AN EXAMPLE OF CATASTROPHIC SELF-FOCUSING IN NONLINEAR OPTICS?

Éric Dumas
Laboratoire de Mathématiques et Physique Théorique
Parc de Grandmont, 37200 TOURS - FRANCE
dumas@gargan.math.univ-tours.fr

Abstract: As the wavelength ε goes to zero, the slowly varying envelope approximation from nonlinear optics allows one to replace the fields (solutions to Maxwell equations) by profiles solutions to a nonlinear Schrödinger equation (NLS). Depending on the model, this equation may be critical and focusing, and then admits explosive solutions. In this case, the approximation breaks down, and, for ε fixed, the fields may be globally defined in time, and smooth. This happens in the case of Maxwell-Bloch equations [6] and of the anharmonic oscillator with saturated nonlinearity [9].

We analyse the question of self-focusing for a wave equation in space dimension 2 –the same techniques apply to usual models in greater dimension. We give a new representation of the fields in terms of oscillating profiles, ruled by focusing rays. For non-saturated nonlinearities, we prove that the approximation by an explosive solution of NLS is valid up to a time of the order of a negative power of $\ln(1/\varepsilon)$ before explosion –this exhibits an amplification of the fields by a positive power of $\ln(1/\varepsilon)$, between $t = 0$ and that time.

Key words: Nonlinear diffractive optics, nonlinear critical Schrödinger equation, self-focusing.

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Introduction

A standard model for describing the propagation of an electromagnetic wave is the coupling of Maxwell equations with an anharmonic oscillator (*cf.* [3], [2]):

$$(1) \qquad \begin{cases} \partial_t E = -\text{curl } B - \partial_t P \\ \partial_t B = \text{curl } E \\ \varepsilon^2 \partial_t^2 P + \nabla_P V(P) = \gamma E. \end{cases}$$

Here, (E, B) is the electromagnetic field, and P is the polarization of the medium. The physically relevant fields also satisfy $div\,(E+P) = div\, B = 0$, which is true for all times as soon as it is at one given time. The response of matter is given by a nonlinear spring force, with the same frequency $1/\varepsilon$ as the wave.

The slowly varying envelope approximation (see [4], [11]) leads to a *nonlinear Schrödinger equation* (NLS). The potential V is replaced by its Taylor expansion at the origin,

$$V(P) \simeq \alpha |P|^2 - \beta |P|^4,$$

and the vector $u = (E, B, P, \varepsilon \partial_t P)$ is approximated by $u_{app} = \varepsilon \mathcal{U}(\varepsilon t, y_1, y_2) e^{i\frac{y_3+t}{\varepsilon}}$, $\mathcal{U} = (\mathcal{E}, \mathcal{B}, \mathcal{P}, \mathcal{Q})$. The fields must be polarized,

$$\mathcal{E} = \begin{pmatrix} K \\ L \\ 0 \end{pmatrix}, \ \mathcal{B} = c_1 \begin{pmatrix} L \\ -K \\ 0 \end{pmatrix}, \ \mathcal{P} = c_2 \begin{pmatrix} K \\ L \\ 0 \end{pmatrix}, \ \mathcal{Q} = i c_3 \begin{pmatrix} K \\ L \\ 0 \end{pmatrix},$$

and the amplitudes $K(T, y_1, y_2), L(T, y_1, y_2)$ are solution to:

$$(2) \qquad i\partial_T \begin{pmatrix} K \\ L \end{pmatrix} - C_1 \Delta_{y_1,y_2} \begin{pmatrix} K \\ L \end{pmatrix} - C_2(|K|^2 + |L|^2) \begin{pmatrix} K \\ L \end{pmatrix} = 0.$$

This is a critical NLS equation in space dimension 2, possibly with *focusing* nonlinearity ($C_1 C_2 > 0$), depending on the details of the model.

This approximation can be rigorously justified (see [5]), for ε small enough, on any time interval $[0, T_1/\varepsilon]$ such that the solution of (2) remains smooth on $[0, T_1]$. However, a time $T_\star$ of explosion for a solution to (2) is usually thought as an indication of *self-focusing*: a variation of the refractive index of the medium induces curved light rays, which concentrate in the region of maximal refractive index.

First, the slowly varying envelope assumption (from which (2) is derived) is violated in this region, where fields become too large. Second, the electromagnetic field may be globally defined in time (and smooth), even if the profile $\mathcal{U}$ explodes in finite time: in [6], Donnat and Rauch consider Maxwell-Bloch systems. In [9], Joly, Métivier and Rauch deal with (1), when the potential V is *saturated*: if the second and third derivatives of V are bounded, $H^2(\mathbb{R}^3)$ initial data generate global solutions to (1).

