtained using GAP code in Appendix A.1. If the free monoid generated by X is equipped with the homogeneous lexicographic order induced by the well order $x_{1,2} \prec x_{1,3} \prec x_{2,3} \prec x_{1,4} \prec x_{2,4} \prec x_{3,4}$ on X, then a Gröbner basis G_A of the ideal (R) in the algebra $\mathbb{T}(V)$ is given by the following 30 elements

$$x_{1,2}^2, x_{1,3}^2, x_{2,3}x_{1,2} - x_{1,3}x_{2,3} - x_{1,2}x_{1,3}, x_{2,3}x_{1,3} + x_{1,3}x_{1,2} - x_{1,2}x_{2,3}, x_{2,3}^2, x_{1,4}x_{2,3} - x_{2,3}x_{1,4},$$
$$x_{1,4}^2, x_{2,4}x_{1,2} - x_{1,4}x_{2,4} - x_{1,2}x_{1,4}, x_{2,4}x_{1,3} - x_{1,3}x_{2,4}, x_{2,4}x_{1,4} + x_{1,4}x_{1,2} - x_{1,2}x_{2,4}, x_{2,4}^2,$$
$$x_{3,4}x_{1,2} - x_{1,2}x_{3,4}, x_{3,4}x_{1,3} - x_{1,4}x_{3,4} - x_{1,3}x_{1,4}, x_{3,4}x_{2,3} - x_{2,4}x_{3,4} - x_{2,3}x_{2,4},$$
$$x_{3,4}x_{1,4} + x_{1,4}x_{1,3} - x_{1,3}x_{3,4}, x_{3,4}x_{2,4} + x_{2,4}x_{2,3} - x_{2,3}x_{3,4}, x_{3,4}^2, x_{1,3}x_{1,2}x_{1,3} + x_{1,2}x_{1,3}x_{1,2},$$
$$x_{1,4}x_{1,2}x_{1,4} + x_{1,2}x_{1,4}x_{1,2}, x_{1,4}x_{1,3}x_{1,2} - x_{1,4}x_{1,2}x_{2,3} + x_{2,3}x_{1,4}x_{1,3},$$
$$x_{1,4}x_{1,3}x_{2,3} + x_{1,4}x_{1,2}x_{1,3} - x_{2,3}x_{1,4}x_{1,2}, x_{1,4}x_{1,3}x_{1,4} + x_{1,3}x_{1,4}x_{1,3},$$
$$x_{2,4}x_{2,3}x_{1,4} + x_{1,4}x_{1,2}x_{2,3} - x_{1,2}x_{2,4}x_{2,3}, x_{2,4}x_{2,3}x_{2,4} + x_{2,3}x_{2,4}x_{2,3},$$
$$x_{1,4}x_{1,2}x_{1,3}x_{2,3} - x_{2,3}x_{1,4}x_{1,2}x_{2,3}, x_{1,4}x_{1,2}x_{1,3}x_{1,4} + x_{1,3}x_{1,4}x_{1,2}x_{1,3} + x_{1,2}x_{1,3}x_{1,4}x_{1,2},$$
$$x_{1,4}x_{1,2}x_{2,3}x_{1,4} + x_{1,2}x_{1,4}x_{1,2}x_{2,3}, x_{1,4}x_{1,2}x_{1,3}x_{1,2}x_{2,3} + x_{2,3}x_{1,4}x_{1,2}x_{1,3}x_{1,2},$$
$$x_{1,4}x_{1,2}x_{1,3}x_{1,2}x_{1,4}x_{1,2} - x_{1,3}x_{1,4}x_{1,2}x_{1,3}x_{1,2}x_{1,4},$$
$$x_{1,4}x_{1,2}x_{1,3}x_{1,2}x_{1,4}x_{1,3} - x_{1,2}x_{1,4}x_{1,2}x_{1,3}x_{1,2}x_{1,4},$$

which are obtained using the GAP code in Appendix A.4. The classes in A of the standard words of $\mathbb{T}(V)$ with respect to G_A thus form a homogeneous $\Bbbk$-basis $\mathscr{B}$ of A. We set $\mathscr{B}_m = \mathscr{B} \cap A_m$ for $m \in [\![0,12]\!]$.

We denote by $\{y_{i,j} = x_{i,j}^* \mid (i,j) \in \mathscr{I}\}$ the basis of V^* dual to the basis $X = \{x_{i,j} \mid (i,j) \in \mathscr{I}\}$ of V. Then, the space of relations $R^\perp \subseteq (V^*)^{\otimes 2}$ of the quadratic dual algebra $A^! = \mathbb{T}(V^*)/(R^\perp) = \oplus_{n\in\mathbb{N}_0} A^!_{-n}$ of A is spanned by the following 19 elements

$$y_{1,2}y_{2,3} + y_{2,3}y_{1,3}, y_{1,3}y_{2,3} + y_{2,3}y_{1,2}, y_{1,2}y_{2,3} + y_{1,3}y_{1,2}, y_{1,2}y_{1,3} + y_{2,3}y_{1,2},$$
$$y_{1,2}y_{2,4} + y_{2,4}y_{1,4}, y_{1,4}y_{2,4} + y_{2,4}y_{1,2}, y_{1,2}y_{2,4} + y_{1,4}y_{1,2}, y_{1,2}y_{1,4} + y_{2,4}y_{1,2},$$
$$y_{1,3}y_{3,4} + y_{3,4}y_{1,4}, y_{1,4}y_{3,4} + y_{3,4}y_{1,3}, y_{1,3}y_{3,4} + y_{1,4}y_{1,3}, y_{1,3}y_{1,4} + y_{3,4}y_{1,3},$$
$$y_{2,3}y_{3,4} + y_{3,4}y_{2,4}, y_{2,4}y_{3,4} + y_{3,4}y_{2,3}, y_{2,3}y_{3,4} + y_{2,4}y_{2,3}, y_{2,3}y_{2,4} + y_{3,4}y_{2,3},$$
$$y_{1,2}y_{3,4} + y_{3,4}y_{1,2}, y_{1,3}y_{2,4} + y_{2,4}y_{1,3}, y_{2,3}y_{1,4} + y_{1,4}y_{2,3}.$$

Using the GAP code in Appendix A.4, we get a Gröbner basis G_B of the ideal $(R^\perp)$ in $\mathbb{T}(V^*)$

given by the following 31 elements

$$\begin{aligned}
&y_{1,3}y_{1,2}+y_{1,2}y_{2,3},\ y_{1,3}y_{2,3}-y_{1,2}y_{1,3},\ y_{2,3}y_{1,2}+y_{1,3}y_{2,3},\ y_{2,3}y_{1,3}+y_{1,2}y_{2,3},\ y_{1,4}y_{1,2}+y_{1,2}y_{2,4},\\
&y_{1,4}y_{1,3}+y_{1,3}y_{3,4},\ y_{1,4}y_{2,3}+y_{2,3}y_{1,4},\ y_{1,4}y_{2,4}-y_{1,2}y_{1,4},\ y_{1,4}y_{3,4}-y_{1,3}y_{1,4},\ y_{2,4}y_{1,2}+y_{1,4}y_{2,4},\\
&y_{2,4}y_{1,3}+y_{1,3}y_{2,4},\ y_{2,4}y_{2,3}+y_{2,3}y_{3,4},\ y_{2,4}y_{1,4}+y_{1,2}y_{2,4},\ y_{2,4}y_{3,4}-y_{2,3}y_{2,4},\ y_{3,4}y_{1,2}+y_{1,2}y_{3,4},\\
&y_{3,4}y_{1,3}+y_{1,4}y_{3,4},\ y_{3,4}y_{2,3}+y_{2,4}y_{3,4},\ y_{3,4}y_{1,4}+y_{1,3}y_{3,4},\ y_{3,4}y_{2,4}+y_{2,3}y_{3,4},\ y_{1,2}y_{2,3}^2-y_{1,2}y_{1,3}^2,\\
&y_{1,2}y_{2,4}^2-y_{1,2}y_{1,4}^2,\ y_{1,3}y_{3,4}^2-y_{1,3}y_{1,4}^2,\ y_{2,3}y_{3,4}^2-y_{2,3}y_{2,4}^2,\ y_{1,2}y_{1,3}^3-y_{1,2}^3y_{1,3},\\
&y_{1,2}y_{1,3}y_{2,4}^2-y_{1,2}y_{1,3}y_{1,4}^2,\ y_{1,2}y_{2,3}y_{2,4}^2-y_{1,2}y_{2,3}y_{1,4}^2,\ y_{1,2}y_{1,4}^3-y_{1,2}^3y_{1,4},\ y_{1,3}y_{1,4}^3-y_{1,3}^3y_{1,4},\\
&y_{2,3}y_{2,4}^3-y_{2,3}^3y_{2,4},\ y_{1,2}y_{1,3}^2y_{2,4}^2-y_{1,2}y_{1,3}^2y_{1,4}^2,\ y_{1,2}y_{2,3}y_{1,4}^3-y_{1,2}^3y_{2,3}y_{1,4}.
\end{aligned}\tag{3.1}$$

Let $\mathscr{B}^!_0=\{1\}\subseteq \Bbbk$, let $\mathscr{B}^!_1=\{y_{i,j}\mid (i,j)\in\mathscr{I}\}\subseteq V^*$, let $\mathscr{B}^!_2\subseteq A^!_{-2}$ be the set formed by the following 17 elements

$$\begin{aligned}
&y_{1,2}^2,\ y_{1,2}y_{1,3},\ y_{1,2}y_{2,3},\ y_{1,2}y_{1,4},\ y_{1,2}y_{2,4},\ y_{1,2}y_{3,4},\ y_{1,3}^2,\ y_{1,3}y_{1,4},\ y_{1,3}y_{2,4},\ y_{1,3}y_{3,4},\ y_{2,3}^2,\ y_{2,3}y_{1,4},\\
&y_{2,3}y_{2,4},\ y_{2,3}y_{3,4},\ y_{1,4}^2,\ y_{2,4}^2,\ y_{3,4}^2,
\end{aligned}$$

let $\mathscr{B}^!_3\subseteq A^!_{-3}$ be the set formed by the following 30 elements

$$\begin{aligned}
&y_{1,2}^3,\ y_{1,2}^2y_{1,3},\ y_{1,2}^2y_{2,3},\ y_{1,2}^2y_{1,4},\ y_{1,2}^2y_{2,4},\ y_{1,2}^2y_{3,4},\ y_{1,2}y_{1,3}^2,\ y_{1,2}y_{1,3}y_{1,4},\ y_{1,2}y_{1,3}y_{2,4},\\
&y_{1,2}y_{1,3}y_{3,4},\ y_{1,2}y_{2,3}y_{1,4},\ y_{1,2}y_{2,3}y_{2,4},\ y_{1,2}y_{2,3}y_{3,4},\ y_{1,2}y_{1,4}^2,\ y_{1,2}y_{3,4}^2,\ y_{1,3}^3,\ y_{1,3}^2y_{1,4},\ y_{1,3}^2y_{2,4},\\
&y_{1,3}^2y_{3,4},\ y_{1,3}y_{1,4}^2,\ y_{1,3}y_{2,4}^2,\ y_{2,3}^3,\ y_{2,3}^2y_{1,4},\ y_{2,3}^2y_{2,4},\ y_{2,3}^2y_{3,4},\ y_{2,3}y_{1,4}^2,\ y_{2,3}y_{2,4}^2,\ y_{1,4}^3,\ y_{2,4}^3,\ y_{3,4}^3,
\end{aligned}$$

and let $\mathscr{B}^!_4\subseteq A^!_{-4}$ be the set formed by the following 38 elements

$$\begin{aligned}
&y_{1,2}^4,\ y_{1,2}^3y_{1,3},\ y_{1,2}^3y_{2,3},\ y_{1,2}^3y_{1,4},\ y_{1,2}^3y_{2,4},\ y_{1,2}^3y_{3,4},\ y_{1,2}^2y_{1,3}^2,\ y_{1,2}^2y_{1,3}y_{1,4},\ y_{1,2}^2y_{1,3}y_{2,4},\\
&y_{1,2}^2y_{1,3}y_{3,4},\ y_{1,2}^2y_{2,3}y_{1,4},\ y_{1,2}^2y_{2,3}y_{2,4},\ y_{1,2}^2y_{2,3}y_{3,4},\ y_{1,2}^2y_{1,4}^2,\ y_{1,2}^2y_{3,4}^2,\ y_{1,2}y_{1,3}^2y_{1,4},\ y_{1,2}y_{1,3}^2y_{2,4},\\
&y_{1,2}y_{1,3}^2y_{3,4},\ y_{1,2}y_{1,3}y_{1,4}^2,\ y_{1,2}y_{2,3}y_{1,4}^2,\ y_{1,2}y_{3,4}^3,\ y_{1,3}^4,\ y_{1,3}^3y_{1,4},\ y_{1,3}^3y_{2,4},\ y_{1,3}^3y_{3,4},\ y_{1,3}^2y_{1,4}^2,\\
&y_{1,3}^2y_{2,4}^2,\ y_{1,3}y_{2,4}^3,\ y_{2,3}^4,\ y_{2,3}^3y_{1,4},\ y_{2,3}^3y_{2,4},\ y_{2,3}^3y_{3,4},\ y_{2,3}^2y_{1,4}^2,\ y_{2,3}^2y_{2,4}^2,\ y_{2,3}y_{1,4}^3,\ y_{1,4}^4,\ y_{2,4}^4,\ y_{3,4}^4.
\end{aligned}$$

Moreover, for every integer $n\geqslant 5$, define $\mathscr{B}^!_n=\mathscr{U}^!_n\cup\mathscr{C}^!_n$, where the set $\mathscr{U}^!_n\subseteq A^!_{-n}$ consists of the following 24 elements

$$\begin{aligned}
&y_{1,2}^{n-1}y_{1,3}, y_{1,2}^{n-1}y_{2,3}, y_{1,2}^{n-1}y_{1,4}, y_{1,2}^{n-1}y_{2,4}, y_{1,2}^{n-2}y_{1,3}^2, y_{1,2}^{n-2}y_{1,3}y_{1,4}, y_{1,2}^{n-2}y_{1,3}y_{2,4}, y_{1,2}^{n-2}y_{1,3}y_{3,4},\\
&y_{1,2}^{n-2}y_{2,3}y_{1,4}, y_{1,2}^{n-2}y_{2,3}y_{2,4}, y_{1,2}^{n-2}y_{2,3}y_{3,4}, y_{1,2}^{n-2}y_{1,4}^2, y_{1,2}^{n-3}y_{1,3}^2y_{1,4}, y_{1,2}^{n-3}y_{1,3}^2y_{2,4}, y_{1,2}^{n-3}y_{1,3}^2y_{3,4},\\
&y_{1,2}^{n-3}y_{1,3}y_{1,4}^2, y_{1,2}^{n-3}y_{2,3}y_{1,4}^2, y_{1,2}^{n-4}y_{1,3}^2y_{1,4}^2, y_{1,3}^{n-1}y_{1,4}, y_{1,3}^{n-1}y_{3,4}, y_{1,3}^{n-2}y_{1,4}^2, y_{2,3}^{n-1}y_{2,4}, y_{2,3}^{n-1}y_{3,4}, y_{2,3}^{n-2}y_{2,4}^2,
\end{aligned}$$

and $\mathscr{C}^!_n\subseteq A^!_{-n}$ is the set of $3(n+1)$ elements given by

$$\mathscr{C}^!_n=\big\{y_{1,2}^{n-r}y_{3,4}^r,\ y_{1,3}^{n-r}y_{2,4}^r,\ y_{2,3}^{n-r}y_{1,4}^r\mid r\in[\![0,n]\!]\big\}.$$

The following result is proved directly from the explicit description of the Gröbner basis G_B given in (3.1) for the ideal $(R^\perp)\subseteq\mathbb{T}(V^*)$.

Fact 3.1. *The set $\mathscr{B}^!_n$ is a basis of $A^!_{-n}$ for $n\in\mathbb{N}_0$, consisting of standard words with respect to the Gröbner basis G_B. In consequence, $\#(\mathscr{B}^!_n)=3n+27$ for $n\geqslant 5$, and the Hilbert series $h(t)$ of $A^!$ is given by*

$$h(t)=1+6t+17t^2+30t^3+38t^4+\sum_{n=5}^{\infty}(3n+27)t^n=\frac{1+4t+6t^2+2t^3-5t^4-4t^5-t^6}{(t-1)^2}.$$

The following result describes several identities expressing products of the generators of the quadratic dual algebra $A^!$ in terms of the basis $\mathscr{B}^!=\cup_{n\in\mathbb{N}_0}\mathscr{B}^!_n$. The proof is a straightforward but rather lengthy verification, which we leave to the reader.

Fact 3.2. *We have the following identities*

$$y_{i,j}^{n-r}y_{k,l}^{r}y_{i,j} = (-1)^{r}y_{i,j}^{n-r+1}y_{k,l}^{r}, \quad y_{i,j}^{n-r}y_{k,l}^{r}y_{k,l} = y_{i,j}^{n-r}y_{k,l}^{r+1} \tag{3.2}$$

and

$$y_{i,j}y_{i,j}^{n-r}y_{k,l}^{r} = y_{i,j}^{n-r+1}y_{k,l}^{r}, \quad y_{k,l}y_{i,j}^{n-r}y_{k,l}^{r} = (-1)^{n-r}y_{i,j}^{n-r}y_{k,l}^{r+1}$$

in $A^!$, for all integers $n \geqslant 2$, $r \in [\![1, n-1]\!]$, $(i,j) \in \mathscr{I}_1$, $(k,l) \in \mathscr{I}$ with $\#\{i,j,k,l\} = 4$. Moreover, we also have the identities

$$\begin{aligned}
y_{1,2}^{n-r}y_{3,4}^{r}y_{1,3} &= \chi_r y_{1,2}^{n-2}y_{1,3}y_{1,4}^{2} - \chi_{r+1}y_{1,2}^{n-1}y_{1,3}y_{1,4},\\
y_{1,2}^{n-r}y_{3,4}^{r}y_{2,3} &= \chi_r y_{1,2}^{n-2}y_{2,3}y_{1,4}^{2} - \chi_{r+1}y_{1,2}^{n-1}y_{2,3}y_{2,4},\\
y_{1,2}^{n-r}y_{3,4}^{r}y_{1,4} &= \chi_r y_{1,2}^{n-2}y_{1,3}^{2}y_{1,4} - \chi_{r+1}y_{1,2}^{n-1}y_{1,3}y_{3,4},\\
y_{1,2}^{n-r}y_{3,4}^{r}y_{2,4} &= \chi_r y_{1,2}^{n-2}y_{1,3}^{2}y_{2,4} - \chi_{r+1}y_{1,2}^{n-1}y_{2,3}y_{3,4},\\
y_{1,3}^{n-r}y_{2,4}^{r}y_{1,2} &= \chi_n\chi_r y_{1,2}^{n-3}y_{1,3}^{2}y_{1,4}^{2} - \chi_{n+1}\chi_r y_{1,2}^{n-2}y_{2,3}y_{1,4}^{2} + \chi_n\chi_{r+1}y_{1,2}^{n-1}y_{2,3}y_{1,4}\\
&\quad - \chi_{n+1}\chi_{r+1}y_{1,2}^{n-2}y_{1,3}^{2}y_{1,4},\\
y_{1,3}^{n-r}y_{2,4}^{r}y_{2,3} &= \chi_n\chi_r y_{1,2}^{n-2}y_{2,3}y_{1,4}^{2} + \chi_{n+1}\chi_r y_{1,2}^{n-2}y_{1,3}y_{1,4}^{2} - \chi_n\chi_{r+1}y_{1,2}^{n-1}y_{1,3}y_{3,4}\\
&\quad - \chi_{n+1}\chi_{r+1}y_{1,2}^{n-1}y_{2,3}y_{3,4},\\
y_{1,3}^{n-r}y_{2,4}^{r}y_{1,4} &= \chi_n\chi_r y_{1,2}^{n-2}y_{1,3}^{2}y_{1,4} + \chi_{n+1}\chi_r y_{1,2}^{n-1}y_{1,3}y_{1,4} + \chi_n\chi_{r+1}y_{1,2}^{n-1}y_{2,3}y_{2,4}\\
&\quad - \chi_{n+1}\chi_{r+1}y_{1,2}^{n-2}y_{1,3}^{2}y_{2,4},\\
y_{1,3}^{n-r}y_{2,4}^{r}y_{3,4} &= \chi_n\chi_r y_{1,2}^{n-2}y_{1,3}^{2}y_{3,4} + \chi_{n+1}\chi_r y_{1,2}^{n-1}y_{1,3}y_{3,4} + \chi_n\chi_{r+1}y_{1,2}^{n-1}y_{1,3}y_{2,4}\\
&\quad + \chi_{n+1}\chi_{r+1}y_{1,2}^{n-1}y_{2,3}y_{2,4},\\
y_{2,3}^{n-r}y_{1,4}^{r}y_{1,2} &= \chi_n\chi_r y_{1,2}^{n-3}y_{1,3}^{2}y_{1,4}^{2} - \chi_{n+1}\chi_r y_{1,2}^{n-2}y_{1,3}y_{1,4}^{2} + \chi_n\chi_{r+1}y_{1,2}^{n-1}y_{1,3}y_{2,4}\\
&\quad - \chi_{n+1}\chi_{r+1}y_{1,2}^{n-2}y_{1,3}^{2}y_{2,4},\\
y_{2,3}^{n-r}y_{1,4}^{r}y_{1,3} &= \chi_n\chi_r y_{1,2}^{n-2}y_{1,3}y_{1,4}^{2} - \chi_{n+1}\chi_r y_{1,2}^{n-2}y_{2,3}y_{1,4}^{2} + \chi_n\chi_{r+1}y_{1,2}^{n-1}y_{2,3}y_{3,4}\\
&\quad - \chi_{n+1}\chi_{r+1}y_{1,2}^{n-1}y_{1,3}y_{3,4},\\
y_{2,3}^{n-r}y_{1,4}^{r}y_{2,4} &= \chi_n\chi_r y_{1,2}^{n-2}y_{1,3}^{2}y_{2,4} + \chi_{n+1}\chi_r y_{1,2}^{n-1}y_{2,3}y_{2,4} - \chi_n\chi_{r+1}y_{1,2}^{n-1}y_{1,3}y_{1,4}\\
&\quad + \chi_{n+1}\chi_{r+1}y_{1,2}^{n-2}y_{1,3}^{2}y_{1,4},\\
y_{2,3}^{n-r}y_{1,4}^{r}y_{3,4} &= \chi_n\chi_r y_{1,2}^{n-2}y_{1,3}^{2}y_{3,4} + \chi_{n+1}\chi_r y_{1,2}^{n-1}y_{2,3}y_{3,4} - \chi_n\chi_{r+1}y_{1,2}^{n-1}y_{2,3}y_{1,4}\\
&\quad + \chi_{n+1}\chi_{r+1}y_{1,2}^{n-1}y_{1,3}y_{1,4},
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
y_{1,3}y_{1,2}^{n-r}y_{3,4}^{r} &= \chi_n\chi_r y_{1,2}^{n-2}y_{1,3}y_{1,4}^{2} - \chi_n\chi_{r+1}y_{1,2}^{n-1}y_{2,3}y_{3,4} - \chi_{n+1}\chi_r y_{1,2}^{n-2}y_{2,3}y_{1,4}^{2}\\
&\quad + \chi_{n+1}\chi_{r+1}y_{1,2}^{n-1}y_{1,3}y_{3,4},\\
y_{2,4}y_{1,2}^{n-r}y_{3,4}^{r} &= \chi_n\chi_r y_{1,2}^{n-2}y_{1,3}^{2}y_{2,4} - \chi_n\chi_{r+1}y_{1,2}^{n-1}y_{1,3}y_{1,4} - \chi_{n+1}\chi_r y_{1,2}^{n-2}y_{1,3}^{2}y_{1,4}\\
&\quad + \chi_{n+1}\chi_{r+1}y_{1,2}^{n-1}y_{2,3}y_{2,4},\\
y_{2,3}y_{1,2}^{n-r}y_{3,4}^{r} &= \chi_n\chi_r y_{1,2}^{n-2}y_{2,3}y_{1,4}^{2} - \chi_n\chi_{r+1}y_{1,2}^{n-1}y_{1,3}y_{3,4} - \chi_{n+1}\chi_r y_{1,2}^{n-2}y_{1,3}y_{1,4}^{2}\\
&\quad + \chi_{n+1}\chi_{r+1}y_{1,2}^{n-1}y_{2,3}y_{3,4},\\
y_{1,4}y_{1,2}^{n-r}y_{3,4}^{r} &= \chi_n\chi_r y_{1,2}^{n-2}y_{1,3}^{2}y_{1,4} - \chi_n\chi_{r+1}y_{1,2}^{n-1}y_{2,3}y_{2,4} - \chi_{n+1}\chi_r y_{1,2}^{n-2}y_{1,3}^{2}y_{2,4}\\
&\quad + \chi_{n+1}\chi_{r+1}y_{1,2}^{n-1}y_{1,3}y_{1,4},\\
y_{1,2}y_{1,3}^{n-r}y_{2,4}^{r} &= \chi_n\chi_r y_{1,2}^{n-3}y_{1,3}^{2}y_{1,4}^{2} + \chi_n\chi_{r+1}y_{1,2}^{n-1}y_{1,3}y_{2,4} + \chi_{n+1}\chi_r y_{1,2}^{n-2}y_{1,3}y_{1,4}^{2}\\
&\quad + \chi_{n+1}\chi_{r+1}y_{1,2}^{n-2}y_{1,3}^{2}y_{2,4},\\
y_{3,4}y_{1,3}^{n-r}y_{2,4}^{r} &= \chi_n\chi_r y_{1,2}^{n-2}y_{1,3}^{2}y_{3,4} + \chi_n\chi_{r+1}y_{1,2}^{n-1}y_{2,3}y_{1,4} - \chi_{n+1}\chi_r y_{1,2}^{n-1}y_{1,3}y_{1,4}\\
&\quad - \chi_{n+1}\chi_{r+1}y_{1,2}^{n-1}y_{2,3}y_{3,4},\\
y_{2,3}y_{1,3}^{n-r}y_{2,4}^{r} &= (-1)^{n}\chi_r y_{1,2}^{n-2}y_{2,3}y_{1,4}^{2} + (-1)^{n+1}\chi_{r+1}y_{1,2}^{n-1}y_{2,3}y_{2,4},
\end{aligned} \tag{3.4}$$

$$
\begin{aligned}
y_{1,4}y_{1,3}^{n-r}y_{2,4}^{r} &= \chi_{n-r}y_{1,2}^{n-2}y_{1,3}^{2}y_{1,4} + (-1)^{n}\chi_{n-r+1}y_{1,2}^{n-1}y_{1,3}y_{3,4},\\
y_{1,2}y_{2,3}^{n-r}y_{1,4}^{r} &= \chi_{n}\chi_{r}y_{1,2}^{n-3}y_{1,3}^{2}y_{1,4}^{2} + \chi_{n}\chi_{r+1}y_{1,2}^{n-1}y_{2,3}y_{1,4} + \chi_{n+1}\chi_{r}y_{1,2}^{n-2}y_{2,3}y_{1,4}^{2}\\
&\quad + \chi_{n+1}\chi_{r+1}y_{1,2}^{n-2}y_{1,3}^{2}y_{1,4},\\
y_{3,4}y_{2,3}^{n-r}y_{1,4}^{r} &= \chi_{n}\chi_{r}y_{1,2}^{n-2}y_{1,3}^{2}y_{3,4} - \chi_{n}\chi_{r+1}y_{1,2}^{n-1}y_{1,3}y_{2,4} - \chi_{n+1}\chi_{r}y_{1,2}^{n-1}y_{2,3}y_{2,4}\\
&\quad - \chi_{n+1}\chi_{r+1}y_{1,2}^{n-1}y_{1,3}y_{3,4},\\
y_{1,3}y_{2,3}^{n-r}y_{1,4}^{r} &= \chi_{r}y_{1,2}^{n-2}y_{1,3}y_{1,4}^{2} + \chi_{r+1}y_{1,2}^{n-1}y_{1,3}y_{1,4},\\
y_{2,4}y_{2,3}^{n-r}y_{1,4}^{r} &= (-1)^{n}\chi_{n-r}y_{1,2}^{n-2}y_{1,3}^{2}y_{2,4} - \chi_{n-r+1}y_{1,2}^{n-1}y_{2,3}y_{3,4},
\end{aligned}
$$

together with

$$
\begin{aligned}
&y_{1,3}y_{1,2}^{n} = \chi_{n}y_{1,2}^{n}y_{1,3} - \chi_{n+1}y_{1,2}^{n}y_{2,3}, && y_{1,3}y_{3,4}^{n} = \chi_{n}y_{1,3}^{n-1}y_{1,4}^{2} + \chi_{n+1}y_{1,3}^{n}y_{3,4},\\
&y_{2,4}y_{1,2}^{n} = \chi_{n}y_{1,2}^{n}y_{2,4} - \chi_{n+1}y_{1,2}^{n}y_{1,4}, && y_{2,4}y_{3,4}^{n} = y_{2,3}^{n}y_{2,4},\\
&y_{2,3}y_{1,2}^{n} = \chi_{n}y_{1,2}^{n}y_{2,3} - \chi_{n+1}y_{1,2}^{n}y_{1,3}, && y_{2,3}y_{3,4}^{n} = \chi_{n}y_{2,3}^{n-1}y_{2,4}^{2} + \chi_{n+1}y_{2,3}^{n}y_{3,4},\\
&y_{1,4}y_{1,2}^{n} = \chi_{n}y_{1,2}^{n}y_{1,4} - \chi_{n+1}y_{1,2}^{n}y_{2,4}, && y_{1,4}y_{3,4}^{n} = y_{1,3}^{n}y_{1,4},\\
&y_{1,2}y_{1,3}^{n} = \chi_{n}y_{1,2}^{n-1}y_{1,3}^{2} + \chi_{n+1}y_{1,2}^{n}y_{1,3}, && y_{1,2}y_{2,4}^{n} = \chi_{n}y_{1,2}^{n-1}y_{1,4}^{2} + \chi_{n+1}y_{1,2}^{n}y_{2,4},\\
&y_{3,4}y_{1,3}^{n} = \chi_{n}y_{1,3}^{n}y_{3,4} - \chi_{n+1}y_{1,3}^{n}y_{1,4}, && y_{3,4}y_{2,4}^{n} = (-1)^{n}y_{2,3}^{n}y_{3,4},\\
&y_{2,3}y_{1,3}^{n} = (-1)^{n}y_{1,2}^{n}y_{2,3}, && y_{2,3}y_{2,4}^{n} = \chi_{n}y_{2,3}^{n-1}y_{2,4}^{2} + \chi_{n+1}y_{2,3}^{n}y_{2,4},\\
&y_{1,4}y_{1,3}^{n} = \chi_{n}y_{1,3}^{n}y_{1,4} - \chi_{n+1}y_{1,3}^{n}y_{3,4}, && y_{1,4}y_{2,4}^{n} = y_{1,2}^{n}y_{1,4},\\
&y_{1,2}y_{2,3}^{n} = \chi_{n}y_{1,2}^{n-1}y_{1,3}^{2} + \chi_{n+1}y_{1,2}^{n}y_{2,3}, && y_{1,2}y_{1,4}^{n} = \chi_{n}y_{1,2}^{n-1}y_{1,4}^{2} + \chi_{n+1}y_{1,2}^{n}y_{1,4},\\
&y_{3,4}y_{2,3}^{n} = \chi_{n}y_{2,3}^{n}y_{3,4} - \chi_{n+1}y_{2,3}^{n}y_{2,4}, && y_{3,4}y_{1,4}^{n} = (-1)^{n}y_{1,3}^{n}y_{3,4},\\
&y_{1,3}y_{2,3}^{n} = y_{1,2}^{n}y_{1,3}, && y_{1,3}y_{1,4}^{n} = \chi_{n}y_{1,3}^{n-1}y_{1,4}^{2} + \chi_{n+1}y_{1,3}^{n}y_{1,4},\\
&y_{2,4}y_{2,3}^{n} = \chi_{n}y_{2,3}^{n}y_{2,4} - \chi_{n+1}y_{2,3}^{n}y_{3,4}, && y_{2,4}y_{1,4}^{n} = (-1)^{n}y_{1,2}^{n}y_{2,4},
\end{aligned}
\tag{3.5}
$$

in $A^!$, for all integers $n \geqslant 2$ and $r \in [\![1, n-1]\!]$.

We also provided further identities in $A^!$ given in Appendix A.2, which are also straightforward to verify.

Recall that the graded dual $(A^!)^{\#} = \oplus_{n\in\mathbb{N}_0}(A^!_{-n})^*$ is a graded bimodule over $A^!$ via the identity $(ufv)(w) = f(vwu)$ for $u, v, w \in A^!$ and $f \in (A^!)^{\#}$. Let $\mathscr{B}_n^{!*}$ be the dual basis to the basis $\mathscr{B}_n^!$ for $n \in \mathbb{N}_0$. We write $\mathscr{B}_0^{!*} = \{\epsilon^!\}$ and $z_{n_1}^{i_1,j_1}\dots z_{n_r}^{i_r,j_r} = (y_{i_1,j_1}^{n_1}\dots y_{i_r,j_r}^{n_r})^* \in \mathscr{B}_n^{!*}$ for $y_{i_1,j_1}^{n_1}\dots y_{i_r,j_r}^{n_r} \in \mathscr{B}_n^!$, where $n = n_1 + \cdots + n_r$, $n, r, n_1, \dots, n_r \in \mathbb{N}$ and $(i_1, j_1), \dots, (i_r, j_r) \in \mathscr{I}$. We will omit the index n_j for $j \in [\![1, r]\!]$ if $n_j = 1$ in the element $z_{n_1}^{i_1,j_1}\dots z_{n_r}^{i_r,j_r}$ or $y_{i_1,j_1}^{n_1}\dots y_{i_r,j_r}^{n_r}$. Obviously, $y_{i,j}z^{i,j} = \epsilon^!$ for $(i, j) \in \mathscr{I}$ and the other actions of $\mathscr{B}_1^!$ on $\mathscr{B}_1^{!*}$ vanish.

Recall that $(K_\bullet(A), d_\bullet)$ denotes the Koszul complex of A in the category of bounded below graded (right) A-modules and $\epsilon : K_0(A) \to \Bbbk$ is the canonical projection. The differential $d_n : K_n(A) \to K_{n-1}(A)$ for $n \in \mathbb{N}$ is given by the multiplication of $\sum_{(i,j)\in\mathscr{I}} y_{i,j} \otimes x_{i,j}$ on the left. To reduce space, we will simply write K_n instead of $K_n(A)$ for $n \in \mathbb{N}_0$ and we will typically use vertical bars instead of the tensor product symbols $\otimes$.

The differential $d_\bullet$ of the Koszul complex of A can be explicitly described in the following result. Its proof is a straightforward but lengthy verification, using the identities listed in Fact 3.2 and in Appendix A.2.

Fact 3.3. *Let $d_n : K_n \to K_{n-1}$ be the differential of the Koszul complex of A for $n \in \mathbb{N}$. It can be explicitly described as follows. First, $d_1(z^{i,j}|1) = \epsilon^!|x_{i,j}$ for $(i, j) \in \mathscr{I}$, and*

$$
d_n(z_{n-r}^{i,j}z_r^{k,l}|1) = (-1)^r z_{n-r-1}^{i,j}z_r^{k,l}|x_{i,j} + z_{n-r}^{i,j}z_{r-1}^{k,l}|x_{k,l}, \tag{3.6}
$$

for $n \geqslant 2$, $r \in [\![0, n]\!]$, $(i, j) \in \mathscr{I}_1$, $(k, l) \in \mathscr{I}$ with $\#\{i, j, k, l\} = 4$, where we follow the convention that $z_n^{i,j}z_0^{k,l} = z_n^{i,j}$, $z_0^{i,j}z_n^{k,l} = z_n^{k,l}$, $z_n^{i,j}z_{-1}^{k,l} = 0$ and $z_{-1}^{i,j}z_n^{k,l} = 0$ for $n \in \mathbb{N}$. Moreover, for $n \geqslant 5$, the differential d_{n+1} is given by (3.6) and

$$
\begin{aligned}
z_n^{1,2}z^{1,3}|1 \mapsto &-(z_{n-1}^{1,2}z^{2,3} + \chi_{n+1}z_n^{2,3})|x_{1,2} + (z_n^{1,2} + z_{n-2}^{1,2}z_2^{1,3} + \chi_n z_n^{2,3})|x_{1,3}\\
&+ (z_{n-1}^{1,2}z^{1,3} + \chi_{n+1}z_n^{1,3})|x_{2,3},
\end{aligned}
$$

$$
\begin{aligned}
z_n^{1,2} z^{2,3}|1 \mapsto & -\{z_{n-1}^{1,2} z^{1,3} + \chi_{n+1} z_n^{1,3}\}|x_{1,2} - \{z_{n-1}^{1,2} z^{2,3} + \chi_{n+1} z_n^{2,3}\}|x_{1,3} \\
& + \{z_n^{1,2} + z_{n-2}^{1,2} z_2^{1,3} + \chi_n z_n^{1,3}\}|x_{2,3}, \\
z_n^{1,2} z^{1,4}|1 \mapsto & -\{z_{n-1}^{1,2} z^{2,4} + \chi_{n+1} z_n^{2,4}\}|x_{1,2} + \{z_n^{1,2} + z_{n-2}^{1,2} z_2^{1,4} + \chi_n z_n^{2,4}\}|x_{1,4} \\
& + \{z_{n-1}^{1,2} z^{1,4} + \chi_{n+1} z_n^{1,4}\}|x_{2,4}, \\
z_n^{1,2} z^{2,4}|1 \mapsto & -\{z_{n-1}^{1,2} z^{1,4} + \chi_{n+1} z_n^{1,4}\}|x_{1,2} - \{z_{n-1}^{1,2} z^{2,4} + \chi_{n+1} z_n^{2,4}\}|x_{1,4} \\
& + \{z_n^{1,2} + z_{n-2}^{1,2} z_2^{1,4} + \chi_n z_n^{1,4}\}|x_{2,4}, \\
z_{n-1}^{1,2} z_2^{1,3}|1 \mapsto & \{z_{n-2}^{1,2} z_2^{1,3} + \chi_n (z_n^{1,3} + z_n^{2,3})\}|x_{1,2} + z_{n-1}^{1,2} z^{1,3}|x_{1,3} + z_{n-1}^{1,2} z^{2,3}|x_{2,3}, \\
z_{n-1}^{1,2} z^{1,3} z^{1,4}|1 \mapsto & \{z_{n-2}^{1,2} z^{2,3} z^{2,4} + \chi_n z_{n-1}^{2,3} z^{2,4}\}|x_{1,2} \\
& - \left\{z_{n-3}^{1,2} z_2^{1,3} z^{3,4} + \chi_{n+1} z_{n-1}^{2,3} z^{3,4} + \sum_{s=1}^{\lfloor \frac{n}{2} \rfloor} z_{n-2s+1}^{1,2} z_{2s-1}^{3,4}\right\}|x_{1,3} \\
& - \{z_{n-2}^{1,2} z^{1,3} z^{1,4} + \chi_n z_{n-1}^{1,3} z^{1,4}\}|x_{2,3} \\
& + \left\{z_{n-1}^{1,2} z^{1,3} + z_{n-3}^{1,2} z^{1,3} z_2^{1,4} + \chi_{n+1} \sum_{s=1}^{\frac{n-1}{2}} z_{n-2s}^{1,3} z_{2s}^{2,4}\right\}|x_{1,4} \\
& - \left\{z_{n-2}^{1,2} z^{2,3} z^{1,4} + \chi_n \sum_{s=1}^{\frac{n}{2}} z_{n-2s+1}^{2,3} z_{2s-1}^{1,4}\right\}|x_{2,4} \\
& + \left\{z_{n-1}^{1,2} z^{1,4} + z_{n-3}^{1,2} z_2^{1,3} z^{1,4} + \chi_{n+1} \sum_{s=1}^{\frac{n-1}{2}} z_{n-2s+1}^{2,3} z_{2s-1}^{1,4}\right\}|x_{3,4}, \\
z_{n-1}^{1,2} z^{1,3} z^{2,4}|1 \mapsto & \left\{z_{n-2}^{1,2} z^{2,3} z^{1,4} + \chi_n \sum_{s=1}^{\frac{n}{2}} z_{n-2s+1}^{2,3} z_{2s-1}^{1,4}\right\}|x_{1,2} \\
& - \{z_{n-1}^{1,2} z^{2,4} + z_{n-3}^{1,2} z_2^{1,3} z^{2,4} + \chi_{n+1} z_{n-1}^{2,3} z^{2,4}\}|x_{1,3} - \{z_{n-2}^{1,2} z^{1,3} z^{3,4} + \chi_n z_{n-1}^{1,3} z^{3,4}\}|x_{2,3} \\
& + \{z_{n-2}^{1,2} z^{2,3} z^{2,4} + \chi_n z_{n-1}^{2,3} z^{2,4}\}|x_{1,4} + \{z_{n-1}^{1,2} z^{1,3} + z_{n-3}^{1,2} z^{1,3} z_2^{1,4} + \chi_{n+1} z_{n-2}^{1,3} z_2^{1,4}\}|x_{2,4} \\
& + \left\{z_{n-2}^{1,2} z^{1,3} z^{2,4} + \chi_n \sum_{s=1}^{\frac{n}{2}} z_{n-2s+1}^{1,3} z_{2s-1}^{2,4}\right\}|x_{3,4}, \\
z_{n-1}^{1,2} z^{1,3} z^{3,4}|1 \mapsto & \{z_{n-2}^{1,2} z^{2,3} z^{3,4} + \chi_n z_{n-1}^{2,3} z^{3,4}\}|x_{1,2} \\
& - \left\{z_{n-1}^{1,2} z^{1,4} + z_{n-3}^{1,2} z_2^{1,3} z^{1,4} + \chi_{n+1} \sum_{s=1}^{\frac{n-1}{2}} z_{n-2s+1}^{2,3} z_{2s-1}^{1,4}\right\}|x_{1,3} \\
& - \left\{z_{n-2}^{1,2} z^{1,3} z^{2,4} + \chi_n \sum_{s=1}^{\frac{n}{2}} z_{n-2s+1}^{1,3} z_{2s-1}^{2,4}\right\}|x_{2,3} \\
& - \left\{z_{n-3}^{1,2} z_2^{1,3} z^{3,4} + \chi_{n+1} z_{n-1}^{2,3} z^{3,4} + \sum_{s=1}^{\lfloor \frac{n}{2} \rfloor} z_{n-2s+1}^{1,2} z_{2s-1}^{3,4}\right\}|x_{1,4} \\
& - \{z_{n-2}^{1,2} z^{1,3} z^{3,4} + \chi_n z_{n-1}^{1,3} z^{3,4}\}|x_{2,4} \\
& + \left\{z_{n-1}^{1,2} z^{1,3} + z_{n-3}^{1,2} z^{1,3} z_2^{1,4} + \chi_{n+1} \sum_{s=1}^{\frac{n-1}{2}} z_{n-2s}^{1,3} z_{2s}^{2,4}\right\}|x_{3,4}, \\
z_{n-1}^{1,2} z^{2,3} z^{1,4}|1 \mapsto & \left\{z_{n-2}^{1,2} z^{1,3} z^{2,4} + \chi_n \sum_{s=1}^{\frac{n}{2}} z_{n-2s+1}^{1,3} z_{2s-1}^{2,4}\right\}|x_{1,2} + \{z_{n-2}^{1,2} z^{2,3} z^{3,4} + \chi_n z_{n-1}^{2,3} z^{3,4}\}|x_{1,3} \\
& - \{z_{n-1}^{1,2} z^{1,4} + z_{n-3}^{1,2} z_2^{1,3} z^{1,4} + \chi_{n+1} z_{n-1}^{1,3} z^{1,4}\}|x_{2,3} \\
& + \{z_{n-1}^{1,2} z^{2,3} + z_{n-3}^{1,2} z^{2,3} z_2^{1,4} + \chi_{n+1} z_{n-2}^{2,3} z_2^{2,4}\}|x_{1,4} - \{z_{n-2}^{1,2} z^{1,3} z^{1,4} + \chi_n z_{n-1}^{1,3} z^{1,4}\}|x_{2,4} \\
& - \left\{z_{n-2}^{1,2} z^{2,3} z^{1,4} + \chi_n \sum_{s=1}^{\frac{n}{2}} z_{n-2s+1}^{2,3} z_{2s-1}^{1,4}\right\}|x_{3,4}, \\
z_{n-1}^{1,2} z^{2,3} z^{2,4}|1 \mapsto & \{z_{n-2}^{1,2} z^{1,3} z^{1,4} + \chi_n z_{n-1}^{1,3} z^{1,4}\}|x_{1,2} + \{z_{n-2}^{1,2} z^{2,3} z^{2,4} + \chi_n z_{n-1}^{2,3} z^{2,4}\}|x_{1,3} \\
& - \left\{z_{n-3}^{1,2} z_2^{1,3} z^{3,4} + \chi_{n+1} z_{n-1}^{1,3} z^{3,4} + \sum_{s=1}^{\lfloor \frac{n}{2} \rfloor} z_{n-2s+1}^{1,2} z_{2s-1}^{3,4}\right\}|x_{2,3}
\end{aligned}
$$

