Global existence for Maxwell-Bloch systems

Éric Dumas

Université Grenoble 1 - Institut Fourier - 100, rue des mathématiques - BP 74 - 38402 Saint Martin d'Hères Cédex - France

Abstract

Maxwell-Bloch equations describe the propagation of an electromagnetic wave through a quantum medium. For any number of quantum levels, in space dimension 3, we show the global existence of weak (L^2) solutions to the initial-value problem. In the case of smoother electromagnetic fields (with curl in L^2), the solution is unique. For smooth data (H^s, $s \geq 2$), the solutions remain smooth for all times.

Key words: Maxwell equations, Bloch equation, energy estimates, compensated compactness, Strichartz estimates
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1 Introduction

1.1 Presentation of the problem

Maxwell-Bloch equations describe the propagation of an electromagnetic wave (the magnetic field is denoted H, the electric field E) in a quantum medium, modelized by a density matrix ρ (see [10]) with N energy levels. The system then reads

$$\begin{cases} \mu \partial_t H + \operatorname{curl} E = 0, \\ \varepsilon \partial_t E - \operatorname{curl} H = -\partial_t P, \\ i \partial_t \rho = [\Omega - E \cdot \Gamma, \rho]. \end{cases} \tag{1}$$

The space-time variables are $(t, x) \in \mathbb{R}^{1+3}$. The fields E and H take values in $\mathbb{R}^3$. The functions μ and ε are positive, and denote magnetic permeability and

Email address: `edumas@ujf-grenoble.fr` (Éric Dumas).
URL: `http://www-fourier.ujf-grenoble.fr/~edumas` (Éric Dumas).

electric permittivity, respectively. The response of the matter to the fields is through a polarization P, given by the constitutive law,

$$P = \mathrm{Tr}\,(\Gamma\rho).$$

The dipole moment operator Γ is a $N \times N$ Hermitian matrix, with entries in $\mathbb{C}^3$, and depends on the material considered. The $N \times N$ Hermitian symmetric matrix Ω, with entries in $\mathbb{C}$, represents the (electromagnetic field-) free Hamiltonian of the medium. The density matrix ρ is Hermitian, non-negative, has size $N \times N$ and entries in $\mathbb{C}$. In the system's eigenstates basis, its n-th diagonal entry is the proportion of quantum states at the n-th energy level, so that

$$\rho_{nn} \geq 0, \quad \text{and} \int_{\mathbb{R}^3} \sum_n \rho_{nn}\, \mathrm{d}x = 1. \tag{2}$$

The off-diagonal entry ρ_{jk} is linked to the transition probability from level j to level k.

Finally, physically, conservation of current and charge must be satisfied,

$$\mathrm{div}(\mu H) = 0, \quad \mathrm{div}(\varepsilon E + P) = 0 \tag{3}$$

–these relations hold at least formally for all time if they do initially.

System (1) is symmetric hyperbolic. Thus, for smooth enough initial data (in $H^s(\mathbb{R}^3)$, with $s > 3/2$), local existence of solutions is guaranteed, on a time interval depending *a priori* on the size of the data. In the present paper, we address the subsequent natural questions:
Q1) May these solutions be defined globally in time? This is motivated in particular by the fact that relevant time scales in quantum optics are "large" (see [9]).
Q2) May solutions be defined with the "natural" regularity given in Proposition 1 below?
Q3) Since these are weak solutions of (1), what about uniqueness? What regularity is sufficient for uniqueness to hold?

By standard energy estimates (see also Proposition 19), after mollification, one gets the following conservations and *a priori* estimates.

Proposition 1 *Let $\Gamma, \Omega \in L^\infty(\mathbb{R}^3)$. Let $\mu, \varepsilon \in L^\infty(\mathbb{R}^3)$, with some $\varepsilon_0 > 0$ such that $\varepsilon \geq \varepsilon_0$ almost everywhere. If $U = (E, H, \rho) \in \mathcal{C}([0, +\infty[, L^2(\mathbb{R}^3))$ is solution to (1), (3) in the sense of distributions, then:*
(i) For almost all $x \in \mathbb{R}^3$, $\mathrm{Tr}\,\rho^\lambda(t,x)$ and $|\rho(t,x)| := (\mathrm{Tr}\,(\rho(t,x)^2))^{1/2}$ are constant in t.
(ii) There is $C = C(\varepsilon_0, \|\Omega\|_{L^\infty}, \|\Gamma\|_{L^\infty})$ such that, for all time t,

$$\mathcal{E}(t) := \|\sqrt{\varepsilon}E(t)\|^2_{L^2} + \|\sqrt{\mu}H(t)\|^2_{L^2} + \|\rho(t)\|^2_{L^2} \leq e^{Ct}\mathcal{E}(0).$$

In the sequel, all norms on finite dimensional spaces will be denoted as above by $|\cdot|$.

Remark 2 *The physically relevant decay in space for ρ, in view of (2), is $\rho(t) \in L^1(\mathbb{R}^3)$, but this is too weak for us to treat the mathematical question. We really need the L^∞ bound, as well as the only natural energy bound, the L^2 one.*

Before answering the questions above , we recall previous results on the subject from which we have drawn some inspiration.

• In the case of 2 levels Maxwell-Bloch systems ($N = 2$), Donnat and Rauch have proved in [2] global existence for the smooth solutions ($H^s(\mathbb{R}^3)$, with $s \geq 2$). Their proof is based on the usual continuation argument, thanks to energy estimates, using the *a priori* estimates of Proposition 1, the decomposition of the fields into their irrotational and divergence-free parts, and Judovic-type estimates. The 2 levels case benefits special symmetries, so that Bloch equation may be written as a system for the polarization P and the levels populations difference $n := \rho_{11} - \rho_{22}$ only,

$$\begin{cases} \partial_t^2 P + \dfrac{1}{T_1}\partial_t P + \omega^2 P = C_1 n E, \\ \partial_t n + \dfrac{n - n_0}{T_2} = -C_2 \partial_t P \cdot E. \end{cases}$$

The particular structure of nonlinearities leads to cancellations, and to conservation of L^2 energy, as well as control of H^1 energy.

• In the case of electromagnetic propagation through a ferromagnetic medium, with magnetization $M(t,x)$,

$$\begin{cases} \mu \partial_t H + \operatorname{curl} E = -\partial_t M, \\ \varepsilon \partial_t E - \operatorname{curl} H = 0, \\ \partial_t M = \alpha \left(M \wedge H + \dfrac{\beta}{|M|} M \wedge (M \wedge H) \right), \end{cases} \tag{4}$$

Joly, Métivier and Rauch proved in [6] the global existence of weak solutions (of the kind of the ones in Proposition 1) in space dimension 3, with constant coefficients ε and μ. This is achieved constructing smooth approximate solutions of (4), with *a priori* estimates analoguous to the ones of Proposition 1, and compensated compactness gives the limit of the nonlinear terms. They get uniqueness in the case when the fields have rotationals in L^2, thanks to an almost L^∞ control of the fields provided by a Strichartz estimate for the wave equation (to which the divergence free part of the magnetic field is solution). This Strichartz estimate, together with a Judovic estimate, allows them also to show the global existence of smooth (H^s, $s \geq 2$) solutions. Haddar obtained similar results (in [5]) in space dimension 2, with electric permittivity $\varepsilon = \varepsilon(x) \in L^\infty(\mathbb{R}^2)$. His strategy follows the same lines, but in space dimension 2, the defect of the injection of H^1 into L^∞ is described by a simpler inequality than the Strichartz estimate of [6].

1.2 The results

Some structure is common to the ferromagnetic system (4) and to Maxwell-Bloch equations: pointwise conservation of the density matrix (of the magnetization, in (4)), which is linked to the irrotational part of the fields, propagation of the divergence free part of the fields according to a wave equation, interaction terms depending linearly on this part of the fields... Thus, we adapt the methods described above to Maxwell-Bloch system in space dimension 3, for any (finite) number of levels, in the case of constant coefficients ε and μ and variable operators Ω and Γ.

Theorem 3 (Global weak solutions) *Let $\Omega, \Gamma \in L^\infty(\mathbb{R}^3)$, and assume that ε, μ are constant. When $U_0 = (H_0, E_0, \rho_0) \in L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times (L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3))$ satisfies the constraint equations* (3)*, there exists $U \in \mathcal{C}([0, +\infty[, L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times (L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)))$, global solution to Maxwell-Bloch system* (1), (3) *in the distributional sense, with U_0 as initial data.*

Theorem 4 (Uniqueness) *Under the assumptions of Theorem 3, if in addition* $\operatorname{curl} H_0$, $\operatorname{curl} E_0 \in L^2(\mathbb{R}^3)$*, then* $\operatorname{curl} H$, $\operatorname{curl} E \in \mathcal{C}([0, +\infty[, L^2(\mathbb{R}^3))$*, and U is unique.*

Theorem 5 (Global smooth solutions) *Let $s \in \mathbb{N}$, $s \geq 2$. Let $\Omega, \Gamma \in \mathcal{C}^s(\mathbb{R}^3)$ be bounded, as well as their derivatives up to order s, and assume that ε, μ are constant. When $U_0 \in H^s(\mathbb{R}^3)$ satisfies the constraint equations* (3)*, there exists a unique $U \in \mathcal{C}([0, +\infty[, H^s(\mathbb{R}^3))$, global solution to* (1), (3) *with U_0 as initial data.*

Remark 6 *(i) In view of the finite speed of propagation property of system* (1)*, the global boundedness assumption on Ω, Γ and ρ_0 is not necessary: for each given time $T > 0$, we could localize the data on a compact domain $\{|x| \leq R\}$ with $R > T$ to get the same results on $\{(t, x) \mid |x| + t \leq R,\ 0 \leq t \leq T\}$ –from the modelling viewpoint, it is also reasonnable to consider that ρ has compact support, namely the space occupied by the matter.*

(ii) It would be interesting to generalize these results to an infinite number of quantum levels. The density matrix is then a trace class operator on $l^2(\mathbb{N})$. Unfortunately, we are unable to control the trace norm of ρ –just as we cannot deal with ρ in L^1 only.

(iii) On the contrary, the analogue to Theorem 5 with μ and ε variable seems tractable, but it requires to prove a precise Strichartz estimate for variable coefficient wave equations (the analogue to Proposition 30).