In this paper, we investigate further the mechanism of self-focusing, evaluating more precisely how long the Schrödinger approximation is valid. We consider a simpler model, a scalar wave equation in space dimension 2:

$$\Box u + iF(\partial_t u) = 0, \text{ with } F(z) = |z|^4 z. \tag{3}$$

Theorem 0.1. *Fix $t_\star > 0$, and define*

$$a_0(t,Y) := (t_\star - t)^{-1/2} e^{i\frac{Y^2-1}{2(t_\star - t)}} R\left(\frac{Y}{t_\star - t}\right), \text{ with } R(Y) = \frac{3^{1/4}}{\sqrt{\mathrm{ch}\,(2Y)}}. \tag{4}$$

There are $\varepsilon_0, C > 0$ such that for all $\varepsilon \in]0, \varepsilon_0]$, the initial value problem associated with (3), for initial data

$$\begin{cases} u^\varepsilon_{|t=0} = \varepsilon a_0\left(0, y_2/\sqrt{\varepsilon}\right) e^{iy_1/\varepsilon}, \\ \partial_t u^\varepsilon_{|t=0} = \dfrac{i}{\varepsilon} u^\varepsilon_{|t=0} + \mathcal{O}(\varepsilon) \text{ in } \mathcal{S}(\mathbb{T}_\varepsilon \times \mathbb{R}), \text{ with } \mathbb{T}_\varepsilon \text{ the torus } \mathbb{R}/(2\pi\varepsilon), \end{cases}$$

admits a unique smooth solution $u^\varepsilon \in \mathcal{C}^1(\mathcal{S}(\mathbb{T}_\varepsilon \times \mathbb{R}))$ for $t \in [0, t_\star - C(\ln 1/\varepsilon)^{-1/3}]$. Furthermore, as $\varepsilon \to 0$, we have the approximation:

$$\left\| \partial_t u^\varepsilon - i a_0\left(t, y_2/\sqrt{\varepsilon}\right) e^{i\frac{y_1+t}{\varepsilon}} \right\|_{L^\infty_y} = o\left(\|a_0(t)\|_{L^\infty}\right).$$

The link with the physical context above will be clearer after a few remarks:
i) The same kind of result is valid in higher dimension, and for the system (1): see Remark 1.1.
ii) From the profile $a_0(t,Y)e^{i\theta}$, in which we substitute $Y = y_2/\sqrt{\varepsilon}, \theta = (y_1 + t)/\varepsilon$, we recover the long-time setting leading to (2) after rescaling: replace (t,y) by $\sqrt{\varepsilon}(t,y)$, and $\sqrt{\varepsilon}$ by ε. We have chosen these scales so as to give an alternative description of u^ε in terms of curved phases and focusing rays. See Section 3, and [7].

iii) Equation (3) preserves only $|\nabla u|_{L^2}$. This is not sufficient to guarantee global existence of u. When the maximal existence time $t^\star$ of a smooth solution u is finite, we have $\|u(t)\|_{L^\infty} \underset{t\to t^\star}{\longrightarrow} 0$. The theorem doesn't prove that u^ε (with ε fixed) explodes at $t = t_\star$, but shows an amplification of $\|u\|_{L^\infty}$ by a factor $(\ln 1/\varepsilon)^{-1/6}$ between $t = 0$ and $t = t_\star - C(\ln 1/\varepsilon)^{-1/3}$. This eliminates standard proofs of global existence, like for small data (see [12]).
iv) Global existence is achieved in the other limit: ε fixed, $t_\star \to 0$, which corresponds to small initial data. Saturation, replacing the nonlinearity $F(z)$ by $G^\varepsilon(z) = \dfrac{|z|^4 z}{1+\varepsilon z^3}$, also ensures global existence of u^ε, as in [9]. Thus, the mechanism of catastrophic self-focusing seems to be: first, a concentration due to linear focusing of rays ("self-focusing"); second, activation of nonlinear effects effects by this amplification. Blow-up ("catastrophic" self-focusing) then depends on the strength of the nonlinearity: saturation stops the development of the singularity, but without saturation, blow-up may occur. That's why we try to measure how close the exact solution is to the explosive approximate solution.

The paper is organized as follows:
Section 1: (formal) definition of the explosive profile a_0 via the conformal invariance of NLS.
Section 2: Wentzel-Kramers-Brillouin asymptotics. A corrector a_c is defined to get a better approximation.
Section 3: thanks to non-uniqueness of the profile representation, we give an alternative description of u^ε based on focusing non-planar phases.
Section 4: proof of Theorem 0.1. We first change scales (Paragraph 4.1) and look at $U(x) = u(t, y_1/\varepsilon, y_2/\sqrt{\varepsilon})$, 2π-periodic in y_1. In Paragraph 4.2, we write down energy estimates for the residual $\partial_t V = \partial_t U - \partial_t U_{app}$ ($\|\partial_t V\|_{L^\infty}$ is then controlled by Sobolev's inequality). This is the notable difference between this work and [10], where the authors need global existence of the (small) approximate solution, whereas we "follow" the explosive approximate solution up to some boundary layer before $t_\star$. This boundary layer appears when requiring the corrector a_c to remain small compared to a_0 (Paragraph 4.3) and in the bootstrap argument showing $\|\partial_t V\|_{L^\infty} \ll \|\partial_t U_{app}\|_{L^\infty}$ (Paragraph 4.4).

