A Handbook of Mathematical Discourse

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Preface

You may click on words and phrases in red for more information. In this version if a phrase in red is split across two lines you must click on the part on the first line. I hope to repair this for the final version.

About this Handbook

Overview

This Handbook is a report on mathematical discourse. Mathematical discourse includes the special dialect of English mathematicians use to communicate mathematical reasoning and the vocabulary that describes the behavior of mathematicians and students when doing mathematics as well as their attitudes towards various aspects of mathematics.

The book is a report on language usage in mathematics and on the difficulties students have with the language; it is not a text on how to write mathematics. The usage is determined by citations, quotations from the literature, the method used by all reputable dictionaries. The descriptions of the problems students have are drawn from the mathematics education literature and the author's own observations.

The Handbook is intended for *teachers of college-level mathematics*, to provide some insight into some of the difficulties their students have with mathematical language, and for *graduate students and upper-level undergraduates* who may find clarification of some of the difficulties they are having as they learn higher-level mathematics.

The earliest dictionaries of the English language listed only "difficult" words (see [Landau, 1989] , pages 38–43). Dictionaries such as Dr. Johnson's that attempted completeness came later. This Handbook is more like the earlier dictionaries, with a focus on usages that cause problems for those who are just beginning to learn how to do abstract mathematics. It is not like a proper dictionary in another way as well: I include words and phrases describing behavior while doing mathematics and attitudes towards various aspects of mathematics that I think should be more widely used.

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Someday, I hope, there will be a complete dictionary based on extensive scientific observation of written and spoken mathematical English, created by a collaborative team of mathematicians, linguists and lexicographers. This Handbook points the way to such an endeavor. However, its primary reason for being is to provide information about the language to instructors and students that will make it easier for them to explain, learn and use mathematics.

In the remainder of the Preface, I discuss some special aspects of book in more detail. Several phrases are used that are described in more detail under that heading in the body of the book. In particular, be warned that the definitions in the Handbook are dictionary-style definitions, not mathematical definitions, and that some familiar words are used with with technical meanings from logic, rhetoric or linguistics.

Point of view

This Handbook is grounded in the following beliefs.

The mathematical register   Mathematicians speak and write in a special register suitable for communicating mathematical arguments. In this book it is called the mathematical register. The mathematical register uses special technical words, as well as ordinary words, phrases and grammatical constructions with special meanings that may be different from their meaning in ordinary English.

The standard interpretation   There is a standard interpretation of the mathematical register, in the sense that at least most of the time most mathematicians would agree on the meaning of most statements made in the register. Students have various other interpretations of particular constructions used in the mathematical register, and one of their tasks as students is to learn how to extract the standard interpretation from what is said and written. One of the tasks of instructors is to teach them how to do that.

Coverage

The words and phrases listed in the Handbook are heterogeneous. The following list describes the main types of entries in more detail.

Technical vocabulary of mathematics: Words and phrases that name mathematical objects, relations or properties. This is not a dictionary of mathematics, and most such

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words ("semigroup", "Hausdorff space") are *not* included. What are included are words that cause students difficulties and that occur in courses through first year graduate mathematics. *Examples:* divide, equivalence relation, function, include, positive. I have also included briefer references to words and phrases with multiple meanings.

Logical signalers: Words, phrases and more elaborate syntactic constructions of the mathematical register that communicate the logical structure of a mathematical argument. *Examples:* if, let, thus.

Types of prose: Descriptions of the types of mathematical prose, with discussions of special usages concerning them. *Examples:* definitions, theorems, labeled style.

Technical vocabulary from other disciplines: Some **technical** words and phrases from rhetoric, linguistics and mathematical logic used in explaining the usage of other words in the list. These are included for completeness. *Examples:* metaphor, register, disjunction, universal quantifier.

Warning: The words used from other disciplines often have ordinary English meanings as well. In general, if you see a familiar word in red, you probably should look it up to see what I mean by it before you flame me based on a misunderstanding of my intention! Some words for which this may be worth doing are: context, elementary, formal, identifier, interpretation, name, precondition, representation, symbol, term, type, variable.

Cognitive and psychological phenomena   Names of the phenomena connected with learning and doing mathematics. *Examples:* mental representation, malrule, reification. Much of this (but not all) is the terminology of the cognitive science or mathematical education community. The entries attitudes, behaviors, and myths list phenomena for which I have not been clever enough to find or invent names.

In contrast to computer people, mathematicians rarely make up words and phrases that describe our attitudes, behavior and mistakes, and I think that is a fault to be remedied. I gave an argument for naming types of behavior in [Wells, 1995] .

Words mathematicians **should** *use:*   This category overlaps the preceding categories. Some of them are my own invention and some come from math education and other disciplines. Words I introduce are always marked as such.

General academic words: Phrases such as "on the one hand ... on the other hand" are familiar parts of a general academic register and are not special to mathematics.

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These are generally not included. However, the boundaries for what to include are certainly fuzzy, and I have erred on the side of inclusivity.

Although the entries are of different types, they are all in one list with lots of cross references. This mixed-bag sort of list is suited to the purpose of the Handbook, to be an aid to instructors and students. The "complete dictionary based on scientific observation" mentioned here may very well be restricted quite properly to the mathematical register.

Descriptive and Prescriptive

Linguists distinguish between " **descriptive**" and " **prescriptive**" treatments of language. A descriptive treatment is intended to describe the language as it is actually used, whereas a prescriptive treatment provides rules for how the author thinks it should be used. This text is mostly descriptive. It is an attempt to describe accurately the language actually used by American mathematicians in the mathematical register as well as in other aspects of communicating mathematics, rather than some ideal form of the language that they should use. Occasionally I give opinions about usage; they are carefully marked as such.

In particular, it follows that it misses the point of the Handbook to complain that some usage should not be included because it is wrong.

Citations

Entries are supported when possible by " **citations**", that is, quotations from textbooks and articles about mathematics. This is in accordance with standard dictionary practice ([Landau, 1989] , pages 151ff). The sources are mostly at the college and early graduate level. They are included in the book in a numbered list beginning on page 503. A reference to a citation (n) means to the nth citation in that list. (In the hypertext version you can jump directly to a citation by clicking on the reference.)

I found more than half the citations on **JSTOR**, a server on the web that provides on-line access to eleven mathematical journals. I obtained access to JSTOR via the server at Case Western Reserve University.

Most of the examples given in the entries are *not* citations. My intent is that the examples be easy to understand for students beginning to study abstract mathematics, as

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well as free of extraneous details.

Presentation

Superscripted cross references

In the printed version of this book, every word or phrase in the text in this sans serif typeface (except this one!) is defined or discussed on the page(s) listed in the superscript. The page listed in that index shows the place where the item is actually discussed. For example, as you can see, writing "the" takes you to the entry for definite article, not the entry for "the", which says merely, "See definite article".

The hypertext versions, of course, don't show the superscripted cross reference; in those versions you can click on the marked words in this paragraph to see the definition. In particular, if you click on "the", you will see the discussion of the definite article.

The TEX and Awk code that accomplishes the superscripted cross references and the hypertext versions will be made available when it is sufficiently presentable to be seen in public.

Acknowledgments

I am grateful for help from many sources:
- Case Western Reserve University, which granted the sabbatical leave during which I prepared the first version of the book, and which has continued to provide me with electronic and library services, especially JSTOR, after I retired.
- Oberlin College, which has made me an affiliate scholar; I have made extensive use of the library privileges this status gave me.
- Helpful information and corrections from or discussions with Marcia Barr, Anne Brown, Gerard Buskes, Laurinda Brown, Christine Browning, Iben M. Christiansen, Geddes Cureton, Tommy Dreyfus, Susanna Epp, Jeffrey Farmer, Cathy Kessel, Leslie Lamport, Leon Sterling, Gary Tee, Owen Thomas Jerry Uhl, Guo Qiang Zhang, and especially Atish Bagchi and Michael Barr.

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- Many of my friends, colleagues and students who have (often unwittingly) served as informants or guinea pigs.
- The many interesting discussions that have been taking place on the **mathedu mailing list**. You can subscribe to `mathedu` by sending email to `majordomo@warwick.ac.uk` saying: `subscribe mathedu` ⟨type in your email address here⟩.

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Entries

a, an See indefinite article.

abstraction   An **abstraction** of a concept C is a concept C' that includes all instances of C and that is constructed by taking as axioms certain statements that are true of all instances of C.

Example 1   The concept of "group" is an abstraction of the concept of the set of all symmetries of an object. The group axioms are all true statements about symmetries when the binary operation is taken to be composition of symmetries.

Other examples are given under model and in Remark **??** under bound variable. See also the discussions under definition, generalization and representation.

References [Dreyfus, 1992] , [Thompson, 1985] .

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abuse of notation   A phrase used to refer to various types of notation that is incomplete or wrong if one insists that all symbolic expressions have compositional semantics. Two common types of abuse of notation are suppression of parameters and synecdoche (which overlap), and examples are given under those headings. Other usage is sometimes referred to as abuse of notation, for example identifying two structures along an isomorphism between them.

  Citations (BusFisMar89.8), (Tei91.494).

  Remark 1   The phrase "abuse of notation' appears to me (but not to everyone) to be deprecatory or at least apologetic, but in fact some of the uses, particularly suppression of parameters, are necessary for readability. The phrase may be a calque on a French phrase, but I don't know its history. The English word "abuse" is stronger than the French word "abus".

Acknowledgments Marcia Barr.

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affirming the consequent   The fallacy of deducing P from $P \Rightarrow Q$ and Q. Also called the **converse error**. This is a fallacy in mathematical reasoning.

Example 1   The student knows that if a function is differentiable, then it is continuous. He concludes that the absolute value function is differentiable, since it is clearly continuous. This is incorrect mathematical reasoning.

Citation (Epp95.36), (Ful89.248).

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algebra   This word has many different meanings in the school system of the USA, and college math majors in particular may be confused by the differences.

High school algebra is primarily algorithmic and concrete in nature.

College algebra is the name given to a college course, perhaps remedial, covering the material covered in high school algebra.

Linear algebra may be a course in matrix theory or a course in linear transformations in a more abstract setting.

A college course for math majors called **algebra**, **abstract algebra**, or perhaps **modern algebra**, is an introduction to groups, rings, fields and perhaps modules. It is for many students the first course in abstract mathematics and may play the role of a filter course. In some departments, linear algebra plays the role of the first course in abstraction.

Universal algebra is a subject math majors don't usually see until graduate school. It is the general theory of structures with n-ary operations subject to equations, and is quite different in character from abstract algebra.

algorithm   An **algorithm** is a specific set of actions that when carried out on data (input) of the allowed type will produce an output. This is the meaning in mathematical discourse. In texts on the subject of algorithm, the word is generally given a mathematical definition, turning an algorithm into a mathematical object (compare the uses of proof). Here, I will refer to that second sense of algorithm as **program** or **code** in order to keep the two ideas separate. This is not in any way standard usage.

Example 1   One can write a program in Pascal and another one in C to take a list with at least three entries and swap the second and third entries. There is a sense in which the two programs, although different as programs, implement the "same" abstract algorithm.

Example 2   One might express a simpleminded algorithm for calculating a zero of a function $f(x)$ using Newton's Method by saying

"Start with a guess x and calculate $\dfrac{f(x)}{f'(x)}$ repeatedly until the answers get sufficiently close together or the process has gone on too long."

One could spell this out in more detail this way:

1. Choose a stopping number ϵ, the maximum number of iterations N, and a guess s.
2. Let $n = 0$.
3. If $f'(s) = 0$ then stop with the message "error".
4. Replace n by $n + 1$.
5. If $n > N$, then stop with the message "too many iterations".
6. Let $r = \dfrac{f(s)}{f'(s)}$.
7. If $|r - s| < \epsilon$ then stop; otherwise go to step 3 with s replaced by r.

Observe that neither description of the algorithm is in a programming language, but that the second one is precise enough that it could be translated into most programming languages quite easily. Nevertheless, it is not a program.

Citation (Bur86.423). Note that the algorithm in the citation is intermediate in precision between the two forms of the Newton algorithm given in Example 2.

Remark 1   It appears to me that it is the naive concept of abstract (if you will) algorithm given in the preceding examples that is referred to by the word "algorithm" as used in mathematical discourse. In particular, the mathematical definitions of algorithm than have been given in the theoretical computer science literature all introduce a mass of syntactic detail that is irrelevant for understanding particular algorithms, although the precise syntax is probably necessary for *proving theorems* about algorithms, such as Turing's theorem on the existence of a noncomputable function.

This following statement by Pomerance [1996] (page 1482) is evidence for this view on the use of the word "algorithm": "This discrepancy was due to fewer computers being used on the project and some 'down time' while code for the final stages of the algorithm was being written." Pomerance clearly distinguishes the algorithm from the code, although he might not agree with all the points made in this article.

Remark 2   A perhaps more controversial point can be made concerning Example 1. A computer program that swaps the second and third entries of a list might do it by changing the values of pointers or alternatively by physically moving the entries. (Compare the discussion under alias). They might even use one method for some types of data (varying-length data such as strings, for example) and the other for other types (fixed-length data). I contend that in some naive sense the two methods still implement the same algorithm at some level of abstraction.

Remark 3   An "algorithm" in the meaning given here appears to be a type of process as that word is used in the APOS approach to describing mathematical understanding. It seems to me that any algorithm fits their notion of process. The converse would not be true because a process need not always terminate, but I would argue that that is the only reason the converse would not be true.

See also overloaded notation.

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algorithm addiction   Many students have the attitude that a problem must be solved or a proof constructed by a algorithm. They become quite uncomfortable when faced with problem solutions that involve guessing or conceptual proofs that involve little or no calculation.

Example 1   Recently I gave a problem in my Theoretical Computer Science class which to solve required finding the largest integer n for which $n! < 10^9$. Most students solved it correctly, but several wrote apologies on their paper for doing it by trial and error. Of course, trial and error *is* a method.

See also Example 1 under look ahead and the examples under conceptual.

Remark 1   Students at a more advanced level may feel insecure in the case where they are faced with solving a problem for which they know there is no known feasible algorithm (something which happens mostly in senior and graduate level classes). I have seen this reaction when I asked whether two particular groups are isomorphic. I have even known graduate students who reacted badly to this situation, but none of them got through qualifiers.

alias   The symmetry of the square illustrated by the figure below can be described in two different ways.
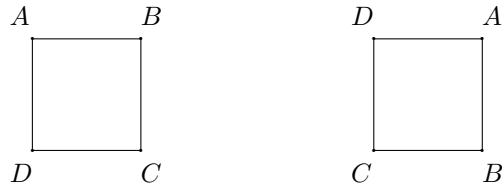
a) The corners of the square are *relabeled*, so that what was labeled A is now labeled D. This is called the **alias** interpretation of the symmetry.

b) The square is *turned*, so that the corner labeled A is now in the upper right instead of the upper left. This is the **alibi** interpretation of the symmetry.

References   These names originated in [Mac Lane and Birkhoff, 1993] , where they are applied to linear transformations.

See also permutation.

Acknowledgments   Michael Barr.

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alibi   See alias.

all   Used to indicate the universal quantifier. Examples are given under universal quantifier.

   Remark 1 [Krantz, 1997] , page 36, warns against using "all" in a sentence such as "All functions have a maximum", which suggests that every function has the same maximum. He suggests using each or every instead. (Other writers on mathematical writing give similar advice.) The point here is that the sentence means $\forall f \exists m(m$ is a maximum for $f)$, not $\exists m \forall f(m$ is a maximum for $f)$. See order of quantifiers and Vulcanism.

   Citation (Pow74.264).

   I have not found a citation of the form "All X have a Y" that *does* mean every X has the same Y, and I am inclined to doubt that this is ever done. ("All" is however used to form a collective plural – see under collective plural for examples.) This does not mean that Krantz's advice is bad.

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always   Used in some circumstances to indicate universal quantification.

Example 1

"A prime greater than 2 is always odd."

This means, "Every prime greater than 2 is odd."

Unlike words such as all and every, the word "always" is attached to the predicate instead of to the noun phrase. See also never.

Remark 1   As the Oxford English Dictionary shows, this is a very old usage in English.

Citation (GibBra85.691).

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analogy   An analogy between two situations is a similarity between some part of one and some part of the other. Analogy, like metaphor, is a form of conceptual blend.

Mathematics often arises out of analogy: Problems are solved by analogy with other problems and new theories are created by analogy with older ones. Sometimes a perceived analogy can be put in a formal setting and becomes a theorem. Analogy in problem solving is discussed in [Hofstadter, 1995] .

Remark 1   An **argument by analogy** is the claim that because of the similarity between certain parts there must also be a similarity between some other parts. Analogy is a powerful tool that *suggests* further similarities; to use it to argue for further similarities is a fallacy.

anaphora   See coreference.

and   The word "and" between two assertions P and Q produces the conjunction of P and Q.

    Example 1   The assertion

    "x is positive and x is less than 10."

is true if both these statements are true: x is positive, x is less than 10.

    The word "and" can also be used between two verb phrases to assert both of them about the same subject.

    Example 2   The assertion of Example 1 is equivalent to the assertion

    "x is positive and less than 10."

See also both.

    Citations (BalYou77.451), (VanLutPrz77.435).

    The word "and" may occur between two noun phrases as well. In that case the translation from English statement to logical statement involves subtleties.

    Example 3   "I like red or white wine" means "I like red wine and I like white wine". So does "I like red and white wine". But consider also "I like red and white candy canes"!

    Example 4   "John and Mary go to school" means the same thing as "John goes to school and Mary goes to school". "John and Mary own a car" (probably) does not mean "John owns a car and Mary owns a car". Consider also the possible meanings of "John and Mary own cars".

    Example 5   In an urn filled with balls, each of a single color, "the set of red and white balls" is the same as "the set of red or white balls".

    See also the discussion under or.

    Remark 1   The preceding examples illustrate that mnemonics of the type "when you see 'and' it means intersection" *cannot work*; the translation problem requires genuine understanding of both the situation being described and the mathematical structure.

    In sentences dealing with physical objects, "and" also may imply a temporal order (he lifted the weight and dropped it, he dropped the weight and lifted it), so that in contrast to the situation in mathematical assertions, "and" may not be commutative in talking about other things. This may be because mathematical objects are eternal.

As this discussion shows, to describe the relationship between English sentences involving "and" and their logical meaning is quite involved and is the main subject of [Kamp and Reyle, 1993] , Section 2.4. Things are even more confusing when the sentences involve coreference, as many examples in [Kamp and Reyle, 1993] illustrate.

Acknowledgments The examples given above were suggested by those in the book just referenced, those in [Schweiger, 1996] , and in comments by Atish Bagchi and Michael Barr.

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angle bracket   **Angle brackets** are the symbols "$\langle$" and "$\rangle$". They are used as outfix notation to denote various constructions, most notably an inner product as in $\langle v, w \rangle$.

Terminology   Angle brackets are also called   **pointy brackets**.
    Citation (Mea93.387).

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anonymous notation   See structural notation.

another planet   Sometimes an author or teacher will give a different definition to a term that has acquired a reasonably standard meaning. This may even be done without warning the reader or student that the definition is deviant. I would say that the person doing this is on **another planet**; that author has no sense of being in a community of scholars who expect to have a common vocabulary.

Example 1   In recent years, authors of high school and lower-level college texts commonly write $A \subseteq B$ to mean that A is included in B. *Citation:* (Str93.3). Some of these write $A \subset B$ to mean that A is properly included in B (citations (GraTre96.234), (Sol95.144)), thereby clashing with the usage in research literature. This was probably the result of formal analogy.

Using $A \subset B$ to mean A is properly included in B seems to be much less common that the usage of "$\subseteq$" and in my opinion should be deprecated.

Another example is Example 2 under semantic contamination.

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antecedent   The hypothesis of a conditional assertion.

any   Used to denote the universal quantifier. See also arbitrary.

APOS   The **APOS** approach to the way students learn mathematics analyzes a student's understanding of a mathematical concept as developing in (at least) three stages: **action**, **process**, **object**. The "S" in "APOS" stands for   **schema**. I will describe these four ideas in terms of computing the value of a function, but the ideas are applied more generally than in that way. This discussion is oversimplified but, I believe, does convey the basic ideas in rudimentary form. The discussion draws heavily on [DeVries, 1997] .

A student's understanding is at the *action stage* when she can carry out the computation of the value of a function in the following sense: after performing each step she knows what the next step is. The student is at the *process stage* when she can conceive of the process as a whole, as an algorithm, without actually carrying it out. She is at the *mathematical object* stage when she can conceive of the function as a entity in itself on which mathematical operations (for example differentiation) can be performed. A student's **schema** for any piece of mathematics is a coherent collection of actions, processes and objects that she can bring to bear on problems in that area (but see compartmentalization).

A brief overview of this theory is in [DeVries, 1997] , and it is discussed in detail in [Thompson, 1994] , pp. 26ff and [Asiala *et al.*, 1996] , pp. 9ff. See also [Sfard, 1992] (who gives a basic discussion of mathematical objects in the context of functions), [Carlson, 1998] , [Dubinsky and Harel, 1992] , the discussion of object and process in [Hersh, 1997b] , pages 77ff, and [Piere and Kieren, 1989] .

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arbitrary   Used to emphasize that there is no restriction on the mathematical structure referred to by the noun that follows. One could usually use any in this situation instead of "arbitrary".

Example 1   "The equation $x^r x^s = x^{r+s}$ holds in an arbitrary semigroup, but the equation $x^r y^r = (xy)^r$ requires commutativity."

Citations (KupPri84.86), (MorShaVal93.749).

In a phrase such as "Let S be an arbitrary set" the word **arbitrary** typically signals an expectation of an upcoming universal generalization. "Any" could be used here as well.

Citation (Str93.7), where the phrase begins a proof of the associative law for intersection.

Difficulties Students are frequently bothered by constructions that seem arbitrary. Some examples are discussed under yes it's weird.

argument [1]   The input to a function may be called the **argument**.
   Citation (Zal75.813).

argument [2]   A proof may be called an **argument**.

Remark 1   The colloquial meaning of "argument" is a disagreement, perhaps with a connotation of unpleasantness. I am not aware that this has caused semantic contamination among students.

Citation (AldDia86.333), (Oss79.5).

arity   The **arity** of a function is the number of arguments taken by the function. The word is most commonly used for symbols denoting functions.

Example 1   The arity of sin is one.

Example 2   The arity of $+$ is two. It takes two arguments.

Remark 1   A function that takes n inputs is also called a **function of** n **variables**. In using the notation given here the order in which the variables are listed is important; for example, one cannot assume in general that $f(2,3) = f(3,2)$.

Remark 2   A function of two variables may be analyzed as a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ where $\mathbb{R} \times \mathbb{R}$ is the cartesian product of $\mathbb{R}$ with itself. In that sense it is a function with *one* input, which must be an ordered pair. I take that point of view in my class notes [Wells, 1997] ; students in my class from time to time ask me why I don't write $f((x,y))$.

Remark 3   One sometimes finds functions with variable arity. For example, one might use MAX for the maximum of a list, and write $\mathrm{MAX}(3,8,5) = 8$ or $\mathrm{MAX}(9,9,-2,5) = 9$.

Example 3   Computer languages such as Lisp and Mathematica® have some functions with variable arity. The expression $+(3,5,5)$ in Lisp evaluates to 13, and so does the expression `Sum[3,5,5]` in Mathematica.

Of course, one might take a point of view here analogous to that of Remark 2 and say that MAX has *one* input which must be a list.

In general, variable arity is possible only for functions written in prefix or postfix notation *with* delimiters. When the symbol for addition (and similar symbols) is written in infix, Polish or reverse Polish notation, the symbol must have exactly two arguments. Thus the symbol $+$ in Mathematica has arity fixed at 2.

Citations (ArbMan74.169).

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article   The **articles** in English are the indefinite article "a" (with variant "an") and the definite article "the". Most of the discussion of articles is under those heads.

   Remark 1   Both articles can cause difficulties with students whose native language does not have anything equivalent. A useful brief discussion aimed at such students is given by [Kohl, 1995] . The discussion in this Handbook is restricted to uses that cause special difficulty in mathematics.

assertion   An **assertion** or **statement** is a symbolic expression or English sentence (perhaps containing symbolic expressions), that may contain variate identifiers, which becomes definitely true or false if determinate identifiers are substituted for all the variate ones. If the assertion is entirely symbolic it is called a **symbolic assertion** or (in mathematical logic) a **formula** or a **predicate**. Contrast with term.

The pronunciation of a symbolic assertion may vary with its position in the discourse. See parenthetic assertion.

Example 1   "2+2=4" is an assertion. It contains no variate identifiers. In mathematical logic such an assertion may be called a **sentence** or proposition.

Example 2   "$x > 0$" is an assertion. The only variate identifier is x. The assertion becomes a true statement if 3 is substituted for x and a false statement if -3 is substituted for x.

By contrast, "$x + 2y$" is not an assertion, it is a term; it does not become true or false when numbers are substituted for x and y, it merely becomes an expression denoting a number.

Example 3   The sentence

"Either $f(x)$ is positive or $f(2x)$ is negative."

is an assertion. It is not a symbolic assertion, which in this Handbook means one that is entirely symbolic. The variables are f and x (this is discussed further under variable.) The assertion becomes true if cos is substituted for f and $\pi/2$ is substituted for x. It becomes false if sin is substituted for f and 0 is substituted for x.

Remark 1   It is useful to think of an assertion as a function with "true" and "false" as values, defined on a complicated domain consisting of statements and the possible values of their free variables.

assume   See let.

assumption   The hypothesis of a conditional assertion is sometimes called the **assumption**.

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at most   For real numbers x and y, the phrase "x is at most y" means $x \leq y$.

Difficulties   Many students, including native English speakers (how many depends in part on the educational institution), do not understand this phrase. Some of them also don't understand "at least" and "not more than".

   Citation (Bry93.62), (Mau78.575).

attitudes   Instructors, students and laymen have certain **attitudes** towards mathematics and its presentation that I think deserve names. A few are listed in this Handbook:

algorithm addiction	Luddism	Vulcanism
another planet	mathematical mind	walking blindfolded
fundamentalist	myths	yes it's weird
guessing	osmosis theory	
light bulb mistake	Platonism	

Here are some attitudes that needs names:

(a) I never would have thought of that   Example 1 under look ahead discusses the example in [Olson, 1998] of deriving a trig identity from the Pythagorean identity. One student, faced with the first step in the derivation, dividing the equation by c, said, "How would I ever know to divide by c?" I have noticed that it is common for a student to be bothered by a step that he feels he could not have thought of. My response in class to this is to say: Nevertheless, you *can* understand the proof, and *now you know a new trick.*

(b) No expertise required   There seem to be subjects about which many educated people both have strong opinions and apparently are not aware that there is a body of knowledge connected with the subject. English usage is such a subject in the USA: many academicians who have never read a style book and know nothing about the discoveries concerning grammar and usage that have been made in this century nevertheless are eloquent in condemning or upholding split infinitives, commas after the penultimate entry in a series, and the like.

Happily or not, mathematics is not one of these bodies of knowledge. Non-mathematicians typically don't believe they know much about mathematics (some engineers are an exception).

On the other hand, many mathematicians take this attitude towards certain subjects; programming is one, and another is mathematics education.

(c) I had to learn it so they should learn it   It is noticeable that in curriculum committees professors strenuously resist relaxing a requirement that was in effect when they were

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students. In mathematical settings this tends to be expressed in sentences such as, "It is inconceivable that anyone could call himself a math major who has never had to integrate $\cos^3 x$" (or whatever). This is clearly related to the you don't know shriek and to Luddism.

(d) I can't even balance my checkbook Many people who have had little association with mathematics believe that mathematics is about numbers and that mathematicians spend their time calculating numbers.

See also behaviors and myths.

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back formation   One may misread a word, perhaps derived from some root by some (often irregular) rule, as having been derived from some other nonexistent root in a more regular way. Using the nonexistent root creates a word called a **back formation**.

   Example 1   The student who refers to a "matricee" has engaged in back formation from "matrices", which is derived irregularly from "matrix". See plural for more examples.

bad at math   See mathematical mind.

bar   A line drawn over a single-symbol identifier is pronounced "bar". For example, $\bar{x}$ is pronounced "x bar". Other names for this symbol are "macron" and "vinculum".

Acknowledgments   Atish Bagchi.

barred arrow notation   A notation for specifying a function. It uses a barred arrow with an identifier for the input variable on the left and an expression or name that describes the value of the function on the right.

Example 1   "The function $x \mapsto x^2$ has exactly one critical point." Compare lambda notation.

Citation (MacBir93.43).

Remark 1   One can substitute input values of the correct type into barred arrow expressions, in contrast to lambda expressions (see bound variable).

Example 2   One can say

"Under the function $x \mapsto x^2$, one may calculate that $2 \mapsto 4$."

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be   The verb "to be" has many uses in the English language. Here I mention a few common usages in mathematical texts.

(a) Has a property Examples given under property.

(b) To give a definition In a statement such as these:
   1.   "A group is **Abelian** if $xy = yx$ for all elements x and y."
   2.   "A semigroup is a set with an associative multiplication defined on it."
   "is" connects a definiendum with the conditions defining it. See mathematical definition for other examples.
        Citations (Cho99.444), (Gal94.352).

(c) Is identical to The word "is" in the statement

   "An idempotent function has the property that its image is its set of fixed points."

asserts that two mathematical descriptions ("its image" and "its set of fixed points") denote the same mathematical object. This is the same as the meaning of "=", and is a special case of meaning (a).
        Citations (Bar96.627), (Bri93.782), (Duk97.193).

(d) Asserting existence See existential quantifier for examples.
        Citation (LewPap98.20), (Rib95.391).

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behaviors   Listed here are a number of behaviors that occur among mathematicians and students. Some of these phenomena have names (in some cases I have named them) and are discussed under that name. Many phenomena that need names are listed below. See also attitudes and myths.

Computer programmers have many names for both productive and unproductive behaviors and attitudes involving programming, many of them detailed in [Raymond, 1991] (see "creationism", "mung" and "thrash" for example). Mathematicians should emulate them. Having a name for a phenomenon makes it more likely that you will be aware of it in situations where it might occur and it makes it easier for a teacher to tell a student what went wrong.

I don't know of any literature from the mathematical education community about the idea that naming dysfunctional behavior makes it easier to avoid. (See [Wells, 1995] .)

(1) Behaviors that have names

affirming the consequent	insight
another planet	jump the fence
compartmentalization	malrule
covert curriculum	sanity check
denying the hypothesis	self-monitoring
enthymeme	semantic contamination
existential bigamy	symbolitis
extrapolate	synecdoche
formal analogy	yes it's weird
grasshopper	you don't know shriek

(2) Behaviors that need names

(a) Excluding special cases Usually, a generalization of a mathematical concept will be defined in such a way as to include the special case it generalizes. Thus a square is a rectangle and a group is a semigroup. Students sometimes exclude the special case, saying "rectangle" to mean that the figure is *not* a square, or asking something such as "Is it a group or a semigroup?" I have seen a discussion of this in the mathematical education

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literature but have lost the reference. I understand also that some high school texts specifically state that a square is not a rectangle; I would appreciate verification of this. **References needed.**

(b) Missing relational arguments Using a binary relation word with only one argument. For an example, see disjoint. Students often do this with "relatively prime".

(c) Forgetting to check trivial cases

  Example 1   A proof about positive integers that begins,

  "Let p be a prime divisor of n."

The integer 1 has no prime divisors.

(d) Proving a conditional assertion backward When asked to prove $P \Rightarrow Q$ a student may come up with a proof beginning "If Q ... " and ending " ... therefore P", thus proving $Q \Rightarrow P$. This is distressingly common among students in discrete mathematics and other courses where I teach beginning mathematical reasoning. I suspect it comes from proving equations in high school, starting with the equation to be proved.

(e) Reading variable names as labels An assertion such as "There are six times as many students as professors" is translated by some students as $6s = p$ instead of $6p = s$ (where p and s have the obvious meanings). This is discussed in [Nesher and Kilpatrick, 1990] , pages 101–102. People in mathematical education refer to this as the **student-professor problem**, but I don't want to adopt that as a name; in some sense *every* problem in teaching is a student-professor problem. See sanity check.

(f) Unclassifiable This particular incident has happened to me twice, with two different students: The student became quite upset (much more than merely puzzled) when I said, "Let p be an odd prime." He was bothered because there is only one prime that is *not* odd. The student has some expectation that is being violated but I have no idea what it is.

binary operation   See operation.

boldface   A style of printing that **looks like this**. Section headings are often in boldface, and some authors put a definiendum in boldface. See definition.

both   Used as an intensifier with and to express a conjunction of two assertions.

Example 1   "2 is both even and a prime."

Citations (Ros93.293), (Kra95.40).

"Both" is also used with or to emphasize that it is inclusive.

Example 2   "If m is even and $m = rs$ then either r or s (or both) is even."

Citation (BumKocWes93b.499).

bound identifier   An identifier is **bound** if it occurs in a phrase that translates directly into a symbolic expression in which the identifier becomes a bound variable. This typically occurs with the use of English quantifiers such as all and every, as well as phrases describing sums, products and integrals. An identifier that is not bound is a **free identifier**.

Example 1   "Any increasing function has a positive derivative." The phrase "increasing function" is bound. This sentence could be translated into the symbolic expression "$\forall f\,(\text{INC}(f) \Rightarrow f' > 0)$".

Remark 1   The definition given here reduces the concept of bound noun phrases to what "bound" means in symbolic expressions. I know of no precedent for this in the literature.

Remark 2   Modern linguists use a formalism for studying generalized quantifiers that interpret a bound noun phrase as a set of sets. See [Chierchia and McConnell-Ginet, 1990] , Chapter 9.

See also variate identifier.

bound variable   A variable is **bound** in a symbolic expression if it is within the scope of an operator that turns the symbolic expression into something referring collectively to all the values of the variable (perhaps within limits). The operator is said to **bind** the variable. The operators that can do this include the existential and universal quantifiers, the integral sign, the sum and product notations Σ and Π, and various notations for functions. (See also bound identifier.) A variable that is not bound is free.

Terminology   Bound variables are also called **dummy variables**. The latter phrase has low status.

A key property of a bound variable is that one is not allowed to substitute for it (but see Example 3).

Example 1   In the expression $x^2 + 1$, the x is a free variable. However, in $\int_3^5 x^2 + 1 \, dx$ it is bound by the integral sign.

Example 2   In the symbolic assertion $x > 7$, x is free. In $\forall x (x > 7)$ it is bound by the universal quantifier (resulting in a false statement).

Example 3   This example is more subtle. In the following sentence, intended to define a function,

"Let $f(x) = x^2 + 1$."

the variable x is bound. It is true that one can substitute for the x in the equation to get, for example $f(2) = 5$, but that substitution *changes the character of the statement*, from the defining equation of a function to a statement about one of its values. It is clearer that the variable x is bound in this statement

"Let the function $f(x)$ be defined by $f(x) = x^2 + 1$."

which could not be transformed into

"Let the function $f(2)$ be defined by $f(2) = 2^2 + 1$."

These remarks apply also to the variables that occur in lambda notation, but see Example 2 under barred arrow notation.

Difficulties   Students find it difficult to learn how to use bound variables correctly. They may allow variable clash. They may not understand that which bound variable it is does not matter (except for variable clash); thus $\int_2^5 x^2 \, dx$ and $\int_2^5 t^2 \, dt$ are the same *by their*

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form. They may move a bound variable out of its binder, for example changing $\sum_{i=1}^{n} i^2$ to $i \sum_{i=1}^{n} i$ (which makes it easy to "solve"!). And they may substitute for it, although in my teaching experience that is uncommon. I am not aware of any mathematical education literature on this. **References needed.**

Remark 1 The discussion in Remark 2 under free variable applies to bound variables as well.

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brace   **Braces** are the symbols "{" and "}".