A simpler case arises with the common assumption (see [9]) that the dipole moment operator is "polarized",

$$\Gamma = \gamma \otimes e, \tag{5}$$

with γ a $N \times N$ Hermitian symmetric matrix and e a vector in $\mathbb{C}^3$. The current density $\partial_t P$ then depends on ρ only, thanks to the relation $\mathrm{Tr}\,(A[A, B]) = 0$:

$$-\partial_t P = -\mathrm{Tr}\,(\Gamma \partial_t \rho) = i\mathrm{Tr}\,(\Gamma[\Omega, \rho]) - i\mathrm{Tr}\,(\gamma[\gamma, \rho])(E \cdot e)e = i\mathrm{Tr}\,(\Gamma[\Omega, \rho]).$$

This enables us to prove existence and uniqueness of weak solutions with variable coefficients ε and μ.

Theorem 7 (Polarized case) *Let $\Omega, \mu, \varepsilon \in L^\infty(\mathbb{R}^3)$, with some $\mu_0, \varepsilon_0 > 0$ such that $\mu \geq \mu_0$ and $\varepsilon \geq \varepsilon_0$ almost everywhere. Assume that Γ has the form (5), with $\gamma \in L^\infty(\mathbb{R}^3, \mathrm{Herm}_N)$ and $e \in L^\infty(\mathbb{R}^3, \mathbb{C}^3)$. When $U_0 \in L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times (L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3))$ satisfies (3) with $\mathrm{curl}\, H_0, \mathrm{curl}\, E_0 \in L^2(\mathbb{R}^3)$, there exists a unique global (distributional) solution U to (1), (3) with U_0 as initial data. When μ and ε are constant, the same holds for $U_0 \in L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times (L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3))$ only.*

The article is structured as follows. In Section 2, we give a general class of systems (Maxwell's equations coupled to some dissipative ODE), including (1) (and also the "regularized" ferromagnetic systems (4) with $\beta = 0$, or $\beta/|M|$ replaced with $\beta/\sqrt{|M|^2 + \delta^2}$), for which Theorems 3, 4, 5 are valid. There, we sketch the proofs given in the next sections: Section 3 is devoted to existence of global weak solutions; 4, to uniqueness; Section 5, to smooth solutions.

2 A general setting for Maxwell-Bloch type systems

We consider system (1) as a particular case of some class of systems, so as to take into account other kinds of interactions. For example, we have in mind adding "transverse relaxation" terms (see [1]): $i\partial_t \rho = [\Omega - E \cdot \Gamma, \rho] - i\gamma \rho_{od}$, with ρ_{od} the off-diagonal part of ρ, and $\gamma(x)$ a non-negative L^∞ function.

Set $u = (u_1, u_2) = (H, E)$,

$$L = \partial_t + \begin{pmatrix} 0 & \mathrm{curl} \\ -\mathrm{curl} & 0 \end{pmatrix}, \tag{6}$$

and consider ρ as a real vector

$$v = (\mathcal{R}\mathrm{e}\,\rho_{k,l}, \mathcal{I}\mathrm{m}\,\rho_{k,l}, \rho_{m,m})_{1\le k<l\le N, 1\le m\le N} \in \mathbb{R}^n,$$

where $n = N^2$ (but the following works for all $n \in \mathbb{N}$). Then, (1) (with $\varepsilon = \mu = 1$) takes the form

$$\begin{cases} Lu = l(x)F(x,v,u), \\ \partial_t v = F(x,v,u), \end{cases} \tag{7}$$

with l and F satisfying the following assumptions (with $n = N^2$).

Hypothesis 8 *There is a scalar product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$, a constant $C > 0$, and for all $R > 0$, there is a constant $C(R)$ such that:*

(1) The function $F : \mathbb{R}^3 \times \mathbb{R}^n \times \mathbb{R}^6 \to \mathbb{R}^n$ is affine in u, and is written $F(x,v,u) = F^0(x,v) + F^1(x,v)u$. Here, F^0 takes its values in $\mathbb{R}^n$, and F^1, in the space of linear operators $\mathcal{L}(\mathbb{R}^6,\mathbb{R}^n)$.
(2) The function F^0 is measurable in x and $\mathcal{C}^0$ in v (Caratheodory regularity). The function F^1 is measurable in x and $\mathcal{C}^1$ in v.
(3) For all $x \in \mathbb{R}^3, u \in \mathbb{R}^6, v \in \mathbb{R}^n$, $\langle F(x,v,u), v\rangle \le 0$.
(4) For $j = 0,1$, for all $x \in \mathbb{R}^3$, $F^j(x,0) = 0$.
(5) For $j = 0,1$, for all $x \in \mathbb{R}^3$, $v, v' \in \mathbb{R}^n$, $|F^j(x,v) - F^j(x,v')| \le C(R)|v - v'|$ when $|v|, |v'| \le R$.
(6) The function l is L^∞ in x, with values in $\mathcal{L}(\mathbb{R}^n,\mathbb{R}^6)$. We denote l_1 and l_2 its $H(= u_1)$ and $E(= u_2)$ coordinates.

The crucial assumptions here are the dissipation property (3) and the fact (1) that F is affine in u. The first one provides a pointwise control on v (see Proposition 10 below), while the second one allows the use of compensated compactness. Concerning Lipschitz and growth conditions, from assumptions 1, 4 and 5, one easily deduces the following estimates for F, for all $x \in \mathbb{R}^3$, $u, u' \in \mathbb{R}^6$, $v, v' \in \mathbb{R}^n$ such that $|v|, |v'| \le R$.

$$|F(x,v,u)| \le C(R)(1 + |u|)|v|, \tag{8}$$
$$|F(x,v,u) - F(x,v',u)| \le C(R)(1 + |u|)|v - v'|, \tag{9}$$
$$|F(x,v,u) - F(x,v,u')| \le C(R)|u - u'|. \tag{10}$$

We take into account the natural L^2 regularity and the conservations for the fields through the following definition of "finite energy solutions".

Definition 9 *Denote by L_{div} the space of $U = (u,v) \in L^2(\mathbb{R}^3) \times (L^2 \cap L^\infty)(\mathbb{R}^3)$ satisfying*

$$\mathrm{div}(u_j - l_j v) = 0 \text{ for } j = 1,2 \text{ in } \mathcal{D}'(\mathbb{R}^3). \tag{11}$$

We call finite energy solution *any* $U \in \mathcal{C}([0,+\infty[, L_{\mathrm{div}})$ *solution to* (7) *in the sense of distributions.*

In this setting, Proposition 1 becomes

Proposition 10 *Under assumptions 8, when* U *is a finite energy solution to* (7)*, there is* $C = C(F, l, \|v_0\|_{L^\infty})$ *such that,*
(i) For almost all $x \in \mathbb{R}^3$, $|v(t,x)|$ *decreases in* t.
(ii) For all time t, $\|U(t)\|_{L^2} \leq e^{Ct}\|U(0)\|_{L^2}$.

Theorems 3,4 and 5 are then corollaries of the following ones.

Theorem 11 *Under assumptions 8, when* $U_0 = (u_0, v_0) \in L_{\mathrm{div}}$*, there exists a (global) finite energy solution* U *to* (7) *with* U_0 *as initial data.*

This result is proved in Section 3. We first construct smooth approximate solutions to (7) through high-frequency cut-offs. These approximate solutions satisfy bounds analogue to the ones of Proposition 10, thus weakly converge (up to a subsequence). The limit in the nonlinear terms (and, in fact, the strong convergence of the approximate solutions) is obtained by compensated compactness, splitting the fields u into their irrotational part $u_\parallel$ (related to v, *via* the constraint (11)) and divergence free part $u_\perp$ (solution to a wave equation).

In Section 3.3, we sketch the proof of Theorem 7. Since the response of the matter to the fields then depends on the density matrix only, the system is "less nonlinear", and after a slightly different regularization procedure, elementary energy estimates and Gronwall's Lemma are enough to pass to the limit. In the case when the coefficients ε and μ are variable, some compactness is needed, under the form of a L^2 control of the curl of the fields, provided by the time derivatives (using $\mathrm{curl}\, E = -\mu\partial_t H$, for example).

In Section 4, we are interested in uniqueness of the energy solutions. Towards this end, we need some L^∞ (in space) control on the fields. *Via*the conservation relation (11), the irrotational part of the fields is linked to v, for which we have a L^∞ bound. Since $H^1(\mathbb{R}^3)$ is "not far" from L^∞, the first step consists in showing the propagation of H^1 regularity of the divergence free part $u_\perp$ of u, or $H(\mathrm{curl})$ regularity of u (Section 4.1). We thus need to reinforce assumption 8(2).

The functions F^0 and F^1 are measurable in x and $\mathcal{C}^1$ in v. (2')

Theorem 12 *If in addition* $\mathrm{curl}\, u_{0,1}$, $\mathrm{curl}\, u_{0,2} \in L^2(\mathbb{R}^3)$*, and assumption 8* (2) *is replaced by* (2′)*, then* $\mathrm{curl}\, u_1$, $\mathrm{curl}\, u_2 \in \mathcal{C}([0,+\infty[, L^2(\mathbb{R}^3))$.

In Section 4.2, we take advantage of this regularity to get the desired L^∞ control on the divergence free part of the fields, *via* a Strichartz estimate for the wave equation (such a L^∞ estimate is false in general, but holds for functions whose Fourier transform have bounded support, as proved in [6]). Furthermore, we assume that the matter (v) does not interact directly with one of the fields ($u_1 = H$ or $u_2 = E$). *A priori*,

$$\partial_t^2 (u_1)_\perp - \Delta (u_1)_\perp = (l_1 dF(v,u) \cdot (F, l_1 F - \operatorname{curl} u_2, l_2 F + \operatorname{curl} u_1) - \operatorname{curl}(l_2 F))_\perp,$$

so that, in order to get a L^2_x control on the source term, we let $l_2 F(v,u)$ vanish. This provides us with an (approximate) L^∞ control of $(u_1)_\perp$. Uniqueness then follows from a simple energy estimate on the system (7), at least if the L^∞ norm of u_2 is not involved, that is, if F does not depend on u_2.