1 From the wave equation to explosive solutions of NLS

A classical technique for constructing explosive solutions to NLS comes from the pseudo-conformal invariance of this equation (see [8]): if $b(t,Y)$ is a solution to

$$(5) \qquad 2i\partial_t u - \partial_Y^2 u - |u|^4 u = 0,$$

then, for all $t_\star \in \mathbb{R}$, we define another solution by:

$$a(t,Y) := \left(t^{1/2} e^{iY^2/2t} b\right)\left(\frac{1}{t_\star - t}, \frac{Y}{t_\star - t}\right) = (t_\star - t)^{-1/2} e^{iY^2/2(t_\star - t)} b\left(\frac{1}{t_\star - t}, \frac{Y}{t_\star - t}\right).$$

When seeking a solution u to (3) in the 3-scales form $u^\varepsilon(x) = \varepsilon \mathcal{U}^\varepsilon(t, y_2/\sqrt{\varepsilon}, (y_1 + t)/\varepsilon)$ with a profile $\mathcal{U}^\varepsilon(t,Y,\theta) \in \mathcal{C}^2([0,t_\star[\times \mathbb{R} \times \mathbb{T})$ (periodic w.r.t. the last variable, θ), the chain rule leads to the following sufficient equation:

$$(6) \qquad [2\partial_t\partial_\theta - \partial_Y^2]\mathcal{U}^\varepsilon + \varepsilon \partial_t^2 \mathcal{U}^\varepsilon + i|\partial_\theta \mathcal{U}^\varepsilon + \varepsilon \partial_t \mathcal{U}^\varepsilon|^4 (\partial_\theta \mathcal{U}^\varepsilon + \varepsilon \partial_t \mathcal{U}^\varepsilon) = 0.$$

In order to let these quantities vanish at first order (see the WKB expansions in Paragraph2), it is then natural to look for a profile $u_0(t,Y,\theta)$ such that:

$$(7) \qquad [2\partial_t\partial_\theta - \partial_Y^2]u_0 + i|\partial_\theta u_0|^4 \partial_\theta u_0 = 0.$$

There are explicit solutions to this equation. When $u_0 = b(t,Y)e^{i\theta}$, it is equivalent to require that b satisfies (5), and $b(t,Y) = e^{-it/2}R(Y)$ is a solution, with $R(Y) = 3^{1/4}(\mathrm{ch}\ (2Y))^{-1/2}$ (the unique positive solution of $R'' - R + R^5 = 0$, up to translation).

Now, use the pseudo-conformal invariance of (5) to get a solution $u_0(t,Y,\theta) = a_0(t,Y)e^{i\theta}$ to (7):

$$(8) \qquad a_0(t,Y) = (t_\star - t)^{-1/2} e^{i(Y^2-1)/2(t_\star - t)} R\left(\frac{Y}{t_\star - t}\right).$$

For all $t \in [0, t_\star[$, $a_0(t) \in \mathcal{S}(\mathbb{R})$, and a_0 explodes at $t = t_\star$ ($\|a_0(t)\|_{L^\infty} = 3^{1/4}(t_\star - t)^{-1/2}$)

Remark 1.1. *The same construction is possible in higher dimension N: the pseudo-conformal transform is $u(t,x) \mapsto t^{-N/2} e^{-i|x|^2/2t}\overline{u}(1/t, x/t)$. Then, $-\Delta R + R - R^{1+4/N} = 0$ also has a solution in $\mathcal{S}$, and the same is valid for the equation $-\Delta R + mR + g(R) = 0$, $m = cst$, under suitable behavior of g at the origin and at infinity (see [1]).*

2 WKB expansions and initial data for u^ε

If we want to deduce from an approximate solution $\varepsilon\mathcal{U}^\varepsilon(t, y_2/\sqrt{\varepsilon}, (y_1+t)/\varepsilon)$ the existence of an exact solution to (3), we must construct a solution to (6) to higher order than $\varepsilon u_0(t, y_2/\sqrt{\varepsilon}, (y_1+t)/\varepsilon)$. That's why we need a corrector to the first profile u_0. The general form for $\mathcal{U}^\varepsilon_{app}$ (from [5]) has two such correctors: $\mathcal{U}^\varepsilon_{app} = \varepsilon(u_0 + \sqrt{\varepsilon}u_1 + \varepsilon u_2)$. Here, when u_1 vanishes at $t=0$, we can let it vanish for all times. Thus, we set:

$$u^\varepsilon_{app}(x) = \varepsilon\mathcal{U}^\varepsilon_{app}(t, y_2/\sqrt{\varepsilon}, (y_1+t)/\varepsilon),\ \mathcal{U}^\varepsilon_{app} = u_0 + \varepsilon u_c = (a_0 + \varepsilon a_c)(t, Y)e^{i\theta}.$$

Proposition 2.1. *We have the (formal) WKB expansion:*

$$\Box u^\varepsilon_{app} + i|\partial_t u^\varepsilon_{app}|\partial_t u^\varepsilon_{app} = (\mathcal{E}_0 + \varepsilon\mathcal{E}_1 + \mathcal{R}^\varepsilon)(t, y_2)e^{i\frac{y_1+t}{\varepsilon}}, \tag{9}$$

with $\mathcal{E}_0(t,Y) = 2i\partial_t a_0 - \partial_Y^2 a_0 - |a_0|^4 a_0,$

$$\mathcal{E}_1(t,Y) = 2i\partial_t a_c - \partial_Y^2 a_c + \partial_t^2 a_0 + G(a_0, a_c),$$

$$G(a_0,a_c) = -|a_0|^4(\partial_t a_0 + ia_c) + 4ia_0|a_0|^2 Re(a_0(\partial_t\overline{a_0} + i\overline{a_c})),$$

$$\mathcal{R}^\varepsilon(t,Y) = \varepsilon^2\partial_t^2 a_c + iF((i+\varepsilon\partial_t)(a_0+\varepsilon a_c)) + F(a_0) - \varepsilon G(a_0, a_1).$$