A very common use of braces is in setbuilder notation.

Example 1   The set $\{(x, y) \mid y = x^2\}$ is a parabola in the plane.

Citation (Bri93.782).

They are also used occasionally as bare delimiters and as outfix notation for functions.

Example 2   The expression $6/\{(1^2 + 3^2) - 2^2\}$ evaluates to 1.

Example 3   The fractional part of a real number r is denoted by $\{r\}$.

Citations (Loe91.243), (RabGil93.168), (Sta70a.774).

A left brace may be used by itself in a definition by cases (see the example under cases).

Terminology   Braces are sometimes called **curly brackets**.

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bracket   This word has several related usages.

(a) Certain delimiters In common mathematical usage, **brackets** are any of the delimiters in the list

$$( ) [ ] \{ \} \langle \rangle$$

Some American dictionaries restrict the meaning to square brackets or angle brackets.

(b) Operation The word "bracket" is used in various mathematical specialties as the name of an operation (for example, Lie bracket, Toda bracket, Poisson bracket) in an algebra (often of operators) with a value in another structure. The operation called bracket may use use square brackets, braces or angle brackets to denote the operation, but the usage for a particular operation may be fixed as one of these. Thus the Lie bracket of v and w is denoted by by $[v, w]$. On the other hand, notation for the Poisson and Toda brackets varies.

   Citations (Wal93.786), (GraPugShu94.304), (BenDav94.295).

(c) Quantity The word "bracket" may be used to denote the quantity inside a pair of brackets (in the sense of (a)).

   Example 1   If the expression $(x^2 - 2x + 1) + (e^{2x} - 5)^3$ is zero, then the two brackets are opposite in sign.

   Citation (Ver91.501).

but

(a) And with contrast As a conjunction, "but" typically means the same as "and", with an indication that what follows is surprising or in contrast to what precedes it. This is a standard usage in English, not peculiar to the mathematical register.

Example 1 "5 is odd, but 6 is even."

(b) Introduces new property Mathematical authors may begin a sentence with "But" to indicate that the subject under discussion has an additional property that will now be mentioned, typically because it leads to the next step in the reasoning. This usage may carry with it no thought of contrast or surprise. The property may be one that is easy to deduce or one that has already been derived or assumed. Of course, in this usage "but" still means "and" as far as the logic goes; it is the connotations that are different from the usage in (a).

Example 2 "We have now shown that $m = pq$, where p and q are primes. But this implies that m is composite."

Example 3 (In a situation where we already know that $x = 7$):
  "... We find that $x^2 + y^2 = 100$. But x is 7, so $y = \sqrt{51}$."

See also just.

References [Chierchia and McConnell-Ginet, 1990] , pages 283–284.
  Citations (Ant84.121), (Epp95.2), (Hol95.206).

Acknowledgments Atish Bagchi

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calculate   To **calculate** is to perform symbol manipulation on an expression to arrive at another, perhaps more satisfactory, expression.

Example 1   "Let us calculate the roots of the equation $x^2 - 4x + 1 = 0$."

Example 2   "An easy calculation shows that the equation $x^3 - 5x = 0$ factors into linear factors over the reals."

Example 3

"We may calculate that $\neg(\forall x \exists y(x > y^2))$ is equivalent to $\exists x \forall y(x \leq y^2)$."

Remark 1   Calculation most commonly involves algebraic manipulation, but the rules used may be in some other system, as Example 3 exhibits (first order logic in that case).

Remark 2   It is my impression that nonmathematicians, including many mathematics students, restrict the word "calculation" to symbol manipulation that comes up with a numerical answer. This may be merely the result of the belief that mathematicians deal primarily with numbers (see item (d) under attitudes).

In contrast, I have heard mathematicians refer to calculating some object when the determination clearly involved conceptual reasoning, not symbol manipulation, as in the citation listed below.

See also compute.

Citations (EdeKos95.7),

call   Used to form a definition.

Example 1   "A monoid is called a **group** if every element has an inverse." *Citation:* (Str93.3)

Example 2   "Let $g = h^{-1}fh$. We call g the *conjugate* of f by h." *Citation:* (BasKul90.845), (Cur93.790), (Mar92.739).

Example 3   "We call an integer **even** if it is divisible by 2." *Citation:* (Fre90.705), (LewPap98.20).

Remark 1   Some object to the usage in Example 3, saying "call" should be used only when you are giving a name to the object as in Examples 1 and 2. However, the usage with adjectives has been in the language for centuries. *Citation:* (Luk49.56).

See also algorithm addiction.

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cardinality   The **cardinality** of a finite set is the number of elements of the set. This terminology is extended to infinite sets either by referring to the set as infinite or by using more precise words such as "countably infinite" or "uncountable".

The cardinality of a group or other structure is the cardinality of its underlying set.

Difficulties   Infinite cardinality behaves in a way that violates the expectation of students. More about this under snow. The book [Lakoff and Núñez, 2000] gives a deep discussion of the metaphors underlying the concept of infinity in Chapters 8–10.

Citations (Lor71.753).

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case   The Roman alphabet, the Greek alphabet, and the Cyrillic alphabet have two forms of letters, "capital" or **uppercase**, A, B, C, etc, and **lowercase**, a, b, c, etc. As far as I can tell, case distinction always matters in mathematics. For example, one may use a capital letter to name a mathematical structure and the same letter in lowercase to name an element of the structure.

Remark 1   The font used may also be significant.

Difficulties   American students at the freshman calculus level or below quite commonly do not distinguish uppercase from lowercase when taking notes.

Citations (Oso94.760), (KolBusRos96.111).

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cases   A concept is **defined by cases** if it is dependent on a parameter and the defining expression is different for different values of the parameter. This is also called a **disjunctive definition** or **split definition**.

Example 1   Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & x > 0 \\ -1 & x \le 0 \end{cases}$$

Citation: (MarDanSep98.116).

Difficulties Students often find disjunctive definitions unnatural. This may be because real life definitions are rarely disjunctive. (One exception is the concept of "strike" in baseball.) This requires further analysis.

category   The word "category" is used with two unrelated meanings in mathematics (Baire and Eilenberg-Mac Lane). It is used with a third meaning by some cognitive scientists, meaning roughly what in this book is usually called "concept". What I would call an instance of the concept they call a member of the category, so the focus of the word is a bit more extensional whereas "concept" is more intensional.

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character A **character** is a typographical symbol such as the letter "a" and the digit "3". A symbol in the sense of this Handbook may consist of more than one character.

Example 1 The expression "sin" as in "$\sin \pi = 0$" is a symbol in the sense of this Handbook composed of three characters.

Remark 1 Of course, "character" also has a mathematical meaning.

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check   The symbol "˘" over a letter is commonly pronounced "check" by mathematicians. For example, $\check{x}$ is pronounced "x check". The typographical name for this symbol is "háček".

circumflex   The symbol ˆ is a **circumflex**. Mathematicians commonly pronounce it **hat**: thus $\hat{x}$ is pronounced "x hat".

closed under   A set is **closed under** an operation if the image of the operation is a subset of the set.

Example 1   The set of positive integers is closed under addition but not under subtraction. *Citation:* (Hol95.222).

Acknowledgments   Guo Qiang Zhang.

code   See algorithm.

codomain   The codomain of a function must be a set that contains every value of the function. It may be named in any way that sets are named.

Authors vary much more in the treatment of the codomain than they do in the treatment of the domain. Many authors do not require that a function have a specified codomain; others don't make it clear whether they require it or not.

Even when authors do require specification of the codomain, the specification is often an empty gesture since the text fudges the question of whether two functions with the same domains and same graphs but different codomains are really different. As an example, consider two sets A and B with $A \subseteq B$, and consider the identity function from A to A and the inclusion function $i : A \to B$ defined by $i(a) = a$. Are they the same function? Some texts treat them as different, others don't make it clear whether they are the same or different.

See range and image.

Citation (Bri85.184).

cognitive dissonance   **Cognitive dissonance** is a term introduced in [Festinger, 1957] . It concerns conflicting understandings of some aspect of the world caused by two different modes of learning. The conflict may be resolved by suppressing the results of one of the modes of learning. Two types of cognitive dissonance are discussed in their own articles: formal analogy and semantic contamination.

Cognitive dissonance also occurs when the definition allows behavior unfamiliar from examples; see Example 2 under prototype.

Remark 1   The book [Festinger, 1957] exhibits some astonishingly bigoted attitudes.

References Cognitive dissonance is discussed further in [Bagchi and Wells, 1998a] , [Brown and Dobson, 1996] , [Carkenord and Bullington, 1993] .

Acknowledgments Thanks to Geddes Cureton and Laurinda Brown for suggesting references.

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collective plural   Using the plural of an identifier to refer to the entire collection of items designated by the identifier.

Example 1

"Let H be a subgroup of G. The left cosets of H are a partition of G."
I do not have a citation for this sort of wording , although I have heard people use it.

Example 2   "Let $\mathbb{Q}$ be the rational numbers." *Citation:* (DavPri90.3), (Ros95.504).

Remark 1   It appears to me that the usage shown in the two examples above is uncommon. It probably should be deprecated. Usually a word such as "form" or "constitute" is used, or else one refers to the set of cosets.

Example 3   "The left cosets of H constitute a partition of G."

Example 4   "The rational numbers form a dense subset of the reals."

Example 5   "The set of all cosets of H is a partition of G."

See distributive plural.

Citation (Niv56.83).

References [Lønning, 1997] , [Kamp and Reyle, 1993] , pages 320ff.

college In the United States, a **college** is an institution one attends after graduating from high school (secondary school) that gives (usually) a B.S. or B.A. degree. A **university** also grants these degrees; the name "university" usually, but not always, connotes that the institution also grants other, higher, degrees. The usage is different in most other countries.

In this text, the phrase **college mathematics** denotes what in most other countries could be called "university mathematics". This is not quite correct, since much of the content of American freshman calculus would probably be taught in secondary school (or in a school that one attends between secondary school and university) in other countries.

colon equals   The expression "$:=$" means "(is) defined to be equal to".

Example 1   "$S := \{1, 2, 3\}$ is a finite set." This is a short way of saying:

"Define S to be the set $\{1, 2, 3\}$. Then S is finite."

This usage is not very common, but my impression is that it is gaining ground.

Remark 1   In citations this seems to occur mostly in parenthetic assertions. This may be because it is hard to make an independent assertion that both is non-redundant and does not start with a symbol. Consider

"Let $S := \{1, 2, 3\}$."

(Or "Define ... ") The word "Let" already tells you we are defining S.

Citation (Bar96.631); (Pow96.879). Note that although the colon equals usage is borrowed from computer languages, these two citations come from works in areas outside computing.

Acknowledgments   Gary Tee.

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combination   An r-combination of a set S is an r-element subset of S. "Combination" is the word used in combinatorics. Everywhere else in mathematics, a subset is called a subset.

  Citation (Str93.119).

comma   In symbolic expressions, a comma between symbolic assertions may denote and.

Example 1   The set

$$\{m \mid m = n^2,\, n \in \mathbb{Z}\}$$

denotes the set of squares of integers. The defining condition is: $m = n^2$ and n is an integer.

Citations (Dra95.258), (Gri99.128).

Remark 1   The comma is used the same way in standard written English. Consider "A large, brown bear showed up at our tent".

The comma may also be used to indicate many-to-one coreference,

Example 2   "Let $x, y \neq 0$."

Citation (Niv56.41), (Oso94.760).

Acknowledgments   Michael Barr.

compartmentalization   A student may have several competing ways of understanding a concept which may even be inconsistent with each other. For example, when doing calculus homework, she may think of functions exclusively in terms of defining expressions, in spite of the fact that she can repeat the ordered-pairs definition when asked and may even be able to give an example of a function in terms of ordered pairs, not using a defining expression. In other words, defining expressions are for doing homework except when the question is "give the definition of 'function' "!

This phenomenon is called **compartmentalization**. The student has not constructed a coherent schema for (in this case) "function".

References [Tall and Vinner, 1981] , [Vinner and Dreyfus, 1989] .

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componentwise   See coordinatewise.

compositional   The meaning of a symbolic expression in mathematics is normally determined by the meanings of the symbols that make it up and by their arrangement (the syntax). Some examples are given under symbolic expression; see also Remark 1 under syntax. Such semantics is said to be **compositional** or **synthetic** or **syntax-driven**.

In contrast, the meaning of a word cannot usually be synthesized from its spelling; the relationship between spelling and meaning is essentially arbitrary. As an example, consider the different roles of the letter "i" in the symbol "sin" and in the expression "$3 - 2i$".

The symbolic language of mathematics generally has compositional semantics. Most of the examples of failure of compositionality that I have been able to find are examples of one of these four phenomena (which overlap, but no one of them includes another):

a) context sensitivity.
b) conventions.
c) suppression of parameters.
d) synecdoche.

Examples are given under those headings.

Remark 1   Some symbolic expressions are multivalued, for example

$$\int x^2 \, dx$$

which is determined only up to an added constant. I don't regard this as failure of compositionality; the standard meaning of the expression is multivalued.

The English-language part of the mathematical register also fails to be compositional in certain cases.

Example 1   Texts commonly define an **ordering** to be a reflexive, antisymmetric and transitive relation, and a **strict ordering** using trichotomy. The consequence is that a strict ordering is not an ordering. This sort of thing is common in natural language; see radial concept.

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compute   "Compute" is used in much the same way as calculate, except that it is perhaps more likely to imply that a computer was used.

Remark 1   As in the case of calculate, research mathematicians often refer to computing an object when the process involves conceptual reasoning as well as symbol manipulation.

Citation (How93.8).

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concept   Mathematical concepts given by mathematical definitions always have the following property: an object is an instance of the concept if and only if it has all the attributes required of it by the definition. An object either satisfies the definition or not, and all objects that satisfy the definition have equal logical status.

Mathematical concepts are thus defined by an **accumulation of attributes**. Most definitions in science writing outside of mathematics are *not* by accumulation of attributes. Scientific definitions are discussed in detail in [Halliday and Martin, 1993] , who clearly regard accumulation of attributes as a minor and exceptional method of definition; they mention this process in Example 13 on page 152 almost as an afterthought.

It is a more familiar fact that mathematical concepts are also **crisp**, as opposed to **fuzzy**. An algebraic structure is either a group or it is not, but one can argue about whether Australia is a continent or a large island.

Most human concepts are not given by accumulation of attributes and are not crisp. They typically have internal structure. See [Lakoff, 1986] , especially the discussion in Section 1, and [Pinker and Prince, 1999] . The latter reference distinguishes between **family resemblance categories** and **classical categories**; the latter are those that in my terminology are defined by accumulation of attributes.

Some aspects of human concepts are discussed under prototype and radial concept. See also the discussion under category.

Remark 1   Of course every student's and every mathematician's mental representation of a mathematical concept has more internal structure than merely the accumulation of attributes. Some instances loom large as prototypical and others are called by rude names such as pathological because they are unpleasant in some way.

Students may erroneously expect to reason with mathematical concepts using prototypes the way they (usually unconsciously) reason about everyday concepts. (See generalization.) On the other hand, students with some skill in handling mathematical concepts can shift psychologically between this extra internal structure and the bare structure given by accumulation of attributes, using the first for motivation and new ideas and the second in proofs. This shifting in the general context of human reasoning is discussed in [Pinker and Prince, 1999] , section 10.4.4.

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Remark 2   Many mathematical concepts are abstractions of a prior, non-mathematical concept that may be fuzzy, and one can argue about whether the mathematical definition captures the prior concept. Note the discussions in subsections (3) and (4) under definition.

See also indefinite article.

References [Bagchi and Wells, 1998a] , [Bagchi and Wells, 1998b] , [Vinner, 1992] . Definitions in science in general are discussed by [Halliday and Martin, 1993] pages 148–150, 170ff, 209ff.

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concept image   See mental representation.

conceptual   A proof is **conceptual** if it is an argument that makes use of one's mental representation or geometric insight. It is opposed to a proof by symbol manipulation.

    Example 1   Let m and n be positive integers, and let r be m mod n. One can prove that $\mathrm{GCD}(m, n) = \mathrm{GCD}(m, r)$ by showing that the set of common divisors of m and n is the same as the set of common divisors of n and r (easy); the result follows because the GCD of two numbers is the *greatest* common divisor, that is, the maximum of the set of common divisors of the two numbers.

    I have shown my students this proof many times, but they almost never reproduce it on an examination.

    Example 2   Now I will provide three proofs of the same assertion, adapted from [Wells, 1995] .

    The statement to prove is that for all x, y and z,

$$(x > z) \Rightarrow ((x > y) \vee (y > z)) \tag{1}$$

(a) Conceptual proof We may visualize x and z on the real line as in this picture:
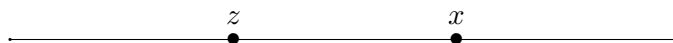
    There are three different regions into which we can place y. In the left two, $x > y$ and in the right two, $y > z$. End of proof.

    This proof is written in English, not in symbolic notation, and it refers to a particular mental representation of the structure in question (the usual ordering of the real numbers).

(b) Symbolic Proof This proof is due to David Gries (private communication) and is in the format advocated in [Gries and Schneider, 1993] . The proof is based on these principles:

P.1 (Contrapositive) The equivalence of $P \Rightarrow Q$ and $\neg Q \Rightarrow \neg P$.

P.2 (DeMorgan) The equivalence of $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$.

P.3 The equivalence in any totally ordered set of $\neg(x > y)$ and $x \leq y$.

In this proof, "$\neg$" denotes negation.

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Proof:

$$(x > z) \Rightarrow ((x > y) \vee (y > z))$$
$$\Leftrightarrow \quad \text{by P.1}$$
$$\neg ((x > y) \vee (y > z)) \Rightarrow \neg(x > z)$$
$$\Leftrightarrow \quad \text{by P.2}$$
$$(\neg(x > y) \wedge \neg(y > z)) \Rightarrow \neg(x > z)$$
$$\Leftrightarrow \quad \text{by P.3 three times}$$
$$((x \leq y) \wedge (y \leq z)) \Rightarrow (x \leq z)$$

which is true by the transitive law.

This proof involves symbol manipulation using logical rules and has the advantage that it is easy to check mechanically. It also shows that the proof works in a wider context (any totally ordered set).

(c) Another conceptual proof The conceptual proof given in (a) provides a geometric visualization of the situation required by the hypothesis of the theorem, and this visualization makes the truth of the theorem obvious. But there is a sense of "conceptual", related to the idea of conceptual definition given under elementary, that does not have a geometric component. This is the idea that a proof is conceptual if it appeals to concepts and theorems at a high level of abstraction.

To a person familiar with the elementary rules of first order logic, the symbolic proof just given becomes a conceptual proof (this happened to me): "Why, in a totally ordered set that statement is nothing but the contrapositive of transitivity!" Although this statement is merely a summary of the symbolic proof, it is enough to enable anyone conversant with simple logic to generate the symbolic proof. Furthermore, in my case at least, it provides an aha experience.

Citations (Rub89.421), (BieGro86.425).

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conceptual blend   A cognitive structure (concept, mental representation or imagined situation) is a **conceptual blend** if it consists of features from two different cognitive structures, with some part of one structure merged with or identified with an isomorphic part of the other structure.

   Example 1   An experienced mathematician may conceive of the function $x \mapsto x^2$ as represented by the parabola that is its graph, or as a machine that given x produces its square (one may even have a particular algorithm in mind). In visualizing the parabola, she may visualize a geometric object, a curve of a certain shape placed in the plane in a certain way, and she will keep in mind that its points are parametrized (or identified with) the set $\{(x, y) \mid y = x^2\}$. The cognitive structure involved with the machine picture will include the set of paired inputs and outputs of the machine. Her complex mental representation of the functions includes all these objects, but in particular the pairs that parametrize the parabola and the input-output pairs of the machine are visualized as being the *same pairs*, the elements of the set $\{(x, y) \mid y = x^2\}$.

   Example 2   A monk starts at dawn at the bottom of a mountain and goes up a path to the top, arriving there at dusk. The next morning at dawn he begins to go down the path, arriving at dusk at the place he started from on the previous day. Prove that there is a time of day at which he is at the same place on the path on both days. Proof: Envision both events occurring on the *same* day, with a monk starting at the top and another starting at the bottom and doing the same thing the monk did on different days. They are on the same path, so they must meet each other. The time at which they meet is the time required. This visualization of both events occurring on the same day is an example of conceptual blending.

   Analogies and metaphors are types of conceptual blends. See also identify.

   Remark 1   A conceptual blend is like an amalgamated sum or a pushout.

References Conceptual blending, analogical mappings, metaphors and metonymies (these words overlap and different authors do not agree on their definitions) are hot topics in current cognitive science. These ideas have only just begun to be applied to the study of mathematical learning. Some of these references are general; those by Presmeg and by Lakoff and Núñez are specific to mathematics: [Lakoff, 1986] , [Fauconnier, 1997] ,

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[Presmeg, 1997b] , [Lakoff and Núñez, 1997] , [Lakoff and Núñez, 1998] , [Lakoff and Núñez, 2000] , [Katz *et al.*, 1998] .

Acknowledgments   The monk example is adapted from [Fauconnier, 1997] , page 151.

conditional assertion   A **conditional assertion** $A \Rightarrow B$ (pronounced A **implies** B) is an assertion formed from two assertions A and B, satisfying the following truth table:

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

The operation "$\Rightarrow$" is called **implication**. Warning: a conditional assertion is often called an implication, as well. *Citation:* (Ros91.6).

In the mathematical register, $A \Rightarrow B$ may be written in many ways. Here are some examples where A is "4 divides n" and B is "2 divides n".

a) If 4 divides n, then 2 divides n. *Citation:* (Bur94.24), (HenLarMarWoo94.213), (VanLutPrz77.435).

b) 2 divides n if 4 divides n. *Citation:* (Bru93.370)

c) 4 divides n only if 2 divides n. *Citation:* (HofTer94.630), (Str89.107).

d) 4 divides n implies 2 divides n. *Citation:* (Kra95.40), (Wit90.144).

e) Suppose [or Assume] 4 divides n. Then 2 divides n.
        Citations (Cop93.480), (GraEntSze94.665).

f) Let 4 divide n. Then 2 divides n. *Citation:* (Ros93.208)

g) A necessary condition for 4 to divide n is that 2 divide n.*Citation:* (HenLarMarWoo94.213), (BuhEisGraWri94.513).

h) A sufficient condition for 2 to divide n is that 4 divide n. *Citation:* (HenLarMarWoo94.213), (BuhEisGraWri94.513).

Remark 1   The word "if" in sentences (a), (b), and (1) can be replaced by "when" or (except for (1)) by "whenever". Note that if has other uses, discussed under that word. The situation with let, assume, and suppose are discussed further in those entries.

Remark 2   Many other English constructions may be *translated* into (are equivalent to) conditional assertions. For example the statement $P \Leftrightarrow$ "Every cyclic group is commutative" is equivalent to the statement "If G is cyclic then it is Abelian" (in a

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context where G is of type "group"). But the statement P is not itself a conditional assertion. See universal quantifier.

Difficulties Students have many difficulties with implication, mostly because of semantic contamination with the usual way "if ... then" and "implies" are used in ordinary English. Some aspects of this are described here.

In the first place, one way conditional sentences are used in ordinary English is to give rules. The effect is that "If P then Q means "P if and only if Q".

Example 1 The sentence

  "If you eat your dinner, you may have dessert."

means in ordinary discourse that if you don't eat your dinner you may not have dessert. A child told this presumably interprets the statement as being in some sort of command mode, with different rules about "if" than in other types of sentences (compare the differences in the use of "if" in definitions and in theorems in the mathematical register.)

Perhaps as a consequence of the way they are used in ordinary English, students often take conditional sentences to be equivalences or even simply read them backward.

Example 2 A student may remember the fact "If a function is differentiable then it is continuous" as saying that being differentiable and being continuous are the same thing, or simply may remember it backward.

Example 3 When asked to prove $P \Rightarrow Q$, some students assume Q and deduce P. This may have to do with the way students are taught to solve equations in high school.

Remark 3 I would recommend that in expository writing about mathematics, if you state a mathematical fact in the form of a conditional assertion, you *always* follow it immediately by a statement explaining whether its converse is true, false or unknown.

Remark 4 Students have particularly difficulty with only if and vacuous implication, discussed under those headings. See also false symmetry.

References The sentence about dessert is from [Epp, 1995] . An analysis of conditionals in ordinary English is given by McCawley [1993] , section 3.4 and Chapter 15. Other more technical approaches are in Section 2.1 of [Kamp and Reyle, 1993] and in Chapter 6 of [Chierchia and McConnell-Ginet, 1990] .

conjunction   A **conjunction** is an assertion P formed from two assertions A and B with the property that P is true if and only if A and B are true. It is defined by the following truth table:

A	B	P
T	T	T
T	F	F
F	T	F
F	F	F

Do not confuse this usage of "conjunction" with the part of speech called "conjunction". Here, a conjunction is a whole sentence.

In the mathematical register, the conjunction of two assertions is usually signaled by connecting the two assertions with and. Examples are given under and.

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connective   In mathematical logic, a **connective** or **logical connective** is a binary operation that takes pairs of assertions to an assertion. The connectives discussed in this text are and, equivalent, imply, and or. Note that some of these connectives are represented in English by conjunctions and others in more complex ways.

  Remark 1   Unary operations such as not are sometimes called connectives as well.

consequent   In a conditional assertion of the form $P \Rightarrow Q$, Q is the **consequent**.

consider   The command "Consider ... " introduces a (usually variable) mathematical object together with notation for the objects and perhaps some of its structure.

      Citation (Mea93.387), (Mol97.531).

constructivism   In mathematics education, this is the name given to the point of view that a student **constructs** her understanding of mathematical concepts from her experience, her struggles with the ideas, and what instructors and fellow students say. It is in opposition to the idea that the instructor in some sense pours knowledge into the student.

Of course, all I have given here are metaphors. However, constructivists draw conclusions concerning teaching and learning from their point of view.

Remark 1   Constructivism is the name of a point of view in the philosophy of mathematics as well, but there is no connection between the two ideas.

Remark 2   "Constructivism" as a philosophy of education may also connote other attitudes, including the idea that scientific knowledge does not or should not have a privileged position in teaching (or perhaps in philosophy). The remarks in the Preface on page 10 about the standard interpretation of discourse in the mathematical register differ from this view.

References   A brief description of constructivism in mathematics education may be found in [Selden and Selden, 1997] . Two very different expositions of constructivism are given by Ernest [1998] and Hersh [1997b] ; these two books are reviewed in [Gold, 1999] .

contain   If A and B are sets, the assertion A contains B can mean one of two things:

a) $B \subseteq A$.

b) $B \in A$.

A similar remark can be made about the sentence "B is contained in A."

Remark 1   Halmos, in [Steenrod *et al.*, 1975] , page 40, recommends using "contain" for the membership relation and "include" for the inclusion relation. However, it appears to me that "contain" is used far more often to mean "include".

Citations (Fre90.705), (GucJoh90.72).

context   The **context** of an assertion includes the interpretation currently holding of the global identifiers and local identifiers. Definitions change the context on the fly, so to speak. An experienced reader of mathematical discourse will be aware of the meanings of the various identifiers and their changes as she reads.

Example 1   Before a phrase such as "Let $x = 3$", x may be known only as an integer variable; after the phrase, it means specifically 3.

Example 2   An indefinite description also changes the context.

"On the last test I used a polynomial whose derivative had four distinct zeroes."

After such a sentence is said or written, definite descriptions such as "that polynomial" may refer specifically to the polynomial mentioned in the sentence just quoted.

Remark 1   The effect of each statement in mathematical discourse can thus be interpreted as a function from context to context. This is described for one particular formalism (but not specifically for mathematical discourse) in [Chierchia and McConnell-Ginet, 1990] , which has further references. See also [de Bruijn, 1994] , page 875 and [Muskens, van Benthem and Visser, 1997] .

Remark 2   The definition given here is a narrow meaning of the word "context" and is analogous to its used in programming language semantics. (Computer scientists also call it the "state".) The word has a broader meaning in ordinary discourse, typically referring to the physical or social surroundings.

context-dependent

(a) Context-dependent interpretation The interpretation of a symbolic expression is **context-dependent** if it depends on the place of the expression in the sentence containing it. The pronunciation of the expression may also vary with its place in the sentence.

Example 1 In speaking of a group G one might say both

"G is commutative."

and

"Every odd number is an element of G."

In the first sentence, the reference is to the binary operation implicitly or explicitly given in the definition of "group", or perhaps to the whole structure, but certainly not to the set G. In the second, the reference is to the underlying set of the structure.

Thus the meaning of the symbol G (but not in this case its pronunciation) is context-dependent; its significance depends on the sentence it occurs in. Observe that this significance does not depend on the narrow meaning of context given under that heading.

Example 2 The pronunciation of many symbolic expressions in mathematics, particularly symbolic assertions, depends on how they are used in the sentence. The most common way this happens is in the case of parenthetic assertions (under which examples are given).

(b) Context-dependent syntax The symbolic language of mathematics has context-dependent *syntax* in certain constructions.

Example 3 The rule for applying functions would put most functions on the left of the argument, but the **factorial** notation "!" goes on the right. So how you write an expression exhibiting the application of a function depends on what you call the function. (If you called the factorial function F, you would presumably write $F(n)$, not nF.)

Example 4 The notation for derivatives is context-dependent. The prime notation can be used for functions of one variable but not for functions of more than one variable. Similarly, the notation involving d is used for derivatives of functions of one variable; for more than one variable one must change it to ∂.

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Remark 1 The two notations in the example have different problems in some respects. The prime notation cannot be used for more than one variable because it would become ambiguous. The difference between d and ∂, on the other hand, is a matter of convention.

Remark 2 Formal languages, including those of the various forms of mathematical logic. are generally specified by recursive definitions that define which strings of symbols are correctly formed expressions. Such recursive definitions, called **grammars**, are typically **context-free**, meaning roughly that any correctly formed expression of a given type can be placed in a "slot" of that type.

The following observation is aimed at those who know something about grammars: The remarks in Example 3 do not indicate that the symbolic language is not context-free: One could simply have different grammatical categories for function symbols applied on the left, function symbols applied on the right, function symbols applied on *top* (like the dot notation for derivatives), outfix notation, and so on. On the other hand, Example 4 produce evidence (not proof) that the symbolic language cannot be context-free, since whether a prime or a d may be used depends on whether there are other variables "arbitrarily elsewhere" in the expression.

References Grammars are defined mathematically in [Lewis and Papadimitriou, 1998] . Linguists on the one hand and computer scientists on the other use very different notation and terminology for grammars.

contrapositive   The **contrapositive** of a conditional assertion $P \Rightarrow Q$ is the statement (not Q) $\Rightarrow$ (not P). In mathematical arguments, the conditional assertion and its contrapositive are equivalent. In particular, to prove $P \Rightarrow Q$ it is enough to prove that (not Q) $\Rightarrow$ (not P), and once you have done that, *no further argument is needed.* I have attended lectures where further argument *was* given, leading me to suspect that the lecturer did not fully understand the contrapositive, but I have not discovered an instance in print that would indicate that. See proof by contradiction.

   Remark 1   The fact that a conditional assertion and its contrapositive are logically equivalent means that a proof can be organized as follows, and in fact many proofs in texts are organized like this:
   a) Theorem: P implies Q.
   b) Assume not Q.
   c) Argument that not P follows.
   d) Hence not P.
   e) End of proof.
Note that the reader is given no hint as to the form of the proof; she must simply recognize the pattern. (See pattern recognition.)

Difficulties   In contrast to the situation in mathematical reasoning, the contrapositive of a conditional sentence in ordinary English about everyday topics of conversation does not in general mean the same thing as the direct sentence. This causes semantic contamination.

   Example 1   The sentence
      "If it rains, I will carry my umbrella."
does not mean the same thing as
      "If I don't carry my umbrella, it won't rain."
   McCawley [1993] , section 3.4 and Chapter 15, discusses the contrapositive and other aspects of conditional sentences in English. See also the remarks under only if.

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convention   A **convention** in mathematical discourse is notation or terminology used with a special meaning in certain contexts or in certain fields.

Example 1   The use of if to mean "if and only if" in a definition is a convention. This is controversial and is discussed under if.

Example 2   Constants or parameters are conventionally denoted by a, b, ... , functions by f, g, ... and variables by x, y, ...

Example 3   Referring to a group (or other mathematical structure) and its underlying set by the same name is a convention.

Example 4   The meaning of $\sin^n x$ is the inverse sine (arcsin) if $n = -1$ but the multiplicative power for positive n $(\sin^n x = (\sin x)^n)$. This is a common convention in calculus texts, usually explicit. It is not an example of context sensitivity since the n is not in the context, it is part of the symbolic expression itself.

Remark 1   Examples 3 and 4 exhibit failure of compositionality.

Remark 2   Examples 1 through 4 differ in how pervasive they are and in whether they are made explicit or not. The convention in Example 1 is so pervasive it is almost never mentioned (it is just beginning to be mentioned in textbooks aimed at first courses in abstract mathematics). That is almost, but not quite, as true of the second convention. The third and fourth conventions are quite common but often made explicit.

Any given culture has some customs and taboos that almost no one inside the culture is aware of, others that only some who are particularly sensitive to such issues (or who have traveled a lot) are aware of, and still others that everyone is aware of because it is regarded as a mark of their subculture (such as grits in the American south). One aspect of this Handbook is an attempt to uncover features of the way mathematicians talk that mathematicians are not generally aware of.

Example 5   Some conventions are pervasive among mathematicians but different conventions hold in other subjects that use mathematics. An example is the use of i to denote the imaginary unit. In electrical engineering it is commonly denoted j instead, a fact that many mathematicians are unaware of. I first learned about it when a student asked me if i was the same as j. *Citation:* (Poo00.4).

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Example 6   Other conventions are pervasive in one country but may be different in another. For example, in the USA one calculates the sine function on the unit circle by starting at $(1, 0)$ and going counterclockwise, but in texts in other countries one may start at $(0, 1)$ and go clockwise. I learned of this also from students, but have no citations.

See also positive and Remark 2 under real number.

converse   The **converse** of a conditional assertion $P \Rightarrow Q$ is $Q \Rightarrow P$. Students often fall into the trap of assuming that if the assertion is true then so is its converse; this is the fallacy of affirming the consequent. See also false symmetry.

coordinatewise   A function $F : A \to B$ induces a function often called F^* from lists of elements of A to lists of elements of B or from A^n to B^n for a fixed positive integer n by defining

$$F^*(a_1, \dots, a_n) = (F(a_1), \dots, F(a_n))$$

One says that this defines F^* **coordinatewise** or   **componentwise**.

Example 1   "In the product of two groups, multiplication is defined coordinatewise."

One can say that assertions are defined coordinatewise, as well. (See Remark 1 under assertion.)

Example 2   "The product of two ordered sets becomes an ordered set by defining the order relation coordinatewise."

Citations (BriPre92.146), (Kop88.93), (Bla79.122).

copy   When one is discussing a mathematical structure, say the ring of integers, one sometimes refers to "a copy of the integers", meaning a structure isomorphic to the integers. This carries the connotation that there is a preferred copy of the mathematical object called the integers (see specific mathematical object); I suspect that some who use this terminology don't believe in such preferred copies. Our language, with its definite descriptions and proper nouns, is not particularly suited to discussing things defined unique up to isomorphism.

    Citations (Tal90.9),(DowKni92.546).

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coreference   **Coreference** is the use of a word or phrase in discourse to denote the same thing as some other word or phrase. In English, third person pronouns (he, she, it, they), demonstratives (this, that, these, those), and the word "do" are commonly used for coreference.