$$
\begin{aligned}
&+\left\{z^{1,2}_{n-2}z^{1,3}z^{2,4}+\chi_n\sum_{s=1}^{\frac{n}{2}}z^{1,3}_{n-2s+1}z^{2,4}_{2s-1}\right\}|x_{1,4}\\
&+\left\{z^{1,2}_{n-1}z^{2,3}+z^{1,2}_{n-3}z^{2,3}z^{1,4}_2+\chi_{n+1}\sum_{s=1}^{\frac{n-1}{2}}z^{2,3}_{n-2s}z^{1,4}_{2s}\right\}|x_{2,4}\\
&+\left\{z^{1,2}_{n-1}z^{2,4}+z^{1,2}_{n-3}z^{1,3}_2z^{2,4}+\chi_{n+1}\sum_{s=1}^{\frac{n-1}{2}}z^{1,3}_{n-2s+1}z^{2,4}_{2s-1}\right\}|x_{3,4},\\
z^{1,2}_{n-1}z^{2,3}z^{3,4}|1\mapsto{}&\{z^{1,2}_{n-2}z^{1,3}z^{3,4}+\chi_nz^{1,3}_{n-1}z^{3,4}\}|x_{1,2}+\left\{z^{1,2}_{n-2}z^{2,3}z^{1,4}+\chi_n\sum_{s=1}^{\frac{n}{2}}z^{2,3}_{n-2s+1}z^{1,4}_{2s-1}\right\}|x_{1,3}\\
&-\left\{z^{1,2}_{n-1}z^{2,4}+z^{1,2}_{n-3}z^{1,3}_2z^{2,4}+\chi_{n+1}\sum_{s=1}^{\frac{n-1}{2}}z^{1,3}_{n-2s+1}z^{2,4}_{2s-1}\right\}|x_{2,3}\\
&+\{z^{1,2}_{n-2}z^{2,3}z^{3,4}+\chi_nz^{2,3}_{n-1}z^{3,4}\}|x_{1,4}\\
&-\left\{z^{1,2}_{n-3}z^{1,3}_2z^{3,4}+\chi_{n+1}z^{1,3}_{n-1}z^{3,4}+\sum_{s=1}^{\lfloor\frac{n}{2}\rfloor}z^{1,2}_{n-2s+1}z^{3,4}_{2s-1}\right\}|x_{2,4}\\
&+\left\{z^{1,2}_{n-1}z^{2,3}+z^{1,2}_{n-3}z^{2,3}z^{1,4}_2+\chi_{n+1}\sum_{s=1}^{\frac{n-1}{2}}z^{2,3}_{n-2s}z^{1,4}_{2s}\right\}|x_{3,4},\\
z^{1,2}_{n-1}z^{1,4}_2|1\mapsto{}&\{z^{1,2}_{n-2}z^{1,4}_2+\chi_n(z^{1,4}_n+z^{2,4}_n)\}|x_{1,2}+z^{1,2}_{n-1}z^{1,4}|x_{1,4}+z^{1,2}_{n-1}z^{2,4}|x_{2,4}, \qquad (3.7)\\
z^{1,2}_{n-2}z^{1,3}_2z^{1,4}|1\mapsto{}&-\left\{z^{1,2}_{n-3}z^{1,3}_2z^{2,4}+\chi_{n+1}\left(z^{2,3}_{n-1}z^{2,4}+\sum_{s=1}^{\frac{n-1}{2}}z^{1,3}_{n-2s+1}z^{2,4}_{2s-1}\right)\right\}|x_{1,2}-z^{1,2}_{n-2}z^{1,3}z^{3,4}|x_{1,3}\\
&-z^{1,2}_{n-2}z^{2,3}z^{1,4}|x_{2,3}\\
&+\left\{z^{1,2}_{n-2}z^{1,3}_2+z^{1,2}_{n-4}z^{1,3}_2z^{1,4}_2+\chi_n\left(z^{2,3}_{n-2}z^{2,4}_2+\sum_{s=1}^{\frac{n-2}{2}}z^{1,3}_{n-2s}z^{2,4}_{2s}\right)+\sum_{s=1}^{\lfloor\frac{n-1}{2}\rfloor}z^{1,2}_{n-2s}z^{3,4}_{2s}\right\}|x_{1,4}\\
&+\left\{z^{1,2}_{n-3}z^{1,3}_2z^{1,4}+\chi_{n+1}\left(z^{1,3}_{n-1}z^{1,4}+\sum_{s=1}^{\frac{n-1}{2}}z^{2,3}_{n-2s+1}z^{1,4}_{2s-1}\right)\right\}|x_{2,4}+z^{1,2}_{n-2}z^{1,3}z^{1,4}|x_{3,4},\\
z^{1,2}_{n-2}z^{1,3}_2z^{2,4}|1\mapsto{}&-\left\{z^{1,2}_{n-3}z^{1,3}_2z^{1,4}+\chi_{n+1}\left(z^{1,3}_{n-1}z^{1,4}+\sum_{s=1}^{\frac{n-1}{2}}z^{2,3}_{n-2s+1}z^{1,4}_{2s-1}\right)\right\}|x_{1,2}-z^{1,2}_{n-2}z^{1,3}z^{2,4}|x_{1,3}\\
&-z^{1,2}_{n-2}z^{2,3}z^{3,4}|x_{2,3}-\left\{z^{1,2}_{n-3}z^{1,3}_2z^{2,4}+\chi_{n+1}\left(z^{2,3}_{n-1}z^{2,4}+\sum_{s=1}^{\frac{n-1}{2}}z^{1,3}_{n-2s+1}z^{2,4}_{2s-1}\right)\right\}|x_{1,4}\\
&+\left\{z^{1,2}_{n-2}z^{1,3}_2+z^{1,2}_{n-4}z^{1,3}_2z^{1,4}_2+\chi_n\left(z^{1,3}_{n-2}z^{1,4}_2+\sum_{s=1}^{\frac{n-2}{2}}z^{2,3}_{n-2s}z^{1,4}_{2s}\right)+\sum_{s=1}^{\lfloor\frac{n-1}{2}\rfloor}z^{1,2}_{n-2s}z^{3,4}_{2s}\right\}|x_{2,4}\\
&+z^{1,2}_{n-2}z^{2,3}z^{2,4}|x_{3,4},\\
z^{1,2}_{n-2}z^{1,3}_2z^{3,4}|1\mapsto{}&-\{z^{1,2}_{n-3}z^{1,3}_2z^{3,4}+\chi_{n+1}(z^{1,3}_{n-1}z^{3,4}+z^{2,3}_{n-1}z^{3,4})\}|x_{1,2}-z^{1,2}_{n-2}z^{1,3}z^{1,4}|x_{1,3}\\
&-z^{1,2}_{n-2}z^{2,3}z^{2,4}|x_{2,3}-z^{1,2}_{n-2}z^{1,3}z^{3,4}|x_{1,4}-z^{1,2}_{n-2}z^{2,3}z^{3,4}|x_{2,4}\\
&+\left\{z^{1,2}_{n-2}z^{1,3}_2+z^{1,2}_{n-2}z^{1,4}_2+z^{1,2}_{n-4}z^{1,3}_2z^{1,4}_2+\chi_n\sum_{s=1}^{\frac{n-2}{2}}(z^{1,3}_{n-2s}z^{2,4}_{2s}+z^{2,3}_{n-2s}z^{1,4}_{2s})\right\}|x_{3,4},\\
z^{1,2}_{n-2}z^{1,3}z^{1,4}_2|1\mapsto{}&-\left\{z^{1,2}_{n-3}z^{2,3}z^{1,4}_2+\chi_{n+1}\left(z^{2,3}_{n-2}z^{2,4}_2+\sum_{s=1}^{\frac{n-1}{2}}z^{2,3}_{n-2s}z^{1,4}_{2s}\right)\right\}|x_{1,2}\\
&+\left\{z^{1,2}_{n-2}z^{1,4}_2+z^{1,2}_{n-4}z^{1,3}_2z^{1,4}_2+\chi_n\left(z^{2,3}_{n-2}z^{2,4}_2+\sum_{s=1}^{\frac{n-2}{2}}z^{2,3}_{n-2s}z^{1,4}_{2s}\right)+\sum_{s=1}^{\lfloor\frac{n-1}{2}\rfloor}z^{1,2}_{n-2s}z^{3,4}_{2s}\right\}|x_{1,3}\\
&+\left\{z^{1,2}_{n-3}z^{1,3}z^{1,4}_2+\chi_{n+1}\left(z^{1,3}_{n-2}z^{1,4}_2+\sum_{s=1}^{\frac{n-1}{2}}z^{1,3}_{n-2s}z^{2,4}_{2s}\right)\right\}|x_{2,3}\\
&+z^{1,2}_{n-2}z^{1,3}z^{1,4}|x_{1,4}+z^{1,2}_{n-2}z^{1,3}z^{2,4}|x_{2,4}+z^{1,2}_{n-2}z^{1,3}z^{3,4}|x_{3,4},
\end{aligned}
$$

$$
\begin{aligned}
z^{1,2}_{n-2}z^{2,3}z^{1,4}_2|1 \mapsto & -\Big\{z^{1,2}_{n-3}z^{1,3}z^{1,4}_2 + \chi_{n+1}\Big(z^{1,3}_{n-2}z^{1,4}_2 + \sum_{s=1}^{\frac{n-1}{2}} z^{1,3}_{n-2s}z^{2,4}_{2s}\Big)\Big\}|x_{1,2} \\
& - \Big\{z^{1,2}_{n-3}z^{2,3}z^{1,4}_2 + \chi_{n+1}\Big(z^{2,3}_{n-2}z^{2,4}_2 + \sum_{s=1}^{\frac{n-1}{2}} z^{2,3}_{n-2s}z^{1,4}_{2s}\Big)\Big\}|x_{1,3} \\
& + \Big\{z^{1,2}_{n-2}z^{1,4}_2 + z^{1,2}_{n-4}z^{1,3}_2z^{1,4}_2 + \chi_n\Big(z^{1,3}_{n-2}z^{1,4}_2 + \sum_{s=1}^{\frac{n-2}{2}} z^{1,3}_{n-2s}z^{2,4}_{2s}\Big) + \sum_{s=1}^{\lfloor\frac{n-1}{2}\rfloor} z^{1,2}_{n-2s}z^{3,4}_{2s}\Big\}|x_{2,3} \\
& + z^{1,2}_{n-2}z^{2,3}z^{1,4}|x_{1,4} + z^{1,2}_{n-2}z^{2,3}z^{2,4}|x_{2,4} + z^{1,2}_{n-2}z^{2,3}z^{3,4}|x_{3,4}, \\
z^{1,2}_{n-3}z^{1,3}_2z^{1,4}_2|1 \mapsto & \Big\{z^{1,2}_{n-4}z^{1,3}_2z^{1,4}_2 + \chi_n\Big(z^{1,3}_{n-2}z^{1,4}_2 + z^{2,3}_{n-2}z^{2,4}_2 + \sum_{s=1}^{\frac{n-2}{2}}(z^{1,3}_{n-2s}z^{2,4}_{2s} + z^{2,3}_{n-2s}z^{1,4}_{2s})\Big)\Big\}|x_{1,2} \\
& + z^{1,2}_{n-3}z^{1,3}z^{1,4}_2|x_{1,3} + z^{1,2}_{n-3}z^{2,3}z^{1,4}_2|x_{2,3} + z^{1,2}_{n-3}z^{1,3}_2z^{1,4}|x_{1,4} + z^{1,2}_{n-3}z^{1,3}_2z^{2,4}|x_{2,4} \\
& + z^{1,2}_{n-3}z^{1,3}_2z^{3,4}|x_{3,4}, \\
z^{1,3}_n z^{1,4}|1 \mapsto & -\{z^{1,3}_{n-1}z^{3,4} + \chi_{n+1}z^{3,4}_n\}|x_{1,3} + \{z^{1,3}_n + z^{1,3}_{n-2}z^{1,4}_2 + \chi_n z^{3,4}_n\}|x_{1,4} \\
& + \{z^{1,3}_{n-1}z^{1,4} + \chi_{n+1}z^{1,4}_n\}|x_{3,4}, \\
z^{1,3}_n z^{3,4}|1 \mapsto & -\{z^{1,3}_{n-1}z^{1,4} + \chi_{n+1}z^{1,4}_n\}|x_{1,3} - \{z^{1,3}_{n-1}z^{3,4} + \chi_{n+1}z^{3,4}_n\}|x_{1,4} \\
& + \{z^{1,3}_n + z^{1,3}_{n-2}z^{1,4}_2 + \chi_n z^{1,4}_n\}|x_{3,4}, \\
z^{1,3}_{n-1}z^{1,4}_2|1 \mapsto & \{z^{1,3}_{n-2}z^{1,4}_2 + \chi_n(z^{1,4}_n + z^{3,4}_n)\}|x_{1,3} + z^{1,3}_{n-1}z^{1,4}|x_{1,4} + z^{1,3}_{n-1}z^{3,4}|x_{3,4}, \\
z^{2,3}_n z^{2,4}|1 \mapsto & -\{z^{2,3}_{n-1}z^{3,4} + \chi_{n+1}z^{3,4}_n\}|x_{2,3} + \{z^{2,3}_n + z^{2,3}_{n-2}z^{2,4}_2 + \chi_n z^{3,4}_n\}|x_{2,4} \\
& + \{z^{2,3}_{n-1}z^{2,4} + \chi_{n+1}z^{2,4}_n\}|x_{3,4}, \\
z^{2,3}_n z^{3,4}|1 \mapsto & -\{z^{2,3}_{n-1}z^{2,4} + \chi_{n+1}z^{2,4}_n\}|x_{2,3} - \{z^{2,3}_{n-1}z^{3,4} + \chi_{n+1}z^{3,4}_n\}|x_{2,4} \\
& + \{z^{2,3}_n + z^{2,3}_{n-2}z^{2,4}_2 + \chi_n z^{2,4}_n\}|x_{3,4}, \\
z^{2,3}_{n-1}z^{2,4}_2|1 \mapsto & \{z^{2,3}_{n-2}z^{2,4}_2 + \chi_n(z^{2,4}_n + z^{3,4}_n)\}|x_{2,3} + z^{2,3}_{n-1}z^{2,4}|x_{2,4} + z^{2,3}_{n-1}z^{3,4}|x_{3,4}.
\end{aligned}
$$

3.2 The main result about FK(4)

We will now define some quadratic A-modules M^i for $i \in [\![1,3]\!]$. Let M^1 be the A-module generated by two homogeneous elements a_1, a_2 of degree zero, subject to the following 6 relations

$$
a_1x_{1,2} + a_2x_{1,2}, a_1x_{1,3}, a_2x_{2,3}, a_2x_{1,4}, a_1x_{2,4}, a_1x_{3,4} + a_2x_{3,4}. \tag{3.8}
$$

Let M^2 be the A-module generated by the set $\{h_i \mid i \in [\![1,7]\!]\}$ of seven homogeneous elements of degree zero, subject to the following 24 relations

$$
\begin{aligned}
& h_1x_{1,2}, h_1x_{1,3}, h_1x_{2,3}, h_2x_{1,2}, h_2x_{1,4}, h_2x_{2,4}, h_3x_{1,3}, h_3x_{1,4}, h_3x_{3,4}, h_4x_{2,3}, h_4x_{2,4}, h_4x_{3,4}, \\
& h_1x_{2,4} - h_3x_{2,4} - h_5x_{1,3}, h_2x_{1,3} - h_4x_{1,3} + h_5x_{2,4}, h_5x_{3,4} - h_6x_{1,2}, h_1x_{1,4} - h_4x_{1,4} + h_6x_{2,3}, \\
& h_2x_{2,3} - h_3x_{2,3} - h_6x_{1,4}, h_5x_{1,2} + h_6x_{3,4}, h_1x_{3,4} + h_2x_{3,4} + h_7x_{1,2}, h_6x_{2,4} + h_7x_{1,3}, \\
& h_5x_{1,4} + h_7x_{2,3}, h_5x_{2,3} - h_7x_{1,4}, h_6x_{1,3} - h_7x_{2,4}, h_3x_{1,2} + h_4x_{1,2} + h_7x_{3,4}.
\end{aligned} \tag{3.9}
$$

Finally, let M^3 be the A-module generated by the set $\{e_i \mid i \in [\![1,8]\!]\}$ of eight homogeneous elements of degree zero, subject to the following 24 relations

$$
\begin{aligned}
& e_1x_{1,2} + e_2x_{3,4}, e_1x_{3,4} - e_2x_{1,2}, e_3x_{1,2} - e_4x_{3,4}, e_3x_{3,4} + e_4x_{1,2}, e_4x_{1,3} + e_2x_{2,4}, e_4x_{2,4} - e_2x_{1,3}, \\
& e_3x_{1,3} + e_1x_{2,4}, e_3x_{2,4} - e_1x_{1,3}, e_1x_{2,3} - e_4x_{1,4}, e_1x_{1,4} + e_4x_{2,3}, e_3x_{2,3} - e_2x_{1,4}, e_3x_{1,4} + e_2x_{2,3}, \\
& e_5x_{1,2}, e_5x_{1,3}, e_5x_{2,3}, e_6x_{1,2}, e_6x_{1,4}, e_6x_{2,4}, e_7x_{1,3}, e_7x_{1,4}, e_7x_{3,4}, e_8x_{2,3}, e_8x_{2,4}, e_8x_{3,4}.
\end{aligned} \tag{3.10}
$$

Since the previous modules are finite dimensional, we use GAP to obtain a homogeneous $\Bbbk$-basis of M^i, and in particular, the Hilbert series of M^i, for $i \in [\![1,3]\!]$. See Appendix A.3 for a basis of M^1.

Fact 3.4. *Given $i \in [\![1,3]\!]$, the Hilbert series $h_{M^i}(t)$ of the quadratic A-module M^i introduced in the previous paragraph is given by*

$$\begin{aligned} h_{M^1}(t) &= 2 + 6t + 11t^2 + 12t^3 + 11t^4 + 6t^5 + 2t^6, \\ h_{M^2}(t) &= 7 + 18t + 32t^2 + 42t^3 + 40t^4 + 30t^5 + 16t^6 + 6t^7 + 1t^8, \\ h_{M^3}(t) &= 8 + 24t + 48t^2 + 72t^3 + 80t^4 + 72t^5 + 48t^6 + 24t^7 + 8t^8. \end{aligned}$$

We now present the second main result of this article, which states that the Fomin-Kirillov algebra of index 4 has a connected resolving datum. The proof of this theorem will follow from several intermediate results that we will provide in the following two subsections (see Subsection 3.5).

Theorem 3.5. *Let $\mathscr{M} = \{M^0 = \Bbbk, M^1, M^2, M^3\}$ be the family of quadratic A-modules introduced in the first paragraph of this subsection, and let $\hbar : [\![0,N]\!]^2 \times \mathbb{N}^2 \to \mathbb{N}_0^2$ be the map given by*

$$\begin{aligned} &\hbar(0,2,3,6) = \hbar(0,0,3,6) = \hbar(1,2,1,4) = (1,0), \\ &\hbar(0,0,3,8) = \hbar(0,1,4,8) = \hbar(0,0,5,16) = \hbar(1,0,1,6) = \hbar(1,0,1,8) = \hbar(2,0,1,4) \\ &\qquad = \hbar(2,0,1,6) = \hbar(2,1,2,6) = \hbar(2,3,3,6) = \hbar(3,3,3,6) = (0,1), \end{aligned}$$

and $\hbar(i,j,k,\ell)$ vanishes on other (i,j,k,ℓ). Then this gives a connected resolving datum on A, whose associated resolving quiver is given in Figure 1, where we denote by ${}_j\alpha_i^{d',d''}$ the unique arrow from M^i to M^j having bidegree (d',d''). In this case, the strict partial order on the arrows is given by ${}_0\alpha_0^{4,8} < {}_0\alpha_0^{4,6}, {}_2\alpha_0^{4,6}$, and ${}_0\alpha_1^{2,6}, {}_0\alpha_1^{2,8} < {}_2\alpha_1^{2,4}$. The arrows ${}_1\alpha_0^{5,8}$ and ${}_1\alpha_2^{3,6}$ of odd difference degrees appear in red.

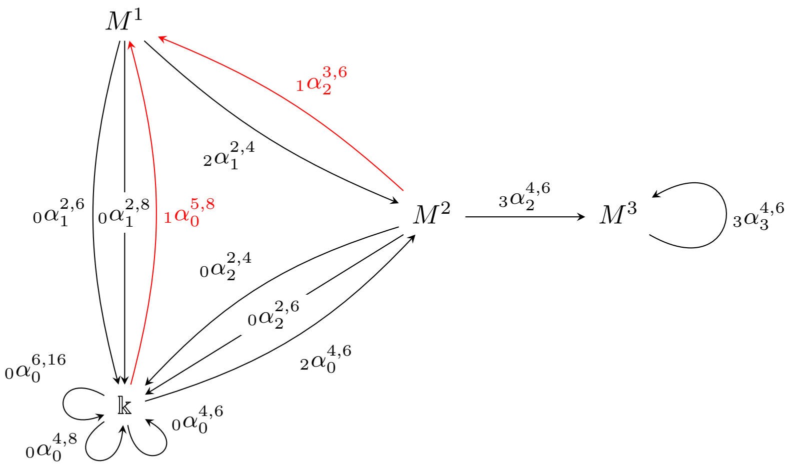

Figure 1: Resolving quiver of $\mathrm{FK}(4)$.

As an explicit application of the previous results, we have also implemented a routine in GAP to compute the dimensions of the Yoneda algebra $\mathrm{Ext}^{\bullet}_{\mathrm{FK}(4)}(\Bbbk,\Bbbk)$, using the projective resolution given by Theorem 2.10 for the resolving datum of $\mathrm{FK}(4)$ in Theorem 3.5. They appear in Table 3.1.

We do not present the precise GAP code used in the computation for Table 3.1, since it is not used to prove any particular theorem. However, it shows that the projective resolution in Theorem 2.10 can be used for even rather hard computations: it took GAP around 23 hours to compute these values in a standard laptop computer, whereas we could not manage to compute any of them by means of the package QPA in GAP that uses a projective resolution constructed using Gröbner bases. The entry for $n = 4$ and $m = 6$ in Table 3.1 was mentioned without proof in the last remark of [13].

$m \backslash n$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1		6									
2			17								
3				30							
4					38						
5						42					
6					8		45				
7						30		48			
8					1		58		51		
9						6		78		54	
10							18		86		57
11								36		90	
12							1		62		93
13								6		96	
14									25		129
15										60	
16									1		97
17										6	
18											26
19											
20											1

Table 3.1: Dimension of $\mathrm{Ext}^{n,-m}_{\mathrm{FK}(4)}(\Bbbk, \Bbbk)$, where n is the cohomological degree and $-m$ the internal degree. The entries that are not explicitly indicated are zero.

3.3 The homology of the Koszul complex of $\mathrm{FK}(4)$

3.3.1 The dimensions of the homology groups

Recall that $(K_\bullet, d_\bullet)$ is the Koszul complex of the trivial module $\Bbbk$ in the category of bounded below graded (right) A-modules. Let $K_{n,m} = (A^!_{-n})^* \otimes A_m$, $d_{n,m} = d_n|_{K_{n,m}} : K_{n,m} \to K_{n-1,m+1}$, $B_{n,m} = \mathrm{Im}(d_{n+1,m-1})$, $D_{n,m} = \mathrm{Ker}(d_{n,m})$, $H_{n,m} = D_{n,m}/B_{n,m}$ for $n \in \mathbb{N}_0$ and $m \in [\![0, 12]\!]$. Let $H_n = \oplus_{m\in[\![0,12]\!]} H_{n,m}$ for $n \in \mathbb{N}_0$. We can compute the dimension of $B_{n,m}$ using GAP for n less than some arbitrary positive integer and $m \in [\![1, 12]\!]$ by using the code in Appendix A.4 together with the following simple routine.

```
for j in [0..11] do
    for i in [1..12] do
        Print(j, " ", i, " ", RankMat(FF(0,j,i)), "\n");
    od;
od;
```

For the rest of the section, we will only indicate the extra code added to the one in Appendix A.4 for every computation, and, for the reader's convenience, we will often indicate the output of many of the intermediate commands in the corresponding successive line: it will be preceded by a pound sign #.

The dimension of $B_{n,m}$ for $n \in [\![0, 11]\!]$ and $m \in [\![0, 12]\!]$ is displayed in Table 3.2.

n \ m	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	6	19	42	71	96	106	96	71	42	19	6	1
1	0	17	72	181	330	470	540	505	384	233	108	35	6
2	0	30	142	384	737	1092	1297	1248	974	606	288	96	17
3	0	38	186	515	1020	1550	1890	1866	1494	956	468	162	30
4	0	42	207	576	1146	1752	2151	2142	1731	1122	558	198	38
5	0	45	222	618	1230	1881	2310	2301	1860	1206	600	213	42
6	0	48	237	660	1314	2010	2469	2460	1989	1290	642	228	45
7	0	51	252	702	1398	2139	2628	2619	2118	1374	684	243	48
8	0	54	267	744	1482	2268	2787	2778	2247	1458	726	258	51
9	0	57	282	786	1566	2397	2946	2937	2376	1542	768	273	54
10	0	60	297	828	1650	2526	3105	3096	2505	1626	810	288	57
11	0	63	312	870	1734	2655	3264	3255	2634	1710	852	303	60

Table 3.2: Dimension of $B_{n,m}$.

Since $\dim D_{n,m} = \dim K_{n,m} - \dim B_{n-1,m+1}$, using Table 3.2 we get the dimension of $D_{n,m}$ for $n \in [\![0,5]\!]$ and $m \in [\![0,12]\!]$, which is displayed in Table 3.3.

n \ m	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	6	19	42	71	96	106	96	71	42	19	6	1
1	0	17	72	181	330	470	540	505	384	233	108	35	6
2	0	30	142	384	737	1092	1297	1248	974	606	288	96	17
3	0	38	186	523	1038	1583	1932	1906	1524	972	474	163	30
4	0	42	207	576	1148	1758	2162	2154	1742	1128	560	198	38
5	0	45	222	618	1230	1881	2310	2301	1860	1206	600	214	42

Table 3.3: Dimension of $D_{n,m}$.

Finally, since $\dim H_{n,m} = \dim D_{n,m} - \dim B_{n,m}$, using Tables 3.2 and 3.3 we get the dimension of $H_{n,m}$ for $n \in [\![0,5]\!]$ and $m \in [\![0,12]\!]$, which appears in Table 3.4.

n \ m	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	8	18	33	42	40	30	16	6	1	0
4	0	0	0	0	2	6	11	12	11	6	2	0	0
5	0	0	0	0	0	0	0	0	0	0	0	1	0

Table 3.4: Dimension of $H_{n,m}$.

More generally, we have the following result.

Proposition 3.6. *For $n \geqslant 5$, the dimension of $B_{n,m}$ is given by*

$$\dim B_{n,m} = \begin{cases} 0, & \text{if } m = 0, \\ 3n + 30, & \text{if } m = 1, \\ 15n + 147, & \text{if } m = 2, \\ 42n + 408, & \text{if } m = 3, \\ 84n + 810, & \text{if } m = 4, \\ 129n + 1236, & \text{if } m = 5, \\ 159n + 1515, & \text{if } m = 6, \\ 159n + 1506, & \text{if } m = 7, \\ 129n + 1215, & \text{if } m = 8, \\ 84n + 786, & \text{if } m = 9, \\ 42n + 390, & \text{if } m = 10, \\ 15n + 138, & \text{if } m = 11, \\ 3n + 27, & \text{if } m = 12. \end{cases} \tag{3.11}$$

Before giving the proof of the previous proposition, let us first give a direct consequence.

Corollary 3.7. *We have* $\dim H_n = 0$ *for* $n \in \mathbb{N} \setminus \{3,4,5\}$*. Moreover, the dimension of* $H_{n,m}$ *for* $n = 0,3,4,5$ *and* $m \in [\![0,12]\!]$ *is the one given in Table 3.4.*

Proof. By $\dim D_{n,m} = \dim K_{n,m} - \dim B_{n-1,m+1}$, Proposition 3.6, together with Tables 3.2 and 3.3, we have $\dim D_{n,m} = \dim B_{n,m}$ for $n \in \mathbb{N} \setminus \{3,4,5\}$ and $m \in [\![0,12]\!]$. Then the corollary holds. □

In order to prove Proposition 3.6, we need some preparatory results. Let $\mathscr{C}_n = \cup_{(i,j)\in\mathscr{I}_1} \mathscr{C}_n^{i,j}$, where

$$\mathscr{C}_n^{i,j} = \left\{ z_{n-r}^{i,j} z_r^{k,l} \mid (k,l) \in \mathscr{I} \text{ such that } \#\{i,j,k,l\} = 4, r \in [\![0,n]\!] \right\} \subseteq \mathscr{B}_n^{!*}$$

for $(i,j) \in \mathscr{I}_1$ and $n \in \mathbb{N}$, and let $\mathscr{U}_n = \mathscr{B}_n^{!*} \setminus \mathscr{C}_n$ for $n \in \mathbb{N}$. Note that the pair $(k,l) \in \mathscr{I}$ is uniquely determined in the definition of $\mathscr{C}_n^{i,j}$. Given $m,n \in \mathbb{N}$, let $C_{n,m}$ be the subspace of $\Bbbk\mathscr{C}_n \otimes A_m$ spanned by $\{d_{n+1}(z|x) \mid z \in \mathscr{C}_{n+1}, x \in A_{m-1}\}$, $C_{n,m}^{i,j}$ be the subspace of $C_{n,m}$ spanned by $\{d_{n+1}(z|x) \mid z \in \mathscr{C}_{n+1}^{i,j}, x \in A_{m-1}\}$ for $(i,j) \in \mathscr{I}_1$, and $U_{n,m}$ be the subspace of $B_{n,m}$ spanned by $\{d_{n+1}(z|x) \mid z \in \mathscr{U}_{n+1}, x \in A_{m-1}\}$.

Fixing the order $x_{1,2} \prec x_{3,4} \prec x_{1,3} \prec x_{2,3} \prec x_{1,4} \prec x_{2,4}$ (resp., $x_{1,3} \prec x_{2,4} \prec x_{1,2} \prec x_{2,3} \prec x_{1,4} \prec x_{3,4}$, $x_{2,3} \prec x_{1,4} \prec x_{1,2} \prec x_{1,3} \prec x_{2,4} \prec x_{3,4}$), the corresponding basis of A consisting of standard words will be denoted by $W^{1,2}$ (resp., $W^{1,3}$, $W^{2,3}$). It can be explicitly computed using GAP (see Appendix A.1 for $W^{1,2}$). For $(i,j) \in \mathscr{I}_1$, let $(k,l) \in \mathscr{I}$ such that $\#\{i,j,k,l\} = 4$, set $W_m^{i,j} = W^{i,j} \cap A_m$. Set $E_m^{i,j}$ as the subset of $W_m^{i,j}$ containing elements whose first element is not $x_{i,j}$, and set $\tilde{E}_m^{i,j}$ as the subset of $W_m^{i,j}$ containing elements whose first element is neither $x_{i,j}$ nor $x_{k,l}$. Let $\mathfrak{a}_m^{i,j} = \#E_m^{i,j}$ and $\mathfrak{b}_m^{i,j} = \#\tilde{E}_m^{i,j}$ for $m \in [\![0,11]\!]$. The integers $\mathfrak{a}_m^{i,j}$ and $\mathfrak{b}_m^{i,j}$ are easily computed from the explicit description of the bases $W_m^{i,j}$, they are independent of (i,j), so they will be denoted simply by $\mathfrak{a}_m$ and $\mathfrak{b}_m$, respectively, and are given in Table 3.5.

m	0	1	2	3	4	5	6	7	8	9	10	11
$\mathfrak{a}_m$	1	5	14	28	43	53	53	43	28	14	5	1
$\mathfrak{b}_m$	1	4	10	18	25	28	25	18	10	4	1	0

Table 3.5: Values of $\mathfrak{a}_m$ and $\mathfrak{b}_m$.

Lemma 3.8. *We have* $C_{n,m} = \bigoplus_{(i,j)\in\mathscr{I}_1} C_{n,m}^{i,j}$ *and the dimension of* $C_{n,m}^{i,j}$ *is given by*

$$\dim C_{n,m}^{i,j} = \begin{cases} n+2, & \text{if } m = 1, \\ 5n+9, & \text{if } m = 2, \\ 14n+24, & \text{if } m = 3, \\ 28n+46, & \text{if } m = 4, \\ 43n+68, & \text{if } m = 5, \\ 53n+81, & \text{if } m = 6, \\ 53n+78, & \text{if } m = 7, \\ 43n+61, & \text{if } m = 8, \\ 28n+38, & \text{if } m = 9, \\ 14n+18, & \text{if } m = 10, \\ 5n+6, & \text{if } m = 11, \\ n+1, & \text{if } m = 12, \end{cases}$$

for all $(i,j) \in \mathscr{I}_1$ *and* $n \in \mathbb{N}$*. Else* $\dim C_{n,m}^{i,j} = 0$.

Proof. Given $(i,j) \in \mathscr{I}_1$, fix $(k,l) \in \mathscr{I}$ such that $\#\{i,j,k,l\} = 4$. Then, the maps $\Bbbk E_{m-1}^{i,j} \to A_m$ and $\Bbbk\tilde{E}_{m-1}^{i,j} \to A_m$ given by left multiplication by $x_{i,j}$ and by left multiplication by $x_{k,l}$, respectively, are injective for $m \in [\![1,12]\!]$. Hence, using (3.6), we see that the set formed by the elements $(-1)^r z_{n-r}^{i,j} z_r^{k,l}|x_{i,j}x + z_{n-r+1}^{i,j} z_{r-1}^{k,l}|x_{k,l}x$, for $x \in E_{m-1}^{i,j}$ and $r \in [\![0,n]\!]$, together with the elements $z_n^{k,l}|x_{k,l}y$ for $y \in \tilde{E}_{m-1}^{i,j}$ gives a basis of $C_{n,m}^{i,j}$. Then, $\dim C_{n,m}^{i,j} = \mathfrak{a}_{m-1}(n+1) + \mathfrak{b}_{m-1}$, which together with Table 3.5 proves the claim. □

Lemma 3.9. *We have* $\dim U_{n,m} = \dim U_{n+2,m}$ *and* $\dim(U_{n,m} \cap C_{n,m}) = \dim(U_{n+2,m} \cap C_{n+2,m})$ *for* $n \geqslant 5$ *and* $m \in [\![1, 12]\!]$.

Proof. For $n \geqslant 5$, set

$$u_n^{i,j} = \sum_{\substack{r \in [\![1, n-1]\!], \\ r \text{ odd}}} z_{n-r}^{i,j} z_r^{k,l} \quad \text{and} \quad v_n^{i,j} = \sum_{\substack{r \in [\![1, n-1]\!], \\ r \text{ even}}} z_{n-r}^{i,j} z_r^{k,l},$$

for $(i, j) \in \mathscr{I}_1$, $(k, l) \in \mathscr{I}$, $\#\{i, j, k, l\} = 4$, and

$$\mathbb{Q}_n = \mathscr{U}_n \cup \{z_n^{i,j} \mid (i,j) \in \mathscr{I}\} \cup \{u_n^{i,j}, v_n^{i,j} \mid (i,j) \in \mathscr{I}_1\} \subseteq (A^!_{-n})^*.$$

Let $\mathscr{U}_n^{i,j}$ be the subset of $\mathscr{U}_n$ consisting of elements whose first element is $z^{i,j}$ for $(i,j) \in \mathscr{I}_1$. There is an isomorphism $\mathfrak{f}_n : \Bbbk\mathbb{Q}_n \to \Bbbk\mathbb{Q}_{n+2}$ of vector spaces defined by $\mathfrak{f}_n(z) = z_2^{i,j} z$ for $z \in \mathscr{U}_n^{i,j}$ and $(i,j) \in \mathscr{I}_1$, $\mathfrak{f}_n(u_n^{i,j}) = u_{n+2}^{i,j}$, $\mathfrak{f}_n(v_n^{i,j}) = v_{n+2}^{i,j}$, and $\mathfrak{f}_n(z_n^{i,j}) = z_{n+2}^{i,j}$ for $(i,j) \in \mathscr{I}$. Then, the map $\mathfrak{g}_n = \mathfrak{f}_n \otimes \mathrm{id}_A : \Bbbk\mathbb{Q}_n \otimes A \to \Bbbk\mathbb{Q}_{n+2} \otimes A$ is a linear isomorphism. By (3.7), $U_{n,m} \subseteq \Bbbk\mathbb{Q}_n \otimes A_m$ and $\mathfrak{g}_n(U_{n,m}) = U_{n+2,m}$, giving an isomorphism $U_{n,m} \cong U_{n+2,m}$ of vector spaces for $n \geqslant 5$ and $m \in [\![1, 12]\!]$. This proves the first part of the lemma.

Set $F_{n,m} = (\Bbbk\mathbb{Q}_n \otimes A_m) \cap C_{n,m}$ and define $L_{n,m}^{i,j} = \Bbbk\{z_n^{i,j}, z_n^{k,l}, u_n^{i,j}, v_n^{i,j}\} \otimes A_m$ as the subspace of $\Bbbk\mathscr{C}_n^{i,j} \otimes A_m$, where $(i,j) \in \mathscr{I}_1$, $(k,l) \in \mathscr{I}$ with $\#\{i,j,k,l\} = 4$. It is clear that $F_{n,m} = \oplus_{(i,j) \in \mathscr{I}_1} (L_{n,m}^{i,j} \cap C_{n,m}^{i,j})$. Fix $(i,j) \in \mathscr{I}_1$, $(k,l) \in \mathscr{I}$ with $\#\{i,j,k,l\} = 4$. Let $\xi^{i,j} \in C_{n,m}^{i,j}$. Then $\xi^{i,j}$ is of the form

$$\xi^{i,j} = \sum_{\substack{r \in [\![0, n]\!], \\ x \in E_{m-1}^{i,j}}} \lambda_{r,x} \{(-1)^r z_{n-r}^{i,j} z_r^{k,l} | x_{i,j} x + z_{n-r+1}^{i,j} z_{r-1}^{k,l} | x_{k,l} x\} + \sum_{y \in \tilde{E}_{m-1}^{i,j}} \mu_y z_n^{k,l} | x_{k,l} y \tag{3.12}$$

for $\lambda_{r,x}, \mu_y \in \Bbbk$. If $\xi^{i,j} \in L_{n,m}^{i,j}$, then $\xi^{i,j}$ is of the form

$$\xi^{i,j} = \sum_{w \in W_m^{i,j}} (\alpha_w z_n^{i,j} | w + \beta_w z_n^{k,l} | w + \gamma_w u_n^{i,j} | w + \eta_w v_n^{i,j} | w) \tag{3.13}$$

for $\alpha_w, \beta_w, \gamma_w, \eta_w \in \Bbbk$. Comparing the coefficients in (3.12) and (3.13), we obtain

$$\begin{aligned}
&\alpha_{x_{i,j}x} = \lambda_{0,x}, \quad \alpha_{x_{k,l}y} = \lambda_{1,y}, \quad \beta_{x_{i,j}x} = (-1)^n \lambda_{n,x}, \quad \beta_{x_{k,l}y} = \mu_y, \\
&\gamma_{x_{i,j}x} = -\lambda_{p,x} \text{ for } p \in [\![1, n-1]\!] \text{ with } p \text{ odd}, \\
&\gamma_{x_{k,l}y} = \lambda_{q,y} \text{ for } q \in [\![2, n]\!] \text{ with } q \text{ even}, \\
&\eta_{x_{i,j}x} = \lambda_{q,x} \text{ for } q \in [\![2, n-1]\!] \text{ with } q \text{ even}, \\
&\eta_{x_{k,l}y} = \lambda_{p,y} \text{ for } p \in [\![3, n]\!] \text{ with } p \text{ odd},
\end{aligned}$$

where $x \in E_{m-1}^{i,j}$ and $y \in \tilde{E}_{m-1}^{i,j}$. Hence, if n is even, the space $L_{n,m}^{i,j} \cap C_{n,m}^{i,j}$ is spanned by $z_n^{i,j} | x_{i,j} x$, $(v_n^{i,j} | x_{k,l} - u_n^{i,j} | x_{i,j} + z_n^{i,j} | x_{k,l}) x$ for $x \in E_{m-1}^{i,j}$, $z_n^{k,l} | x_{k,l} y$, $(u_n^{i,j} | x_{k,l} + v_n^{i,j} | x_{i,j} + z_n^{k,l} | x_{i,j}) y$ for $y \in \tilde{E}_{m-1}^{i,j}$, $v_n^{i,j} | x_{i,j} w$ and $z_n^{k,l} | x_{i,j} w$ for $w \in E_{m-1}^{i,j} \setminus \tilde{E}_{m-1}^{i,j}$. If n is odd, the space $L_{n,m}^{i,j} \cap C_{n,m}^{i,j}$ is spanned by $z_n^{i,j} | x_{i,j} x$, $(u_n^{i,j} | x_{k,l} + v_n^{i,j} | x_{i,j}) x$ for $x \in E_{m-1}^{i,j}$, $z_n^{k,l} | x_{k,l} y$, $(v_n^{i,j} | x_{k,l} - u_n^{i,j} | x_{i,j} + z_n^{i,j} | x_{k,l} - z_n^{k,l} | x_{i,j}) y$ for $y \in \tilde{E}_{m-1}^{i,j}$, $u_n^{i,j} | x_{i,j} w$ and $z_n^{k,l} | x_{i,j} w$ for $w \in E_{m-1}^{i,j} \setminus \tilde{E}_{m-1}^{i,j}$. We finally note that $U_{n,m} \cap C_{n,m} = U_{n,m} \cap F_{n,m}$ and $\mathfrak{g}_n(F_{n,m}) = F_{n+2,m}$. Hence, $U_{n,m} \cap C_{n,m} \cong U_{n+2,m} \cap C_{n+2,m}$ as vector spaces. This proves the second part of the lemma. □

Proof of Proposition 3.6. By Table 3.2, we obtain that (3.11) holds for $(n,m) \in [\![5, 6]\!] \times [\![0, 12]\!]$. On the other hand, by Lemma 3.9, we get that $\dim B_{n+2,m} - \dim B_{n,m} = \dim C_{n+2,m} - \dim C_{n,m}$ for $n \geqslant 5$ and $m \in [\![1, 12]\!]$. The statement then follows. □

Using GAP, we can easily compute the dimension of $U_{n,m}$ for $n \geqslant 3$ and $m \in [\![1, 12]\!]$, which is given in Table 3.6.

n \ m	1	2	3	4	5	6	7	8	9	10	11	12
3	23	138	422	896	1428	1800	1815	1468	947	466	162	30
$n \geqslant 4$ with n even	24	136	408	850	1344	1690	1716	1406	924	466	168	34
$n \geqslant 5$ with n odd	24	144	434	912	1452	1836	1872	1536	1008	504	180	36

Table 3.6: Dimension of $U_{n,m}$.

Remark 3.10. *Lemma 3.8 tells us that the subcomplex of $\mathrm{K}_\bullet$ formed by the submodules $\Bbbk\mathscr{C}_n \otimes A$ for $n \in \mathbb{N}$ is exact for $n \geqslant 2$.*

3.3.2 The A-module structure of the homology groups

Lemma 3.11. *We have the following isomorphisms*

$$\mathrm{H}_n(\Bbbk) \cong \begin{cases} M^1(-8), & \text{if } n = 4, \\ \Bbbk(-16), & \text{if } n = 5, \\ 0, & \text{if } n \in \mathbb{N} \setminus [\![3,5]\!], \end{cases} \tag{3.14}$$

of graded A-modules, as well as the non-split short exact sequence

$$0 \to M^2(-6) \oplus \Bbbk(-6) \to \mathrm{H}_3(\Bbbk) \to \Bbbk(-8) \to 0 \tag{3.15}$$

of graded A-modules.