Hypothesis 13 *There is an index $j \in \{1,2\}$ such that $F(v,u)$ does not depend on u_{3-j}, and $l_{3-j}(x)F(x,v,u)$ identically vanishes.*

Theorem 14 *Under the assumptions of Theorem 12 and assumption 13, the finite energy solution U is unique.*

Finally, we address the question of the global existence of smooth solutions. We thus replace the Caratheodory regularity of F by $\mathcal{C}^s$ regularity. Furthermore, the u-linearity of F is not necessary as an algebraic condition, and is more to be considered as a growth condition.

Hypothesis 15 *Let $s \in \mathbb{N}$ be greater or equal to 2.*
- *The function $F \in \mathcal{C}^s(\mathbb{R}^3 \times \mathbb{R}^n \times \mathbb{R}^6)$ enjoys the affine and dissipation properties (1) and (3) of assumption 8. For all ω, relatively compact subset of $\mathbb{R}^n \times \mathbb{R}^6$, it is bounded on $\mathbb{R}^3 \times \omega$, as well as its derivatives up to order s.*
- *The derivatives $\partial_x F$ and $\partial_x^2 F$ satisfy (8).*
- *For all $x \in \mathbb{R}^3$, $v \in \mathbb{R}^n$, $u \in \mathbb{R}^6$, $k = 1$ or 2,*

$$|\partial_v^k F(x,v,u)| \leq C(R)(1 + |u|) \text{ when } |v| \leq R.$$

- *The function $l \in \mathcal{C}^\infty(\mathbb{R}^3, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^6))$ is bounded, as well as its derivatives up to order s.*

Theorem 16 *Under assumptions 15 and 13 ($s \in \mathbb{N}$ is then given, $s \geq 2$), consider $U_0 \in H^s(\mathbb{R}^3)$ satisfying (11). Then, there exists a unique $U \in \mathcal{C}([0,+\infty[, H^s(\mathbb{R}^3))$, global solution to (7), (11) with U_0 as initial data.*

The proof is based on the usual continuation argument, through subquadratic control of the H^2 norm. Again, this is achieved by a L^∞ control of the fields, thanks to a Judovic estimate for the irrotational part, and the same Strichartz estimate as above for the divergence free part.

3 Existence of global weak solutions

3.1 Regularization

We use the Fourier multiplier S^λ, with symbol $\chi_\lambda := \chi(\cdot/\lambda)$, where the cut-off function $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d, [0,1])$ takes value 1 when $|\xi| \le 1/2$, and 0 when $|\xi| \ge 1$. *Via* the Fourier transform, the following properties are immediate.

Lemma 17 *For all $s \in \mathbb{N}$, there is $C_s > 0$ ($C_0 = 1$) such that*

$$\begin{aligned} S^\lambda \textit{ is a continuous map } &\textit{from } L^2(\mathbb{R}^3) \textit{ to } H^s(\mathbb{R}^3), \textit{ with norm } C_s\lambda^s, \\ &\textit{from } L^2(\mathbb{R}^3) \textit{ to } L^\infty(\mathbb{R}^3), \textit{ with norm } \lambda^{3/2}, \\ &\textit{from } L^p(\mathbb{R}^3) \textit{ to } L^p(\mathbb{R}^3) \textit{ for all } p \in [1,\infty[, \\ &\qquad\qquad\qquad \textit{with norm } \|\hat{\chi}\|_{L^1}. \end{aligned}$$

Furthermore, as λ goes to infinity, S^λ converges strongly (in the space of bounded linear operators on $L^2(\mathbb{R}^d)$) towards the identity.

Define an approximate solution U^λ to (7) by

$$\begin{cases} Lu^\lambda = S^\lambda l(x) F(x, v^\lambda, u^\lambda), \\ \partial_t v^\lambda = F(x, v^\lambda, u^\lambda), \\ U^\lambda_{|t=0} = (S^\lambda u_0, v_0). \end{cases} \tag{12}$$

This system may be written as an ODE $\frac{\mathrm{d}}{\mathrm{dt}} U^\lambda = G^\lambda(U^\lambda)$ on the Banach space $L^2_\lambda \times (L^2 \cap L^\infty)(\mathbb{R}^3)$, where L^2_λ is the subspace of $L^2(\mathbb{R}^3)$ of functions whose Fourier transform has support contained in the ball $\{|\xi| \le \lambda\}$.

Lemma 18 *Under assumptions 8, for all $\lambda > 0$, the map*

$$G^\lambda : U = (u, v) \mapsto \left(S^\lambda l(x) F(x, v, u) - \begin{pmatrix} 0 & \mathrm{curl} \\ -\mathrm{curl} & 0 \end{pmatrix} u, F(x, v, u) \right)$$

is locally Lipschitz continuous on $L^2_\lambda \times (L^2 \cap L^\infty)(\mathbb{R}^3)$.

Proof: Thanks to the support condition defining L^2_λ, the curl linear part is bounded on this space.

When $(u, v) \in L^2_\lambda \times (L^2 \cap L^\infty)(\mathbb{R}^3)$, measurability of $F(x, v, u)$ is ensured by the Caratheodory regularity of F (assumption 8(2)). The growth in (8) implies that $F(x, v, u) \in (L^2 \cap L^\infty)(\mathbb{R}^3)$, and $S^\lambda l(x)$ maps L^2 to L^2_λ. Thus, G^λ maps $L^2_\lambda \times (L^2 \cap L^\infty)(\mathbb{R}^3)$ to itself. Finally, the locally Lipschitz property is inherited from the Lipschitz estimates (9), (10) on F. □

This yields local existence of the approximate solution, and as usual, global existence is given by *a priori* bounds.

Proposition 19 *Under assumptions 8, for all* $U_0 = (u_0, v_0) \in L_{\mathrm{div}}$ *and* $\lambda > 0$, *there exists a unique* $U^\lambda \in \mathcal{C}^1([0, +\infty[, L^2_\lambda \times (L^2 \cap L^\infty)(\mathbb{R}^3))$ *solution to* (12).

Furthermore, there is a constant $C = C(F, l, \|v_0\|_{L^\infty})$ *such that*
(i) For almost all $x \in \mathbb{R}^3$, $|v^\lambda(t, x)|$ *decreases in time.*
(ii) For all times t, $\mathrm{div}(u^\lambda_j(t) - S^\lambda l_j v^\lambda(t)) = 0$ *for* $j = 1, 2$.
(iii) For all times t, $\mathcal{E}(U^\lambda(t)) := \|u^\lambda(t)\|^2_{L^2} + \|v^\lambda(t)\|^2_{L^2} \le e^{Ct}\mathcal{E}(U_0)$.

Proof: To obtain *(i)*, form the scalar product $(\langle\cdot,\cdot\rangle)$ of v^λ with the second equation in (12) and use the dissipation assumption 8(3).

Take the divergence of the first equation in (12) and replace $F(v^\lambda, u^\lambda)$ by $\partial_t v^\lambda$ to get *(ii)*.

Finally, taking the scalar product of u^λ with the first equation in (12), integrating in space and using the relation $u_1 \cdot \mathrm{curl}\, u_2 - u_2 \cdot \mathrm{curl}\, u_1 = \mathrm{div}(u_2 \wedge u_1)$, we have

$$\partial_t \|u^\lambda\|^2_{L^2} \le C(\|v_0\|_{L^\infty}) \left(\|v^\lambda\|^2_{L^2} + \|v^\lambda\|_{L^\infty} \|u^\lambda\|_{L^2} \|v^\lambda\|_{L^2} \right),$$

thanks to (8). Thus, we add the inequality $\partial_t \|v^\lambda\|^2_{L^2} \le 0$ obtained from *(i)*, and Gronwall's Lemma concludes. □

3.2 Proof of Theorem 11: strong convergence

Fix some $T > 0$, and denote $\Omega_T = [0, T] \times \mathbb{R}^3$. From the bounds above, we know that, up to a subsequence of λ's, U^λ weakly converges (in $L^2(\Omega_T)$, and weak-$\star$ in $L^\infty([0, +\infty[\times\mathbb{R}^3)$ for v^λ) to some U^∞. Now, we show (up to a subsequence again) the strong convergence in $\mathcal{C}([0, T], L^2(\mathbb{R}^3))$ (and thus in $\mathcal{C}([0, T], L_{\mathrm{div}})$, passing to the limit in the relation *(ii)* of Proposition 19).

The main step is the strong convergence of v^λ. First, perform an energy estimate on the difference of the equations for v^λ and v^μ, introducing the limit u^∞.

$$\begin{aligned}\partial_t(|v^\lambda - v^\mu|^2) =& 2\langle F(v^\lambda, u^\lambda) - F(v^\mu, u^\mu), v^\lambda - v^\mu\rangle \\ =& 2\langle F^0(v^\lambda) - F^0(v^\mu) + (F^1(v^\lambda) - F^1(v^\mu))u^\infty, v^\lambda - v^\mu\rangle \\ & + 2\langle F^1(v^\lambda)(u^\lambda - u^\infty) - F^1(v^\mu)(u^\mu - u^\infty), v^\lambda - v^\mu\rangle \\ \leq & C(\|v_0\|_{L^\infty})(1+|u^\infty|)|v^\lambda - v^\mu|^2 \\ & + 2\langle F^1(v^\lambda)(u^\lambda - u^\infty) - F^1(v^\mu)(u^\mu - u^\infty), v^\lambda - v^\mu\rangle\end{aligned}$$

thanks to assumption 8(5).

Now, we introduce the measurable, almost eveywhere (on Ω_T) finite function

$$a(t,x) := C(\|v_0\|_{L^\infty}) \int_0^t (1 + |u^\infty(t')|)dt', \tag{13}$$

so that

$$\partial_t(e^{-a}|v^\lambda - v^\mu|^2) \leq 2e^{-a}\langle F^1(v^\lambda)(u^\lambda - u^\infty) - F^1(v^\mu)(u^\mu - u^\infty), v^\lambda - v^\mu\rangle. \tag{14}$$

To get strong convergence, we need a weight sufficiently decreasing at infinity (see Lemma 22 and Lemma 24 below).