Linguists have formulated some of the rules that govern the use of coreference in English. Typically, the rules produce some syntactic restrictions on what can be referred to, which in some cases determine the reference uniquely, but in many other cases the meaning must be left ambiguous to be disambiguated (if possible) by the situation in which it is uttered. This is not the place to describe those rules; a native English speaker has the rules built into his or her language understanding mechanism. I will give a few examples.

Example 1   In the sentence

"Assume, for the moment, that $a \neq 0$. Then the equation $ax + by + cz + d = 0$ can be rewritten as $a(x + (d/a)) + by + cz = 0$. But this is a point-normal form of the plane passing through the point $(-d/a, 0, 0)$ and having $\mathbf{n} = (a, b, c)$ as a normal."

from citation (Ant84.121), "this" refers to the preceding equation, not the equation before it,

Example 2   Consider the sentences, "Oscar kissed his mother" and "He kissed Oscar's mother". In the first one we assume Oscar kissed his own mother. In the second, we assumed the person doing the kissing did *not* kiss his own mother. In the first case, and perhaps in the second, the circumstances under which the sentences were uttered could alter these assumptions. This shows that both the English syntax and the circumstances constrain how the sentences are interpreted.

Example 3   Some years ago the following question appeared in my classnotes [Wells, 1997] :

"Cornwall Computernut has 5 computers with hard disk drives and one without. Of these, several have speech synthesizers, including the one without hard disk. Several have Pascal, including those with synthesizers. Exactly 3 of the computers with hard disk have Pascal. How many have Pascal?"

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   Some students did not understand that the phrase "including those with synthesizers" meant "including *all* those with synthesizers" (this misunderstanding removes the uniqueness of the answer). They were a minority, but some of them were quite clear that "including those with synthesizers" means *some or all* of those with synthesizers have Pascal; if I wanted to require that all of them have Pascal I would have to say "including all those with synthesizers". A survey of a later class elicited a similar minority response.

   I do not know of any literature in linguistics that addresses this specific point.

References needed.

   Citation (Kan97.260).

   See also respectively.

   Remark 1   The phenomenon of coreference is also called "anaphora", a word borrowed from rhetoric which originally meant something else. Many (but not all) linguists restrict "anaphora" to backward coreference and use "cataphora" for forward reference.

References   The place to start in reading about the approach of modern linguistics to anaphora is probably [Fiengo and May, 1996] , where I found the Oscar sentences of Example 2. See also [Kamp and Reyle, 1993] , pp. 66ff, [Chierchia, 1995] , [McCarthy, 1994] , and [Halliday, 1994] , pp. 312ff.

corollary   A **corollary** of a theorem is a fact that follows easily from the theorem.

   Citations (BleMccSel98.535), (Bur94.24), (JanHam90.300), (Niv56.41) (corollary of two theorems).

   Remark 1   "Easily" may mean by straightforward calculations, as in (BleMccSel98.535), where some of the necessary calculations occur in the proof of the theorem, and in (Niv56.41), or the corollary may be simply an instance of the theorem as in (JanHam90.300).

counterexample   A counterexample to an universally quantified assertion is an instance of the assertion for which it is false.

Example 1   A counterexample to the assertion

"For all real x, $x^2 > x$"

is $x = 1/2$. See also example,

Difficulties Students sometimes attempt to prove a universally quantified statement by giving an example. They sometimes specifically complain that the instructor uses examples, so why can't they? There are several possibilities for why this happens:

a) The students have seen the instructor use examples and don't have a strong sensitivity to when one is carrying out a proof and when one is engaged in an illuminatory discussion.

b) The student has seen counterexamples used to disprove universal statements, and expects to be able to prove such statements by a kind of false symmetry.

c) The student is thinking of the example as generic and is carrying out a kind of universal generalization.

d) The problem may have expressed the universal quantifier as in Example 1 under indefinite article.

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covert curriculum   The **covert curriculum** (or **hidden curriculum**) consists of the skills we expect math students to acquire without our teaching the skills or even mentioning them. What is in the covert curriculum depends to some extent on the teacher, but for students in higher level math courses it generally includes the ability to read mathematical texts and follow mathematical proofs. (We do try to give the students explicit instruction, usually somewhat offhandedly, in how to *come up with* a proof, but generally not in how to read and follow one.) This particular skill is one that this Handbook is trying to make overt. There are undoubtedly other things in the covert curriculum as well.

References [Vallance, 1977] .

Acknowledgments I learned about this from Annie Selden. Christine Browning provided references.

dash See prime.

defined in   A function F is **defined in**, **defined on** or **defined over** a set A if its domain is A.

   Citations (BelEvaHum79.121), (HasRee93.772), (RicRic93.475).

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defining condition   See setbuilder notation.

defining equation   See function.

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definite article   The word "**the**" is called the **definite article**. It is used in forming definite descriptions.

(1) The definite article as universal quantifier

Both the indefinite article and the definite article can have the force of universal quantification. Examples are given under universal quantifier.

(2) The definite article and setbuilder notation

A set $\{x \mid P(x)\}$ in setbuilder notation is often described with a phrase such as "the set of x such that $P(x)$". In particular, this set is the set of *all* x for which $P(x)$ is true.

   Example 1   The set described by the phrase "the set of even integers" is the set of *all* even integers.

Difficulties   Consider this test question:

   "Let E be the set of even integers. Show that the sum of any two elements
   of E is even."

Students have given answers such as this:

   "Let $E = \{2, 4, 6\}$. Then $2 + 4 = 6$, $2 + 6 = 8$ and $4 + 6 = 10$, and 6, 8 and 10
   are all even."

This misinterpretation has been made in my classes by both native and non-native speakers of English.

(3) Definite article in definitions

The definiendum of a definition may be a definite description.

   Example 2

   "The **sum** of vectors (a_1, a_2) and (b_1, b_2) is $(a_1 + b_1, a_2 + b_2)$."

I have known this to cause difficulty with students in the case that the definition is not clearly marked as such. The definite description makes the student believe that he or she should know what it refers to. In the assertion in Example 2, the only clue that it is a definition is that "sum" is in boldface. This is discussed further under definition.

   Citations (Cur93.790), (Gal94.352).

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definite description   If [N] is a noun phrase, "the [N]" is a **definite description** or a **definite** noun phrase; it refers to the object (presumed uniquely determined) described by [N]. The assumption is that the object referred to is already known to the speaker and the listener or has already been referred to.

   Example 1   If you overheard a person at the blackboard say to someone
       "The function is differentiable, so ... "
you would probably assume that that person is referring to a function that speaker and listener both already know about. It may be a specific function, but it does not have to be; they could be in the middle of a proof of a theorem about functions of a certain type and "the function" could be a variable function that they named for the purposes of proving the theorem.

   This example shows that in the mathematical register, whether a description is definite or indefinite is independent of whether the identifier involved is determinate or variate.

   Example 2
       "Let G be a group. Show that the identity of G is idempotent."
This example shows that the presumptive uniquely determined object ("the identity") can depend on a parameter, in this case G. As another example, in the phrase "the equation of a plane" the parameter is the plane. See context.

   Citation (Ant84.121).

References   See [Kamp and Reyle, 1993] , Section 3.7.5.

   I need a reference to a discussion of definite and indefinite at the expository level.
References needed.

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definition [1] (*Mathematical definitions*)

A **mathematical definition** prescribes the meaning of a symbol, word, or phrase, called the **definiendum** here, as a mathematical object satisfying all of a set of requirements. The definiendum will be either an adjective that denotes a property that mathematical objects may have, or it may be a noun that denotes a type of mathematical object with certain properties. (Mathematical texts sometimes define other parts of speech, but that possibility will be ignored here.)

Remark 1   A mathematical definition is quite different from other sorts of definitions, a fact that is not widely appreciated by mathematicians. The differences are dicussed under concept and under dictionary definition.

(1) Syntax of definitions

Definitions of nouns and of adjectives have similar syntax, with some variations. Every definition will contain a definiendum and a **definiens**, which is a set of properties an object must have to be correctly named by the definiendum. The definiens may be syntactically scattered throughout the definition. In particular, a definition may have any or all of the following structures:

1. A **precondition**, occurring before the definiendum, which typically gives the type of structure that the definition applies to and may give other conditions.
2. A **defining phrase**, a list of conditions on the definiendum occurring in the same sentence as the definiendum.
3. A postcondition, required conditions occurring after the ostensible definition which appear to be an afterthought. The postcondition commonly begins with "where" and some examples are given under that heading.

Example 1   One can define "domain" in point set topology directly by saying

"A **domain** is a connected open set."

(See be.) The definiendum is "domain" and the defining phrase (which constitutes the entire definiens) is "is a connected open set". Similarly:

"'An **even integer** is an integer that is divisible by 2."

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Citation: (Bel75.476), (BleMccSel98.529). In both these cases the definiendum is the subject of the sentence.

   Remark 2   The definition of "domain" in Example 1 involves a suppressed parameter, namely the ambient topological space.

   It is more common to word the definition using "if", in a conditional sentence. In this case the subject of the sentence is a noun phrase giving the type of object being defined and the definiendum is given in the predicate of the conclusion of the conditional sentence. The subject of the sentence may be a definite noun phrase or an indefinite one. The conditional sentence, like any such, may be worded with hypothesis first or with conclusion first. All this is illustrated in the list of examples following, which is not exhaustive.

1. [Indefinite noun phrase, definiendum with no proper name.] A set is a **domain** if it is open and connected. Or: If a set is open and connected, it is a **domain**. Similarly: An integer is **even** if it is divisible by 2. *Citation:* (Fre90.705).

2. [Indefinite noun phrase, definiendum given proper name.] A set D is a **domain** if D is open and connected. An integer n is **even** if n is divisible by 2. (In both cases and in similar named cases below the second occurrence of the name could be replaced by "it".) *Citation:* (Gie71.37), (JonPea00.95).

3. [Definite noun phrase.] The set D is a **domain** if D is open and connected. Similarly: The integer n is **even** if n is divisible by 2. Using the definite form is much less common than using the indefinite form, and seems to occur most often in the older literature. It requires that the definiendum be given a proper name. *Citation:* (App71.56), (Bae55.16).

4. [Using "let" in a precondition.] Let D be a set. Then D is a **domain** if it is open and connected. Similarly: Let n be an integer. Then n is **even** if it is divisible by 2. "Let" is commonly used to establish notation.

5. [Using "if" in a precondition] If n is an integer, then it is **even** if it is divisible by 2. *Citation:* (Bel75.476).

   Remark 3   All the definitions are given with the definiendum marked by being in boldface. Many other forms of marking are possible; see marking below.

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A symbolic expression may be defined by using phrases similar to those just given.

Example 2   "For an integer n, $\sigma(n)$ is the sum of the positive divisors of n."

Sometimes "define" is used instead of "let" in the sense of section (a) under let.

Example 3

"Define $f(x)$ to be $x^2 + 1$. What is the derivative of f?"

Students sometimes wonder what they are supposed to do when they read a sentence such as "Define $f(x)$ to be $x^2 + 1$", since they take it as a command. *Citation:* (KloAleLar93.757), (Kon00.902).

Other ways of giving a definition use call, put, and set, usually in the imperative the way "define" is used in Example 3. See also say. Many other forms of syntax are used, but most of them are either a direct definition such as in Example 1 or a definition using a conditional, with variations in syntax that are typical of academic prose.

Remark 4   Some authors have begun using "if and only if" in definitions instead of "if". More about this in the entry for if. See also convention and let.

(2) Marking

The definiendum may be put in italics or quotes or some other typeface instead of boldface, or may not be **marked** at all. When it is not marked, one often uses signaling phrases such as "is defined to be", "is said to be", or "is called", to indicate what the definiendum is. A definition may be delineated, with a label "Definition".

Citations (Gie71.37), (IpsMey95.905) (formally marked as definition); (LewPap98.20), where it is signaled as definition by the sentence beginning "We call two sets . . . "; (Epp95.534) and (Fra82.41), where the only clue that it is a definition is that the word is in boldface; (BleMccSel98.529), where the clue is that the word is in italics.

Remark 5   Words and phrases such as "We have defined. . . " or "recall" may serve as a valuable clue that what follows is *not* a definition.

Remark 6   Some object to the use of boldface to mark the definiendum. I know of no such object in print; this observation is based on my experience with referees.

(3) Definitions and concepts

The definition of a concept has a special logical status. It is the fundamental fact about the concept from which all other facts about it must ultimately be deduced. I have found this special logical status one of the most difficult concepts to get across to students beginning to study abstract mathematics (in a first course in linear algebra, discrete mathematics or abstract algebra). There is more about this under concept, mental representation, rewrite using definitions and trivial. See also unwind.

There is of course a connection among the following three ideas:
  a) The uses of the word "function" in the mathematical register.
  b) The mathematical definition of function.
  c) The mental representation associated with "function".

To explicate this connection (for all mathematical concepts, not just "function") is a central problem in the philosophy of mathematics.

References [Tall and Vinner, 1981] , [Vinner, 1992] , [Vinner and Dreyfus, 1989] , [Wood, 1999] .

(4) Definition as presentation of a structure

A mathematical definition of a concept is spare by intent: it will generally provide a minimal or nearly minimal list of data and of relationships that must hold among the data that imply all the other data and relationships belonging to the concept. Data or properties that follow from other given items are generally not included intentionally in a definition (some exceptions are noted under redundant) and when they are the author may feel obligated to point out the redundancy.

As a result, a mathematical definition hides the richness and complexity of the concept and as such may not be of much use to students who want to understand it (gain a rich mental representation of it). Moreover, a person not used to the minimal nature of a mathematical definition may gain an exaggerated idea of the importance of the items that the definition *does* include. See also fundamentalist.

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definition [2] (*Dictionary definition*) An explanation, typically in a dictionary or glossary, of the meaning of a word. This is not the same as a mathematical definition (meaning (1) above). To distinguish, this Handbook will refer to a definition of the sort discussed here as a **dictionary definition**. The entries in this Handbook are (for the most part) dictionary definitions.

  Example 1   The entry for "function" given in this Handbook describes how the word "function" and related words are used in the mathematical register. The definition of function given in a typical mathematical textbook (perhaps as a set of ordered pairs with certain properties) specified what kind of mathematical object is to be called a function. See Remark 2 under free variable for a discussion of this issue in a particular case.

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definition by cases   . See cases.

definition proper   See mathematical definition.

degenerate   An example of a type of mathematical structure is in some disciplines called **degenerate** if either (i) some parts of the structure that are distinct in the definition of that type coincide (I call this **collapsing**), or (ii) some parameter is zero. The converse, that if a structure satisfies (i) or (ii) then it is called degenerate, is far from being correct; the word seems to be limited to certain specific disciplines.

Example 1   A line segment can be seen as a degenerate isosceles triangle – two sides coincide and the third has zero length. Note that this fits both (i) and (ii).

Example 2   The concept of degenerate critical point has a technical definition (a certain matrix has zero determinant) and is responsible for a sizeable fraction of the occurrences I found on JSTOR. A small perturbation turns a degenerate critical point into several critical points, so this can be thought of as a kind of collapsing.

Remark 1   The definition of degenerate given here is based on reading about thirty examples of the use of the word on JSTOR. Sometimes the word has a mathematical definition specific to the particular discipline of the paper and sometimes it appears to be used informally.

Citations (Ran95.641), (Rot81.12).

delimiter   **Delimiters** consist of pairs of symbols used in the symbolic language for grouping. A pair of delimiters may or may not have significance beyond grouping; if they do not they are **bare delimiters**. The three types of characters used as bare delimiters in mathematics are parentheses, brackets, and braces. Typically, parentheses are the standard delimiters in symbolic expressions. Brackets or braces may be used to aid parsing when parentheses are nested or when the expression to be enclosed is large, but brackets and braces are occasionally used alone as bare delimiters as well.

Example 1   The expression $\left((x+1)^2 - (x-2)^2\right)^n$ contains nested parentheses and might alternatively be written as $\left[(x+1)^2 - (x-2)^2\right]^n$.

Parentheses, brackets and braces may also be used with additional significance; such uses are discussed with examples under their own headings.

Other symbols also are used to carry meaning and also act as delimiters, such as angle brackets, the symbols for absolute value and norm, or the integral sign and its matching dx.

Example 2   The integral sign and the dx in the expression $\int_a^b x^2 + 1\, dx$ delimit the integrand and also provide other information.

Citations (DarGof70.729), (Sta70a.774), (Sta70b.884), (Tew70.730).

delineated   A piece of text is **delineated** if it is set off typographically, perhaps as a display or by being enclosed in a rectangle. Delineated text is often labeled, as well.

   Example 1

   " **Theorem** *An integer n that is divisible by* 4 *is divisible by* 2."

The label is "Theorem".

denote   To say that an expression A **denotes** a specific object B means that A refers to B; in a sentence containing a description of B, the description can be replaced by A and the truth value of the sentence remains the same.

Example 1   The symbol π denotes the ratio of the circumference of a circle to its diameter.

Citations (RabGil93.168), (Cur93.790).

Remark 1 [Krantz, 1997] , page 38, objects to the use of "denote" when the expression being introduced refers (in my terminology) to a variable mathematical object, for example in a sentence such as "Let f denote a continuous function".

Remark 2   Some authors also object to the usage exemplified by "the ratio of the circumference of a circle to its diameter is denoted π"; they say it should be "denoted by". *Citation* (Pow74.264), (Str93.3), (Epp95.76).

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denying the hypothesis   The fallacy of deducing not Q from $P \Rightarrow Q$ and not P. Also called   **inverse error**.

Example 1   You are asked about a certain subgroup H of a non-abelian group G. You "know" H is not normal in G because you know the theorem that if a group is Abelian, then every subgroup is normal in it.

In contrast, consider Example 1 under conditional assertion.

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dependent variable notation   This is a method of referring to a function that uses the pattern

"Let y be a function of x."

where x is an identifier for the input and y is an identifier for the output. In this case, one says that y is **dependent** on x. The rule for the function is typically not given.

In this usage, the value of the unnamed function at x is sometimes denoted $y(x)$. Note that this does not qualify as structural notation since the notation does not determine the function. *Citation:* (Kno81.235), (Sto95.614).

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determinate   A free identifier is **determinate** if it refers to a specific mathematical object.

Example 1   The symbol "3" is determinate; it refers to the unique integer 3. But see Remark 1 under mathematical object.

An extended discussion of determinate and variate identifiers may be found under variate.

discourse   Connected meaningful speech or writing. Connected meaningful writing is also called   **text**.

Discourse analysis is the name for the branch of linguistics that studies how one extracts meaning from sequences of sentences in natural language.   [Kamp and Reyle, 1993]   provides a mathematical model that may explain how people extract logic from connected discourse, but it does not mention the special nature of mathematical exposition. A shorter introduction to discourse analysis is   [van Eijck and Kamp, 1997] .

disjoint   Two sets are **disjoint** if their intersection is empty.

> *Example 1*   "$\{1, 2\}$ and $\{3, 4, 5\}$ are disjoint."
>
> The word may be used with more than two sets, as well:
>
> *Example 2*   "Let $\mathcal{F}$ be a family of disjoint sets."
>
> *Example 3*   "Let A, B and C be disjoint sets."
>
> *Citation* (Fre90.715), (Oxt77.155).

Difficulties Students sometimes say things such as: "Each set in a partition is disjoint". This is an example of a missing relational argument (see Section (b) under behaviors).

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disjunction   A **disjunction** is an assertion P formed from two assertions A and B with the property that P is true if and only if at least one A and B is true. It is defined by the following truth table:

A	B	P
T	T	T
T	F	T
F	T	T
F	F	F

In the mathematical register, the disjunction of two assertions is usually signaled by connecting the two assertions with "or". Difficulties with disjunctions are discussed under or.

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disjunctive definition   See cases.

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display   A symbolic expression is **displayed** if it is put on a line by itself. Displays are usually centered. The word "displayed" is usually used only for symbolic expressions. See delineated.

distinct   When several new identifiers are introduced at once, the word **distinct** is used to require that no two of them can have the same value.

Example 1   "Let m and n be distinct integers."

This means that in the following argument, one can assume that $m \neq n$.

Difficulties   Students may not understand that without a word such as "distinct", the variables may indeed have the same value. Thus

"Let m and n be integers."

allows $m = n$. In [Rota, 1996] , page 19, it is reported that E. H. Moore was sufficiently bothered by this phenomenon to say,

"Let m be an integer and let n be an integer."

Citation (KloAleLar93.758), (Her64.2), (Mar92.739).

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distributive plural   The use of a plural as the subject of a sentence in such a way that the predicate applies individually to each item referred to in the subject.

Example 1   "The multiples of 4 are even." (or "All the multiples of 4 are even" – see universal quantifier.)

This phenomenon is given a theoretical treatment in [Kamp and Reyle, 1993] , pages 320ff. See also collective plural and each.

divide

(a) Divides for integers An integer m **divides** an integer n (or: m is a **divisor** or factor of n) if there is an integer q for which $n = qm$. Some authors require that q be uniquely determined, which has the effect of implying that no integer divides 0. (0 does not divide any other integer in any case.) This definition, with or without the requirement for uniqueness, appears to be standard in texts in discrete mathematics and number theory.

(b) Divides for commutative rings If a and b are elements of a commutative ring R, then a divides b if there is an element c of R with the property that $b = ac$. This appears to be the standard definition in texts in abstract algebra. I am not aware of any such text that requires uniqueness of c.

Of course, the second meaning is a generalization of the first one. I have known this to cause people to assert that every nonzero integer divides every integer, which of course is true in the second meaning, taking the commutative ring to be the ring of rationals or reals.

Acknowledgments John S. Baltutis.

domain   The **domain** of a function must be a set and may be named in any way that sets are named. The domain is frequently left unspecified. It may be possible to deduce it from what is stated; in particular, in cases where the rule of the function is a symbolic expression the domain may be implicitly or explicitly assumed to be the set of all values for which the expression is defined.

Citation (Sto95.614).

The following phrases may be used to state that a set S is the domain of a function f:

a) f is a function with domain S. *Citation:* (Fis82.445).

b) $\text{dom} f = S$. *Citation:* (Fis82.445).

c) f is a function on S. *Citation:* (Bar96.626).

d) $f : S \to T$. This is read "f is a function from S to T" if it is an independent clause and "f from S to T" if it is parenthetic. This expression also says that the codomain of f is T. *Citation:* (Thi53.260).

See also defined in.

For most authors, a function must be defined at every element of the domain, if the domain is specified. A **partial function** is a mathematical object defined in the same way as a function except for the requirement that it be defined for every element of the domain.

Remark 1   The word "domain" is also used in topology (connected open set) and in computing (lattice satisfying various conditions) with meanings unrelated to the concept of domain of a function (or to each other). See multiple meanings.

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dummy variable   Same as bound variable.

each Generally can be used in the same way as all, every, and any to form a universal quantifier.

Example 1

"Each multiple of 4 is even."

Remark 1 It appears to me that this direct use of "each" is uncommon. When it is used this way, it always indicates a distributive plural, in contrast to all.

"Each" is more commonly used before a noun that is the object of a preposition, especially after "for", to have the same effect as a distributive plural.

Example 2 "For each even number n there is an integer k for which $n = 2k$."

Example 3 "A *binary operation* $*$ *on a set* is a rule that assigns to each ordered pair of elements of the set some element of the set." (from [Fraleigh, 1982] , page 11).

Example 4 Some students do not understand a postposited "each" as in the sentence below.

"Five students have two pencils each."

This usage occurs in combinatorics, for example.

Citations (Bri93.782), (DorHoh78.166), (JacTho90.71).

element   If S is a set, the expression "$x \in S$" is pronounced in English in any of the following ways:

a) "x is in S". *Citation:* (Epp95.534), (Fis82.445).

b) "x is an **element** of S" [or " "in S"]. *Citation:* (Her64.2), (Kra95.40), (Sen90.330).

c) "x is a **member** of S". *Citation:* (BelBluLewRos66.48).

d) "S contains x" or "x is contained in S". *Citation:* (Fre90.705), (DevDur91.222).

   Remark 1   Sentence (d) could also mean x is contained in S as a subset. See contain.

   Remark 2   A common myth among students is that there are two kinds of mathematical objects: "sets" and "elements". This can cause confusion when they are faced with the idea of a set being an element of a set. The word "element" is used by experienced mathematicians only in a phrase involving both a mathematical object and a set. In particular, being an element is not a property that some mathematical objects have and some don't.

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elementary   In everyday English, an explanation is "elementary" if it is easy and if it makes use of facts and principles known to most people. Mathematicians use the word "elementary" with several other meanings as well.

(a) Elementary proofs   A proof of a theorem is **elementary** if it uses only ideas from the same field as the theorem. Rota [Rota, 1996] , pages 113 ff., discusses the case of the prime number theorem in depth; the first proofs around 1900 used complex function theory, but it was given an elementary proof much later. That proof was quite long and complicated, not at all elementary in the non-mathematician's sense. (A simpler one was found much later.)

(b) Elementary definitions   Mathematicians sometimes use "elementary" in another sense whose meaning is not quite clear to me. It is apparently in opposition to conceptual. Here are two possible definitions; we need citations to clear this up.

a) A definition of a type of mathematical structure is elementary if it involves quantifying only over the elements of the underlying set(s) of the structure. In particular it does not involve quantifying over sets or over functions. This is the meaning used by Vought [1973] , page 3.

b) A definition of a type of structure is elementary if it does not make use of other definitions at the same level of abstraction. Thus it is unwound.

Example 1   The usual definition of a topological space as a set together with a set of subsets with certain properties can be expressed in an elementary way according to definition (b) but not in a direct way according to definition (a). (But see the next remark.)

Remark 1   An elementary definition in the sense of (a) is also called **first order**, because the definition can be easily translated into the language of first order logic in a direct way. However, by incorporating the axioms of Zermelo-Fraenkel set theory into a first order theory, one can presumably state most mathematical definitions in first order logic. How this can be done is described in Chapter 7 of [Ebbinghaus, Flum and Thomas, 1984] .

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In spite of the fact that the ZF axioms are first order, one often hears mathematicians refer to a definition that involves quantifying over sets or over functions (as in Example 1) as non-elementary.

Example 2 Here is a conceptual definition of a left R-module for a ring R: It is an Abelian group M together with a homomorphism $\phi : R \to \mathrm{End}(M)$, where $\mathrm{End}(M)$ denotes the ring of endomorphisms of M.

Now here is a more elementary definition obtained by unwinding the previous one: It is an Abelian group M together with an operation $(r, m) \mapsto rm : R \times M \to M$ for which

a) $1m = m$ for $m \in M$, where 1 is the unit element of R.

b) $r(m + n) = rm + rn$ for $r \in R$, $m, n \in M$.

c) $(rs)m = r(sm)$ for $r, s \in R$, $m \in M$.

d) $(r + s)m = rm + sm$ for $r, s \in R$, $m \in M$.

One could make this a completely elementary definition by spelling out the axioms for an Abelian group. The resulting definition is elementary in both senses given above.

Remark 2 The conceptual definition makes the puzzling role of "left" clear in the phrase "left R-module". A right R-module would require a homomorphism from the *opposite ring* of R to $\mathrm{End}(M)$. On the other hand, computations on elements of the module will require knowing the laws spelled out in the elementary definition.

The concept of 2-category is given both an elementary and a conceptual definition in [Barr and Wells, 1995] , Section 4.8.

Acknowledgments Michael Barr and Colin McLarty.

empty set   The **empty set** is the unique set with no elements. It is denoted by the symbols $\emptyset$ or $\{\,\}$.

   Citations (HenLarMarWoo94.213), (Bry93.42), (Ros87.342).

   Remark 1   The symbol $\emptyset$ for the empty set is not the Greek letter ϕ (phi). I remember reading many years ago that the person who invented the symbol meant it to be a *circle* with a slash through it, but both [Knuth, 1986] (page 128) and [Schwartzman, 1994] say it is a *zero* with a slash through it. Typographically a zero is not a circle. More information about this would be appreciated. **Information needed.**

Difficulties   Students may be puzzled by the proof that the empty set is included in every set, which is an example of vacuous implication. Students also circulate a myth among themselves that the empty set is an *element of* every set.

   Other difficulties some students have include the belief that the empty set may be denoted by $\{\emptyset\}$ as well as by $\emptyset$, and not understanding that the empty set is *something* rather than *nothing*, so that for example the set $\{\emptyset, 3, 5\}$ contains three elements, not two. These two confusions are probably related to each other.

   See myths and set.

Acknowledgments   Atish Bagchi

encapsulation   See reification.

endowed   See equipped.

enthymeme   An **enthymeme** is an argument based partly on unexpressed beliefs. Beginners at the art of writing proofs often produce enthymemes.

Example 1   In the process of showing that the intersection of two equivalence relations E and E' is also an equivalence relation, a student may write

"$E \cap E'$ is transitive because E and E' are both transitive."

This is an enthymeme; it omits stating, much less proving, that the intersection of transitive relations is transitive. The student may "know" that it is obvious that the intersection of transitive relations is transitive, having never considered the similar question of the *union* of transitive relations. It is very possible that the student possesses (probably subconsciously) a malrule to the effect that for *any* property P the union or intersection of relations with property P also has property P. The instructor very possibly suspects this. For some students, of course, the suspicion will be unjustified, but for which ones? This sort of thing is a frequent source of tension between student and instructor.

Terminology   "Enthymeme" is a classical rhetorical term [Lanham, 1991] .

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entification   See reification.

equation   An **equation** has the form $e_1 = e_2$, where e_1 and e_2 are terms. The usual extensional meaning of such an equation is that e_1 and e_2 denote the same mathematical object. The intensional semantics varies with the equation and the reader.

   Example 1   The intensional meaning of the equation $2 \times 3 = 6$ for a grade school student may be a multiplication fact. The intensional meaning of $6 = 2 \times 3$ is typically information about a factorization. And $2 \times 3 = 3 \times 2$ is may be perceived as an instance of the commutative law.

Acknowledgments   The example comes from [Schoenfeld, 1985] , page 66.

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equipped   Used to associate the structure attached to a set to make up a mathematical structure. Also **endowed**.

Example 1   A semigroup is a set equipped with [endowed with] an associative binary operation.

Citation (BriPre92.146), (Kra95.55).

Acknowledgments Atish Bagchi.

equivalence relation   An equivalence relation E on a set S is a binary relation on S that is reflexive, symmetric and transitive.

Difficulties   Students at first find it difficult to reify the equivalence classes of an equivalence relation. It is a standard tool in higher mathematics to take the classes of an equivalence relation and make them elements of a structure, points in a space, and so on. This type of construction may very well be the pons asinorum of higher mathematics. (But there is the difference between continuity and uniform continuity, and other problems students have with order of quantifiers, as well.)

See also fundamentalist.

equivalent [1] (*Equivalence of assertions*) Two assertions are **equivalent** (sometimes **logically equivalent**) if they necessarily have the same truth values.

Example 1   There are many ways to say that two assertions are equivalent. Some are listed here, all for the same assertions.

  a) A real number has a real square root if and only if it is nonnegative.
     Citation (NeiAnn96.642), (Ros93.223).
  b) If a real number has a real square root then it is nonnegative. Conversely, if it is nonnegative, then it has a real square root.
     Citation (Ros93.208).
  c) A real number having a real square root is equivalent to its being nonnegative.

Other phrases are used in special cases: in other words, that is, or equivalently, and the following are equivalent.

Remark 1   If P and Q are assertions, most authors write either $P \Leftrightarrow Q$ or $P \equiv Q$ to say that the two statements are equivalent. But be warned: there is a Boolean operation, often denoted by $\leftrightarrow$, with truth table

A	B	$A \leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

This is an *operation*, not a relation, and the difference between "$\leftrightarrow$" and "$\Leftrightarrow$" matters. In particular, the assertion that three statements P, Q and R are (pairwise) equivalent is sometimes expressed by using if and only if or iff in the form "P iff Q iff R". This could be translated by "$P \Leftrightarrow Q \Leftrightarrow R$". Now, the connective $\leftrightarrow$ is associative, so that

$$((P \leftrightarrow Q) \leftrightarrow R) \Leftrightarrow (P \leftrightarrow (Q \leftrightarrow R))$$

but the assertion "$(P \leftrightarrow Q) \leftrightarrow R$" does not have the same meaning as "P iff Q iff R" (consider the case where P and Q are false and R is true).

In texts on discrete mathematics, [Grimaldi, 1999] , [Rosen, 1991] , and [Ross and Wright, 1992] all use $\leftrightarrow$ for the connective and $\Leftrightarrow$ for the relation. The text [Gries and Schneider, 1993] uses $\equiv$ for the connective (and avoids the relation altogether) and [Epp,

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1995] uses $\equiv$ for the relation. It appears to me that most books on logic avoid using the relation.

Acknowledgments Susanna Epp, Owen Thomas.

equivalent [2] (*By an equivalence relation*) A phrase of the form "x is equivalent to y" is also used to mean that x and y are related by an equivalence relation. If the equivalence relation is not clear from context a phrase such as "by the equivalence relation E" or "under E" may be added.

> *Citation* (Exn96.35).

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establish notation   Mathematicians frequently say

"Let's establish some notation."

meaning they will introduce a methodical way of using certain symbols to refer to a particular type of mathematical object. This is a type of definition on the fly, so to speak. See also fix and let.

Citation (CulSha92.235).

evaluate   To **evaluate** a function f at an argument x is to determine the value $f(x)$. See function. ***Citations needed.***

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every   See universal quantifier.

example   An **example** of a kind of mathematical object is a mathematical object of that kind. One also may talk about an example of a theorem; but this is often called an illustration and is discussed under that heading.

This article will provide a rough taxonomy of types of examples. The types given overlap, and whether an example is an instance of a particular type of example may depend on the circumstances (for example, the background of the reader or the student in a class).

(a) Easy example   An **easy example** is one that can be immediately verified with the information at hand or that is already familiar to the reader. Easy examples may be given just before or after a definition.

Example 1   An introduction to group theory may give as examples the integers on addition or the cyclic group of order 2, the last (I hope) presented as the group of symmetries of an isosceles triangle as well as via modular arithmetic.

(b) Motivating example   A **motivating example** is an example given before the definition of the concept, with salient features point out. Such an example gives the student something to keep in mind when reading the definition.

Example 2   A teacher could discuss the symmetries of the square and point out that symmetries compose and are reversible, then define "group".

Remark 1   I have occasionally known students who object strenuously to giving an example of a concept before it is defined, on the grounds that one can't think about how it fits the definition when one doesn't know the definition. Students who feel this way are in my experience always A students.

(c) Delimiting example   A **delimiting example** (called also a **trivial example**) is one with the least possible number of elements or with degenerate structure.

Example 3   An example of a continuous function on $\mathbb{R}$ that is zero at every integer is the constant zero function. Many students fail to come up with examples of this sort ([Selden and Selden, 1998]).

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(d) Consciousness-raising example A **consciousness-raising example** of a kind of mathematical object is an example that makes the student realize that there are more possibilities for that kind of thing that he or she had thought of. In particular, a consciousness-raising example may be a counterexample to an unconscious assumption on the student's part.

Example 4   The function

$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is an example that helps the student realize that the "draw it without lifting the chalk" criterion for continuity is inadequate.

Example 5   Example 1 of [Richter and Wardlaw, 1990] provides a diagonizable integral matrix whose diagonal form over $\mathbb{Z}_6$ is not unique up to ordering. This shows that the usual assumption in diagonalization theorems that one is working over a field cannot be relaxed with impunity.

(e) Inventory examples Many mathematicians will check a conjecture about a type of mathematical object against a small number of examples they keep in mind when considering such objects. This could be called a list of **inventory examples**. The quaternion group of order 8 is usefully kept on a list of inventory examples for finite groups, for example.

(f) Pathological example A research mathematician will typically come up with a definition of a new type of mathematical structure based on some examples she knows about. Then further thought or conversation with colleagues will product examples of structures that fit the definition that she had not thought of and furthermore that she doesn't *want* to be the kind of thing she was thinking of. Often the definition is modified. Sometimes, no suitable modification seems practical and one must accept these new examples as valid. In that case they are often referred to by rude names such as **pathological** or **monster**. This was the attitude of many nineteenth-century mathematicians toward the space-filling curves, for example.