Proof. The isomorphism in (3.14) for $n \in \mathbb{N} \setminus [\![3,5]\!]$ follows from Corollary 3.7. Similarly, the isomorphism in (3.14) for $n = 5$ follows immediately from Table 3.4. Recall that we write H_n instead of $\mathrm{H}_n(\Bbbk)$ for $n \in \mathbb{N}$ to simplify the notation.

Let us prove the isomorphism in (3.14) for $n = 4$. The following GAP code shows that the dimension vector of the submodule of H_4 generated by two basis elements a'_1, a'_2 of $H_{4,4}$ is $(2, 6, 11, 12, 11, 6, 2)$. So, Table 3.4 tells us that H_4 is generated by a'_1, a'_2 as an A-module.

```
Imm:=Im(0,4,4);;
RankMat(Imm);
# 1146
gene:=geneMH(0,4,4);;
Append(Imm,gene);
RankMat(Imm);
# 1148
Uh:=UU(gene,4);; Vh:=VV(gene,4);; Wh:=WW(gene,4);;
for r in [5..10] do
    hxr:=HXR(0,Uh,Vh,Wh,4,4,r-4);
    Im4r:=Im(0,4,r);
    Append(Im4r, hxr);
    Print(r, " ", RankMat(Im4r)-RankMat(Im(0,4,r)), "\n");
od;
# 5 6
# 6 11
# 7 12
# 8 11
# 9 6
# 10 2
```

On the other hand, it is direct to check that the generators a'_1, a'_2 of H_4 satisfy the quadratic relations (3.8) defining M^1. Indeed, the following code shows that the dimension of the subspace generated by $B_{4,5}$ together with the elements of the form (3.8) with a'_i instead of a_i coincides with the dimension of $B_{4,5}$.

```
gene:=geneMH(0,4,4);;
Uh:=UU(gene,4);; Vh:=VV(gene,4);; Wh:=WW(gene,4);;
hx:=HXR(0,Uh,Vh,Wh,4,4,1);;
cc:=0*[1..6];;
cc[1]:=hx[1]+hx[7];; cc[2]:=hx[2];; cc[3]:=hx[5];; cc[4]:=hx[6]+hx[12];;
cc[5]:=hx[9];; cc[6]:=hx[10];;
Imm:=Im(0,4,5);;
RankMat(Imm);
# 1752
Append(Imm,cc);
```

```
RankMat(Imm);
# 1752
```

Hence, there is a surjective morphism $M^1(-8) \to H_4$ of graded A-modules. Since the dimension vector of M^1 is $(2, 6, 11, 12, 11, 6, 2)$ by Fact 3.4, we have $H_4 \cong M^1(-8)$ as graded A-modules, as claimed.

Let us now prove the existence of the short exact sequence (3.15). The following GAP code shows that the dimension vector of the submodule of H_3 generated by the basis elements $c'_i, i \in [\![1, 8]\!]$ of $H_{3,3}$ is $(8, 18, 32, 42, 40, 30, 16, 6, 1)$.

```
Imm:=Im(0,3,3);;
RankMat(Imm);
# 515
gene:=geneMH(0,3,3);;
Append(Imm,gene);
RankMat(Imm);
# 523
Uh:=UU(gene,3);; Vh:=VV(gene,3);; Wh:=WW(gene,3);;
for r in [4..11] do
   hxr:=HXR(0,Uh,Vh,Wh,3,3,r-3);
   Im3r:=Im(0,3,r);
   Append(Im3r, hxr);
   Print(r, " ", RankMat(Im3r)-RankMat(Im(0,3,r)), "\n");
od;
# 4 18
# 5 32
# 6 42
# 7 40
# 8 30
# 9 16
# 10 6
# 11 1
```

Let M^4 be the quadratic module generated by the set $\{c_i \mid i \in [\![1, 8]\!]\}$ of eight homogeneous elements of degree zero, subject to the following 30 relations

$$\begin{aligned}
&c_1x_{1,2}, c_1x_{1,3}, c_1x_{2,3}, c_2x_{1,2}, c_2x_{1,4}, c_2x_{2,4}, c_3x_{1,3}, c_3x_{1,4}, c_3x_{3,4}, c_4x_{2,3}, c_4x_{2,4}, c_4x_{3,4},\\
&c_5x_{1,3} - c_1x_{2,4} + c_3x_{2,4}, c_5x_{2,4} + c_2x_{1,3} - c_4x_{1,3}, c_6x_{2,3} + c_1x_{1,4} - c_4x_{1,4},\\
&c_6x_{1,4} - c_2x_{2,3} + c_3x_{2,3}, c_7x_{1,2} + c_1x_{3,4} + c_2x_{3,4}, c_7x_{3,4} + c_3x_{1,2} + c_4x_{1,2}, c_5x_{1,2} + c_6x_{3,4},\\
&c_5x_{3,4} - c_6x_{1,2}, c_6x_{1,3} - c_7x_{2,4}, c_6x_{2,4} + c_7x_{1,3}, c_5x_{1,4} + c_7x_{2,3}, c_5x_{2,3} - c_7x_{1,4},\\
&c_8x_{1,2}, c_8x_{1,3}, c_8x_{2,3}, c_8x_{1,4}, c_8x_{2,4}, c_8x_{3,4}.
\end{aligned} \tag{3.16}$$

Using GAP we get that the dimension vector of M^4 is $(8, 18, 32, 42, 40, 30, 16, 6, 1)$. It is direct to check that the elements $c'_i, i \in [\![1, 8]\!]$ of H_3 satisfy the quadratic relations (3.16). Indeed, the following code shows that the dimension of the subspace generated by $B_{3,4}$ together with the elements of the form (3.16) with c'_i instead of c_i coincides with the dimension of $B_{3,4}$.

```
gene:=geneMH(0,3,3);;
Uh:=UU(gene,3);; Vh:=VV(gene,3);; Wh:=WW(gene,3);;
hx:=HXR(0,Uh,Vh,Wh,3,3,1);;
cc:=0*[1..30];;
cc[1]:=hx[1];; cc[2]:=hx[2];; cc[3]:=hx[3];; cc[4]:=hx[7];; cc[5]:=hx[10];;
cc[6]:=hx[11];; cc[7]:=hx[14];; cc[8]:=hx[16];; cc[9]:=hx[18];; cc[10]:=hx[21];;
cc[11]:=hx[23];; cc[12]:=hx[24];; cc[13]:=hx[5]-hx[17]-hx[26];;
cc[14]:=hx[8]-hx[20]+hx[29];; cc[15]:=hx[30]-hx[31];; cc[16]:=hx[4]-hx[22]+hx[33];;
cc[17]:=hx[9]-hx[15]-hx[34];; cc[18]:=hx[25]+hx[36];; cc[19]:=hx[6]+hx[12]+hx[37];;
cc[20]:=hx[35]+hx[38];; cc[21]:=hx[28]+hx[39];; cc[22]:=hx[27]-hx[40];;
cc[23]:=hx[32]-hx[41];; cc[24]:=hx[13]+hx[19]+hx[42];; cc[25]:=hx[43];;
cc[26]:=hx[44];; cc[27]:=hx[45];; cc[28]:=hx[46];; cc[29]:=hx[47];; cc[30]:=hx[48];;
Imm:=Im(0,3,4);;
RankMat(Imm);
# 1020
Append(Imm,cc);
RankMat(Imm);
# 1020
```

Hence, there is a morphism $M^4(-6) \to H_3$ of graded A-modules whose image is the submodule of H_3 generated by $c'_i, i \in [\![1, 8]\!]$. Since the dimension vectors of M^4 and the submodule of H_3

generated by $c'_i, i \in [\![1,8]\!]$ are the same, the previous morphism is injective. Moreover, the submodule of M^4 generated by $c_i, i \in [\![1,7]\!]$ is isomorphic to M^2 via the map given by $c_i \mapsto h_i$ for $i \in [\![1,7]\!]$, and the submodule of M^4 generated by c_8 is isomorphic to the trivial A-module $\Bbbk$. It is direct to check that these submodules have trivial intersection, by degree reasons. By comparing the Hilbert series of M^4, M^2 and $\Bbbk$ we obtain the isomorphism $M^4 \cong M^2 \oplus \Bbbk$ of graded A-modules. In consequence, there is an injective morphism $M^2(-6) \oplus \Bbbk(-6) \to H_3$. By a direct dimension and grading argument using Table 3.4, its cokernel is exactly $\Bbbk(-8)$.

Finally, we prove that the short exact sequence (3.15) is non-split. Let c_i for $i \in [\![1,33]\!]$ be the basis elements of space $H_{3,5}$ and $p : \mathrm{H}_3(\Bbbk) \to \Bbbk(-8)$ the surjection in (3.15), satisfying that $p(\mathsf{c}_i) = 0$ for $i \in [\![1,32]\!]$, and $p(\mathsf{c}_{33}) = \mathsf{e}_1$, where e_1 is the identity element of $\Bbbk(-8)$. The short exact sequence (3.15) is split if and only if there exists a morphism $s : \Bbbk(-8) \to \mathrm{H}_3(\Bbbk)$ of graded A-modules such that the composition ps is the identity map. Assume that there exists such a map s. Let $m = s(\mathsf{e}_1) \in H_{3,5}$. Then m is of the form $\sum_{i=1}^{32} \lambda_i \mathsf{c}_i + \mathsf{c}_{33}$ for $\lambda_i \in \Bbbk$, and $m.x = s(\mathsf{e}_1).x = s(\mathsf{e}_1.x) = s(0) = 0$ for all $x \in A_+$. In particular, $\sum_{i=1}^{32} \lambda_i \mathsf{c}_i x_{1,2} + \mathsf{c}_{33} x_{1,2} = 0$ for some $\lambda_i \in \Bbbk$, *i.e.* $\mathsf{c}_{33} x_{1,2}$ is a linear combination of $\mathsf{c}_i x_{1,2}$ for $i \in [\![1,32]\!]$. Using GAP, we choose suitable representative elements $\mathsf{c}'_i \in D_{3,5}$ of c_i for $i \in [\![1,33]\!]$, and get that the dimension of the space spanned by $\mathsf{c}'_i x_{1,2}$ for $i \in [\![1,33]\!]$ and elements in $B_{3,6}$, is strictly larger than the dimension of the space spanned by $\mathsf{c}'_i x_{1,2}$ for $i \in [\![1,32]\!]$ and elements in $B_{3,6}$, as the following code shows.

```
gene:=geneMH(0,3,3);;
Uh:=UU(gene,3);; Vh:=VV(gene,3);; Wh:=WW(gene,3);;
hx:=HXR(0,Uh,Vh,Wh,3,3,2);;
hxx:=0*[1..33];;
hxx[1]:=hx[14];; hxx[2]:=hx[15];; hxx[3]:=hx[16];; hxx[4]:=hx[17];; hxx[5]:=hx[18];;
hxx[6]:=hx[19];; hxx[7]:=hx[25];; hxx[8]:=hx[26];; hxx[9]:=hx[27];; hxx[10]:=hx[29];;
hxx[11]:=hx[31];; hxx[12]:=hx[32];; hxx[13]:=hx[39];; hxx[14]:=hx[40];;
hxx[15]:=hx[41];; hxx[16]:=hx[42];; hxx[17]:=hx[50];; hxx[18]:=hx[51];;
hxx[19]:=hx[58];; hxx[20]:=hx[59];; hxx[21]:=hx[60];; hxx[22]:=hx[61];;
hxx[23]:=hx[65];; hxx[24]:=hx[67];; hxx[25]:=hx[77];; hxx[26]:=hx[78];;
hxx[27]:=hx[79];; hxx[28]:=hx[80];; hxx[29]:=hx[88];; hxx[30]:=hx[89];;
hxx[31]:=hx[91];; hxx[32]:=hx[93];; hxx[33]:=Ker(0,3,5)[79];;
Imm:=Im(0,3,5);;
RankMat(Imm);
# 1550
Append(Imm, hxx);
RankMat(Imm);
# 1583
gene:=hxx;;
Uh:=UU(gene,5);; Vh:=VV(gene,5);; Wh:=WW(gene,5);;
cc:=HXR(0,Uh,Vh,Wh,3,5,1);;
cc12:=0*[1..32];;
for i in [1..32] do
    cc12[i]:=cc[6*i-5];
od;
Imm:=Im(0,3,6);;
RankMat(Imm);
# 1890
Append(Imm,cc12);
RankMat(Imm);
# 1910
Append(Imm,[cc[6*33-5]]);
RankMat(Imm);
# 1911
```

This shows that $\mathsf{c}_{33} x_{1,2} \neq 0$, and it is not a linear combination of $\mathsf{c}_i x_{1,2}$ for $i \in [\![1,32]\!]$, which is a contradiction. So, (3.15) is non-split. □

3.4 The homology of the Koszul complex of M^i for $i \in \{1,2,3\}$

For a quadratic A-module M, we write the quadratic dual module $M^{!_m}$ simply by $M^!$, and write the Koszul complex of M by $(\mathrm{K}_\bullet(M), d_\bullet(M))$, where $\mathrm{K}_n(M) = (M^!_{-n})^* \otimes A$ for $n \in \mathbb{N}_0$ and the differential $d_n(M)$ for $n \in \mathbb{N}$ is given by the multiplication of $\sum_{(i,j)\in\mathcal{J}} y_{i,j} \otimes x_{i,j}$ on the left. Let $\mathrm{K}_{n,m}(M) = (M^!_{-n})^* \otimes A_m$, $d_{n,m}(M) = d_n(M)|_{\mathrm{K}_{n,m}(M)} : \mathrm{K}_{n,m}(M) \to \mathrm{K}_{n-1,m+1}(M)$, $B^M_{n,m} = \mathrm{Im}(d_{n+1,m-1}(M))$, $D^M_{n,m} = \mathrm{Ker}(d_{n,m}(M))$ and $\mathrm{H}_{n,m}(M) = D^M_{n,m}/B^M_{n,m}$ for $n \in \mathbb{N}_0$ and $m \in [\![0,12]\!]$. Let $\mathrm{H}_n(M) = \oplus_{m\in[\![0,12]\!]} \mathrm{H}_{n,m}(M)$ for $n \in \mathbb{N}_0$. Using GAP, we can also compute the dimension of the $\mathrm{H}_{n,m}(M^i)$ for n less than some arbitrary positive integer, $m \in [\![1, 12]\!]$ and $i \in [\![1, 3]\!]$.

3.4.1 Homology of the Koszul complex of M^1

In this subsubsection, we compute $\mathrm{H}_n(M^1)$ for all $n \in \mathbb{N}_0$.

3.4.1.1 The dimensions of the homology groups

Recall that $M^1 = (W \otimes A)/(I)$, where W is the 2-dimensional vector space spanned by a_1, a_2, and I is the subspace of $W \otimes V$ spanned by (3.8). The quadratic dual $(M^1)^! = \oplus_{n\in\mathbb{N}_0}(M^1)^!_{-n} = (U \otimes A^!)/(J)$ of M^1 is an $A^!$-module, where U is the 2-dimensional vector space spanned by b_1, b_2 (for $\{b_1, b_2\}$ the dual basis to $\{a_1, a_2\}$), and J is the subspace of $U \otimes V^*$ spanned by

$$\{b_1 y_{1,2} - b_2 y_{1,2}, b_2 y_{1,3}, b_1 y_{2,3}, b_1 y_{1,4}, b_2 y_{2,4}, b_1 y_{3,4} - b_2 y_{3,4}\}. \tag{3.17}$$

Lemma 3.12. *Recall that $\mathscr{B}^! = \cup_{n\in\mathbb{N}_0}\mathscr{B}^!_n$ is the basis of $A^!$. Let $u, v \in \mathscr{B}^!$ and*

$$Y_{1,2} = \{\pm y_{1,2}^{r_1} y_{3,4}^{r_2} | r_1, r_2 \in \mathbb{N}_0\}, Y_{1,3} = \{\pm y_{1,3}^{r_1} y_{2,4}^{r_2} | r_1, r_2 \in \mathbb{N}_0\}, Y_{2,3} = \{\pm y_{2,3}^{r_1} y_{1,4}^{r_2} | r_1, r_2 \in \mathbb{N}_0\}. \tag{3.18}$$

If $uv \in Y_{i,j}$ for $(i, j) \in \mathscr{I}_1$, then $u, v \in Y_{i,j}$.

Proof. We will prove the lemma by induction on the degree of v. Let $u \in \mathscr{B}^!_m$ and $v \in \mathscr{B}^!_n$ for $m, n \in \mathbb{N}_0$. Obviously, the lemma holds for $n = 0$ and $m \in \mathbb{N}_0$. Assume that $v = v'y$ for $y \in \{y_{s,t} | (s, t) \in \mathscr{I}\}$ and $v' \in \mathscr{B}^!_{n-1}$. Note that $uv' = \pm c$, where $c \in \mathscr{B}^!_{m+n-1}$. By Tables A.1 - A.7 together with (3.2) and (3.3), $cy \in Y_{i,j}$ implies that $c, y \in Y_{i,j}$. Then, by induction hypothesis we get that $u, v' \in Y_{i,j}$. In consequence, $v = v'y \in Y_{i,j}$, as was to be shown. □

Lemma 3.13. *Set $T_n = \{b_1 y_{1,2}^k y_{3,4}^{n-k}, b_1 y_{1,3}^k y_{2,4}^{n-k}, b_2 y_{2,3}^k y_{1,4}^{n-k} \mid k \in [\![0, n]\!]\} \subseteq (M^1)^!_{-n}$ for $n \in \mathbb{N}_0$. Note that T_n has cardinal $3(n + 1)$ for $n \in \mathbb{N}$, and cardinal 2 for $n = 0$, since $T_0 = \{b_1, b_2\}$. Then, T_n is a basis of the space $(M^1)^!_{-n}$ for $n \in \mathbb{N}_0$.*

Proof. Note that the space $(M^1)^!_{-n}$ is spanned by $\{b_1 y, b_2 y \mid y \in \mathscr{B}^!_n\}$ for $n \in \mathbb{N}_0$. It is easy to check that

$$\begin{aligned}
b_j y_{1,2}^m y_{1,3} &= \chi_m b_1 y_{2,3}^m y_{1,3} - \chi_{m+1} b_1 y_{2,3}^m y_{1,2} = 0,\\
b_j y_{1,2}^m y_{2,3} &= \chi_m b_2 y_{1,3}^m y_{2,3} - \chi_{m+1} b_2 y_{1,3}^m y_{1,2} = 0,\\
b_j y_{1,2}^m y_{1,4} &= \chi_m b_2 y_{2,4}^m y_{1,4} - \chi_{m+1} b_2 y_{2,4}^m y_{1,2} = 0,\\
b_j y_{1,2}^m y_{2,4} &= \chi_m b_1 y_{1,4}^m y_{2,4} - \chi_{m+1} b_1 y_{1,4}^m y_{1,2} = 0,\\
b_1 y_{1,3}^m y_{1,4} &= \chi_m b_1 y_{3,4}^m y_{1,4} + \chi_{m+1} b_1 y_{1,4}^m y_{3,4} = \chi_m b_2 y_{3,4}^m y_{1,4} + \chi_{m+1} b_1 y_{1,4}^m y_{3,4}\\
&= \chi_m b_2 y_{1,3}^m y_{1,4} + \chi_{m+1} b_1 y_{1,4}^m y_{3,4} = 0,\\
b_1 y_{1,3}^m y_{3,4} &= \chi_m b_1 y_{1,4}^m y_{3,4} - \chi_{m+1} b_1 y_{1,4}^m y_{1,3} = 0,\\
b_2 y_{2,3}^m y_{2,4} &= \chi_m b_2 y_{3,4}^m y_{2,4} + \chi_{m+1} b_2 y_{2,4}^m y_{3,4} = \chi_m b_1 y_{3,4}^m y_{2,4} + \chi_{m+1} b_2 y_{2,4}^m y_{3,4}\\
&= \chi_m b_1 y_{2,3}^m y_{2,4} + \chi_{m+1} b_2 y_{2,4}^m y_{3,4} = 0,\\
b_2 y_{2,3}^m y_{3,4} &= \chi_m b_2 y_{2,4}^m y_{3,4} - \chi_{m+1} b_2 y_{2,4}^m y_{2,3} = 0,
\end{aligned}$$

for $j \in [\![1, 2]\!]$ and $m \in \mathbb{N}$. Together with (3.17), we get that the space $(M^1)^!_{-n}$ is spanned by T_n for $n \in \mathbb{N}_0$.

It is clear that T_0 is linearly independent. Next, we prove that the elements in T_n are linearly independent for $n \in \mathbb{N}$. Suppose that

$$\sum_{k\in[\![0,n]\!]} \alpha_k b_1 y_{1,2}^k y_{3,4}^{n-k} + \sum_{k\in[\![0,n]\!]} \beta_k b_1 y_{1,3}^k y_{2,4}^{n-k} + \sum_{k\in[\![0,n]\!]} \gamma_k b_2 y_{2,3}^k y_{1,4}^{n-k} = 0$$

in $(M^1)^!_{-n}$, where $\alpha_k, \beta_k, \gamma_k \in \Bbbk$ for $k \in [\![0, n]\!]$. Then

$$\begin{aligned}
&\sum_{k\in[\![0,n]\!]} \alpha_k b_1 y_{1,2}^k y_{3,4}^{n-k} + \sum_{k\in[\![0,n]\!]} \beta_k b_1 y_{1,3}^k y_{2,4}^{n-k} + \sum_{k\in[\![0,n]\!]} \gamma_k b_2 y_{2,3}^k y_{1,4}^{n-k} \\
&= \sum_{u\in\mathscr{B}^!_{n-1}} \lambda_{1,u}(b_1 y_{1,2} - b_2 y_{1,2})u + \sum_{u\in\mathscr{B}^!_{n-1}} \lambda_{2,u} b_2 y_{1,3} u + \sum_{u\in\mathscr{B}^!_{n-1}} \lambda_{3,u} b_1 y_{2,3} u \\
&\quad + \sum_{u\in\mathscr{B}^!_{n-1}} \lambda_{4,u} b_1 y_{1,4} u + \sum_{u\in\mathscr{B}^!_{n-1}} \lambda_{5,u} b_2 y_{2,4} u + \sum_{u\in\mathscr{B}^!_{n-1}} \lambda_{6,u}(b_1 y_{3,4} - b_2 y_{3,4})u \in U \otimes A^!,
\end{aligned}$$

where $\lambda_{i,u} \in \Bbbk$ for $i \in [\![1, 6]\!]$ and $u \in \mathscr{B}^!_{n-1}$. So,

$$\begin{aligned}
\sum_{k\in[\![0,n]\!]} \gamma_k b_2 y_{2,3}^k y_{1,4}^{n-k} = &- \sum_{u\in\mathscr{B}^!_{n-1}} \lambda_{1,u} b_2 y_{1,2} u + \sum_{u\in\mathscr{B}^!_{n-1}} \lambda_{2,u} b_2 y_{1,3} u + \sum_{u\in\mathscr{B}^!_{n-1}} \lambda_{5,u} b_2 y_{2,4} u \\
&- \sum_{u\in\mathscr{B}^!_{n-1}} \lambda_{6,u} b_2 y_{3,4} u \in \Bbbk\{b_2\} \otimes A^! \cong A^!
\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}
&\sum_{k\in[\![0,n]\!]} \alpha_k b_1 y_{1,2}^k y_{3,4}^{n-k} + \sum_{k\in[\![0,n]\!]} \beta_k b_1 y_{1,3}^k y_{2,4}^{n-k} \\
&= \sum_{u\in\mathscr{B}^!_{n-1}} \lambda_{1,u} b_1 y_{1,2} u + \sum_{u\in\mathscr{B}^!_{n-1}} \lambda_{3,u} b_1 y_{2,3} u + \sum_{u\in\mathscr{B}^!_{n-1}} \lambda_{4,u} b_1 y_{1,4} u + \sum_{u\in\mathscr{B}^!_{n-1}} \lambda_{6,u} b_1 y_{3,4} u \\
&\in \Bbbk\{b_1\} \otimes A^! \cong A^!.
\end{aligned} \tag{3.20}$$

Lemma 3.12 and (3.19) imply that

$$\sum_{k\in[\![0,n]\!]} \gamma_k y_{2,3}^k y_{1,4}^{n-k} = \sum_{u\in\mathscr{B}^!_{n-1}\cap Y_{1,2}} \lambda_{1,u} y_{1,2} u + \sum_{u\in\mathscr{B}^!_{n-1}\cap Y_{1,2}} \lambda_{6,u} y_{3,4} u = 0$$

in $A^!$, whereas Lemma 3.12 and (3.20) imply that

$$\sum_{k\in[\![0,n]\!]} \alpha_k y_{1,2}^k y_{3,4}^{n-k} = \sum_{u\in\mathscr{B}^!_{n-1}\cap Y_{1,2}} \lambda_{1,u} y_{1,2} u + \sum_{u\in\mathscr{B}^!_{n-1}\cap Y_{1,2}} \lambda_{6,u} y_{3,4} u, \quad \sum_{k\in[\![0,n]\!]} \beta_k y_{1,3}^k y_{2,4}^{n-k} = 0$$

in $A^!$. Hence, $\alpha_k = \beta_k = \gamma_k = 0$ for $k \in [\![0, n]\!]$. The lemma is thus proved. □

Given $n \in \mathbb{N}$, we will denote by $T_n^* = \{x^* \mid x \in T_n\}$ the dual basis of T_n. Note that the differential $d_1(M^1) : \mathrm{K}_1(M^1) \to \mathrm{K}_0(M^1)$ is given by

$$\begin{aligned}
&(b_1 y_{1,2})^*|1 \mapsto b_1^*|x_{1,2} + b_2^*|x_{1,2}, (b_1 y_{1,3})^*|1 \mapsto b_1^*|x_{1,3}, (b_1 y_{2,4})^*|1 \mapsto b_1^*|x_{2,4}, \\
&(b_1 y_{3,4})^*|1 \mapsto b_1^*|x_{3,4} + b_2^*|x_{3,4}, (b_2 y_{2,3})^*|1 \mapsto b_2^*|x_{2,3}, (b_2 y_{1,4})^*|1 \mapsto b_2^*|x_{1,4},
\end{aligned}$$

where $b_s y_{i,j} \in T_1$ and $(b_s y_{i,j})^* \in T_1^*$ is the dual element of $b_s y_{i,j}$. The differential $d_n(M^1) : \mathrm{K}_n(M^1) \to \mathrm{K}_{n-1}(M^1)$ for $n \geqslant 2$ is given by

$$(b_s y_{i,j}^{n-r} y_{k,l}^r)^*|1 \mapsto (-1)^r (b_s y_{i,j}^{n-1-r} y_{k,l}^r)^*|x_{i,j} + (b_s y_{i,j}^{n-r} y_{k,l}^{r-1})^*|x_{k,l},$$

where $s \in [\![1, 2]\!]$, $r \in [\![0, n]\!]$, $(i, j) \in \mathscr{I}_1$, $(k, l) \in \mathscr{I}$ with $\#\{i, j, k, l\} = 4$, $b_s y_{i,j}^{n-r} y_{k,l}^r \in T_n$ and $x^* \in ((M^1)^!_{-n})^* \in T_n^*$ is the dual element of $x \in T_n \subseteq (M^1)^!_{-n}$.

Proposition 3.14. *We have* $\dim \mathrm{H}_n(M^1) = 0$ *for integers* $n \geqslant 2$.

Proof. It is clear that there is an isomorphism $((M^1)^!_{-n})^* \otimes A \to \Bbbk\mathscr{C}_n \otimes A$ of chain complex of graded A-modules given by $(b_s y_{i,j}^{n-r} y_{k,l}^r)^*|x \mapsto z_{n-r}^{i,j} z_r^{k,l}|x$, where $x \in A$ and $n \in \mathbb{N}$. So, $\dim B_{n,m}^{M^1} = \dim C_{n,m}$ for $m \in [\![0, 12]\!]$ and $n \in \mathbb{N}$, where $\dim C_{n,m}$ is given by Lemma 3.8. The result now follows from the fact that the Koszul complex $\mathrm{K}_\bullet(M^1)$ is isomorphic to the complex $\Bbbk\mathscr{C}_\bullet \otimes A$ for $\bullet \in \mathbb{N}$ and Remark 3.10. □

Corollary 3.15. *The dimension of $B^{M^1}_{n,m}$ for $n \in \mathbb{N}_0$ and $m \in [\![0,12]\!]$ is given by*

$n \diagdown m$	0	1	2	3	4	5	6	7	8	9	10	11	12
$n=0$	0	6	27	72	131	186	210	192	142	84	38	12	2
$n \in \mathbb{N}$	0	$3n+6$	$15n+27$	$42n+72$	$84n+138$	$129n+204$	$159n+243$	$159n+234$	$129n+183$	$84n+114$	$42n+54$	$15n+18$	$3n+3$

Table 3.7: Dimension of $B^{M^1}_{n,m}$.

Proof. The last row of Table 3.7 follows from Lemma 3.8, since $\dim B^{M^1}_{n,m} = \dim C_{n,m}$ for $n \in \mathbb{N}$ and $m \in [\![0,12]\!]$, as explained in the proof of Proposition 3.14. For the remaining case, note that $\dim B^{M^1}_{0,m} = \dim D^{M^1}_{0,m} = \dim(((M^1)^!_0)^* \otimes A_m) - \dim H_{4,m+4} = 2\dim A_m - \dim H_{4,m+4}$ for $m \in [\![0,12]\!]$. The result now follows. □

Corollary 3.16. *The dimension of $D^{M^1}_{1,m}$ for $m \in [\![0,12]\!]$ is given by*

m	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim D^{M^1}_{1,m}$	0	9	42	121	240	366	444	434	342	214	102	34	6

Table 3.8: Dimension of $D^{M^1}_{1,m}$.

Hence, the dimension of $\mathrm{H}_{1,m}(M^1)$ for $m \in [\![0,12]\!]$ is exactly given in Table 3.9, by $\dim \mathrm{H}_{1,m}(M^1) = \dim D^{M^1}_{1,m} - \dim B^{M^1}_{1,m}$. In particular, $\dim \mathrm{H}_1(M^1) = 194$.

Proof. The result follows directly from $\dim D^{M^1}_{1,m} = \dim(((M^1)^!_{-1})^* \otimes A_m) - \dim B^{M^1}_{0,m+1} = 6\dim A_m - \dim B^{M^1}_{0,m+1}$ for $m \in [\![0,12]\!]$, together with Corollary 3.15. □

We can also use GAP to get the dimension of the homology $\mathrm{H}_n(M^1)$ of the Koszul complex of M^1 for n less than some positive integer. In particular, using Appendix A.4 and the following routine in GAP

```
for j in [0..8] do
    for i in [1..12] do
        Print(j, " ", i, " ", RankMat(FF(1,j,i)), "\n");
    od;
od;
```

we obtain the dimension of $B^{M^1}_{n,m}$ for $n \in [\![0,8]\!]$ and $m \in [\![1,12]\!]$. The dimension of homology of the Koszul complex of M^1 for $n = 1$ and $m \in [\![0,12]\!]$ is given in Table 3.9. The dimensions that are not listed in the following table are zeros.

$n \diagdown m$	0	1	2	3	4	5	6	7	8	9	10	11	12
1				7	18	33	42	41	30	16	6	1	

Table 3.9: Dimension of $\mathrm{H}_{n,m}(M^1)$.

3.4.1.2 The A-module structure of the homology groups

Lemma 3.17. *We have the isomorphism*

$$\mathrm{H}_n(M^1) = 0 \tag{3.21}$$

of graded A-modules for $n \geqslant 2$, as well as the non-split short exact sequence of graded A-modules of the form

$$0 \to M^2(-4) \to \mathrm{H}_1(M^1) \to \Bbbk(-6) \oplus \Bbbk(-8) \to 0. \tag{3.22}$$

Proof. The isomorphism in (3.21) for $n \in \mathbb{N} \setminus \{1\}$ follows from Proposition 3.14. It remains to show the existence of the non-split short exact sequence.

The following GAP code shows that the dimension vector of the A-submodule of $\mathrm{H}_1(M^1)$ generated by basis elements $h'_i, i \in [\![1,7]\!]$ of $\mathrm{H}_{1,3}(M^1)$ is $(7,18,32,42,40,30,16,6,1)$.

```
Imm:=Im(1,1,3);;
RankMat(Imm);
# 114
gene:=geneMH(1,1,3);;
Append(Imm,gene);
RankMat(Imm);
# 121
Uh:=UU(gene,3);; Vh:=VV(gene,3);; Wh:=WW(gene,3);;
for r in [4..11] do
   hxr:=HXR(1,Uh,Vh,Wh,1,3,r-3);
   Im1r:=Im(1,1,r);
   Append(Im1r, hxr);
   Print(r, " ", RankMat(Im1r)-RankMat(Im(1,1,r)), "\n");
od;
# 4 18
# 5 32
# 6 42
# 7 40
# 8 30
# 9 16
# 10 6
# 11 1
```

Moreover, it is direct to check that the elements $h'_i, i \in [\![1,7]\!]$ of $\mathrm{H}_1(M^1)$ satisfy the quadratic relations (3.9) defining M^2. Indeed, the following code shows that the dimension of the subspace generated by $B^{M^1}_{1,4}$ together with the elements of the form (3.9) with h'_i instead of h_i coincides with the dimension of $B^{M^1}_{1,4}$.

```
gene:=geneMH(1,1,3);;
Uh:=UU(gene,3);; Vh:=VV(gene,3);; Wh:=WW(gene,3);;
hx:=HXR(1,Uh,Vh,Wh,1,3,1);;
cc:=0*[1..24];;
cc[1]:=hx[1];; cc[2]:=hx[2];; cc[3]:=hx[3];; cc[4]:=hx[7];; cc[5]:=hx[10];;
cc[6]:=hx[11];; cc[7]:=hx[14];; cc[8]:=hx[16];; cc[9]:=hx[18];; cc[10]:=hx[21];;
cc[11]:=hx[23];; cc[12]:=hx[24];; cc[13]:=hx[5]-hx[17]-hx[26];;
cc[14]:=hx[8]-hx[20]+hx[29];; cc[15]:=hx[30]-hx[31];; cc[16]:=hx[4]-hx[22]+hx[33];;
cc[17]:=hx[9]-hx[15]-hx[34];; cc[18]:=hx[25]+hx[36];; cc[19]:=hx[6]+hx[12]+hx[37];;
cc[20]:=hx[35]+hx[38];; cc[21]:=hx[28]+hx[39];; cc[22]:=hx[27]-hx[40];;
cc[23]:=hx[32]-hx[41];; cc[24]:=hx[13]+hx[19]+hx[42];;
Imm:=Im(1,1,4);;
RankMat(Imm);
# 222
Append(Imm,cc);
RankMat(Imm);
# 222
```

Hence, there is a surjective morphism from $M^2(-4)$ to the submodule of $\mathrm{H}_1(M^1)$ generated by $h'_i, i \in [\![1,7]\!]$, which is an isomorphism of graded A-modules since the dimension vector of M^2 is also $(7,18,32,42,40,30,16,6,1)$. Namely, there is an injective morphism $M^2(-4) \to \mathrm{H}_1(M^1)$ of graded modules. A simple argument using dimensions and grading together with Table 3.9 tells us that the cokernel of this injective morphism is exactly the graded A-module $\Bbbk(-6) \oplus \Bbbk(-8)$, as was to be shown.

We finally show that (3.22) is non-split. Let c_i for $i \in [\![1,33]\!]$ be the basis elements of space $\mathrm{H}_{1,5}(M^1)$ and $p : \mathrm{H}_1(M^1) \to \Bbbk(-6) \oplus \Bbbk(-8)$ the surjection in (3.22), satisfying that $p(\mathsf{c}_i) = 0$ for $i \in [\![1,32]\!]$, and $p(\mathsf{c}_{33}) = \mathsf{e}_1$, where e_1 is the identity element of $\Bbbk(-6)$. The short exact sequence (3.22) is split if and only if there exists a morphism $s : \Bbbk(-6) \oplus \Bbbk(-8) \to \mathrm{H}_1(M^1)$ of graded A-modules such that the composition ps is the identity map. Assume there is such a map s. Let $m = s(\mathsf{e}_1) \in \mathrm{H}_{1,5}(M^1)$. Then m is of the form $\sum_{i=1}^{32} \lambda_i \mathsf{c}_i + \mathsf{c}_{33}$ for $\lambda_i \in \Bbbk$, and $m.x = s(\mathsf{e}_1).x = s(\mathsf{e}_1.x) = s(0) = 0$ for all $x \in A_+$. In particular, $\sum_{i=1}^{32} \lambda_i \mathsf{c}_i x_{1,2} + \mathsf{c}_{33} x_{1,2} = 0$ for some $\lambda_i \in \Bbbk$, *i.e.* $\mathsf{c}_{33} x_{1,2}$ is a linear combination of $\mathsf{c}_i x_{1,2}$ for $i \in [\![1,32]\!]$. Using GAP, we choose suitable representative elements $\mathsf{c}'_i \in D^{M^1}_{1,5}$ of c_i for $i \in [\![1,33]\!]$, and get that the dimension of the space spanned by $\mathsf{c}'_i x_{1,2}$ for $i \in [\![1,33]\!]$ and elements in $B^{M^1}_{1,6}$, is strictly larger than the dimension of the space spanned by $\mathsf{c}'_i x_{1,2}$ for $i \in [\![1,32]\!]$ and elements in $B^{M^1}_{1,6}$, as the following code shows.

```
gene:=geneMH(1,1,3);;
Uh:=UU(gene,3);; Vh:=VV(gene,3);; Wh:=WW(gene,3);;
hx:=HXR(1,Uh,Vh,Wh,1,3,2);;
hxx:=0*[1..33];;
hxx[1]:=hx[14];; hxx[2]:=hx[15];; hxx[3]:=hx[16];; hxx[4]:=hx[17];; hxx[5]:=hx[18];;
hxx[6]:=hx[19];; hxx[7]:=hx[25];; hxx[8]:=hx[26];; hxx[9]:=hx[27];; hxx[10]:=hx[29];;
hxx[11]:=hx[31];; hxx[12]:=hx[32];; hxx[13]:=hx[39];; hxx[14]:=hx[40];;
hxx[15]:=hx[41];; hxx[16]:=hx[42];; hxx[17]:=hx[50];; hxx[18]:=hx[51];;
hxx[19]:=hx[58];; hxx[20]:=hx[59];; hxx[21]:=hx[60];; hxx[22]:=hx[61];;
hxx[23]:=hx[65];; hxx[24]:=hx[67];; hxx[25]:=hx[77];; hxx[26]:=hx[78];;
hxx[27]:=hx[79];; hxx[28]:=hx[80];; hxx[29]:=hx[88];; hxx[30]:=hx[89];;
hxx[31]:=hx[91];; hxx[32]:=hx[93];; hxx[33]:=Ker(1,1,5)[76];;
Imm:=Im(1,1,5);;
RankMat(Imm);
# 333
Append(Imm, hxx);
RankMat(Imm);
# 366
gene:=hxx;;
Uh:=UU(gene,5);; Vh:=VV(gene,5);; Wh:=WW(gene,5);;
cc:=HXR(1,Uh,Vh,Wh,1,5,1);;
cc12:=0*[1..32];;
for i in [1..32] do
    cc12[i]:=cc[6*i-5];
od;
Imm:=Im(1,1,6);;
RankMat(Imm);
# 402
Append(Imm,cc12);
RankMat(Imm);
# 422
Append(Imm,[cc[6*33-5]]);
RankMat(Imm);
# 423
```

This shows that $\mathsf{c}_{33}x_{1,2} \neq 0$, and it is not a linear combination of $\mathsf{c}_i x_{1,2}$ for $i \in [\![1, 32]\!]$, which is a contradiction. So, (3.22) is non-split. $\square$

3.4.2 Homology of the Koszul complex of M^2

3.4.2.1 The dimensions of the homology groups

Recall the definition of the quadratic module M^2 given in Subsection 3.2. Let $\{g_i \mid i \in [\![1, 7]\!]\}$ be the dual basis to the basis $\{h_i \mid i \in [\![1, 7]\!]\}$ of the space of generators of M^2. Then, it is easy to see that the $A^!$-module $(M^2)^!$ is generated by $g_i, i \in [\![1, 7]\!]$, subject to the following 18 relations

$$\begin{aligned}
&g_1y_{3,4} - g_2y_{3,4}, g_3y_{1,2} - g_4y_{1,2}, g_5y_{1,2} - g_6y_{3,4}, g_5y_{3,4} + g_6y_{1,2}, g_1y_{3,4} - g_7y_{1,2}, g_3y_{1,2} - g_7y_{3,4},\\
&g_1y_{2,4} + g_3y_{2,4}, g_2y_{1,3} + g_4y_{1,3}, g_6y_{1,3} + g_7y_{2,4}, g_6y_{2,4} - g_7y_{1,3}, g_1y_{2,4} + g_5y_{1,3}, g_2y_{1,3} - g_5y_{2,4},\\
&g_1y_{1,4} + g_4y_{1,4}, g_2y_{2,3} + g_3y_{2,3}, g_5y_{2,3} + g_7y_{1,4}, g_5y_{1,4} - g_7y_{2,3}, g_1y_{1,4} - g_6y_{2,3}, g_2y_{2,3} + g_6y_{1,4}.
\end{aligned} \tag{3.23}$$

Using GAP we get the basis of $(M^2)^!_{-n}$ for $n \in [\![0, 3]\!]$ given in Appendix A.5. Let $\mathscr{U}_n^{!,M^2}$ be the subset of $(M^2)^!_{-n}$ consisting of the following 24 elements

$$\begin{aligned}
&g_1y_{1,2}^{n-1}y_{1,3}, g_1y_{1,2}^{n-1}y_{2,3}, g_1y_{1,2}^{n-1}y_{1,4}, g_1y_{1,2}^{n-1}y_{2,4}, g_1y_{1,2}^{n-2}y_{1,3}^2, g_1y_{1,2}^{n-2}y_{1,3}y_{1,4}, g_1y_{1,2}^{n-2}y_{1,3}y_{2,4},\\
&g_1y_{1,2}^{n-2}y_{1,3}y_{3,4}, g_1y_{1,2}^{n-2}y_{2,3}y_{1,4}, g_1y_{1,2}^{n-2}y_{2,3}y_{2,4}, g_1y_{1,2}^{n-2}y_{2,3}y_{3,4}, g_1y_{1,2}^{n-2}y_{1,4}^2, g_1y_{1,2}^{n-3}y_{1,3}^2y_{3,4},\\
&g_1y_{1,2}^{n-3}y_{1,3}y_{1,4}^2, g_1y_{1,2}^{n-3}y_{2,3}y_{1,4}^2, g_2y_{1,2}^{n-1}y_{1,4}, g_2y_{1,2}^{n-1}y_{2,4}, g_2y_{1,2}^{n-2}y_{1,4}^2, g_3y_{1,3}^{n-1}y_{1,4}, g_3y_{1,3}^{n-1}y_{3,4},\\
&g_3y_{1,3}^{n-2}y_{1,4}^2, g_4y_{2,3}^{n-1}y_{2,4}, g_4y_{2,3}^{n-1}y_{3,4}, g_4y_{2,3}^{n-2}y_{2,4}^2,
\end{aligned} \tag{3.24}$$

and $\mathscr{C}_n^{!,M^2}$ the subset of $(M^2)^!_{-n}$ consisting of the following $3n + 21$ elements

$$\begin{aligned}
&g_1y_{1,2}^n, g_1y_{1,2}^{n-r}y_{3,4}^r, g_1y_{3,4}^n, g_2y_{1,2}^n, g_3y_{1,2}y_{3,4}^{n-1}, g_3y_{3,4}^n, g_4y_{3,4}^n, g_5y_{1,2}^n, g_5y_{1,2}^{n-1}y_{3,4},\\
&g_1y_{1,3}^n, g_1y_{1,3}^{n-r}y_{2,4}^r, g_1y_{2,4}^n, g_2y_{1,3}y_{2,4}^{n-1}, g_2y_{2,4}^n, g_3y_{1,3}^n, g_4y_{2,4}^n, g_6y_{1,3}^n, g_6y_{1,3}^{n-1}y_{2,4},\\
&g_1y_{2,3}^n, g_1y_{2,3}^{n-r}y_{1,4}^r, g_1y_{1,4}^n, g_2y_{2,3}y_{1,4}^{n-1}, g_2y_{1,4}^n, g_3y_{1,4}^n, g_4y_{2,3}^n, g_5y_{2,3}^n, g_5y_{2,3}^{n-1}y_{1,4},
\end{aligned} \tag{3.25}$$

where $r \in [\![1, n-1]\!]$ and $n \geqslant 4$.