Proposition 20 *Set $b(t,x) = a(t,x) + |x|^2$ with a from* (13)*. Then, under the assumptions of Proposition 19, for all $T > 0$, $\left(v^\lambda\right)_{\lambda>0}$ is a Cauchy sequence in $L^2(\Omega_T, e^{-b}dtdx)$.*

Before we give the proof of this proposition, we show how it implies Theorem 11, and in particular, the

Corollary 21 *The sequence $(u^\lambda, v^\lambda)_{\lambda>0}$ strongly converges in $\mathcal{C}([0,T], L^2(\mathbb{R}^3))$ towards a finite energy solution to* (7)*.*

Proof:

Convergence of v^λ almost everywhere and in $\mathcal{C}([0,T], L^2(\mathbb{R}^3))$: up to a subsequence, $\left(v^\lambda\right)_{\lambda>0}$ converges almost everywhere on Ω_T (for the measure $e^{-b}dtdx$, or $dtdx$, since e^{-b} is positive). Now, thanks to the pointwise estimate *(i)* of Proposition 19 and dominated convergence, we get strong convergence in $L^2(\Omega_T)$. In particular, the (sub)sequence of applications $t \mapsto \|v^\lambda - v^\mu\|_{L^2}(t)$ converges to zero as $\lambda, \mu \to \infty$ for almost all $t \in [0,T]$. Finally, the equation on v^λ shows that $\partial_t v^\lambda$ is bounded in $L^\infty L^2$ (thanks to the bound (8) on F), so that $(\|v^\lambda - v^\mu\|_{L^2})_{\lambda,\mu}$ is equicontinuous, and thus converges in $\mathcal{C}([0,T],\mathbb{R})$ by Ascoli's Theorem.

Convergence of the nonlinear terms in $L^1(0,T,L^2(\mathbb{R}^3))$: the uniform bound (Proposition 19*(i)*) and strong convergence of v^λ are enough for $F(x, v^\lambda, u^\lambda)$

to converge in $\mathcal{D}'(]0,T[\times\mathbb{R}^3)$ to $F(x,v^\infty,u^\infty)$, since $F(x,v,u) = F^0(x,v) + F^1(x,v)u$ is affine in u and each F^j is locally Lipschitz in v, uniformly in x (assumption 8(5)). This allows to pass to the limit in equations (12) in $\mathcal{D}'$, but we need more to get strong convergence (in $\mathcal{C}([0,T],L^2(\mathbb{R}^3))$) of u^λ.

Thanks to the Lipschitz properties (9),(10) of F, we have

$$|F(v^\lambda,u^\lambda) - F(v^\infty,u^\infty)| \leq C(\|v_0\|_{L^\infty})\left(|u^\lambda - u^\infty| + (1+|u^\infty|)|v^\lambda - v^\infty|\right).$$

We already know that v^λ converges to v^∞ in $\mathcal{C}([0,T],L^2(\mathbb{R}^3))$. The product $u^\infty(v^\lambda - v^\infty)$ converges to zero almost everywhere, and is dominated by $2\|v_0\|_{L^\infty}|u^\infty| \in L^2(\Omega_T)$, thus converges to zero in $L^2(\Omega_T)$ by Lebesgue's dominated convergence theorem. Therefore, we get

$$\|F(v^\lambda,u^\lambda) - F(v^\infty,u^\infty)\|_{L^1L^2} \leq C(\|v_0\|_{L^\infty})\|u^\lambda - u^\infty\|_{L^1L^2} + o(1), \qquad (15)$$

and next we prove simultaneously the convergence of u^λ in $\mathcal{C}([0,T],L^2(\mathbb{R}^3))$ and of $F(v^\lambda,u^\lambda)$ in $L^1(0,T,L^2(\mathbb{R}^3))$.

Convergence of the fields u^λ in $\mathcal{C}([0,T],L^2(\mathbb{R}^3))$: the classical energy estimate gives

$$\begin{aligned}\|u^\lambda - u^\infty\|_{L^2}(t) &\leq \|(1-S^\lambda)u_0\|_{L^2} + C\int_0^t \|F(v^\lambda,u^\lambda) - F(v^\infty,u^\infty)\|_{L^2}dt' \\ &\leq C\int_0^t \|u^\lambda - u^\infty\|_{L^2}(t')dt' + o(1),\end{aligned}$$

in view of(15), so that Gronwall's Lemma concludes the proof of Theorem 11. □

Proof of Proposition 20:

Integrate on Ω_t the product of (14) and $e^{-|x|^2}$ to get

$$\begin{aligned}&\|e^{-b/2}(v^\lambda - v^\mu)(t)\|_{L^2}^2 \\ &\leq 2\int_{\Omega_t} e^{-b}\langle F^1(v^\lambda)(u^\lambda - u^\infty) - F^1(v^\mu)(u^\mu - u^\infty), v^\lambda - v^\mu\rangle dxdt'.\end{aligned} \qquad (16)$$

Thus we study the two terms

$$\int_{\Omega_t} e^{-b}\langle F^1(v^\lambda)(u^\nu - u^\infty), v^\lambda - v^\mu\rangle dxdt', \quad \nu = \lambda,\mu. \qquad (17)$$

Introduce the Fourier multipliers $\pi_\parallel$ and $\pi_\perp$, with symbols $(\xi/|\xi|,.)\xi/|\xi|$ and $(\xi/|\xi|\wedge .)\wedge\xi/|\xi|$, respectively. The symbols are homogeneous of degree zero, so that the operators map $L^p(\mathbb{R}^3)$ into itself continuously, for all finite p [11]. Furthermore, they are the (L^2) orthogonal projectors onto irrotational and divergence free vector fields, respectively. This is the Hodge decomposition, which we use to split $(u^\nu - u^\infty)$ in (17) into two parts. For notational convenience, define

$$\Pi_\| = \begin{pmatrix} \pi_\| & 0 \\ 0 & \pi_\| \end{pmatrix}, \quad \Pi_\perp = \begin{pmatrix} \pi_\perp & 0 \\ 0 & \pi_\perp \end{pmatrix}.$$

Now, (17) splits into the sum of

$$\begin{aligned} I &= \int_{\Omega_t} e^{-b} \langle F^1(v^\lambda)\Pi_\|(u^\nu - u^\infty), v^\lambda - v^\mu \rangle dx dt' \\ \text{and} \quad II &= \int_{\Omega_t} e^{-b} \langle F^1(v^\lambda)\Pi_\perp(u^\nu - u^\infty), v^\lambda - v^\mu \rangle dx dt'. \end{aligned} \tag{18}$$

Lemma 22 *The integral I from* (18) *satisfies, as λ, μ go to infinity,*

$$I \leq C(\|v_0\|_{L^\infty}) \left(\|e^{-b/2}(v^\lambda - v^\infty)\|^2_{L^2(\Omega_t)} + \|e^{-b/2}(v^\lambda - v^\mu)\|^2_{L^2(\Omega_t)} \right) + o(1).$$

Proof: The divergence relation *(ii)* in Proposition 19 is equivalent to

$$\Pi_\| u^\nu = \Pi_\| S^\nu l(x) v^\nu,$$

and becomes in the (weak) limit $\lambda \to \infty$

$$\Pi_\| u^\infty = \Pi_\| l(x) v^\infty.$$

Furthermore, thanks to the strong convergence of S^λ to id in $\mathcal{L}(L^2)$, we can replace $l(x)v^\infty$ with $S^\nu l(x) v^\infty$. This yields

$$\begin{aligned} I &= \int_{\Omega_t} e^{-b} \langle F^1(v^\lambda)\Pi_\|(S^\nu l(x) v^\nu - l(x) v^\infty), v^\lambda - v^\mu \rangle dx dt' \\ &= \int_{\Omega_t} e^{-b} \langle F^1(v^\lambda)\Pi_\| S^\nu l(x)(v^\nu - v^\infty), v^\lambda - v^\mu \rangle dx dt' + o(1). \end{aligned}$$

Now, another $o(1)$ error is added when commuting $e^{-b/2}$ with $\Pi_\| S^\nu$, as shows the following version of Rellich's Theorem (proved below).

Lemma 23 *For all $p > 2$, the operators $[S^\nu \Pi_\|, e^{-b/2}]$ mapping $(L^2 \cap L^p)(\Omega_T)$ into $L^2(\Omega_T)$ are compact, uniformly w.r.t. ν:*
When $w^\nu \rightharpoonup 0$ in $(L^2 \cap L^p)(\Omega_T)$ weak, $\quad [S^\nu \Pi_\|, e^{-b/2}] w^\nu \underset{\nu \to \infty}{\longrightarrow} 0$ *in $L^2(\Omega_T)$.*

We use the growth properties of F^1 (assumption 8, (4) and (5)), together with the bounds $l \in L^\infty(\mathbb{R}^3)$, $|e^{-b/2}| \leq 1$, and $\|\Pi_\| S^\nu\|_{\mathcal{L}(L^2)} = 1$, to bound

$$\begin{aligned} &\left| \int_{\Omega_t} \langle F^1(v^\lambda)[\Pi_\| S^\nu, e^{-b/2}] l(x)(v^\nu - v^\infty), e^{-b/2}(v^\lambda - v^\mu) \rangle dx dt' \right| \\ &\qquad \leq C(\|v_0\|_{L^\infty}) \|[\Pi_\| S^\nu, e^{-b/2}](v^\nu - v^\infty)\|_{L^2(\Omega_T)} \|v^\lambda - v^\mu\|_{L^2(\Omega_T)}, \end{aligned}$$

which goes to zero as λ, μ go to infinity, thanks to Lemma 23. Thus, we obtain

$$I = \int_{\Omega_t} \langle F^1(v^\lambda)\Pi_\| S^\nu l(x) e^{-b/2}(v^\nu - v^\infty), e^{-b/2}(v^\lambda - v^\mu)\rangle dxdt' + o(1),$$

which immediately implies Lemma 22. $\square$

Proof of Lemma 23:

Since $b(t,x) \geq |x|^2$, we have $e^{-b/2} \in L^p(\Omega_T)$ for all $p \in [1, \infty]$. Next, $\Pi_\| S^\nu$ is a bounded family of continuous operators on $L^p(\Omega_T)$ for all $p \in [2, \infty[$, so that, when $\phi \in \mathcal{C}_c^\infty(]0,T[\times\mathbb{R}^3)$,

$$\begin{aligned} &\|[S^\nu \Pi_\|, e^{-b/2}]w^\nu - [S^\nu \Pi_\|, \phi]w^\nu\|_{L^2(\Omega_T)} \\ &\quad \leq \left(\|S^\nu \Pi_\|\|_{\mathcal{L}(L^2(\Omega_T))} + \|S^\nu \Pi_\|\|_{\mathcal{L}(L^4(\Omega_T))}\right) \|e^{-b/2} - \phi\|_{L^4(\Omega_T)} \|w^\nu\|_{L^4(\Omega_T)}, \end{aligned}$$

and we may replace $e^{-b/2}$ by an L^4 approximation $\phi \in \mathcal{C}_c^\infty(]0,T[\times\mathbb{R}^3)$.