We can construct a_0 and a_c such that $\mathcal{E}_0 = \mathcal{E}_1 = 0$: such an $a_0 \in \mathcal{C}^\infty([0,t_\star[\times\mathbb{R})$ is given by (8), and $\mathcal{E}_1 = 0$ is a linear Schrödinger equation, which has a unique solution for any $a_{c|_{t=0}} \in L^2(\mathbb{R})$.

Our goal is to show the existence of an exact solution u^ε to (3) close to $\varepsilon u_0(t, y_2/\sqrt{\varepsilon}, (y_1+t)/\varepsilon)$. Towards this end, we choose

$$a_{c|_{t=0}} = 0, \tag{10}$$

which provides us with a (unique) corrector $a_c \in \mathcal{C}^\infty([0,t_\star[\times\mathbb{R})$. Next, we take the simplest initial data for u^ε, in view of evaluating $u^\varepsilon - u^\varepsilon_{app}$:

$$\begin{cases} {u^\varepsilon}_{|_{t=0}} = u^\varepsilon_{app|_{t=0}}, \\ \partial_t {u^\varepsilon}_{|_{t=0}} = \partial_t u^\varepsilon_{app|_{t=0}}. \end{cases} \tag{11}$$

Remark 2.1.
i) We can compute $\partial_t u^\varepsilon_{app|_{t=0}}$ in terms of the function R, since a_0 is known explicitly, and $\partial_t a_c$ is given by the equation $\mathcal{E}_1 = 0$, so that:

$$\partial_t a_{c|_{t=0}} = \frac{i}{2}\left[\partial_t^2 a_0 - |a_0|^4\partial_t a_0 + 2ia_0|a_0|^2(a_0\partial_t\overline{a_0} + \overline{a_0}\partial_t a_0\right]_{|_{t=0}}. \tag{12}$$

ii) Since the data are $2\pi\varepsilon$-periodic in y_1, the standard uniqueness argument shows that so is $u^\varepsilon(t)$ for each time.

3 Linear focusing

We can give an alternative profile description of the data in (11): ${u^\varepsilon}_{|t=0}$ also has a representation via $\tilde{u}_0^0 \in \mathcal{S}(\mathbb{R}\times\mathbb{T})$,

$$ {u^\varepsilon}_{|t=0} = \varepsilon \tilde{u}_0^0 \left(\frac{y_2}{\sqrt{\varepsilon}}, \frac{y_1 + y_2^2/2t_\star}{\varepsilon} \right), \text{ where } \tilde{u}_0^0(Y,\theta) = t_\star^{-1/2} e^{-i/2t_\star} R\left(\frac{Y}{t_\star}\right) e^{i\theta}. $$

Similarly,

$$ \begin{aligned} \partial_t {u^\varepsilon}_{|t=0} = & \left(i t_\star^{-1/2} R\left(\frac{Y}{t_\star}\right) + \varepsilon \left[\frac{i}{2} t_\star^{-5/2} Y^2 R\left(\frac{Y}{t_\star}\right) + t_\star^{-5/2} Y R'\left(\frac{Y}{t_\star}\right) \right.\right. \\ & \left.\left. + \frac{1}{2} t_\star^{-3/2}(1 - i t_\star^{-1}) R\left(\frac{Y}{t_\star}\right) \right] e^{-i/2t_\star} e^{i\theta} \right)_{|Y = y_2/\sqrt{\varepsilon}, \theta = (y_1 + y_2^2/2t_\star)/\varepsilon} \\ & + \varepsilon^2 \partial_t {a_c}_{|t=0, Y = y_2/\sqrt{\varepsilon}} e^{iy_1/\varepsilon}, \end{aligned} $$

and $e^{i\frac{y_1 + y_2^2/2t_\star}{\varepsilon}}$ also factors in the last term (see (12) in Remark 2.1).

Thus, we have a new profile representation of the initial data, with oscillations involving the *curved* phase $y_1 + y_2^2/2t_\star$. Now, defining v^ε such that

$$ \begin{cases} \Box v^\varepsilon + iF(\partial_t v^\varepsilon) = 0, \\ {v^\varepsilon}_{|t=0} = {u^\varepsilon}_{|t=0}, \\ \partial_t {v^\varepsilon}_{|t=0} = i\sqrt{1 + (y_2/t_\star)^2} {v^\varepsilon}_{|t=0}, \end{cases} $$

we have two different ways of analyzing v^ε:

1- Plane phases: $\partial_t {v^\varepsilon}_{|t=0} = i\left(\sqrt{1+\varepsilon(Y/t_\star)^2} u_{0|t=0}\right)(y_2/\sqrt{\varepsilon}, y_1/\varepsilon)$
$= i\left(u_{0|t=0} + \mathcal{O}_{\mathcal{S}(\mathbb{R}\times\mathbb{T})}(\varepsilon)\right)(y_2/\sqrt{\varepsilon}, y_1/\varepsilon)$,

and from [5], for each $\underline{t} < t_\star$, when ε is small enough,

$$ u^\varepsilon = \varepsilon \mathcal{U}^\varepsilon \left(t, \frac{y_2}{\sqrt{\varepsilon}}, \frac{y_1 + t}{\varepsilon}\right), v^\varepsilon = \varepsilon \mathcal{V}^\varepsilon \left(t, \frac{y_2}{\sqrt{\varepsilon}}, \frac{y_1 + t}{\varepsilon}\right), \text{ and } \|\mathcal{U}^\varepsilon - \mathcal{V}^\varepsilon\|_{\cap H^s} \underset{\varepsilon \to 0}{\longrightarrow} 0, $$

so that $\|\partial_t u^\varepsilon - \partial_t v^\varepsilon\|_{L^\infty} \underset{\varepsilon\to 0}{\longrightarrow} 0$ uniformly on $[0,\underline{t}]$.
2- Curved phases: since

$$\begin{cases} v^\varepsilon{}_{|t=0} = \varepsilon\tilde{u}_0^0\left(y_2/\sqrt{\varepsilon},\phi_0/\varepsilon\right) \\ \partial_t v^\varepsilon{}_{|t=0} = i|\partial_y\phi_0|\tilde{u}_0^0\left(y_2/\sqrt{\varepsilon},\phi_0/\varepsilon\right) \end{cases}$$

with $\phi_0 = y_1 + y_2^2/2t_\star$, on each time interval $[0,\underline{t}]$, [7] ensures the representation $v^\varepsilon = \varepsilon\tilde{\mathcal{V}}^\varepsilon\left(t,\frac{y_2}{\sqrt{\varepsilon}},\frac{\phi}{\varepsilon}\right)$, where ϕ is characteristic for the d'Alembertian operator:

$$\partial_t\phi = |\partial_y\phi|, \text{ and } \phi_{|t=0} = y_1 + y_2^2/2t_\star.$$

This phase is implicitly determined by the "ray method": $\phi(t,y) = \phi_0(z)$, where z is the origin (at $t=0$) of the ray through (t,y) (which here is a straight line):

$$y - z + t\frac{\nabla\phi_0(z)}{|\nabla\phi_0(z)|} = 0.$$

Direct computations show that the rays focus exactly at time $t = t_\star$ (generating a caustic, where ϕ is no more smooth).

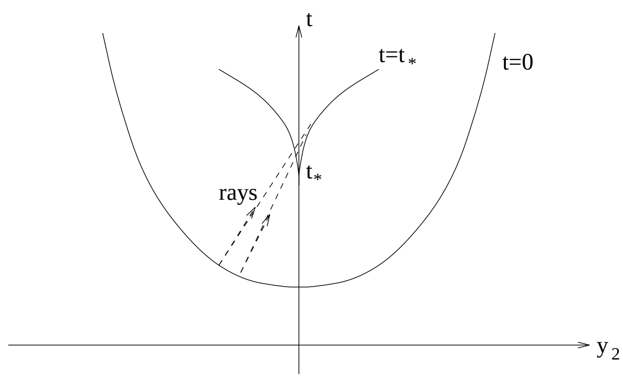

Figure 1: The graph of $\phi(t,y_1,.)$ at $t=0$ and $t=t_\star$.

Remark 3.1. *Here, ϕ can be determined explicitly (before $t=t_\star$) by solving analytically a fourth degree algebraic equation. However, the region where it is of interest is the set of the "first" rays to focus, corresponding to $y_2 = 0$. On these rays, one easily computes that the gradient (w.r.t. t,y) of ϕ is the* *same as the one of the linear phase y_1+t. This indicates that planar and non-planar phases representations correspond to similar oscillations, and give two different ways of understanding the amplitude's behavior.*

4 Existence and approximation of u^ε

We prove a slightly stronger approximation than in Theorem 0.1:

Theorem 4.1. *When $C > 0$ is sufficiently large, and for all $\alpha > 0$, as $\varepsilon \to 0$,*

$$\left\|\partial_t u^\varepsilon - \partial_t u^\varepsilon_{app}\right\| = o(\varepsilon^{2-\alpha}) \text{ in } L^\infty((0, t_\star - C(\ln 1/\varepsilon)^{-1/3}) \times \mathbb{R}^2).$$

4.1 Rescaling

So as to evaluate the lifespan of u^ε, we look for the times during which $\|\partial_t u^\varepsilon\|_{L^\infty}$ is finite. With the idea of giving an approximation of u^ε, we define $v^\varepsilon := u^\varepsilon - u^\varepsilon_{app}$, and try to verify $\|\partial_t v^\varepsilon\|_{L^\infty} \ll \left\|\partial_t u^\varepsilon_{app}\right\|_{L^\infty}$.