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References The discussion of examples herein is drawn from [Michener, 1978] and [Bagchi and Wells, 1998b] .

Difficulties We construct our mental representations of the concept primarily through examples. (Indeed, one of the most effective ways to learn a new mathematical concept is to generate examples. See [Dahlberg and Housman, 1997] .) Experienced mathematicians know that this mental representation must always be adjusted to conform to the definition, but students often let examples mislead them (see generalization).

existential bigamy   A mistake beginning abstract mathematics students make that occurs in connection with a property P of an mathematical object x that is defined by requiring the existence of an item y with a certain relationship to x. When students have a proof that assumes that there are two items x and x' with property P, they sometimes assume that the same y serves for both of them.

Example 1   Let m and n be integers. By definition, m divides n if there is an integer q such that $n = qm$. Suppose you are asked to prove that if m divides both n and p, then m divides $n + p$. If you begin the proof by saying, "Let $n = qm$ and $p = qm \ldots$ " then you are committing existential bigamy.

Terminology   The name is my own. The fact that Muriel and Bertha are both married (there is a person to whom Muriel is married and there is a person to whom Bertha is married) doesn't mean they are married to the same person. See behaviors.

References [Wells, 1995] .

Acknowledgments   Laurent Siebenmann.

existential instantiation   When $\exists(x)$ is known to be true (see existential quantifier), one may choose a symbol c and assert $P(c)$. The symbol c then denotes a variable mathematical object that satisfies P. That this is a legitimate practice is a standard rule of inference in mathematical logic.

existential quantifier   For a predicate P, an assertion of the form $\exists x\, P(x)$ means that there is at least one mathematical object c of the type of x for which the assertion $P(c)$ is true. If the assertion is true, there may be only one object c for which $P(c)$ is true, there may be many c for which $P(c)$ is true, and in fact $P(x)$ may be true for every x of the appropriate type.

Example 1   Let n be of type integer and suppose $P(n)$ is the predicate "n is divisible by 6". Then the assertion $\exists n\, P(n)$ can be expressed in the mathematical register in these ways:

a)   **There is** an integer divisible by 6. *Citation:* (LewPap98.20), (Mea93.387).
b)   There are integers divisible by 6.
        Citations (HenLarMarWoo94.213), (Ros93.293).
c)   Some integer is divisible by 6. *Citation:* (Kra95.40).
d)   Some integers are divisible by 6.

Note that if $P(n)$ is the assertion, "n is prime and less than 3", then the assertion $\exists n\, P(n)$ can be expressed in the same ways.

Remark 1   It follows from the discussion above that in mathematical English, the assertion, "Some of the computers have sound cards", allows as a possibility that only one computer has a sound card, and it also allows as a possibility that all the computers have sound cards. *Neither of these interpretations reflect ordinary English usage.*

In particular, in mathematical discourse, the assertion

   "Some primes are less than 3."

is *true*, even though there is exactly one prime less than 3. However, I do not have an unequivocal citation for this.

It would be a mistake to regard such a statement as false since we often find ourselves making existential statements in cases where we do not know how many witnesses there are. **Citations needed.**

In general, the passage from the quantifying English expressions to their interpretations as quantifiers is fraught with difficulty. Some of the basic issues are discussed in [Chierchia and McConnell-Ginet, 1990] , Chapter 3; see also [Kamp and Reyle, 1993] , [Gil, 1992] and [Wood and Perrett, 1997] , page 12 (written for students).

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See also universal quantifier, order of quantifiers, and Example 2 under indefinite article.

expansive generalization   See generalization.

explicit assertion   An assertion not requiring pattern recognition.

>   *Example 1*   Some calculus students memorize rules in the form of explicit assertions:
>   "The derivative of the square of a function is 2 times the function times the
>   derivative of the function."

A form of this rule that *does* require pattern recognition is:

>   "The derivative of $(f(x))^2$ is $2f(x)f'(x)$."

>   *Remark 1*   Most definitions and theorems in mathematics do require pattern
recognition and many would be difficult or impossible to formulate clearly without it.

>   *Remark 2*   The process of converting a definition requiring pattern recognition into
one that does not require it bears a striking resemblance to the way a compiler converts a
mathematical expression into computer code..

Terminology   The terminology "explicit assertion" with this meaning is my own.

expression　See symbolic expression.

extensional   See semantics.

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extrapolate   To assume (often incorrectly) that an assertion involving a certain pattern in a certain system holds for expressions of similar pattern in other systems.

Example 1   The derivative of x^n is nx^{n-1}, so the derivative of e^x is xe^{x-1}. Of course, the patterns here are only superficially similar; but that sort of thing is precisely what causes problems for beginning abstract mathematics students.

Example 2   The malrule invented by many first year calculus students that transforms $\frac{d(uv)}{dx}$ to $\frac{du}{dx}\frac{dv}{dx}$ may have been generated by extrapolation from the correct rule

$$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

by changing addition to multiplication. The malrule

$$\sqrt{x+y} = \sqrt{x} + \sqrt{y}$$

might have been extrapolated from the distributive law

$$a(x+y) = ax + ay$$

Another example is given under infinite.

References [Resnick, Cauzinille-Marmeche and Mathieu, 1987] .

Terminology   "Extrapolation" is the name given for this phenomenon in the mathematical education literature. It is a type of generalization, but the latter word is overworked and not used in that meaning here.

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factor   If an expression e is the product of several expressions e_i, in other words

$$e = \Pi_{i=1}^n e_i$$

then each e_i is a **factor** of e. A divisor of an integer is also called a factor of the integer.

"Factor" is also used as a verb. To factor an expression is to represent it as the product of several expressions; similarly, to factor an integer (more generally an element of a structure with an associative binary operation) is to represent it as a product of integers. The use of the word typically carries with it the understanding that the factors of the product must be of a specific type, for example primes in the case of integers.

See also term.

Citations (Sog89.494), (Sur77.728), (Sny82.463).

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fallacy   A **fallacy** is an error in reasoning. Two fallacies with standard names that are commonly committed by students are affirming the consequent and denying the hypothesis.

Terminology   The meaning given here is that used in this Handbook. It is widely used with a looser meaning and often connotes deliberate deception, which is not intended here.

false symmetry   A student may fall into the trap of thinking that some valid method or true statement can be rearranged in some sense and still be valid or true. The examples below illustrate what I mean.

The fallacy of affirming the consequent is a kind of false symmetry, and one might argue that extrapolation is another kind. The examples below are intended to illustrate other types of false symmetry. See also counterexample.

I have observed all these errors in my own classes.

Example 1   The product of any two rational numbers is a rational number, so if x is rational and $x = yz$ then y and z must be rational.

Example 2   If V is a vector space with subspace W, then any basis of W is included in a basis of V. This means that any basis of V contains a basis of W as a subset.

Example 3   Any subgroup of an Abelian group is normal, so any Abelian subgroup of a group must be normal.

Remark 1   It would be desirable to come up with a better description of this process than "rearranged in some sense"! There may, of course, be more than one process involved.

family   A **family of sets** sometimes means an indexed set of sets (so differently indexed members may be the same) and sometimes merely a set of sets.

[Ross and Wright, 1992] , page 686 and [Fletcher and Patty, 1988] , pages 41–42 both define a family to be a set; the latter book uses "indexed family" for a tuple or sequence of sets.

Citations (New67.911), (Oxt77.155), (Rot97.1440).

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field   A **field** is both a type of object in physics and a type of object in abstract algebra. The two meanings are unrelated.

find   Used in problems in much the same way as give.

Example 1   "Find a function of x whose value at 0 is positive" means "Give [an example of] a function ... "

Also used in phrases such as "we find" to mean that there is an instance of what is described after the phrase.

Example 2   "Since $\lim_{x \to \infty} f(x) = \infty$, we may find a number x such that $f(x) > 10^4$."

Citations (BumKocWes93.796), (Bir93.279).

first order logic   See mathematical logic.

fix   A function f **fixes** a point p if $f(p) = p$.

Remark 1   This is based on this metaphor: you fix an object if you make it hold still (she fixed a poster to the wall). In my observation, Americans rarely use "fix" this way; the word nearly always means "repair".

Remark 2   "Fix" is also used in sentences such as "In the following we fix a point p one unit from the origin", which means that we will be talking about *any* point one unit from the origin (a *variable* point!) and we have established the notation p to refer to that point.

Citations (Axl95.140), (Mar92.741).

Acknowledgments   Guo Qiang Zhang.

follow   The statement that an assertion Q **follows** from an assertion P means that P implies Q.

The word "follow" is also used to indicate that some statements after the current one are to be grouped with the current one, or (as in "the following are equivalent) are to be grouped with each other.

Example 1   "A set G with a binary operation is a **group** if it satisfies the following axioms ... " This statement indicates that the axioms that follow are part of the definition currently in progress.

Citations (Epp95.36), (BelBluLewRos66.48), (BruMarWei92.140), (FarJon89.272), (Kar72.706).

following are equivalent   The phrase " **the following are equivalent**" (or " **TFAE**") is used to assert the equivalence of the following assertions (usually more than two and presented in a list).

   Citation (MorShaVal93.751).

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for all   See universal quantifier.

form   See collective plural.

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formal

(a) Carefully written mathematics Describes prose or speech that directly presents a mathematical definition or argument, as in "formal proof". This is the terminology used by Steenrod in [Steenrod *et al.*, 1975] . In this Handbook such formal assertions are said to be in the mathematical register.

   Citation (AkiDav85.243).

(b) Use in mathematical logic The phrase "formal proof" is also used to mean a proof in the sense of mathematical logic; see proof.

References [Grassman and Tremblay, 1996] , pages 46–48 define formal proof as in logic.

(c) Opposite of colloquial The word "formal" also describes a style of writing which is elevated, the opposite of colloquial. It is not used in that meaning in this book.

formal analogy   A student may expect that a notation is to be used in a certain way by analogy with other notation based on similarity of form, whereas the definition of the notation requires a different use.

   Example 1   Given real numbers r and s with s nonzero, one can form the real number r/s. Given vectors $\vec{v}$ and $\vec{w}$, students have been known to write $\vec{v}/\vec{w}$ by **formal analogy**.

   Example 2   In research articles in mathematics the assertion $A \subset B$ usually means A is included as a subset in B. It carries no implication that A is different from B. However, the difference between "$m < n$" and "$m \leq n$" often causes students to expect that $A \subset B$ should mean A is a *proper* subset of B and that one should express the idea that A is included in and possibly equal to B by writing $A \subseteq B$. The research mathematical usage thus fails to be parallel to the usage for inequalities, which can cause cognitive dissonance.

   This formal analogy has resulted in a change of usage discussed further under another planet.

   Remark 1   I would conjecture that in Example 2, the same process is at work that is called **leveling** by linguists: that is the process that causes small children to say "goed" instead of "went".

References   This discussion is drawn from [Bagchi and Wells, 1998a] .

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formal language   A set of symbolic expressions defined by a mathematical definition. The definition is usually given recursively.

Example 1   Pascal, like other modern programming languages, is a formal language. The definition, using Backus-Naur notation (a notation that allows succinct recursive definitions), may be found in [Jensen and Wirth, 1985] .

Example 2   The languages of mathematical logic are formal languages. Thus terms and expressions are defined recursively on pages 14 and 15 of [Ebbinghaus, Flum and Thomas, 1984] .

Example 3   The traditional symbolic language of mathematics is not a formal language; this is discussed under that entry.

See also context-dependent.

formal logic Mathematical logic.

formula A symbolic expression that is an assertion. See the discussion under semantic contamination.

forward reference   A **forward reference** occurs when a pronoun refers to something named later in the text.

Example 1   This is a problem I gave on a test:

 "Describe how to tell from its last digit in base 8 whether an integer is even."

In this sentence "its" refers to "an integer", which occurs later in the sentence.

Remark 1   That problem and other similar problems have repeatedly caused a few of my students to ask what it meant. These included native English speakers. Of course, this problem is not specific to mathematical discourse.

Terminology   I took this terminology from computer science. Linguists refer to forward reference as "cataphoric reference" [Halliday, 1994] (page 314) or "backward dependency" [Chierchia, 1995] .

fraktur   An alphabet formerly used for writing German that is sometimes used for mathematical symbols. It appears to me that its use is dying out in mathematics. Many of the forms are confusing and are mispronounced by younger mathematicians. In particular, 𝔄 may be mispronounced "U".

Also called gothic.

𝔄, 𝔞	A, a	𝔍, 𝔧	J, j	𝔖, 𝔰	S, s
𝔅, 𝔟	B, b	𝔎, 𝔨	K, k	𝔗, 𝔱	T, t
ℭ, 𝔠	C, c	𝔏, 𝔩	L, l	𝔘, 𝔲	U, u
𝔇, 𝔡	D, d	𝔐, 𝔪	M, m	𝔙, 𝔳	V, v
𝔈, 𝔢	E, e	𝔑, 𝔫	N, n	𝔚, 𝔴	W, w
𝔉, 𝔣	F, f	𝔒, 𝔬	O, o	𝔛, 𝔵	X, x
𝔊, 𝔤	G, g	𝔓, 𝔭	P, p	𝔜, 𝔶	Y, y
ℌ, 𝔥	H, h	𝔔, 𝔮	Q, q	ℨ, 𝔷	Z, z
ℑ, 𝔦	I, i	ℜ, 𝔯	R, r		

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free variable   A variable in a symbolic assertion is **free** if it is possible to substitute the identifier of a specific mathematical object and get a meaningful statement. In particular, if one substitutes identifiers of specific mathematical objects for every free variable in a symbolic assertion one should get a statement that is definitely true or definitely false. In that sense, an assertion with free variables in it is parametrized; choosing values for the parameters gives a specific statement. Similarly, one may substitute an identifier of a specific mathematical object of the correct type for each free variable in a term; doing this turns the parametrized term into an expression that denotes a specific mathematical object.

In contrast, one cannot substitute for bound variables.

Example 1   The assertion
   "$x^2 - 1 > 0$"
is not definitely true or false. However, if you substitute 2 for x you get $3 > 0$ which is true, and if you substitute 0 you get a false statement.

Remark 1   Observes that if we change the assertion in Example 1 to "$x^2 + 1 > 0$", the result is definitely true (assuming x of type real) before substitution is made. Nevertheless, you can substitute a real number for x in the assertion and get a statement that is definitely true or definitely false (namely definitely true), so x is free. See open sentence.

Example 2   The term $x^2 + y^2$ becomes an expression denoting 13 if 2 is substituted for x and 3 is substituted for y.

Example 3   The term $\Sigma_{k=1}^n k$ becomes an expression denoting 6 if 3 is substituted for n. But when one substitutes a number for k, getting for example $\Sigma_{5=1}^n 5$, one gets nonsense; k is not a free variable in the expression "$\Sigma_{k=1}^n k$", it is a bound variable.

Remark 2   The preceding discussion gives a kind of behavioral definition of how free variables are used in the mathematical register; this definition is in the spirit of a dictionary definition. In texts on mathematical logic and on formal languages, freeness is generally given a recursive mathematical definition based on the formal recursive definition of the language. That sort of definition constitutes a abstraction of the concept of free variable defined here.

It is necessary to give such a mathematical definition of "free variable" if one is going to prove theorems about them. However, students need to know the intuition or metaphor underlying the concept if they are going to make fluent use of it. Most modern logic books do attempt to provide some such explanation.

function   The concept of function in mathematics is as complex and important as any mathematical idea, though perhaps such concepts as space have more subtleties. This long article discusses the syntax we use in talking about functions, the many metaphors behind the idea, and the difficulties connected with it.

(1) Objects associated with a function

When a function is discussed in the mathematical register, some or all of the following data will be referred to.

 a) An identifier for the function.
 b) The domain of the function (discussed under domain).
 c) The codomain of the function (discussed under codomain).
 d) An element of the domain at which the function is evaluated.
 e) The value of the function at an element of the domain.
 f) The **rule** of the function, which is an expression or algorithm that provides a means of determining the value of the function.

There is no single item in the preceding list that a discussion of a function *must* refer to. We list many of the possibilities for referring to these data and the common restrictions on their use.

(2) The identifier of a function

(a) Name Functions may have names, for example "sine" or the exponential function. The name in English and the symbol for the function may be different; e.g. "sine" and "sin", "exponential function" and "exp". See also definition.

(b) Local identifier A function may be given a local identifier. This is by convention a single letter, often drawn from the Roman letters f through h or one of many Greek letters.

(c) Anonymous reference A function may be specified without an identifier, using some form of structural notation. One form is to use the defining expression (discussed below). Other types of structural notation include barred arrow notation and lambda notation, discussed under those entries.

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(d) Naming a function by its value at x   It is common to refer to a function with identifier f (which may or may not be a name) as $f(x)$ (of course some other variable may be used instead of x). This is used with functions of more than one variable, too.

Example 1   "Let $f(x)$ be a continuous function."

Example 2   "The function $\sin x$ is bounded."

Citations (GraKnuPat89.71), (Mor95.716).

The defining expression as the name of a function   It is very common to refer to a function whose rule is given by an expression $f(x)$ by simply mentioning the expression, which is called its **defining expression**. This is a special case of naming a function by its value.

Example 3   "The derivative of x^3 is always nonnegative."

Remark 1   It is quite possible that this usage should be analyzed as simply referring to the expression, rather than a function.

(e) Using the name to refer to all the values   The name of function can be used to stand for all values. *Examples:*

"$f \geq 0$."

"x^2 is nonnegative."

Compare collective plural.

Citation (Pow96.879)

(3) The input

The element of the domain at which the function is evaluated may be called the argument. I have heard lecturers call it the **input**, but in print this usage seems to be limited to inputs to operators (usually functions themselves) or to algorithms.

See also arity.

(4) The value

The object that is the result of evaluating a function at an element x of its domain is called the **value** or **output** of the function at x. *Citation:* (Fis82.445), (Jac34.70).

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If the function is denoted by f, then the value at x is denoted by $f(x)$, or less commonly using one of many other arrangements, including:

a) fx (Polish notation).
b) xf (reverse Polish notation).
c) $(x)f$ (postfix notation). These are discussed in their own entries with examples.
d) f_x (mostly for integer functions — see subscript). **Citations needed.**
e) $f[x]$. This notation is used by Mathematica®; parentheses are reserved for grouping. (But many mathematicians regard the parentheses in the expression $f(x)$ as an example of grouping, presumably because they use Polish notation.)

See parentheses for more about their usage with function values.

Citations (Niv56.41), (Oso94.760).

More elaborate possibilities exist for functions with more than one input. See infix notation, prefix notation, postfix notation, outfix notation.

Remark 2   The word "value" is also used to refer to the number denoted by a literal expression. *Citation:* (Mol97.531), (Oxt77.155).

Remark 3   Adjectives applied to a function often refer to its outputs. *Examples:* The phrase "real function" means that the outputs of the function are real (but many authors would prefer "real-valued function"). "F is a positive function" means that $F(x) > 0$ for every x in its domain.

Citations (BelBluLewRos66.186); (BelEvaHum79.121); (GraKnuPat89.71). However: "rational function" doesn't mean rational output! *Citation:* (Pow96.879).

(5) The rule for evaluation

The rule for evaluation of a function can be a symbolic expression or an algorithm, expressed informally or in a formal language. When the rule is given by an expression $e(x)$, the definition of the function often includes the statement

$$y = e(x)$$

which is called the   **defining equation** of the function.

Example 4

"Consider the function given by $y = x^2 + 1$."

The defining equation is "$y = x^2 + 1$".

Example 5

"Consider the function $f(x) = x^2 + 1$."

Note that this gives the definition equation as a parenthetic assertion.

Citation (Mol97.531)

Difficulties An expression involving disjunctions can confuse students, who don't recognize it as one expression defining one function.

Example 6   "Let $f(x) = \begin{cases} x + 1 & \text{if } x > 2 \\ 2x - 1 & \text{otherwise.} \end{cases}$ "

Because of the practice of using defining equations, students often *regard* a function as an equation [Thompson, 1994] , pp 24ff. So do teachers [Norman, 1992] .

(6) Variations in terminology

It appears to me that many mathematicians avoid using the word "function" for functions that do not act on numbers, perhaps for reasons of readability. Instead, they use words such as functional, operator, or operation. I have heard secondhand stories of mathematicians who objected to using the word "function" for a binary operation such as addition on the integers, but I have never seen that attitude expressed in print.

In this text functions are *not* restricted to operating on numbers. See also mapping.

(7) How one thinks of functions

A mathematician's mental representation of a function is generally quite rich and involves many different metaphors. Some of the more common ways are noted here. These points of view have blurry edges!

(a) Expression to evaluate Function as expression to evaluate. This is the image behind statements such as "the derivative of x^3 is $3x^2$" mentioned above. Note Example 1 under semantics.

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(b) Graph Function as graph. This provides a *picture* of the function as a *relation* between argument and value; of course it is a special kind of relation.

(c) Dependency relation Function as a **dependency relation**. This is the metaphor behind such descriptions as "let x depend smoothly on t". It is related to the graph point of view, but may not carry an explicit picture; indeed, an explicit picture may be impossible.

(d) Transformer Function as **transformer**, or machine that takes an object and turns it into another object. In this picture, the function $F(x) = x^3$ transforms 2 into 8. This is often explicitly expressed as a " **black box**" interpretation, meaning that all that matters is input and output and not how it is performed. This idea is revealed by such language as "2 becomes 8 under f".

(e) Algorithm Function as algorithm or set of rules that tell you how to take an input and convert it into an output. This is a metaphor related to those of function as expression and as transformer, but the actual process is *implicit* in the expression view (in the intensional semantics of the expression) and *hidden* in the transformer (black box) view.

(f) Relocator Function as **relocator**. In this version, the function $F(x) = x^3$ moves the point at 2 over to the location labeled 8. This is the "alibi" interpretation of [Mac Lane and Birkhoff, 1993] (page 256). It is revealed by such language as "f takes 2 to 8".

(g) Map Function as map. This is one of the most powerful metaphors in mathematics. It takes the point of view that the function $F(x) = x^3$ renames the point labeled 2 as 8. A clearer picture of a function as a map is given by some function that maps the unit circle onto, say, an ellipse in the plane. The ellipse is a map of the unit circle in the same way that a map of Ohio has a point corresponding to each point in the actual state of Ohio (and preserving shapes in some approximate way). This is something like the "alias" interpretation of [Mac Lane and Birkhoff, 1993] : The point on the map labeled "Oberlin", for example, has been *renamed* "Oberlin".

References [Lakoff and Núñez, 1997] , [Selden and Selden, 1992]

(8) Mathematical definitions of function

Texts in calculus and discrete mathematics often define the concept of function as a mathematical object. There are two nonequivalent definitions in common use. One defines a function to be a set of ordered pairs with the functional property (pairs with the same first coordinate have the same second coordinate). The other specifies a domain and a codomain as well as a set of ordered pairs with the functional property, and requires that the domain be exactly the set of first coordinates and the codomain include all the second coordinates. In the latter case the set of ordered pairs is the **graph** of the function.

(9) Difficulties

Typically, the definition of "function" does not correspond very well with actual usage. For example, one generally does not see the function expressed in terms of ordered pairs, one more commonly uses the $f(x)$ notation instead. To avoid this discrepancy, I suggested in [Wells, 1995] the use of a specification for functions instead of a definition. Another discrepancy is noted under codomain.

These discrepancies probably cause some difficulty for students, but for the most part students' difficulties are related to their inability to reify the concept of function or to their insistence on maintaining just one mental representation of a function (for example as a set of ordered pairs, a graph, an expression or a defining equation).

There is a large literature on the difficulties functions cause students, I am particularly impressed with [Thompson, 1994] . Another important source is the book [Harel and Dubinsky, 1992] and the references therein, especially [Dubinsky and Harel, 1992] , [Norman, 1992] , [Selden and Selden, 1992] , [Sfard, 1992] . See also [Vinner and Dreyfus, 1989] , [Eisenberg, 1992] and [Carlson, 1998] . [Hersh, 1997a] discusses the confusing nature of the word "function" itself.

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functional A function whose inputs are functions and whose outputs are not functions (typically the outputs are elements of some field). *Citations needed.*

fundamentalist   A **fundamentalist**, or **literalist**, or **reductionist** believes that the
formalism used to give a mathematical definition or to axiomatize a set of mathematical
phenomena should be taken as the "real meaning" of the idea and in extreme cases even
as the primary way one should think about the concepts involved.

   Example 1   In the study of the foundations of mathematics, one of the problems is
to show that mathematics is consistent. One standard way to do this is to define
everything in terms of sets (so that math is consistent if set theory is consistent). In
particular, a function is defined as a set of ordered pairs, an ordered pair (a, b) is defined
to be something like $\{a, \{a, b\}\}$, and the number 3 may be defined as $\{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}$.
A **fundamentalist** will insist that this means that an ordered pair and the number 3 *really
are* the sets just described, thus turning a perfectly legitimate consistency proof into a
pointless statement about reality.

   That sort of behavior is not damaging as long as one does not engage in it in front of
students (except in a foundations class). Is it a good idea to send students out in the
world who believe the statement "2 is even" is on a par with the statement "2 is an
element of 3 but not an element of 1"? What matters about natural numbers and about
ordered pairs is their specification.

   We now consider two different definitions of the same type of object.

   Example 2   A partition Π of a set S is a set of nonempty subsets of S which are
pairwise disjoint and whose union is all of S. Here the only data are S and the set Π of
subsets and the only requirements are those listed.

   Example 3   An equivalence relation on a set S is a reflexive, symmetric, transitive
relation on S. Here the data are S and the relation and the properties are those named.

   The definitions given in these examples provide *exactly the same class of structures*.
Example 2 takes the set of equivalence classes as given data and Example 3 uses the
relation as given data. Each aspect determines the other uniquely. Each definition is a
different way of *presenting* the same type of structure.

   Thus a partition is the same thing as an equivalence relation. G.-C. Rota [Rota,
1997] exhibits this point of view when he says (on page 1440) "The family of all
partitions of a set (also called equivalence relations) is a lattice when partitions are

ordered by refinement". Fundamentalists object to regarding an equivalence relation and its associated partition as the "same structure". They say things like "How can a set of subsets be the same thing as a relation?" It seems to me that this fundamentalist attitude is an *obstruction to understanding the concept.*

The mature mathematician thinks of a type of structure as a whole rather than always coming back to one of the defining aspects. Students don't always get to that stage quickly. The set of subsets and the relation are merely *data* used to describe the structure. To understand the structure properly requires understanding the important objects and concepts (such as a function being compatible with the partition) involved in these structures and all the important things that are true of them, on an equal footing (as in the concept of clone in universal algebra), and the ability to focus on one or another aspect as needed.

Example 4 First order logic is a mathematical model of mathematical reasoning. The fundamentalist attitude would say: Then the expression of our mathematical reasoning should look like first order logic. This is Vulcanism.

Example 5 Fundamentalists may also object to phrases such as "incomplete proof" and "this function is not well-defined". See radial concept.

See also the discussions under mathematical definition, mathematical logic and mathematical structure.

Remark 1 One could argue that "fundamentalist" should be restricted to being literal-minded about foundational definitions, but not necessarily other definitions in mathematics, and that "literalist" should be used for the more general meaning.

Acknowledgments Peter Freyd, Owen Thomas and also [Lewis and Papadimitriou, 1998] , page 9, where I got the word "fundamentalist". Lewis and Papadimitriou did not use the word in such an overtly negative way as I have.

References [Benaceraff, 1965] , [Lakoff and Núñez, 1997] , pages 369–374, and [Makkai, 1999] .

generalization

(1) Legitimate generalization

To generalize a mathematical concept C is to find a concept C' with the property that instances of C are also instances of C'.

(a) Expansive generalization One may generalize a concept by changing a datum of C to a parameter. This is **expansive generalization**.

Example 1 $\mathbb{R}^n$, for arbitrary positive integer n, is a generalization of $\mathbb{R}^2$.

(b) Reconstructive generalization A generalization may require a substantial cognitive reconstruction of the concept. This is **reconstructive generalization**.

Example 2 The relation of the concept of abstract vector space to R^n is an example of a reconstructive generalization.

Remark 1 The relation between reconstructive generalization and abstraction should be studied further.

The names "expansive" and "reconstructive" are due to [Tall, 1992a] .

(2) Generalization from examples

The idea of generalization just discussed is part of the legitimate methodology of mathematics. There is another process often called generalization, namely **generalization from examples**. This process is a special case of extrapolation and can lead to incorrect results.

Example 3 All the limits of sequences a student knows may have the property that the limit is not equal to any of the terms in the sequences, so the student generalizes this behavior with some (erroneous!) assertion such as, "A sequence gets close to its limit but never equals it". See also extrapolate and myths.

References This example is from the discussion in [Tall, 1992b] , Sections 1.5 and 1.6. See also [Pimm, 1983] .

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Terminology It appears to me that the usual meaning of the word "generalization" in colloquial English is generalization from examples. Indeed, in colloquial English the word is often used in a derogatory way. The contrast between this usage and the way it is used in mathematics may be a source of cognitive dissonance.

generally See in general.

give "Give" is used in many ways in the mathematical register, often with the same sense it would be used in any academic text ("we give a proof ... ", "we give a construction ... "). One particular mathematical usage: to **give** a mathematical object means to describe it sufficiently that it is uniquely determined. Thus a phrase of the form "give an X such that P" means describe a object of type X that satisfies predicate P. The description may be by providing a determinate identifier or it may be a definition of the object in the mathematical register.

Example 1 "Problem: Give a function of x that is positive at $x = 0$." A correct answer to this problem could be "the cosine function" (provide an identifier), or "the function $f(x) = x^2 + 1$" (in the calculus book dialect of the mathematical register).

"Given" may be used to introduce an expression that defines an object.

Example 2 One could provide an answer for the problem in the preceding example by saying:

"the function $f : \mathbb{R} \to \mathbb{R}$ **given** by $f(x) = 2x + 1$."

The form **given** is also used like if.

Example 3 "Given sets S and T, the **intersection** $S \cap T$ is the set of all objects that are elements of both S and T."

See also find.

Citations (Bur94.17), (Epp95.534), (GelOlm90.65), (JenMul00.634), (Len92.216), (Str93.17).

global See local.

global identifier   A **global identifier** in a mathematical text is an identifier that has the same meaning throughout the text. These may be classified into three types:

(a) Global to all of mathematics Some global identifiers are used by nearly all authors, for example "$=$", mostly without definition. Some global identifiers such as π and e are sometimes overridden in a particular text. Even "$=$" is sometimes overridden; for example, one may define the rationals as equivalence classes of ordered pairs of integers, and say we write a/b for (a,b) and $a/b = a'/b'$ if (a,b) is equivalent to (a',b').

(b) Global to a field Some are used by essentially all authors in a given field and generally are defined only in the most elementary texts of that field.

   Example 1   The integral sign is global to any field that uses the calculus. This seems never to be overridden in the context of calculus, but it does have other meanings in certain special fields (ends and coends in category theory, for example).

(c) Global to a text A global identifier may be particular to a given book or article and defined at the beginning of that text.

   An identifier defined only in a section or paragraph is a local identifier.

   Remark 1   The classification just given is in fact an arbitrary division into three parts of a continuum of possibilities.

Difficulties Global identifiers specific to a given text impose a burden on the memory that makes the text more difficult to read, especially for grasshoppers. It helps to provide a glossary or list of symbols, and to use type labeling. Steenrod [1975] says global symbols specific to a text should be limited to five.

   Mnemonic global identifiers of course put less burden on the reader.

Acknowledgments Thanks to Michael Barr, who made valuable suggestions concerning an earlier version of this article.

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global parameters   Many texts have **global** parameters, not always made explicit, that determine which of the many sine functions (and other trigonometric functions) and log functions they are using.

gothic   The German fraktur alphabet is sometimes called gothic, as is an alphabet similar to fraktur but easier to read that is used as newspaper titles. Certain sans-serif typefaces are also called gothic.

graph   The word "graph" has two unrelated meanings in undergraduate mathematics:

a) The set $\{(x, f(x) \mid x \in D\}$ for a function f with domain D.
   Citation (Bau78.644).

b) A structure consisting of nodes with directed or undirected edges that connect the nodes (actual mathematical definitions in the literature vary a bit).
   Citation (Wil89.704).

Moreover, in both cases the word "graph" may also be used for *drawings* of (often only part of) the mathematical objects just described.

Citations (Dan78.539), (Cur90.524), (BakDewSzi74.835).

grasshopper   A reader who starts reading a book or article at the point where it discusses what he or she is interested in, then jumps back and forth through the text finding information about the ideas she meets. Contrasted with someone who starts at the beginning and reads straight through.

Terminology   The terminology is due to Steenrod [1975] . Steenrod calls the reader who starts at the beginning and reads straight through a **normal** reader, a name which this particular grasshopper resents.

Greek alphabet   Every letter of the Greek alphabet except omicron (O,*o*) is used in mathematics. Knuth [1986] , page 434) said upsilon (Υ, υ) was not used, but it occurs in [Nyikos, 1984] , page 663. (Thanks to Gary Tee for this reference.)

All the lowercase forms and all those uppercase forms that are not identical with the Roman alphabet are used. Students and young mathematicians very commonly mispronounce some of them. The letters are listed here with pronunciations and with some comments on usage. Some information about the common uses of many of these letters is given in [Schwartzman, 1994] .

Pronunciation key: ăt, āte, bĕt, ēve, pĭt, rīde, cŏt, gō, fōōd, fŏŏt, bŭt, mūte, ą the neutral unaccented vowel as in ago (ągō) or focus (fōkąs). A prime after a syllable indicates primary accent; double prime secondary accent, as in secretary (sĕ′krątă″rĭ) (American pronunciation). (Br) indicates that the pronunciation is used chiefly outside the USA.

A, α Alpha, ăl′fą. *Citation:* (Bil73.1107).

B, β Beta, bā′tą or bē′tą (Br). *Citation:* (Bil73.1107).

Γ, γ Gamma, gă′mą. *Citation:* (EdeKos95.7), (Mor92.91).

Δ, δ Delta, dĕl′tą. *Citation:* (Bar96.626), (BieGro86.425).

E, ϵ or ε Epsilon, ĕp′sąląn, ĕp′sąlŏn″, or ĕpsī′ląn. Note that the symbol $\in$ for elementhood is not an epsilon. *Citation:* (DarGof70.729), (Sur90.321).

Z, ζ Zeta, zā′tą or zē′tą (Br). *Citation:* (Sny82.463).

H, η Eta, ā′tą or ē′tą (Br). *Citation:* (FarJon89.272).

Θ, θ or ϑ Theta, thā′tą or thē′tą (Br). *Citation:* (IpsMey95.905), (Har93a.639), (Sur90.321).

I, ι Iota, īo′tą. *Citation:* (New67.911).

K, κ Kappa, kăp′ą. *Citation:* (Car82.316).

Λ, λ Lambda, lăm′dą. *Citation:* (Car82.316).

M, μ Mu, mū. *Citation:* (KupPri84.86).

N, ν Nu, nōō or nū. *Citation:* (Gil60.622).

Ξ, ξ Xi. I have heard ksē, sī and zī. Note that the pronunciation sī is also used for ψ (discussed further there). *Citation:* (DebHol91.795)

O, o Omicron, ŏ′mĭkrŏn″ or ō′mĭkrŏn″.

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Π, π   Pi, pī. To the consternation of some students beginning abstract mathematics, π is very commonly used to mean all sorts of things besides the ratio of the circumference of a circle to its diameter. *Citation:* (Har93a.639), (Lef77.643), (Oss79.18).

P, ρ   Rho, rō. *Citation:* (Ost71.624), (Sri81.640).