Now, denoting $m^\nu(\xi)$ the symbol of $S^\nu \Pi_\|$, for all cut-off function $\gamma \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ such that $\gamma \equiv 1$ in a neighborhood of 0, the symbol $(1-\gamma)m^\nu$ has bounded semi-norms. Hence $T^\nu = [S^\nu \Pi_\|, \phi]$ is a bounded family of pseudodifferential operators on Ω_T, with degree -1. Consider $\psi_1, \psi_2 \in \mathcal{C}_c^\infty(]0,T[\times\mathbb{R}^3)$ satisfying $\psi_1 \equiv 1$ on suppϕ and $\psi_2 \equiv 1$ on suppψ_1, so that $\psi_1\phi = \phi$ and $\psi_2\psi_1 = \psi_1$. Then,

$$T^\nu w^\nu = \psi_2 T^\nu w^\nu + (1-\psi_2)T^\nu w^\nu.$$

Thanks to Rellich's Theorem, the first term on the right-hand side strongly tends to zero in $L^2(\Omega_T)$. In view of the conditions on the supports of ψ_1 and ψ_2, the second term writes

$$\begin{aligned} (1-\psi_2)[S^\nu \Pi_\|, \phi]w^\nu =& (1-\psi_2)S^\nu \Pi_\| \phi w^\nu \\ =& (1-\psi_2)S^\nu \Pi_\| \psi_1 \phi w^\nu \\ =& (1-\psi_2)[S^\nu \Pi_\|, \psi_1]\phi w^\nu. \end{aligned}$$

Here again, $(1-\psi_2)[S^\nu \Pi_\|, \psi_1]$ is a bounded family of pseudodifferential operators on Ω_T, with degree -1, and ϕw^ν strongly converges to zero in $H^{-1}(\Omega_T)$ (by Rellich's Theorem), so that the product converges to zero in $L^2(\Omega_T)$. $\square$

For the second integral II, we use compensated compactness.

Lemma 24 *The integral II from* (18) *has limit zero as λ, μ go to infinity, uniformly w.r.t. $t \in [0,T]$.*

Proof: The integral II is equal to

$$II = \int_{\Omega_t} e^{-b} Q(x, v^\lambda, v^\mu) \cdot \Pi_\perp(u^\nu - u^\infty)\ dxdt',$$

where $Q(x, v, w)$ is a $\mathbb{R}^6$ valued function, defined by

$$Q(x, v, w) \cdot \xi = \langle F^1(x, v)\xi, v - w\rangle, \quad \forall \xi \in \mathbb{R}^6.$$

First, we know that $Q(v^\lambda, v^\mu)$ is bounded in $L^2(\Omega_T)$, thanks to the $L^2 \cap L^\infty$ bound on v^λ, v^μ, and the control $|F^1(x, v)| \le C(|v|)|v|$ on F^1. Since $\Pi_\perp(u^\nu - u^\infty)$ is also bounded in $L^2(\Omega_T)$, and e^{-b} takes its values in $[0, 1]$, the family of integrals II, as functions of t, is uniformly (in λ, μ) equicontinuous on $[0, T]$. As a consequence, it is sufficient to prove Lemma 24 for t fixed.

Next, since the weight e^{-b} goes to zero as $|x|$ goes to infinity, we may localize II: for all $\delta > 0$, there is $R = R(\delta)$ such that

$$II \le \int_{[0,t]\times\{|x|\le R\}} e^{-b}\langle F^1(v^\lambda)\Pi_\perp(u^\nu - u^\infty), v^\lambda - v^\mu\rangle dxdt' + \delta. \tag{19}$$

Now, observe that the divergence free part $\Pi_\perp u^\nu$ of the fields is solution to a wave equation (thanks to the relation $\operatorname{curl}\operatorname{curl}\pi_\perp = -\Delta\pi_\perp$),

$$(\partial_t^2 - \Delta)\Pi_\perp u^\nu = \Pi_\perp S^\nu\Big(\partial_t(l_1F(v^\nu, u^\nu)) - \operatorname{curl}(l_2F(v^\nu, u^\nu)),$$
$$\partial_t(l_2F(v^\nu, u^\nu)) + \operatorname{curl}(l_1F(v^\nu, u^\nu))\Big),$$

so that $(\partial_t^2 - \Delta)\Pi_\perp(u^\nu - u^\infty)$ is bounded in $H^{-1}(\Omega_T)$.

We apply the results of compensated compactness [3], [12] on $[0, t]\times\{|x| \le R\}$. The operators $\Box$ and ∂_t have non-intersecting characteristic varieties

$$\mathcal{C}_\Box := \{\tau^2 - |\xi|^2 = 0\} \setminus \{0\} \text{ and } \mathcal{C}_{\partial_t} := \{\tau = 0\} \setminus \{0\},$$

and $\Pi_\perp(u^\nu - u^\infty)$ is bounded in $L^2(\Omega_T)$, with $\Box\Pi_\perp(u^\nu - u^\infty)$ relatively compact in $H^{-2}(\Omega_T)$. Since $Q(v^\lambda, v^\mu)e^{-b}$ is bounded in $L^2(\Omega_T)$, it is sufficient to check that $\partial_t\left(Q(v^\lambda, v^\mu)e^{-b}\right)$ is relatively compact in $H^{-1}(\Omega_T)$ to conclude that the integral in (19) goes to zero as λ, μ go to infinity. But we have

$$\partial_t\left(Q(v^\lambda, v^\mu)e^{-b}\right) = \Big((\partial_{v,w}Q)(v^\lambda, v^\mu)\cdot(\partial_t v^\lambda, \partial_t v^\mu) - (\partial_t b)Q(v^\lambda, v^\mu)\Big)\, e^{-b}.$$

Here, $e^{-b} \in L^\infty(\Omega_T)$, since $b \ge |x|^2$. Furthermore, $(\partial_{v,w}Q)(v^\lambda, v^\mu)$ is bounded in $L^\infty(\Omega_T)$ (by the Lipschitz assumption 8(5) on F^1), and $\partial_t v^\lambda = F(v^\lambda, u^\lambda)$ is bounded in $L^2(\Omega_T)$. For the second term, $\partial_t b = C(\|v_0\|_{L^\infty})(1 + |u^\infty|) \in L^\infty(\Omega_T) + L^2(\Omega_T)$, and $Q(v^\lambda, v^\mu)e^{-b}$ is bounded in $L^2(\Omega_T) \cap L^\infty(\Omega_T)$. This shows that $\partial_t\left(Q(v^\lambda, v^\mu)e^{-b}\right)$ is bounded in $L^2(\Omega_T)$, and Lemma 24 is proved. $\Box$

Add the estimates of Lemma 22 and Lemma 24 for $\nu = \lambda, \mu$. From (16), Gronwall's Lemma implies that for all $t \in [0,T]$,

$$\|e^{-b/2}(v^\lambda - v^\mu)(t)\|_{L^2}^2 \le C(\|v_0\|_{L^\infty}, T) \int_0^t \|e^{-b/2}(v^\lambda - v^\infty)(t')\|_{L^2}^2 dt' + o(1). \quad (20)$$

For all $t \in [0,T]$, $(v^\lambda(t))_{\lambda>0}$ is bounded in L^2. The equation $\partial_t v^\lambda = F(v^\lambda, u^\lambda)$ shows that $(v^\lambda)_{\lambda>0}$ is an L^2 weak valued equicontinuous family. Thus, by Ascoli's Theorem, up to a subsequence, $v^\lambda(t)$ converges (in L^2 weak) for all t. Taking the limit $\mu \to \infty$ in (20) and applying Gronwall's Lemma again completes the proof of Proposition 20. □

3.3 Proof of Theorem 7: the polarized case

In this section, we sketch the proof of Theorem 7, when system (1) reduces to

$$\begin{cases} \mu\partial_t H + \operatorname{curl} E = 0, \\ \varepsilon\partial_t E - \operatorname{curl} H = i\mathrm{Tr}\,(\Gamma[\Omega, \rho]), \\ i\partial_t \rho = [\Omega - E \cdot \Gamma, \rho]. \end{cases} \quad (21)$$

For the sake of simplicity, we don't give any similar statement for general systems, even if we only use here: hyperbolicity of the system (any space dimension is allowed, as well as variable coefficients), the $L^2 \cap L^\infty$ *a priori* bound on the density matrix ρ, the (local in ρ, global in (H,E)) Lipschitz property of the nonlinear terms, and the dependence of the source term in Maxwell's equations on x and ρ only (in particular, neither the conservations (3) are needed, nor the decomposition of the fields into irrotational and divergence free parts).

First, regularize the system in a slightly different way from that of Section 3.1 to define an ODE with locally Lipschitz nonlinearity on the Banach space $B = L^2(\mathbb{R}^3) \times L^\infty(\mathbb{R}^3) \times (L^2(\mathbb{R}^3) \cap L^2(\mathbb{R}^3))$,

$$\begin{cases} \mu\partial_t H^\lambda + \operatorname{curl} S^\lambda E^\lambda = 0, \\ \varepsilon\partial_t E^\lambda - \operatorname{curl} S^\lambda H^\lambda = i\mathrm{Tr}\,(\Gamma[\Omega, \rho^\lambda]), \\ i\partial_t \rho^\lambda = [\Omega - S^\lambda E^\lambda \cdot \Gamma, \rho^\lambda], \\ (H^\lambda, E^\lambda, \rho^\lambda)_{|t=0} = (H_0, E_0, \rho_0) \in B. \end{cases} \quad (22)$$

We recover the same bounds as in Proposition 19. To these, we add an energy estimate for the time derivatives, using the $H(\mathrm{curl})$ regularity. It serves in the sequel for controlling the $\operatorname{curl} S^\lambda$ terms (another avatar of this trick is present in Section 4.1).