Lemma 3.18. *The set* $T_n^{M^2} = \mathscr{U}_n^{!,M^2} \cup \mathscr{C}_n^{!,M^2}$ *is a basis of* $(M^2)^!_{-n}$ *for* $n \geqslant 4$. *Moreover,* $\dim(M^2)^!_0 = 7$, $\dim(M^2)^!_{-1} = 24$, $\dim(M^2)^!_{-2} = 43$ *and* $\dim(M^2)^!_{-n} = 3n + 45$ *for* $n \geqslant 3$.

Proof. We will prove that the set $T_n^{M^2}$ is a basis of $(M^2)^!_{-n}$ for $n \geqslant 4$. Firstly, using GAP, $T_n^{M^2}$ is a basis of $(M^2)^!_{-n}$ for $n \in [\![4,7]\!]$. Note that the space $(M^2)^!_{-n}$ is spanned by $\{g_i y \mid i \in [\![1,7]\!], y \in \mathscr{B}^!_n\}$ for $n \in \mathbb{N}_0$. Moreover, the following identities are straightforward to verify and are left to the reader:

$$
\begin{aligned}
g_1 y_{1,2}^{n-3} y_{1,3}^2 y_{1,4} &= g_1 y_{1,2}^{n-3} y_{1,4} y_{1,3}^2 = -\chi_n g_1 y_{2,4}^{n-3} y_{1,2} y_{1,3}^2 + \chi_{n+1} g_1 y_{2,4}^{n-3} y_{1,4} y_{1,3}^2 \\
&= \chi_n g_5 y_{1,3} y_{2,4}^{n-4} y_{1,2} y_{1,3}^2 - \chi_{n+1} g_5 y_{1,3} y_{2,4}^{n-4} y_{1,4} y_{1,3}^2 = -g_5 y_{1,2}^3 y_{2,3} y_{2,4}^{n-4} \\
&= -g_6 y_{3,4} y_{1,2}^2 y_{2,3} y_{2,4}^{n-4} = g_6 y_{2,3} y_{1,2}^2 y_{2,4}^{n-3} = g_1 y_{1,4} y_{1,2}^2 y_{2,4}^{n-3} = g_1 y_{1,2}^{n-1} y_{1,4}, \\
g_1 y_{1,2}^{n-3} y_{1,3}^2 y_{2,4} &= g_1 y_{1,2}^{n-3} y_{2,4} y_{1,3}^2 = -\chi_n g_1 y_{1,4}^{n-3} y_{1,2} y_{1,3}^2 + \chi_{n+1} g_1 y_{1,4}^{n-3} y_{2,4} y_{1,3}^2 \\
&= -\chi_n g_6 y_{2,3} y_{1,4}^{n-4} y_{1,2} y_{1,3}^2 + \chi_{n+1} g_6 y_{2,3} y_{1,4}^{n-4} y_{2,4} y_{1,3}^2 \\
&= \chi_n g_6 y_{1,2}^3 y_{1,3} y_{1,4}^{n-4} - \chi_{n+1} g_6 y_{1,2}^{n-2} y_{1,3} y_{1,4} \\
&= -\chi_n g_5 y_{3,4} y_{1,2}^2 y_{1,3} y_{1,4}^{n-4} + \chi_{n+1} g_5 y_{3,4} y_{1,2}^{n-3} y_{1,3} y_{1,4} \\
&= \chi_n g_5 y_{1,3} y_{1,2}^2 y_{1,4}^{n-3} - \chi_{n+1} g_5 y_{1,3} y_{1,4} y_{1,2}^{n-3} y_{1,4} \\
&= -\chi_n g_1 y_{2,4} y_{1,2}^2 y_{1,4}^{n-3} + \chi_{n+1} g_1 y_{2,4} y_{1,4} y_{1,2}^{n-3} y_{1,4} = g_1 y_{1,2}^{n-1} y_{2,4}, \\
g_1 y_{1,3}^{n-1} y_{1,4} &= -\chi_n g_1 y_{3,4}^{n-1} y_{1,3} + \chi_{n+1} g_1 y_{3,4}^{n-1} y_{1,4} = -\chi_n g_7 y_{1,2} y_{3,4}^{n-2} y_{1,3} + \chi_{n+1} g_7 y_{1,2} y_{3,4}^{n-2} y_{1,4} \\
&= -\chi_n g_7 y_{1,4}^2 y_{1,2}^{n-3} y_{1,3} + \chi_{n+1} g_7 y_{2,3} y_{1,2} y_{3,4} y_{1,2}^{n-3} \\
&= \chi_n g_5 y_{2,3} y_{1,4} y_{1,2}^{n-3} y_{1,3} + \chi_{n+1} g_5 y_{1,4} y_{1,2} y_{3,4} y_{1,2}^{n-3} \\
&= -\chi_n g_5 y_{1,3}^2 y_{1,2} y_{2,4} y_{1,2}^{n-4} - \chi_{n+1} g_5 y_{1,3} y_{1,4} y_{1,2}^{n-2} \\
&= \chi_n g_1 y_{2,4} y_{1,3} y_{1,2} y_{2,4} y_{1,2}^{n-4} + \chi_{n+1} g_1 y_{2,4} y_{1,4} y_{1,2}^{n-2} \\
&= \chi_n g_1 y_{1,2}^{n-2} y_{1,3} y_{1,4} + \chi_{n+1} g_1 y_{1,2}^{n-1} y_{1,4}, \\
g_1 y_{1,3}^{n-1} y_{3,4} &= -\chi_n g_1 y_{3,4} y_{1,4} y_{1,3}^{n-2} + \chi_{n+1} g_1 y_{3,4} y_{1,3}^{n-1} = -\chi_n g_7 y_{1,2} y_{1,4} y_{1,3}^{n-2} + \chi_{n+1} g_7 y_{1,2} y_{1,3}^{n-1} \\
&= \chi_n g_7 y_{2,4} y_{1,2} y_{1,3}^{n-2} + \chi_{n+1} g_7 y_{1,3}^2 y_{1,2}^{n-2} \\
&= -\chi_n g_6 y_{1,3} y_{1,2} y_{1,3}^{n-2} + \chi_{n+1} g_6 y_{2,4} y_{1,3} y_{1,2}^{n-2} \\
&= -\chi_n g_6 y_{2,3}^3 y_{1,3}^{n-3} + \chi_{n+1} g_6 y_{2,3} y_{1,3} y_{1,4} y_{1,2}^{n-3} \\
&= -\chi_n g_1 y_{1,4} y_{2,3}^2 y_{1,3}^{n-3} + \chi_{n+1} g_1 y_{1,4} y_{1,3} y_{1,4} y_{1,2}^{n-3} \\
&= \chi_n g_1 y_{1,2}^{n-2} y_{1,3} y_{3,4} + \chi_{n+1} g_1 y_{1,2}^{n-3} y_{1,3}^2 y_{3,4}, \\
g_1 y_{2,3}^{n-1} y_{2,4} &= g_1 y_{2,4} y_{3,4}^{n-1} = -g_5 y_{1,3} y_{3,4}^{n-1} = \chi_n g_5 y_{1,4} y_{1,3}^{n-1} - \chi_{n+1} g_5 y_{1,4}^2 y_{1,3}^{n-2} \\
&= \chi_n g_7 y_{2,3} y_{1,3}^{n-1} - \chi_{n+1} g_7 y_{2,3} y_{1,4} y_{1,3}^{n-2} = -\chi_n g_7 y_{1,2}^{n-1} y_{2,3} - \chi_{n+1} g_7 y_{1,2}^{n-2} y_{2,3} y_{3,4} \\
&= -\chi_n g_1 y_{3,4} y_{1,2}^{n-2} y_{2,3} - \chi_{n+1} g_1 y_{3,4} y_{1,2}^{n-3} y_{2,3} y_{3,4} \\
&= \chi_n g_1 y_{1,2}^{n-2} y_{2,3} y_{2,4} + \chi_{n+1} g_1 y_{1,2}^{n-3} y_{1,3}^2 y_{2,4} = \chi_n g_1 y_{1,2}^{n-2} y_{2,3} y_{2,4} + \chi_{n+1} g_1 y_{1,2}^{n-1} y_{2,4}, \\
g_1 y_{2,3}^{n-1} y_{3,4} &= (-1)^{n+1} g_1 y_{3,4} y_{2,4}^{n-1} = (-1)^{n+1} g_7 y_{1,2} y_{2,4}^{n-1} \\
&= -\chi_n g_7 y_{2,4}^2 y_{1,2} y_{2,4}^{n-3} + \chi_{n+1} g_7 y_{2,4}^{n-1} y_{1,2} \\
&= \chi_n g_6 y_{1,3} y_{2,4} y_{1,2} y_{2,4}^{n-3} - \chi_{n+1} g_6 y_{1,3} y_{2,4}^{n-2} y_{1,2} \\
&= -\chi_n g_6 y_{2,3} y_{1,3} y_{1,4} y_{2,4}^{n-3} + \chi_{n+1} g_6 y_{2,3} y_{1,3}^{n-2} y_{1,4} \\
&= -\chi_n g_1 y_{1,4} y_{1,3} y_{1,4} y_{2,4}^{n-3} + \chi_{n+1} g_1 y_{1,4} y_{1,3}^{n-2} y_{1,4} \\
&= \chi_n g_1 y_{1,2}^{n-2} y_{2,3} y_{3,4} + \chi_{n+1} g_1 y_{1,3}^{n-1} y_{3,4} = \chi_n g_1 y_{1,2}^{n-2} y_{2,3} y_{3,4} + \chi_{n+1} g_1 y_{1,2}^{n-3} y_{1,3}^2 y_{3,4}, \\
g_2 y_{1,3}^2 &= g_5 y_{2,4} y_{1,3} = -g_5 y_{1,3} y_{2,4} = g_1 y_{2,4}^2, \ g_2 y_{2,3}^2 = -g_6 y_{1,4} y_{2,3} = g_6 y_{2,3} y_{1,4} = g_1 y_{1,4}^2, \\
g_2 y_{1,2}^{n-1} y_{1,3} &= -\chi_n g_2 y_{2,3}^3 y_{1,2}^{n-3} + \chi_{n+1} g_2 y_{2,3}^2 y_{1,3} y_{1,2}^{n-3} \\
&= -\chi_n g_1 y_{1,4}^2 y_{2,3} y_{1,2}^{n-3} + \chi_{n+1} g_1 y_{1,4}^2 y_{1,3} y_{1,2}^{n-3} = g_1 y_{1,2}^{n-3} y_{1,3} y_{1,4}^2, \\
g_2 y_{1,2}^{n-1} y_{2,3} &= -\chi_n g_2 y_{1,3}^3 y_{1,2}^{n-3} + \chi_{n+1} g_2 y_{1,3}^2 y_{2,3} y_{1,2}^{n-3} \\
&= -\chi_n g_1 y_{2,4}^2 y_{1,3} y_{1,2}^{n-3} + \chi_{n+1} g_1 y_{2,4}^2 y_{2,3} y_{1,2}^{n-3} = g_1 y_{1,2}^{n-3} y_{2,3} y_{1,4}^2,
\end{aligned}
$$

$$\begin{aligned}
g_3y_{1,2}^2 &= g_7y_{3,4}y_{1,2} = -g_7y_{1,2}y_{3,4} = -g_1y_{3,4}^2,\\
g_5y_{3,4}^2 &= -g_6y_{1,2}y_{3,4} = g_6y_{3,4}y_{1,2} = g_5y_{1,2}^2,\ g_5y_{1,4}^2 = g_7y_{2,3}y_{1,4} = -g_7y_{1,4}y_{2,3} = g_5y_{2,3}^2,\\
g_5y_{1,2}^{n-1}y_{1,3} &= \chi_n g_5y_{1,3}y_{2,3}y_{1,2}^{n-2} + \chi_{n+1}g_5y_{1,3}y_{1,2}^{n-1} = -\chi_n g_1y_{2,4}y_{2,3}y_{1,2}^{n-2} - \chi_{n+1}g_1y_{2,4}y_{1,2}^{n-1}\\
&= \chi_n g_1y_{1,2}^{n-2}y_{2,3}y_{3,4} - \chi_{n+1}g_1y_{1,2}^{n-1}y_{2,4},\\
g_5y_{1,2}^{n-1}y_{2,3} &= -\chi_n g_5y_{1,3}y_{1,2}^{n-1} + \chi_{n+1}g_5y_{1,3}^2y_{2,3}y_{1,2}^{n-3}\\
&= \chi_n g_1y_{2,4}y_{1,2}^{n-1} - \chi_{n+1}g_1y_{2,4}y_{1,3}y_{2,3}y_{1,2}^{n-3}\\
&= -\chi_n g_1y_{1,2}^{n-1}y_{1,4} - \chi_{n+1}g_1y_{1,2}^{n-2}y_{1,3}y_{3,4},\\
g_5y_{1,2}^{n-1}y_{1,4} &= \chi_n g_5y_{3,4}^2y_{1,2}^{n-3}y_{1,4} = \chi_n g_5y_{1,3}^2y_{1,4}y_{2,4}y_{1,2}^{n-4} + \chi_{n+1}g_5y_{1,3}^2y_{1,4}y_{1,2}^{n-3}\\
&= -\chi_n g_1y_{2,4}y_{1,3}y_{1,4}y_{2,4}y_{1,2}^{n-4} - \chi_{n+1}g_1y_{2,4}y_{1,3}y_{1,4}y_{1,2}^{n-3}\\
&= \chi_n g_1y_{1,2}^{n-3}y_{2,3}y_{1,4}^2 + \chi_{n+1}g_1y_{1,2}^{n-2}y_{2,3}y_{2,4},\\
g_5y_{1,2}^{n-1}y_{2,4} &= \chi_n g_5y_{3,4}^2y_{1,2}^{n-3}y_{2,4} = -\chi_n g_5y_{1,3}^2y_{1,4}y_{1,2}^{n-3} - \chi_{n+1}g_5y_{1,3}^2y_{1,4}y_{2,4}y_{1,2}^{n-4}\\
&= \chi_n g_1y_{2,4}y_{1,3}y_{1,4}y_{1,2}^{n-3} + \chi_{n+1}g_1y_{2,4}y_{1,3}y_{1,4}y_{2,4}y_{1,2}^{n-4}\\
&= -\chi_n g_1y_{1,2}^{n-2}y_{1,3}y_{1,4} + \chi_{n+1}g_1y_{1,2}^{n-3}y_{1,3}y_{1,4}^2,\\
g_5y_{2,3}^{n-1}y_{2,4} &= g_5y_{1,4}^2y_{2,3}^{n-3}y_{2,4} = \chi_n g_5y_{1,3}^2y_{2,3}y_{2,4}y_{2,3}^{n-4} - \chi_{n+1}g_5y_{1,3}^2y_{2,3}y_{3,4}y_{2,3}^{n-4}\\
&= -\chi_n g_1y_{2,4}y_{1,3}y_{2,3}y_{2,4}y_{2,3}^{n-4} + \chi_{n+1}g_1y_{2,4}y_{1,3}y_{2,3}y_{3,4}y_{2,3}^{n-4}\\
&= \chi_n g_1y_{1,2}^{n-2}y_{1,3}y_{3,4} + \chi_{n+1}g_1y_{1,2}^{n-3}y_{1,3}y_{1,4}^2,\\
g_5y_{2,3}^{n-1}y_{3,4} &= g_5y_{1,4}^2y_{2,3}^{n-3}y_{3,4} = \chi_n g_5y_{1,3}^2y_{2,3}y_{3,4}y_{2,3}^{n-4} + \chi_{n+1}g_5y_{1,3}^2y_{3,4}y_{2,3}^{n-3}\\
&= -\chi_n g_1y_{2,4}y_{1,3}y_{2,3}y_{3,4}y_{2,3}^{n-4} - \chi_{n+1}g_1y_{2,4}y_{1,3}y_{3,4}y_{2,3}^{n-3}\\
&= -\chi_n g_1y_{1,2}^{n-3}y_{1,3}y_{1,4}^2 + \chi_{n+1}g_1y_{1,2}^{n-2}y_{1,3}y_{2,4},\\
g_6y_{2,4}^2 &= g_7y_{1,2}y_{2,4} = -g_7y_{2,4}y_{1,3} = g_6y_{1,3}^2,\\
g_6y_{1,3}^{n-1}y_{1,4} &= -\chi_n g_6y_{3,4}y_{1,3}^{n-1} + \chi_{n+1}g_6y_{1,4}y_{1,3}^{n-1} = -\chi_n g_5y_{1,2}y_{1,3}^{n-1} - \chi_{n+1}g_2y_{2,3}y_{1,3}^{n-1}\\
&= -\chi_n g_1y_{1,2}^{n-2}y_{2,3}y_{3,4} - \chi_{n+1}g_1y_{1,2}^{n-3}y_{2,3}y_{1,4}^2,\\
g_6y_{1,3}^{n-1}y_{3,4} &= -\chi_n g_6y_{1,4}y_{1,3}^{n-1} + \chi_{n+1}g_6y_{3,4}y_{1,3}^{n-1} = \chi_n g_2y_{2,3}y_{1,3}^{n-1} + \chi_{n+1}g_5y_{1,2}y_{1,3}^{n-1}\\
&= -\chi_n g_1y_{1,2}^{n-3}y_{2,3}y_{1,4}^2 + \chi_{n+1}g_1y_{1,2}^{n-2}y_{2,3}y_{1,4},
\end{aligned}$$

for $n \geqslant 5$. Using the previous identities together with (3.23) we see that the space $(M^2)^!_{-n}$ is spanned by $T_n^{M^2}$ for $n \geqslant 8$.

We will next prove that the elements in $T_n^{M^2}$ for $n \geqslant 8$ are linearly independent. Suppose that we have the identity

$$\sum_{i\in[\![1,24]\!]} \alpha_i t_i + \sum_{i\in[\![1,n+7]\!]} \alpha_i^{1,2}t_i^{1,2} + \sum_{i\in[\![1,n+7]\!]} \alpha_i^{1,3}t_i^{1,3} + \sum_{i\in[\![1,n+7]\!]} \alpha_i^{2,3}t_i^{2,3} = \sum_{\substack{i\,\in\,[\![1,\,18]\!],\\ u\,\in\,\mathscr{B}^!_{n-1}}} \lambda_u^i r_i u, \tag{3.26}$$

in $\Bbbk\{g_i | i \in [\![1,7]\!]\} \otimes A^!$, where t_i is the i-th element in (3.24) for $i \in [\![1,24]\!]$, $t_i^{1,2}$ is the i-th element in the first line of (3.25), $t_i^{1,3}$ is the i-th element in the second line of (3.25), and $t_i^{2,3}$ is the i-th element in the last line of (3.25) for $i \in [\![1,n+7]\!]$, r_i is the i-th element in (3.23), and $\alpha_i, \alpha_i^{1,2}, \alpha_i^{1,3}, \alpha_i^{2,3}, \lambda_u^i \in \Bbbk$. We need to prove that the coefficients α_i vanish for all $i \in [\![1,24]\!]$, as well as that $\alpha_i^{1,2}$, $\alpha_i^{1,3}$ and $\alpha_i^{2,3}$ vanish for all $i \in [\![1,n+7]\!]$. By Lemma 3.12, (3.26) implies that

$$\sum_{i\in[\![1,n+7]\!]} \alpha_i^{1,2}t_i^{1,2} = \sum_{\substack{i\,\in\,[\![1,\,6]\!],\\ u\,\in\,\mathscr{B}^!_{n-1}\,\cap\,Y_{1,2}}} \lambda_u^i r_i u, \tag{3.27}$$

$$\sum_{i\in[\![1,n+7]\!]} \alpha_i^{1,3}t_i^{1,3} = \sum_{\substack{i\,\in\,[\![7,\,12]\!],\\ u\,\in\,\mathscr{B}^!_{n-1}\,\cap\,Y_{1,3}}} \lambda_u^i r_i u, \tag{3.28}$$

$$\sum_{i\in[\![1,n+7]\!]} \alpha_i^{2,3}t_i^{2,3} = \sum_{\substack{i\,\in\,[\![13,\,18]\!],\\ u\,\in\,\mathscr{B}^!_{n-1}\,\cap\,Y_{2,3}}} \lambda_u^i r_i u, \tag{3.29}$$

and

$$\sum_{i\in[\![1,24]\!]}\alpha_i t_i=\sum_{\substack{i\,\in\,[\![1,6]\!],\\ u\,\in\,\mathscr{B}^!_{n-1}\setminus Y_{1,2}}}\lambda^i_u r_i u+\sum_{\substack{i\,\in\,[\![7,12]\!],\\ u\,\in\,\mathscr{B}^!_{n-1}\setminus Y_{1,3}}}\lambda^i_u r_i u+\sum_{\substack{i\,\in\,[\![13,18]\!],\\ u\,\in\,\mathscr{B}^!_{n-1}\setminus Y_{2,3}}}\lambda^i_u r_i u, \tag{3.30}$$

in $\Bbbk\{g_i|i\in[\![1,7]\!]\}\otimes A^!$. By (3.27), we get $\alpha_i^{1,2}=0$ for $i\in[\![1,n+7]\!]$. Indeed, since there is no $g_2y_{1,2}^n$, $g_3y_{3,4}^n$, $g_4y_{3,4}^n$ on the right side of (3.27), we get that $\alpha^{1,2}_{n+2}=\alpha^{1,2}_{n+4}=\alpha^{1,2}_{n+5}=0$. Furthermore, as there is no $g_4y_{1,2}^{n-r}y_{3,4}^r$ for $n-r\in\mathbb{N}$ on the left side of (3.27), we see that $\lambda^2_u=0$ for $u\in\mathscr{B}^!_{n-1}\cap Y_{1,2}$. Moreover, since there is no $g_7y_{3,4}^n$ on the left side of (3.27), we obtain that $\lambda^6_{y_{3,4}^{n-1}}=0$. This implies that $\alpha^{1,2}_{n+3}=0$ and $\lambda^6_u=0$ for $u\in\mathscr{B}^!_{n-1}\cap Y_{1,2}$. Finally, since there is no g_2u and g_7u for $u\in\mathscr{B}^!_n$ on the left side of (3.27), we have that $\lambda^1_u=\lambda^5_u=0$ for $u\in\mathscr{B}^!_{n-1}\cap Y_{1,2}$. In consequence, we get $\alpha_i^{1,2}=0$ for $i\in[\![1,n+1]\!]$. Now, we have that

$$\alpha^{1,2}_{n+6}g_5y_{1,2}^n+\alpha^{1,2}_{n+7}g_5y_{1,2}^{n-1}y_{3,4}=\sum_{r\in[\![0,n-1]\!]}\mathscr{a}_r(g_5y_{1,2}-g_6y_{3,4})u+\sum_{r\in[\![0,n-1]\!]}\mathscr{b}_r(g_5y_{3,4}+g_6y_{1,2})u$$

in $\Bbbk\{g_i|i\in[\![1,7]\!]\}\otimes A^!$, where $\mathscr{a}_r=\lambda^3_{y_{1,2}^{n-1-r}y_{3,4}^r}$ and $\mathscr{b}_r=\lambda^4_{y_{1,2}^{n-1-r}y_{3,4}^r}$ for $r\in[\![0,n-1]\!]$. Hence,

$$\begin{aligned}&\alpha^{1,2}_{n+6}g_5y_{1,2}^n+\alpha^{1,2}_{n+7}g_5y_{1,2}^{n-1}y_{3,4}\\&=\mathscr{a}_0g_5y_{1,2}^n+\sum_{r\in[\![1,n-1]\!]}\Big(\mathscr{a}_r+\big((-1)^r\chi_n+(-1)^{r-1}\chi_{n+1}\big)\mathscr{b}_{r-1}\Big)g_5y_{1,2}^{n-r}y_{3,4}^r+\mathscr{b}_{n-1}g_5y_{3,4}^n\\&\quad+\mathscr{b}_0g_6y_{1,2}^n+\sum_{r\in[\![1,n-1]\!]}\Big(\mathscr{b}_r+\big((-1)^{r-1}\chi_n+(-1)^r\chi_{n+1}\big)\mathscr{a}_{r-1}\Big)g_6y_{1,2}^{n-r}y_{3,4}^r-\mathscr{a}_{n-1}g_6y_{3,4}^n\end{aligned}$$

in $\Bbbk\{g_i|i\in[\![1,7]\!]\}\otimes A^!$. Comparing the coefficients, it is easy to see that $\alpha^{1,2}_{n+6}=\alpha^{1,2}_{n+7}=0$ and $\mathscr{a}_r=\mathscr{b}_r=0$ for $r\in[\![0,n-1]\!]$. Similarly, (3.28) implies $\alpha_i^{1,3}=0$ for $i\in[\![1,n+7]\!]$, and (3.29) implies $\alpha_i^{2,3}=0$ for $i\in[\![1,n+7]\!]$. By regarding the coefficients of g_i in (3.30) for $i\in[\![1,7]\!]$, we get that (3.30) is tantamount to

$$\begin{aligned}g_1\big(y_{3,4}\Delta^1+y_{3,4}\Delta^5+y_{2,4}\Delta^7+y_{2,4}\Delta^{11}+y_{1,4}\Delta^{13}+y_{1,4}\Delta^{17}\big)&=\sum_{i\in[\![1,15]\!]}\alpha_it_i,\\ g_2\big(-y_{3,4}\Delta^1+y_{1,3}\Delta^8+y_{1,3}\Delta^{12}+y_{2,3}\Delta^{14}+y_{2,3}\Delta^{18}\big)&=\sum_{i\in[\![16,18]\!]}\alpha_it_i,\\ g_3\big(y_{1,2}\Delta^2+y_{1,2}\Delta^6+y_{2,4}\Delta^7+y_{2,3}\Delta^{14}\big)&=\sum_{i\in[\![19,21]\!]}\alpha_it_i,\\ g_4\big(-y_{1,2}\Delta^2+y_{1,3}\Delta^8+y_{1,4}\Delta^{13}\big)&=\sum_{i\in[\![22,24]\!]}\alpha_it_i,\\ g_5\big(y_{1,2}\Delta^3+y_{3,4}\Delta^4+y_{1,3}\Delta^{11}-y_{2,4}\Delta^{12}+y_{2,3}\Delta^{15}+y_{1,4}\Delta^{16}\big)&=0,\\ g_6\big(-y_{3,4}\Delta^3+y_{1,2}\Delta^4+y_{1,3}\Delta^9+y_{2,4}\Delta^{10}-y_{2,3}\Delta^{11}+y_{1,4}\Delta^{12}\big)&=0,\\ g_7\big(-y_{1,2}\Delta^5-y_{3,4}\Delta^6+y_{2,4}\Delta^9-y_{1,3}\Delta^{10}+y_{1,4}\Delta^{15}-y_{2,3}\Delta^{16}\big)&=0,\end{aligned}\tag{3.31}$$

in $\Bbbk\{g_i\}\otimes A^!$ for $i\in[\![1,7]\!]$ respectively, where $\Delta^j=\sum_{u\in\mathscr{B}^!_{n-1}\setminus Y_{1,2}}\lambda^j_uu$ for $j\in[\![1,6]\!]$, $\Delta^j=\sum_{u\in\mathscr{B}^!_{n-1}\setminus Y_{1,3}}\lambda^j_uu$ for $j\in[\![7,12]\!]$, $\Delta^j=\sum_{u\in\mathscr{B}^!_{n-1}\setminus Y_{2,3}}\lambda^j_uu$ for $j\in[\![13,18]\!]$ and $Y_{i,j}$ is defined in (3.18). In consequence, we see that the elements in $T_n^{M^2}$ are linearly independent if and only if equation (3.31) implies that $\alpha_i=0$ for $i\in[\![1,24]\!]$.

Let

$$a_0^i=\lambda^i_{y_{1,2}^{n-1}},\ a_0'^i=\lambda^i_{y_{3,4}^{n-1}},\ a_1^i=\sum_{\substack{r\,\in\,[\![1,\,n\,-\,2]\!],\\ r\ odd}}\lambda^i_{y_{1,2}^{n-1-r}y_{3,4}^r},\ a_2^i=\sum_{\substack{r\,\in\,[\![1,\,n\,-\,2]\!],\\ r\ even}}\lambda^i_{y_{1,2}^{n-1-r}y_{3,4}^r},$$

$$b_0^j=\lambda^j_{y_{1,3}^{n-1}},\ b_0'^j=\lambda^j_{y_{2,4}^{n-1}},\ b_1^j=\sum_{\substack{r\,\in\,[\![1,\,n\,-\,2]\!],\\ r\ odd}}\lambda^j_{y_{1,3}^{n-1-r}y_{2,4}^r},\ b_2^j=\sum_{\substack{r\,\in\,[\![1,\,n\,-\,2]\!],\\ r\ even}}\lambda^j_{y_{1,3}^{n-1-r}y_{2,4}^r},$$

$$c_0^k = \lambda^k_{y_{2,3}^{n-1}}, c_0'^k = \lambda^k_{y_{1,4}^{n-1}}, c_1^k = \sum_{\substack{r\in [\![1,n-2]\!],\\ r\ odd}} \lambda^k_{y_{2,3}^{n-1-r}y_{1,4}^r}, c_2^k = \sum_{\substack{r\in [\![1,n-2]\!],\\ r\ even}} \lambda^k_{y_{2,3}^{n-1-r}y_{1,4}^r},$$

for $i \in [\![7,18]\!]$, $j \in [\![1,6]\!] \cup [\![13,18]\!]$ and $k \in [\![1,12]\!]$. From (3.31) as well as the products (3.4) and (3.5) in $A^!$, we get a system E_n of linear equations in the field $\Bbbk$, which contains $24 \times 7 = 168$ linear equations and $24 + 24 \times 18 + 4 \times 12 \times 3 = 600$ variables α_i, λ_u^j for $u \in \mathscr{U}^!_{n-1}$, a_0^i, $a_0'^i$, a_1^i, a_2^i, b_0^j, $b_0'^j$, b_1^j, b_2^j, c_0^k, $c_0'^k$, c_1^k, c_2^k. Moreover, the linear independence of $T_n^{M^2}$ (or, equivalently, the fact that (3.31) implies that $\alpha_i = 0$ for $i \in [\![1,24]\!]$) is equivalent to the fact that the linear system E_n implies that $\alpha_i = 0$ for $i \in [\![1,24]\!]$. Note that E_n has the same form when n increases by 2. Using GAP, the elements in $T_n^{M^2}$ are linearly independent for $n \in \{8,9\}$, so the lemma holds for all integers $n \geqslant 8$. $\square$

Let $\mathscr{U}_n^{M^2}$ be the dual basis to $\mathscr{U}_n^{!,M^2}$, $\mathscr{C}_n^{M^2}$ the dual basis to $\mathscr{C}_n^{!,M^2}$, and $\mathscr{C}_n^{M^2} = \cup_{(i,j)\in\mathscr{I}_1} \mathscr{C}_n^{i,j,M^2}$, where $\mathscr{C}_n^{i,j,M^2}$ is the subset of $\mathscr{C}_n^{M^2}$ consisting of elements of the form $(g_s y_{i,j}^{n-\bullet} y_{k,l}^{\bullet})^*$ for $(i,j) \in \mathscr{I}_1$, $(k,l) \in \mathscr{I}$ with $\#\{i,j,k,l\} = 4$. Given $n, m \in \mathbb{N}$, let $C_{n,m}^{M^2}$ be the subspace of $\Bbbk\mathscr{C}_n^{M^2} \otimes A_m$ spanned by

$$\{d_{n+1}(M^2)(z|x) \mid z \in \mathscr{C}_{n+1}^{M^2}, x \in A_{m-1}\},$$

$C_{n,m}^{i,j,M^2}$ the subspace of $C_{n,m}^{M^2}$ spanned by

$$\{d_{n+1}(M^2)(z|x) \mid z \in \mathscr{C}_{n+1}^{i,j,M^2}, x \in A_{m-1}\}$$

for $(i,j) \in \mathscr{I}_1$, and $U_{n,m}^{M^2}$ the subspace of $B_{n,m}^{M^2}$ spanned by

$$\{d_{n+1}(M^2)(z|x) \mid z \in \mathscr{U}_{n+1}^{M^2}, x \in A_{m-1}\}.$$

Using the actions listed in Appendix A.6, it is direct but lengthy to check that the differential in the subcomplex $\Bbbk\mathscr{C}_{n+1}^{1,2,M^2} \otimes A$ of the Koszul complex is given by

$$\begin{aligned}
(g_1 y_{1,2}^{n+1})^*|1 &\mapsto (g_1 y_{1,2}^n)^*|x_{1,2},\\
(g_1 y_{1,2}^n y_{3,4})^*|1 &\mapsto -(g_1 y_{1,2}^{n-1} y_{3,4})^*|x_{1,2} + (g_1 y_{1,2}^n)^*|x_{3,4} + (g_2 y_{1,2}^n)^*|x_{3,4},\\
(g_1 y_{1,2}^{n+1-r} y_{3,4}^r)^*|1 &\mapsto (-1)^r (g_1 y_{1,2}^{n-r} y_{3,4}^r)^*|x_{1,2} + (g_1 y_{1,2}^{n+1-r} y_{3,4}^{r-1})^*|x_{3,4} \text{ for } r \in [\![2,n-1]\!],\\
(g_1 y_{1,2} y_{3,4}^n)^*|1 &\mapsto (-1)^n (g_1 y_{3,4}^n)^*|x_{1,2} + (g_1 y_{1,2} y_{3,4}^{n-1})^*|x_{3,4},\\
(g_1 y_{3,4}^{n+1})^*|1 &\mapsto (g_1 y_{3,4}^n)^*|x_{3,4} + (-1)^n (g_3 y_{1,2} y_{3,4}^{n-1})^*|x_{1,2},\\
(g_2 y_{1,2}^{n+1})^*|1 &\mapsto (g_2 y_{1,2}^n)^*|x_{1,2},\\
(g_3 y_{1,2} y_{3,4}^n)^*|1 &\mapsto (-1)^n (g_3 y_{3,4}^n)^*|x_{1,2} + (-1)^n (g_4 y_{3,4}^n)^*|x_{1,2} + (g_3 y_{1,2} y_{3,4}^{n-1})^*|x_{3,4},\\
(g_3 y_{3,4}^{n+1})^*|1 &\mapsto (g_3 y_{3,4}^n)^*|x_{3,4},\\
(g_4 y_{3,4}^{n+1})^*|1 &\mapsto (g_4 y_{3,4}^n)^*|x_{3,4},\\
(g_5 y_{1,2}^{n+1})^*|1 &\mapsto (g_5 y_{1,2}^n)^*|x_{1,2} + (g_5 y_{1,2}^{n-1} y_{3,4})^*|x_{3,4},\\
(g_5 y_{1,2}^n y_{3,4})^*|1 &\mapsto -(g_5 y_{1,2}^{n-1} y_{3,4})^*|x_{1,2} + (g_5 y_{1,2}^n)^*|x_{3,4},
\end{aligned}$$

for $n \geqslant 4$. Similarly, the differential in $\Bbbk\mathscr{C}_{n+1}^{1,3,M^2} \otimes A$ is given by

$$\begin{aligned}
(g_1 y_{1,3}^{n+1})^*|1 &\mapsto (g_1 y_{1,3}^n)^*|x_{1,3},\\
(g_1 y_{1,3}^n y_{2,4})^*|1 &\mapsto -(g_1 y_{1,3}^{n-1} y_{2,4})^*|x_{1,3} + (g_1 y_{1,3}^n)^*|x_{2,4} - (g_3 y_{1,3}^n)^*|x_{2,4},\\
(g_1 y_{1,3}^{n+1-r} y_{2,4}^r)^*|1 &\mapsto (-1)^r (g_1 y_{1,3}^{n-r} y_{2,4}^r)^*|x_{1,3} + (g_1 y_{1,3}^{n+1-r} y_{2,4}^{r-1})^*|x_{2,4} \text{ for } r \in [\![2,n-1]\!],\\
(g_1 y_{1,3} y_{2,4}^n)^*|1 &\mapsto (-1)^n (g_1 y_{2,4}^n)^*|x_{1,3} + (g_1 y_{1,3} y_{2,4}^{n-1})^*|x_{2,4},\\
(g_1 y_{2,4}^{n+1})^*|1 &\mapsto (g_1 y_{2,4}^n)^*|x_{2,4} - (-1)^n (g_2 y_{1,3} y_{2,4}^{n-1})^*|x_{1,3},\\
(g_2 y_{1,3} y_{2,4}^n)^*|1 &\mapsto (-1)^n (g_2 y_{2,4}^n)^*|x_{1,3} - (-1)^n (g_4 y_{2,4}^n)^*|x_{1,3} + (g_2 y_{1,3} y_{2,4}^{n-1})^*|x_{2,4},\\
(g_2 y_{2,4}^{n+1})^*|1 &\mapsto (g_2 y_{2,4}^n)^*|x_{2,4},\\
(g_3 y_{1,3}^{n+1})^*|1 &\mapsto (g_3 y_{1,3}^n)^*|x_{1,3},
\end{aligned}$$

$$
\begin{aligned}
(g_4y_{2,4}^{n+1})^*|1 &\mapsto (g_4y_{2,4}^{n})^*|x_{2,4},\\
(g_6y_{1,3}^{n+1})^*|1 &\mapsto (g_6y_{1,3}^{n})^*|x_{1,3} + (g_6y_{1,3}^{n-1}y_{2,4})^*|x_{2,4},\\
(g_6y_{1,3}^{n}y_{2,4})^*|1 &\mapsto -(g_6y_{1,3}^{n-1}y_{2,4})^*|x_{1,3} + (g_6y_{1,3}^{n})|x_{2,4},
\end{aligned}
$$

for $n \geqslant 4$, whereas the differential in $\Bbbk\mathscr{C}_{n+1}^{2,3,M^2} \otimes A$ is given by

$$
\begin{aligned}
(g_1y_{2,3}^{n+1})^*|1 &\mapsto (g_1y_{2,3}^{n})^*|x_{2,3},\\
(g_1y_{2,3}^{n}y_{1,4})^*|1 &\mapsto -(g_1y_{2,3}^{n-1}y_{1,4})^*|x_{2,3} + (g_1y_{2,3}^{n})^*|x_{1,4} - (g_4y_{2,3}^{n})^*|x_{1,4},\\
(g_1y_{2,3}^{n+1-r}y_{1,4}^{r})^*|1 &\mapsto (-1)^r(g_1y_{2,3}^{n-r}y_{1,4}^{r})^*|x_{2,3} + (g_1y_{2,3}^{n+1-r}y_{1,4}^{r-1})^*|x_{1,4} \text{ for } r \in [\![2, n-1]\!],\\
(g_1y_{2,3}y_{1,4}^{n})^*|1 &\mapsto (-1)^n(g_1y_{1,4}^{n})^*|x_{2,3} + (g_1y_{2,3}y_{1,4}^{n-1})^*|x_{1,4},\\
(g_1y_{1,4}^{n+1})^*|1 &\mapsto (g_1y_{1,4}^{n})^*|x_{1,4} - (-1)^n(g_2y_{2,3}y_{1,4}^{n-1})^*|x_{2,3},\\
(g_2y_{2,3}y_{1,4}^{n})^*|1 &\mapsto (-1)^n(g_2y_{1,4}^{n})^*|x_{2,3} - (-1)^n(g_3y_{1,4}^{n})^*|x_{2,3} + (g_2y_{2,3}y_{1,4}^{n-1})^*|x_{1,4},\\
(g_2y_{1,4}^{n+1})^*|1 &\mapsto (g_2y_{1,4}^{n})^*|x_{1,4},\\
(g_3y_{1,4}^{n+1})^*|1 &\mapsto (g_3y_{1,4}^{n})^*|x_{1,4},\\
(g_4y_{2,3}^{n+1})^*|1 &\mapsto (g_4y_{2,3}^{n})^*|x_{2,3},\\
(g_5y_{2,3}^{n+1})^*|1 &\mapsto (g_5y_{2,3}^{n})^*|x_{2,3} + (g_5y_{2,3}^{n-1}y_{1,4})^*|x_{1,4},\\
(g_5y_{2,3}^{n}y_{1,4})^*|1 &\mapsto -(g_5y_{2,3}^{n-1}y_{1,4})^*|x_{2,3} + (g_5y_{2,3}^{n})^*|x_{1,4},
\end{aligned}
$$

for $n \geqslant 4$.

Recall that the sets $W_m^{i,j}$, $E_m^{i,j}$ and $\tilde{E}_m^{i,j}$ for $(i,j) \in \mathscr{I}_1$ are defined in the paragraph before Table 3.5. For $(i,j) \in \mathscr{I}_1$, $(k,l) \in \mathscr{I}$ with $\#\{i,j,k,l\} = 4$, let $\hat{E}_m^{i,j}$ be the subset of $W_m^{i,j}$ containing elements whose first element is $x_{i,j}$ and second element is not $x_{k,l}$. Let $E_m'^{i,j}$ be the subset of $W_m^{i,j}$ containing elements whose first element is $x_{i,j}$ and the second element is $x_{k,l}$. The left multiplication of $x_{k,l}$ from $\Bbbk\hat{E}_{m-1}^{i,j}$ to $\Bbbk E_m'^{i,j}$ is isomorphic. It is easy to check that $\#(\hat{E}_m^{i,j} \cup \tilde{E}_m^{i,j}) = \mathfrak{a}_m$, where $\mathfrak{a}_m$ is given in Table 3.5. A basis of $C_{n,m}^{1,2,M^2}$ is given by $d_{n+1}(M^2)(t_i|x)$ for $i \in [\![1, n+1]\!] \cup \{n+3, n+7, n+8\}$ and $x \in E_{m-1}^{1,2}$, $d_{n+1}(M^2)(t_i|x)$ for $i \in [\![n+4, n+6]\!]$ and $x \in \hat{E}_{m-1}^{1,2} \cup \tilde{E}_{m-1}^{1,2}$, $d_{n+1}(M^2)((g_1y_{3,4}^{n+1})^*|x)$ for $x \in \tilde{E}_{m-1}^{1,2}$, where $t_i \in \mathscr{C}_{n+1}^{1,2,M^2}$ is the i-th element in the following sequence

$$
\begin{aligned}
&(g_1y_{1,2}^{n+1})^*, (g_1y_{1,2}^{n+1-r}y_{3,4}^{r})^* \text{ for } r \in [\![1,n]\!], (g_1y_{3,4}^{n+1})^*, (g_2y_{1,2}^{n+1})^*, (g_3y_{1,2}y_{3,4}^{n})^*, (g_3y_{3,4}^{n+1})^*,\\
&(g_4y_{3,4}^{n+1})^*, (g_5y_{1,2}^{n+1})^*, (g_5y_{1,2}^{n}y_{3,4})^*.
\end{aligned} \tag{3.32}
$$

A basis of $C_{n,m}^{1,3,M^2}$ is given by $d_{n+1}(M^2)(t_i|x)$ for $i \in [\![1, n+1]\!] \cup \{n+5, n+7, n+8\}$ and $x \in E_{m-1}^{1,3}$, $d_{n+1}(M^2)(t_i|x)$ for $i \in \{n+3, n+4, n+6\}$ and $x \in \hat{E}_{m-1}^{1,3} \cup \tilde{E}_{m-1}^{1,3}$, $d_{n+1}(M^2)((g_1y_{2,4}^{n+1})^*|x)$ for $x \in \tilde{E}_{m-1}^{1,3}$, where $t_i \in \mathscr{C}_{n+1}^{1,3,M^2}$ is the i-th element in the following sequence

$$
\begin{aligned}
&(g_1y_{1,3}^{n+1})^*, (g_1y_{1,3}^{n+1-r}y_{2,4}^{r})^* \text{ for } r \in [\![1,n]\!], (g_1y_{2,4}^{n+1})^*, (g_2y_{1,3}y_{2,4}^{n})^*, (g_2y_{2,4}^{n+1})^*, (g_3y_{1,3}^{n+1})^*,\\
&(g_4y_{2,4}^{n+1})^*, (g_6y_{1,3}^{n+1})^*, (g_6y_{1,3}^{n}y_{2,4})^*.
\end{aligned} \tag{3.33}
$$

A basis of $C_{n,m}^{2,3,M^2}$ is given by $d_{n+1}(M^2)(t_i|x)$ for $i \in [\![1, n+1]\!] \cup [\![n+6, n+8]\!]$ and $x \in E_{m-1}^{2,3}$, $d_{n+1}(M^2)(t_i|x)$ for $i \in [\![n+3, n+5]\!]$ and $x \in \hat{E}_{m-1}^{2,3} \cup \tilde{E}_{m-1}^{2,3}$, $d_{n+1}(M^2)((g_1y_{1,4}^{n+1})^*|x)$ for $x \in \tilde{E}_{m-1}^{2,3}$, where $t_i \in \mathscr{C}_{n+1}^{2,3,M^2}$ is the i-th element in the following sequence

$$
\begin{aligned}
&(g_1y_{2,3}^{n+1})^*, (g_1y_{2,3}^{n+1-r}y_{1,4}^{r})^* \text{ for } r \in [\![1,n]\!], (g_1y_{2,3}y_{1,4}^{n})^*, (g_1y_{1,4}^{n+1})^*, (g_2y_{1,4}^{n+1})^*, (g_3y_{1,4}^{n+1})^*,\\
&(g_4y_{2,3}^{n+1})^*, (g_5y_{2,3}^{n+1})^*, (g_5y_{2,3}^{n}y_{1,4})^*.
\end{aligned} \tag{3.34}
$$

So, $\dim C_{n,m}^{i,j,M^2} = \mathfrak{a}_{m-1}(n+7) + \mathfrak{b}_{m-1}$, where $\mathfrak{a}_m$ and $\mathfrak{b}_m$ are given in Table 3.5.