Σ, σ   Sigma, sĭg′mą. *Citation:* (Wit90.144), (Cur93.790), (Mea93.387), (Tal86.257).

T, τ   Tau, pronounced to rhyme with cow or caw. *Citation:* (Sog89.494).

Υ, υ   Upsilon. The first syllable can be pronounced ōōp or ŭp and the last like the last syllable of epsilon. *Citation:* (Ree86.509), (Zan72.102).

Φ, φ or φ   Phi, fī or fē. For comments on the symbol for the empty set, see empty set. *Citation:* (Lef77.643), (Oso94.760).

X, χ   Chi, pronounced kī. I have never heard anyone say kē while speaking English (that would be the expected vowel sound in European languages). German speakers may pronounce the first consonant like the ch in "Bach". *Citation:* (Sri81.640).

Ψ, ψ   Psi, pronounced sī, sē, psī or psē. I have heard two different young mathematicians give lectures containing both φ and ψ who pronounced one of them fī and the other fē. I am sorry to say that I did not record which pronunciation was associated to which letter. I have also been to lectures in which both letters were pronounced in exactly the same way. *Citation:* (Ost71.624), (Sur90.321), (Tal86.257).

Ω, ω   Omega, ōmā′gą or ōmē′gą. *Citation:* (App71.56), (Fin99.774), (Mcc89.1328) , (Tal86.257).

Remark 1   Most Greek letters are pronounced differently in modern Greek; β for example is pronounced vē′ta (last vowel as "a" in father).

grounding metaphor   See metaphor.

grouping   Various syntactical devices are used to indicate that several statements in the mathematical register belong together as one logical unit (usually as a definition or theorem). In the symbolic language this is accomplished by delimiters. In general mathematical prose various devices are used. The statement may be delineated or labeled, or phrases from the general academic register such as "the following" may be used. Examples are given under delimiter and follow.

guessing   If the definition of a mathematical object determines it uniquely, then **guessing** at the answer to a problem and then using the definition or a theorem to prove it is correct is legitimate, but many students don't believe this.

Example 1   It is perfectly appropriate to guess at an antiderivative and then prove that it is correct by differentiating it. Many students become uncomfortable if a professor does that in class.

This attitude is a special case of algorithm addiction.

hanging theorem   A theorem stated at the point where its proof is completed, in contrast to the more usual practice of stating the theorem and then giving the proof.

References   The name is due to Halmos [Steenrod *et al.*, 1975] , page 34, who deprecates the practice, as does [Krantz, 1997] , page 68.

hat See circumflex.

hidden curriculum Covert curriculum.

hold An assertion P about mathematical objects of type X **holds** for an instance i of X if P becomes true when P is instantiated at i.

   Example 1 Let the type of x be real and let P be the predicate

$$f(x) > -1$$

Then P holds when f is instantiated as the sine function and x is instantiated as 0. Typical usage in the mathematical register would be something like this: "P holds for $f = \sin$ and $x = 0$."

   "Hold" is perhaps most often used when the instance i is bound by a quantifier.

   Example 2 "$x^2 + 1$ is positive for all x."

   Citation (Bar96.631). (FarJon89.272).

hypothesis   The hypothesis of a conditional assertion of the form $P \Rightarrow Q$ is P. Also called **antecedent** or **assumption**.

❚ The symbol I is often used to denote the unit interval, the set of real numbers x for which $0 \leq x \leq 1$. For some authors, I or $\mathbb{I}$ is defined to be the set of integers.

identifier   An **identifier** is a word, phrase or symbol used as the name of a mathematical object. An identifier may be a symbol or a name. Symbols and names are defined in their own entries; each of these words has precise meanings in this Handbook that do not coincide with common use.

We discuss the distinction between name and symbol here. A name is an English noun phrase. A symbol is a part of the symbolic language of mathematics.

Example 1   The expressions i, π and sin can be used in symbolic expressions and so are symbols for certain objects. The phrase "the sine function" is a name. If a citation is found for "sine" used in a symbolic expression, such as "sine(π)", then for that author, "sine" is a symbol.

Remark 1   The number π does not appear to have a nonsymbolic name in common use; it is normally identified by its symbol in both English discourse and symbolic expressions. The complex number i is also commonly referred to by its symbol, but it can also be called the   **imaginary unit**. *Citation:* (HarJor67.559), (Poo00.4).

Remark 2   I have not found examples of an identifier that is not clearly either a name or a symbol. The symbolic language and the English it is embedded in seem to be quite sharply distinguished.

Terminology   I have adopted the distinction between name and symbol from [Beccari, 1997] , who presumably is following the usage of [ISO, 1982] which at this writing I have not seen yet.

identify   To **identify** an object A with another object B is to regard them as identically the same object. This may be done via some formalism such as an amalgamated product or a pushout in the sense of category theory, but it may also be done in a way that suppresses the formalism (as in Example 1 below).

Example 1   The Möbius strip may be constructed by identifying the edge

$$\{(0, y) \mid 0 \leq y \leq 1\}$$

of the unit square with the edge

$$\{(1, y) \mid 0 \leq y \leq 1\}$$

in such a way that $(0, y)$ is identified with $(1, 1 - y)$.

Remark 1   One may talk about identifying one *structure* (space) with another, or about identifying individual *elements* of one structure with another. The word is used both ways as the citations illustrate. Example 1 uses the word both ways in the same construction.

Remark 2   One often identifies objects without any formal construction and even without comment. That is an example of conceptual blending; examples are given there.

Citations (Maz93.29), (Mor92.91).

identity   This word has three common meanings.

(a) Equation that always holds An **identity** is an equation that holds between two expressions for any valid values of the variables in the expressions. Thus, for real numbers, the equation $(x + 1)^2 = x^2 + 2x + 1$ is an identity.

(b) Identity element of an algebraic structure If $x \, \Delta \, e = e \, \Delta \, x$ for all x in an algebraic structure with binary operation Δ, then e is an **identity** or **identity element** for the structure.

(c) Identity function For a given set S, the function from S to S that takes every element of S to itself is called the **identity function**. This is an example of a polymorphic definition.

if

(a) Introduces conditional assertion The many ways in which "if" is used in translating conditional assertions are discussed under conditional assertion.

(b) In definitions It is a convention that the word if used to introduce the definiens in a definition means "if and only if".

> *Example 1* "An integer is **even** if it is divisible by 2."
> *Citation* (GraTre96.105); (LewPap98.20), definition of "equinumerous". Some authors regularly use "if and only if" or "iff". *Citation:* (Epp95.534), (Sol95.144).
> This is discussed (with varying recommendations) in [Gillman, 1987] , page 14; [Higham, 1993] , page 16; [Krantz, 1997] , page 71; [Bagchi and Wells, 1998a] .

(c) In the precondition of a definition "If" can be used in the precondition of a definition to introduce the structures necessary to make the definition, in much the same way as let. See Example 5 under definition.

> *Citations* (Ant84.90), (KolBusRos96.109).
> See also the discussion under let.

if and only if   This phrase denotes the relation equivalent that may hold between two assertions.

iff　Abbreviation of if and only if.
　　　Citation (DorHoh78.166), (Pin64.108).

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illustration   A drawing or computer rendering of a curve or surface may be referred to as an **illustration**. Thus a drawing of (part of) the graph of the equation $y = x^2$ would be called an illustration.

The word is also used to refer to an instance of an object that satisfies the hypotheses and conclusion of a theorem. (This is also called an example of the theorem.)

Example 1   A professor could illustrate the theorem that a function is increasing where its derivative is positive by referring to a drawing of the graph of $y = x^2$.

Example 2   The fact that subgroups of an Abelian group are normal could be illustrated by calculating the cosets of the two-element subgroup of Z_6. This calculation might not involve a picture or drawing but it could still be called an illustration of the theorem.

Citations (Bil73.1107), (BruMarWei92.140).

image   Used to refer to the mental representation of a concept. This word also has a polymorphic mathematical meaning discussed under overloaded notation.

implication   See conditional assertion.

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in general   The phrase "in general" occurs in at least two ways in mathematical statements. (One may often use "generally" with the same meaning.)

Example 1   "The equation $x^2 - 1 = (x-1)(x+1)$ is true in general."

Example 2   "In general, not every subgroup of a group is normal."

Example 1 asserts that the equation in question is always true. Example 2 does not make the analogous claim. In searching for citations I have found many uses of the phrase where it was quite difficult to know which meaning the author intended: one must have a good grasp of the subject matter to determine that. The examples suggest that syntax may give a clue as to the author's intentions. This phrase requires further investigation. ***Citations needed.***

This phrase should probably be deprecated.

Acknowledgments   Owen Thomas.

in other words   This phrase means that what follows is equivalent to what precedes. Usually used when the equivalence is easy to see.

    Citation (MacBir93.43).

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in particular   Used to specify that the following statement is an instantiation of the preceding statement, or more generally a consequence of some of the preceding statements. The following statement may indeed be equivalent to the preceding one, although that flies in the face of the usual meaning of "particular".

> *Example 1*   "We now know that f is differentiable. In particular, it is continuous."
> *Citations* (Duk97.193), (Kra95.40), (Mau78.575).

> *Remark 1*   In the literature search I have found examples where what follows did not seem to be a consequence of what preceded in any reasonable sense, for example in (Zal80.162), but this may be the result of my ignorance.

in your own words   Students are encouraged in high school to describe things "in your own words". When they do this in mathematics class, the resulting reworded definition or theorem can be seriously misleading or wrong. It might be reasonable for a teacher to encourage students to rewrite mathematical statements in their own words and then *submit them to the teacher*, who would scrutinize them for dysfunctionality.

   Example 1   Students frequently use the word "unique" inappropriately. A notorious example concerns the definition of function and the definition of injective, both of which students may reword *using the same words*:

   "A function is a relation where is a unique output for every input."

   "An injective function is one where is a unique output for every input."

include   For sets A and B, B **includes** A, written $A \subseteq B$, if every element of A is an element of B. See the discussions under contain and formal analogy.

indefinite article   The word "**a**" or "**an**" is the **indefinite article**, one of two articles in English.

(a) Generic use In mathematical writing, the indefinite article may be used in the subject of a clause with an identifier of a type of mathematical object (producing an indefinite description) to indicate an arbitrary object of that type. Note that plural indefinite descriptions do not use an article. This usage occurs outside mathematics as well and is given a theoretical treatment in [Kamp and Reyle, 1993] , section 3.7.4.

> *Example 1*
>
> "Show that an integer that is divisible by four is divisible by two."

Correct interpretation: Show that *every* integer that is divisible by four is divisible by two. *Incorrect interpretation:* Show that some integer that is divisible by four is divisible by two. Thus in a sentence like this it the indefinite article has the force of a universal quantifier. Unfortunately, this is also true of the definite article in some circumstances; more examples are given in the article on universal quantifier.

> *Citation* (Gie71.37), (Ros93.208) (for the indefinite article); (Ros93.223) (for the definite article).

> *Remark 1*   This usage is deprecated by Gillman [1987] , page 7. Hersh [1997a] makes the point that if a student is asked the question above on an exam and answers, "24 is divisible by 4, and it is divisible by 2", the student should realize that with that interpretation the problem is too trivial to be on the exam.

> *Remark 2*   An indefinite description apparently has the force of universal quantification only in the subject of the clause. Consider:

> a) "A number divisible by 4 is even." (Subject of sentence.)
> b) "Show that a number divisible by 4 is even." (Subject of subordinate clause.)
> c) "Problem: Find a number divisible by 4." (Object of verb.) This does not mean find *every* number divisible by 4; one will do.

> *Remark 3*   In ordinary English sentences, such as
>
> "A wolf takes a mate for life."

([Kamp and Reyle, 1993] , page 294), the meaning is that the assertion is true for a *typical* individual (typical wolf in this case). In mathematics, however, the assertion is required to be true without exception. See concept.

The universal force of an indefinite description can be changed by context, so that one cannot always tell merely from syntax whether the universal quantification is intended.

(b) Existential meaning An indefinite description may have existential force.

Example 2

"A prime larger than 100 was found in 2700 B.C. by Argh P. Ugh."

This does *not* mean that Mr. Ugh found *every* prime larger than 100. In this case the indefinite description is the subject of a *passive* verb, but in ordinary English indefinite subjects of active verbs can have existential force, too, as in "A man came to the door last night selling toothbrushes". I have found it difficult to come up with an analogous example in the mathematical register. This needs further analysis.

indefinite description   An **indefinite description** is a noun phrase that is marked with the indefinite article in the singular and no article in the plural. It typically refers to something not known from prior discourse or the physical context.

Example 1   Consider this passage:

"There is a finite group with the property that for some proper divisor n of its order a subgroup of order n does not exist. However, groups also exists that have subgroups of every possible order."

The phrases "a finite group", "a subgroup", and "subgroups" are all indefinite descriptions.

An indefinite description may have a generic use, discussed under indefinite article.

Remark 1   This description of indefinite descriptions does not do justice to the linguistic subtleties of the concept. See [Kamp and Reyle, 1993] , section 1.1.3. I would appreciate knowing of a reference to a less technical exposition of the subject.
References needed.

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infinite   The concept of infinity causes trouble for students in various ways.

(a) Failure of intuition concerning size Students expect their intuition on size to work for infinite sets, but it fails badly. For example, a set and a proper subset can have the same cardinality, and so can a set and its Cartesian product with itself. (As Atish Bagchi pointed out to me, the intuition of experienced mathematicians on this subject failed miserably in the nineteenth century!) This is discussed further under snow.

(b) Infinite vs. unbounded Students may confuse "infinite" with "unbounded". Computer science students learn about the set A^* of strings of finite length of characters from an alphabet A. There is an *infinite number* of such strings, each one is of *finite length*, and there is *no limit* on how long they can be (except to be finite). I have seen students struggle with this idea many times.

(c) Treating "∞" as a number. Of course, mathematicians treat this symbol like a number in *some* respects but not in others. Thus we sometimes say that $1/\infty = 0$ and we can get away with it. Students then assume we can treat it like a number in other ways and write $\infty/\infty = 1$, which we cannot get away with. This is an example of extrapolation.

References The mathematical concepts of infinity are discussed very perceptively in [Lakoff and Núñez, 2000] , Chapter 8.

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infix notation   A function of two variables may be written with its name between the two arguments. Thus one writes $3 + 5$ rather than $+(3, 5)$. Usually used with binary operations that have their own nonalphabetical symbol. See prefix notation and postfix notation.

injective   A function f is **injective** if $f(x) \neq f(y)$ whenever x and y are in the domain of f and $x \neq y$. Also called **one-to-one**.

Remark 1   When proving statements using this concept, the contrapositive form of the definition is often more convenient.

Remark 2   Students often confuse this concept with the univalent property of functions. See in your own words.

Remark 3   I recall that in the sixties there were older mathematicians who became quite incensed if I said "injective" instead of "one-to-one". At the time I connected this attitude with an anti-Bourbaki stance.

The last one who had this attitude (that I can remember) died recently. That is how language changes.

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insight   You have an **insight** into some mathematical phenomenon if you have a sudden jump in your understanding of the phenomenon. This may be accompanied by ejaculations such as " **aha**!" or " **eureka**!". The jump may be in incremental (but not gradual!) increase in understanding (worthy of "aha!") or a complete leap from incomprehension to clarity ("eureka!").

Remark 1   In my experience, the clarity that you feel after a Eureka insight tends to become a bit cloudy as you become aware of subtleties you didn't originally notice.

Example 1   The geometric diagram below proves that

$$a^2 - b^2 = (a - b)(a + b)$$

at least for positive real numbers a and b with $b < a$.
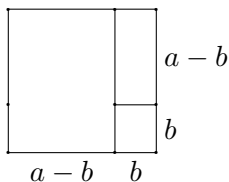
This causes many who have not seen it before to have a feeling something like: "Aha! Now I really understand it!" or at least, "Aha! Now I have a better grasp of why it is true." Even if you don't feel that way about this proof, you may have experienced a similar feeling about another theorem, perhaps one whose proof by symbol manipulation was more obscure.

Compare this with the proof given under symbol manipulation. Another example of the aha experience is given under conceptual. See also light bulb mistake. In many cases, the gain in insight is irreversible, an instance of the ratchet effect.

instantiate   To **instantiate** a variable in an expression is to replace it with an identifier of a specific mathematical object of the appropriate type. If all the variables in an expression are instantiated, the expression should denote a specific object.

Example 1   If you instantiate x at 5 in the expression $2x + 1$ you get an expression denoting 11.

integer   A whole number, positive, negative or zero.

Remark 1   I have no citation in which "integer" means nonnegative integer or positive integer. However, students quite commonly assume that the word means nonnegative or positive integer.

Remark 2   Many computer languages are arranged so that an integer is not a real number. This may be indicated by requiring that every number be explicitly declared as one or the other, or by the convention that a number is real only if it is represented using a decimal point. Students often assume that mathematicians follow that convention and need to be explicitly told that they don't.

Example 1   In Mathematica®, "32" is an integer and "32.0" is a real number.
See divide.

integral   An **integral** of a function is an antiderivative of the function. The word is also used to denote a solution of a more general differential equation and as an adjective to require that the modified noun phrase denote an integer (8 is an integral power of 2). I have known students to be confused by both these variants.

interpretation   An interpretation of a text is the current assignment of a value (possibly a variable object) to each identifier used in the discourse. With a given semantics, the text with that interpretation may result in statements about the values of the identifiers which may be true or false or (if some identifiers are variables) indeterminate. See context and standard interpretation.

In mathematical logic the language is a formal language and the values lie in some mathematical structure defined for the purpose.

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intensional   See semantics.

isomorphic   Each type of mathematical structure has its own definition of "isomorphism". The categorists' definition of isomorphism (a morphism that has an inverse) has all these definitions as special cases.

Difficulties Students frequently don't catch on to the fact that, if M and N are isomorphic structures of some type, there can be *many* isomorphisms between M and N.
     See copy and up to.

italics   A style of printing that *looks like this.* Many texts put a definiendum in italics. See definition.

jump the fence   If you are working with an expression whose variables are constrained to certain values, and you instantiate the expression at a value that violates the constraint, you **jump the fence**.

    Example 1   A student, in dealing with a sum of Fibonacci numbers, might write

$$\Sigma_{k=0}^{n} f(k) = \Sigma_{k=0}^{n} f(k-1) + \Sigma_{k=0}^{n} f(k-2)$$

not noticing that the sums on the right involve $f(-1)$ and $f(-2)$, which may not have been defined when the definition of Fibonacci number was given.

Terminology   The name "jump the fence" is my variation of the "fencepost error" discussed in [Raymond, 1991] .

just   One use of the word "just" in mathematical discourse is to indicate that what precedes satisfies the statement that occurs after the word "just".

Example 1   (Assuming r and s are known to be integers greater than 1).

" ... Then $m = rs$. But that is just the definition of "composite"."

(Or "That just means that m is composite".)

Remark 1   My own perception of this usage before I looked for citations is that the word "just" meant that what followed was *equivalent* to what preceded, but in many citations what follows is only a consequence of what precedes. Indeed, in some citations it is completely redundant.

Citations (Shp95.1303),(Mor88.814),(Put73.82).

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just in case This phrase means that what follows is logically equivalent to what precedes.

      Example 1 "An integer is even just in case it is divisible by 2."

      Citation (Put73.82),

juxtaposition   Two symbols are **juxtaposed** if they are written down one after the other. This most commonly indicates the numerical product, but is also used to denote other binary operations, in particular the concatenate of strings.

Citations (Cli59.106), (Fea82.161).

labeled style   The **labeled style** of writing mathematics requires labeling essentially everything that is written according to its intent: definition, theorem, proof, remark, example, discussion, and so on. Opposed to narrative style. This Handbook is written in a fairly strongly labeled style.

References   The labeled style was named and discussed in [Bagchi and Wells, 1998b] .

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lambda notation   A notation for referring to a function. The function is denoted by $\lambda x.e(x)$, where e is some expression that allows one to calculate the value at x. The x is bound in the expression $\lambda x.e(x)$.

   Example 1   "The function $\lambda x.x^2$ has exactly one critical point." This notation is used in mathematical logic, computer science, and linguistics, but not generally by mathematicians.

   Citation (Bez89.271).

   Compare barred arrow notation.

larger   A text that says one set is **larger** than another may be referring to ordering by inclusion, or may be referring to cardinality.

    Citation (Van92.35); note that the authors feel obligated to explain that they mean cardinality, not inclusion.

lemma   A theorem. One may typically expect that a lemma is not of interest for itself, but is useful in proving other theorems. However, some lemmas (König's Lemma, Schanuel's Lemma, Zorn's Lemma) have become quite famous.

Acknowledgments   Owen Thomas.

lemmata   Lemmas. An obsolete plural.

let   "Let" is used in several different ways in the mathematical register. What follows is a tentative classification. As remarked below, some of the variations in usage (as in Examples 1 and 2) make no difference to the logical argument that the usage expresses. This may make the classification seem excessively picky. I am not aware of research on students' misunderstandings in these situations.

   Assume and Suppose   In many cases, **assume** and **suppose** can be used instead of "let". There are subtle differences between the way they are used and the way "let" is used that need further investigation. Note that one says "Let x *be* … " but "Assume [Suppose] x *is* … " If can also be used in some of these situations. The grammar varies here: "If $x = 1$" cannot be a complete sentence, but "Let $x = 1$" can be.

(a) Introducing a new interpretation   The most common use of "let" is to introduce a new symbol, or change the interpretation of one or more symbols or of names. This, of course, is a species of definition.

   Example 1   Consider the theorem
      "An integer divisible by 4 is divisible by 2."
A proof could begin this way:
      "Let n be an integer divisible by 4."
This introduces a new variable symbol n and constrains it to be divisible by 4.

   Example 2   Suppose the theorem of the preceding example had been stated this way:
      "*Let n be an integer. If n is divisible by* 4 *then it is divisible by* 2."
Then the proof could begin
      "Let n be divisible by 4."
In this sentence, n is introduced in the theorem and is further constrained in the proof.

   Remark 1   These two examples illustrate that whether a new symbol is introduced or a previous symbol is given a new interpretation is a minor matter of wording; the underlying logical structure of the argument is the same.

   Remark 2   "Define" is sometimes used in this sense of "let"; see Example 3 under mathematical definition. Of course, there is no logical distinction between this use of "let" and a formal definition; the difference apparently concerns whether the newly introduced

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expression is for temporary use or global and whether it is regarded as important or not. Further investigation is needed to spell the distinction out.

There are several distinct possible purposes in introducing a new symbol. They are distinguished here because it may prove useful to make these distinctions overt to students.

(b) To introduce a hypothesis

Example 3 "Let n be an integer divisible by 4. Then n is even." If, assume and suppose can be used here, with requisite changes in syntax: "is" instead of "be" for assume and suppose, and the sentences must be combined into one sentence with "if".

Citations (BhaSer97.502), (GibBra85.691), (Bru93.370).

(c) To consider successive cases

Example 4 "Let $n > 0$. ... Now let $n \le 0$. ... " If, assume and suppose seem to be more common that "let" in this use. See now.

Citation (Ant84.121), (Djo82.233), (Kra95.40).

(d) To introduce the precondition of a definition

Example 5 "**Definition** Let n be an integer. Then n is **even** if n is divisible by 2." If, assume and suppose can be used here.

Citation (Kra95.57).

(e) To introduce an arbitrary object To pick an unrestricted object from a collection with the purpose of proving an assertion about all elements in the collection using universal generalization. Example 1 above is an example of this use. Often used with arbitrary. If, assume and suppose can be used here.

(f) To name a witness To provide a local identifier for an arbitrary object from a collection of objects known to be nonempty. Equivalently, to choose a witness to an existential assertion that is known to be true. If, assume and suppose can be used here.

Example 6 In proving a theorem about a differentiable function that is increasing on some interval and decreasing on some interval, one might write:

"Let a and b be real numbers for which $f'(a) > 0$ and $f'(b) < 0$."

These numbers exist by hypothesis.

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Example 7   In the context that G is known to be a noncommutative group:

"Let x and y be elements of G for which $xy \neq yx \ldots$"

The following is a more explicit version of the same assertion:

"Let the noncommutative group G be given. Since G is noncommutative, the collection $\{(x, y) \in G \times G \mid xy \neq yx\}$ is nonempty. Hence we may choose a member (x, y) of this set $\ldots$ "

A variation on this is *parametrized choice*: Given that $(\forall x)(\exists y)Q(x, y)$ and given c, let d be an object such that $Q(c, d)$.

Example 8   Assuming c is a complex number:

"Let d be an nth root of c."

(g) "Let" in definitions   Let can be used in a definition proper.

Example 9   "Let an integer be **even** if it is divisible by 2."

Remark 3   This usage strikes me as unidiomatic. It sounds like a translation of a French ("Soit $\ldots$ ") or German ("Sei $\ldots$ ") subjunctive. If, assume and suppose cannot be used here. **Citations needed.**

Remark 4   The way "let", assume, suppose and if differ in their usage needs much more analysis.

References   This article follows the discussion in [Bagchi and Wells, 1998a] .

Acknowledgments   Atish Bagchi, Owen Thomas.

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lg See logarithm.

light bulb mistake   The **light bulb mistake** is the belief that mathematical discovery, by researchers and students, should normally consist of a series of (in my words) major insights. In fact, in much research, the insights are uncommon and often tiny, coming after tiresome calculation and much frustration.

The name comes from [Rota, 1996] , pages 130 ff. He refers specifically to insights which come with a feeling of having experienced something beautiful.

linking metaphor   See metaphor.

ln   See logarithm.

local   With respect to a structure $\mathfrak{M}$, an object is defined **locally** if it is in some sense defined only on a substructure of $\mathfrak{M}$. It is defined **globally** if it is defined on all of $\mathfrak{M}$. This usage is usually informal, but in some cases the word "local" or "global" has a formal definition.

Example 1   The phrases local identifier and global identifier in this text (borrowed from computer science) are examples of informal usage of the terms.

The words may be used in settings outside the mathematical register. For example, one might complain that one understands a proof "locally but not globally", meaning that one can follow the individual steps but has no overall grasp of the proof.

Citations needed.

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local identifier   A local identifier in a segment of a mathematical text is an identifier for a particular mathematical object that has that meaning only in the current block of text. The block of text for which that meaning is valid is called the **scope** of the identifier.

The scope may be only for the paragraph or subsection in which it is defined, with no explicit specification of the scope given. If the scope is at the chapter level or higher the author may make it explicit.

Example 1

 "Throughout this chapter f will be a continuous function."

Citations needed.

See also global identifier.

logarithm   The expression "$\log x$" has a suppressed parameter, namely the base being used. My observation is that in pure mathematics the base is normally e, in texts by scientists it may be 10, and in computer science it may be 2, and that in all these cases the base may not be explicitly identified.

Students in particular need to know that this means there are *three different functions* in common use called "log". See also global parameters and trigonometric functions.

Remark 1   In calculus texts, $\log_e$ may be written " **ln**", and in computer science $\log_2$ may be written " **lg**".

Citation (Bar84.429), (GreHoo98.36), (LleTovTri88.913).

Acknowledgments Owen Thomas.

look ahead   When performing a calculation to solve a problem, one may **look ahead** to the form the solution must take to guide the manipulations one carries out.

Example 1   Given a right triangle with legs a and b and hypoteneuse c, one can derive the Pythagorean Theorem $a^2 + b^2 = c^2$ from the identity

$$\sin^2 \theta + \cos^2 \theta = 1$$

by rewriting it as

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$$

and then multiplying by c^2. Olson [1998] discovered that when asked to reverse the process to derive the trig identity from the Pythagorean Theorem, some students balked at the first step, which is to divide the equation $a^2 + b^2 = c^2$ by c^2, because "there is no reason to divide by c^2":

The students apparently could think of no method or algorithm which said to do this. Of course there *is* a method — **look ahead** to see what form of the equation you need. More about this example in section (a) under attitudes. This is related to walking blindfolded.

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lowercase   See case.

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Luddism   **Luddism** is an unreasoning opposition to all technological innovation. Luddites appear in mathematics, most noticeable lately concerning the use of calculators and computers by students. There is also resistance to new terminology or notation (see Remark 3 under injective).

Remark 1   There is a legitimate debate over such questions as: Should calculators be withheld from students until they can do long division rapidly and accurately? Should Mathematica be withheld from students until they can carry out formal integration rapidly and accurately?

Not everyone opposed to calculators or computers is a Luddite. Unfortunately, professors by their nature tend to be skilled in argumentation, so it may take long anthropological observation to distinguish a Luddite from a rational opponent of a particular piece of technology.

Remark 2   The two questions in the preceding remark do not have to be answered the same way. Nor do they have to be answered the same way for math majors and for other students.

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macron   See bar.

malrule   A **malrule** is an incorrect rule for syntactic transformation of a mathematical expression. Examples are given in the entry for extrapolate.

This name comes from the mathematics education literature.

matchfix notation   Same as outfix notation.

map   Also **mapping**. Some texts use it interchangeably with the word "function". Others distinguish between the two, for example requiring that a mapping be a continuous function. See also section (g) under function.

　　Citation (RosWri92.19).

Acknowledgments Michael Barr.

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mathematical discourse   is the discourse used by mathematicians and students of mathematics for communicating mathematical ideas in a broad sense, including not only definitions and proofs but also approaches to problem solving, typical errors, and attitudes and behaviors connected with doing mathematics.

This is my own definition; I have no citations for it.

mathematical education   One purpose of this Handbook is to raise mathematicians' awareness of what specialists in mathematical education have found out in recent years. The following entries discuss that and have pointers to the literature.

abstraction	function
and	generalization
APOS	if
behaviors	malrule
cognitive dissonance	mathematical object
compartmentalization	mental representation
concept	order of quantifiers
conceptual blend	proof
constructivism	reification
covert curriculum	representation
definition	self-monitoring
equations	symbolic expression
example	syntax
extrapolate	universal quantifier

The Advanced Mathematical Thinking website at `http://www.soton.ac.uk/~amt` provides an introduction to research in mathematics education at the level of college and university students with an extensive list of references. Other useful websites are `http://www.bham.ac.uk/ctimath/talum/newsletter` and `http://forum.swarthmore.edu/library/ed_topics`.

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mathematical logic   **Mathematical logic** is any one of a number of mathematical structures that models many of the assertions spoken and written in the mathematical register; such a structure typically is provided with rules for proof and rules for giving meaning to items in the structure (semantics). The phrases **formal logic** and **symbolic logic** are also used; the latter appears to me to be obsolescent. ***Citations needed.***

First Order Logic The most familiar form of mathematical logic is **first order logic**, in which, as in many other forms of logic, sentences are represented as strings of symbols. For example,

      "There is an m such that for all n, $n < m$"

could be represented as "$\exists m \forall n \, (n < m)$".

    First order logic is a useful codification of many aspects of mathematical formalism, but it is not the natural or inevitable result of any attempt of formalizing mathematics. For example, Hintikka [1996] has developed a logic, somewhat stronger than first order logic (but without its nice completeness properties), in which the dependencies among variables does not depend on the order of the quantifiers. The approach of category theory to model theory, as expounded in [Makkai and Reyes, 1977] , [Makkai and Paré, 1990] , and [Adámek and Rosičky, 1994] , produces formal systems that are very different in character from standard first order logic and that vary in strength (in both directions) from first order logic.

    This situation is in contrast to the facts about the theory of computation, where various attempts to give a mathematical definition of "computable function" have all given the same class of functions.

    Nevertheless, first order logic has better formal properties than many other codifications of mathematical reasoning, and many mathematicians operate in the belief that the assertions and proofs they give in the mathematical register can in principle be translated into first order logic. This is desirable because a purported proof in the formal symbolism of mathematical logic can be mechanically checked for correctness.

    In practice no substantial proof gets so far as to be expressed in logical symbolism; in fact to do so would probably be impossibly time-consuming and the resulting proof not mechanically checkable because it would be too large. What does happen is that someone

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will challenge a step in a proof and the author will defend it by expanding the step into a proof containing more detail, and this process continues until everyone is satisfied. The mathematicians mentioned in the preceding paragraph may believe that if this expansion process is continued long enough the proof will become a proof in the sense of mathematical logic, at least in the sense that every step is directly translatable into logical formalism.

The best place to see the argument that every mathematical proof can *in principle* be translated into first order logic is the book [Ebbinghaus, Flum and Thomas, 1984] (read the beginning of Chapter XI). In particular, proofs involving quantification over sets can be expressed in first order logic by incorporating some set of axioms for set theory. Even if this is so, caveats must be attached:

1. First order logic may be optimal for mathematical reasoning, but not for reasoning in everyday life or in other sciences.

2. First order logic is clearly not the ideal language for *communicating* mathematical arguments, which are most efficiently and most clearly communicated in the mathematical register using a mixture of English and the symbolic language.

Aside from those caveats there is a more controversial point. Consider the proof involving the monk given in Example 2 under conceptual blend. This proof can probably be transformed into a proof in first order logic (making use of continuous mappings and the intermediate value theorem), but the resulting proof *would not be the same proof* in some sense. In particular, it loses its physical immediacy. Many geometric proofs as well have a (physical? visual?) immediacy that is lost when they are translated into first order logic.

One could defend the proposition that all proofs can be translated into first order logic by either denying that the monk proof (and a pictorial geometric proof) is a mathematical proof, or by denying that the translation into first order logic changes the proof. The first approach says Euclid was not doing mathematics. The second violates my own understanding of how one does mathematics, because what is lost in the translation is for me *the heart of the proof.* Specifically, the checking one could do on the first order logic form of the proof would not check the physical or geometric content.

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I do not deny that the translation process correctly models one sort of proof as another sort of proof, and that it is a Good Thing that this can be done, as it usually is when one kind of mathematics is modeled in another. The point is that the two kinds of proof are different and both must be regarded as mathematics.

See order of quantifiers, translation problem, and Vulcanism.

References   First order logic is presented in the textbooks [Mendelson, 1987] , [Ebbinghaus, Flum and Thomas, 1984] , [van Dalen, 1989] . The formalisms in these books are different but equivalent.

Acknowledgments   Discussions with Colin McLarty.

mathematical mind   People who have tried higher level mathematical courses and have become discouraged often say, "I just don't have a mathematical mind" or "I am **bad at math**". Some possible reasons for this attitude are discussed under ratchet effect, trivial and yes it's weird. Reasons for people being discouraged about mathematics (or hating it) are discussed in [Kenschaft, 1997] .

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mathematical object   Mathematical objects are what we refer to when we do mathematics. Mathematicians talk about mathematical objects using most of the same grammatical constructions in English that they use when talking about physical objects.

Mathematical objects are like physical objects in that our experience with them is *repeatable*: If you ask many mathematicians about a property of some particular mathematical object that is not too hard to verify, they will generally agree on what they say about it, and when there is disagreement they commonly discover that someone made a mistake or misunderstood the problem.

This is not the place to consider what a mathematical object "really is" or even to give a proper definition of one. However, I will make some distinctions and give examples.

(a) Types of mathematical objects It is useful to distinguish between **specific** mathematical objects and **variable** ones.

Example 1   The number 3 is a specific mathematical object. So is the sine function (once you decide whether you are using radians or degrees).

Example 2   Consider the discourse:
  "Let G be a group with identity element e and an element a for which $a^2 = e$.
  Then $a = a^{-1}$."
The author or speaker may go on to give a proof, talking about G, e, and a with the same syntax used to refer to physical objects and to specific mathematical objects such as 3 or the sine function (see also Platonism). Because of the way the proof is written, the writer will appear to have in mind not any specific group, and not all possible groups, but a *nonspecific* or *variable* group. So the natural interpretation of G is as a variable or generic mathematical object.

Remark 1   The statement above that 3 is a specific mathematical object would not be accepted by everyone. As Michael Barr pointed out in a response to a previous version of this article, there are various possible definitions of the natural numbers and each one has its own element called 3. Nevertheless, it appears to me that mathematicians normally speak and think of the number 3 as one specific mathematical object, and it is customary usage that this Handbook is concerned with. See fundamentalist and unique.