Proposition 25 *There exists a unique* $U^\lambda = (u^\lambda, \rho^\lambda) \in \mathcal{C}^1([0,+\infty[, B)$ *solution to* (22)*. Furthermore,*
(i) For almost all $x \in \mathbb{R}^3$, $|\rho^\lambda(t,x)|$ *is constant in time.*
(ii) There is a constant $C = C(\mu_0, \varepsilon_0, \|\Omega\|_{L^\infty}, \|\Gamma\|_{L^\infty})$ *such that, for all times* t, $\|U^\lambda(t)\|_{L^2} \leq e^{Ct}\|U_0(t)\|_{L^2}$.
(iii) If in addition $\operatorname{curl} H_0, \operatorname{curl} E_0 \in L^2(\mathbb{R}^3)$*, then there is a constant* $C = C(\mu_0, \varepsilon_0, \|\Omega\|_{L^\infty}, \|\Gamma\|_{L^\infty}, \|\rho_0\|_{L^\infty}, \|\operatorname{curl} H_0\|_{L^2}, \|\operatorname{curl} E_0\|_{L^2})$ *such that, for all times* t, $\|\partial_t u^\lambda(t)\|_{L^2} \leq C(1+t)$.

We then show that the whole sequence $(U^\lambda)_{\lambda>0}$ converges in $\mathcal{C}([0,T], L^2(\mathbb{R}^3))$ for all $T > 0$.

Proposition 26 *For all* $T > 0$*, the solution* U^λ *to* (22) *converges in* $\mathcal{C}([0,T], L^2(\mathbb{R}^3))$ *towards the unique* $U \in \mathcal{C}([0,T], B)$ *solution to* (21) *in* $\mathcal{D}'$ *with* U_0 *as initial data.*

Proof:

To show that U^λ is a Cauchy sequence in $\mathcal{C}([0,T], L^2(\mathbb{R}^3))$, consider the difference of systems (22) for U^λ and U^ν, and take the scalar product in L^2 of the equations with $H^\lambda - H^\nu$, $E^\lambda - E^\nu$ and $\rho^\lambda - \rho^\nu$, respectively. The $u = (H, E)$ equations yield

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}\left(\|\sqrt{\mu}(H^\lambda - H^\nu)\|_{L^2}^2 + \|\sqrt{\varepsilon}(E^\lambda - E^\nu)\|_{L^2}^2\right) \\
=&2i\int \operatorname{Tr}(\Gamma[\Omega, \rho^\lambda - \rho^\nu])(E^\lambda - E^\nu) \\
&- 2\int (\operatorname{curl}(S^\lambda E^\lambda - S^\nu E^\nu) \cdot (H^\lambda - H^\nu) \\
&+ 2\int (\operatorname{curl}(S^\lambda H^\lambda - S^\nu H^\nu) \cdot (E^\lambda - E^\nu).
\end{aligned} \tag{23}$$

From Proposition 25, $\partial_t u^\lambda$ is bouded in $L^\infty(0,T,L^2)$, thus so are $\operatorname{curl} S^\lambda E^\lambda$ and $\operatorname{curl} S^\lambda H^\lambda = \varepsilon \partial_t E^\lambda - i\operatorname{Tr}(\Gamma[\Omega, \rho^\lambda])$. Since S^λ strongly converges to the identity as an operator on L^2, we write $H^\lambda - H^\nu = S^\lambda H^\lambda - S^\nu H^\nu + o_{L^2}(1)$, and the last two terms on the r.h.s. of (23) go to zero as $\lambda, \nu \to \infty$. The rest of the estimates is standard, and passing to the limit in the system is immediate.

Uniqueness of the limit is obtained in the same way, with S^λ replaced by 1: as in Proposition 1, energy estimates are obtained after mollification, using Friedrich's Lemma. □

Remark 27 *In the case of constant coefficients* μ *and* ε*, we use the same regularization as in* (12)*, so that no bound on the curl of the fields is needed.*

4 Uniqueness of weak solutions

Let U and U' be two finite energy solutions to (7) with the same initial data. The difference $\delta U := U' - U$ is solution to a symmetric hyperbolic system

$$\begin{aligned} &M(\delta U) = (l\delta F, \delta F), \\ &\text{where } \delta F = F^0(v') - F^0(v) + (F^1(v') - F^1(v))u + F^1(v')(u' - u). \end{aligned} \tag{24}$$

When the fields u belong to $L^\infty(\Omega_T)$, an energy estimate provides $\|\delta U(t)\|_{L^2} \leq e^{C(\|u\|_{L^\infty})t}\|\delta U(0)\|_{L^2}$, which implies $U' = U$. Unfortunately, this *a priori* estimate is false for general finite energy solution. It becomes true when cutting off the high frequencies of u, and we control the cut-off error thanks to the H^1 norm of u.

4.1 Proof of Theorem 12: propagation of $H(\text{curl})$ *regularity*

First note that for any $u \in L^2(\mathbb{R}^3, \mathbb{R}^3)$, the divergence free part $\pi_\perp u$ is in H^1 iff the curl of u belongs to L^2. We use Maxwell's equations to convert space derivatives of the fields into time derivatives, and we control the latter by energy estimates. Now, since the function v is already defined, we write Maxwell's equations as a linear system for $u_\perp := \Pi_\perp(u_1, u_2)$ (with L from (6)),

$$Lu_\perp = \Pi_\perp B u_\perp + \Pi_\perp f, \tag{25}$$

where $B(t,x) = l(x)F^1(x, v(t,x)), \quad f(t,x) = l(x)F(x, v(t,x), \Pi_\| l(x)v(t,x))$,

using the constraint (11) to get the forcing f. Thus, B is a 6×6 matrix with coefficients in $\mathcal{C}([0, +\infty[, (L^2 \cap L^\infty)(\mathbb{R}^3))$, and time derivatives in $\mathcal{C}([0, +\infty[, L^p(\mathbb{R}^3))$ for all $p \in [2, \infty[$. In the same way, $f \in \mathcal{C}([0, +\infty[, (L^2 \cap L^\infty)(\mathbb{R}^3))$ and $\partial_t f \in \mathcal{C}([0, +\infty[, L^p(\mathbb{R}^3))$ for all $p \in [2, \infty[$. Since the restriction of L to the range of $\Pi_\perp$ is symmetric hyperbolic, the Cauchy problem associated with (25) and any initial data $u_{\perp,0} = \Pi u_{\perp,0} \in L^2(\mathbb{R}^3)$ has a unique solution $u_\perp = \Pi_\perp u_\perp \in \mathcal{C}([0, +\infty[, L^2(\mathbb{R}^3))$.

Proposition 28 *Let* $f, \partial_t f \in \mathcal{C}([0, +\infty[, L^2(\mathbb{R}^3))$, $B \in \mathcal{C}([0, +\infty[, L^\infty(\mathbb{R}^3))$ *and* $\partial_t B \in \mathcal{C}([0, +\infty[, L^3(\mathbb{R}^3))$. *Then, for all* $u_{\perp,0} = \Pi u_{\perp,0} \in H^1(\mathbb{R}^3)$, *the solution* $u_\perp$ *to the Cauchy problem associated with* (25) *belongs to* $\mathcal{C}([0, +\infty[, H^1(\mathbb{R}^3))$. *Furthermore, for all* $T > 0$, *there are constants* $K_1 = K_1(\|u_{\perp,0}\|_{H^1}, \|B\|_{L^\infty(\Omega_T)}, \|f\|_{W^{1,\infty}([0,T],L^2))})$, $K_2 = K_2(\|B\|_{L^\infty(\Omega_T)}, \|\partial_t B\|_{L^\infty([0,T],L^3))})$, *such that for all* $t \in [0, T]$,

$$\|u_\perp(t)\|_{H^1} \leq K_1 + K_2 e^{K_2 t}(1 - e^{-K_1 t}). \tag{26}$$

Proof: Fix $T > 0$. The classical fixed-point argument shows that $u_\perp$ is the limit, in $\mathcal{C}([0,T], L^2(\mathbb{R}^3))$, of the sequence $(u_\perp^n)_{n\in\mathbb{N}}$ defined by $u_\perp^0 = u_{\perp,0}$ and $u_\perp^{n+1} = \mathcal{T} u_\perp^n$, where $\mathcal{T} y =: z$ is solution to $Lz = \Pi_\perp By + \Pi_\perp f$, $z_{|t=0} = u_{\perp,0}$.

Begin with the inhomogeneous equation $Lz = \Pi_\perp f \in L^1([0,T], L^2)$, and the usual energy estimate

$$\|z(t)\|_{L^2} \leq \|z(0)\|_{L^2} + 2\int_0^t \|f(t')\|_{L^2} dt'. \tag{27}$$

Now, $L = \partial_t + iP(D_x)$, where the symbol of the Fourier multiplier $P(D_x)$ has eigenvalues $\pm|\xi|$ with constant multiplicities. Denote by $\Pi_\pm$ the associated projections. The wave z splits up into

$$z = z_+ + z_-, \qquad \text{where } \widehat{z_\pm}(t,\xi) = e^{\pm it|\xi|}\Pi_\pm \hat{z_0}(\xi) + \int_0^t e^{\pm i(t-t')|\xi|}\Pi_\pm \hat{f}(t',\xi) dt'.$$

Since $f \in \mathcal{C}([0,T], L^2(\mathbb{R}^3))$ and $\partial_t f \in \mathcal{C}([0,T], L^2(\mathbb{R}^3))$, integrating by parts gives

$$\begin{aligned}\frac{1}{i}\widehat{\partial_x z_\pm}(t,\xi) =& e^{\pm it|\xi|}\xi \Pi_\pm \widehat{z_0}(\xi) \\ & \pm i \left[e^{\pm i(t-t')|\xi|}\Pi_\pm \hat{f}(t',\xi)\right]_0^t \pm i \int_0^t e^{\pm i(t-t')|\xi|}\frac{\xi}{|\xi|}\Pi_\pm \widehat{\partial_t f}(t',\xi)dt',\end{aligned}$$

from which we deduce

$$\|\partial_x z(t)\|_{L^2} \leq \|\partial_x z(0)\|_{L^2} + 2\|f\|_{\mathcal{C}(L^2)} + 2\int_0^t \|\partial_t f(t')\|_{L^2} dt'. \tag{28}$$

Furthermore, the relation $\partial_t z = -iP(D_x)z + \Pi_\perp f$ implies

$$\|\partial_t z(t)\|_{L^2} \leq C\|\partial_x z(t)\|_{L^2} + \|f(t)\|_{L^2}. \tag{29}$$

When defining z by $Lz = \Pi_\perp By + \Pi_\perp f$ with $y \in \mathcal{C}([0,T], H^1) \cap \mathcal{C}^1([0,T], L^2)$, according to (28) and (29), we recover a z in the same space, as soon as we control $\partial_t(By)$ ($By \in L^1(L^2)$ is immediate). But for a fixed t,

$$\begin{aligned}\|\partial_t(By)\|_{L^2} &\leq \|(\partial_t B)y\|_{L^2} + \|B\partial_t y\|_{L^2} \\ &\leq \|\partial_t B\|_{L^3}\|y\|_{L^6} + \|B\|_{L^\infty}\|\partial_t y\|_{L^2} \\ &\leq C\|\partial_t B\|_{L^3}\|\partial_x y\|_{L^2} + \|B\|_{L^\infty}\|\partial_t y\|_{L^2},\end{aligned} \tag{30}$$

thanks to Gagliardo-Nirenberg-Sobolev's inequality.