Lemma 3.19. *We have* $C^{M^2}_{n,m} = \bigoplus_{(i,j)\in\mathscr{I}_1} C^{i,j,M^2}_{n,m}$ *and the dimension of* $C^{i,j,M^2}_{n,m}$ *is given by*

$$\dim C^{i,j,M^2}_{n,m} = \begin{cases} n+8, & \text{if } m=1,\\ 5n+39, & \text{if } m=2,\\ 14n+108, & \text{if } m=3,\\ 28n+214, & \text{if } m=4,\\ 43n+326, & \text{if } m=5,\\ 53n+399, & \text{if } m=6,\\ 53n+396, & \text{if } m=7,\\ 43n+319, & \text{if } m=8,\\ 28n+206, & \text{if } m=9,\\ 14n+102, & \text{if } m=10,\\ 5n+36, & \text{if } m=11,\\ n+7, & \text{if } m=12, \end{cases}$$

for all $(i,j)\in\mathscr{I}_1$ *and* $n\geqslant 4$. *Moreover, if* $(i,j)\in\mathscr{I}_1$, $n\geqslant 4$ *and* $m\geqslant 13$, $\dim C^{i,j,M^2}_{n,m} = 0$.

Lemma 3.20. *We have* $\dim U^{M^2}_{n,m} = \dim U^{M^2}_{n+2,m}$ *and* $\dim(U^{M^2}_{n,m}\cap C^{M^2}_{n,m}) = \dim(U^{M^2}_{n+2,m}\cap C^{M^2}_{n+2,m})$ *for* $n\geqslant 4$ *and* $m\in[\![1,12]\!]$.

Proof. Let

$$u^{i,j}_n = \sum_{\substack{r\,\in\,[\![1,\,n\,-\,1]\!],\\ r\ odd}} (g_1 y^{n-r}_{i,j} y^r_{k,l})^*, \quad v^{i,j}_n = \sum_{\substack{r\,\in\,[\![1,\,n\,-\,1]\!],\\ r\ even}} (g_1 y^{n-r}_{i,j} y^r_{k,l})^*,$$

for $(i,j)\in\mathscr{I}_1$, $(k,l)\in\mathscr{I}$ with $\#\{i,j,k,l\}=4$. Let

$$\mathbb{Q}^{i,j}_n = \big(\mathscr{C}^{i,j,M^2}_n \setminus \{t^{i,j,n}_r | r\in[\![2,n]\!]\}\big)\cup\{u^{i,j}_n, v^{i,j}_n\}$$

for $(i,j)\in\mathscr{I}_1$, where $t^{i,j,n}_r = (g_1 y^{n-r+1}_{i,j} y^{r-1}_{k,l})^* \in \mathscr{C}^{i,j,M^2}_n$ for $r\in[\![2,n]\!]$. Let

$$\mathbb{Q}_n = \mathscr{U}^{M^2}_n \cup (\cup_{(i,j)\in\mathscr{I}_1}\mathbb{Q}^{i,j}_n).$$

It is clear that there is an isomorphism $\mathfrak{f}_n : \Bbbk\mathbb{Q}_n \to \Bbbk\mathbb{Q}_{n+2}$ of vector spaces. Consider thus the linear isomorphism $\mathfrak{g}_n = \mathfrak{f}_n\otimes \mathrm{id}_A : \Bbbk\mathbb{Q}_n\otimes A \to \Bbbk\mathbb{Q}_{n+2}\otimes A$. Then $U^{M^2}_{n,m}\subseteq \Bbbk\mathbb{Q}_n\otimes A_m$ and $\mathfrak{g}_n(U^{M^2}_{n,m}) = U^{M^2}_{n+2,m}$ for $n\geqslant 4$. Hence, $U^{M^2}_{n,m}\cong U^{M^2}_{n+2,m}$ as vector spaces for $n\geqslant 4$.

Let $F^{i,j}_{n,m} = (\Bbbk\mathbb{Q}^{i,j}_n\otimes A_m)\cap C^{i,j,M^2}_{n,m}$ for $(i,j)\in\mathscr{I}_1$. Then

$$U^{M^2}_{n,m}\cap C^{M^2}_{n,m} = U^{M^2}_{n,m}\cap\big(\Bbbk\mathbb{Q}_n\otimes A_m\big)\cap C^{M^2}_{n,m} = U^{M^2}_{n,m}\cap\bigg(\bigoplus_{(i,j)\in\mathscr{I}_1} F^{i,j}_{n,m}\bigg).$$

To prove that $\dim(U^{M^2}_{n,m}\cap C^{M^2}_{n,m}) = \dim(U^{M^2}_{n+2,m}\cap C^{M^2}_{n+2,m})$, it is sufficient to show that $\mathfrak{g}_n(F^{i,j}_{n,m}) = F^{i,j}_{n+2,m}$ for $(i,j)\in\mathscr{I}_1$. This follows directly from the next simple facts, whose proof is left to the reader. If n is even, $F^{i,j}_{n,m}$ is spanned by the elements

(E.1) $(g_1 y^n_{i,j})^*|x_{i,j}x$, $(v^{i,j}_n|x_{k,l} - u^{i,j}_n|x_{i,j} + (g_1 y^n_{i,j})^*|x_{k,l} + \xi^{i,j}|x_{k,l})x$ for $x\in E^{i,j}_{m-1}$,

(E.2) $((g_1 y^n_{k,l})^*|x_{k,l} + \eta^{i,j}|x_{i,j})y$, $(v^{i,j}_n|x_{i,j} + u^{i,j}_n|x_{k,l} + (g_1 y^n_{k,l})^*|x_{i,j})y$ for $y\in\tilde{E}^{i,j}_{m-1}$,

(E.3) $v^{i,j}_n|x_{i,j}w$, $(g_1 y^n_{k,l})^*|x_{i,j}w$ for $w\in E^{i,j}_{m-1}\setminus\tilde{E}^{i,j}_{m-1}$,

(E.4) $d_{n+1}(M^2)(t^{i,j}_r|x)$ for $(r,x)\in\Delta^{i,j}$,

whereas, if n is odd, $F^{i,j}_{n,m}$ is spanned by the elements

(O.1) $(g_1 y^n_{i,j})^*|x_{i,j}x$, $(u^{i,j}_n|x_{k,l} + v^{i,j}_n|x_{i,j})x$ for $x\in E^{i,j}_{m-1}$, (O.2) $((g_1y^n_{k,l})^*|x_{k,l} + \eta^{i,j}|x_{i,j})y$, $(v^{i,j}_n|x_{k,l} - u^{i,j}_n|x_{i,j} + (g_1y^n_{i,j})^*|x_{k,l} + \xi^{i,j}|x_{k,l} - (g_1y^n_{k,l})^*|x_{i,j})y$ for $y \in \tilde{E}^{i,j}_{m-1}$,

(O.3) $u^{i,j}_n|x_{i,j}w$, $(g_1y^n_{k,l})^*|x_{i,j}w$ for $w \in E^{i,j}_{m-1} \setminus \tilde{E}^{i,j}_{m-1}$,

(O.4) $d_{n+1}(M^2)(t^{i,j}_r|x)$ for $(r,x) \in \Delta^{i,j}$.

Here, $t^{1,2}_r$ (resp., $t^{1,3}_r$, $t^{2,3}_r$) is the r-th element in (3.32) (resp., (3.33), (3.34)), and $\xi^{1,2} = (g_2y^n_{1,2})^*$, $\xi^{1,3} = -(g_3y^n_{1,3})^*$, $\xi^{2,3} = -(g_4y^n_{2,3})^*$, $\eta^{1,2} = (-1)^n(g_3y_{1,2}y^{n-1}_{3,4})^*$, $\eta^{1,3} = (-1)^{n+1}(g_2y_{1,3}y^{n-1}_{2,4})^*$, $\eta^{2,3} = (-1)^{n+1}(g_2y_{2,3}y^{n-1}_{1,4})^*$, $\Delta^{1,2} = (\{n+3, n+7, n+8\} \times E^{1,2}_{m-1}) \cup ([\![n+4, n+6]\!] \times (\hat{E}^{1,2}_{m-1} \cup \tilde{E}^{1,2}_{m-1}))$, $\Delta^{1,3} = (\{n+5, n+7, n+8\} \times E^{1,3}_{m-1}) \cup (\{n+3, n+4, n+6\} \times (\hat{E}^{1,3}_{m-1} \cup \tilde{E}^{1,3}_{m-1}))$, and $\Delta^{2,3} = ([\![n+6, n+8]\!] \times E^{2,3}_{m-1}) \cup ([\![n+3, n+5]\!] \times (\hat{E}^{2,3}_{m-1} \cup \tilde{E}^{2,3}_{m-1}))$. □

Recall that $B^{M^2}_{n,m}$ (resp., $D^{M^2}_{n,m}$) is the image (resp., kernel) concentrated in homological degree n and internal degree $m+n$ of the Koszul complex of M^2.

Proposition 3.21. *The dimension of $B^{M^2}_{n,m}$ is given by*

$$\dim B^{M^2}_{n,m} = \begin{cases} 0, & \text{if } m = 0, \\ 3n + 48, & \text{if } m = 1, \\ 15n + 237, & \text{if } m = 2, \\ 42n + 660, & \text{if } m = 3, \\ 84n + 1314, & \text{if } m = 4, \\ 129n + 2010, & \text{if } m = 5, \\ 159n + 2469, & \text{if } m = 6, \\ 159n + 2460, & \text{if } m = 7, \\ 129n + 1989, & \text{if } m = 8, \\ 84n + 1290, & \text{if } m = 9, \\ 42n + 642, & \text{if } m = 10, \\ 15n + 228, & \text{if } m = 11, \\ 3n + 45, & \text{if } m = 12, \end{cases}$$

for $n \geqslant 3$.

Proof. By Lemma 3.20, we have $\dim B^{M^2}_{n+2,m} - \dim B^{M^2}_{n,m} = \dim C^{M^2}_{n+2,m} - \dim C^{M^2}_{n,m}$ for $n \geqslant 4$ and $m \in [\![1, 12]\!]$. Using GAP we get the value of $\dim B^{M^2}_{n,m}$ for $n \in [\![3, 5]\!]$ and $m \in [\![1, 12]\!]$. □

Corollary 3.22. *We have $\mathrm{H}_n(M^2) = 0$ for $n \geqslant 4$.*

Proof. The result follows from $\dim D^{M^2}_{n,m} = (3n + 45)\dim A_m - \dim B^{M^2}_{n-1,m+1}$ for $n \geqslant 4$ and $m \in [\![0, 12]\!]$, and $\dim \mathrm{H}_{n,m}(M^2) = \dim D^{M^2}_{n,m} - \dim B^{M^2}_{n,m}$. □

By Appendix A.4 and the following code

```
for j in [0..8] do
    for i in [1..12] do
        Print(j, " ", i, " ", RankMat(FF(2,j,i)), "\n");
    od;
od;
```

we obtain the dimension of $B^{M^2}_{n,m}$ for $n \in [\![0, 8]\!]$ and $m \in [\![1, 12]\!]$. Then the dimension of $\mathrm{H}_{n,m}(M^2)$ for $n \in [\![1, 8]\!]$ and $m \in [\![0, 12]\!]$ is given by Table 3.10. The dimensions that are not listed in the following table are zeros.

$n \backslash m$	0	1	2	3	4	5	6	7	8	9	10	11	12
1				1		1							
2					2	6	11	12	11	6	2		
3				8	24	48	72	80	72	48	24	8	

Table 3.10: Dimension of $\mathrm{H}_{n,m}(M^2)$.

3.4.2.2 The A-module structure of the homology groups

Lemma 3.23. *We have the following isomorphisms of graded A-modules*

$$\mathrm{H}_n(M^2) \cong \begin{cases} \Bbbk(-4) \oplus \Bbbk(-6), & \text{if } n = 1, \\ M^1(-6), & \text{if } n = 2, \\ M^3(-6), & \text{if } n = 3, \\ 0, & \text{if } n \geqslant 4. \end{cases} \tag{3.35}$$

Proof. The isomorphisms in (3.35) for all integers $n \geqslant 4$ follow immediately from Corollary 3.22. A simple argument using dimension and grading together with Table 3.10 gives the isomoprhism in (3.35) for $n = 1$.

We prove that the space $\mathrm{H}_2(M^2)$ is a quadratic module, which is isomorphic to $M^1(-6)$. The following GAP code shows that the dimension vector of the submodule of $\mathrm{H}_2(M^2)$ generated by two basis elements a''_1, a''_2 of $\mathrm{H}_{2,4}(M^2)$ is $(2, 6, 11, 12, 11, 6, 2)$. So, $\mathrm{H}_2(M^2)$ is generated by the two elements as an A-module.

```
Imm:=Im(2,2,4);;
RankMat(Imm);
# 1474
gene:=geneMH(2,2,4);;
Append(Imm,gene);
RankMat(Imm);
# 1476
Uh:=UU(gene,4);; Vh:=VV(gene,4);; Wh:=WW(gene,4);;
for r in [5..10] do
   hxr:=HXR(2,Uh,Vh,Wh,2,4,r-4);
   Im2r:=Im(2,2,r);
   Append(Im2r, hxr);
   Print(r, " ", RankMat(Im2r)-RankMat(Im(2,2,r)), "\n");
od;
# 5 6
# 6 11
# 7 12
# 8 11
# 9 6
# 10 2
```

Furthermore, it is direct to check that the generators a''_1, a''_2 of $\mathrm{H}_2(M^2)$ satisfy the quadratic relations (3.8) defining M^1. Indeed, the following code shows that the dimension of the subspace generated by $B^{M^2}_{2,5}$ together with the elements of the form (3.8) with a''_i instead of a_i coincides with the dimension of $B^{M^2}_{2,5}$.

```
gene:=geneMH(2,2,4);;
Uh:=UU(gene,4);; Vh:=VV(gene,4);; Wh:=WW(gene,4);;
hx:=HXR(2,Uh,Vh,Wh,2,4,1);;
cc:=0*[1..6];;
cc[1]:=hx[1]+hx[7];; cc[2]:=hx[2];; cc[3]:=hx[5];; cc[4]:=hx[6]+hx[12];;
cc[5]:=hx[9];; cc[6]:=hx[10];;
Imm:=Im(2,2,5);;
RankMat(Imm);
# 2244
Append(Imm,cc);
RankMat(Imm);
# 2244
```

Hence, there is a surjective morphism $M^1(-6) \to \mathrm{H}_2(M^2)$ of graded A-modules. Since the dimension vector of M^1 is $(2, 6, 11, 12, 11, 6, 2)$, we have $\mathrm{H}_2(M^2) \cong M^1(-6)$ as graded A-modules, as claimed.

Next, we prove that the space $\mathrm{H}_3(M^2)$ is also a quadratic module, which is isomorphic to $M^3(-6)$. The following code shows that the dimension vector of the submodule of $\mathrm{H}_3(M^2)$ generated by basis elements $e'_i, i \in [\![1,8]\!]$ of $\mathrm{H}_{3,3}(M^2)$ is $(8,24,48,72,80,72,48,24,8)$. So, $\mathrm{H}_3(M^2)$ is generated by the eight elements $e'_i, i \in [\![1,8]\!]$ as an A-module.

```
Imm:=Im(2,3,3);;
RankMat(Imm);
# 786
gene:=geneMH(2,3,3);;
Append(Imm,gene);
RankMat(Imm);
# 794
Uh:=UU(gene,3);; Vh:=VV(gene,3);; Wh:=WW(gene,3);;
for r in [4..11] do
   hxr:=HXR(2,Uh,Vh,Wh,3,3,r-3);
   Im3r:=Im(2,3,r);
   Append(Im3r, hxr);
   Print(r, " ", RankMat(Im3r)-RankMat(Im(2,3,r)), "\n");
od;
# 4 24
# 5 48
# 6 72
# 7 80
# 8 72
# 9 48
# 10 24
# 11 8
```

Moreover, it is direct to check that the generators $e'_i, i \in [\![1,8]\!]$ of $\mathrm{H}_3(M^2)$ satisfy the quadratic relations (3.10). Indeed, the following code shows that the dimension of the subspace generated by $B^{M^2}_{3,4}$ together with the elements of the form (3.10) with e'_i instead of e_i coincides with the dimension of $B^{M^2}_{3,4}$.

```
gene:=geneMH(2,3,3);;
Uh:=UU(gene,3);; Vh:=VV(gene,3);; Wh:=WW(gene,3);;
hx:=HXR(2,Uh,Vh,Wh,3,3,1);;
cc:=0*[1..24];;
cc[1]:=hx[1]+hx[12];; cc[2]:=hx[6]-hx[7];; cc[3]:=hx[13]-hx[24];;
cc[4]:=hx[18]+hx[19];; cc[5]:=hx[20]+hx[11];; cc[6]:=hx[23]-hx[8];;
cc[7]:=hx[14]+hx[5];; cc[8]:=hx[17]-hx[2];; cc[9]:=hx[3]-hx[22];;
cc[10]:=hx[4]+hx[21];; cc[11]:=hx[15]-hx[10];; cc[12]:=hx[16]+hx[9];;
cc[13]:=hx[25];; cc[14]:=hx[26];; cc[15]:=hx[27];; cc[16]:=hx[31];; cc[17]:=hx[34];;
cc[18]:=hx[35];; cc[19]:=hx[38];; cc[20]:=hx[40];; cc[21]:=hx[42];; cc[22]:=hx[45];;
cc[23]:=hx[47];; cc[24]:=hx[48];;
Imm:=Im(2,3,4);;
RankMat(Imm);
# 1566
Append(Imm,cc);
RankMat(Imm);
# 1566
```

Hence, there is a surjective morphism $M^3(-6) \to \mathrm{H}_3(M^2)$ of graded A-modules. Since the dimension vector of M^3 is $(8,24,48,72,80,72,48,24,8)$, we have $\mathrm{H}_3(M^2) \cong M^3(-6)$ as graded A-modules, as claimed. $\square$

3.4.3 Homology of the Koszul complex of M^3

3.4.3.1 The dimensions of the homology groups

Note first that $M^3 \cong N \oplus (\oplus_{k\in[\![1,4]\!]} S_k)$ as graded A-modules, where N is the submodule of M^3 generated by $e_i, i \in [\![1,4]\!]$, and S_k is the submodule generated by e_{k+4} for $k \in [\![1,4]\!]$. Let $\{f_i \mid i \in [\![1,8]\!]\}$ be the dual basis to $\{e_i \mid i \in [\![1,8]\!]\}$. It is easy to see that the $A^!$-module $(M^3)^!$ is generated by f_i for $i \in [\![1,8]\!]$, subject to the following 24 relations

$$f_1y_{1,2} - f_2y_{3,4}, f_1y_{3,4} + f_2y_{1,2}, f_3y_{1,2} + f_4y_{3,4}, f_3y_{3,4} - f_4y_{1,2}, f_4y_{1,3} - f_2y_{2,4}, f_4y_{2,4} + f_2y_{1,3},$$
$$f_3y_{1,3} - f_1y_{2,4}, f_3y_{2,4} + f_1y_{1,3}, f_1y_{2,3} + f_4y_{1,4}, f_1y_{1,4} - f_4y_{2,3}, f_3y_{2,3} + f_2y_{1,4}, f_3y_{1,4} - f_2y_{2,3},$$
$$f_5y_{1,4}, f_5y_{2,4}, f_5y_{3,4}, f_6y_{1,3}, f_6y_{2,3}, f_6y_{3,4}, f_7y_{1,2}, f_7y_{2,3}, f_7y_{2,4}, f_8y_{1,2}, f_8y_{1,3}, f_8y_{1,4}.$$

Using GAP, a basis of $(M^3)^!_{-1}$ is given by the 24 elements

$$f_1y_{1,2}, f_1y_{1,3}, f_1y_{2,3}, f_1y_{1,4}, f_1y_{2,4}, f_1y_{3,4}, f_2y_{1,3}, f_2y_{2,3}, f_2y_{1,4}, f_2y_{2,4}, f_3y_{1,2}, f_3y_{3,4}, f_5y_{1,2}, f_5y_{1,3}, f_5y_{2,3}, f_6y_{1,2}, f_6y_{1,4}, f_6y_{2,4}, f_7y_{1,3}, f_7y_{1,4}, f_7y_{3,4}, f_8y_{2,3}, f_8y_{2,4}, f_8y_{3,4},$$

and a basis of $(M^3)^!_{-2}$ is given by the 40 elements

$$f_1y^2_{1,2}, f_1y_{1,2}y_{1,3}, f_1y_{1,2}y_{2,3}, f_1y_{1,2}y_{1,4}, f_1y_{1,2}y_{2,4}, f_1y_{1,2}y_{3,4}, f_1y^2_{1,3}, f_1y_{1,3}y_{1,4}, f_1y_{1,3}y_{2,4}, f_1y_{1,3}y_{3,4}, f_1y^2_{2,3}, f_1y_{2,3}y_{1,4}, f_1y_{2,3}y_{2,4}, f_1y_{2,3}y_{3,4}, f_2y^2_{1,3}, f_2y_{1,3}y_{2,4}, f_2y^2_{2,3}, f_2y_{2,3}y_{1,4}, f_3y^2_{1,2}, f_3y_{1,2}y_{3,4}, f_5y^2_{1,2}, f_5y_{1,2}y_{1,3}, f_5y_{1,2}y_{2,3}, f_5y^2_{1,3}, f_5y^2_{2,3}, f_6y^2_{1,2}, f_6y_{1,2}y_{1,4}, f_6y_{1,2}y_{2,4}, f_6y^2_{1,4}, f_6y^2_{2,4}, f_7y^2_{1,3}, f_7y_{1,3}y_{1,4}, f_7y_{1,3}y_{3,4}, f_7y^2_{1,4}, f_7y^2_{3,4}, f_8y^2_{2,3}, f_8y_{2,3}y_{2,4}, f_8y_{2,3}y_{3,4}, f_8y^2_{2,4}, f_8y^2_{3,4}.$$

Remark 3.24. *Let $k \in [\![1,4]\!]$ and $f_{k+4}y_{i_1,j_1}\dots y_{i_s,j_s}$ be a monomial in $(S_k)^!$, where $s \in \mathbb{N}$ and $(i_1,j_1),\dots,(i_s,j_s) \in \mathcal{J}$. If $5-k \in \{i_1,j_1,\dots,i_s,j_s\}$, then $f_{k+4}y_{i_1,j_1}\dots y_{i_s,j_s} = 0 \in (S_k)^!$.*

Remark 3.25. *Let $k \in [\![1,4]\!]$, $y_{i_1,j_1}\dots y_{i_s,j_s} = y_{i'_1,j'_1}\dots y_{i'_s,j'_s}$ in $A^!$ for $(i_1,j_1),\dots,(i_s,j_s) \in \mathcal{J}$, $(i'_1,j'_1),\dots,(i'_s,j'_s) \in \mathcal{J}$ and $s \in \mathbb{N}$. If $k \in \{i_1,j_1,\dots,i_s,j_s\}$, then $k \in \{i'_1,j'_1,\dots,i'_s,j'_s\}$.*

Lemma 3.26. *Let*

$$\begin{aligned}
T^{S_1}_n &= \{f_5y^n_{1,2}, f_5y^{n-1}_{1,2}y_{1,3}, f_5y^{n-1}_{1,2}y_{2,3}, f_5y^{n-2}_{1,2}y^2_{1,3}, f_5y^n_{1,3}, f_5y^n_{2,3}\},\\
T^{S_2}_n &= \{f_6y^n_{1,2}, f_6y^{n-1}_{1,2}y_{1,4}, f_6y^{n-1}_{1,2}y_{2,4}, f_6y^{n-2}_{1,2}y^2_{1,4}, f_6y^n_{1,4}, f_6y^n_{2,4}\},\\
T^{S_3}_n &= \{f_7y^n_{1,3}, f_7y^{n-1}_{1,3}y_{1,4}, f_7y^{n-1}_{1,3}y_{3,4}, f_7y^{n-2}_{1,3}y^2_{1,4}, f_7y^n_{1,4}, f_7y^n_{3,4}\},\\
T^{S_4}_n &= \{f_8y^n_{2,3}, f_8y^{n-1}_{2,3}y_{2,4}, f_8y^{n-1}_{2,3}y_{3,4}, f_8y^{n-2}_{2,3}y^2_{2,4}, f_8y^n_{2,4}, f_8y^n_{3,4}\},
\end{aligned}$$

for $n \geqslant 3$. Then $T^{S_k}_n$ is a basis of $(S_k)^!_{-n}$ for $k \in [\![1,4]\!]$ and $n \geqslant 3$. Note that $T^{S_k}_n$ has cardinal 6 for $k \in [\![1,4]\!]$ and $n \geqslant 3$.

Proof. The space $(S_k)^!_{-n}$ is spanned by $\{f_{k+4}y \mid y \in \mathscr{B}^!_n\}$. By Remark 3.24, the space $(S_k)^!_{-n}$ is spanned by $T^{S_k}_n$ for $n \geqslant 3$. By Remark 3.25, it is easy to see that the elements of $T^{S_k}_n$ are linearly independent. Indeed, let $\sum_{i\in[\![1,6]\!]}\alpha_iq_i = \sum_{u\in\mathscr{B}^!_{n-1}}\lambda^1_uf_5y_{1,4}u + \sum_{u\in\mathscr{B}^!_{n-1}}\lambda^2_uf_5y_{2,4}u + \sum_{u\in\mathscr{B}^!_{n-1}}\lambda^3_uf_5y_{3,4}u$ in $\Bbbk\{f_5\}\otimes A^!$, where $\alpha_i, \lambda^j_u \in \Bbbk$, and q_i is the i-th element of $T^{S_1}_n$. By Remark 3.25, the right side of the equation is a linear combination of elements of form $f_5y_{i_1,j_1}\dots y_{i_n,j_n} \in f_5\mathscr{B}^!_n$ for $4 \in \{i_1,j_1,\dots,i_n,j_n\}$. This implies $\alpha_i = 0$ for $i \in [\![1,6]\!]$. Hence, $T^{S_1}_n$ are linearly independent. The other cases are similar. □

Lemma 3.27. *The set T^N_n consisting of the following 24 elements*

$$\begin{aligned}
&f_1y^n_{1,2}, f_1y^{n-1}_{1,2}y_{3,4}, f_1y^n_{1,3}, f_1y^{n-1}_{1,3}y_{2,4}, f_1y^n_{2,3}, f_1y^{n-1}_{2,3}y_{1,4}, f_2y^n_{1,3}, f_2y^{n-1}_{1,3}y_{2,4}, f_2y^n_{2,3},\\
&f_2y^{n-1}_{2,3}y_{1,4}, f_3y^n_{1,2}, f_3y^{n-1}_{1,2}y_{3,4}, f_1y^{n-1}_{1,2}y_{1,3}, f_1y^{n-1}_{1,2}y_{2,3}, f_1y^{n-1}_{1,2}y_{1,4}, f_1y^{n-1}_{1,2}y_{2,4},\\
&f_1y^{n-2}_{1,2}y^2_{1,3}, f_1y^{n-2}_{1,2}y_{1,3}y_{1,4}, f_1y^{n-2}_{1,2}y_{1,3}y_{2,4}, f_1y^{n-2}_{1,2}y_{1,3}y_{3,4}, f_1y^{n-2}_{1,2}y_{2,3}y_{1,4},\\
&f_1y^{n-2}_{1,2}y_{2,3}y_{2,4}, f_1y^{n-2}_{1,2}y_{2,3}y_{3,4}, f_1y^{n-3}_{1,2}y^2_{1,3}y_{3,4}
\end{aligned} \tag{3.36}$$

is a basis of $N^!_{-n}$ for $n \geqslant 3$.

Proof. Firstly, using GAP, T^N_n is a basis of $N^!_{-n}$ for $n \in [\![3,5]\!]$. Note that the space $N^!_{-n}$ is spanned by $\{f_iy \mid i \in [\![1,4]\!], y \in \mathscr{B}^!_n\}$ for $n \in \mathbb{N}_0$. By the dual relations, it is easy to see that $N^!_{-n}$ is spanned by

$$f_1y^n_{1,2}, f_1y^{n-1}_{1,2}y_{3,4}, f_1y^n_{1,3}, f_1y^{n-1}_{1,3}y_{2,4}, f_1y^n_{2,3}, f_1y^{n-1}_{2,3}y_{1,4}, f_2y^n_{1,3}, f_2y^{n-1}_{1,3}y_{2,4}, f_2y^n_{2,3}, f_2y^{n-1}_{2,3}y_{1,4}, f_3y^n_{1,2}, f_3y^{n-1}_{1,2}y_{3,4}, f_iy,$$

for $i \in [\![1,3]\!]$, $y \in \mathscr{U}^!_n$ and $n \geqslant 2$. Note that $y^2_{i,j}$ is central in $A^!$ and $f_sy^2_{i,j} = f_sy^2_{k,l}$ for $s \in [\![1,4]\!]$ and $(i,j),(k,l) \in \mathcal{J}$ with $\#\{i,j,k,l\} = 4$. For $n \geqslant 5$ and $i \in [\![1,3]\!]$,

$$f_iy^{n-2}_{1,2}y^2_{1,4} = f_iy^{n-2}_{1,2}y^2_{2,3} = f_iy^{n-2}_{1,2}y^2_{1,3},\ f_iy^{n-3}_{1,2}y^2_{1,3}y_{1,4} = f_iy^{n-3}_{1,2}y^2_{2,4}y_{1,4} = f_iy^{n-1}_{1,2}y_{1,4},$$

$$
\begin{aligned}
&f_iy_{1,2}^{n-3}y_{1,3}^2y_{2,4}=f_iy_{1,2}^{n-3}y_{2,4}^3=f_iy_{1,2}^{n-1}y_{2,4},\ f_iy_{1,2}^{n-3}y_{1,3}y_{1,4}^2=f_iy_{1,2}^{n-3}y_{1,3}y_{2,3}^2=f_iy_{1,2}^{n-1}y_{1,3},\\
&f_iy_{1,2}^{n-3}y_{2,3}y_{1,4}^2=f_iy_{1,2}^{n-3}y_{2,3}^3=f_iy_{1,2}^{n-1}y_{2,3},\ f_iy_{1,2}^{n-4}y_{1,3}^2y_{1,4}^2=f_iy_{1,2}^{n-4}y_{1,3}^2y_{2,3}^2=f_iy_{1,2}^{n-2}y_{1,3}^2,
\end{aligned}
$$

$$
\begin{aligned}
f_iy_{1,3}^{n-1}y_{1,4}&=\chi_nf_iy_{1,3}y_{2,4}^{n-2}y_{1,4}+\chi_{n+1}f_iy_{2,4}^{n-1}y_{1,4}=\chi_nf_iy_{1,3}y_{1,2}^{n-2}y_{1,4}+\chi_{n+1}f_iy_{1,2}^{n-1}y_{1,4}\\
&=\chi_nf_iy_{1,2}^{n-2}y_{1,3}y_{1,4}+\chi_{n+1}f_iy_{1,2}^{n-1}y_{1,4},\\
f_iy_{1,3}^{n-1}y_{3,4}&=f_iy_{1,3}^{n-3}y_{2,4}^2y_{3,4}=f_iy_{1,3}^{n-3}y_{2,3}^2y_{3,4}=\chi_nf_iy_{1,2}^{n-3}y_{1,3}y_{2,3}y_{3,4}+\chi_{n+1}f_iy_{1,2}^{n-3}y_{2,3}^2y_{3,4}\\
&=\chi_nf_iy_{1,2}^{n-2}y_{1,3}y_{3,4}+\chi_{n+1}f_iy_{1,2}^{n-3}y_{1,3}^2y_{3,4},\\
f_iy_{2,3}^{n-1}y_{2,4}&=\chi_nf_iy_{2,3}y_{1,4}^{n-2}y_{2,4}+\chi_{n+1}f_iy_{1,4}^{n-1}y_{2,4}=\chi_nf_iy_{2,3}y_{1,2}^{n-2}y_{2,4}+\chi_{n+1}f_iy_{1,2}^{n-1}y_{2,4}\\
&=\chi_nf_iy_{1,2}^{n-2}y_{2,3}y_{2,4}+\chi_{n+1}f_iy_{1,2}^{n-1}y_{2,4},\\
f_iy_{2,3}^{n-1}y_{3,4}&=f_iy_{2,3}^{n-3}y_{1,4}^2y_{3,4}=f_iy_{2,3}^{n-3}y_{1,3}^2y_{3,4}=-\chi_nf_iy_{1,2}^{n-3}y_{2,3}y_{1,3}y_{3,4}+\chi_{n+1}f_iy_{1,2}^{n-3}y_{1,3}^2y_{3,4}\\
&=\chi_nf_iy_{1,2}^{n-2}y_{2,3}y_{3,4}+\chi_{n+1}f_iy_{1,2}^{n-3}y_{1,3}^2y_{3,4}.
\end{aligned}
$$

Moreover, by the dual relation $f_2y_{1,2}=-f_1y_{3,4}$, and

$$
\begin{aligned}
f_3y_{1,2}^{n-1}y_{1,3}&=\chi_nf_3y_{1,2}y_{1,3}y_{1,2}^{n-2}+\chi_{n+1}f_3y_{1,3}y_{1,2}^{n-1}=-\chi_nf_3y_{2,3}y_{1,2}y_{1,2}^{n-2}+\chi_{n+1}f_3y_{1,3}y_{1,2}^{n-1}\\
&=\chi_nf_2y_{1,4}y_{1,2}^{n-1}+\chi_{n+1}f_1y_{2,4}y_{1,2}^{n-1},\\
f_3y_{1,2}^{n-1}y_{2,3}&=\chi_nf_3y_{1,2}y_{2,3}y_{1,2}^{n-2}+\chi_{n+1}f_3y_{2,3}y_{1,2}^{n-1}=-\chi_nf_3y_{2,3}y_{1,3}y_{1,2}^{n-2}+\chi_{n+1}f_3y_{2,3}y_{1,2}^{n-1}\\
&=\chi_nf_2y_{1,4}y_{1,3}y_{1,2}^{n-2}-\chi_{n+1}f_2y_{1,4}y_{1,2}^{n-1},\\
f_3y_{1,2}^{n-1}y_{1,4}&=\chi_nf_3y_{1,2}y_{1,4}y_{1,2}^{n-2}+\chi_{n+1}f_3y_{1,4}y_{1,2}^{n-1}=-\chi_nf_3y_{2,4}y_{1,2}y_{1,2}^{n-2}+\chi_{n+1}f_3y_{1,4}y_{1,2}^{n-1}\\
&=\chi_nf_1y_{1,3}y_{1,2}^{n-1}+\chi_{n+1}f_2y_{2,3}y_{1,2}^{n-1},\\
f_3y_{1,2}^{n-1}y_{2,4}&=\chi_nf_3y_{1,2}y_{2,4}y_{1,2}^{n-2}+\chi_{n+1}f_3y_{2,4}y_{1,2}^{n-1}=-\chi_nf_3y_{1,4}y_{1,2}y_{1,2}^{n-2}+\chi_{n+1}f_3y_{2,4}y_{1,2}^{n-1}\\
&=-\chi_nf_2y_{2,3}y_{1,2}y_{1,2}^{n-2}-\chi_{n+1}f_1y_{1,3}y_{1,2}^{n-1},
\end{aligned}
$$

for $n\geqslant 3$, the space $N^!_{-n}$ is spanned by T^N_n for $n\geqslant 5$.

Next, we prove that the elements in T^N_n for $n\geqslant 6$ are linearly independent. Suppose that we have the identity

$$
\begin{aligned}
\sum_{i\in[\![1,24]\!]}\alpha_it_i=&\sum_{u\in\mathscr{B}^!_{n-1}}\lambda^1_u(f_1y_{1,2}-f_2y_{3,4})u+\sum_{u\in\mathscr{B}^!_{n-1}}\lambda^2_u(f_1y_{3,4}+f_2y_{1,2})u\\
&+\sum_{u\in\mathscr{B}^!_{n-1}}\lambda^3_u(f_3y_{1,2}+f_4y_{3,4})u+\sum_{u\in\mathscr{B}^!_{n-1}}\lambda^4_u(f_3y_{3,4}-f_4y_{1,2})u\\
&+\sum_{u\in\mathscr{B}^!_{n-1}}\lambda^5_u(f_4y_{1,3}-f_2y_{2,4})u+\sum_{u\in\mathscr{B}^!_{n-1}}\lambda^6_u(f_4y_{2,4}+f_2y_{1,3})u\\
&+\sum_{u\in\mathscr{B}^!_{n-1}}\lambda^7_u(f_3y_{1,3}-f_1y_{2,4})u+\sum_{u\in\mathscr{B}^!_{n-1}}\lambda^8_u(f_3y_{2,4}+f_1y_{1,3})u\\
&+\sum_{u\in\mathscr{B}^!_{n-1}}\lambda^9_u(f_1y_{2,3}+f_4y_{1,4})u+\sum_{u\in\mathscr{B}^!_{n-1}}\lambda^{10}_u(f_1y_{1,4}-f_4y_{2,3})u\\
&+\sum_{u\in\mathscr{B}^!_{n-1}}\lambda^{11}_u(f_3y_{2,3}+f_2y_{1,4})u+\sum_{u\in\mathscr{B}^!_{n-1}}\lambda^{12}_u(f_3y_{1,4}-f_2y_{2,3})u,
\end{aligned}
\tag{3.37}
$$

in $\Bbbk\{f_1,f_2,f_3,f_4\}\otimes A^!$, where $\alpha_i,\lambda^j_u\in\Bbbk$, and t_i is the i-th element in (3.36). We need to show that $\alpha_i=0$ for all $i\in[\![1,24]\!]$. By inspecting the coefficients of the term $f_sy_{i,j}^{n-r}y_{k,l}^r$ for $\#\{i,j,k,l\}=4$, it is easy to see that $\alpha_i=0$ for $i\in[\![1,12]\!]$. Then (3.37) is equivalent to

$$
\begin{aligned}
&f_1\big(y_{1,2}\Delta^1+y_{3,4}\Delta^2+y_{1,3}\Delta^8-y_{2,4}\Delta^7+y_{2,3}\Delta^9+y_{1,4}\Delta^{10}\big)=\sum_{i\in[\![13,24]\!]}\alpha_it_i,\\
&f_2\big(y_{1,2}\Delta^2-y_{3,4}\Delta^1+y_{1,3}\Delta^6-y_{2,4}\Delta^5-y_{2,3}\Delta^{12}+y_{1,4}\Delta^{11}\big)=0,\\
&f_3\big(y_{1,2}\Delta^3+y_{3,4}\Delta^4+y_{1,3}\Delta^7+y_{2,4}\Delta^8+y_{2,3}\Delta^{11}+y_{1,4}\Delta^{12}\big)=0,\\
&f_4\big(-y_{1,2}\Delta^4+y_{3,4}\Delta^3+y_{1,3}\Delta_5+y_{2,4}\Delta^6-y_{2,3}\Delta^{10}+y_{1,4}\Delta^9\big)=0,
\end{aligned}
\tag{3.38}
$$

in $\Bbbk\{f_i\} \otimes A^!$ for $i \in [\![1,4]\!]$ respectively, where $\Delta^j = \sum_{u\in\mathscr{B}^!_{n-1}\setminus Y_{1,2}} \lambda^j_u u$ for $j \in [\![1,4]\!]$, $\Delta^j = \sum_{u\in\mathscr{B}^!_{n-1}\setminus Y_{1,3}} \lambda^j_u u$ for $j \in [\![5,8]\!]$, $\Delta^j = \sum_{u\in\mathscr{B}^!_{n-1}\setminus Y_{2,3}} \lambda^j_u u$ for $j \in [\![9,12]\!]$ and $Y_{i,j}$ is defined in (3.18). In particular, we see that the elements in T^N_n are linearly independent if and only if equation (3.38) implies that $\alpha_i = 0$ for all $i \in [\![13,24]\!]$.

Let

$$a^j_0 = \lambda^j_{y^{n-1}_{1,2}}, a'^j_0 = \lambda^j_{y^{n-1}_{3,4}}, a^j_1 = \sum_{\substack{r\in[\![1,n-2]\!],\\ r \text{ odd}}} \lambda^j_{y^{n-1-r}_{1,2} y^r_{3,4}}, a^j_2 = \sum_{\substack{r\in[\![1,n-2]\!],\\ r \text{ even}}} \lambda^j_{y^{n-1-r}_{1,2} y^r_{3,4}} \text{ for } j \in [\![5,12]\!],$$

$$b^j_0 = \lambda^j_{y^{n-1}_{1,3}}, b'^j_0 = \lambda^j_{y^{n-1}_{2,4}}, b^j_1 = \sum_{\substack{r\in[\![1,n-2]\!],\\ r \text{ odd}}} \lambda^j_{y^{n-1-r}_{1,3} y^r_{2,4}}, b^j_2 = \sum_{\substack{r\in[\![1,n-2]\!],\\ r \text{ even}}} \lambda^j_{y^{n-1-r}_{1,3} y^r_{2,4}} \text{ for } j \in [\![1,4]\!] \cup [\![9,12]\!],$$

$$c^j_0 = \lambda^j_{y^{n-1}_{2,3}}, c'^j_0 = \lambda^j_{y^{n-1}_{1,4}}, c^j_1 = \sum_{\substack{r\in[\![1,n-2]\!],\\ r \text{ odd}}} \lambda^j_{y^{n-1-r}_{2,3} y^r_{1,4}}, c^j_2 = \sum_{\substack{r\in[\![1,n-2]\!],\\ r \text{ even}}} \lambda^j_{y^{n-1-r}_{2,3} y^r_{1,4}} \text{ for } j \in [\![1,8]\!].$$

Using (3.38) together with the products (3.4) and (3.5), in $A^!$, we get a system of linear equations E_n, which contains $24\times 4 = 96$ linear equations and $12+24\times 12+4\times 8\times 3 = 396$ variables α_i, λ^j_u for $u \in \mathscr{U}^!_{n-1}$, a^j_0, a'^j_0, a^j_1, a^j_2, b^j_0, b'^j_0, b^j_1, b^j_2, c^j_0, c'^j_0, c^j_1, c^j_2. Hence, the linear independence of T^N_n (or, equivalently, the fact that equation (3.38) implies that $\alpha_i = 0$ for all $i \in [\![13,24]\!]$) is tantamount to the fact that the linear system E_n implies that $\alpha_i = 0$ for all $i \in [\![13,24]\!]$. Furthermore, it is easy to see that E_n has the same form as E_{n+2}. We then use GAP to check that the elements in T^N_n are linearly independent for $n \in [\![6,7]\!]$, and conclude that the lemma holds for all integers $n \geqslant 6$. □

Corollary 3.28. *We have* $\mathrm{H}_n(M^3) = 0$ *for* $n \in \mathbb{N} \setminus \{3\}$.