(b) More about variable mathematical objects There are various approaches via
mathematical logic or category theory to giving a formal mathematical definition of
"variable object". In classical logic an interpretation of discourse such as that in
Example 2 assigns a specific group to G, its identity element to e, an element of that
group to a, and so on. An assertion containing identifiers of variable mathematical objects
is said to be true if it is true in all interpretations. I will call this the **logician's semantics**
of variables. It would be reasonable to identify the variable object with the symbol in the
formal language (such as G in the example above) corresponding to it. Another possibility
would be to identify the variable object with the set of all possible interpretations,
although to do that correctly would require dealing with the fact that that "set" might
actually be a proper class. I don't know whether anyone has worked out this point of
view. ***References needed.***

Categorists have another approach to the concept of variable mathematical object.
One defines a **theory**, which is a specific category (the theory for groups, for example).
The theory contains a specific object g. Every group is the value at g of a certain type of
functor based on that theory. It is natural to interpret the object g of the theory (or,
perhaps better, the entire theory) as the object denoted by the identifier G in Example 2
above.

The fact that a variable real number (for example) is neither positive nor negative,
and that a variable group is neither Abelian nor non-Abelian, is a sign that in reasoning
about variable objects *the logic you use must be restricted*, in particular missing the law of
the excluded middle. That sort of thing is worked out explicitly for the categorical
approach in [Fourman, 1977] , [Makkai and Reyes, 1977] , [Fourman and Vickers, 1986] ,
[Lambek and Scott, 1986] .

The approaches suggested so far are *general* ways of understanding variable objects.
Certain *specific* constructions for particular types of variable objects have been known for
years, for example the construction of the variable x in the polynomial ring of a field as an
infinite sequence that is all 0's except for a 1 in the second place.

See also determinate, variate and Platonism.

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(c) The nature of mathematical objects Some of the difficulties students have when reasoning about mathematical objects may have to do with the properties we regard them as having. Specifically, as most mathematicians think of them, mathematical objects are **inert** (don't interact with other objects, even other mathematical objects) and **eternal** (time passing does not affect them in any way).

Remark 2 A dentist may tell you that he has a hole in his schedule at 3PM next Monday, would you like to come then? That hole in his schedule is certainly not a physical object. It is an abstract object. But it is not a mathematical object; it interacts with physical objects (people!) and it changes over time.

(d) Difficulties A central difficulty for students beginning the study of mathematics is being able to conceive of objects such as the sine function *as an object*, thus reaching the third stage of the APOS theory. This is the problem of reification. Students also confuse a mathematical object with the symbols denoting it. [Pimm, 1987] discusses this in children, pages 17ff, and much of the mathematical education literature concerning function mentions that problem, too, as well as the more severe problem of reification. See also Example 2 under definition.

The difficulties students have with conditional sentences may be related to the inert and eternal nature of mathematical objects discussed above under (c); this is discussed further under only if and contrapositive.

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mathematical register   This is a special register of the English language used for communicating mathematical definitions, theorems, proofs and examples. It includes a special version of English as well as the symbolic language of mathematics, mixed together.

Distinctive features of the mathematical register of English include

a) Ordinary words used in a technical sense, such as "function", "include", "integral", and "group".

b) Technical words special to the subject, such as "topology", "polynomial", and "homeomorphism".

c) Syntactic structures used to communicate the logic of an argument that are similar to those in ordinary English but with *differences in meaning*. Examples: "all", "if ... then", "let", "or", "there is", "the", ...

Any register belonging to a technical subject has items such as (a) and (b). Some words like these are listed in this Handbook, including words that cause special problems to students and words that are used with multiple meanings.

The syntactic structures mentioned in (c) are major stumbling block for students. It appears to me that these structures make the mathematical register quite unusual even among technical registers in general in how far its semantics deviates from the semantics of ordinary English. (However, every tribe thinks it is "more different" than any other tribe ...) Some of these syntactic structures involve expressions that are used with meanings that are subtly different from their meanings in ordinary English or even in the general scientific register. Some of these are discussed in detail in the following entries:

arbitrary	if	order of quantifiers
but	indefinite article	some
conditional assertion	just	such that
contrapositive	larger	universal generalization
definite article	negation	universal quantifier
disjunction	only if	vacuous implication
existential quantifier	or	

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References   There seem to be very few articles that specifically study the mathematical register. Some aspects are described in [Epp, 1999a] , [Pimm, 1987] , [Schweiger, 1994a] , [Schweiger, 1994b] , [Schweiger, 1996] . Steenrod [1975] , page 1, distinguishes between the mathematical register (which he calls the "formal structure") and other registers.

N. J. de Bruijn [1994] introduces the concept of the **mathematical vernacular**. He says it is "the very precise mixture of words and formulas used by mathematicians in their better moments". He excludes some things, for example proof by instruction ([de Bruijn, 1994] , page 267), which I would include in the mathematical register. He makes a proposal for turning a part of the mathematical vernacular into a formal system and in the process provides a detailed study of part of (what I call the) mathematical register as well as other types of mathematical writing.

Remark 1   Many mathematical texts include discussions of history, intuitive descriptions of phenomena and applications, and so on, that are in a general scientific register rather than the mathematical register. Some attempts to classify such other types of mathematical writing may be found in [Bagchi and Wells, 1998b] , [de Bruijn, 1994] , and in Steenrod's article in [Steenrod *et al.*, 1975] .

References   Much of the current discussion is drawn from [Bagchi and Wells, 1998b] .

Acknowledgments   Cathy Kessel.

mathematical structure   A **mathematical structure** is a set (or sometimes several sets) with various associated mathematical objects such as subsets, sets of subsets, operations of various arities, and relations, all of which must satisfy various requirements. The associated mathematical objects are called the **structure** and the set is called the **underlying set**. Two examples of definitions of mathematical structures may be found under equivalence relation.

Example 1   A topological space is a set S together with a set T of subsets of S satisfying certain requirements.

Remark 1   The definition above of mathematical structure is not a mathematical definition. To give a proper mathematical definition of "mathematical object" results in an unintuitive and complicated construction and the only reason for giving the idea a mathematical definition is for purposes of foundations. In this category theorist's mind, the concept of set-
with-
structure is anyway the wrong way to do foundations of mathematics.

For example, this approach doesn't fit the naive picture of a space to start with a set of points. A space ought to be a chunk with parts, not a collection of points. The points ought to be *hard to see*, not the first thing you start with in the definition. This point of view has been developed by category theorists such as William Lawvere. [Lawvere and Schanuel, 1997] provides an easy introduction to the categorical approach.

Difficulties Presenting a complex mathematical idea as a mathematical structure involves finding a minimal set of associated objects (the structure) and a minimal set of conditions on those objects from which the theorems about the structure follow. The minimal set of objects and conditions may not be the most important aspects of the structure for applications or for one's mental representation.

Example 2   A function is commonly defined as a set of ordered pairs with a certain property. A mathematician's picture of a function has many facets: how it models some covariation (for example, velocity), its limiting behavior, algorithms for calculating it, and so on. The set of ordered pairs is not what first comes to mind, except perhaps when one is thinking of the function's graph.

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Remark 2 Another aspect of definitions of structure is that the same structure can have two very different looking definitions. An example is given under fundamentalist.

mean

(a) To form a definition "Mean" may be used in forming a definition. *Citation:* (Adl77.619).

> *Example 1*   "To say that an integer is **even** means that it is divisible by 2."

(b) Implies To say that a statement P means a statement Q may mean that P implies Q. *Citation:* (Kra95.40).

> *Example 2*   "We have proved that 4 divides n. This means in particular that n is even."

member   See element.

mental representation   One's **mental representation** (also "internal representation" of a particular mathematical concept is the cognitive structure associated with the concept, including associated metaphors, mental pictures, examples, properties and processes.

   The mental representation is called the **concept image** by many writers in mathematical education. The definition just given is in fact a modification of the definition of concept image given by Tall[1992b] , page 7. The way human concepts are organized, as described by cognitive scientists such as George Lakoff [Lakoff, 1986] , includes much of the structure of the mental representation of the concept in my sense. This is discussed further under concept, prototype and radial concept.

   In written or spoken mathematical discourse, discussion of some aspect of the mental representation of a concept is often signaled by such phrases as "intuitively" or "you can think of . . . ". *Citations needed.*

(a) Mental representations and definitions   The contrast between a student's mental representation of a concept and its mathematical definition is a source of cognitive dissonance; students may avoid the disparity by ignoring the definition. The disparity comes about from inappropriate learning strategies such as generalization and extrapolation.

   Professional mathematicians who are learning a subject know they must adjust their mental representation to the definition. In contrast, in doing research they often quite correctly *adjust the definition* instead of their mental representation. That is a primary theme of [Lakatos, 1976] .

References   Many articles in the book [Tall, 1992a] discuss mental representation (under various names often including the word "image") in depth, particularly [Tall, 1992b] , [Dreyfus, 1992] and [Harel and Kaput, 1992] . See also [Dieudonné, 1992] V.6, page 163, [Kieran, 1990] , [Meel, 1998] (expecially pages 168–170), [Piere and Kieren, 1989] , [Presmeg, 1997a] , [Tall and Vinner, 1981] , [Wells, 1995] , [Wheatley, 1997] .

   Mental imagery is discussed from a philosophical point of view, with many references to the literature, by Dennett[1991] , Chapter 10. The book [Lakoff, 1986] is concerned with concepts in general, with more of a linguistic emphasis.

   A sophisticated mental representation of an important concept will have various

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formalisms and mental pictures that fit together by conceptual blending or metaphor. [Lakoff and Núñez, 1997] regard metaphor as central to understanding what mathematics is all about.

Remark 1   I have known logicians and computer scientists (but not many, and no mathematicians) who deny having any nonsymbolic mental representations of mathematical concepts. Some of them have claimed to be entirely syntax directed; all they think of is symbols. Perhaps some of these colleagues do have mental representations in the broad sense, but not pictorial or geometric ones. Possibly the phrase "mental image" should be restricted to cases where there is geometric content.

See also aha, conceptual, mathematical object, Platonism and representation.

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metaphor   A metaphor is an implicit identification of part of one situation with part of another. Synecdoche is a form of metaphor. Metaphor, like analogy, is an aspect of conceptual blending.

I am using "metaphor" here to describe a type of thought configuration, a form of conceptual blend. It is also of course used to refer to a figure of speech that communicates such a thought configuration. Other figures do this, too, for example similes. See [Lanham, 1991] for figures of speech and [Lakoff and Núñez, 2000] , Chapter 2, for an introduction to metaphors in cognitive science.

(a) Names from metaphors Many names in the mathematics register arose as metaphors.

Example 1   The interior of a closed curve is called that because it is like the interior in the everyday sense of a bucket or a house. The fact that the circle is two-dimensional instead of three-dimensional illustrates my description of a metaphor as identifying *part* of one situation with part of another. One aspect is emphasized; another aspect (where they differ) is ignored.

Example 2   Presumably the name "group" was grounded in the metaphor of a group of things as a thing. This is not a suggestive metaphor; after all, most mathematical structures involve a collection of things. The fact that groups have a name that means a collection of things is merely an unfortunate historical accident. The name is essentially a **dead metaphor** which does not normally surface to my consciousness when I think about groups.

(b) Grounding and linking metaphors Lakoff and Núñez [1997] , [1998] , [2000] divide metaphors in mathematics into two fundamental types: **grounding metaphors**, based on everyday experience, and **linking metaphors** that link one branch of mathematics to another. The following examples are derived from their work.

Example 3   The name "set" is grounded in the metaphor of "set as container".

Example 4   In college level mathematics we have another metaphor: set as object which can be the subject of operations. This is a linking metaphor (set as element of an algebra). This causes difficulties for students, particularly "set as element of a set"; see reification.

Example 5   The representation of a number as a location on a line, and more generally tuples of numbers as locations in a space, links numbers to geometry.

Example 6   The insight in the previous example got turned around in the late nineteenth century to create the metaphor of space as a set of points. Topology, differential geometry, and other branches of mathematics were invented to turn this metaphor into a mathematical definition, which both made the study of spaces more rigorous and also created unexpected structures such as space filling curves. See [Lakoff and Núñez, 1997] , [Lakoff and Núñez, 2000] for insightful discussion of these matters. See also conceptual blend, snow and Remark 1 under mathematical structure.

(c) Difficulties   Most important mathematical concepts are based on several metaphors; for example see the discussion under function. The daily use of these metaphors by mathematicians cause enormous trouble to students, because each metaphor provides a way of thinking about an A as a kind of B *in some respects.* The student naturally thinks about A as a kind of B in inappropriate respects as well.

Example 7   One metaphor for the real line is that it is a set of points (as in Example 6.) It is natural to think of points as tiny little dots; that is the way we use the word outside mathematics. This makes it natural to think that to the left and right of each point there is another one, and to go on and wonder whether two such neighboring points touch each other. It is valuable to think of the real line as a set of points, but the properties of a "line of points" just described *must be ignored* when thinking of the real line. In the real line there is no point next to a given one, and the question of two points touching brings inappropriate physical considerations into an abstract structure.

This example comes from [Lakoff and Núñez, 2000] .

Remark 1   The discussion in Example 7 is the tip of an iceberg. It may be that *most* difficulties students have, especially with higher-level mathematics (past calculus) are based on not knowing which aspects of a given metaphor are applicable in a given situation, indeed, on not being consciously aware that one has to restrict the applicability of the mental pictures that come with a metaphor.

Why not tell them? It would be appropriate for textbooks to devote considerable space to how mathematicians think of each concept, complete with a discussion of which

aspects of a metaphor are apt and which are not.

References   Besides the references cited above, see [English, 1997] , [Pimm, 1988] , [Sfard, 1994] , [Sfard, 1997] .

mnemonic   A **mnemonic** identifier is one that suggests what it is naming.

Example 1   Mathematical mnemonic identifiers usually consist of the initial letter of the word the identifier suggests, as f for a function, G for a group, and so on.

Example 2   Category theorists use "**Ab**" for the category of Abelian groups.

[Bagchi and Wells, 1998a]   urge the use of mnemonic identifiers and that use is exemplified in their research articles [Bagchi and Wells, 1997a]   and [Bagchi and Wells, 1997b] .

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model   A **model** of a phenomenon is a mathematical object that represents the phenomenon. In fact, the mathematical object is often called a representation of the phenomenon (this is discussed further under representation).

   Example 1   A moving physical object has a location at each instant; this is modeled by a function. One then observes that there is a relation between the derivative of the function and the average velocity of the physical object that allows one to define the instantaneous velocity of the object.

   Example 2   Mathematical logic is a mathematical model of some of the best-behaved part of mathematical discourse. Thus one models the assertion

   "For every x there is a y for which $xy = 1$"

by the formula $\forall x \exists y \, (xy = 1)$. See symbolic expression.

   Remark 1   Mathematical logic itself uses a concept of model (which has a precise mathematical definition) that is an abstraction of the notion of model discussed here.

multiple meanings   Many names and symbols in the mathematical register have more
than one meaning.

  Example 1   I recall as a graduate student being puzzled at the two meanings of
domain that I then knew, with the result that I spent a (mercifully short) time trying to
prove that the domain of a continuous function had to be an connected open set.

  Following is a list of entries in this Handbook of words and symbols that have two or
more distinct meanings. I have generally restricted this to cases where students are likely
to meet both usages by the time they are first year graduate students in mathematics.

algebra	field	map	result
argument	formal	model	revise
category	function	or	subscript
constructivism	graph	order	superscript
contain	identity	parenthesis	term
divide	if	permutation	trigonometric
domain	image	power	functions
elementary	integral	prime	
equivalent	larger	proposition	
family	logarithm	range	

must   One frequently finds "must be" used in the mathematical register when "is" would give the same meaning. I presume this is to emphasize that the fact being asserted can be proved from facts known in the context.

    Example 1   "If m is a positive integer and $2^m - 1$ is prime, then m must be prime." This example is from citation (Ros93.224).

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myths   Students in mathematics courses have many false beliefs about the subject which are perpetuated explicitly from class to class in their discussions with each other in attempting to explain a concept "in their own words". Some of the myths, sadly, are perpetuated by high school teachers. I list two here; it would be helpful to give them names as discussed under behaviors. Another example is given under element.

(a) The empty set Students in my discrete math classes frequently believe that the empty set is an element of every set. Readers of early versions of this book have told me that many high school teachers and even some college-level mathematicians believe this myth.
   Other problems with the empty set are discussed in the article about them.

(b) Limits Many students believe that a sequence with a limit "approaches the limit but never gets there". This is discussed under cognitive dissonance.

   Remark 1   Dysfunctional behavior is included as a "myth" in this Handbook only if it is the result of belief in statements made explicitly by the students. See attitudes.

N   The symbol ℕ usually denotes the set of natural numbers.
Citation (Epp95.76), (DavPri90.3).

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name   The **name** of a mathematical object is an English word or phrase used as an identifier of the object. It may be a determinate identifier or variate. It should be distinguished from a symbol used as an identifier. The distinction between name and symbol is discussed under identifier.

Sources of names

   Common words as names   A **suggestive name** is a a common English word or phrase, chosen to suggest its meaning. Thus it is a metaphor.

   Example 1   "Slope" (of a curve), or "connected subspace" (of a topological space). See the discussion of suggestive names in [Wells, 1995] and [Bagchi and Wells, 1998a] .

   Remark 1   Example 2 under semantic contamination shows the dangerous side of a name being suggestive.

   Learnèd names   A name may be a new word coined from (usually) Greek or Latin roots. Such an identifier is a **learned name**. (Pronounce "learned" with two syllables.)

   Example 2   "Homomorphism".

   Personal names   A concept may be named after a person.

   Example 3   L'Hôpital's Rule, Hausdorff space.

Difficulties   The possible difficulties students may have with common words used as identifiers is discussed under formal analogy and semantic contamination. See also cognitive dissonance and multiple meanings.

References   This discussion is drawn from [Bagchi and Wells, 1998a] . [Hersh, 1997a] gives many examples of dissonance between the mathematical meaning and the ordinary meaning of mathematical words.

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namely   Used to indicate that what follows is an explication (often a repetition of the definition) of what precedes.

Example 1   "Let G be an Abelian group, namely a group whose multiplication is commutative."

Example 2   "We now consider a specific group, namely $\mathcal{S}_3$".

The word is also used after an existence claim to list those things that are claimed to exist. (Of course, this is a special case of explication.)

Example 3   "12 has two prime factors, namely 2 and 3."

Citations (MacBer93.237), (MacBer93.241), (Nev94.875).

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narrative style   The **narrative style** of writing mathematics is a style involving infrequent labeling; most commonly, the only things labeled are definitions, theorems, proofs, and major subsections a few paragraphs to a few pages in length. The reader must deduce the logical status of each sentence from connecting phrases and bridge sentences. This is the way most formal mathematical prose is written.

Contrast labeled style.

This style is named and discussed in [Bagchi and Wells, 1998b] .

References [Bagchi and Wells, 1998b] .

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natural number   For some authors, a **natural number** is a positive integer. For others it is a nonnegative integer, and for others it is any integer. It appears to me that the most common meaning these days is that a natural number is a nonnegative integer.

   Citation (Epp95.76), (DavPri90.3), (Hat87.162), (New81.39).

   Remark 1   As the citations show, the disagreement concerning the meaning of this phrase dates back to the nineteenth century.

necessary Q is **necessary** for P if P implies Q. Examples are given under conditional
  assertion.

negation   The **negation** of an assertion P is an assertion that denies P. In some circumstances that is the effect of the English word **not**.

(a) Negation of quantified statements If $P(x)$ is a predicate possibly containing the variable x, then the negation of the assertion $\forall x\, P(x)$ is $\exists x\, \neg P(x)$. Similarly, the negation of the assertion $\exists x\, P(x)$ is $(\forall x)\neg P(x)$. Here, the symbol "$\neg$" means "not".

Remark 1   Both of these rules cause difficulty in translating to and from English. It is my experience that many students need to be explicitly taught these rules and how to express them in English.

Example 1   The negation of the assertion

   "All multiples of 4 are even."

is not

   "All multiples of 4 are not even."

but rather

   "Multiples of 4 are not all even."

or, equivalently,

   "Not all multiples of 4 are even."

Remark 2   This illustrates the fact that simply putting a "not" into a sentence may very well give the wrong results.

Example 2   In colloquial English as spoken by many people (including students!), the sentence

   "All multiples of 3 are not odd."

means that *some* multiples of 3 are not odd (a true statement). A similar remark holds for "Every multiple of 3 is not odd." I believe that most mathematicians would interpret it as meaning that no multiple of 3 is odd (a false statement). See Vulcanism. **Citations needed.**

Remark 3   This phenomenon quite possibly interferes with students' understanding of negating quantifiers, but I have no evidence of this.

never   An assertion about a variable mathematical object of the form "A is never B" means that for all A, A is not B. An assertion of that form when A is a function means that no value of A is B.

Example 1   "A real number never has a negative square."

Example 2

"The sine function is never greater than 1."

Citation (Duk97.193), (Pom96.1478). **Citations needed.**

See also always and universal quantifier.

not   See negation.

notation   **Notation** is a system of signs and symbols not belonging to a natural language used as a representation of something. The symbolic language of mathematics is a system of notation. See establish notation.

now

(a) Change of subject "Now" may indicate a change of subject. In this use, it may have the effect of canceling assumptions made in the preceding text in order to begin a new argument. *Citation:* (Kra95.40).

Example 1 "We have now shown that if $x \in A$, then $x \in B$. Now suppose $x \in B$. ..."

(b) Bring up a fact that is needed "Now" may be used to point out a fact that is already known or easily deduced and that will be used in the next step of the proof.

Example 2 In a situation where we already know that $x = 7$, one could say:

"... We get that $x^2 + y^2 = 100$. Now, x is 7, so $y = \sqrt{51}$."

This is similar to meaning (b) of but.

Citations needed.

(c) Superfluous word "Now" is commonly a filler word, often omissible without any effect. *Citation:* (Ade97.806).

Remark 1 Usages (a) and (c) are not always easy to distinguish.

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number theory   In spite of what the phrase suggests, **number theory** is the study of the integers, particularly with respect to properties of prime numbers.

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object See mathematical object.

obtain   Most commonly, "obtain" means "get", as in ordinary English.

  Example 1   "Set $x = 7$ in $x^2 + y^2 = 100$ and we obtain $y = \sqrt{51}$."

  In the mathematical register, "obtain" may also be used in much the same way as hold. This usage appears uncommon.

  Example 2

  "Let G be a group in which $g^2 = e$ obtains for every element g."

  Citations (Bau78.644), (FurMar91.842), (GelOlm90.85), (Zab95.486).

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ratchet effect   Once you acquire an insight, you may not be able to understand how someone else can't understand it. It becomes **obvious**, or **trivial** to prove. That is the **ratchet effect**.

This process probably involves synthesizing a new concept, as discussed by Dreyfus [1992] , section 3.2. See also [Thurston, 1990] .

Remark 1   It is distressingly common that a mathematician for whom a concept has become obvious because of the ratchet effect will then tell students that the concept is obvious or trivial. This is the phenonemon discussed in Remark 1 under trivial. It is a major point made in [Kenschaft, 1997] , page 30.

one to one   Injective.

only if   In the mathematical register, if P and Q are assertions, "P only if Q" means P implies Q. The phrase "only if" is not often used this way in ordinary English discourse.

   Example 1   The sentence
      "4 divides n only if 2 divides n"
means the same thing as the sentence
      "If 4 divides n, then 2 divides n."
   Example 2   The sentence
      "I will carry my umbrella only if it rains."
does not mean the same thing as
      "If I carry my umbrella, it will rain."

Difficulties   Students often get sentence in Example (1) backward, taking it to mean

$$(2 \text{ divides } n) \Rightarrow (4 \text{ divides } n)$$

Some of them flatly refuse to believe me when I tell them the correct interpretation. This is a classic example of semantic contaimination, a form of cognitive dissonance – two sources of information appear to contradict each other, in this case the professor and a lifetime of intimate experience with the English language, with the consequence that one of them is rejected or suppressed. It is hardly suprising that many students prefer to suppress the professor's unnatural and unmotivated claims.

   McCawley [1993] also rejects the equivalence of "A only if B" with "If A, then B", for ordinary discourse, but in the mathematical register the sentence must be taken to be equivalent to the others. This difference may have come about because conditional assertions in ordinary English carry connotations of *causality* and *time dependence*. Because mathematical objects are thought of as inert and eternal, the considerations that distinguish the two sentences in the example do not apply to statements such as the sentence in Example (1); the truth of the statement is determined entirely by the truth table for implication.

   The remarks in the preceding paragraph may explain some of the difficulties students have with the contrapositive, as well.

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onto Surjective.

open sentence   An **open sentence** is an assertion containing free variables. In many circumstances such an assertion is taken as being true for all instantiates of its variables. *Citation* (GraTre96.105). See universal quantifier for examples.

operation   Used to refer to a function of two variables that is written in infix notation. May be called a **binary operation**.

    Example 1   The operation of addition on the set of real numbers is a binary operation.

    Citation (Fra82.11); (MacBir93.43). Some authors use "operation" in certain contexts to refer to any function. *Citation:* (Sto95.619).

operator   **Operator** means function. Most authors seem to use "operator" only in certain restricted situations. It is often used when the domain is a set of functions or when the operator is a function from a space to itself.

Citation (Buc97.60). (But a "linear operator" can be between different spaces.)

The text [Grassman and Tremblay, 1996] uses "operator" to refer to a binary operation used in infix notation (see the discussion on pages 104 through 108). The text [Gries and Schneider, 1993] takes a similar approach (page 7 and page 387). The word is used to refer both to the symbol and to the function. This usage may be associated with authors having a background in computer science.

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or   **Or** placed between two assertions produces the disjunction of the assertions.

> *Example 1*   "x is nonnegative or $x < 0$".

Difficulties   As the truth table for disjunction indicates, "P or Q" allows both P and Q to be true, although they cannot both be true in the example just given. The assertion

> " $x > 0$ or $x < 2$ "

is *true* for any real number x. A student may feel discomfort at this assertion, perhaps because in many assertions involving "or" both cases cannot happen. Authors often emphasize the inclusiveness by saying something such as "or both".

> See [Hersh, 1997a] for more examples.

> *Citation* (Ant84.91), (BumKocWes93b.499), (Mol97.531).

> Students also have trouble negating conjunctions and disjunctions. A statement such
as

> "x is not $(P$ or $Q)$"

means

> "x is not P *and* x is not Q."

So does

> "x is neither P nor Q."

> See also both and Example 3 under yes it's weird.

or equivalently This phrase means that what follows is equivalent to what precedes. It is usually used when the equivalence is easy to see. This usage has no relation to the connective "or".

    Citation (BhaSer97.503).

order

(a) Ordering　"Order" may be a variant of "ordering".

　　Example 1　"Let $\leq$ be the usual order on the real numbers."
　　Citation (DavPri90.3).

(b) Cardinality　The **order** of a structure such as a group is the cardinality of (the underlying set of) the structure.

　　Citations (Sen90.330). But the meaning can be more devious than that: See (Lam91.305).

(c) Parameter　The word "order" may refer to a nonnegative integer parameter of the structure. Of course, the cardinality meaning just mentioned is a special case of this.

　　Example 2　The **order** of a differential equation is the highest derivative occurring in the equation.

　　Remark 1　The word　**degree** is also used in this way, but the uses are not interchangeable. Indeed, a structure may have both an order and a degree, for example a permutation group.

　　Citations (HawKis95.333), (Ros95.504).

order of quantifiers   The distinction between continuity and uniform continuity shows that the order in which quantifiers are applied can make a crucial difference in the meaning of a definition.

Example 1   The statement

$$\forall x \exists y(xy = e)$$

about elements x and y of a group with identity e, says that every element has a right inverse and is satisfied by all groups. In contrast, the statement

$$\exists x \forall y(xy = e)$$

is not satisfied by all groups. The idea is that the element y in the first sentence depends on the element x, and that according to the customary interpretation of sentences in mathematical logic, this is signaled by the fact that the x comes first. (See Vulcanize for more about this example.)

Example 2   The definition of continuity is commonly begun this way:

"For every $\epsilon > 0$, there is a $\delta > 0$ for which. . ."

Here δ depends on ϵ, but in contrast to the preceding example, the dependence is not functional. This is discussed in [Bagchi and Wells, 1998a] .

Remark 1   In ordinary English the way quantifiers are ordered does not always obey these rules. A student might say, "there is an inverse for every element" and be understood in much the same way as one would understand a statement such as "there is an ice cream cone for every child". The latter statement, translated mindlessly into first order logic, brings up the picture of n children licking one cone. But no one in everyday discourse would understand it that way, and only a few Vulcanists would think it bad English. Nevertheless, in writing mathematical arguments in English, such constructions should be avoided (see Vulcanize).

It appears to me that the meaning of sentences such as "There is an ice cream cone for every child" is extracted using a mechanism similar to that for a distributive plural, but I have not found anything in the linguistics literature about this. ***Information needed.***

See also all, and and mathematical logic.

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References [Dubinsky, 1997] .

orthogonal   A system of notation is **orthogonal** if any construction possible in the notation can be used anywhere it is appropriate.

Example 1   The notation for derivatives is not orthogonal. The prime notation can be used for functions of one variable but not for functions of more than one variable.

Example 2   The notation involving d is used for functions of one variable; for more than one variable one must change it to ∂.

Remark 1   These two examples are of different sorts. The prime notation cannot be used for more than one variable because it would become ambiguous. The difference between d and ∂, on the other hand, is a matter of convention.

Example 3   Early forms of Fortran were not orthogonal; one could use an arithmetic expression (for example, $i + 2j^2$) that evaluated to an integer in most places where one could use an integer — but not in the subscript of an array. In other words, one could write the equivalent of A_i but not of A_{i+2j^2}. This context is where I first met the word "orthogonal" used.

Terminology   I borrowed this terminology from the usage in computer language design. Some computer scientists have told me they never heard the word used in this way, but I heard it used in a talk recently. ***Citations needed.***

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osmosis theory   The **osmosis theory** of teaching is this attitude: We should not have to teach students to understand the way mathematics is written, or the finer points of logic (for example how quantifiers are negated). They should be able to figure these things on their own — "learn it by osmosis". If they cannot do that they are not qualified to major in mathematics.

Citation I have seen this attitude expressed in a letter to the editor in the *Notices*, but have lost the reference. ***References needed.***

Remark 1 We learned our native language(s) as children by osmosis. That does not imply that college students should learn mathematical reasoning that way. It does not even mean that college students should learn a foreign language that way.

outfix notation   A function is displayed in **outfix notation** if its symbol consists of characters or expressions put on both sides of the argument.

Example 1   The most familiar example is indefinite integral notation: The indefinite integral of a function f is denoted by $\int f(x)\,dx$. The definite integral is more complicated, since it has *three* arguments (two numbers and a function) placed in three different locations.

Example 2   The absolute value of a number r is denoted $|r|$.

Example 3   The greatest integer in x is sometimes denoted by $\lfloor x \rfloor$.

Other examples are described under brace, angle bracket and bracket.

Also called **matchfix notation**. The latter name is used in Macsyma.

Citations (Ant84.121).

overloaded notation   This phrase usually applies to a symbol or a name for a function
that takes on different meanings depending on which type of element it is evaluated at.
Such a function is also called **polymorphic**.

Example 1   The identity function is a polymorphic name; in the usual formalism
there is a different identity function on each set.

Example 2   A familiar example is the symbol $\times$, which is overloaded in college
mathematics courses. When a and b are numbers, $a \times b$ is their product. When A and B
are matrices, $A \times B$ is the matrix product. When v and w are 3-vectors, $v \times w$ is their
vector product.

Example 3   Another example is the common treatment of the image for arbitrary
functions: Let $F : S \to T$ be a function.
   a) If $x \in S$, $F(x)$ is the value of F applied to x. It is called the **image of** x **under** F.
   b) If A is a subset of S, then $F(A) = \{F(x) \mid x \in A\}$ (see setbuilder notation). It is
      called the **image of** A **under** F.
   c) The image of F is the set of all t in T for which there is an $x \in S$ such that
      $F(x) = t$, which is the image in the sense of (b) of the domain of F. The word
      "range" is also used for this meaning.

Remark 1   The preceding example is in a way fake. One could simply stipulate that
every function $F : S \to T$ has values defined for every element of S and (in the way
illustrated above) for every subset of S. However, the phrase "the image of F" would still
overload the word "image".

Example 4   A functor F from a category $\mathcal{C}$ to a category $\mathcal{D}$ is defined on both
objects and arrows of $\mathcal{C}$. This, too is a fake example, since the value of the functor at
identity arrows determines its value on objects.

Example 5   A text on vector spaces will very likely use $+$ for addition of vectors in
every vector space. Similarly, some texts on group theory will use e or 1 for the identity
element for any group and juxtaposition for the binary operation of any group.

Remark 2   Example 5 illustrates the common case of using the same symbol in every
model for a particular operation in an axiomatically defined mathematical structure.

Remark 3   The operation $\times$ does not require the same algorithm on matrices as it does on 3-vectors. This is the sort of phenomenon computer scientists call **ad-hoc polymorphism**. It is contrasted with **parametric polymorphism**, an example of which is the algorithm "swap the two entries in an ordered pair", which applies to ordered pairs of any type of element. (The parameter that gives rise to the name "parametric" is the type of element.) See algorithm. The identity function provides a trivial example of parametric polymorphism.

Many mathematicians think and speak informally of a parametrically polymorphic function as one single function. (" ... the identity function is injective").

Remark 4   The concept "overloaded" is natural in computer science because operations on different data types are typically implemented differently. For example, addition of integers is implemented differently from addition of floating point numbers in most computer languages. The concept is less natural in mathematics, where you could define the operation on the disjoint union of all the sets under consideration (for $\times$, the set might be $\mathbb{R}$ plus the set of all 3-dimensional real vectors plus the set of all $n \times n$ real matrices for each n). Even there, however, the implementation algorithm differs for some of the subsets. (See cases.)

Remark 5   When students start taking college mathematics, the sort of phenomena mentioned here means that they have to *read the surrounding text* to understand what a symbolic expression means: the expression is no longer self-sufficient. When I first came across this aspect of mathematics in a matrix theory course at Texas Southmost College, I felt that I had been ejected from paradise.

See also superscript.

parameter   A **parameter** is a variable used in the definition of a mathematical object. When the parameters are all instantiated, the object becomes specific. The parameters may or may not be shown explicitly in the identifier for the object; see synecdoche and suppression of parameters. See also Example 2 under definite description.

Example 1   Let $[a, b]$ be a closed interval. Here the parameters are a and b. A particular instantiation gives the specific closed interval $[\pi, 2\pi]$.

Example 2   Consider the polynomial $x^2 + ax + b$. The parameters are again a and b. (See Remark 1 below.)

Example 3   Consider the function $f(x) = x^2 + ax + b$. Again a and b are parameters and x is not.

Remark 1   A parameter in a symbolic expression is necessarily a free variable, but the converse is not true. In the polynomial $x^2 + ax + b$ mentioned in Example 2, for example, all the variables a, b and x are free. One calls a and b parameters because the expression $x^2 + ax + b$ is called a polynomial and because the convention is that the variable required by the definition of polynomial is usually represented by a letter near the end of the alphabet. Similarly, in Example 3, the variable x is not free because the definition of polynomial requires a placeholder and the convention for defining expressions uses letters late in the alphabet for the placeholder.

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parenthesis   **Parentheses** are the symbols "(" and ")". Parentheses are used in various ways in expressions.

(a) Grouping   Parentheses are very commonly used as bare delimiters to group subexpressions.