Estimates (27) to (30) show that $\mathcal{T}$ maps continuously $\mathcal{C}([0,T], H^1) \cap \mathcal{C}^1([0,T], L^2)$ into itself. In addition, when $z \in \mathcal{C}([0,T], H^1)$ is solution to $z = \mathcal{T}z$ ($\in$

$\mathcal{C}^1([0,T],L^2))$, we have

$$\|z(t)\|_{H^1} \le \|z(0)\|_{H^1} + C(\|B\|_{L^\infty(\Omega_t)}, \|\partial_t B\|_{L^\infty([0,t],L^3)})) \int_0^t \|z(t')\|_{H^1} dt' \\ + C(\|f\|_{L^\infty([0,t],L^2))}, \|B\|_{L^\infty(\Omega_t)}, \|\partial_t f\|_{L^\infty([0,t],L^2))}),$$

so that Gronwall's Lemma implies (26).

Proceed in the same way with the difference $u_\perp^{n+1} - u_\perp^n$ to get, for $n \ge 1$,

$$\|\partial_{t,x}(u_\perp^{n+1} - u_\perp^n)(t)\|_{L^2} \le C\Big(\|u_\perp^n - u_\perp^{n-1}\|_{\mathcal{C}([0,T],L^2)} \\ + \int_0^t \|\partial_{t,x}(u_\perp^n - u_\perp^{n-1})(t')\|_{L^2} dt' \Big).$$

For T_1 small enough ($CT_1 < 1/2$, which depends on B, but not on the initial data), this inequality implies that $(u_\perp^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0,T_1],H^1) \cap \mathcal{C}^1([0,T_1],L^2)$. Iterate this on $[T_1, 2T_1]$, $[2T_1, 3T_1]\dots$ to obtain convergence on the whole $[0,T]$. □

4.2 Proof of Theorem 14: uniqueness of $H(\mathrm{curl})$ solutions

Consider U and U', finite energy solutions to (7) with the same initial data, such that $\mathrm{curl}\, u_j$, $\mathrm{curl}\, u'_j$ belong to $\mathcal{C}([0,+\infty[, L^2(\mathbb{R}^3))$ for $j = 1,2$. The difference $\delta U := U' - U$ is solution to the symmetric hyperbolic system (24). Under the assumptions of Theorem 14, there is $j \in \{1,2\}$ such that,

$$l_{3-j}\delta F = 0, \quad \delta F = \delta F(v, v', u_j, u'_j).$$

We construct an L^∞ approximation of the fields u (analogous to the ones of [6], Lemma 6.2, and [5], Lemma 2.7).

Lemma 29 *Under the assumptions of Theorem 14, let U be a finite energy solution to (7) such that $\mathrm{curl}\, u_j \in \mathcal{C}([0,+\infty[, L^2(\mathbb{R}^3))$, $j = 1,2$. Then, for all $T > 0$, $\lambda \ge e$, there are $u_j^\lambda \in L^\infty(\Omega_T)$, $\alpha_\lambda \in L^2(0,T)$, $\beta_\lambda \in L^\infty(0,T), C > 0$ such that, for all $t \in [0,T]$,*

$$\|u_j^\lambda(t)\|_{L^\infty} \le \alpha_\lambda(t) + \beta_\lambda(t) \quad \textit{and} \quad \|(u_j - u_j^\lambda)(t)\|_{L^2} \le C/\lambda, \\ \textit{with} \quad \|\alpha_\lambda\|_{L^2} \le C\sqrt{\ln \lambda} \quad \textit{and} \quad \|\beta_\lambda\|_{L^\infty} \le C \ln \lambda.$$

Before we give a proof of Lemma 29, we finish the proof of Theorem 14. From (24), we have

$$\begin{aligned}\|\delta F(t)\|_{L^2} &\leq C(\|v_0\|_{L^\infty})\Big(\|\delta v(t)\|_{L^2} + (\alpha_\lambda+\beta_\lambda)(t)\|\delta v(t)\|_{L^2} \\ &\qquad + \frac{C}{\lambda}\|\delta v(t)\|_{L^\infty} + \|\delta u_j(t)\|_{L^2}\Big) \\ &\leq C(\|v_0\|_{L^\infty})\Big((1+\alpha_\lambda+\beta_\lambda)(t)\|\delta U(t)\|_{L^2} + 1/\lambda\Big),\end{aligned}$$

so that the energy estimate, together with Gronwall's Lemma, gives

$$\|\delta U(t)\|_{L^2} \leq C\frac{t}{\lambda}e^{C\int_0^t(1+\alpha_\lambda+\beta_\lambda)(t')dt'}.$$

Since $C\int_0^t(1+\alpha_\lambda+\beta_\lambda)(t')dt' \leq C(T)\ln\lambda$ with $C(T)\underset{T\to 0}{\longrightarrow}0$, we choose T_0 small enough (in order to have $C(T_0)<1$), and let λ go to infinity. This shows that $\delta U(t)$ vanishes on $[0,T_0]$. Repeat this procedure on intervals of size T_0 to get the desired conclusion. □

Proof of Lemma 29:

First split u_j into its irrotational and divergence free parts,

$$u_j = \pi_\perp u_j + \pi_\| u_j.$$

The divergence free part is linked to v *via* the conservations (11): $\pi_\| u_j = \pi_\|(l_j v)$.

Approximation of the divergence free part. Here, we don't need the specific form of (7) required in Theorem 14. Since $\pi_\|$ is a homogeneous Fourier multiplier of degree 0, it defines for all finite p a bounded endomorphism on $L^p(\mathbb{R}^3)$, with norm less than $C_0 p$ (see [11]). Using $l\in L^\infty$ and the pointwise estimate on v (Proposition 10*(ii)*),

$$\|\pi_\| u_j(t)\|_{L^p} \leq C_0'\ p\ \|v(t)\|_{L^2\cap L^\infty} \leq C_0'\ p\ \|v(0)\|_{L^2\cap L^\infty}. \tag{31}$$

Then, define

$$u_{j\|}^\lambda(t,x) := \left|\begin{array}{l} u_{j\|}(t,x) \text{ if } |\pi_\| u_j(t,x)| \leq C\ln\lambda \\ \qquad \text{(where the constant } C \text{ has to be chosen)}, \\ 0 \text{ otherwise,}\end{array}\right.$$

so that,

$$\begin{aligned}\|(\pi_\| u_j - u^\lambda_{j\|})(t)\|^2_{L^2} &= \int_{\{|\pi_\| u_j|\geq C\ln\lambda\}} |\pi_\| u_j(t)|^2 dx \\ &\leq (C\ln\lambda)^{2-p} \|\pi_\| u_j(t)\|^p_{L^p} \\ &\leq \frac{(C'_0\ p\ \|v(0)\|_{L^2\cap L^\infty})^p}{(C\ln\lambda)^{p-2}} = (C\ln\lambda)^2 \lambda^{\alpha\ln\left(\alpha\frac{C'_0}{C}\|v(0)\|_{L^2\cap L^\infty}\right)},\end{aligned}$$

when setting $p = \alpha\ln\lambda$. Choosing $\alpha = 2$, for C big enough, this last quantity is less than C'/λ^2, for $\lambda \geq e$. Thus, set

$$\beta_\lambda(t) := \|u^\lambda_{j\|}\|_{L^\infty} \leq C\ln\lambda.$$

Approximation of the irrotational part.

Thanks to the assumptions $F = F(x, v, u_j)$ and $l_{3-j}F(x, v, u_j) = 0$, we get from (7),

$$\begin{aligned}(\partial_t^2 - \Delta)\pi_\perp u_j &= \pi_\perp l_j dF(v, u_j) \cdot \Big(F(v, u_j), l_j F(x, v, u_j) + (-1)^j \operatorname{curl} u_{3-j}\Big) \\ &= \pi_\perp l_j \Big[\mathrm{d}F^0(v) \cdot F(v, u_j) + (\mathrm{d}F^1(v) \cdot F(v, u_j))u_j \\ &\qquad + F^1(v)(l_j F(x, v, u_j) + (-1)^j \operatorname{curl} u_{3-j})\Big].\end{aligned}$$

Since $v \in \mathcal{C}([0,T], L^2 \cap L^\infty)$ and $|F(v, u_j)| \leq C(1 + |u_j|)|v|$ from (8), the first term on the r.h.s. belong to $\mathcal{C}([0,T], L^2)$. The same holds for $|u_j|^2$, for $u_j = \pi_\perp u_j + \pi_\| u_j \in \mathcal{C}([0,T], H^1) + \mathcal{C}([0,T], L^4) \hookrightarrow \mathcal{C}([0,T], L^4)$ by Sobolev's embedding. Finally,

$$\|\Box\pi_\perp u_j(t)\|_{L^2} \leq C(\|v_0\|_{L^\infty})(1 + \|u_j(t)\|_{L^2} + \|\Pi_\perp u(t)\|_{H^1} + \|\pi_\| u_j(t)\|^2_{L^4}), \quad (32)$$

which is easily bounded (in $\mathcal{C}([0,T])$) in terms of T, $\|u_{|t=0}\|_{L^2}$, $\|\Pi_\perp u_{|t=0}\|_{H^1}$ and $\|v_{|t=0}\|_{L^2\cap L^\infty}$, thanks to the basic L^2 estimate, Proposition 28 and (31), respectively.