Proof. By Tables A.6 and A.7, and the reductions in the proof of Lemma 3.27, the differential at homological degree n in the Koszul complex N or S_k has the same form when $n \geqslant 4$ increases by 2. Then $\mathrm{H}_{n+2}(M^3) = \mathrm{H}_n(M^3)$ for $n \geqslant 4$. Using GAP, $\mathrm{H}_n(M^3) = 0$ for $n \in [\![1,5]\!] \setminus \{3\}$. By induction on n, $\mathrm{H}_n(M^3) = 0$ for $n \in \mathbb{N} \setminus \{3\}$. □

By Appendix A.4 and the code

```
for j in [0..8] do
    for i in [1..12] do
        Print(j, " ", i, " ", RankMat(FF(3,j,i)), "\n");
    od;
od;
```

we obtain the dimension of $B^{M^3}_{n,m}$ for $n \in [\![0,8]\!]$ and $m \in [\![1,12]\!]$. Then the dimension of $\mathrm{H}_{n,m}(M^3)$ is given by Table 3.11 for $n \in [\![1,8]\!]$ and $m \in [\![0,12]\!]$. The dimensions that are not listed in the following table are zeros.

n \ m	0	1	2	3	4	5	6	7	8	9	10	11	12
3				8	24	48	72	80	72	48	24	8	

Table 3.11: Dimension of $\mathrm{H}_{n,m}(M^3)$.

3.4.3.2 The A-module structure of the homology groups

Lemma 3.29. *We have the following isomorphisms of graded* A*-modules*

$$\mathrm{H}_n(M^3) \cong \begin{cases} M^3(-6), & \text{if } n = 3,\\ 0, & \text{if } n \in \mathbb{N}\setminus\{3\}. \end{cases} \tag{3.39}$$

Proof. The isomorphism in (3.39) for $n \in \mathbb{N} \setminus \{3\}$ immediately follows from Corollary 3.28. We prove that the space $\mathrm{H}_3(M^3)$ is isomorphic to $M^3(-6)$. The following GAP code shows that the dimension vector of the submodule of $\mathrm{H}_3(M^3)$ generated by basis elements $e''_i, i \in [\![1,8]\!]$ of $\mathrm{H}_{3,3}(M^3)$ is $(8, 24, 48, 72, 80, 72, 48, 24, 8)$. So, $\mathrm{H}_3(M^3)$ is generated by the eight elements as an A-module.

```
Imm:=Im(3,3,3);;
RankMat(Imm);
# 672
gene:=geneMH(3,3,3);;
Append(Imm,gene);
RankMat(Imm);
# 680
Uh:=UU(gene,3);; Vh:=VV(gene,3);; Wh:=WW(gene,3);;
for r in [4..11] do
   hxr:=HXR(3,Uh,Vh,Wh,3,3,r-3);
   Im3r:=Im(3,3,r);
   Append(Im3r, hxr);
   Print(r, " ", RankMat(Im3r)-RankMat(Im(3,3,r)), "\n");
od;
# 4 24
# 5 48
# 6 72
# 7 80
# 8 72
# 9 48
# 10 24
# 11 8
```

Furthermore, it is direct to check that the generators $e''_i, i \in [\![1,8]\!]$ of $\mathrm{H}_3(M^3)$ satisfy the quadratic relations (3.10). Indeed, the following code shows that the dimension of the subspace generated by $B^{M^3}_{3,4}$ together with the elements of the form (3.10) with e''_i instead of e_i coincides with the dimension of $B^{M^3}_{3,4}$.

```
gene:=geneMH(3,3,3);;
Uh:=UU(gene,3);; Vh:=VV(gene,3);; Wh:=WW(gene,3);;
hx:=HXR(3,Uh,Vh,Wh,3,3,1);;
cc:=0*[1..24];;
cc[1]:=hx[1]+hx[12];; cc[2]:=hx[6]-hx[7];; cc[3]:=hx[13]-hx[24];;
cc[4]:=hx[18]+hx[19];; cc[5]:=hx[20]+hx[11];; cc[6]:=hx[23]-hx[8];;
cc[7]:=hx[14]+hx[5];; cc[8]:=hx[17]-hx[2];; cc[9]:=hx[3]-hx[22];;
cc[10]:=hx[4]+hx[21];; cc[11]:=hx[15]-hx[10];; cc[12]:=hx[16]+hx[9];;
cc[13]:=hx[25];; cc[14]:=hx[26];; cc[15]:=hx[27];; cc[16]:=hx[31];; cc[17]:=hx[34];;
cc[18]:=hx[35];; cc[19]:=hx[38];; cc[20]:=hx[40];; cc[21]:=hx[42];; cc[22]:=hx[45];;
cc[23]:=hx[47];; cc[24]:=hx[48];;
Imm:=Im(3,3,4);;
RankMat(Imm);
# 1344
Append(Imm,cc);
RankMat(Imm);
# 1344
```

Hence, we see that there is a surjective morphism $M^3(-6) \to \mathrm{H}_3(M^3)$ of graded A-modules. Since the dimension vector of M^3 is $(8, 24, 48, 72, 80, 72, 48, 24, 8)$, we have $\mathrm{H}_3(M^3) \cong M^3(-6)$ as graded A-modules, as claimed. □

3.5 Proof of Theorem 3.5

Proof of Theorem 3.5. The result is a direct consequence of Lemmas 3.11, 3.17, 3.23 and 3.29. □

A Some computations

In this Appendix, we list some computations about the Fomin-Kirillov algebra $\mathrm{FK}(4)$ of index 4. As before, we will usually denote $\mathrm{FK}(4)$ simply by A.

A.1 A basis of $\mathrm{FK}(4)$

We present here the GAP code as well the result to compute the basis $W^{1,2}$ (consisting of standard words) of A under the order $x_{1,2} \prec x_{3,4} \prec x_{1,3} \prec x_{2,3} \prec x_{1,4} \prec x_{2,4}$.

```
LoadPackage("GBNP");
A:=FreeAssociativeAlgebraWithOne(Rationals,"x12","x34","x13","x23","x14","x24");;
x12:=A.x12;; x13:=A.x13;; x23:=A.x23;; x14:=A.x14;; x24:=A.x24;; x34:=A.x34;;
oA:=One(A);;
```

```
relationsA:=[x12^2, x13^2, x23^2, x14^2, x24^2, x34^2, x12*x23-x23*x13-x13*x12,
 x23*x12-x12*x13-x13*x23, x12*x24-x24*x14-x14*x12, x24*x12-x12*x14-x14*x24,
 x13*x34-x34*x14-x14*x13, x34*x13-x13*x14-x14*x34, x23*x34-x34*x24-x24*x23,
 x34*x23-x23*x24-x24*x34, x12*x34-x34*x12, x13*x24-x24*x13, x14*x23-x23*x14];;
relsANP:=GP2NPList(relationsA);;
GA:=Grobner(relsANP);;
GBNP.ConfigPrint(A);
PrintNPList(GA);

 x12^2
 x34x12 - x12x34
 x34^2
 x13^2
 x23x12 - x13x23 - x12x13
 x23x13 + x13x12 - x12x23
 x23^2
 x14x34 + x13x14 - x34x13
 x14x13 - x13x34 + x34x14
 x14x23 - x23x14
 x14^2
 x24x12 - x14x24 - x12x14
 x24x34 + x23x24 - x34x23
 x24x13 - x13x24
 x24x23 - x23x34 + x34x24
 x24x14 + x14x12 - x12x24
 x24^2
 x13x12x13 + x12x13x12
 x13x34x13 - x34x13x34
 x23x34x23 - x34x23x34
 x14x12x34 + x13x14x12 - x34x13x12
 x14x12x13 - x23x14x12 + x13x34x23 - x34x23x14
 x14x12x23 + x23x34x14 + x34x14x12 - x12x23x34
 x14x12x14 + x12x14x12
 x23x34x13x12 - x13x34x23x34 + x34x23x34x13 - x12x13x34x23
 x23x34x13x34 + x13x12x34x13 - x12x23x34x13
 x23x34x13x23 + x13x23x34x13 - x34x13x23x34
 x13x12x34x13x12 + x34x13x12x34x13
 x13x12x34x13x34 + x12x13x12x34x13

W:=BaseQA(GA,6,0);;
PrintNPList(W);
```

The basis $W^{1,2}$ is given by the following 576 elements

$1, x_{1,2}, x_{3,4}, x_{1,3}, x_{2,3}, x_{1,4}, x_{2,4}, x_{1,2}x_{3,4}, x_{1,2}x_{1,3}, x_{1,2}x_{2,3}, x_{1,2}x_{1,4}, x_{1,2}x_{2,4}, x_{3,4}x_{1,3}, x_{3,4}x_{2,3},$
$x_{3,4}x_{1,4}, x_{3,4}x_{2,4}, x_{1,3}x_{1,2}, x_{1,3}x_{3,4}, x_{1,3}x_{2,3}, x_{1,3}x_{1,4}, x_{1,3}x_{2,4}, x_{2,3}x_{3,4}, x_{2,3}x_{1,4}, x_{2,3}x_{2,4}, x_{1,4}x_{1,2},$
$x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}, x_{1,2}x_{3,4}x_{2,3}, x_{1,2}x_{3,4}x_{1,4}, x_{1,2}x_{3,4}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}, x_{1,2}x_{1,3}x_{3,4}, x_{1,2}x_{1,3}x_{2,3},$
$x_{1,2}x_{1,3}x_{1,4}, x_{1,2}x_{1,3}x_{2,4}, x_{1,2}x_{2,3}x_{3,4}, x_{1,2}x_{2,3}x_{1,4}, x_{1,2}x_{2,3}x_{2,4}, x_{1,2}x_{1,4}x_{1,2}, x_{1,2}x_{1,4}x_{2,4},$
$x_{3,4}x_{1,3}x_{1,2}, x_{3,4}x_{1,3}x_{3,4}, x_{3,4}x_{1,3}x_{2,3}, x_{3,4}x_{1,3}x_{1,4}, x_{3,4}x_{1,3}x_{2,4}, x_{3,4}x_{2,3}x_{3,4}, x_{3,4}x_{2,3}x_{1,4},$
$x_{3,4}x_{2,3}x_{2,4}, x_{3,4}x_{1,4}x_{1,2}, x_{3,4}x_{1,4}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}, x_{1,3}x_{1,2}x_{2,3}, x_{1,3}x_{1,2}x_{1,4}, x_{1,3}x_{1,2}x_{2,4},$
$x_{1,3}x_{3,4}x_{2,3}, x_{1,3}x_{3,4}x_{1,4}, x_{1,3}x_{3,4}x_{2,4}, x_{1,3}x_{2,3}x_{3,4}, x_{1,3}x_{2,3}x_{1,4}, x_{1,3}x_{2,3}x_{2,4}, x_{1,3}x_{1,4}x_{1,2},$
$x_{1,3}x_{1,4}x_{2,4}, x_{2,3}x_{3,4}x_{1,3}, x_{2,3}x_{3,4}x_{1,4}, x_{2,3}x_{3,4}x_{2,4}, x_{2,3}x_{1,4}x_{1,2}, x_{2,3}x_{1,4}x_{2,4}, x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}, x_{1,2}x_{3,4}x_{1,3}x_{2,3}, x_{1,2}x_{3,4}x_{1,3}x_{1,4}, x_{1,2}x_{3,4}x_{1,3}x_{2,4},$
$x_{1,2}x_{3,4}x_{2,3}x_{3,4}, x_{1,2}x_{3,4}x_{2,3}x_{1,4}, x_{1,2}x_{3,4}x_{2,3}x_{2,4}, x_{1,2}x_{3,4}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{1,4}x_{2,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{3,4}, x_{1,2}x_{1,3}x_{1,2}x_{2,3}, x_{1,2}x_{1,3}x_{1,2}x_{1,4}, x_{1,2}x_{1,3}x_{1,2}x_{2,4}, x_{1,2}x_{1,3}x_{3,4}x_{2,3},$
$x_{1,2}x_{1,3}x_{3,4}x_{1,4}, x_{1,2}x_{1,3}x_{3,4}x_{2,4}, x_{1,2}x_{1,3}x_{2,3}x_{3,4}, x_{1,2}x_{1,3}x_{2,3}x_{1,4}, x_{1,2}x_{1,3}x_{2,3}x_{2,4},$
$x_{1,2}x_{1,3}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{1,4}x_{2,4}, x_{1,2}x_{2,3}x_{3,4}x_{1,3}, x_{1,2}x_{2,3}x_{3,4}x_{1,4}, x_{1,2}x_{2,3}x_{3,4}x_{2,4},$
$x_{1,2}x_{2,3}x_{1,4}x_{1,2}, x_{1,2}x_{2,3}x_{1,4}x_{2,4}, x_{1,2}x_{1,4}x_{1,2}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}, x_{3,4}x_{1,3}x_{1,2}x_{2,3},$
$x_{3,4}x_{1,3}x_{1,2}x_{1,4}, x_{3,4}x_{1,3}x_{1,2}x_{2,4}, x_{3,4}x_{1,3}x_{3,4}x_{2,3}, x_{3,4}x_{1,3}x_{3,4}x_{1,4}, x_{3,4}x_{1,3}x_{3,4}x_{2,4},$
$x_{3,4}x_{1,3}x_{2,3}x_{3,4}, x_{3,4}x_{1,3}x_{2,3}x_{1,4}, x_{3,4}x_{1,3}x_{2,3}x_{2,4}, x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{3,4}x_{1,3}x_{1,4}x_{2,4},$
$x_{3,4}x_{2,3}x_{3,4}x_{1,3}, x_{3,4}x_{2,3}x_{3,4}x_{1,4}, x_{3,4}x_{2,3}x_{3,4}x_{2,4}, x_{3,4}x_{2,3}x_{1,4}x_{1,2}, x_{3,4}x_{2,3}x_{1,4}x_{2,4},$
$x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}, x_{1,3}x_{1,2}x_{3,4}x_{2,3}, x_{1,3}x_{1,2}x_{3,4}x_{1,4}, x_{1,3}x_{1,2}x_{3,4}x_{2,4},$
$x_{1,3}x_{1,2}x_{2,3}x_{3,4}, x_{1,3}x_{1,2}x_{2,3}x_{1,4}, x_{1,3}x_{1,2}x_{2,3}x_{2,4}, x_{1,3}x_{1,2}x_{1,4}x_{1,2}, x_{1,3}x_{1,2}x_{1,4}x_{2,4},$

$x_{1,3}x_{3,4}x_{2,3}x_{3,4}, x_{1,3}x_{3,4}x_{2,3}x_{1,4}, x_{1,3}x_{3,4}x_{2,3}x_{2,4}, x_{1,3}x_{3,4}x_{1,4}x_{1,2}, x_{1,3}x_{3,4}x_{1,4}x_{2,4},$
$x_{1,3}x_{2,3}x_{3,4}x_{1,3}, x_{1,3}x_{2,3}x_{3,4}x_{1,4}, x_{1,3}x_{2,3}x_{3,4}x_{2,4}, x_{1,3}x_{2,3}x_{1,4}x_{1,2}, x_{1,3}x_{2,3}x_{1,4}x_{2,4},$
$x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{2,3}x_{3,4}x_{1,3}x_{2,4}, x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{2,3}x_{3,4}x_{1,4}x_{2,4},$
$x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{1,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{1,4}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}, x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}, x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{1,2},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}, x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}, x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{2,4},$
$x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3},$
$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{3,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{1,4}, x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{1,2}x_{1,4}x_{2,4},$
$x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{3,4}, x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{1,4}, x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{2,4}, x_{1,2}x_{1,3}x_{3,4}x_{1,4}x_{1,2},$
$x_{1,2}x_{1,3}x_{3,4}x_{1,4}x_{2,4}, x_{1,2}x_{1,3}x_{2,3}x_{3,4}x_{1,3}, x_{1,2}x_{1,3}x_{2,3}x_{3,4}x_{1,4}, x_{1,2}x_{1,3}x_{2,3}x_{3,4}x_{2,4},$
$x_{1,2}x_{1,3}x_{2,3}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{2,3}x_{1,4}x_{2,4}, x_{1,2}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4},$
$x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{2,4}, x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{1,2}x_{2,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}, x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{1,4}, x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{1,4}x_{1,2},$
$x_{3,4}x_{1,3}x_{1,2}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}, x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{1,4}, x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{2,4},$
$x_{3,4}x_{1,3}x_{3,4}x_{1,4}x_{1,2}, x_{3,4}x_{1,3}x_{3,4}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}, x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4},$
$x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{2,4}, x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{1,2}, x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{2,4}, x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{2,4},$
$x_{3,4}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,4},$
$x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}, x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}, x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}x_{1,4}x_{1,2},$
$x_{1,3}x_{1,2}x_{3,4}x_{1,4}x_{2,4}, x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}, x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4}, x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{2,4},$
$x_{1,3}x_{1,2}x_{2,3}x_{1,4}x_{1,2}, x_{1,3}x_{1,2}x_{2,3}x_{1,4}x_{2,4}, x_{1,3}x_{1,2}x_{1,4}x_{1,2}x_{2,4}, x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3},$
$x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}, x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{2,4}, x_{1,3}x_{3,4}x_{2,3}x_{1,4}x_{1,2}, x_{1,3}x_{3,4}x_{2,3}x_{1,4}x_{2,4},$
$x_{1,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{2,4}, x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2},$
$x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{1,3}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4},$
$x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{1,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{1,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3},$
$x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}, x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{2,4}, x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2},$
$x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{2,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,4}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}, x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{1,4}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}, x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}, x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{2,4}, x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{1,4}x_{1,2},$
$x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{1,4}x_{2,4}, x_{1,2}x_{1,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{1,2}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{2,4},$
$x_{1,2}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{1,2}x_{1,3}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2},$
$x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{2,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,4}x_{1,2}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}, x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{1,4}x_{1,2}, x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{1,4}x_{1,2}x_{2,4},$
$x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}, x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}, x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{2,4}, x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{1,4}x_{1,2},$
$x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{2,4},$
$x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2},$

$x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4},$

$x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3},$

$x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}, x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{1,2}, x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{2,4},$

$x_{1,3}x_{1,2}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{2,4}, x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{1,2},$

$x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{1,3}x_{1,2}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{2,4},$

$x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{1,3}x_{3,4}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2},$

$x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$

$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,4},$

$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{2,4},$

$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3},$

$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{1,4}x_{1,2},$

$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3},$

$x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{1,4}x_{1,2},$

$x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4},$

$x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{2,4},$

$x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4},$

$x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4},$

$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{2,4},$

$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{2,4},$

$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,4}x_{1,2}x_{2,4},$

$x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{1,2},$

$x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4},$

$x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{2,4}, x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{2,4},$

$x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4},$

$x_{1,2}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4},$

$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{1,2},$

$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4},$

$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{1,2}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{2,4},$

$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{2,4},$

$x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{1,4}x_{1,2}x_{2,4},$

$x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{2,4}, x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2},$

$x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2},$

$x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$

$x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{2,4},$

$x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{1,2}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$

$x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2},$

$x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2},$

$x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2},$

$x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$

$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{2,4},$

$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3},$

$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{1,2},$

$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4},$

$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{2,4},$

$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{2,4},$

$x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{1,4}x_{1,2}x_{2,4},$

$x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4},$

$x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{2,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{2,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{1,2}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{2,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{2,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4},$
$x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4},$
$x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{2,4},$
$x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$

$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{1,2},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{2,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{2,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2},$
$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{2,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{2,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2},$
$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4},$
$x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4},$
$x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}, x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2},$

$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4},$
$x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4},$
$x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}, x_{1,2}x_{3,4}x_{1,3}x_{1,2}x_{3,4}x_{1,3}x_{2,3}x_{3,4}x_{1,3}x_{1,4}x_{1,2}x_{2,4}.$

A.2 Products in $\mathrm{FK}(4)$

It is easy to check the products in $A^!$, listed in Table A.1-A.4, by using GAP or by computing them directly, and to check the products listed in A.6-A.9 by induction on integers $n \geqslant 5$. In Tables A.1-A.4, A.6 and A.7, the entry appearing in the row indexed by y and the column indexed by y' is the product yy'. In Tables A.8 and A.9, the entry appearing in the column indexed by y' and the row indexed by y is the product $y'y$. To reduce space, in Table A.3, we write the product yy' by $\pm m$, where $m \in [\![55, 92]\!]$ is the integer appearing in the first column of Table A.4, and indicating the element in the second column of Table A.4 that is in the same row as it. In Table A.4, we write the product yy' by $\pm m$, where $m \in [\![93, 134]\!]$ is the integer appearing in the first column of Table A.5, and indicating the element in the second column of Table A.5 that is in the same row as it. In Table A.8 and A.9, we write the product yy' by $\pm m$, where $m \in [\![1, 24]\!]$ is the integer appearing in the first column of Table A.8 (or A.9), and indicating the element a_m^{n+1}, where

$$\begin{array}{llll}
\mathsf{a}_1^n = y_{1,2}^{n-1}y_{1,3}, & \mathsf{a}_2^n = y_{1,2}^{n-2}y_{1,3}^2, & \mathsf{a}_3^n = y_{1,2}^{n-1}y_{2,3}, & \mathsf{a}_4^n = y_{1,2}^{n-1}y_{1,4}, \\
\mathsf{a}_5^n = y_{1,2}^{n-2}y_{1,3}y_{1,4}, & \mathsf{a}_6^n = y_{1,2}^{n-3}y_{1,3}^2y_{1,4}, & \mathsf{a}_7^n = y_{1,3}^{n-1}y_{1,4}, & \mathsf{a}_8^n = y_{1,2}^{n-2}y_{2,3}y_{1,4}, \\
\mathsf{a}_9^n = y_{1,2}^{n-2}y_{1,4}^2, & \mathsf{a}_{10}^n = y_{1,2}^{n-3}y_{1,3}y_{1,4}^2, & \mathsf{a}_{11}^n = y_{1,2}^{n-4}y_{1,3}^2y_{1,4}^2, & \mathsf{a}_{12}^n = y_{1,3}^{n-2}y_{1,4}^2, \\
\mathsf{a}_{13}^n = y_{1,2}^{n-3}y_{2,3}y_{1,4}^2, & \mathsf{a}_{14}^n = y_{1,2}^{n-1}y_{2,4}, & \mathsf{a}_{15}^n = y_{1,2}^{n-2}y_{1,3}y_{2,4}, & \mathsf{a}_{16}^n = y_{1,2}^{n-3}y_{1,3}^2y_{2,4}, \\
\mathsf{a}_{17}^n = y_{1,2}^{n-2}y_{2,3}y_{2,4}, & \mathsf{a}_{18}^n = y_{2,3}^{n-1}y_{2,4}, & \mathsf{a}_{19}^n = y_{2,3}^{n-2}y_{2,4}^2, & \mathsf{a}_{20}^n = y_{1,2}^{n-2}y_{1,3}y_{3,4}, \\
\mathsf{a}_{21}^n = y_{1,2}^{n-3}y_{1,3}^2y_{3,4}, & \mathsf{a}_{22}^n = y_{1,3}^{n-1}y_{3,4}, & \mathsf{a}_{23}^n = y_{1,2}^{n-2}y_{2,3}y_{3,4}, & \mathsf{a}_{24}^n = y_{2,3}^{n-1}y_{3,4},
\end{array}$$

for $n \geqslant 5$ and $m \in [\![1, 24]\!]$.

1	$y \backslash y'$	$y_{1,2}$	$y_{1,3}$	$y_{2,3}$	$y_{1,4}$	$y_{2,4}$	$y_{3,4}$
2	$y_{1,2}$	$y_{1,2}^2$	$y_{1,2}y_{1,3}$	$y_{1,2}y_{2,3}$	$y_{1,2}y_{1,4}$	$y_{1,2}y_{2,4}$	$y_{1,2}y_{3,4}$
3	$y_{1,3}$	$-y_{1,2}y_{2,3}$	$y_{1,3}^2$	$y_{1,2}y_{1,3}$	$y_{1,3}y_{1,4}$	$y_{1,3}y_{2,4}$	$y_{1,3}y_{3,4}$
4	$y_{2,3}$	$-y_{1,2}y_{1,3}$	$-y_{1,2}y_{2,3}$	$y_{2,3}^2$	$y_{2,3}y_{1,4}$	$y_{2,3}y_{2,4}$	$y_{2,3}y_{3,4}$
5	$y_{1,4}$	$-y_{1,2}y_{2,4}$	$-y_{1,3}y_{3,4}$	$-y_{2,3}y_{1,4}$	$y_{1,4}^2$	$y_{1,2}y_{1,4}$	$y_{1,3}y_{1,4}$
6	$y_{2,4}$	$-y_{1,2}y_{1,4}$	$-y_{1,3}y_{2,4}$	$-y_{2,3}y_{3,4}$	$-y_{1,2}y_{2,4}$	$y_{2,4}^2$	$y_{2,3}y_{2,4}$
7	$y_{3,4}$	$-y_{1,2}y_{3,4}$	$-y_{1,3}y_{1,4}$	$-y_{2,3}y_{2,4}$	$-y_{1,3}y_{3,4}$	$-y_{2,3}y_{3,4}$	$y_{3,4}^2$

Table A.1: Products yy'.

1	$y \backslash y'$	$y_{1,2}$	$y_{1,3}$	$y_{2,3}$	$y_{1,4}$	$y_{2,4}$	$y_{3,4}$
8	$y_{1,2}^2$	$y_{1,2}^3$	$y_{1,2}^2 y_{1,3}$	$y_{1,2}^2 y_{2,3}$	$y_{1,2}^2 y_{1,4}$	$y_{1,2}^2 y_{2,4}$	$y_{1,2}^2 y_{3,4}$
9	$y_{1,2} y_{1,3}$	$-y_{1,2}^2 y_{2,3}$	$y_{1,2} y_{1,3}^2$	$y_{1,2}^2 y_{1,3}$	$y_{1,2} y_{1,3} y_{1,4}$	$y_{1,2} y_{1,3} y_{2,4}$	$y_{1,2} y_{1,3} y_{3,4}$
10	$y_{1,2} y_{2,3}$	$-y_{1,2}^2 y_{1,3}$	$-y_{1,2}^2 y_{2,3}$	$y_{1,2} y_{1,3}^2$	$y_{1,2} y_{2,3} y_{1,4}$	$y_{1,2} y_{2,3} y_{2,4}$	$y_{1,2} y_{2,3} y_{3,4}$
11	$y_{1,2} y_{1,4}$	$-y_{1,2}^2 y_{2,4}$	$-y_{1,2} y_{1,3} y_{3,4}$	$-y_{1,2} y_{2,3} y_{1,4}$	$y_{1,2} y_{1,4}^2$	$y_{1,2}^2 y_{1,4}$	$y_{1,2} y_{1,3} y_{1,4}$
12	$y_{1,2} y_{2,4}$	$-y_{1,2}^2 y_{1,4}$	$-y_{1,2} y_{1,3} y_{2,4}$	$-y_{1,2} y_{2,3} y_{3,4}$	$-y_{1,2}^2 y_{2,4}$	$y_{1,2} y_{1,4}^2$	$y_{1,2} y_{2,3} y_{2,4}$
13	$y_{1,2} y_{3,4}$	$-y_{1,2}^2 y_{3,4}$	$-y_{1,2} y_{1,3} y_{1,4}$	$-y_{1,2} y_{2,3} y_{2,4}$	$-y_{1,2} y_{1,3} y_{3,4}$	$-y_{1,2} y_{2,3} y_{3,4}$	$y_{1,2} y_{3,4}^2$
14	$y_{1,3}^2$	$y_{1,2} y_{1,3}^2$	$y_{1,3}^3$	$y_{1,2}^2 y_{2,3}$	$y_{1,3}^2 y_{1,4}$	$y_{1,3}^2 y_{2,4}$	$y_{1,3}^2 y_{3,4}$
15	$y_{1,3} y_{1,4}$	$y_{1,2} y_{2,3} y_{2,4}$	$-y_{1,3}^2 y_{3,4}$	$-y_{1,2} y_{1,3} y_{1,4}$	$y_{1,3} y_{1,4}^2$	$-y_{1,2} y_{2,3} y_{1,4}$	$y_{1,3}^2 y_{1,4}$
16	$y_{1,3} y_{2,4}$	$y_{1,2} y_{2,3} y_{1,4}$	$-y_{1,3}^2 y_{2,4}$	$-y_{1,2} y_{1,3} y_{3,4}$	$y_{1,2} y_{2,3} y_{2,4}$	$y_{1,3} y_{2,4}^2$	$y_{1,2} y_{1,3} y_{2,4}$
17	$y_{1,3} y_{3,4}$	$y_{1,2} y_{2,3} y_{3,4}$	$-y_{1,3}^2 y_{1,4}$	$-y_{1,2} y_{1,3} y_{2,4}$	$-y_{1,3}^2 y_{3,4}$	$-y_{1,2} y_{1,3} y_{3,4}$	$y_{1,3} y_{1,4}^2$
18	$y_{2,3}^2$	$y_{1,2} y_{1,3}^2$	$y_{1,2}^2 y_{1,3}$	$y_{2,3}^3$	$y_{2,3}^2 y_{1,4}$	$y_{2,3}^2 y_{2,4}$	$y_{2,3}^2 y_{3,4}$
19	$y_{2,3} y_{1,4}$	$y_{1,2} y_{1,3} y_{2,4}$	$y_{1,2} y_{2,3} y_{3,4}$	$-y_{2,3}^2 y_{1,4}$	$y_{2,3} y_{1,4}^2$	$-y_{1,2} y_{1,3} y_{1,4}$	$-y_{1,2} y_{2,3} y_{1,4}$
20	$y_{2,3} y_{2,4}$	$y_{1,2} y_{1,3} y_{1,4}$	$y_{1,2} y_{2,3} y_{2,4}$	$-y_{2,3}^2 y_{3,4}$	$y_{1,2} y_{1,3} y_{2,4}$	$y_{2,3} y_{2,4}^2$	$y_{2,3}^2 y_{2,4}$
21	$y_{2,3} y_{3,4}$	$y_{1,2} y_{1,3} y_{3,4}$	$y_{1,2} y_{2,3} y_{1,4}$	$-y_{2,3}^2 y_{2,4}$	$y_{1,2} y_{2,3} y_{3,4}$	$-y_{2,3}^2 y_{3,4}$	$y_{2,3} y_{2,4}^2$
22	$y_{1,4}^2$	$y_{1,2} y_{1,4}^2$	$y_{1,3} y_{1,4}^2$	$y_{2,3} y_{1,4}^2$	$y_{1,4}^3$	$y_{1,2}^2 y_{2,4}$	$y_{1,3}^2 y_{3,4}$
23	$y_{2,4}^2$	$y_{1,2} y_{1,4}^2$	$y_{1,3} y_{2,4}^2$	$y_{2,3} y_{2,4}^2$	$y_{1,2}^2 y_{1,4}$	$y_{2,4}^3$	$y_{2,3}^2 y_{3,4}$
24	$y_{3,4}^2$	$y_{1,2} y_{3,4}^2$	$y_{1,3} y_{1,4}^2$	$y_{2,3} y_{2,4}^2$	$y_{1,3}^2 y_{1,4}$	$y_{2,3}^2 y_{2,4}$	$y_{3,4}^3$

Table A.2: Products yy'.

1	$y \backslash y'$	$y_{1,2}$	$y_{1,3}$	$y_{2,3}$	$y_{1,4}$	$y_{2,4}$	$y_{3,4}$
25	$y_{1,2}^3$	55	56	57	58	59	60
26	$y_{1,2}^2 y_{1,3}$	−57	61	56	62	63	64
27	$y_{1,2}^2 y_{2,3}$	−56	−57	61	65	66	67
28	$y_{1,2}^2 y_{1,4}$	−59	−64	−65	68	58	62
29	$y_{1,2}^2 y_{2,4}$	−58	−63	−67	−59	68	66
30	$y_{1,2}^2 y_{3,4}$	−60	−62	−66	−64	−67	69
31	$y_{1,2} y_{1,3}^2$	61	56	57	70	71	72
32	$y_{1,2} y_{1,3} y_{1,4}$	66	−72	−62	73	−65	70
33	$y_{1,2} y_{1,3} y_{2,4}$	65	−71	−64	66	73	63
34	$y_{1,2} y_{1,3} y_{3,4}$	67	−70	−63	−72	−64	73
35	$y_{1,2} y_{2,3} y_{1,4}$	63	67	−70	74	−62	−65
36	$y_{1,2} y_{2,3} y_{2,4}$	62	66	−72	63	74	71
37	$y_{1,2} y_{2,3} y_{3,4}$	64	65	−71	67	−72	74
38	$y_{1,2} y_{1,4}^2$	68	73	74	58	59	72
39	$y_{1,2} y_{3,4}^2$	69	73	74	70	71	75
40	$y_{1,3}^3$	−57	76	56	77	78	79
41	$y_{1,3}^2 y_{1,4}$	−71	−79	−65	80	70	77
42	$y_{1,3}^2 y_{2,4}$	−70	−78	−67	−71	81	66
43	$y_{1,3}^2 y_{3,4}$	−72	−77	−66	−79	−67	80
44	$y_{1,3} y_{1,4}^2$	−74	80	73	77	63	79
45	$y_{1,3} y_{2,4}^2$	−74	81	73	62	82	64
46	$y_{2,3}^3$	−56	−57	83	84	85	86
47	$y_{2,3}^2 y_{1,4}$	−71	−64	−84	87	70	62
48	$y_{2,3}^2 y_{2,4}$	−70	−63	−86	−71	88	85
49	$y_{2,3}^2 y_{3,4}$	−72	−62	−85	−64	−86	88
50	$y_{2,3} y_{1,4}^2$	−73	−74	87	89	66	67
51	$y_{2,3} y_{2,4}^2$	−73	−74	88	65	85	86
52	$y_{1,4}^3$	−59	−79	−89	90	58	77
53	$y_{2,4}^3$	−58	−82	−86	−59	91	85
54	$y_{3,4}^3$	−75	−77	−85	−79	−86	92

Table A.3: Products yy'.

1	$y \backslash y'$	$y_{1,2}$	$y_{1,3}$	$y_{2,3}$	$y_{1,4}$	$y_{2,4}$	$y_{3,4}$
55	$y_{1,2}^4$	93	94	95	96	97	98
56	$y_{1,2}^3y_{1,3}$	-95	99	94	100	101	102
57	$y_{1,2}^3y_{2,3}$	-94	-95	99	103	104	105
58	$y_{1,2}^3y_{1,4}$	-97	-102	-103	106	96	100
59	$y_{1,2}^3y_{2,4}$	-96	-101	-105	-97	106	104
60	$y_{1,2}^3y_{3,4}$	-98	-100	-104	-102	-105	107
61	$y_{1,2}^2y_{1,3}^2$	99	94	95	108	109	110
62	$y_{1,2}^2y_{1,3}y_{1,4}$	104	-110	-100	111	-103	108
63	$y_{1,2}^2y_{1,3}y_{2,4}$	103	-109	-102	104	111	101
64	$y_{1,2}^2y_{1,3}y_{3,4}$	105	-108	-101	-110	-102	111
65	$y_{1,2}^2y_{2,3}y_{1,4}$	101	105	-108	112	-100	-103
66	$y_{1,2}^2y_{2,3}y_{2,4}$	100	104	-110	101	112	109
67	$y_{1,2}^2y_{2,3}y_{3,4}$	102	103	-109	105	-110	112
68	$y_{1,2}^2y_{1,4}^2$	106	111	112	96	97	110
69	$y_{1,2}^2y_{3,4}^2$	107	111	112	108	109	113
70	$y_{1,2}y_{1,3}^2y_{1,4}$	-109	-102	-103	114	108	100
71	$y_{1,2}y_{1,3}^2y_{2,4}$	-108	-101	-105	-109	114	104
72	$y_{1,2}y_{1,3}^2y_{3,4}$	-110	-100	-104	-102	-105	114
73	$y_{1,2}y_{1,3}y_{1,4}^2$	-112	114	111	100	101	102
74	$y_{1,2}y_{2,3}y_{1,4}^2$	-111	-112	114	103	104	105
75	$y_{1,2}y_{3,4}^3$	-113	-100	-104	-102	-105	115
76	$y_{1,3}^4$	99	116	95	117	118	119
77	$y_{1,3}^3y_{1,4}$	104	-119	-100	120	-103	117
78	$y_{1,3}^3y_{2,4}$	103	-118	-102	104	121	101
79	$y_{1,3}^3y_{3,4}$	105	-117	-101	-119	-102	120
80	$y_{1,3}^2y_{1,4}^2$	114	120	112	117	109	119
81	$y_{1,3}^2y_{2,4}^2$	114	121	112	108	122	110
82	$y_{1,3}y_{2,4}^3$	103	-122	-102	104	123	101
83	$y_{2,3}^4$	99	94	124	125	126	127
84	$y_{2,3}^3y_{1,4}$	101	105	-125	128	-100	-103
85	$y_{2,3}^3y_{2,4}$	100	104	-127	101	129	126
86	$y_{2,3}^3y_{3,4}$	102	103	-126	105	-127	129
87	$y_{2,3}^2y_{1,4}^2$	114	111	128	130	109	110
88	$y_{2,3}^2y_{2,4}^2$	114	111	129	108	126	127
89	$y_{2,3}y_{1,4}^3$	101	105	-130	131	-100	-103
90	$y_{1,4}^4$	106	120	131	132	97	119
91	$y_{2,4}^4$	106	123	129	96	133	127
92	$y_{3,4}^4$	115	120	129	117	126	134

Table A.4: Products yy'.

1	
93	$y_{1,2}^5$
94	$y_{1,2}^4y_{1,3}$
95	$y_{1,2}^4y_{2,3}$
96	$y_{1,2}^4y_{1,4}$
97	$y_{1,2}^4y_{2,4}$
98	$y_{1,2}^4y_{3,4}$
99	$y_{1,2}^3y_{1,3}^2$
100	$y_{1,2}^3y_{1,3}y_{1,4}$
101	$y_{1,2}^3y_{1,3}y_{2,4}$
102	$y_{1,2}^3y_{1,3}y_{3,4}$
103	$y_{1,2}^3y_{2,3}y_{1,4}$
104	$y_{1,2}^3y_{2,3}y_{2,4}$
105	$y_{1,2}^3y_{2,3}y_{3,4}$
106	$y_{1,2}^3y_{1,4}^2$
107	$y_{1,2}^3y_{3,4}^2$
108	$y_{1,2}^2y_{1,3}^2y_{1,4}$
109	$y_{1,2}^2y_{1,3}^2y_{2,4}$
110	$y_{1,2}^2y_{1,3}^2y_{3,4}$
111	$y_{1,2}^2y_{1,3}y_{1,4}^2$
112	$y_{1,2}^2y_{2,3}y_{1,4}^2$
113	$y_{1,2}^2y_{3,4}^3$
114	$y_{1,2}y_{1,3}^2y_{1,4}^2$
115	$y_{1,2}y_{3,4}^4$
116	$y_{1,3}^5$
117	$y_{1,3}^4y_{1,4}$
118	$y_{1,3}^4y_{2,4}$
119	$y_{1,3}^4y_{3,4}$
120	$y_{1,3}^3y_{1,4}^2$
121	$y_{1,3}^3y_{2,4}^2$
122	$y_{1,3}^2y_{2,4}^3$
123	$y_{1,3}y_{2,4}^4$
124	$y_{2,3}^5$
125	$y_{2,3}^4y_{1,4}$
126	$y_{2,3}^4y_{2,4}$
127	$y_{2,3}^4y_{3,4}$
128	$y_{2,3}^3y_{1,4}^2$
129	$y_{2,3}^3y_{2,4}^2$
130	$y_{2,3}^2y_{1,4}^3$
131	$y_{2,3}y_{1,4}^4$
132	$y_{1,4}^5$
133	$y_{2,4}^5$
134	$y_{3,4}^5$

Table A.5: Elements in $\mathscr{B}_5^!$.