    Example 1   Parentheses are used for grouping in the expressions $(x^2 + 1)^2$ and $x(y + z)$.

(b) Tuples and matrices   Parentheses may be used to denote an ordered n-tuple, as in $(3, 1, 2)$, and are the standard notation for matrices.

(c) Open interval   The symbol (a, b) may denote the real interval $\{x \mid a < x < b\}$. *Citation:* (Fra98.609).

(d) Greatest common divisor   The symbol (m, n) may denote the greatest common divisor of the integers m and n. *Citation:* (Fra98.609). Note that the citations for this and the last usage come from the same sentence. It appears to me quite unlikely that any experienced mathematician would be confused by that sentence. Students are another matter.

(e) Function values   It is not clear whether the use of parentheses to delimit the argument in denoting the value of a function, in for example $f(x + 1)$, is a simple matter of grouping, or whether it is part of a special syntax for function application.

    With some function identifiers the parentheses are conventionally omitted by many authors who otherwise use them. *Examples:*

    "$\sin \pi = 0$."

    "$\log \frac{3}{2} = \log 3 - \log 2$."

    "$n! > 2^n$."

Terminology   Parentheses are also called brackets, but "bracket" may also refer to other delimiters. Sometimes parentheses are called **round parentheses** for emphasis.

    Citation (Yu98.656).

parenthetic assertion   A symbolic assertion is **parenthetic** if it is embedded in a sentence in a natural language in such a way that its pronunciation becomes a phrase (not a clause) embedded in the sentence. In contrast, when a symbolic assertion is a clause it is pronounced in such a way as to be a complete sentence.

Example 1

   "For any $x > 0$ there is a $y > 0$ such that $y < x$."

The assertion "$x > 0$" in isolation is a complete sentence, typically pronounced "x is greater than 0". In the sentence quoted above, however, it is pronounced "x greater than 0" or "x that is greater than 0", becoming a noun phrase embedded in the main sentence. Note that in the quoted sentence, "$x > 0$" and "$y > 0$" are parenthetic but "$y < x$" is a full clause.

   Citation (Bar96.631); (BleMccSel98.535); (Dra95.258); (Pow96.879); (Zul96.227).

   Remark 1   In seeking citations I was struck by the fact the some authors use parenthetic assertions in almost every paragraph and others essentially never do this: the latter typically use symbolic assertions only as complete clauses. Compare the articles [Bartle, 1996] and [Neidinger and Annen III, 1996] , in the same issue of The American Mathematical Monthly.

Example 2

   "...we define a *null set* in $I := [a, b]$ to be a set that can be covered by a
   countable union of intervals with arbitrarily small total length."

This is from [Bartle, 1996] , page 631. It would be read in this way: "...we define a null set in I, which is defined to be $[a, b]$, to be a set...". In other words, the phrase "$I := [a, b]$" is a *definition occurring as a parenthetic assertion.*

Example 3

   "Consider the circle $S^1 \subseteq \mathbf{C} = \mathbf{R}^2$"

This example is adapted from [Zulli, 1996] . Notice that the parenthetic remark contains another parenthetic remark inside it.

   See also context-dependent.

References [Gillman, 1987] , pages 12–13; [Krantz, 1997] , page 25.

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pattern recognition   Mathematicians must recognize abstract patterns that occur in symbolic expressions, geometric figures, and in their own mental representations of mathematical objects. This is one small aspect of human pattern recognition; for a general view, see [Guenther, 1998] , Chapter 3.

On particular type of pattern recognition that students find immensely difficult it *recognizing that a given expression is an instance of a substitution into a known expression.*

Example 1   This Handbook's definition of "at most" says that "x is at most y" means $x \leq y$. To understand this definition requires recognizing the **pattern** "x is at most y" no matter what occurs in place of x and y. For example,

"$\sin x$ is at most 1"

means that $\sin x \leq 1$.

Example 2   The assertion

"$x^2 + y^2 > 0$"

has as a special case

"$(-x^2 - y^2)^2 + (y^2 - x^2)^2 > 0$."

where you must insert appropriate parentheses. Students have trouble with expressions such as this one not only in recognizing it as an instance of substitution but in performing the substitution in the first place (see substitution).

Example 3   Students in postcalculus courses must recognize patterns of proof without being told. Examples are given under contrapositive and proof by contradiction.

Example 4   The rule for differentiation the square of a function is

"The derivative of $(f(x))^2$ is $2f(x)f'(x)$."

Consider the complexities involved in using this rule to calculate the derivatives of these functions:

a) $\frac{1}{x^2}$.
b) $\sin^2 x$ (Exponent not where it is in the pattern, no parentheses around the x.)
c) e^{2x} (What is the function? Do you recognize it as being squared?)

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Remark 1   Some proofs involve recognizing that a symbolic expression or figure fits a pattern *in two different ways*. This is illustrated by the next two examples. I have seen students flummoxed by Example 5, and Example 6 may for all I know be the proof that flummoxed medieval geometry students (see pons asinorum).

Example 5   In set with an associative binary operation and an identity element e, suppose x is an element with an inverse x^{-1}. (In this situation, it is easy to see that x has only one inverse.)

Theorem: $(x^{-1})^{-1} = x$.

Proof: By definition of inverse, y is the inverse of x if and only if

$$xy = yx = e \tag{2}$$

It follows by putting x^{-1} for x and x for y in Equation (2) that we must show that

$$x^{-1}x = xx^{-1} = e \tag{3}$$

But this is true by that same Equation (2), putting x for x and x^{-1} for y.

Example 6   Theorem: If a triangle has two equal angles, then it has two equal sides.

Proof: In the figure below, assume $\angle ABC = \angle ACB$. Then triangle ABC is congruent to triangle ACB since the sides BC and CB are equal and the adjoining angles are equal.
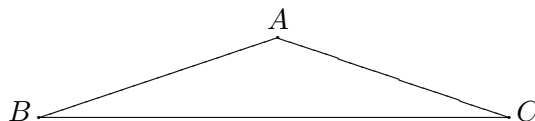
See also explicit assertion.

Remark 2   References to the mathematical education literature on pattern recognition are needed. **References needed.**

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permutation   A permutation is defined in the literature in two different ways:

a) A permutation of an n-element set is a sequence of length n in which each element of the set appears once.

b) A permutation of a set is a bijection from the set to itself.

Remark 1   Of course, the two definitions can be converted into each other, but psychologically they are rather different. Both definitions are given by [Kolman, Busby and Ross, 1996] , pages 75 and 181.

Citation (JacTho90.55), (Str93.27).

Platonism   Often used by mathematicians to refer to the attitude that mathematical objects exist in some manner analogous to the existence of physical objects.

Remark 1   It appears to me that all mathematicians, whether they regard themselves as Platonists or not, refer to mathematical objects using the same grammatical constructions as are used for references to physical objects. For example, one refers to "a continuous function" (indefinite reference) and "the sine function" (definite reference), in the way one refers to "a boy" and "the boss", not in the way one refers to nonmathematical abstract concepts such as "truth" or "gravity" (no article). (This behavior is not limited to mathematical objects: "the orbit of the moon" for example.) Symbols are generally used in the same way as proper nouns.

See also mathematical object and Remark 2 under symbol.

References   I have not discovered studies by linguists of this phenomenon.

Terminology   I doubt that the name "Platonism" is historically justified, but that is true of the names of lots of mathematical concepts that have been given the names of people.

References needed.

plug into   "Plug a into f" means evaluate f at a. Here, f may be a function or an expression, and a may be an expression.

Example 1   "If you plug π into the sine function, you get 0."

Remark 1   Some find the use of this phrase offensive. I judge this to have low status.

Citations (Ken83.166), (Mcc89.1328), (TemTra92.518).

plural   Many authors form the plural of certain learnèd words using endings from the language from which the words originated. Students may get these wrong, and may sometimes meet with ridicule for doing so.

(a) Plurals ending in a vowel Here are some of the common mathematical terms with vowel plurals.

singular	*plural*
automaton	automata
focus	foci
locus	loci
radius	radii

The plurals that end in a are often not recognized as plurals and are therefore used as singulars. (This does not seem to happen with my students with the -i plurals.) Linguists have noted that such plurals seem to be processed differently from s-plurals ([Pinker and Prince, 1999]). In particular, when used as adjectives, most nouns appear in the singular, but vowel-plural nouns appear in the plural: Compare "automata theory" with "group theory".

Interestingly, we no longer form the plural of Latin feminine nouns ending in "a" with "ae"; nowadays one almost always says "formulas" and "parabolas" instead of "formulae" and "parabolae".

(b) Plurals in s with modified roots

singular	*plural*
matrix	matrices
simplex	simplices
vertex	vertices

Students recognize these as plurals but produce new singulars for the words as back formations. For example, one hears "matricee" and "verticee" as the singular for "matrix" and "vertex". I have also heard "vertec".

Remark 1   It is not unfair to say that many scholars insist on using foreign plurals as a form of one-upmanship. But students and young professors need to be aware of these plurals in their own self interest.

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It appears to me that ridicule and put-down for using standard English plurals instead of foreign plurals (and for mispronouncing foreign names) is much less common than it was thirty years ago.

The use of plurals in the mathematical register is discussed under collective plural and distributive plural.

pointwise   See coordinatewise.

pointy bracket   See delimiter.

Polish notation   **Polish notation** consists in using prefix notation without parentheses. This requires that all function names have a single arity, so that which symbols apply to which inputs is unambiguous.

Example 1   In Polish notation,

$$2 \sin x + \sin y$$

would be written

$$+ \ * \ 2 \ \sin \ x \ \sin \ y$$

with $*$ denoting multiplication.

See also reverse Polish notation.

Remark 1   Polish notation originated with the Polish logic school of the 1920's. In particular the phrases "Polish notation" and "reverse Polish notation" originated from that fact and were not intended as ethnic slurs.

Terminology   Some authors use the phrase "Polish notation" even though parentheses are used (they are always redundant but add intelligibility). Polish notation is occasionally called **left Polish notation**.

Citations (Sin78.366), (Mck75.187), (Bau77.318).

polymorphic   See overloaded notation.

pons asinorum   The theorem in plane geometry that if a triangle has two equal angles then it has two equal sides has been called the **pons asinorum** (bridge of donkeys) because some students found its proof impossible to understand.

A candidate for the pons asinorum of post-calculus mathematics is mentioned under equivalence relation.

Remark 1   I assume the "bridge" is an isosceles triangle like that in Example 6 under pattern recognition, which is drawn wider than one typically draws such triangles nowadays. That figure reminds me of the very old arched bridges one sees here and there in Europe, for example in Venice.

positive In most (but not all) North American texts and university courses, the phrase "x is positive" means $x > 0$. In a European setting it may mean $x \geq 0$. See convention and another planet. This may have been an innovation by Bourbaki. ***Citations needed.***

postcondition   A **postcondition** in a definition or statement of a theorem is a condition stated after the definition or theorem.

"If n is divisible by four then it is even. This holds for any integer n."

The second sentence is a postcondition. Another example and a citation is given under where. I do not have citations for postconditions that don't use "where". ***Citations needed.***

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postfix notation   **Postfix notation** consists in writing the name of the function after its arguments.

Example 1   The expression $x + y$ in postfix notation would be $(x, y)+$.

Citations (Yet90.44), (Fra82.41).

Most authors write functions of one variable in prefix notation, but some algebraists use postfix notation. The factorial function is normally written in postfix notation.

See also Polish notation and prefix notation.

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power   The integer 5^3 is a **power** of 5 with exponent 3. One also describes 5^3 as "5 to the third power". I have seen students confused by this double usage. A statement such as "8 is a power of 2" may make the student think of 2^8.

    Citations (Pow96.879).

precedence   If Δ and $*$ are two binary operators, one says that Δ has higher **precedence** than $*$ if the expression $x\Delta y * z$ denotes $(x\Delta y) * z$ rather than $x\Delta(y * z)$.

Example 1   The expression $xy + z$ means $(xy) + z$, not $x(y + z)$, because in the symbolic language, multiplication has higher precedence than addition.

Unary operations (functions with one input) in mathematical writing typically have low precedence.

Example 2   One writes $\sin x$ but $\sin(x + y)$ because $\sin x + y$ may be perceived as either ambiguous or as $(\sin x) + y$. As this example illustrates, in the traditional symbolic language the precedence relationship of some pairs of operations is not necessarily well-defined.

Remark 1   The metaphor behind the word "precedence" is that if one carries out a calculation of the expression, one must apply the operator with higher precedence before the other one. Thus in calculating $(x\Delta y) * z$ one calculates $u = x\Delta y$ and *then* $u * z$.

See delimiter and evaluation.

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predicate   See assertion.

prefix notation   An expression is in **prefix notation** if the function symbols are written on the left of the argument.

Example 1   The expression $x + y$ written in prefix notation would be $+(x, y)$

Remark 1   In the traditional mathematical symbolic language, functions of one variable are used in prefix notation but a few, for example the symbols for the factorial and the greatest integer function, are used in other ways. Most binary operations denoted by special nonalphabetical symbols are written in infix notation, but those with alphabetical symbols are generally written in prefix notation and others such as an inner product may be written in outfix notation.

Citations (Yet90.44), (GraTre96.105), (Ant84.121).

See also postfix notation, Polish notation, reverse Polish notation and outfix notation.

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prescriptivist   A **prescriptivist** is someone who gives rules for which forms and syntax are correct in English (or another language). Prescriptivists are those who say we should not use double negatives, split infinitives, and "ain't". Opposed to **descriptivist**.

Vulcanism is a special form of prescriptivism.

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prime   (The typographical symbol). The symbol "′" is pronounced "prime" or "dash". For example, x' is pronounced "x prime" or "x dash". The pronunciation "dash" is used mostly outside North America.

process   See APOS and algorithm.

program   See algorithm.

pronunciation   Some students have told me that they find it necessary to be able to pronounce an expression that occurs in a text; if they can't, they can't read the text. One student brought this up with the common notation "$F : S \to T$". I would be glad to be informed of references to this phenomenon in the mathematical education literature. ***References needed.***

See also plural, context-dependent and mental representation.

proof   A **proof** is a step by step argument intended to persuade other mathematicians of the correctness of an assertion.

Mathematical logic also has a concept called proof: that is a mathematical object intended to model mathematicians' proofs. Proofs in mathematical logic may be called **formal proofs**, but that phrase is also used to indicate a particularly careful and detailed proof in the ordinary sense.

The individual sentences in a proof can be classified as follows:

Proof steps   A proof will contain formal mathematical statements that follow from previous statements. We call these **proof steps**. They are assertions in the mathematical register, like theorems, but unlike theorems one must deduce from the context the hypotheses that make them true.

Restatements   These state what must be proved, or, part way through a proof, what is left to be proved or what has just been proved.

Pointers   These give the location of pieces of the proof that are out of order, either elsewhere in the current proof or elsewhere in the text or in another text. References to another text are commonly called **citations**.

This discussion is drawn from [Bagchi and Wells, 1998b] . [Hanna, 1992] discusses the role of proofs in mathematics (with lots of references to the literature) and issues for mathematical education. Other discussions of proof in mathematical education may be found in [Epp, 1999b] , [Nardi, 1998] .

proof by contradiction   There are two somewhat different formats for proof that mathematicians refer to as proof by contradiction.

(a) Proof by deducing a false statement To prove P, assume P is false and deduce some assertion Q that is known to be false. This is the form of one well-known proof that $\sqrt{2}$ is irrational; one assumes it is rational and then concludes by violating the known fact that every fraction can be reduced to lowest terms.

Authors, even writing for undergraduates, often give such a proof by contradiction without saying they are doing it. The format of such a proof would be:
   a) Theorem: P implies Q.
   b) Assume P.
   c) Suppose not Q.
   d) Argument that R is true, where R is some statement well known to be false. The argument that R is true will assume that P is true, usually without saying so.
   e) End of proof.
The proof of Theorem 1 in [Herzog, 1998] has this form. The student must recognize the pattern of proof by contradiction without being told that that is what it is. (See pattern recognition.)

(b) Proof by contrapositive A proof that a conditional assertion $P \Rightarrow Q$ is true may have the following format:
   a) Theorem: P implies Q.
   b) Assume that P is true and Q is false.
   c) Argument that not P follows from not Q.
   d) Conclude that P and not P, a contradiction.
   e) End of proof.
Most commonly, P is a conjunction of several hypothesis and one concludes that one of the hypotheses is false (hence the conjunction is also false).

Remark 1   It is usually much simpler to prove the contrapositive directly (as described in the entry under contrapositive) instead of carrying out the procedure described in the preceding paragraph.

Citations (Kra95.40).

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Remark 2   As citation (Bry93.42) exemplifies, a proof $P \Rightarrow Q$ will frequently contain a subproof of some statement R which has the form of proving (not $R \Rightarrow$ not P).

References [Krantz, 1997] , page 68, discusses how to write proofs by contradiction.

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proof by instruction   A **proof by instruction** consists of instructions as to how to write a proof or how to modify a given proof to obtain another one. They come in several types.

   Geometric instructions   As an example, I could have worded the proof in Example 6 under pattern recognition this way: "Flip triangle ABC around the bisector of side BC and you must get the same triangle since a side and two adjoining angles are equal. Thus $AB = AC$."

   Citations needed.

   Algebraic instructions   An example is the instruction in Example 1 under look ahead to divide the Pythagorean identity $a^2 + b^2 = c^2$ by c^2 to obtain the identity $\sin^2 \theta + \cos^2 \theta = 1$.

   Citations (Ant84.113), (Pol65.7).

   Proof modification   This is the sort of instruction such as "interchange the role of x and y in the preceding proof and you get a proof of ... ".

   Citation (Fin99.774). ***Citations needed.***

proper   A subset T of a set S is a **proper** subset if it is not S. This is also used with substructures of a structure (proper subgroup, and so on.)

  Citations (GraTre96.234), (LeeMye99.428).

  See also trivial.

property   A property that an instance of a class of mathematical objects may have determines a subclass of those objects.

Example 1   Being even is a property that integers may have. This property determines a subset of integers, namely the set of even integers.

One states that an object has a property using a form of "to be" and an adjective or a noun.

Example 2   One can say

"4 is even."

or

"4 is an even integer."

Citations (Gro78.537), (KupPri84.86), (Mol97.531).

In some cases the property may also be given by a verb. See vanish for examples.

Remark 1   Some authors and editors object to using a property named after a person as an adjective. Instead of saying "The space S is Hausdorff" they would prefer "S is a Hausdorff space."

Remark 2   Mathematical texts sometimes identify a property with the class of objects having that property. (Similarly one may define a relation as a set of ordered pairs.) In my experience, this causes students quite a bit of difficulty at first.

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proposition   **Proposition** is used as another word for theorem. Some texts distinguish between propositions and theorems, reserving the word "theorem" for those that are considered especially important. This is the practice in [Epp, 1995] , for example (see her discussion on page 129).

   Citation (Epp95.2).

   The word "proposition" is used in some texts to denote an assertion that is definitely true or definitely false.

   Citation (Ros91.6).

prototype   Commonly a human concept has **typical** members, called **prototypes** by
Lakoff.

Example 1   For most of us, a sparrow is a prototypical bird, and a penguin is not.

Example 2   Many students believe the myth that a sequence that has a limit
"approaches the limit but never get there". They have presumably constructed their
prototypical sequence based on the examples they have seen in class or in the text, most
of which behave that way. This phenomenon is discussed in detail by [Cornu, 1992] and
[Tall and Vinner, 1981] .

The concept of "prototype" is subtle; these examples only hints at its depth. See also
radial concept.

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provided that   Used like if to give a definition.

    Example 1   "The integer n is **squarefree** provided that no square of a prime divides n." Also **providing that**.

    Citations (Str93.3). ***Citations needed.***

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put   Used in definitions, mainly to define a symbol.

    Example 1   "Put $f(x) = x^2 \sin x$."

    Citation (BasKul90.845).

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Q   The symbol $\mathbb{Q}$ usually denotes the set of rational numbers.
Citation (DavPri90.3).

quantifier   In this text, a **quantifier** is either the existential quantifier or the universal quantifier. Linguists and logicians study other quantifiers not discussed here.

R   The symbol ℝ usually denotes the set of real numbers.

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radial concept   A **radial concept** or **radial category** is a concept with some central prototypical examples and other examples described by phrases using the basic name of the concept that deviate from the prototypical examples in various ways.

Some members of a radial category deviate only slightly from the prototypes; others are highly metaphorical. The members are not automatically generated from the prototypical examples; membership is to a considerable extent a matter of convention.

Example 1   The concept of "mother" is a radial concept. Various members of the category amond English-speakers include birth mother, adoptive mother, foster mother, earth mother, stepmother, grandmother, mother-in-law, motherboard, mother lode and mother of pearl.

Remark 1   Our mental representation of the world is to a great extent organized around radial categories. The practice of adding new deviant members to a radial category is common and largely unconscious. That explains the origin of phrases such as "incomplete proof", "multivalued function", "left identity" and so on. According to the very special way mathematical concepts are formulated, by accumulation of attributes, an incomplete proof is not a proof, a multivalued function is not a function, and a left identity is not an identity. Fundamentalists may object to such usages, but they are fighting a losing battle against a basic method built into the human brain for organizing our mental representation of the world.

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range   Depending on the text, the **range** of a function can denote either the codomain or the image. The texts [Krantz, 1995] takes the first approach, and [Epp, 1995] and [Grassman and Tremblay, 1996] take the second approach.

real number A **real number** is the sort of number used in freshman calculus courses. The word **real** is frequently used as an adjective, as in "Let x be real."

Remark 1 I have heard students use the phrase "real number" to mean "genuine number", that is, not a variable.

Remark 2 Computer languages typically treat integers as if they were distinct from real numbers. In particular, many languages have the convention that the expression 2 denotes the integer and the expression 2.0 denotes the real number. I have known students who assumed that professors of mathematics were all familiar with this fact (probably nearly true in recent years) and that we all use notation this way (almost never true).

Citations (CalVel93.373), (RabGil93.168).

recall   Used before giving a definition, theorem or proof.

    Example 1   "Recall that an integer is even if it is divisible by 2." The intent seems to be that the author expects that the reader already knows the meaning of the defined term, but just in case here is a reminder. See Remark 5 under mathematical definition.

    Citation (Fou93.377), (Fri95.29).

reconstructive generalization   See generalization.

redundant   A given discourse is **redundant** if it contains words and expressions that could be omitted without changing the meaning. As an example, type labeling is a form of redundancy.

Another form of redundancy occurs in definitions. For example, the definitions of partition and of equivalence relation in Examples 2 and 3 under fundamentalist both mention the set S as part of the basic data. Giving S is redundant in both cases, in the sense that partitions and reflexive relations both determine the set on which they are defined. Including the underlying set of a structure in a definition even when the requirements determine the underlying set is the main instance I know of where it is conventional to include redundant data in a definition. (Rudin [1966] point out this phenomenon on page 21.)

There are some other examples where the definition is redundant and the redundancy cannot be described as a matter of convention. For example, in defining a group one usually requires an identity and that every element have a two-sided inverse; in fact, a left identity and left inverses with respect to the left identity are enough. In this case it is properties, rather than data, that are redundant. See radial concept.

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register   A **register** in linguistics is a choice of grammatical constructions, level of formality and other aspects of the language, suitable for use in a given social context. The **scientific register** is the distinctive register for writing and speaking about science. It is marked in particular by the use of complex nominal phrases connected by verbs that describe relations rather than actions. That register and the difficulties students have with it is discussed in detail in [Halliday and Martin, 1993] . In that book the scientific register is called "scientific English", but the remarks in chapters 3 and 5 make it clear that the authors regard scientific English as a register. A distinctive subregister of the scientific register is used in mathematics, namely the mathematical register.

reification   The mental process whereby a collection of processes and data is conceived of as a single mathematical object capable of being (for example) an element of a set or the input to a function. Also called **entification** or **encapsulation**.

References   This concept is discussed by Sfard [1992] (in connection with understanding functions) and [1994] . She gives many useful references to the literature. Students also have problems with reifying sets; this is discussed in [Lakoff and Núñez, 1997] . See also mathematical object and metaphor.

relation   Texts frequently define a (binary) **relation** on a set S to be a subset of the cartesian product $S \times S$. The relation *in use*, however, behaves like a two-place predicate. This caused much cognitive dissonance among my students. See also property.

    Citations (Epp95.534).

representation   Mathematicians and their students make use of both **external representations** and **internal representations** of mathematical objects. These phrases are used in the mathematical education literature. I take internal representations to be the same thing as mental representations or concept images.

An external representation of a phenomenon is a mathematical or symbolic system intended to allow one to deduce assertions about the phenomenon. Certain aspects of the phenomenon being represented are identified with certain mathematical objects; thus a representation involves a type of conceptual blend.

This is related to and may for some purposes be regarded as the same as the concept of model. The difference is that the word "representation" is more likely to be used (except by logicians) when mathematical objects are the phenomena being represented and "model" is more often used when physical phenomena are being represented by mathematical objects. This distinction must be regarded as preliminary and rough; it is not based on citations.

Example 1   The decimal notation for the integers, together with the grade school algorithms for adding and multiplying them, is an external representation of the integers and of some common operations available for them.

Example 2   Some of the ways in which one may represent functions are: as sets of ordered pairs, as algorithms, as maps (in the everyday sense) or other pictures, and as black boxes with input and output.

Other examples occur under model.

Note from these examples that the internal and external representations of an idea are not sharply distinguished from one another. In particular, the internal representation will in general involve the symbolism and terminology of the external representation, as well as nonverbal and nonsymbolic images and relationships.

The book [Janvier, 1987] is a primary source of information about representations. [Thompson, 1994] discussions confusions in the concept of representation on pages 39ff. See also [Vinner and Dreyfus, 1989] .

Remark 1   Of course, "representation" is also a mathematical word with various definitions in different disciplines. These definitions are generally abstractions of the

concept of representation discussed here.

respectively   Used to indicate term-by-term coreference between two lists of objects. Rarely used with lists with more than two entries.

Example 1   "The smallest prime divisors of 9 and 10 are 3 and 2, respectively."

Citations (Bur94.17), (Kra95.57), (Ros93.293).

See also comma, as well as citation (Niv56.41).

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result   A **result** is a mathematical fact that has been proved. The value produced by a computation is also called a result.

    Citations (Bur97.55), (Mcc89.1328).

    See also example.

reverse Polish notation   A form of postfix notation that is used without parentheses. This requires that the arity of all the symbols used be fixed.

   Example 1   In reverse Polish notation,

$$2 \sin x + \sin y$$

would be written

$$x \sin 2 * y \sin +$$

with $*$ denoting multiplication.

   Citation (Sin78.366).

   Reverse Polish notation is used by some Hewlett-Packard calculators and by the computer languages Forth and Postscript. It has come into prominence because expressions in a reverse Polish language are already in the form needed for an interpreter or compiler to process them.

   See Polish notation. Reverse Polish notation is sometimes called **right Polish notation**.

revise   In the United States, to "revise" a document means to change it, hopefully improving it in the process. Speakers influenced by British English use "revise" to mean "review"; in particular, students may talk about revising for an upcoming test.

 Remark 1   This entry has nothing directly to do with mathematics, but I have several times witnessed the confusion it can cause in academic circles and so thought it worth including here.

rewrite using definitions   One of the secrets of passing a first course in abstract mathematics that teaches proofs (first algebra course, first discrete math course, advanced calculus, and so on) is to take every statement to be proved and first rewrite it using the definitions of the terms in the statement. It is remarkably difficult to convince students to try this. See trivial and unwind.

root   A **root** of an equation $f(x) = 0$ is a value c for which $f(c) = 0$. This value c is also called a root or a **zero** of the function f.

Remark 1   Some hold it to be incorrect to refer c as a "root of f" instead of "zero of f", although I have not been able to find a statement to that effect in print. The practice is quite widespread, particularly when the function is a polynomial. **References needed.**

Citations (Bre71.592), (Pin64.108).

Remark 2   "Root" is of course used with a different but related meaning in phrases such as "square root", "nth root", and so on.

Acknowledgments   Gary Tee.

round parentheses   See delimiter.

sanity check   A simple test to check if something you have formulated makes sense.

    Example 1   If you write down $6s = p$ for the student-professor problem and check your work by plugging in $s = 12$, $p = 2$, you immediately discover your error.

satisfy   A mathematical structure **satisfies** an assertion that contains variables if the assertion makes a meaningful statement about the structure that becomes true for every possible instantiation of the variables.

Example 1   "Every group satisfies the statement $\forall x \exists y (xyx = x)$."
Citation (Fra98.614).

say   Used to signal that a definition is being given.

Example 1   "We say that an integer n is **even** if n is divisible by 2." Variation: "An integer n is said to be even if it is divisible by 2."

The word "say" is also used to introduce notation or to give an example.

Example 2   Let $f(x)$ be a polynomial, say

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n$$

Example 3   Let p be a prime, say 23.

Remark 1   Note that the syntax of the two meanings is different.

Citations (Niv56.41), (GraTre96.105), (Mol97.531).

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schema   See APOS.

scope   In a symbolic expression, a variable is within the **scope** of an operator if its meaning or use is affected by the operator. I will discuss the use of this word here only for operators that bind variables.

Example 1   In the expression

$$\int_a^b x^2 \, dx$$

the variable x is bound by the integral operator.

Example 2   In the expression

$$\int_a^b (x+y)^2 \, dx$$

the x is bound but not the y, so that one would expect the value to be in terms of a, b and y, but not x.

Example 3   In the expression

$$x < 2 \text{ or } \forall x (x > 0)$$

the first occurrence of x is not bound by the $\forall$ operator, but the x in "$x > 0$" is bound. In particular, one can substitute for the first x only. Thus an instance of that expression is the (false) statement

$$5 < 2 \text{ or } \forall x (x > 0)$$

It would not make sense to write

$$5 < 2 \text{ or } \forall 5 (5 > 0)$$

See bound variable.

Remark 1   A mathematical definition of scope, like that of bound variable, requires a formal recursive definition of "symbolic expression". The definition given in this article is a dictionary definition. This is discussed in more detail in Remark 2 under free variable.

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self-monitoring   **Self-monitoring** is the activity a student engages in when she notices that some practice she uses in solving problems is counterproductive (or is helpful) and modifies her behavior accordingly. It is discussed in [Resnick, 1987] , [Schoenfeld, 1987b] , and [Wells, 1995] .

semantic contamination   The connotations of a word or phrase that has been given a mathematical definition sometimes creates an expectation in the reader that the word or phrase has a certain meaning, different from its actual meaning given by the definition. This is **semantic contamination**. It is a form of cognitive dissonance. In this case the two modes of learning in the definition of cognitive dissonance are learning the meaning from the definition and learning the meaning implicitly from connotations of the word used (which is a common mode of learning in the humanities.) A mathematics student may suppress the information given by the definition and rely only on the connotations.

Examples of semantic contamination related to implication are given under conditional assertion, contrapositive and only if. Some other illustrative examples are given here.

Example 1   The word "formula" is used informally to mean an expression such as H_2O that describes the composition of a chemical. However, in many texts in mathematical logic ([Mendelson, 1987] , [Ebbinghaus, Flum and Thomas, 1984] , [van Dalen, 1989]), a formula is a formal expression that in the intended semantics has a truth value when the variables are instantiated; it is thus a symbolic assertion. Thus "$x + 2y$" is not a formula (it is a term), but "$(x > y)$" and $x + y = z$" are formulas. When teaching logic, I have frequently witnessed the difficulties students have had in remembering the difference in meaning between a formula and a term in this context.

Remark 1   In this Handbook the word assertion is usually used instead of "formula".

Example 2   The text [Dym and Ivey, 1980] , page 76, states that the assumption that discrete data (they were referring to samples of noise levels) can be plotted as a continuous curve is referred to as the continuum hypothesis. In this case the authors presumably assumed they knew what the phrase meant without checking the definition at all. This may deserve criticism, even severe criticism. Nevertheless, in reading for college courses in the humanities this a common way for students to pick up on the meanings of unfamiliar words or phrases.

Terminology   The name "semantic contamination" is due to Pimm [1987] , page 88.

References [Hersh, 1997a]   gives many examples of disparities between the ordinary meaning and the mathematical meaning of mathematical words. Any of them could be

the source of semantic contamination.

semantics   A **semantics** is a method of determining the meaning of an expression in a natural or artificial language or in a system of notation.

Example 1   Symbolic expressions in the mathematical register have both **intensional** (note the spelling) and **extensional** semantics. Speaking *very* roughly, the intensional semantics carries information concerning how its meaning is constructed or calculated; the extensional semantics is merely the mathematical object(s) denoted by the expression. For example, the intensional interpretation of

$$\frac{3+5}{2}$$

for a mathematician is something like:

  "The result of adding 5 and 3 and dividing the result by 2."
For a student, the intensional meaning might be imperative:

  "Add 5 to 3 and divide the result by 2."
The extensional interpretation of that fraction is 4. Another example of this is given in Example 1 under equation.

Example 2   The semantics of natural languages is currently the object of intensive study by linguists. Good starting places to find out about this are [Chierchia and McConnell-Ginet, 1990] and [Partee, 1996] . Some of what semanticists have learned sheds light on students' misunderstandings: see for example the related discussions of definite article, indefinite article, universal quantifier and existential quantifier.

Example 3   Mathematical logic typically constructs an interpretation of a text in some formal language. For example, an interpretation of the symbolic assertion $x + 2 = 7$ might take the universe of the interpretation to be the set of integers, and could interpret x as 2. A familiar semantics for algebraic expressions causes it to be interpreted as the assertion that $4 = 7$, and under the usual method of determining truth for that assignment, this statement is "invalid" in that interpretation. If x is interpreted as 5 then the symbolic assertion is valid for that interpretation. One also says that 5 satisfies the assertion but 2 does not.

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Remark 1   Many computer scientists use the word "semantics" to mean interpretation in the sense it is used in the Handbook. Here, a semantics is a *method* of interpretation, not a particular interpretation.

set [1]   (Verb). Use in definitions, usually to define a symbol.

    Example 1   "Set $f(t) = 3t^2$."
    Citation (Ant80.364).

set [2]   (Noun). In abstract mathematics courses one may be tempted to "define" set, only to quail at the prospect of presenting Zermelo-Fränkel set theory. This may result in a total cop-out accompanied by mutterings about collections of things. One way out is to give a specification for sets. Two crucial properties of sets that students need to know are

a) A set is not the same thing as its elements.

b) A set is determined completely by what its elements are.

Most facts about sets as used in undergraduate mathematics courses are made reasonable by knowing these two facts. See also element, empty set and setbuilder notation.

References [Wells, 1995] , [Wells, 1997] .

Difficulties In advanced mathematics course structures such as quotient groups are built on sets whose elements are sets; this requires reifying the sets involved. See [Lakoff and Núñez, 1997] .

Students sometimes express discomfort when faced with sets that seem too arbitrary. See yes it's weird.

setbuilder notation   The expression $\{x \mid P(x)\}$ defines a set. Its elements are exactly those x for which the predicate $P(x)$ is true. (The type of x is often deduced from the context.) This is called **setbuilder notation** (a low-status name) or **set comprehension** (a higher status but confusing name). Setbuilder notation is a form of structural notation.

Difficulties   The basic rule of inference for setbuilder notation is that $P(a)$ is true *if and only if* $a \in \{x \mid P(x)\}$. This means in particular that if $P(a)$ then $a \in \{x \mid P(x)\}$, and if not $P(a)$, then $a \notin \{x \mid P(x)\}$. Students may fail to make use of the latter fact.

Variations   A colon is used by many authors instead of a vertical line.

One may put an expression before the vertical line. This can be misleading.

Example 1   The set $\{x^2 \mid x \in \mathbb{R} \text{ and } x \neq 3\}$ *does* contain 9, because $9 = (-3)^2$.

[Gries and Schneider, 1993] , Chapter 11, give examples that show that putting an expression before the vertical line can be ambiguous. They introduce a more elaborate notation that eliminates the ambiguity.