We now wish to use a Strichartz estimate to control $\|\pi_\perp u_j\|_{L^2(L^\infty)}$ in terms of $\|\Box\pi_\perp u_j\|_{L^1(L^2)}$. The usual estimates allow a $L^r(L^p)$ control for finite p only (see [4], [8], [7]). The limit case only holds when truncating frequencies, and is proved in [6] (Proposition 6.3): using the cut-off of Section 3.1,

Proposition 30 *There is a constant $C > 0$ such that, for all $\lambda, T > 0$ and $u \in \mathcal{C}([0, +\infty[, H^2(\mathbb{R}^3))$,*

$$|S^\lambda u\|_{L^2([0,T],L^\infty(\mathbb{R}^3))} \leq C\sqrt{\ln(1 + \lambda T)}\Big(\|\partial_{t,x} u(0)\|_{L^2(\mathbb{R}^3)} + \|\Box u\|_{L^1([0,T],L^2(\mathbb{R}^3))}\Big).$$

To end the proof, set

$$u_j^\lambda := S^\lambda \pi_\perp u_j + u_{j\|}^\lambda.$$

With $\alpha_\lambda(t) := \|S^\lambda \pi_\perp u_j(t)\|_{L^\infty(\mathbb{R}^3)}$, we get $\|\alpha_\lambda\|_{L^2} \leq C\sqrt{\ln \lambda}$ from (32) and Proposition 30, and finally,

$$\begin{aligned}
\|(\pi_\perp u_j - S^\lambda \pi_\perp u_j)(t)\|_{L^2} &\leq \|\mathbf{1}_{\{|\xi|\geq\lambda\}}\widehat{\pi_\perp u_j}(t)\|_{L^2} \\
&\leq \|\frac{(1+|\xi|^2)^{1/2}}{(1+\lambda^2)^{1/2}}\mathbf{1}_{\{|\xi|\geq\lambda\}}\widehat{\pi_\perp u_j}(t)\|_{L^2} \\
&\leq \frac{C}{(1+\lambda^2)^{1/2}}\|\operatorname{curl} u_j\|_{\mathcal{C}([0,T],L^2)}.
\end{aligned}$$

□

5 Proof of Theorem 16: global smooth solutions

We consider initial data $U_0 \in H^s(\mathbb{R}^3)$ for $s \geq 2$, and the associated maximal smooth solution to (7), $U \in \mathcal{C}([0, T_\star[, H^s(\mathbb{R}^3))$. As is well-known, if $T_\star$ is finite, then the L^∞ (in space) norm of $U(t)$ blows up as t goes to $T_\star$. Hence, arguing by contradiction and assuming that $T_\star$ is finite, it suffices to show that the H^2 norm of $U(t)$ remains bounded on $[0, T_\star[$, thanks to Sobolev's inequality.

Furthermore, it is also a classical fact that, approximating $U_0 \in H^s(\mathbb{R}^3)$ by a sequence of smooth initial data, one generates a sequence of smooth solutions which forms a Cauchy sequence in $\mathcal{C}([0, T], H^s(\mathbb{R}^3))$ and converges to a solution to (7), for all $T > 0$ for which the whole sequence is defined. As a consequence, we only need to prove that solutions corresponding to smooth initial data have H^2 bounds which depend only on the H^2 norm of the initial data. In the sequel, we take $U_0 \in H^3(\mathbb{R}^3)$.

We proceed by means of energy estimates for the equations satisfied by U, $\partial_x U$ and $\partial_x^2 U = (\partial_{x_i}\partial_{x_j} U)_{i,j}$. Since $l \in W^{2,\infty}$, we only have to bound the derivatives $G^k(t,x) := \partial_x^k(F(x, v(t,x), u_j(t,x))$, $k = 0, 1, 2$. Since $U \in \mathcal{C}([0, T_\star[, H^3(\mathbb{R}^3))$, the function $\|U(t)\|_{H^2}^2$ is continuously differentiable on $[0, T_\star[$, and satisfies

$$\frac{d}{dt}\left(\|U(t)\|_{H^2}^2\right) \leq C\sum_{k=0}^{2}\|G^k(t)\|_{L^2}\|U(t)\|_{H^2}. \tag{33}$$

Direct computation, together with the pointwise estimate $|v(t,x)| \leq |v_0(x)|$, give

$$\begin{aligned}
|G^0| =&|F(x,v,u_j)| \le C(v_0)(1+|u_j|)|v|,\\
|G^1| =&|\partial_x F + \partial_v F\cdot\partial_x v + \partial_{u_j} F\cdot \partial_x u_j|\\
\le& C(v_0)((1+|u_j|)|v| + |\partial_x u_j| + (1+|u_j|)|\partial_x v|),\\
|G^2| \le& C(v_0)\Big((1+|u_j|)(|v|+|\partial_x v|+|\partial_x^2 v|)\\
&\qquad + |\partial_x u_j| + |\partial_x^2 u_j| + |\partial_x u_j||\partial_x v| + |\partial_x v|^2|u_j|\Big),
\end{aligned}$$

with a constant $C(v_0)$ depending on $\|v_0\|_{L^\infty}$ only.

Next, the products of first order derivatives $|\partial_x u_j||\partial_x v|$ and $|\partial_x v|^2$ are estimated thanks to Gagliardo-Nirenberg's inequality,

$$\|\partial_x w\|_{L^4} \le C\|w\|_{L^\infty}^{1/2}\|w\|_{H^2}^{1/2}.$$

This shows that for $k=0,1,2$, and $t\in[0,T_\star[$,

$$\|G^k(t)\|_{L^2} \le C(v_0)(1+\|u_j(t)\|_{L^\infty})\|U(t)\|_{H^2}.$$

There remains to estimate $u_j(t)$ in L^∞. To this end, as before, we decompose u_j into its irrotational and divergence free parts.

The irrotational part $\pi_\parallel u_j$ is estimated using $\pi_\parallel u_j = \pi_\parallel(l_j v)$, and Judovic's inequality (see [13]) for the homogeneous Fourier multiplier $\pi_\parallel$ of degree zero, applied to $w = l_j v(t)$,

$$\|\pi_\parallel w\|_{L^\infty} \le C\|w\|_{L^\infty}\ln(2+\|w\|_{H^2}).$$

The divergence free part $\pi_\perp u_j$ is split again into a low-frequency part and a high-frequency part, using the cut-off of Section 3.1,

$$\pi_\perp u_j = S^\lambda \pi_\perp u_j + (1-S^\lambda)\pi_\perp u_j.$$

The low-frequency part $S^\lambda\pi_\perp u_j$ is controlled in $L^2(L^\infty)$ by the Strichartz estimate of Proposition 30, and thus in terms of $T_\star$, $\|u_{|t=0}\|_{L^2}$, $\|\Pi_\perp u_{|t=0}\|_{H^1}$ and $\|v_{|t=0}\|_{L^2\cap L^\infty}$, as noticed in Section 4.2 (see (32)),

$$\|S^\lambda \pi_\perp u_j(t)\|_{L^\infty} \le C(U_0,T_\star)\alpha(t), \quad \alpha\in\mathcal{C}([0,T_\star[),\ \|\alpha\|_{L^2(0,T_\star)} \le \sqrt{\ln\lambda}.$$

Concerning the high-frequency part, we simply have, for $w=\pi_\perp u_j(t)$ and all $\varepsilon>0$,

$$\begin{aligned}
\|(1-S^\lambda)w\|_{L^\infty} \le& \|\mathcal{F}((1-S^\lambda)w)\|_{L^1} \le \int_{|\xi|\ge\lambda}|\hat{w}|d\xi\\
\le& \lambda^{-(1/2-\varepsilon)}\int_{|\xi|\ge\lambda}|\xi|^{-(3/2+\varepsilon)}|\xi|^2|\hat{w}|d\xi \le C_\varepsilon \lambda^{-(1/2-\varepsilon)}\|w\|_{H^2}.
\end{aligned}$$

The value $\varepsilon = 1/4$ is admissible for what follows. Gathering the above estimates and plugging them into (33), we get, for all $\lambda \geq e$, with $C = C(U_0, T_\star)$,

$$\frac{\mathrm{d}}{\mathrm{dt}} \left(\|U(t)\|_{H^2}^2 \right) \leq C \left(\alpha(t) + \ln(2 + \|U(t)\|_{H^2}) + \lambda^{-1/4} \|U(t)\|_{H^2} \right) \|U(t)\|_{H^2}.$$

The weight $\lambda^{-1/4}$ allows to balance the quadratic growth of $\|U(t)\|_{H^2}^2$, while $\|\alpha\|_{L^2(0,T_\star)} \sim \sqrt{\ln \lambda}$ does not blow up too fast. Precisely, choosing

$$\lambda = \max \left(e, \|U(t)\|_{H^2}^4 \right),$$

and setting $\phi(t) = \|U(t)\|_{H^2}^2$, we have

$$\begin{aligned} \phi \in \mathcal{C}^1([0, T_\star[, [0, +\infty[), \quad \phi' &\leq C(\alpha + \ln(2 + \sqrt{\phi}))\sqrt{\phi} \\ &\leq C(\alpha + \ln(2 + \phi))(2 + \phi), \end{aligned}$$

up to changing the constant C. In the same way, we may suppose that $\|\alpha\|_{L^2(0,T_\star)} \leq \sqrt{\ln(2 + \phi)}$. As a consequence,

$$(\ln(2 + \phi))' \leq C(\alpha + \ln(2 + \phi)).$$

Then, Gronwall's Lemma and Cauchy-Schwarz inequality imply

$$\begin{aligned} \ln(2 + \phi(t)) &\leq e^{Ct} \ln(2 + \phi(0)) + C \int_0^t e^{C(t-t')} \alpha(t') dt' \\ &\leq e^{CT_\star} \ln(2 + \phi(0)) + e^{CT_\star} \sqrt{\frac{C}{2}} \sqrt{\ln(2 + \phi(t))}. \end{aligned}$$

This quadratic inequality shows that $\sqrt{\ln(2 + \phi(t))}$ is bounded in terms of C, $T_\star$ and $\phi(0)$ only, and the proof is complete. □

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