$y \backslash y'$	$y_{1,2}$	$y_{1,3}$	$y_{2,3}$
$y_{1,2}^{n}$	$y_{1,2}^{n+1}$	$y_{1,2}^{n}y_{1,3}$	$y_{1,2}^{n}y_{2,3}$
$y_{1,2}^{n-1}y_{1,3}$	$-y_{1,2}^{n}y_{2,3}$	$y_{1,2}^{n-1}y_{1,3}^{2}$	$y_{1,2}^{n}y_{1,3}$
$y_{1,2}^{n-1}y_{2,3}$	$-y_{1,2}^{n}y_{1,3}$	$-y_{1,2}^{n}y_{2,3}$	$y_{1,2}^{n-1}y_{1,3}^{2}$
$y_{1,2}^{n-1}y_{1,4}$	$-y_{1,2}^{n}y_{2,4}$	$-y_{1,2}^{n-1}y_{1,3}y_{3,4}$	$-y_{1,2}^{n-1}y_{2,3}y_{1,4}$
$y_{1,2}^{n-1}y_{2,4}$	$-y_{1,2}^{n}y_{1,4}$	$-y_{1,2}^{n-1}y_{1,3}y_{2,4}$	$-y_{1,2}^{n-1}y_{2,3}y_{3,4}$
$y_{1,2}^{n-2}y_{1,3}^{2}$	$y_{1,2}^{n-1}y_{1,3}^{2}$	$y_{1,2}^{n}y_{1,3}$	$y_{1,2}^{n}y_{2,3}$
$y_{1,2}^{n-2}y_{1,3}y_{1,4}$	$y_{1,2}^{n-1}y_{2,3}y_{2,4}$	$-y_{1,2}^{n-2}y_{1,3}^{2}y_{3,4}$	$-y_{1,2}^{n-1}y_{1,3}y_{1,4}$
$y_{1,2}^{n-2}y_{1,3}y_{2,4}$	$y_{1,2}^{n-1}y_{2,3}y_{1,4}$	$-y_{1,2}^{n-2}y_{1,3}^{2}y_{2,4}$	$-y_{1,2}^{n-1}y_{1,3}y_{3,4}$
$y_{1,2}^{n-2}y_{1,3}y_{3,4}$	$y_{1,2}^{n-1}y_{2,3}y_{3,4}$	$-y_{1,2}^{n-2}y_{1,3}^{2}y_{1,4}$	$-y_{1,2}^{n-1}y_{1,3}y_{2,4}$
$y_{1,2}^{n-2}y_{2,3}y_{1,4}$	$y_{1,2}^{n-1}y_{1,3}y_{2,4}$	$y_{1,2}^{n-1}y_{2,3}y_{3,4}$	$-y_{1,2}^{n-2}y_{1,3}^{2}y_{1,4}$
$y_{1,2}^{n-2}y_{2,3}y_{2,4}$	$y_{1,2}^{n-1}y_{1,3}y_{1,4}$	$y_{1,2}^{n-1}y_{2,3}y_{2,4}$	$-y_{1,2}^{n-2}y_{1,3}^{2}y_{3,4}$
$y_{1,2}^{n-2}y_{2,3}y_{3,4}$	$y_{1,2}^{n-1}y_{1,3}y_{3,4}$	$y_{1,2}^{n-1}y_{2,3}y_{1,4}$	$-y_{1,2}^{n-2}y_{1,3}^{2}y_{2,4}$
$y_{1,2}^{n-2}y_{1,4}^{2}$	$y_{1,2}^{n-1}y_{1,4}^{2}$	$y_{1,2}^{n-2}y_{1,3}y_{1,4}^{2}$	$y_{1,2}^{n-2}y_{2,3}y_{1,4}^{2}$
$y_{1,2}^{n-3}y_{1,3}^{2}y_{1,4}$	$-y_{1,2}^{n-2}y_{1,3}^{2}y_{2,4}$	$-y_{1,2}^{n-1}y_{1,3}y_{3,4}$	$-y_{1,2}^{n-1}y_{2,3}y_{1,4}$
$y_{1,2}^{n-3}y_{1,3}^{2}y_{2,4}$	$-y_{1,2}^{n-2}y_{1,3}^{2}y_{1,4}$	$-y_{1,2}^{n-1}y_{1,3}y_{2,4}$	$-y_{1,2}^{n-1}y_{2,3}y_{3,4}$
$y_{1,2}^{n-3}y_{1,3}^{2}y_{3,4}$	$-y_{1,2}^{n-2}y_{1,3}^{2}y_{3,4}$	$-y_{1,2}^{n-1}y_{1,3}y_{1,4}$	$-y_{1,2}^{n-1}y_{2,3}y_{2,4}$
$y_{1,2}^{n-3}y_{1,3}y_{1,4}^{2}$	$-y_{1,2}^{n-2}y_{2,3}y_{1,4}^{2}$	$y_{1,2}^{n-3}y_{1,3}^{2}y_{1,4}^{2}$	$y_{1,2}^{n-2}y_{1,3}y_{1,4}^{2}$
$y_{1,2}^{n-3}y_{2,3}y_{1,4}^{2}$	$-y_{1,2}^{n-2}y_{1,3}y_{1,4}^{2}$	$-y_{1,2}^{n-2}y_{2,3}y_{1,4}^{2}$	$y_{1,2}^{n-3}y_{1,3}^{2}y_{1,4}^{2}$
$y_{1,2}^{n-4}y_{1,3}^{2}y_{1,4}^{2}$	$y_{1,2}^{n-3}y_{1,3}^{2}y_{1,4}^{2}$	$y_{1,2}^{n-2}y_{1,3}y_{1,4}^{2}$	$y_{1,2}^{n-2}y_{2,3}y_{1,4}^{2}$
$y_{1,3}^{n}$	$\chi_n y_{1,2}^{n-1}y_{1,3}^{2} - \chi_{n+1}y_{1,2}^{n}y_{2,3}$	$y_{1,3}^{n+1}$	$\chi_n y_{1,2}^{n}y_{2,3} + \chi_{n+1}y_{1,2}^{n}y_{1,3}$
$y_{1,3}^{n-1}y_{1,4}$	$\chi_n y_{1,2}^{n-1}y_{2,3}y_{2,4} - \chi_{n+1}y_{1,2}^{n-2}y_{1,3}^{2}y_{2,4}$	$-y_{1,3}^{n}y_{3,4}$	$-\chi_n y_{1,2}^{n-1}y_{1,3}y_{1,4} - \chi_{n+1}y_{1,2}^{n-1}y_{2,3}y_{1,4}$
$y_{1,3}^{n-1}y_{3,4}$	$\chi_n y_{1,2}^{n-1}y_{2,3}y_{3,4} - \chi_{n+1}y_{1,2}^{n-2}y_{1,3}^{2}y_{3,4}$	$-y_{1,3}^{n}y_{1,4}$	$-\chi_n y_{1,2}^{n-1}y_{1,3}y_{2,4} - \chi_{n+1}y_{1,2}^{n-1}y_{2,3}y_{2,4}$
$y_{1,3}^{n-2}y_{1,4}^{2}$	$\chi_n y_{1,2}^{n-3}y_{1,3}^{2}y_{1,4}^{2} - \chi_{n+1}y_{1,2}^{n-2}y_{2,3}y_{1,4}^{2}$	$y_{1,3}^{n-1}y_{1,4}^{2}$	$\chi_n y_{1,2}^{n-2}y_{2,3}y_{1,4}^{2} + \chi_{n+1}y_{1,2}^{n-2}y_{1,3}y_{1,4}^{2}$
$y_{2,3}^{n}$	$\chi_n y_{1,2}^{n-1}y_{1,3}^{2} - \chi_{n+1}y_{1,2}^{n}y_{1,3}$	$\chi_n y_{1,2}^{n}y_{1,3} - \chi_{n+1}y_{1,2}^{n}y_{2,3}$	$y_{2,3}^{n+1}$
$y_{2,3}^{n-1}y_{2,4}$	$\chi_n y_{1,2}^{n-1}y_{1,3}y_{1,4} - \chi_{n+1}y_{1,2}^{n-2}y_{1,3}^{2}y_{1,4}$	$\chi_n y_{1,2}^{n-1}y_{2,3}y_{2,4} - \chi_{n+1}y_{1,2}^{n-1}y_{1,3}y_{2,4}$	$-y_{2,3}^{n}y_{3,4}$
$y_{2,3}^{n-1}y_{3,4}$	$\chi_n y_{1,2}^{n-1}y_{1,3}y_{3,4} - \chi_{n+1}y_{1,2}^{n-2}y_{1,3}^{2}y_{3,4}$	$\chi_n y_{1,2}^{n-1}y_{2,3}y_{1,4} - \chi_{n+1}y_{1,2}^{n-1}y_{1,3}y_{1,4}$	$-y_{2,3}^{n}y_{2,4}$
$y_{2,3}^{n-2}y_{2,4}^{2}$	$\chi_n y_{1,2}^{n-3}y_{1,3}^{2}y_{1,4}^{2} - \chi_{n+1}y_{1,2}^{n-2}y_{1,3}y_{1,4}^{2}$	$\chi_n y_{1,2}^{n-2}y_{1,3}y_{1,4}^{2} - \chi_{n+1}y_{1,2}^{n-2}y_{2,3}y_{1,4}^{2}$	$y_{2,3}^{n-1}y_{2,4}^{2}$
$y_{1,4}^{n}$	$\chi_n y_{1,2}^{n-1}y_{1,4}^{2} - \chi_{n+1}y_{1,2}^{n}y_{2,4}$	$\chi_n y_{1,3}^{n-1}y_{1,4}^{2} - \chi_{n+1}y_{1,3}^{n}y_{3,4}$	$(-1)^{n}y_{2,3}y_{1,4}^{n}$
$y_{2,4}^{n}$	$\chi_n y_{1,2}^{n-1}y_{1,4}^{2} - \chi_{n+1}y_{1,2}^{n}y_{1,4}$	$(-1)^{n}y_{1,3}y_{2,4}^{n}$	$\chi_n y_{2,3}^{n-1}y_{2,4}^{2} - \chi_{n+1}y_{2,3}^{n}y_{3,4}$
$y_{3,4}^{n}$	$(-1)^{n}y_{1,2}y_{3,4}^{n}$	$\chi_n y_{1,3}^{n-1}y_{1,4}^{2} - \chi_{n+1}y_{1,3}^{n}y_{1,4}$	$\chi_n y_{2,3}^{n-1}y_{2,4}^{2} - \chi_{n+1}y_{2,3}^{n}y_{2,4}$

Table A.6: Products yy' for $n \geqslant 5$.

$y \backslash y'$	$y_{1,4}$	$y_{2,4}$	$y_{3,4}$
$y_{1,2}^{n}$	$y_{1,2}^{n}y_{1,4}$	$y_{1,2}^{n}y_{2,4}$	$y_{1,2}^{n}y_{3,4}$
$y_{1,2}^{n-1}y_{1,3}$	$y_{1,2}^{n-1}y_{1,3}y_{1,4}$	$y_{1,2}^{n-1}y_{1,3}y_{2,4}$	$y_{1,2}^{n-1}y_{1,3}y_{3,4}$
$y_{1,2}^{n-1}y_{2,3}$	$y_{1,2}^{n-1}y_{2,3}y_{1,4}$	$y_{1,2}^{n-1}y_{2,3}y_{2,4}$	$y_{1,2}^{n-1}y_{2,3}y_{3,4}$
$y_{1,2}^{n-1}y_{1,4}$	$y_{1,2}^{n-1}y_{1,4}^{2}$	$y_{1,2}^{n}y_{1,4}$	$y_{1,2}^{n-1}y_{1,3}y_{1,4}$
$y_{1,2}^{n-1}y_{2,4}$	$-y_{1,2}^{n}y_{2,4}$	$y_{1,2}^{n-1}y_{1,4}^{2}$	$y_{1,2}^{n-1}y_{2,3}y_{2,4}$
$y_{1,2}^{n-2}y_{1,3}^{2}$	$y_{1,2}^{n-2}y_{1,3}^{2}y_{1,4}$	$y_{1,2}^{n-2}y_{1,3}^{2}y_{2,4}$	$y_{1,2}^{n-2}y_{1,3}^{2}y_{3,4}$
$y_{1,2}^{n-2}y_{1,3}y_{1,4}$	$y_{1,2}^{n-2}y_{1,3}y_{1,4}^{2}$	$-y_{1,2}^{n-1}y_{2,3}y_{1,4}$	$y_{1,2}^{n-2}y_{1,3}^{2}y_{1,4}$
$y_{1,2}^{n-2}y_{1,3}y_{2,4}$	$y_{1,2}^{n-1}y_{2,3}y_{2,4}$	$y_{1,2}^{n-2}y_{1,3}y_{1,4}^{2}$	$y_{1,2}^{n-1}y_{1,3}y_{2,4}$
$y_{1,2}^{n-2}y_{1,3}y_{3,4}$	$-y_{1,2}^{n-2}y_{1,3}^{2}y_{3,4}$	$-y_{1,2}^{n-1}y_{1,3}y_{3,4}$	$y_{1,2}^{n-2}y_{1,3}y_{1,4}^{2}$
$y_{1,2}^{n-2}y_{2,3}y_{1,4}$	$y_{1,2}^{n-2}y_{2,3}y_{1,4}^{2}$	$-y_{1,2}^{n-1}y_{1,3}y_{1,4}$	$-y_{1,2}^{n-1}y_{2,3}y_{1,4}$
$y_{1,2}^{n-2}y_{2,3}y_{2,4}$	$y_{1,2}^{n-1}y_{1,3}y_{2,4}$	$y_{1,2}^{n-2}y_{2,3}y_{1,4}^{2}$	$y_{1,2}^{n-2}y_{1,3}^{2}y_{2,4}$
$y_{1,2}^{n-2}y_{2,3}y_{3,4}$	$y_{1,2}^{n-1}y_{2,3}y_{3,4}$	$-y_{1,2}^{n-2}y_{1,3}^{2}y_{3,4}$	$y_{1,2}^{n-2}y_{2,3}y_{1,4}^{2}$
$y_{1,2}^{n-2}y_{1,4}^{2}$	$y_{1,2}^{n}y_{1,4}$	$y_{1,2}^{n}y_{2,4}$	$y_{1,2}^{n-2}y_{1,3}^{2}y_{3,4}$
$y_{1,2}^{n-3}y_{1,3}^{2}y_{1,4}$	$y_{1,2}^{n-3}y_{1,3}^{2}y_{1,4}^{2}$	$y_{1,2}^{n-2}y_{1,3}^{2}y_{1,4}$	$y_{1,2}^{n-1}y_{1,3}y_{1,4}$
$y_{1,2}^{n-3}y_{1,3}^{2}y_{2,4}$	$-y_{1,2}^{n-2}y_{1,3}^{2}y_{2,4}$	$y_{1,2}^{n-3}y_{1,3}^{2}y_{1,4}^{2}$	$y_{1,2}^{n-1}y_{2,3}y_{2,4}$
$y_{1,2}^{n-3}y_{1,3}^{2}y_{3,4}$	$-y_{1,2}^{n-1}y_{1,3}y_{3,4}$	$-y_{1,2}^{n-1}y_{2,3}y_{3,4}$	$y_{1,2}^{n-3}y_{1,3}^{2}y_{1,4}^{2}$
$y_{1,2}^{n-3}y_{1,3}y_{1,4}^{2}$	$y_{1,2}^{n-1}y_{1,3}y_{1,4}$	$y_{1,2}^{n-1}y_{1,3}y_{2,4}$	$y_{1,2}^{n-1}y_{1,3}y_{3,4}$
$y_{1,2}^{n-3}y_{2,3}y_{1,4}^{2}$	$y_{1,2}^{n-1}y_{2,3}y_{1,4}$	$y_{1,2}^{n-1}y_{2,3}y_{2,4}$	$y_{1,2}^{n-1}y_{2,3}y_{3,4}$
$y_{1,2}^{n-4}y_{1,3}^{2}y_{1,4}^{2}$	$y_{1,2}^{n-2}y_{1,3}^{2}y_{1,4}$	$y_{1,2}^{n-2}y_{1,3}^{2}y_{2,4}$	$y_{1,2}^{n-2}y_{1,3}^{2}y_{3,4}$
$y_{1,3}^{n}$	$y_{1,3}^{n}y_{1,4}$	$y_{1,3}^{n}y_{2,4}$	$y_{1,3}^{n}y_{3,4}$
$y_{1,3}^{n-1}y_{1,4}$	$y_{1,3}^{n-1}y_{1,4}^{2}$	$\chi_{n+1}y_{1,2}^{n-2}y_{1,3}^{2}y_{1,4} - \chi_n y_{1,2}^{n-1}y_{2,3}y_{1,4}$	$y_{1,3}^{n}y_{1,4}$
$y_{1,3}^{n-1}y_{3,4}$	$-y_{1,3}^{n}y_{3,4}$	$-\chi_n y_{1,2}^{n-1}y_{1,3}y_{3,4} - \chi_{n+1}y_{1,2}^{n-1}y_{2,3}y_{3,4}$	$y_{1,3}^{n-1}y_{1,4}^{2}$
$y_{1,3}^{n-2}y_{1,4}^{2}$	$y_{1,3}^{n}y_{1,4}$	$\chi_n y_{1,2}^{n-2}y_{1,3}^{2}y_{2,4} + \chi_{n+1}y_{1,2}^{n-1}y_{1,3}y_{2,4}$	$y_{1,3}^{n}y_{3,4}$
$y_{2,3}^{n}$	$y_{2,3}^{n}y_{1,4}$	$y_{2,3}^{n}y_{2,4}$	$y_{2,3}^{n}y_{3,4}$
$y_{2,3}^{n-1}y_{2,4}$	$\chi_n y_{1,2}^{n-1}y_{1,3}y_{2,4} - \chi_{n+1}y_{1,2}^{n-2}y_{1,3}^{2}y_{2,4}$	$y_{2,3}^{n-1}y_{2,4}^{2}$	$y_{2,3}^{n}y_{2,4}$
$y_{2,3}^{n-1}y_{3,4}$	$\chi_n y_{1,2}^{n-1}y_{2,3}y_{3,4} - \chi_{n+1}y_{1,2}^{n-1}y_{1,3}y_{3,4}$	$-y_{2,3}^{n}y_{3,4}$	$y_{2,3}^{n-1}y_{2,4}^{2}$
$y_{2,3}^{n-2}y_{2,4}^{2}$	$\chi_n y_{1,2}^{n-2}y_{1,3}^{2}y_{1,4} + \chi_{n+1}y_{1,2}^{n-2}y_{2,3}y_{1,4}$	$y_{2,3}^{n}y_{2,4}$	$y_{2,3}^{n}y_{3,4}$
$y_{1,4}^{n}$	$y_{1,4}^{n+1}$	$\chi_n y_{1,2}^{n}y_{2,4} + \chi_{n+1}y_{1,2}^{n}y_{1,4}$	$\chi_n y_{1,3}^{n}y_{3,4} + \chi_{n+1}y_{1,3}^{n}y_{1,4}$
$y_{2,4}^{n}$	$\chi_n y_{1,2}^{n}y_{1,4} - \chi_{n+1}y_{1,2}^{n}y_{2,4}$	$y_{2,4}^{n+1}$	$\chi_n y_{2,3}^{n}y_{3,4} + \chi_{n+1}y_{2,3}^{n}y_{2,4}$
$y_{3,4}^{n}$	$\chi_n y_{1,3}^{n}y_{1,4} - \chi_{n+1}y_{1,3}^{n}y_{3,4}$	$\chi_n y_{2,3}^{n}y_{2,4} - \chi_{n+1}y_{2,3}^{n}y_{3,4}$	$y_{3,4}^{n+1}$

Table A.7: Products yy' for $n \geqslant 5$.

	$y \backslash y'$	$y_{1,2}$	$y_{3,4}$	$y_{1,3}$	$y_{2,4}$	$y_{2,3}$	$y_{1,4}$
1	$y_{1,2}^{n-1}y_{1,3}$	1	5	3	20	−2	15
2	$y_{1,2}^{n-2}y_{1,3}^2$	2	21	1	16	3	6
3	$y_{1,2}^{n-1}y_{2,3}$	3	17	−2	8	−1	23
4	$y_{1,2}^{n-1}y_{1,4}$	4	20	−8	−9	−5	14
5	$y_{1,2}^{n-2}y_{1,3}y_{1,4}$	5	−10	6	−17	−8	21
6	$y_{1,2}^{n-3}y_{1,3}^2y_{1,4}$	6	20	−8	−11	−5	16
7	$y_{1,3}^{n-1}y_{1,4}$	5	−12	7	−17	−8	22
8	$y_{1,2}^{n-2}y_{2,3}y_{1,4}$	8	−15	5	−23	6	−13
9	$y_{1,2}^{n-2}y_{1,4}^2$	9	21	10	14	13	4
10	$y_{1,2}^{n-3}y_{1,3}y_{1,4}^2$	10	5	13	20	−11	15
11	$y_{1,2}^{n-4}y_{1,3}^2y_{1,4}^2$	11	21	10	16	13	6
12	$y_{1,3}^{n-2}y_{1,4}^2$	11	22	12	16	13	7
13	$y_{1,2}^{n-3}y_{2,3}y_{1,4}^2$	13	17	−11	8	−10	23
14	$y_{1,2}^{n-1}y_{2,4}$	14	23	−17	−4	−15	−9
15	$y_{1,2}^{n-2}y_{1,3}y_{2,4}$	15	8	16	−10	−17	20
16	$y_{1,2}^{n-3}y_{1,3}^2y_{2,4}$	16	23	−17	−6	−15	−11
17	$y_{1,2}^{n-2}y_{2,3}y_{2,4}$	17	−13	15	21	16	5
18	$y_{2,3}^{n-1}y_{2,4}$	17	−19	15	24	18	5
19	$y_{2,3}^{n-2}y_{2,4}^2$	11	24	10	18	19	6
20	$y_{1,2}^{n-2}y_{1,3}y_{3,4}$	20	−6	21	−15	−23	−10
21	$y_{1,2}^{n-3}y_{1,3}^2y_{3,4}$	21	−11	−23	−5	−20	−17
22	$y_{1,3}^{n-1}y_{3,4}$	20	−7	22	−15	−23	−12
23	$y_{1,2}^{n-2}y_{2,3}y_{3,4}$	23	−16	20	−13	21	8
24	$y_{2,3}^{n-1}y_{3,4}$	23	−18	20	−19	24	8

Table A.8: Products $y'y$ for $n \geqslant 6$ even.

	$y \backslash y'$	$y_{1,2}$	$y_{3,4}$	$y_{1,3}$	$y_{2,4}$	$y_{2,3}$	$y_{1,4}$
1	$y_{1,2}^{n-1}y_{1,3}$	1	−5	2	−15	−3	−20
2	$y_{1,2}^{n-2}y_{1,3}^2$	2	−21	−3	−6	−1	−16
3	$y_{1,2}^{n-1}y_{2,3}$	3	−17	1	−23	2	−8
4	$y_{1,2}^{n-1}y_{1,4}$	4	−20	5	−14	8	9
5	$y_{1,2}^{n-2}y_{1,3}y_{1,4}$	5	10	8	−21	−6	17
6	$y_{1,2}^{n-3}y_{1,3}^2y_{1,4}$	6	−20	5	−16	8	11
7	$y_{1,3}^{n-1}y_{1,4}$	6	−22	7	−16	8	12
8	$y_{1,2}^{n-2}y_{2,3}y_{1,4}$	8	15	−6	13	−5	23
9	$y_{1,2}^{n-2}y_{1,4}^2$	9	−21	−13	−4	−10	−14
10	$y_{1,2}^{n-3}y_{1,3}y_{1,4}^2$	10	−5	11	−15	−13	−20
11	$y_{1,2}^{n-4}y_{1,3}^2y_{1,4}^2$	11	−21	−13	−6	−10	−16
12	$y_{1,3}^{n-2}y_{1,4}^2$	10	−7	12	−15	−13	−22
13	$y_{1,2}^{n-3}y_{2,3}y_{1,4}^2$	13	−17	10	−23	11	−8
14	$y_{1,2}^{n-1}y_{2,4}$	14	−23	15	9	17	4
15	$y_{1,2}^{n-2}y_{1,3}y_{2,4}$	15	−8	17	−20	−16	10
16	$y_{1,2}^{n-3}y_{1,3}^2y_{2,4}$	16	−23	15	11	17	6
17	$y_{1,2}^{n-2}y_{2,3}y_{2,4}$	17	13	−16	−5	−15	−21
18	$y_{2,3}^{n-1}y_{2,4}$	16	−24	15	19	18	6
19	$y_{2,3}^{n-2}y_{2,4}^2$	13	−18	10	−24	19	−8
20	$y_{1,2}^{n-2}y_{1,3}y_{3,4}$	20	6	23	10	−21	15
21	$y_{1,2}^{n-3}y_{1,3}^2y_{3,4}$	21	11	20	17	23	5
22	$y_{1,3}^{n-1}y_{3,4}$	21	12	22	17	23	7
23	$y_{1,2}^{n-2}y_{2,3}y_{3,4}$	23	16	−21	−8	−20	13
24	$y_{2,3}^{n-1}y_{3,4}$	21	19	20	18	24	5

Table A.9: Products $y'y$ for $n \geqslant 5$ odd.

A.3 A basis of M^1

We present here the GAP code for computing a basis of the quadratic module M^1, defined at the beginning of Subsection 3.2. The code was provided by J.W. Knopper.

```
# Knopper's code
LoadPackage("GBNP");
A:=FreeAssociativeAlgebraWithOne(Rationals,"x12","x13","x23","x14","x24","x34");;
x12:=A.x12;; x13:=A.x13;; x23:=A.x23;; x14:=A.x14;; x24:=A.x24;; x34:=A.x34;;
oA:=One(A);;
relationsA:=[x12^2, x13^2, x23^2, x14^2, x24^2, x34^2, x12*x23-x23*x13-x13*x12,
 x23*x12-x12*x13-x13*x23, x12*x24-x24*x14-x14*x12, x24*x12-x12*x14-x14*x24,
 x13*x34-x34*x14-x14*x13, x34*x13-x13*x14-x14*x34, x23*x34-x34*x24-x24*x23,
 x34*x23-x23*x24-x24*x34, x12*x34-x34*x12, x13*x24-x24*x13, x14*x23-x23*x14];;
relsANP:=GP2NPList(relationsA);;
GBNP.ConfigPrint(A);
GA:=Grobner(relsANP);;

MD:=A^2;;
ab:=GeneratorsOfLeftModule(MD);
c2:=ab[1]; c1:=ab[2];
modrels:=[c1*x13, c1*x24, c2*x23, c2*x14, c1*x12+c2*x12, c1*x34+c2*x34];;
modrelsNP:=GP2NPList(modrels);
PrintNPList(modrelsNP);

GBNP.CheckHom:=function(G,wtv)
    local i,j,k,l,mon,h1,h2,ans;
    mon:=LMonsNP(G);
    ans:=GBNP.WeightedDegreeList(mon,wtv);
    for i in [1..Length(G)] do
        h1:=ans[i];
        l:=Length(G[i][1]);
        for j in [2..l] do
            mon:=G[i][1][j];
            h2:=0;
            for k in [1..Length(mon)] do
                if mon[k]>0 then
                # Don't count module generators, which have a negative index.
                # Only count two-sided generators with index 1 or more.
                       h2:=h2+wtv[mon[k]];
                fi;
            od;
            if h2<>h1 then return(false); fi;
        od;
    od;
    Info(InfoGBNP,1,"Input is homogeneous");
    return(ans);
end;

GBNP.WeightedDegreeMon:=function(mon,lst)
    local i,ans;
    ans:=0;
    for i in mon do
        # Don't count module generators, which have a negative index.
        # Only count two-sided generators with index 1 or more.
        if i>0 then
            ans:=ans+lst[i];
        fi;
    od;
    return(ans);
end;;

SetInfoLevel(InfoGBNP,1);
SetInfoLevel(InfoGBNPTime,1);
combinedrelsNP:=Concatenation(GA,modrelsNP);
GAT:=SGrobnerTrunc(combinedrelsNP, 9, [1,1,1,1,1,1]);
PrintNPList(GAT);

splitGAT:=function(GAT)
    local p, ts, rel, lm;
    # p: list of module or prefix relations, ts: list of two-sided relations,
    # rel: current relation, lm: leading monomial of current relation rel.
```

```
    p:=[];
    ts:=[];
    for rel in GAT do
        # get leading monomial.
        lm := rel[1,1];
        if Length(lm)>1 and lm[1]<0 then
            # module relations start with a negative generator.
            # if 1 is part of the GB then it does not have a generator,
            # furthermore it is two-sided.
            Add(p, rel);
        else
            Add(ts, rel);
        fi;
    od;
    return rec(p:=p, ts:=ts);
end;;

split:=splitGAT(GAT);
GBR:=rec(p:=split.p, pg:=2, ts:=split.ts);
BQM:=BaseQM(GBR,6,2,0);;
PrintNPList(BQM);

[ 0, 1 ]
[ 1 , 0]
[ 0, x12 ]
[ 0, x23 ]
[ 0, x14 ]
[ 0, x34 ]
[ x13 , 0]
[ x24 , 0]
[ 0, x12x13 ]
[ 0, x12x23 ]
[ 0, x12x14 ]
[ 0, x12x24 ]
[ 0, x12x34 ]
[ 0, x23x14 ]
[ 0, x23x24 ]
[ 0, x23x34 ]
[ 0, x14x13 ]
[ 0, x14x34 ]
[ x13x24 , 0]
[ 0, x12x13x14 ]
[ 0, x12x13x24 ]
[ 0, x12x13x34 ]
[ 0, x12x23x14 ]
[ 0, x12x23x24 ]
[ 0, x12x23x34 ]
[ 0, x12x14x13 ]
[ 0, x12x14x34 ]
[ 0, x12x24x23 ]
[ 0, x12x24x34 ]
[ 0, x23x14x12 ]
[ 0, x23x14x34 ]
[ 0, x12x13x14x12 ]
[ 0, x12x13x14x13 ]
[ 0, x12x13x14x24 ]
[ 0, x12x13x14x34 ]
[ 0, x12x13x24x23 ]
[ 0, x12x13x24x34 ]
[ 0, x12x23x14x13 ]
[ 0, x12x23x14x24 ]
[ 0, x12x23x24x34 ]
[ 0, x12x14x13x34 ]
[ 0, x12x24x23x34 ]
[ 0, x12x13x14x12x23 ]
[ 0, x12x13x14x12x24 ]
[ 0, x12x13x14x12x34 ]
[ 0, x12x13x14x13x34 ]
[ 0, x12x13x14x24x23 ]
[ 0, x12x13x24x23x34 ]
[ 0, x12x13x14x12x23x34 ]
[ 0, x12x13x14x12x24x23 ]
```

A.4 Koszul complex of M^i for $i \in \{0,1,2,3\}$

We present here the GAP code for computing the differential of the Koszul complex of the quadratic modules $M^0 = \Bbbk$ and M^i for $i \in [\![1,3]\!]$ defined in Subsection 3.2. We also present a basis of $\mathrm{H}_{n,m}(M^i)$ for some pairs (n,m). In the following code, the matrix FF(i,n,m) represents the linear map $d_{n+1,m-1}(M^i) : \mathrm{K}_{n+1,m-1}(M^i) \to \mathrm{K}_{n,m}(M^i)$, Im(i,n,m) is a basis of the space $B^{M^i}_{n,m}$ and Ker(i,n,m) is a basis of the space $D^{M^i}_{n,m}$. Moreover, geneMH(i,n,m) are some elements in $D^{M^i}_{n,m}$, and we can show that it represents a basis of $\mathrm{H}_{n,m}(M^i)$ since the dimension of the space spanned by $B^{M^i}_{n,m}$ and geneMH(i,n,m) coincides with the dimension of $D^{M^i}_{n,m}$.

```
LoadPackage("GBNP");
A:=FreeAssociativeAlgebraWithOne(Rationals,"x12","x13","x23","x14","x24","x34");;
x12:=A.x12;; x13:=A.x13;; x23:=A.x23;; x14:=A.x14;; x24:=A.x24;; x34:=A.x34;;
oA:=One(A);;
relationsA:=[x12^2, x13^2, x23^2, x14^2, x24^2, x34^2, x12*x23-x23*x13-x13*x12,
 x23*x12-x12*x13-x13*x23, x12*x24-x24*x14-x14*x12, x24*x12-x12*x14-x14*x24,
 x13*x34-x34*x14-x14*x13, x34*x13-x13*x14-x14*x34, x23*x34-x34*x24-x24*x23,
 x34*x23-x23*x24-x24*x34, x12*x34-x34*x12, x13*x24-x24*x13, x14*x23-x23*x14];;
# A/relationsA is the Fomin-Kirillov algbera on 4 generators.
relsANP:=GP2NPList(relationsA);;
GBNP.ConfigPrint(A);
GA:=Grobner(relsANP);; # GA is a Gröbner basis of the ideal in A.
# PrintNPList(GA);
C:=BaseQA(GA,6,0);;  # C is the set of standard words with respect to GA.
# PrintNPList(C);

f:=function(n)
    if n=0 then return 1;
    elif n=1 then return 6;
    elif n=2 then return 19;
    elif n=3 then return 42;
    elif n=4 then return 71;
    elif n=5 then return 96;
    elif n=6 then return 106;
    elif n=7 then return 96;
    elif n=8 then return 71;
    elif n=9 then return 42;
    elif n=10 then return 19;
    elif n=11 then return 6;
    elif n=12 then return 1;
    fi;
end;
# f(n) is the dimension of $A_n$.

g:=function(n)
    if n=-1 then return 0;
    elif n=0 then return f(0);
    elif n>0 then return Sum(List([0..n], s->f(s)));
    fi;
end;
# g(n)-g(n-1)=f(n).

B:=FreeAssociativeAlgebraWithOne(Rationals,"y12","y13","y23","y14","y24","y34");;
y12:=B.y12;; y13:=B.y13;; y23:=B.y23;; y14:=B.y14;; y24:=B.y24;; y34:=B.y34;;
oB:=One(B);;
relationsB:=[y12*y23+y23*y13, y13*y23+y23*y12, y12*y23+y13*y12, y12*y13+y23*y12,
 y12*y24+y24*y14, y14*y24+y24*y12, y12*y24+y14*y12, y12*y14+y24*y12, y13*y34+y34*y14,
 y14*y34+y34*y13, y13*y34+y14*y13, y13*y14+y34*y13, y23*y34+y34*y24, y24*y34+y34*y23,
 y23*y34+y24*y23, y23*y24+y34*y23, y12*y34+y34*y12, y13*y24+y24*y13,
 y23*y14+y14*y23];;
# B/relationsB is the quadratic dual of the Fomin-Kirillov algebra on 4 generators.
relsBNP:=GP2NPList(relationsB);;
wtv:= [1,1,1,1,1,1];;
GBNP.ConfigPrint(B);;
GB:=Grobner(relsBNP);; # GB is a Gröbner basis of the ideal in B.
# PrintNPList(GB);
D:= BaseQATrunc(GB,15,wtv);;
for degpart in D do for mon in degpart do PrintNP([[mon],[1]]); od; od;
DT:=[];
for degpart in D do for mon in degpart do Append(DT,[[[mon],[1]]]); od; od;
```

```
S:=B^8;;
ab:=GeneratorsOfLeftModule(S);;
g8:=ab[1];; g7:=ab[2];; g6:=ab[3];; g5:=ab[4];;
g4:=ab[5];; g3:=ab[6];; g2:=ab[7];; g1:=ab[8];;
modrels:=[g1*y12-g2*y34, g1*y34+g2*y12, g3*y12+g4*y34, g3*y34-g4*y12, g4*y13-g2*y24,
 g4*y24+g2*y13, g3*y13-g1*y24, g3*y24+g1*y13, g1*y23+g4*y14, g1*y14-g4*y23,
 g3*y23+g2*y14, g3*y14-g2*y23, g5*y14, g5*y24, g5*y34, g6*y13, g6*y23, g6*y34, g7*y12,
 g7*y23, g7*y24, g8*y12, g8*y13, g8*y14 ];;
modrelsNP:=GP2NPList(modrels);;
# PrintNPList(modrelsNP);;

GBNP.CheckHom:=function(G,wtv)
    local i,j,k,l,mon,h1,h2,ans;
    mon:=LMonsNP(G);
    ans:=GBNP.WeightedDegreeList(mon,wtv);
    for i in [1..Length(G)] do
        h1:=ans[i];
        l:=Length(G[i][1]);
        for j in [2..l] do
            mon:=G[i][1][j];
            h2:=0;
            for k in [1..Length(mon)] do
                if mon[k]>0 then
                    h2:=h2+wtv[mon[k]];
                fi;
            od;
            if h2<>h1 then return(false); fi;
        od;
    od;
    Info(InfoGBNP,1,"Input is homogeneous");
    return(ans);
end;

GBNP.WeightedDegreeMon:=function(mon,lst)
    local i,ans;
    ans:=0;
    for i in mon do
        if i>0 then
            ans:=ans+lst[i];
        fi;
    od;
    return(ans);
end;;

SetInfoLevel(InfoGBNP,1);;
SetInfoLevel(InfoGBNPTime,1);;
combinedrelsNP:=Concatenation(GB,modrelsNP);;
GBT:=SGrobnerTrunc(combinedrelsNP, 15, [1,1,1,1,1,1]);;
# PrintNPList(GBT);

splitGBT:=function(GBT)
    local p, ts, rel, lm;
    p:=[];
    ts:=[];
    for rel in GBT do
        lm := rel[1,1];
        if Length(lm)>1 and lm[1]<0 then
            Add(p, rel);
        else
            Add(ts, rel);
        fi;
    od;
    return rec(p:=p, ts:=ts);
end;;

split:=splitGBT(GBT);;
GBRM3:=rec(p:=split.p, pg:=8, ts:=split.ts);;
BQMM3:=BaseQM(GBRM3,6,8,650);;
# PrintNPList(BQMM3);

S:=B^7;;
ab:=GeneratorsOfLeftModule(S);;
g7:=ab[1];; g6:=ab[2];; g5:=ab[3];; g4:=ab[4];; g3:=ab[5];; g2:=ab[6];; g1:=ab[7];;
```

```
modrels:=[g1*y14+g4*y14, g1*y24+g3*y24, g1*y34-g2*y34, g2*y13+g4*y13, g2*y23+g3*y23,
 g3*y12-g4*y12, g5*y12-g6*y34, g1*y24+g5*y13, g5*y23+g7*y14, g5*y14-g7*y23,
 g2*y13-g5*y24, g5*y34+g6*y12, g6*y13+g7*y24, g1*y14-g6*y23, g2*y23+g6*y14,
 g6*y24-g7*y13, g1*y34-g7*y12, g3*y12-g7*y34];;
modrelsNP:=GP2NPList(modrels);;
# PrintNPList(modrelsNP);

SetInfoLevel(InfoGBNP,1);;
SetInfoLevel(InfoGBNPTime,1);;
combinedrelsNP:=Concatenation(GB,modrelsNP);;
GBT:=SGrobnerTrunc(combinedrelsNP, 15, [1,1,1,1,1,1]);;
# PrintNPList(GBT);

split:=splitGBT(GBT);;
GBRM2:=rec(p:=split.p, pg:=7, ts:=split.ts);;
BQMM2:=BaseQM(GBRM2,6,7,1000);;
# PrintNPList(BQMM2);

MD:=B^2;;
ab:=GeneratorsOfLeftModule(MD);;
g2:=ab[1];; g1:=ab[2];;
modrels:=[g1*y12-g2*y12, g2*y13, g1*y23, g1*y14, g2*y24, g1*y34-g2*y34];;
modrelsNP:=GP2NPList(modrels);;
# PrintNPList(modrelsNP);

SetInfoLevel(InfoGBNP,1);;
SetInfoLevel(InfoGBNPTime,1);;
combinedrelsNP:=Concatenation(GB,modrelsNP);;
GBT:=SGrobnerTrunc(combinedrelsNP, 15, [1,1,1,1,1,1]);;
# PrintNPList(GBT);

split:=splitGBT(GBT);;
GBRM1:=rec(p:=split.p, pg:=2, ts:=split.ts);;
BQMM1:=BaseQM(GBRM1,6,2,400);;
# PrintNPList(BQMM1);

ff:=function(i,n)
    if i=0 and n=-1 then return 0;
    elif i=0 and n=0 then return 1;
    elif i=0 and n=1 then return 7;
    elif i=0 and n=2 then return 24;
    elif i=0 and n=3 then return 54;
    elif i=0 and n=4 then return 92;
    elif i=0 and n>4 then return (3*n+69)*(n-4)/2+92;

    elif i=1 and n=-1 then return 0;
    elif i=1 and n=0 then return 2;
    elif i=1 and n>0 then return (3*n+9)*n/2+2;

    elif i=2 and n=-1 then return 0;
    elif i=2 and n=0 then return 7;
    elif i=2 and n=1 then return 31;
    elif i=2 and n=2 then return 74;
    elif i=2 and n>2 then return (3*n+99)*(n-2)/2+74;

    elif i=3 and n=-1 then return 0;
    elif i=3 and n=0 then return 8;
    elif i=3 and n=1 then return 32;
    elif i=3 and n=2 then return 72;
    elif i=3 and n>2 then return 48*n-24;
    fi;
end;

FFM0:=function(j,i)
    local F,RDF,H,L,RFA,DFA,rra,dda,s,LAs,k,t;
    RDF:=List([ff(0,j-1)+1..ff(0,j+1)], p -> DT[p]);
    H:=List([1..6], s -> TransposedMat(MatrixQA(s,RDF,GB)));;
    L:=List([1..6], s -> List([ff(0,j)-ff(0,j-1)+1..ff(0,j+1)-ff(0,j-1)],
     q -> List([1..ff(0,j)-ff(0,j-1)], p -> H[s][q][p])));;
    RFA:=List([g(i-1)+1..g(i)], p -> C[p]);
    DFA:=List([g(i-2)+1..g(i-1)], p -> C[p]);
    rra:=Length(RFA);
    dda:=Length(DFA);
```

```
    F:=[];
    for s in [1..6] do
        LAs:=0*[1..dda];
        for k in [1..dda] do
            LAs[k]:=0*[1..rra];
            for t in [1..Length(MulQA(C[s+1], DFA[k], GA)[1])] do
                LAs[k][Position(RFA,[[MulQA(C[s+1], DFA[k], GA)[1][t]],[1]])]:=
                 MulQA(C[s+1], DFA[k], GA)[2][t];
            od;
        od;
        F:=F+KroneckerProduct(L[s],LAs);
    od;
    return F;
end;;

FFM1:=function(j,i)
    local FF,RF,DF,rr,dd,RFA,DFA,rra,dda,s,LLs,LAs,k,t;
    RF:=List([ff(1,j-1)+1..ff(1,j)], p -> BQMM1[p]);
    DF:=List([ff(1,j)+1..ff(1,j+1)], p -> BQMM1[p]);
    rr:=Length(RF);
    dd:=Length(DF);
    RFA:=List([g(i-1)+1..g(i)], p -> C[p]);
    DFA:=List([g(i-2)+1..g(i-1)], p -> C[p]);
    rra:=Length(RFA);
    dda:=Length(DFA);
    FF:=[];
    for s in [1..6] do
        LLs:=0*[1..rr];
        LAs:=0*[1..dda];
        for k in [1..rr] do
            LLs[k]:=0*[1..dd];
            for t in [1..Length(MulQM(RF[k], DT[s+1], GBRM1)[1])] do
                LLs[k][Position(DF,[[MulQM(RF[k], DT[s+1], GBRM1)[1][t]],[1]])]:=
                 MulQM(RF[k], DT[s+1], GBRM1)[2][t];
            od;
        od;
        for k in [1..dda] do
            LAs[k]:=0*[1..rra];
            for t in [1..Length(MulQA(C[s+1], DFA[k], GA)[1])] do
                LAs[k][Position(RFA,[[MulQA(C[s+1], DFA[k], GA)[1][t]],[1]])]:=
                 MulQA(C[s+1], DFA[k], GA)[2][t];
            od;
        od;
        FF:=FF+KroneckerProduct(TransposedMat(LLs),LAs);
    od;
    return FF;
end;

FFM2:=function(j,i)
    local FF,RF,DF,rr,dd,RFA,DFA,rra,dda,s,LLs,LAs,k,t;
    RF:=List([ff(2,j-1)+1..ff(2,j)], p -> BQMM2[p]);
    DF:=List([ff(2,j)+1..ff(2,j+1)], p -> BQMM2[p]);
    rr:=Length(RF);
    dd:=Length(DF);
    RFA:=List([g(i-1)+1..g(i)], p -> C[p]);
    DFA:=List([g(i-2)+1..g(i-1)], p -> C[p]);
    rra:=Length(RFA);
    dda:=Length(DFA);
    FF:=[];
    for s in [1..6] do
        LLs:=0*[1..rr];
        LAs:=0*[1..dda];
        for k in [1..rr] do
            LLs[k]:=0*[1..dd];
            for t in [1..Length(MulQM(RF[k], DT[s+1], GBRM2)[1])] do
                LLs[k][Position(DF,[[MulQM(RF[k], DT[s+1], GBRM2)[1][t]],[1]])]:=
                 MulQM(RF[k], DT[s+1], GBRM2)[2][t];
            od;
        od;
        for k in [1..dda] do
            LAs[k]:=0*[1..rra];
            for t in [1..Length(MulQA(C[s+1], DFA[k], GA)[1])] do
                LAs[k][Position(RFA,[[MulQA(C[s+1], DFA[k], GA)[1][t]],[1]])]:=
```

```
                MulQA(C[s+1], DFA[k], GA)[2][t];
            od;
        od;
        FF:=FF+KroneckerProduct(TransposedMat(LLs),LAs);
    od;
    return FF;
end;

FFM3:=function(j,i)
    local FF,RF,DF,rr,dd,RFA,DFA,rra,dda,s,LLs,LAs,k,t;
    RF:=List([ff(3,j-1)+1..ff(3,j)], p -> BQMM3[p]);
    DF:=List([ff(3,j)+1..ff(3,j+1)], p -> BQMM3[p]);
    rr:=Length(RF);
    dd:=Length(DF);
    RFA:=List([g(i-1)+1..g(i)], p -> C[p]);
    DFA:=List([g(i-2)+1..g(i-1)], p -> C[p]);
    rra:=Length(RFA);
    dda:=Length(DFA);
    FF:=[];
    for s in [1..6] do
        LLs:=0*[1..rr];
        LAs:=0*[1..dda];
        for k in [1..rr] do
            LLs[k]:=0*[1..dd];
            for t in [1..Length(MulQM(RF[k], DT[s+1], GBRM3)[1])] do
                LLs[k][Position(DF,[[MulQM(RF[k], DT[s+1], GBRM3)[1][t]],[1]])]:=
                 MulQM(RF[k], DT[s+1], GBRM3)[2][t];
            od;
        od;
        for k in [1..dda] do
            LAs[k]:=0*[1..rra];
            for t in [1..Length(MulQA(C[s+1], DFA[k], GA)[1])] do
                LAs[k][Position(RFA,[[MulQA(C[s+1], DFA[k], GA)[1][t]],[1]])]:=
                 MulQA(C[s+1], DFA[k], GA)[2][t];
            od;
        od;
        FF:=FF+KroneckerProduct(TransposedMat(LLs),LAs);
    od;
    return FF;
end;

FF:=function(ii,j,i)
    if ii=0 then return FFM0(j,i);
    elif ii=1 then return FFM1(j,i);
    elif ii=2 then return FFM2(j,i);
    elif ii=3 then return FFM3(j,i);
    fi;
end;

Im:=function(ii,j,i)
    local Imm;
    Imm:=TriangulizedMat(BaseMatDestructive(FF(ii,j,i)));
    return Imm;
end;

Ker:=function(ii,j,i)
    local Kerr;
    Kerr:=TriangulizedNullspaceMatDestructive(FF(ii,j-1,i+1));
    return Kerr;
end;

HXR:=function(ii,Uh,Vh,Wh,n,m,r)
    local hxr,Vhxr,CC,s,t,le,yy,VP,j,i,k;
    VP:=[];;
    hxr:=0*[1..Length(Uh)*f(r)];;
    Vhxr:=0*[1..Length(Uh)*f(r)];;
    CC:=List([g(m+r-1)+1..g(m+r)], p -> C[p]);;
    for s in [1..Length(Uh)] do
        for t in [1..f(r)] do
            le:=Length(Uh[s]);;
            yy:=C[g(r-1)+t];;
            hxr[(s-1)*f(r)+t]:=0*[1..(ff(ii,n)-ff(ii,n-1))*f(m+r)];;
            VP:=0*[1..le];;
```

```
            Vhxr[(s-1)*f(r)+t]:=0*[1..le];;
            for j in [1..le] do
                VP[j]:=[ [  ], [  ] ];
                for i in [1..Length(Vh[s][j])] do
                    VP[j]:=AddNP(VP[j],MulQA(C[g(m-1)+Vh[s][j][i]],yy,GA),1,
                     Wh[s][j][i]);
                od;
                Vhxr[(s-1)*f(r)+t][j]:=List([1..Length(VP[j][1])], k ->
                 Position(CC, [ [VP[j][1][k]], [ 1 ] ] ));
                for k in [1..Length(VP[j][1])] do
                    hxr[(s-1)*f(r)+t][f(m+r)*(Uh[s][j]-1)+Vhxr[(s-1)*f(r)+t][j][k]]:=
                     VP[j][2][k];
                od;
            od;
        od;
    od;
    return hxr;
end;

UU:=function(gene,ii)
    local Rest,Uh,Vh,Wh,Post,k,aa,Quo,Res,Sig,Qu,Re,Sg,i,j;
    Rest:=function(n)
        if n mod f(ii) > 0 then return n mod f(ii);
        else return f(ii);
        fi;
    end;
    Uh:=0*[1..Length(gene)];;
    Vh:=0*[1..Length(gene)];;
    Wh:=0*[1..Length(gene)];;
    Post:=[];;
    for k in [1..Length(gene)] do
        Uh[k]:=[]; Vh[k]:=[]; Wh[k]:=[];
        aa:=gene[k];
        Post:=[1..Length(aa)];
        SubtractSet(Post, Positions(aa,0));
        Quo:=List([1..Length(Post)], s->(Post[s]-Rest(Post[s]))/f(ii)+1);
        Res:=List([1..Length(Post)], s->Rest(Post[s]));
        Sig:=List([1..Length(Post)], s->gene[k][Post[s]]);
        Qu:=Set(Quo);
        Re:=0*[1..Length(Qu)];
        Sg:=0*[1..Length(Qu)];
        for i in [1..Length(Qu)] do
            Re[i]:=[];
            Sg[i]:=[];
            for j in [1..Length(Positions(Quo,Qu[i]))] do
                Re[i][j]:=Res[Position(Quo,Qu[i])+j-1];
                Sg[i][j]:=Sig[Position(Quo,Qu[i])+j-1];
            od;
        od;
        Uh[k]:=Qu;
        Vh[k]:=Re;
        Wh[k]:=Sg;
    od;
    return Uh;
end;

VV:=function(gene,ii)
    local Rest,Uh,Vh,Wh,Post,k,aa,Quo,Res,Sig,Qu,Re,Sg,i,j;
    Rest:=function(n)
        if n mod f(ii) > 0 then return n mod f(ii);
        else return f(ii);
        fi;
    end;
    Uh:=0*[1..Length(gene)];;
    Vh:=0*[1..Length(gene)];;
    Wh:=0*[1..Length(gene)];;
    Post:=[];;
    for k in [1..Length(gene)] do
        Uh[k]:=[]; Vh[k]:=[]; Wh[k]:=[];
        aa:=gene[k];
        Post:=[1..Length(aa)];
        SubtractSet(Post, Positions(aa,0));
        Quo:=List([1..Length(Post)], s->(Post[s]-Rest(Post[s]))/f(ii)+1);
```

```
        Res:=List([1..Length(Post)], s->Rest(Post[s]));
        Sig:=List([1..Length(Post)], s->gene[k][Post[s]]);
        Qu:=Set(Quo);
        Re:=0*[1..Length(Qu)];
        Sg:=0*[1..Length(Qu)];
        for i in [1..Length(Qu)] do
            Re[i]:=[];
            Sg[i]:=[];
            for j in [1..Length(Positions(Quo,Qu[i]))] do
                Re[i][j]:=Res[Position(Quo,Qu[i])+j-1];
                Sg[i][j]:=Sig[Position(Quo,Qu[i])+j-1];
            od;
        od;
        Uh[k]:=Qu;
        Vh[k]:=Re;
        Wh[k]:=Sg;
    od;
    return Vh;
end;

WW:=function(gene,ii)
    local Rest,Uh,Vh,Wh,Post,k,aa,Quo,Res,Sig,Qu,Re,Sg,i,j;
    Rest:=function(n)
        if n mod f(ii) > 0 then return n mod f(ii);
        else return f(ii);
        fi;
    end;
    Uh:=0*[1..Length(gene)];;
    Vh:=0*[1..Length(gene)];;
    Wh:=0*[1..Length(gene)];;
    Post:=[];;
    for k in [1..Length(gene)] do
        Uh[k]:=[]; Vh[k]:=[]; Wh[k]:=[];
        aa:=gene[k];
        Post:=[1..Length(aa)];
        SubtractSet(Post, Positions(aa,0));
        Quo:=List([1..Length(Post)], s->(Post[s]-Rest(Post[s]))/f(ii)+1);
        Res:=List([1..Length(Post)], s->Rest(Post[s]));
        Sig:=List([1..Length(Post)], s->gene[k][Post[s]]);
        Qu:=Set(Quo);
        Re:=0*[1..Length(Qu)];
        Sg:=0*[1..Length(Qu)];
        for i in [1..Length(Qu)] do
            Re[i]:=[];
            Sg[i]:=[];
            for j in [1..Length(Positions(Quo,Qu[i]))] do
                Re[i][j]:=Res[Position(Quo,Qu[i])+j-1];
                Sg[i][j]:=Sig[Position(Quo,Qu[i])+j-1];
            od;
        od;
        Uh[k]:=Qu;
        Vh[k]:=Re;
        Wh[k]:=Sg;
    od;
    return Wh;
end;

geneMH:=function(i,n,m)
    if i=0 and n=3 and m=3 then return
    [Ker(0,3,3)[99], Ker(0,3,3)[378], Ker(0,3,3)[164], -Ker(0,3,3)[467],
     Ker(0,3,3)[219], -Ker(0,3,3)[40], Ker(0,3,3)[301],
     Ker(0,3,3)[206]-Ker(0,3,3)[99]-Ker(0,3,3)[378]+Ker(0,3,3)[164]-Ker(0,3,3)[467]];
    elif i=0 and n=3 and m=5 then return [Ker(0,3,5)[79]];
    elif i=0 and n=4 and m=4 then return
    [Ker(0,4,4)[550], Ker(0,4,4)[450]-Ker(0,4,4)[550]];
    elif i=0 and n=5 and m=11 then return [Ker(0,5,11)[90]];
    elif i=1 and n=1 and m=3 then return
    [Ker(1,1,3)[15], Ker(1,1,3)[27], Ker(1,1,3)[53], -Ker(1,1,3)[67],
     Ker(1,1,3)[19], -Ker(1,1,3)[16], Ker(1,1,3)[22]];
    elif i=1 and n=1 and m=5 then return [Ker(1,1,5)[76]];
    elif i=1 and n=1 and m=7 then return [Ker(1,1,7)[64]];
    elif i=2 and n=1 and m=3 then return [Ker(2,1,3)[257]];
    elif i=2 and n=1 and m=5 then return [Ker(2,1,5)[908]];
```

```
    elif i=2 and n=2 and m=4 then return
    [Ker(2,2,4)[783]-Ker(2,2,4)[784], Ker(2,2,4)[784]];
    elif i=2 and n=3 and m=3 then return
    [Ker(2,3,3)[36]-Ker(2,3,3)[193]-Ker(2,3,3)[470]-Ker(2,3,3)[570]-Ker(2,3,3)[658],
     Ker(2,3,3)[200], Ker(2,3,3)[197], -Ker(2,3,3)[16],
     Ker(2,3,3)[193], Ker(2,3,3)[470], Ker(2,3,3)[570], Ker(2,3,3)[658]];
    elif i=3 and n=3 and m=3 then return
    [Ker(3,3,3)[179], -Ker(3,3,3)[185], -Ker(3,3,3)[174], Ker(3,3,3)[176],
     Ker(3,3,3)[355], Ker(3,3,3)[452], Ker(3,3,3)[540], Ker(3,3,3)[628]];
    else return [];
    fi;
end;
```