We make use of the wave equation satisfied by v^ε. It provides us with energy estimates for $\partial_{t,y} v^\varepsilon$, which bound $\|\partial_t v^\varepsilon\|_{L^\infty}$, thanks to Sobolev inequality. However, these direct computations are too crude, because of Remark 2.1*ii)*: u^ε and u^ε_{app} are $2\pi\varepsilon$-periodic in y_1, and so is v^ε. Thus, estimating $\|\partial_t v^\varepsilon\|_{L^\infty}$ by $\|\partial_t v^\varepsilon\|_{H^s}$, we loose a factor ε^s. That's why we finally try to control $\|\partial_t V^\varepsilon\|_{L^\infty}$, where

$$V^\varepsilon(t,y) := v^\varepsilon(t, \varepsilon y_1, \sqrt{\varepsilon} y_2). \tag{13}$$

In the same way, set $(U^\varepsilon, U^\varepsilon_{app}, R^\varepsilon)(x) := (u^\varepsilon, u^e_{app}p, r^\varepsilon)(t, \varepsilon y_1, \sqrt{\varepsilon} y_2)$ (where $r^\varepsilon(x) = \mathcal{R}^\varepsilon(t, y_2/\sqrt{\varepsilon}) e^{i\frac{y_1+t}{\varepsilon}}$ from Proposition 2).

Notation 4.1. *We write $a \preceq b$ when there is a constant C such that $a \leq Cb$.*

4.2 Energy estimates for the error $\partial_t V^\varepsilon$

From the relation (13), subtracting equations (3) and (9), and using Taylor's formula for $F(z) = |z|^4 z$ (as a differentiable function on $\mathbb{R}^2$), we get:

(14)

$$(\partial_t^2 - \varepsilon^{-2}\partial_{y_1}^2 - \varepsilon^{-1}\partial_{y_2}^2) V^\varepsilon = -i \left(\int_0^1 dF\left(\partial_t U^\varepsilon_{app} + r\partial_t V^\varepsilon\right) dr \right) .\partial_t V^\varepsilon - R^\varepsilon.$$

Consider $\int_{(-\pi\times\pi)\times\mathbb{R}} 2\mathrm{Re}\left((14)\times\partial_t\overline{V^\varepsilon}\right)dy$. This gives:

$$\frac{d}{dt}\left(\|\partial_t V^\varepsilon\|_{L^2}^2+\varepsilon^{-2}\|\partial_{y_1}V^\varepsilon\|_{L^2}^2+\varepsilon^{-1}\|\partial_{y_2}V^\varepsilon\|_{L^2}^2\right.$$
$$\preceq\left(\left\|\partial_t U^\varepsilon_{app}\right\|_{L^\infty}^4+\|\partial_t V^\varepsilon\|_{L^\infty}^4\right)\|\partial_t V^\varepsilon\|_{L^2}^2+\|R^\varepsilon\|_{L^\infty}\|\partial_t V^\varepsilon\|_{L^2},$$

so that, writing $N(V^\varepsilon):=\left(\|\partial_t V^\varepsilon\|_{L^2}^2+\varepsilon^{-2}\|\partial_{y_1}V^\varepsilon\|_{L^2}^2+\varepsilon^{-1}\|\partial_{y_2}V^\varepsilon\|_{L^2}^2\right)^{1/2}$ and $I_k:=\left\|\partial_t U^\varepsilon_{app}\right\|_{L^\infty}^k+\|\partial_t V^\varepsilon\|_{L^\infty}^k$:

$$\frac{d}{dt}N(V^\varepsilon)\preceq I_4 N(V^\varepsilon)+\|R^\varepsilon\|_{L^\infty}. \tag{15}$$

Differentiating (14), we get in the same way:

$$\frac{d}{dt}N(\partial_y V^\varepsilon)\preceq I_4 N(\partial_y V^\varepsilon)+I_3\left\|\partial_t\partial_y U^\varepsilon_{app}\right\|_{L^2}\|\partial_t V^\varepsilon\|_{L^\infty}+\|\partial_y R^\varepsilon\|_{L^\infty}, \tag{16}$$

and for any second order derivative ∂_y^2:

(17)
$$\frac{d}{dt}N(\partial_y^2 V^\varepsilon)\preceq I_3\left(\left\|\partial_t\partial_y^2 U^\varepsilon_{app}\right\|_{L^2}\|\partial_t V^\varepsilon\|_{L^\infty}+\left\|\partial_t\partial_y U^\varepsilon_{app}\right\|_{L^\infty}\|\partial_t\partial_y V^\varepsilon\|_{L^2}\right)$$
$$+\left\|\partial_y^2 R^\varepsilon\right\|_{L^\infty}+I_4 N(\partial_y^2 V^\varepsilon).$$

Adding (15)-(17), using Sobolev's inequality and Gronwall's lemma, since $V^\varepsilon_{|t=0}=\partial_t V^\varepsilon_{|t=0}=0$, we have:

$$\|\partial_t V^\varepsilon\|_{L^\infty}\preceq N(V^\varepsilon,\partial_y V^\varepsilon,\partial_y^2 V^\varepsilon)\preceq e^{CJ(t)}\int_0^t\|R^\varepsilon\|_{W^{2,\infty}}dt', \tag{18}$$

where $J(t)=\int_0^t\left[I_4+I_3\left(\left\|\partial_t\partial_y U^\varepsilon_{app}\right\|_{L^2}+\left\|\partial_t\partial_y^2 U^\varepsilon_{app}\right\|_{L^2}+\left\|\partial_t\partial_y U^\varepsilon_{app}\right\|_{L^\infty}\right)\right]dt'$.

4.3 Defining the boundary layer for the corrector

We first define an interval $[0,\underline{t}(\varepsilon)]$ where $U^\varepsilon_{app}\simeq U^\varepsilon_0:=\varepsilon u_0\left(t,y_2,y_1+\dfrac{t}{\varepsilon}\right)$, *i.e.* where $U^\varepsilon_c:=\varepsilon^2 u_c\left(t,y_2,y_1+\frac{t}{\varepsilon}\right)$ is a corrector to this quantity.

Proposition 4.1. *When $t_\star - t > C(\ln 1/\varepsilon)^{-1/3}$ for some C ($\gg 1$),*

$$\|\partial_t U_c^\varepsilon\|_{W^{1,\infty}} \ll \|\partial_t U_{app}^\varepsilon\|_{W^{1,\infty}} \text{ and } \|\partial_t U_c^\varepsilon\|_{H^2} \ll \|\partial_t U_{app}^\varepsilon\|_{H^2}.$$

Proof :

We use the equation $\mathcal{E}_1 = 0$ from Proposition 2 to obtain energy estimates. Since $\partial_t \left[a_c(t, y_2) e^{i(y_1 + t/\varepsilon)}\right] = \left(\partial_t a_c + \frac{i}{\varepsilon} a_c\right) e^{i(y_1+t/\varepsilon)}$, we have to estimate $\|a_c\|_{H^s_Y}$ and $\|\partial_t a_c\|_{H^s_Y}$, $s = 1, 2$:

$$\begin{aligned} \frac{d}{dt}\|a_c\|_{H^1} \preceq & \|\partial_t^2 a_0\|_{H^1} + \||a_0|^4 \partial_t a_0\|_{H^1} \\ & + \left(\|a_0\|_{L^\infty}^4 + \|a_0\|_{L^\infty}^3 \|\partial_Y a_0\|_{L^\infty}\right)\|a_c\|_{H^1}, \end{aligned} \tag{19}$$

$$\begin{aligned} \frac{d}{dt}\|\partial_t a_c\|_{H^1} \preceq & \|\partial_t^3 a_0\|_{H^1} + \||a_0|^3 |\partial_t a_0|^2\|_{H^1} + \||a_0|^4 \partial_t^2 a_0\|_{H^1} \\ & + \left(\|a_0\|_{L^\infty}^3 \|\partial_t a_0\|_{W^{1,\infty}} + \|a_0\|_{L^\infty}^2 \|\partial_t a_0\|_{L^\infty} \|\partial_Y a_0\|_{L^\infty}\right)\|a_c\|_{H^1} \\ & + \left(\|a_0\|_{L^\infty}^4 + \|a_0\|_{L^\infty}^3 \|\partial_Y a_0\|_{L^\infty}\right)\|\partial_t a_c\|_{H^1}, \end{aligned} \tag{20}$$

$$\begin{aligned} \frac{d}{dt}\|a_c\|_{H^2} \preceq & \|\partial_t^2 a_0\|_{H^2} + \||a_0|^4 \partial_t a_0\|_{H^2} + \big(\|a_0\|_{L^\infty}^4 + \|a_0\|_{L^\infty}^3 \|\partial_Y a_0\|_{L^\infty} \\ & + \|a_0\|_{L^\infty}^2 \|\partial_Y a_0\|_{L^\infty}^2 + \|a_0\|_{L^\infty}^3 \left\|\partial_Y^2 a_0\right\|_{L^\infty}\big)\|a_c\|_{H^2}, \end{aligned} \tag{21}$$

$$\begin{aligned} \frac{d}{dt}\|\partial_t a_c\|_{H^2} \preceq & \|\partial_t^3 a_0\|_{H^2} + \||a_0|^3 |\partial_t a_0|^2\|_{H^2} + \||a_0|^4 \partial_t^2 a_0\|_{H^2} \\ & + \big(\|a_0\|_{L^\infty}^3 \|\partial_t a_0\|_{W^{1,\infty}} + \|a_0\|_{L^\infty}^2 \|\partial_t a_0\|_{L^\infty} \|\partial_Y a_0\|_{L^\infty} \\ & + \|a_0\|_{L^\infty} \|\partial_t a_0\|_{W^{1,\infty}} \|\partial_Y a_0\|_{L^\infty}^2 + \|a_0\|_{L^\infty}^2 \|\partial_t a_0\|_{L^\infty} \left\|\partial_Y^2 a_0\right\|_{L^\infty} \\ & + \|a_0\|_{L^\infty}^3 \left\|\partial_t \partial_Y^2 a_0\right\|_{L^\infty}\big)\|a_c\|_{H^2} \\ & + \big(\|a_0\|_{L^\infty}^4 + \|a_0\|_{L^\infty}^3 \|\partial_Y a_0\|_{L^\infty} + \|a_0\|_{L^\infty}^2 \|\partial_Y a_0\|_{L^\infty}^2 \\ & + \|a_0\|_{L^\infty}^3 \left\|\partial_Y^2 a_0\right\|_{L^\infty}\big)\|\partial_t a_c\|_{H^2}. \end{aligned} \tag{22}$$