A.5 A basis of $(M^2)^!$

We present here the GAP code to compute a basis of $(M^2)^!_{-n}$ for n less than some positive integer, where the quadratic module M^2 is defined at the beginning of Subsection 3.2. We also list the basis of $(M^2)^!_{-n}$ for $n \in [\![0, 3]\!]$.

```
LoadPackage("GBNP");
B:=FreeAssociativeAlgebraWithOne(Rationals,"y12","y13","y23","y14","y24","y34");;
y12:=B.y12;; y13:=B.y13;; y23:=B.y23;; y14:=B.y14;; y24:=B.y24;; y34:=B.y34;;
oB:=One(B);;
relationsB:=[y12*y23+y23*y13, y13*y23+y23*y12, y12*y23+y13*y12, y12*y13+y23*y12,
 y12*y24+y24*y14, y14*y24+y24*y12, y12*y24+y14*y12, y12*y14+y24*y12, y13*y34+y34*y14,
 y14*y34+y34*y13, y13*y34+y14*y13, y13*y14+y34*y13, y23*y34+y34*y24, y24*y34+y34*y23,
 y23*y34+y24*y23, y23*y24+y34*y23, y12*y34+y34*y12, y13*y24+y24*y13,
 y23*y14+y14*y23];;
relsBNP:=GP2NPList(relationsB);;
wtv:= [1,1,1,1,1,1];;
GB:=Grobner(relsBNP);;
GBNP.ConfigPrint(B);;
PrintNPList(GB);

D:= BaseQATrunc(GB,12,wtv);;
for degpart in D do
    for mon in degpart do
        PrintNP([[mon],[1]]);
    od;
od;
DT:=[];
for degpart in D do
    for mon in degpart do
        Append(DT,[[[mon],[1]]]);
    od;
od;

S:=B^7;
ab:=GeneratorsOfLeftModule(S);
g7:=ab[1]; g6:=ab[2]; g5:=ab[3]; g4:=ab[4]; g3:=ab[5]; g2:=ab[6]; g1:=ab[7];
modrels:=[g1*y14+g4*y14, g1*y24+g3*y24, g1*y34-g2*y34, g2*y13+g4*y13, g2*y23+g3*y23,
 g3*y12-g4*y12, g5*y12-g6*y34, g1*y24+g5*y13, g5*y23+g7*y14, g5*y14-g7*y23,
 g2*y13-g5*y24, g5*y34+g6*y12, g6*y13+g7*y24, g1*y14-g6*y23, g2*y23+g6*y14,
 g6*y24-g7*y13, g1*y34-g7*y12, g3*y12-g7*y34];
modrelsNP:=GP2NPList(modrels);
PrintNPList(modrelsNP);

GBNP.CheckHom:=function(G,wtv)
    local i,j,k,l,mon,h1,h2,ans;
    mon:=LMonsNP(G);
    ans:=GBNP.WeightedDegreeList(mon,wtv);
    for i in [1..Length(G)] do
        h1:=ans[i];
        l:=Length(G[i][1]);
        for j in [2..l] do
            mon:=G[i][1][j];
            h2:=0;
            for k in [1..Length(mon)] do
                if mon[k]>0 then
                    h2:=h2+wtv[mon[k]];
                fi;
            od;
```

```
            if h2<>h1 then return(false); fi;
        od;
    od;
    Info(InfoGBNP,1,"Input is homogeneous");
    return(ans);
end;

GBNP.WeightedDegreeMon:=function(mon,lst)
    local i,ans;
    ans:=0;
    for i in mon do
        if i>0 then
            ans:=ans+lst[i];
        fi;
    od;
    return(ans);
end;;

SetInfoLevel(InfoGBNP,1);
SetInfoLevel(InfoGBNPTime,1);
combinedrelsNP:=Concatenation(GB,modrelsNP);
GBT:=SGrobnerTrunc(combinedrelsNP, 15, [1,1,1,1,1,1]);
PrintNPList(GBT);

splitGBT:=function(GBT)
    local p, ts, rel, lm;
    p:=[];
    ts:=[];
    for rel in GBT do
        lm := rel[1,1];
        if Length(lm)>1 and lm[1]<0 then
            Add(p, rel);
        else
            Add(ts, rel);
        fi;
    od;
    return rec(p:=p, ts:=ts);
end;;

split:=splitGBT(GBT);
GBR:=rec(p:=split.p, pg:=7, ts:=split.ts);
BQM:=BaseQM(GBR,6,7,500);;
PrintNPList(BQM);

[ 0, 0, 0, 0, 0, 0, 1 ]
[ 0, 0, 0, 0, 0, 1 , 0]
[ 0, 0, 0, 0, 1 , 0, 0]
[ 0, 0, 0, 1 , 0, 0, 0]
[ 0, 0, 1 , 0, 0, 0, 0]
[ 0, 1 , 0, 0, 0, 0, 0]
[ 1 , 0, 0, 0, 0, 0, 0]
[ 0, 0, 0, 0, 0, 0, y12 ]
[ 0, 0, 0, 0, 0, 0, y13 ]
[ 0, 0, 0, 0, 0, 0, y23 ]
[ 0, 0, 0, 0, 0, 0, y14 ]
[ 0, 0, 0, 0, 0, 0, y24 ]
[ 0, 0, 0, 0, 0, 0, y34 ]
[ 0, 0, 0, 0, 0, y12 , 0]
[ 0, 0, 0, 0, 0, y13 , 0]
[ 0, 0, 0, 0, 0, y23 , 0]
[ 0, 0, 0, 0, 0, y14 , 0]
[ 0, 0, 0, 0, 0, y24 , 0]
[ 0, 0, 0, 0, y12 , 0, 0]
[ 0, 0, 0, 0, y13 , 0, 0]
[ 0, 0, 0, 0, y14 , 0, 0]
[ 0, 0, 0, 0, y34 , 0, 0]
[ 0, 0, 0, y23 , 0, 0, 0]
[ 0, 0, 0, y24 , 0, 0, 0]
[ 0, 0, 0, y34 , 0, 0, 0]
[ 0, 0, y12 , 0, 0, 0, 0]
[ 0, 0, y23 , 0, 0, 0, 0]
[ 0, 0, y14 , 0, 0, 0, 0]
[ 0, 0, y34 , 0, 0, 0, 0]
```

```
[ 0, y13 , 0, 0, 0, 0, 0]
[ 0, y24 , 0, 0, 0, 0, 0]
[ 0, 0, 0, 0, 0, 0, y12^2 ]
[ 0, 0, 0, 0, 0, 0, y12y13 ]
[ 0, 0, 0, 0, 0, 0, y12y23 ]
[ 0, 0, 0, 0, 0, 0, y12y14 ]
[ 0, 0, 0, 0, 0, 0, y12y24 ]
[ 0, 0, 0, 0, 0, 0, y12y34 ]
[ 0, 0, 0, 0, 0, 0, y13^2 ]
[ 0, 0, 0, 0, 0, 0, y13y14 ]
[ 0, 0, 0, 0, 0, 0, y13y24 ]
[ 0, 0, 0, 0, 0, 0, y13y34 ]
[ 0, 0, 0, 0, 0, 0, y23^2 ]
[ 0, 0, 0, 0, 0, 0, y23y14 ]
[ 0, 0, 0, 0, 0, 0, y23y24 ]
[ 0, 0, 0, 0, 0, 0, y23y34 ]
[ 0, 0, 0, 0, 0, 0, y14^2 ]
[ 0, 0, 0, 0, 0, 0, y24^2 ]
[ 0, 0, 0, 0, 0, 0, y34^2 ]
[ 0, 0, 0, 0, 0, y12^2 , 0]
[ 0, 0, 0, 0, 0, y12y13 , 0]
[ 0, 0, 0, 0, 0, y12y23 , 0]
[ 0, 0, 0, 0, 0, y12y14 , 0]
[ 0, 0, 0, 0, 0, y12y24 , 0]
[ 0, 0, 0, 0, 0, y13y24 , 0]
[ 0, 0, 0, 0, 0, y23y14 , 0]
[ 0, 0, 0, 0, 0, y14^2 , 0]
[ 0, 0, 0, 0, 0, y24^2 , 0]
[ 0, 0, 0, 0, y12y34 , 0, 0]
[ 0, 0, 0, 0, y13^2 , 0, 0]
[ 0, 0, 0, 0, y13y14 , 0, 0]
[ 0, 0, 0, 0, y13y34 , 0, 0]
[ 0, 0, 0, 0, y14^2 , 0, 0]
[ 0, 0, 0, 0, y34^2 , 0, 0]
[ 0, 0, 0, y23^2 , 0, 0, 0]
[ 0, 0, 0, y23y24 , 0, 0, 0]
[ 0, 0, 0, y23y34 , 0, 0, 0]
[ 0, 0, 0, y24^2 , 0, 0, 0]
[ 0, 0, 0, y34^2 , 0, 0, 0]
[ 0, 0, y12^2 , 0, 0, 0, 0]
[ 0, 0, y12y34 , 0, 0, 0, 0]
[ 0, 0, y23^2 , 0, 0, 0, 0]
[ 0, 0, y23y14 , 0, 0, 0, 0]
[ 0, y13^2 , 0, 0, 0, 0, 0]
[ 0, y13y24 , 0, 0, 0, 0, 0]
[ 0, 0, 0, 0, 0, 0, y12^3 ]
[ 0, 0, 0, 0, 0, 0, y12^2y13 ]
[ 0, 0, 0, 0, 0, 0, y12^2y23 ]
[ 0, 0, 0, 0, 0, 0, y12^2y14 ]
[ 0, 0, 0, 0, 0, 0, y12^2y24 ]
[ 0, 0, 0, 0, 0, 0, y12^2y34 ]
[ 0, 0, 0, 0, 0, 0, y12y13^2 ]
[ 0, 0, 0, 0, 0, 0, y12y13y14 ]
[ 0, 0, 0, 0, 0, 0, y12y13y24 ]
[ 0, 0, 0, 0, 0, 0, y12y13y34 ]
[ 0, 0, 0, 0, 0, 0, y12y23y14 ]
[ 0, 0, 0, 0, 0, 0, y12y23y24 ]
[ 0, 0, 0, 0, 0, 0, y12y23y34 ]
[ 0, 0, 0, 0, 0, 0, y12y14^2 ]
[ 0, 0, 0, 0, 0, 0, y12y34^2 ]
[ 0, 0, 0, 0, 0, 0, y13^3 ]
[ 0, 0, 0, 0, 0, 0, y13^2y24 ]
[ 0, 0, 0, 0, 0, 0, y13^2y34 ]
[ 0, 0, 0, 0, 0, 0, y13y14^2 ]
[ 0, 0, 0, 0, 0, 0, y13y24^2 ]
[ 0, 0, 0, 0, 0, 0, y23^3 ]
[ 0, 0, 0, 0, 0, 0, y23^2y14 ]
[ 0, 0, 0, 0, 0, 0, y23y14^2 ]
[ 0, 0, 0, 0, 0, 0, y23y24^2 ]
[ 0, 0, 0, 0, 0, 0, y14^3 ]
[ 0, 0, 0, 0, 0, 0, y24^3 ]
[ 0, 0, 0, 0, 0, 0, y34^3 ]
[ 0, 0, 0, 0, 0, y12^3 , 0]
```

```
[ 0, 0, 0, 0, 0, y12^2y14 , 0]
[ 0, 0, 0, 0, 0, y12^2y24 , 0]
[ 0, 0, 0, 0, 0, y12y14^2 , 0]
[ 0, 0, 0, 0, 0, y13y24^2 , 0]
[ 0, 0, 0, 0, 0, y23y14^2 , 0]
[ 0, 0, 0, 0, 0, y14^3 , 0]
[ 0, 0, 0, 0, 0, y24^3 , 0]
[ 0, 0, 0, 0, y12y34^2 , 0, 0]
[ 0, 0, 0, 0, y13^3 , 0, 0]
[ 0, 0, 0, 0, y13^2y14 , 0, 0]
[ 0, 0, 0, 0, y13^2y34 , 0, 0]
[ 0, 0, 0, 0, y13y14^2 , 0, 0]
[ 0, 0, 0, 0, y14^3 , 0, 0]
[ 0, 0, 0, 0, y34^3 , 0, 0]
[ 0, 0, 0, y23^3 , 0, 0, 0]
[ 0, 0, 0, y23^2y24 , 0, 0, 0]
[ 0, 0, 0, y23^2y34 , 0, 0, 0]
[ 0, 0, 0, y23y24^2 , 0, 0, 0]
[ 0, 0, 0, y24^3 , 0, 0, 0]
[ 0, 0, 0, y34^3 , 0, 0, 0]
[ 0, 0, y12^3 , 0, 0, 0, 0]
[ 0, 0, y12^2y34 , 0, 0, 0, 0]
[ 0, 0, y23^3 , 0, 0, 0, 0]
[ 0, 0, y23^2y14 , 0, 0, 0, 0]
[ 0, y13^3 , 0, 0, 0, 0, 0]
[ 0, y13^2y24 , 0, 0, 0, 0, 0]
```

A.6 Right action of $\mathrm{FK}(4)^!$ on $(M^2)^!$

We list below the right action of some elements of $A^!$ on $(M^2)^!$, where M^2 is the quadratic (right) A-module defined at the beginning of Subsection 3.2. In Tables A.10-A.13, the entry appearing in the row indexed by y and the column indexed by y' is the product yy'. To reduce space, the integer $m \in [\![1, 24]\!]$, appearing in the third to fifth columns of Tables A.10-A.13 indicates the element b_m^{n+1}, where b_m^n is the m-th element in (3.24) for $n \geqslant 4$ and $m \in [\![1, 24]\!]$.

	$y \backslash y'$	$y_{1,2}$	$y_{1,3}$	$y_{2,3}$
1	$g_1y_{1,2}^{n-1}y_{1,3}$	-2	5	1
2	$g_1y_{1,2}^{n-1}y_{2,3}$	-1	-2	5
3	$g_1y_{1,2}^{n-1}y_{1,4}$	-4	-8	-9
4	$g_1y_{1,2}^{n-1}y_{2,4}$	-3	-7	-11
5	$g_1y_{1,2}^{n-2}y_{1,3}^2$	5	1	2
6	$g_1y_{1,2}^{n-2}y_{1,3}y_{1,4}$	10	-13	-6
7	$g_1y_{1,2}^{n-2}y_{1,3}y_{2,4}$	9	-4	-8
8	$g_1y_{1,2}^{n-2}y_{1,3}y_{3,4}$	11	-3	-7
9	$g_1y_{1,2}^{n-2}y_{2,3}y_{1,4}$	7	11	-3
10	$g_1y_{1,2}^{n-2}y_{2,3}y_{2,4}$	6	10	-13
11	$g_1y_{1,2}^{n-2}y_{2,3}y_{3,4}$	8	9	-4
12	$g_1y_{1,2}^{n-2}y_{1,4}^2$	12	14	15
13	$g_1y_{1,2}^{n-3}y_{1,3}^2y_{3,4}$	-13	-6	-10
14	$g_1y_{1,2}^{n-3}y_{1,3}y_{1,4}^2$	-15	12	14
15	$g_1y_{1,2}^{n-3}y_{2,3}y_{1,4}^2$	-14	-15	12
16	$g_2y_{1,2}^{n-1}y_{1,4}$	-17	-8	-9
17	$g_2y_{1,2}^{n-1}y_{2,4}$	-16	-7	-11
18	$g_2y_{1,2}^{n-2}y_{1,4}^2$	18	14	15
19	$g_3y_{1,3}^{n-1}y_{1,4}$	-10	-20	6
20	$g_3y_{1,3}^{n-1}y_{3,4}$	-11	-19	7
21	$g_3y_{1,3}^{n-2}y_{1,4}^2$	-12	21	-15
22	$g_4y_{2,3}^{n-1}y_{2,4}$	-6	-10	-23
23	$g_4y_{2,3}^{n-1}y_{3,4}$	-8	-9	-22
24	$g_4y_{2,3}^{n-2}y_{2,4}^2$	-12	-14	24
	$g_1y_{1,2}^n$	$g_1y_{1,2}^{n+1}$	1	2
	$g_1y_{1,2}^{n-r}y_{3,4}^r$	$(-1)^rg_1y_{1,2}^{n-r+1}y_{3,4}^r$	$\chi_r14 - \chi_{r+1}6$	$\chi_r15 - \chi_{r+1}10$
	$g_1y_{3,4}^n$	$g_1y_{1,2}y_{3,4}^n$	14	15
	$g_2y_{1,2}^n$	$g_2y_{1,2}^{n+1}$	14	15
	$g_3y_{1,2}y_{3,4}^{n-1}$	$g_1y_{3,4}^{n+1}$	6	10
	$g_3y_{3,4}^n$	$g_3y_{1,2}y_{3,4}^n$	21	-15
	$g_4y_{3,4}^n$	$g_3y_{1,2}y_{3,4}^n$	-14	24
	$g_5y_{1,2}^n$	$g_5y_{1,2}^{n+1}$	-4	-8
	$g_5y_{1,2}^{n-1}y_{3,4}$	$-g_5y_{1,2}^ny_{3,4}$	-11	3
	$g_1y_{1,3}^n$	5	$g_1y_{1,3}^{n+1}$	2
	$g_1y_{1,3}^{n-r}y_{2,4}^r$	$\chi_r12 + \chi_{r+1}9$	$(-1)^rg_1y_{1,3}^{n-r+1}y_{2,4}^r$	$\chi_r15 - \chi_{r+1}8$
	$g_1y_{2,4}^n$	12	$g_1y_{1,3}y_{2,4}^n$	15
	$g_2y_{1,3}y_{2,4}^{n-1}$	9	$-g_1y_{2,4}^{n+1}$	-8
	$g_2y_{2,4}^n$	18	$g_2y_{1,3}y_{2,4}^n$	15
	$g_3y_{1,3}^n$	-12	$g_3y_{1,3}^{n+1}$	-15
	$g_4y_{2,4}^n$	-12	$-g_2y_{1,3}y_{2,4}^n$	24
	$g_6y_{1,3}^n$	-7	$g_6y_{1,3}^{n+1}$	3
	$g_6y_{1,3}^{n-1}y_{2,4}$	-13	$-g_6y_{1,3}^ny_{2,4}$	-10
	$g_1y_{2,3}^n$	5	1	$g_1y_{2,3}^{n+1}$
	$g_1y_{2,3}^{n-r}y_{1,4}^r$	$\chi_r12 + \chi_{r+1}7$	$\chi_r14 + \chi_{r+1}11$	$(-1)^rg_1y_{2,3}^{n-r+1}y_{1,4}^r$
	$g_1y_{1,4}^n$	12	14	$g_1y_{2,3}y_{1,4}^n$
	$g_2y_{2,3}y_{1,4}^{n-1}$	7	11	$-g_1y_{1,4}^{n+1}$
	$g_2y_{1,4}^n$	18	14	$g_2y_{2,3}y_{1,4}^n$
	$g_3y_{1,4}^n$	-12	21	$-g_2y_{2,3}y_{1,4}^n$
	$g_4y_{2,3}^n$	-12	-14	$g_4y_{2,3}^{n+1}$
	$g_5y_{2,3}^n$	9	-4	$g_5y_{2,3}^{n+1}$
	$g_5y_{2,3}^{n-1}y_{1,4}$	-13	-6	$-g_5y_{2,3}^ny_{1,4}$

Table A.10: Products yy' for $n \geqslant 4$ even.

	$y \backslash y'$	$y_{1,4}$	$y_{2,4}$	$y_{3,4}$
1	$g_1y_{1,2}^{n-1}y_{1,3}$	6	7	8
2	$g_1y_{1,2}^{n-1}y_{2,3}$	9	10	11
3	$g_1y_{1,2}^{n-1}y_{1,4}$	12	3	6
4	$g_1y_{1,2}^{n-1}y_{2,4}$	-4	12	10
5	$g_1y_{1,2}^{n-2}y_{1,3}^2$	3	4	13
6	$g_1y_{1,2}^{n-2}y_{1,3}y_{1,4}$	14	-9	3
7	$g_1y_{1,2}^{n-2}y_{1,3}y_{2,4}$	10	14	7
8	$g_1y_{1,2}^{n-2}y_{1,3}y_{3,4}$	-13	-8	14
9	$g_1y_{1,2}^{n-2}y_{2,3}y_{1,4}$	15	-6	-9
10	$g_1y_{1,2}^{n-2}y_{2,3}y_{2,4}$	7	15	4
11	$g_1y_{1,2}^{n-2}y_{2,3}y_{3,4}$	11	-13	15
12	$g_1y_{1,2}^{n-2}y_{1,4}^2$	3	4	13
13	$g_1y_{1,2}^{n-3}y_{1,3}^2y_{3,4}$	-8	-11	12
14	$g_1y_{1,2}^{n-3}y_{1,3}y_{1,4}^2$	6	7	8
15	$g_1y_{1,2}^{n-3}y_{2,3}y_{1,4}^2$	9	10	11
16	$g_2y_{1,2}^{n-1}y_{1,4}$	18	16	6
17	$g_2y_{1,2}^{n-1}y_{2,4}$	-17	18	10
18	$g_2y_{1,2}^{n-2}y_{1,4}^2$	16	17	13
19	$g_3y_{1,3}^{n-1}y_{1,4}$	21	9	19
20	$g_3y_{1,3}^{n-1}y_{3,4}$	-20	8	21
21	$g_3y_{1,3}^{n-2}y_{1,4}^2$	19	-4	20
22	$g_4y_{2,3}^{n-1}y_{2,4}$	-7	24	22
23	$g_4y_{2,3}^{n-1}y_{3,4}$	-11	-23	24
24	$g_4y_{2,3}^{n-2}y_{2,4}^2$	-3	22	23
	$g_1y_{1,2}^n$	3	4	$g_1y_{1,2}^ny_{3,4}$
	$g_1y_{1,2}^{n-r}y_{3,4}^r$	$\chi_r3-\chi_{r+1}8$	$\chi_r4-\chi_{r+1}11$	$g_1y_{1,2}^{n-r}y_{3,4}^{r+1}$
	$g_1y_{3,4}^n$	3	4	$g_1y_{3,4}^{n+1}$
	$g_2y_{1,2}^n$	16	17	$g_1y_{1,2}^ny_{3,4}$
	$g_3y_{1,2}y_{3,4}^{n-1}$	8	11	$g_3y_{1,2}y_{3,4}^n$
	$g_3y_{3,4}^n$	19	-4	$g_3y_{3,4}^{n+1}$
	$g_4y_{3,4}^n$	-3	22	$g_4y_{3,4}^{n+1}$
	$g_5y_{1,2}^n$	10	14	$g_5y_{1,2}^ny_{3,4}$
	$g_5y_{1,2}^{n-1}y_{3,4}$	-15	6	$g_5y_{1,2}^{n+1}$
	$g_1y_{1,3}^n$	3	$g_1y_{1,3}^ny_{2,4}$	13
	$g_1y_{1,3}^{n-r}y_{2,4}^r$	$\chi_r3+\chi_{r+1}10$	$g_1y_{1,3}^{n-r}y_{2,4}^{r+1}$	$\chi_r13+\chi_{r+1}7$
	$g_1y_{2,4}^n$	3	$g_1y_{2,4}^{n+1}$	13
	$g_2y_{1,3}y_{2,4}^{n-1}$	10	$g_2y_{1,3}y_{2,4}^n$	7
	$g_2y_{2,4}^n$	16	$g_2y_{2,4}^{n+1}$	13
	$g_3y_{1,3}^n$	19	$-g_1y_{1,3}^ny_{2,4}$	20
	$g_4y_{2,4}^n$	-3	$g_4y_{2,4}^{n+1}$	23
	$g_6y_{1,3}^n$	-15	$g_6y_{1,3}^ny_{2,4}$	9
	$g_6y_{1,3}^{n-1}y_{2,4}$	-8	$g_6y_{1,3}^{n+1}$	12
	$g_1y_{2,3}^n$	$g_1y_{2,3}^ny_{1,4}$	4	13
	$g_1y_{2,3}^{n-r}y_{1,4}^r$	$g_1y_{2,3}^{n-r}y_{1,4}^{r+1}$	$\chi_r4-\chi_{r+1}6$	$\chi_r13-\chi_{r+1}9$
	$g_1y_{1,4}^n$	$g_1y_{1,4}^{n+1}$	4	13
	$g_2y_{2,3}y_{1,4}^{n-1}$	$g_2y_{2,3}y_{1,4}^n$	-6	-9
	$g_2y_{1,4}^n$	$g_2y_{1,4}^{n+1}$	17	13
	$g_3y_{1,4}^n$	$g_3y_{1,4}^{n+1}$	-4	20
	$g_4y_{2,3}^n$	$-g_1y_{2,3}^ny_{1,4}$	22	23
	$g_5y_{2,3}^n$	$g_5y_{2,3}^ny_{1,4}$	14	7
	$g_5y_{2,3}^{n-1}y_{1,4}$	$g_5y_{2,3}^{n+1}$	-11	12

Table A.11: Products yy' for $n \geqslant 4$ even.

	$y \setminus y'$	$y_{1,2}$	$y_{1,3}$	$y_{2,3}$
1	$g_1 y_{1,2}^{n-1} y_{1,3}$	-2	5	1
2	$g_1 y_{1,2}^{n-1} y_{2,3}$	-1	-2	5
3	$g_1 y_{1,2}^{n-1} y_{1,4}$	-4	-8	-9
4	$g_1 y_{1,2}^{n-1} y_{2,4}$	-3	-7	-11
5	$g_1 y_{1,2}^{n-2} y_{1,3}^2$	5	1	2
6	$g_1 y_{1,2}^{n-2} y_{1,3} y_{1,4}$	10	-13	-6
7	$g_1 y_{1,2}^{n-2} y_{1,3} y_{2,4}$	9	-4	-8
8	$g_1 y_{1,2}^{n-2} y_{1,3} y_{3,4}$	11	-3	-7
9	$g_1 y_{1,2}^{n-2} y_{2,3} y_{1,4}$	7	11	-3
10	$g_1 y_{1,2}^{n-2} y_{2,3} y_{2,4}$	6	10	-13
11	$g_1 y_{1,2}^{n-2} y_{2,3} y_{3,4}$	8	9	-4
12	$g_1 y_{1,2}^{n-2} y_{1,4}^2$	12	14	15
13	$g_1 y_{1,2}^{n-3} y_{1,3}^2 y_{3,4}$	-13	-6	-10
14	$g_1 y_{1,2}^{n-3} y_{1,3} y_{1,4}^2$	-15	12	14
15	$g_1 y_{1,2}^{n-3} y_{2,3} y_{1,4}^2$	-14	-15	12
16	$g_2 y_{1,2}^{n-1} y_{1,4}$	-17	-8	-9
17	$g_2 y_{1,2}^{n-1} y_{2,4}$	-16	-7	-11
18	$g_2 y_{1,2}^{n-2} y_{1,4}^2$	18	14	15
19	$g_3 y_{1,3}^{n-1} y_{1,4}$	4	-20	9
20	$g_3 y_{1,3}^{n-1} y_{3,4}$	13	-19	10
21	$g_3 y_{1,3}^{n-2} y_{1,4}^2$	15	21	-14
22	$g_4 y_{2,3}^{n-1} y_{2,4}$	3	7	-23
23	$g_4 y_{2,3}^{n-1} y_{3,4}$	13	6	-22
24	$g_4 y_{2,3}^{n-2} y_{2,4}^2$	14	15	24
	$g_1 y_{1,2}^n$	$g_1 y_{1,2}^{n+1}$	1	2
	$g_1 y_{1,2}^{n-r} y_{3,4}^r$	$(-1)^r g_1 y_{1,2}^{n-r+1} y_{3,4}^r$	$\chi_r 14 - \chi_{r+1} 6$	$\chi_r 15 - \chi_{r+1} 10$
	$g_1 y_{3,4}^n$	$-g_1 y_{1,2} y_{3,4}^n$	-6	-10
	$g_2 y_{1,2}^n$	$g_2 y_{1,2}^{n+1}$	14	15
	$g_3 y_{1,2} y_{3,4}^{n-1}$	$-g_1 y_{3,4}^{n+1}$	-14	-15
	$g_3 y_{3,4}^n$	$-g_3 y_{1,2} y_{3,4}^n$	-19	10
	$g_4 y_{3,4}^n$	$-g_3 y_{1,2} y_{3,4}^n$	6	-22
	$g_5 y_{1,2}^n$	$g_5 y_{1,2}^{n+1}$	11	-3
	$g_5 y_{1,2}^{n-1} y_{3,4}$	$-g_5 y_{1,2}^n y_{3,4}$	-4	-8
	$g_1 y_{1,3}^n$	-2	$g_1 y_{1,3}^{n+1}$	1
	$g_1 y_{1,3}^{n-r} y_{2,4}^r$	$-\chi_r 15 - \chi_{r+1} 3$	$(-1)^r g_1 y_{1,3}^{n-r+1} y_{2,4}^r$	$\chi_r 14 - \chi_{r+1} 11$
	$g_1 y_{2,4}^n$	-3	$-g_1 y_{1,3} y_{2,4}^n$	-11
	$g_2 y_{1,3} y_{2,4}^{n-1}$	-15	$g_1 y_{2,4}^{n+1}$	14
	$g_2 y_{2,4}^n$	-16	$-g_2 y_{1,3} y_{2,4}^n$	-11
	$g_3 y_{1,3}^n$	15	$g_3 y_{1,3}^{n+1}$	-14
	$g_4 y_{2,4}^n$	3	$g_2 y_{1,3} y_{2,4}^n$	-23
	$g_6 y_{1,3}^n$	-8	$g_6 y_{1,3}^{n+1}$	4
	$g_6 y_{1,3}^{n-1} y_{2,4}$	10	$-g_6 y_{1,3}^n y_{2,4}$	-6
	$g_1 y_{2,3}^n$	-1	-2	$g_1 y_{2,3}^{n+1}$
	$g_1 y_{2,3}^{n-r} y_{1,4}^r$	$-\chi_r 14 - \chi_{r+1} 4$	$-\chi_r 15 - \chi_{r+1} 8$	$(-1)^r g_1 y_{2,3}^{n-r+1} y_{1,4}^r$
	$g_1 y_{1,4}^n$	-4	-8	$-g_1 y_{2,3} y_{1,4}^n$
	$g_2 y_{2,3} y_{1,4}^{n-1}$	-14	-15	$g_1 y_{1,4}^{n+1}$
	$g_2 y_{1,4}^n$	-17	-8	$-g_2 y_{2,3} y_{1,4}^n$
	$g_3 y_{1,4}^n$	4	-20	$g_2 y_{2,3} y_{1,4}^n$
	$g_4 y_{2,3}^n$	14	15	$g_4 y_{2,3}^{n+1}$
	$g_5 y_{2,3}^n$	-11	3	$g_5 y_{2,3}^{n+1}$
	$g_5 y_{2,3}^{n-1} y_{1,4}$	6	10	$-g_5 y_{2,3}^n y_{1,4}$

Table A.12: Products yy' for $n \geqslant 5$ odd.

	$y \backslash y'$	$y_{1,4}$	$y_{2,4}$	$y_{3,4}$
1	$g_1 y_{1,2}^{n-1} y_{1,3}$	6	7	8
2	$g_1 y_{1,2}^{n-1} y_{2,3}$	9	10	11
3	$g_1 y_{1,2}^{n-1} y_{1,4}$	12	3	6
4	$g_1 y_{1,2}^{n-1} y_{2,4}$	-4	12	10
5	$g_1 y_{1,2}^{n-2} y_{1,3}^2$	3	4	13
6	$g_1 y_{1,2}^{n-2} y_{1,3} y_{1,4}$	14	-9	3
7	$g_1 y_{1,2}^{n-2} y_{1,3} y_{2,4}$	10	14	7
8	$g_1 y_{1,2}^{n-2} y_{1,3} y_{3,4}$	-13	-8	14
9	$g_1 y_{1,2}^{n-2} y_{2,3} y_{1,4}$	15	-6	-9
10	$g_1 y_{1,2}^{n-2} y_{2,3} y_{2,4}$	7	15	4
11	$g_1 y_{1,2}^{n-2} y_{2,3} y_{3,4}$	11	-13	15
12	$g_1 y_{1,2}^{n-2} y_{1,4}^2$	3	4	13
13	$g_1 y_{1,2}^{n-3} y_{1,3}^2 y_{3,4}$	-8	-11	12
14	$g_1 y_{1,2}^{n-3} y_{1,3} y_{1,4}^2$	6	7	8
15	$g_1 y_{1,2}^{n-3} y_{2,3} y_{1,4}^2$	9	10	11
16	$g_2 y_{1,2}^{n-1} y_{1,4}$	18	16	6
17	$g_2 y_{1,2}^{n-1} y_{2,4}$	-17	18	10
18	$g_2 y_{1,2}^{n-2} y_{1,4}^2$	16	17	13
19	$g_3 y_{1,3}^{n-1} y_{1,4}$	21	-3	19
20	$g_3 y_{1,3}^{n-1} y_{3,4}$	-20	11	21
21	$g_3 y_{1,3}^{n-2} y_{1,4}^2$	19	-7	20
22	$g_4 y_{2,3}^{n-1} y_{2,4}$	4	24	22
23	$g_4 y_{2,3}^{n-1} y_{3,4}$	8	-23	24
24	$g_4 y_{2,3}^{n-2} y_{2,4}^2$	-9	22	23
	$g_1 y_{1,2}^n$	3	4	$g_1 y_{1,2}^n y_{3,4}$
	$g_1 y_{1,2}^{n-r} y_{3,4}^r$	$\chi_r 3 - \chi_{r+1} 8$	$\chi_r 4 - \chi_{r+1} 11$	$g_1 y_{1,2}^{n-r} y_{3,4}^{r+1}$
	$g_1 y_{3,4}^n$	-8	-11	$g_1 y_{3,4}^{n+1}$
	$g_2 y_{1,2}^n$	16	17	$g_1 y_{1,2}^n y_{3,4}$
	$g_3 y_{1,2} y_{3,4}^{n-1}$	-3	-4	$g_3 y_{1,2} y_{3,4}^n$
	$g_3 y_{3,4}^n$	-20	11	$g_3 y_{3,4}^{n+1}$
	$g_4 y_{3,4}^n$	8	-23	$g_4 y_{3,4}^{n+1}$
	$g_5 y_{1,2}^n$	15	-6	$g_5 y_{1,2}^n y_{3,4}$
	$g_5 y_{1,2}^{n-1} y_{3,4}$	10	14	$g_5 y_{1,2}^{n+1}$
	$g_1 y_{1,3}^n$	6	$g_1 y_{1,3}^n y_{2,4}$	8
	$g_1 y_{1,3}^{n-r} y_{2,4}^r$	$\chi_r 6 - \chi_{r+1} 4$	$g_1 y_{1,3}^{n-r} y_{2,4}^{r+1}$	$\chi_r 8 + \chi_{r+1} 10$
	$g_1 y_{2,4}^n$	-4	$g_1 y_{2,4}^{n+1}$	10
	$g_2 y_{1,3} y_{2,4}^{n-1}$	6	$g_2 y_{1,3} y_{2,4}^n$	8
	$g_2 y_{2,4}^n$	-17	$g_2 y_{2,4}^{n+1}$	10
	$g_3 y_{1,3}^n$	19	$-g_1 y_{1,3}^n y_{2,4}$	20
	$g_4 y_{2,4}^n$	4	$g_4 y_{2,4}^{n+1}$	22
	$g_6 y_{1,3}^n$	-11	$g_6 y_{1,3}^n y_{2,4}$	-15
	$g_6 y_{1,3}^{n-1} y_{2,4}$	14	$g_6 y_{1,3}^{n+1}$	3
	$g_1 y_{2,3}^n$	$g_1 y_{2,3}^n y_{1,4}$	10	11
	$g_1 y_{2,3}^{n-r} y_{1,4}^r$	$g_1 y_{2,3}^{n-r} y_{1,4}^{r+1}$	$\chi_r 10 + \chi_{r+1} 3$	$\chi_r 11 + \chi_{r+1} 6$
	$g_1 y_{1,4}^n$	$g_1 y_{1,4}^{n+1}$	3	6
	$g_2 y_{2,3} y_{1,4}^{n-1}$	$g_2 y_{2,3} y_{1,4}^n$	10	11
	$g_2 y_{1,4}^n$	$g_2 y_{1,4}^{n+1}$	16	6
	$g_3 y_{1,4}^n$	$g_3 y_{1,4}^{n+1}$	-3	19
	$g_4 y_{2,3}^n$	$-g_1 y_{2,3}^n y_{1,4}$	22	23
	$g_5 y_{2,3}^n$	$g_5 y_{2,3}^n y_{1,4}$	8	-14
	$g_5 y_{2,3}^{n-1} y_{1,4}$	$g_5 y_{2,3}^{n+1}$	15	4

Table A.13: Products yy' for $n \geqslant 5$ odd.

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