Remark that we can compute the exact value of the norm of a_0 from the formula (4):

$$(23)\ \ \forall\alpha,\ \ \|(\partial_{t,Y})^\alpha a_0\|_{L^\infty} = C(t_\star - t)^{-1/2-|\alpha|},\ \ \|(\partial_{t,Y})^\alpha a_0\|_{L^2} = C(t_\star - t)^{-|\alpha|}.$$

Thus, from (19)-(23), Gronwall's lemma implies that there are $C > 0$, $\mu \in \mathbb{R}$ such that:

$$\|a_c, \partial_t a_c\|_{H^2_Y} \preceq (t_\star - t)^\mu e^{C(t_\star - t)^{-3}}. \tag{24}$$

Now, simply check that out of the boundary layer $t_\star - t \leq C(\ln 1/\varepsilon)^{-1/3}$ (with C big enough), we have:

$$\begin{aligned}
&\|\partial_t U^\varepsilon_c\|_{L^\infty} \preceq \varepsilon^2 \|\partial_t a_c\|_{H^1} + \varepsilon\|a_c\|_{H^1} \ll \|\partial_t U^\varepsilon_0\|_{L^\infty} \sim (t_\star - t)^{-1/2},\\
&\|\partial_t\partial_y U^\varepsilon_c\|_{L^\infty} \preceq \varepsilon^2 \|\partial_t a_c\|_{H^2} + \varepsilon\|a_c\|_{H^2} \ll \|\partial_t\partial_y U^\varepsilon_0\|_{L^\infty} \sim (t_\star - t)^{-3/2},\\
&\|\partial_t\partial_y U^\varepsilon_c\|_{L^2} \preceq \varepsilon^2 \|\partial_t a_c\|_{H^1} + \varepsilon\|a_c\|_{H^1} \ll \|\partial_t\partial_y U^\varepsilon_0\|_{L^2} \sim (t_\star - t)^{-1}.
\end{aligned}$$

4.4 Endgame: proof of Theorem 4.1

We now take advantage of (18): in view of Proposition 4.1, when $t_\star - t \geq C(\ln 1/\varepsilon)^{-1/3}$,

$$\|\partial_t V^\varepsilon\|_{L^\infty} \preceq e^{C\tilde{J}(t)} \int_0^t \|R^\varepsilon\|_{W^{2,\infty}} dt', \tag{25}$$

where $R^\varepsilon(x) = \mathcal{R}^\varepsilon(t, y_2)e^{i(y_1 + t/\varepsilon)}$ is given by Proposition 2, and

$$\begin{aligned}
\tilde{J}(t) &= \int_0^t \left[\tilde{I}_4 + \tilde{I}_3\left(\|\partial_t\partial_y U^\varepsilon_0\|_{L^2} + \|\partial_t\partial_y U^\varepsilon_0\|_{L^\infty}\right)\right] dt',\\
\tilde{I}_k &= \left\|\partial_t U^\varepsilon_{app}\right\|^k_{L^\infty} + \|\partial_t V^\varepsilon\|^k_{L^\infty}.
\end{aligned}$$

Hence, (25) is a relation of the form $\varphi \leq \psi e^{\varphi^4}$ for $\varphi(t) := C\|\partial_t V^\varepsilon\|_{L^\infty((0,t)\times\mathbb{R}^2)}$. Even if ψ is "small", this does not imply that φ is: it could be very large, on the contrary. But since here $\varphi_{|t=0} = 0$, continuity w.r.t. t forces that *as long as* (25) is valid, φ has to be "small". Thus, for each $\varepsilon \in]0, 1]$, we look for the maximal time $\underline{t}(\varepsilon) \in]0, t_\star - C(\ln 1/\varepsilon)^{-1/3}$ until which

$$\|\partial_t V^\varepsilon\|_{L^\infty} \leq \|\partial_t U^\varepsilon_0\|_{L^\infty} \sim (t_\star - t)^{-1/2}, \tag{26}$$

and we replace $\tilde{I}_k$ by $\hat{I}_k := 2\,\|\partial_t U_0^\varepsilon\|_{L^\infty}^k$.

Since (from (23)) $\hat{I}_k(t) \sim (t_\star - t)^{-2}$,

$$\|\partial_t V^\varepsilon\|_{L^\infty} \leq \varepsilon^2 (t_\star - t)^\mu e^{C(t_\star - t)^{-2}}, \tag{27}$$

and the r.h.S. is much smaller than $(t_\star - t)^{-1/2}$ as soon as $t_\star - t > C'(\ln 1/\varepsilon)^{-1/2}$, with C' sufficiently large. As ε goes to zero, $(\ln 1/\varepsilon)^{-1/2} \ll (\ln 1/\varepsilon)^{-1/3}$, so that the condition $t_\star - t > C(\ln 1/\varepsilon)^{-1/3}$ is the relevant one.

Furthermore, for each $\alpha > 0$, possibly increasing C, (27) shows that (26) is improved to $\|\partial_t V^\varepsilon\|_{L^\infty((0, t_\star - C(\ln 1/\varepsilon)^{-1/3}) \times \mathbb{R}^2)} = o(\varepsilon^{2-\alpha})$.

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