Citation (Fra82.41), (KolBusRos96.109), and (for the colon variation) (Bri93.782).

show   To prove (see proof). Some scientists and possibly some high school teachers use "show" in a meaning that is something like "provide evidence for" or "illustrate". It appears to me that the collegiate level usage is that "show" is synonymous with "prove", but I don't have citations. ***Citations needed.***

Comments on this point would be welcome. See [Maurer, 1991] , page 15. *Citation* (EdgUllWes97.574).

snow   Professors sometimes try to **snow** the students, meaning to confront them with unbelievable or difficult to understand assertions without preparing the ground; this is done in an effort to make them realize just how wonderfully knowledgable the professor is and what worms the students are. (This use of the word "snow" is old slang, probably dying out.)

Notions of infinite cardinality are a favorite tool for such putdowns. Thus it is a scam to try to startle or mystify students with statements such as "There are just as many even integers as integers!" The would-be snower is taking advantage of the mathematician's use of "same number of elements" as a metaphor for infinite sets in bijective correspondence, a metaphor with severe limitations of applicability. (See [Lakoff and Núñez, 2000] , pages 142–144.)

That scam is like asking a student "Please bring me that stick over there on the other blackboard" without mentioning the fact that you have decided to call a piece of chalk a "stick". It is true that there is *some* analogy between a piece of chalk and a stick (more than, say, between a piece of chalk and an elf), but I would expect the student to look confusedly for a long narrow thing made out of wood, not immediately guessing that you meant the piece of chalk.

Remark 1   Math majors are well known for trying to snow each other, too.

some   The word **some** is used in the mathematical register to indicate the existential quantifier. Some examples are given under existential quantifier.

space

specification   A **specification** of a mathematical concept describes the way the concept is used in sufficient detail for the purposes of a particular course or text, but does not give a mathematical definition. Specifications are particularly desirable in courses for students beginning abstract mathematics for concepts such as set, function and "ordered pair" where the standard definitions are either difficult or introduce irrelevant detail. Examples may be found under set and function.

References [Wells, 1995] and [Bagchi and Wells, 1998b] .

   Remark 1   On pages 48ff of [Rota, 1996] the distinction is made between "description" and "definition" in mathematics. As an example of a description which is not a definition, he mentions D. C. Spencer's characterization of a tensor as "an object that transforms according to the following rules". That sounds mighty like a specification to me.

   Remark 2   Definitions in category theory, for example of "product", are often simply precise specifications. That is revealed by the fact that a product of sets in the categorical sense is not a uniquely defined set in the way it appears to be in the classical definition as a set of ordered pairs. Category theory has thus made the practice of specification into a precise and dependable tool.

split definition   A definition by cases.

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square bracket   Square brackets are the delimiters [ ]. They are occasionally used as bare delimiters, and may be used instead of parentheses to enclose the argument to a function in an expression of its value (as in $f[x]$ instead of $f(x)$). They are also used as outfix notation with other special meanings, for example to denote closed intervals. See bracket.

    Citations (DarGof70.729), (Dum95.1419), (FeeMar00.381), (Sta70b.884), (Tew70.730).

squiggle   See tilde.

standard interpretation   The **standard interpretation** of a mathematical discourse is the meaning a mathematician competent in a given field will understand from a discourse belonging to that field. (One aspect of being "competent", of course, is familiarity with the standard interpretation!)

I will state two theses about the standard interpretation here and make some comments.

(a) First thesis   There is such a thing as the standard interpretation and it is a proper subject for study in linguistics.

My evidence for this is that for most mathematical discourse, most mathematicians in the appropriate field who read it will agree on its meaning (and will mark students' papers wrong if they have a nonstandard interpretation). Furthermore, rules for how the interpretation is carried out can be apparently formulated for much of the symbolic part (see the discussion of Mathematica® under symbolic language), and some of the structure of the expressions that communicate logical reasoning is used outside mathematics and has been the subject of intensive study by semanticists; for example, see [Chierchia and McConnell-Ginet, 1990] and [Kamp and Reyle, 1993] .

My claim that most of the time mathematicians agree on the meaning of what they read must be understood in the way that the claims of physics are understood. If an experiment disagrees with an established law, the experimenter can often discover a flaw that explains the disagreement. If mathematicians disagree about the meaning of a text, they often discover a flaw as well: one of them had the wrong definition of a word, they come to agree that the text is genuinely ambiguous, or the author tells them about a typo . . .

(b) Second thesis   One of the major tasks of an instructor in mathematics is to show a student how to extract correctly the standard interpretation of a piece of text.

This thesis is based on my own experience. I have always been sensitive to language-based misunderstandings, and not just in mathematics. I have kept records of such misunderstandings and learned some basic ideas of linguistics as a result of my curiosity about them. It appears to me from my teaching experience that language based misunderstandings are a common cause of problems in learning mathematics at the

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post-calculus level.

My perceptions may be a case of "I have a hammer so everything looks like a nail". And adherents of the osmosis theory will reject my efforts on principle.

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status   I have had a few experiences that lead me to believe that some phrases in the mathematical register are "in" (have **high status**) and others are "out" (**low status**).

Example 1   To some mathematicians, "dummy variable" may sound high-schoolish and low status; it is much more refined to say "bound variable".

Example 2   The phrase "setbuilder notation" may have lower status than "comprehension".

Remark 1   Variations in status no doubt differ in different mathematical disciplines.

Remark 2   I believe that in both examples just given, the low status word is much more likely to be understood by high school and beginning college students in the USA.

Remark 3   A reviewer of a book I wrote said, " . . . and he even referred to 'setbuilder notation' . . . " without any further explanation as to why that was a bad thing. I was mightily puzzled by that remark until it occurred to me that status might be involved.

See also plug into.

structure   See mathematical structure.

structural notation   Structural notation for a mathematical object is a symbolic expression that, in the given context, describes the (possibly variable) mathematical object unambiguously without providing an symbol for it. Also called   **anonymous notation**.

Example 1   The expression $\{1, 2, 4\}$ is structural notation for the unique set that contains the elements 1, 2 and 4 and no other elements.

Example 2   The expression

$$\begin{pmatrix} a^2 & ab \\ -ab & b^2 \end{pmatrix}$$

is structural notation for a matrix, given the parameters a and b.

Example 3   Setbuilder notation is a type of systematic structural notation. So are barred arrow notation and lambda notation for functions.

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subscript   A string of characters is a **subscript** to a character if the string is placed immediately after the character and below the base line of the text.

Example 1   In the expression x_{23}, the string 23 is a subscript to x.

Subscripts are normally used for indexing.

Example 2   The tuple $a = (3, 1, 5)$ is determined by the fact that $a_1 = 3$, $a_2 = 1$, and $a_3 = 5$.

Example 3   The Fibonacci sequence $f_0, f_1, \ldots$ is defined by $f_0 = 0$, $f_1 = 1$, and $f_i = f_{i-1} + f_{i-2}$ for $i > 1$. (Some authors define $f_0 = 1$.)

Difficulties   The tuple in Example 2 can be seen as a function on the set $\{1, 2, 3\}$ ("a tuple is a function on its index set"), and the Fibonacci sequence can be seen as a function on the nonnegative integers. The ith entry of Fibonacci sequence could thus be written indifferently as f_i or $f(i)$. This fact is familiar to working mathematicians, but in a classroom where the Fibonacci function is denoted by f_i a remark such as

"The Fibonacci sequence is an increasing function of i."

can cause considerable confusion to beginners.

Subscripts may also be used to denote partial derivatives.

Example 4   If $F(x, y) = x^2 y^3$ then $F_x = 2xy^3$.

Citations (Bar96.631). ***Citations needed.***

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substitution   To **substitute** an expression e for a variable x that occurs in an expression t is to replace every occurrence of x by e (in a sophisticated way — see the remarks under "Difficulties" below). The expression resulting from the substitution has a possibly different denotation which can generally be determined from the syntax.

Example 1   Let e be $x + y$ and t be $2u$. Then substituting t for x in e yields $2u + y$.

Difficulties   The act of substituting may require insertion of parentheses and other adjustments to the expression containing the variable. In general, substituting is not a mechanical act, but requires understanding the syntax of the expression.

Example 2   Substituting $2u$ for x in $x^2 + 2x + y$ gives $(2u)^2 + 2(2u) + y$; note the changes that have to be made from a straight textual substitution.

Example 3   Substituting 4 for x in the expression $3x$ results in 12, not 34 (!).

Example 4   Suppose $f(x, y) = x^2 - y^2$. What is $f(y, x)$? What is $f(x, x)$? Many students have trouble with this kind of question.

See also pattern recognition.

Remark 1   A fundamental fact about the syntax and semantics of all mathematical expressions (as far as I know) is that *substitution commutes with evaluation*. This means that if you replace a subexpression by its value the value of the containing expression remains the same. For example, if you instantiate the variable x in the expression $3x + y$ with 4 and replace the subexpression $3x$ by its value 12, you get the expression $12 + y$, which must have the same value as $3x + y$ as long as x has the value 4. This is a basic fact about manipulating mathematical expressions.

Acknowledgments   Michael Barr.

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such that   For a predicate P, a phrase of the form "c such that $P(c)$" means that $P(c)$ holds.

Example 1   "Let n be an integer such that $n > 2$." means that in the following assertions that refer to n, one can assume that $n > 2$.

Example 2   "The set of all integers n such that $n > 2$." refers to the set $\{n \mid n > 2\}$. (See setbuilder notation.)

Remark 1   Note that in pronouncing $\exists x P(x)$ the phrase "such that" is usually inserted. This is not done for the universal quantifier.

Example 3   "$\exists x(x > 0)$" is pronounced "There is an x such that x is greater than 0", but "$\forall x(x > 0)$" is pronounced "For all x, x is greater than 0".

Remark 2   Yes, I know that "$\forall x(x > 0)$" is false.

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sufficient P is **sufficient** for Q if P implies Q. Examples are given under conditional assertion.

superscript   A string of characters is a **superscript** to a character if the string is placed immediately after the character and raised above the base line of the text.

   Example 1   In the expression x^{23}, the string 23 is a superscript to x.
   Superscripts are used in many ways:
   a) To indicate a power (including the Cartesian power of a set). *Citation:* (BumKocWes93b.499), (KloAleLar93.757), (Pow96.879).
   b) As an index. A superscript used as an index may indicate contravariance. *Citation:* (Fra98.609).
   c) To indicate the domain of a function space. *Citation:* (Bar96.631).
   d) To indicate the dimension of a space. *Citation:* (Zul96.227).
   e) As a bound on an operator. *Citation:* (Mea93.387), (GelOlm90.65).
   f) A few authors use a superscript to the *left* of the base character, as in ^{23}x, *Citation:* (Tit64.321).

Difficulties Superscripting numbers is a heavily overloaded operation. A serious confusion in lower level college math courses occurs between f^{-1} as the reciprocal of a function and f^{-1} as the inverse.

   Remark 1   Sometimes in my classes students give answers that show they think that the Cartesian power $\{1, 2, 3\}^2$ is $\{1, 4, 9\}$.

suppose   Discussed under let.

suppression of parameters   An identifier or other mathematical notation may omit a parameter on which the meaning of the notation depends.

Example 1   A common form of suppression of parameters is to refer to a mathematical structure by its underlying set. Thus a group with underlying set G and binary operation $*$ may be called G, so that the notation omits the binary operation. This is also an example of synecdoche.

Example 2   A parameter that is suppressed from the notation may or may not be announced explicitly in the text. For example, a text may, by the expression $\log x$, refer to the logarithm with base e, and may or may not announce this fact explicitly. Note that this is *not* an example of synecdoche.

See also abuse of notation.

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surjective   A function $f : A \to B$ is **surjective** if for every element b of B there is an element a of A such that $f(a) = b$. One also says f is **onto** B.

Remark 1   Strictly speaking one should either adopt the stance that every function is equipped with a codomain, or one should always attach a phrase of the form "onto B" to any occurrence of the word "surjective". ***Citations needed.***

See also trivial and Remark 3 under injective.

symbol   A symbol is an identifier used in the symbolic language which is a minimal arrangement of characters. "Minimal" means it is not itself constructed of (mathematical) symbols by the rules of construction of symbolic expressions.

Example 1   The symbol for the ratio between the circumference and the diameter of a circle is "π". This is a mathematical symbol consisting of one character.

Example 2   The symbol for the sine function is sin. This is a symbol made up of three characters. Although one of the characters is i and i is itself a symbol, its role in the symbol "sin" is purely as a character. (One could think "sin" is the product of s, i and n, and indeed students do sometimes assume such things, but that would not be the author's intent.)

This is in contrast to the role of i in the symbolic expression $3i^2$, a compound expression (not called a symbol in this Handbook) whose meaning is determined synthetically by the meanings of the symbols 3, i and 2 and the way they are arranged.

Remark 1   Many authors, for example [Fearnley-Sander, 1982] and [Harel and Kaput, 1992] , use "symbol" to mean what I call symbolic expression. Others use "symbol" to mean character.

Remark 2   The syntax of symbols and symbolic expressions in the mathematical register needs analysis. It appears to me that they are treated like proper nouns: In particular, they don't take the article.

Example 3   Compare "Sym$_3$ is noncommutative" with "Flicka is a horse".

They are also used in apposition like proper nouns.

Example 4   Compare "The group Sym$_3$" with "my friend Flicka" or "that boy Albert".

This applies to variables as well as determinate symbols, as in "the quantity $x^2 + 1$" and "for all integers n".

See also name.

References   This discussion derives in part from [de Bruijn, 1994] , page 876.

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symbolic assertion   See assertion.

symbolic expression   A **symbolic expression** (or just **expression**) is a collection of mathematical symbols arranged

    a) according to the commonly accepted rules for writing mathematics, or

    b) according to some mathematical definition of a formal language.

Every expression is either a term or an symbolic assertion.

    The meaning of a symbolic expression is normally determined synthetically from the arrangement and the meanings of the individual symbols. In particular, every symbol is a symbolic expression.

    Example 1   The expressions x^2 and $\sin^2 \pi$ mentioned under symbol are symbolic expressions. "$\sin^2 \pi$" is an arrangement of three symbols, namely sin, 2 and π. The arrangement itself is meaningful; "$\sin^2 \pi$" is not the same symbolic expression as $2 \sin \pi$ even though they have the same *value*; see semantics.

    Remark 1   As the example indicates, the "arrangement" need not be a string.

    An expression may contain a **subexpression**. The rules for forming expressions and the use of delimiters allow one to determine the subexpressions.

    Example 2   The subexpressions in x^2 are x^2, x and 2. Two of the subexpressions in $(2x + 5)^3$ are $2x$ and $2x + 5$. The rules of algebra require the latter to be inclosed in parentheses, but not the former.

    Example 3   Is $\sin \pi$ a subexpression of $\sin^2 \pi$? This depends on the rules for construction of this expression; there is no book to consult because the rules for symbolic expressions in the mathematical register are not written down anywhere, except possible in the bowels of Mathematica®(see Remark 1 under symbolic language). One could imagine a rule that constructs the function $\sin^2$ from sin and notation for the squaring function, in which case $\sin \pi$ is not a subexpression of $\sin^2 \pi$. On the other hand, one could imagine a system in which one constructs $(\sin \pi)^2$ and than a Chomsky-style transformation converts it to $\sin^2 \pi$. In that case $\sin \pi$ is in some sense a subexpression of $\sin^2 \pi$.

    This example shows that determining subexpressions from the typographical arrangement is not a trivial task. One must understand the rules for forming expressions, implicitly if not explicitly.

Example 4   The set
$$\{f \mid f = \sin^n, n \in \mathbb{N}, n > 0\}$$

could also be written

$$\{f \mid f \text{ is a positive integral power of the sine function}\}$$

showing that English phrases can occur embedded in symbolic expressions. **Citations needed.**

References   Symbols and symbolic expressions are discussed in the context of mathematical education in [Schoenfeld, 1985] , [Harel and Kaput, 1992] , [Tall, 1992c] .

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symbolic language   The **symbolic language** of mathematics is a distinct part of the mathematical register. It consists of symbolic expressions written in the way mathematicians traditionally write them. They may stand as complete sentences or may be incorporated into statements in English. Occasionally statements in English are embedded in symbolic expressions. (See Remark 2 under identifier.)

Example 1   "$\pi > 0$." is a complete assertion (formula) in the symbolic language of mathematics.

Example 2   "If x is any number, then $x^2 \geq 0$." is an assertion in the mathematical register containing two symbolic expressions. Note that "x" is a term and "$x^2 \geq 0$" is a symbolic assertion incorporated into the larger assertion in English. See parenthetic assertion.

Example 3

   "$\{n \mid n \text{ is even}\}$."

This is a term containing an embedded phrase in the mathematical register.

Remark 1   The symbolic language of mathematics has never been given by a mathematical definition (in other words it is not a formal language). There would be difficulties in doing so.

In the first place, the symbolic language is context-dependent (examples are given under that heading). Also, the symbolic language of mathematics has many variants depending on the field and individual idiosyncrasies. Finally, even if one gives a formal definition one would have problems with mechanical parsing because the language contains ambiguities. For example, is "ma" a symbol or is it m times a?

Mathematica® 3.0 has a standardized version (`StandardForm`) of the symbolic expression language of the mathematical vernacular that eliminates ambiguities, and it can also output symbolic expressions in a form called `TraditionalForm` that is rather close to actual usage. (See [Wolfram, 1997] , pages 187ff.) Presumably the implementation of `TraditionalForm` would have involved a definition of it as a formal language.

De Bruijn [1994] proposes modeling a large part of the mathematical vernacular (not just the symbolic language) using a programming language.

symbolic logic   Another name for mathematical logic.

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symbolitis   The excessive use of symbols (as opposed to English words and phrases) in mathematical writing – the meaning of "excessive", of course, depends on the speaker! There seems to be more objection to symbols from mathematical logic such as $\forall$ and $\exists$ than to others.

References [Gillman, 1987] , page 7.

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symbol manipulation   **Symbol manipulation** is the transformation of a symbolic expression by using algebraic or syntactic rules, typically with the intention of producing a more satisfactory expression. Symbol manipulation may be performed as a step in a proof or as part of the process of solving a problem.

Example 1   The proof that $a^2 - b^2 = (a + b)(a - b)$ based on the distributive law, the commutative law for multiplication, and the algebraic laws concerning additive inverses:

$$(a + b)(a - b) = a(a - b) + b(a - b) = a^2 - ab + ba - b^2 = a^2 - ab + ab - b^2 = a^2 - b^2$$

An example of a proof by symbolic manipulation of formulas in mathematical logic is given under conceptual. Proof by symbol manipulation is contrasted with conceptual proof. See also aha.

Difficulties Students often manipulate symbols inappropriately, using rules not valid for the objects being manipulated. This is discussed by Harel [1998] .

synecdoche   **Synecdoche** is naming something by naming a part of it.

Example 1   Referring to a car as "wheels".

Example 2   Naming a mathematical structure by its underlying set. This happens very commonly. This is also a case of suppression of parameters. ***Citations needed.***

Example 3   Naming an equivalence class by a member of the class. Note that this is not an example of suppression of parameters. See well-defined. ***Citations needed.***

See also metaphor.

References [Presmeg, 1997b] .

syntax   The **syntax** of an expression is an analysis of the manner in which the expression has been constructed from its parts.

Example 1   The syntax of the expression $5 + 3$ consists partly of the fact that "5" is placed before "+" and "3" after it, but the syntax is more than that; it also includes the fact that "+" is a binary operation written in infix notation, so that the expression $5 + 3$ is a term and not an assertion. The expression $3 + 5$ is a *different* expression; the semantics usually used for this expression tells us that it has the same value as $5 + 3$.

Example 2   The syntax of the expression $5 > 3$ tells us that it is an assertion; the semantics tells us that it is a true assertion.

Example 3   The syntax of the expression $3x + y$ is different from the syntax of $3(x + y)$. In the common tree notation for syntax the two expressions are parsed as follows:

$$
\begin{array}{ccc}
& + & \\
\diagup \quad \diagdown & \\
* \qquad y & \quad\quad & 3 \qquad + \\
\diagup \quad \diagdown & & \diagup \quad \diagdown \\
3 \qquad x & & x \qquad y \\
3x + y & & 3(x + y)
\end{array}
\tag{4}
$$
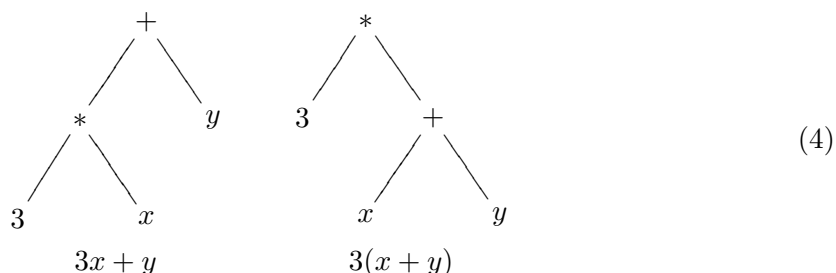
Remark 1   The syntax of an expression gives it structure beyond being merely a string of symbols. The structure must be deduced by the reader with clues given by convention (in the case of Example 3, that multiplication dominates addition), parentheses, and the context. (See also Example 3 under symbolic expression.)

Successful students generally learn to deduce this structure without much explicit instruction, and in many cases without much conscious awareness of the process. For example, college students may be able to calculate $3(x + y)$ and $3x + y$ correctly for given instantiations of x and y, but they may have never consciously noticed that in calculating $3x + y$ you must calculate the product before the sum, and the other way around for $3(x + y)$. (A reverse Polish calculator forces you to notice things like that.) See also compositional.

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The way the order of calculation is determined by the syntactic structure and the observation in Remark 1 under semantics that substitution commutes with evaluation are basic aspects of learning to deal with mathematical expressions that are essentially never made explicit in teaching. (No teacher under whom I studied ever made them explicit.)

Difficulties Students vary widely on how much they are able to use the syntax to decode mathematical expressions. A student may be able to understand a very complicated statement that is in context, but will find meaningless a statement with the same logical structure about abstract objects. Note Example 3 under coreference.

More references to the literature are needed on this subject. **References needed.** See also compositional, substitute.

Acknowledgments Some of this discussion was suggested by [Dubinsky, 1997] . A good reference to the syntax of English is [McCawley, 1988a] , [McCawley, 1988b] . Thanks also to Atish Bagchi.

synthetic　See compositional.

term   A **term** is a symbolic expression that denotes a (possibly variable) mathematical object. This is in contrast to a symbolic assertion.

Example 1   Any symbol that denotes a (possibly variable) mathematical object is a term. Thus π and 3 are terms.

Example 2   The expression $2 + 5$ is a term that denotes 7.

Example 3   The expression $x + 2y$ is a term. It denotes a variable number. If specific numbers are substituted for x and y the resulting expression is a term that (in the usual extensional semantics) denotes a specific number.

Example 4   The expression

$$\int_1^2 x \, dx$$

is a term; it (extensionally) denotes the number $3/2$.

Remark 1   "Term" is used in mathematical logic with this meaning. Some mathematicians use "term" to denote a constituent of a sum, in a way analogous to the use of factors for products. However, in this text we follow the usage in logic; in particular a factor of a product is an example of a term (and so is the product). ***Citations needed.***

Acknowledgments   Owen Thomas.

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TFAE Abbreviation for "the following are equivalent".

that is   Used to indicate that what follows is equivalent to what precedes, usually when the equivalence is essentially a rewording. *Citation:* (Kra95.40).

the   See definite article.

the following are equivalent   See following are equivalent.

then   The word **then** in the mathematical register generally means that what follows can be deduced from the preceding assumption, which is commonly signaled by if or when. See imply.

> *Example 1*
> "If n is divisible by 4, then n is even."
>
> *Remark 1*   Occasionally "then" has a temporal meaning.
> *Citations* (Bar96.631), (Bur94.24).

theorem To call an assertion a **theorem** is to claim that the assertion has been proved.

Remark 1 In texts the proof is often given after the theorem has been stated. In that case (assuming the proof is correct) it is still true that the theorem "has been proved"!

Some authors refer only to assertions they regard as important as theorems, and use the word proposition for less important ones. See also lemma and corollary.

Theorems, along with definitions, are often delineated. See labeled style.

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thus   **Thus** means that what follows is a consequence of (or is equivalent to) what precedes.
Citations (KolBusRos96.109), (Mac71.58).

tilde   The symbol "~" over a letter is pronounced "tilde" (till-day or till-duh). Thus "$\tilde{x}$" is pronounced "x tilde". Also called "twiddle" or "squiggle".

This symbol is also used in web addresses, and I have heard it called "squiggle" in that case as well.

translation problem   The **translation problem** is the name used in this Handbook for the process of discovering the logical structure of a mathematical assertion P that has been stated in the mathematical register. This is essentially equivalent to the problem of finding a statement in mathematical logic that represents P. Learning how to do this is one of the difficult skills students of mathematics have to acquire, even very early with simple word problems. Many of the entries in this Handbook illustrate the complications this involves; see for example conditional assertion and universal quantifier – only two of many.

   As far as I know, there are no extended or theory-based treatments of the translation problem, although many entries in this Handbook refer to discussions in the literature about particular cases of the problem. The text [Kamp and Reyle, 1993] is essentially a study of the analogous problem of discovering the logical structure of statements in ordinary English rather than in the mathematical register. The relationship between the English and the logic is quite complex. See the discussion under and, for example.

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trigonometric functions   It is not always explicitly noted to students that if you write $\sin x$ meaning the sine of x degrees, you are not using the same function as when you write $\sin x$, meaning the sine of x radians. They have different derivatives, for example. The same remark may be made of the other trigonometric functions.

Remark 1   This point is correctly made in [Edwards and Penney, 1998] , page 167.

Remark 2   It appears to me that in postcalculus pure mathematics "$\sin x$" nearly always refers to the sine of x radians (not degrees), often without explicitly noting the fact. This is certainly not true for texts written by nonmathematicians, but the situation is made easier by the customary use of the degree symbol when degrees are intended.

See also logarithm and global parameters.

Acknowledgments   Michael Barr.

trivial

(a) About propositions A fact is said to be **trivial** to prove if the fact follows by rewriting using definitions, or perhaps if the common mental representation of the mathematical objects involved in the fact makes the truth of the fact immediately perceivable. (This needs further analysis. I would tend to use obvious for the second meaning.)

Example 1   A textbook may define the image of a function $F : A \rightarrow B$ to be the set of all elements of B of the form $F(a)$ for some $a \in A$. It then goes on to say that F is surjective if for every element b of B there is an element $a \in A$ with the property that $F(a) = b$. It may then state a theorem, or given an exercise, that says that a function $F : A \rightarrow B$ is surjective if and only if the image of F is B. The proof follows immediately by rewriting using definitions. I have known instructors to refer to such an assertion as "trivial" and to question the worth of including it in the text. In contrast, I have known *many* students in discrete math and abstract algebra classes who were totally baffled when asked to prove such an assertion. *This disparity between the students' and the instructors' perception of what is "trivial" must be taken seriously.*

Remark 1   I suspect that teachers (and hotshot math majors) telling students that an assertion is "obvious" or "trivial" is an important cause (but not the only one) of the feeling much of the American population has that they are "bad at math". In many cases a person who feels that way may have simply not learned to rewrite using definitions, and so finds many proofs impossibly difficult that their instructor calls "trivial".

Remark 2   The teacher's feeling that an assertion is obvious may also come from the ratchet effect.

References [Solow, 1995] is one text with a discussion of image and surjective as described in Example 1.
   Citations needed.

(b) About mathematical objects A function may be called **trivial** if it is the identity function or a constant function, and possibly in other circumstances. ***Citations needed.***

A solution to an equation is said to be **trivial** if it is the identity element for some operation involved in the equation. There may be other situations in which a solution is

called "trivial" as well.

Citations (Len92.216), (Rib95.391).

A mathematical structure is said to be **trivial** if its underlying set is empty or a singleton set. In particular, a subset of a set is **nontrivial** if it is nonempty. See proper.
Citations needed.

Citation (Bri77.189).

Remark 3 "Trivial" and degenerate overlap in meaning but are not interchangeable. A citation search might be desirable, but it is not clear to me that there is a consistent meaning to either word.

turf   If you are defensive about negative comments about your field, or annoyed when another department tries to teach a course you believe belongs in mathematics, you are protecting your **turf**. The use of this word of course is not restricted to mathematicians (nor is the phenomenon it describes).

   Example 1   I have occasionally witnessed irritation by people familiar with one field at the use of a term in that field by people in a different field with a different meaning. This happened on the `mathedu` mailing list when some subscribers started talking about constructivism with the meaning it has in mathematical education rather than the (unrelated) meaning it has in mathematical logic.

twiddle   See tilde.

type   The **type** of a symbol is the kind of value it is allowed to have in the current context.

   Example 1   In the assertion

   "If f is differentiable and $f'(x) = 0$ then x is a critical point of f."

we may deduce that f is of type "function" and x is (probably) of type "real", even if the author does not say this. This sort of type deduction requires both mathematical knowledge and knowledge of conventions; in the present example one convention is that complex numbers are more commonly written z instead of x. Mathematical knowledge (as well as convention) tells us that x cannot be of type integer.

   Remark 1   One could dispense with the concept of type and refer to the set of possible values of the symbol. It appears to me however that "type" is psychologically different from "set". Normally one expects the type to be a natural and homogeneous kind such as "function" or "real number", not an arbitrary kind such as "real number bigger than 3 or integer divisible by 4". One has no such psychological constraint on sets themselves. This needs further investigation.

   Remark 2   Mathematicians do not use the word "type" much. Students commonly make type mistakes (talking about 2π being divisible by 2, for example); it would be helpful to refer to the concept explicitly as a way of raising consciousness. This is discussed in [Wells, 1995] .

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type labeling   Giving the type of a symbol along with the symbol.

Example 1   If it has been established on some early page of a text that S_3 denotes the symmetric group on 3 letters. A later reference to it as "the group S_3" or "the symmetric group S_3" is an example of type labeling.

Remark 1   Russian mathematical authors seem to do this a lot, although that may be because one cannot attach grammatical endings to symbols.

References   Jeffrey Ullman, in a guest appearance in [Knuth, Larrabee and Roberts, 1989] , flatly recommends *always* giving the type of a symbol. Using explicit typing in teaching is advocated in [Wells, 1995] . See also [Bagchi and Wells, 1998a] .

under   Used to name the function by which one has computed the value, or the function being used as an operation.

Example 1   "If the value of x under F is greater than the value of x under G for every x, one says that $F > G$."

Citation (Fra82.41).

Example 2   "The set $\mathbb{Z}$ of integers is a group under addition."

Citation (BriPre92.146), (Max93.27).

Example 3   "If x is related to y under E, we write xEy."

Citation (Exn96.35).

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unique   To say that an object satisfying certain conditions is **unique** means that there is only one object satisfying those conditions.

Citation (Bri93.782), (Str93.17) (where the unique object is variable, dependent on a parameter).

Remark 1   This meaning can have philosophical complications; for example, some mathematicians would say that by "the set of natural numbers" they mean any of the models of the Peano axioms, all of which are isomorphic. Such mathematicians would say that the natural numbers are "unique up to isomorphism". Others would simply assert that there is a unique set of natural numbers. Nevertheless, most mathematicians in ordinary discourse speak of the natural numbers as if they were unique, whatever they believe.

Note that "the symmetric group on n letters" is unique up to isomorphism, but in contrast to the Peano natural numbers, it is not unique up to a *unique* isomorphism.

The word "unique" is misused by students; see in your own words. See also up to.

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universal generalization   If you have proved $P(c)$ for a variable object c of some type, and during the proof have made no restrictions on c, then you are entitled to conclude that $P(x)$ is true for all x of the appropriate type. This process is formalized in mathematical logic as the rule of deduction called **universal generalization**.

universal instantiation   If it is known that $P(x)$ is true of all x of the appropriate type, and c is the identifier of a specific mathematical object of that type, then you are entitled to conclude that $P(c)$ is true. In mathematical logic, the formal version of this is known as **universal instantiation**.

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universal quantifier   An expression in mathematical logic of the form $\forall x\, P(x)$, where P is a predicate, means that $P(x)$ is true for every x of the appropriate type. The symbol $\forall$ is pronounced "for all" and is called the **universal quantifier**.

Expressing universal quantification in the mathematical register   When a universally quantified sentence in the mathematical register is translated into a sentence of the form $\forall x\, P(x)$ in mathematical logic, the assertion $P(x)$ is nearly always in the form of a conditional assertion. Thus in particular all the sentences listed as examples under conditional assertion provide ways of expressing universal quantification in English. However, there are many other ways of doing that that are not conditional assertions in English. To provide examples, let $C(f)$ mean that f is continuous and and $D(f)$ mean that f is differentiable. The assertion $\forall n\, (D(n) \Rightarrow C(n))$ can be said in the following ways:

a) **Every** differentiable function is continuous. *Citation:* (Bar96.631), (MorShaVal93.751).

b) **Any** differentiable function is continuous. *Citation:* (EdgUllWes97.574), (Niv56.83).

c) **All** differentiable functions are continuous. *Citation:* (AldDia86.333), (Ost71.624).

d) Differentiable functions are continuous. *Citation:* (Kau74.429).

e) A differentiable function is continuous. *Citation:* (BalYou77.451), (MacBir93.43).

f) Each differentiable function is continuous. *Citation:* (Bry93.30).

g) The multiples of 4 are even. I changed this example because to me "The differentiable functions are continuous" sounds odd. In any case, I don't have a citation for this. ***Citations needed.***

One can make the assertion an explicit conditional one using the same words:

h) For every function f, if f is differentiable then it is continuous. *Citation:* (BhaSer97.502), (VanLutPrz77.435).

i) For any function f, if f is differentiable then it is continuous. *Citation:* (Bry93.62), (Bla79.122).

j) For all functions f, if f is differentiable then it is continuous. *Citation:* (Cho99.444), (Pow96.879).

In any of these sentences, the "for all" phrase may come after the main clause. The conditional assertion can be varied in the ways described under that listing. See

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also each.

   If the variable is typed, either the definite or the indefinite article may be used:

k) "If the function f is differentiable, then it is continuous."

l) "If a function f is ... ". *Citation:* (RabGil93.168).

Remark 1   Sentences such as (d), (e) and (g) are often not recognized by students as having universal quantification force. Sentence (e) is discussed further under indefinite article, and sentence (f) is discussed further under each.

See also always, distributive plural and negation.

Universal quantification in the symbolic language   The quantifier is sometimes expressed by parentheses in displayed symbolic assertions. The assertion, "The square of any real number is nonnegative" can be written this way:

$$x^2 \geq 0 \qquad\qquad\qquad \text{(all real } x)$$

or less explicitly

$$x^2 \geq 0 \qquad\qquad\qquad (x)$$

Open sentences   Sometimes, the quantifier is not reflected by any symbol or English word. The sentence is then an **open sentence** and is interpreted as universally quantified. The clue that this is the case is that the variables involved have not in the present context been given specific values. Thus in (GraTre96.105):

   "A function f of arity 2 is *commutative* if $f(x, y) = f(y, x)$."

This means that $f(x, y) = f(y, x)$ for all x and all y.

Remark 2   Sometimes an author does not make it clear which variable is being quantified.

   " In fact, every $Q_i(s) \cong 1 \pmod{m}$, since.... "

The context shows that this means

$$\forall i \left(Q_i(s) \cong 1 \pmod{m} \right)$$

(This is from [Neidinger and Annen III, 1996] , page 646.)

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References [Epp, 1999a] , [Wood and Perrett, 1997] , page 12 (written for students). For studies of quantification in English, see [Chierchia and McConnell-Ginet, 1990] and [Keenan and Westerståhl, 1997] .

See also always, never, existential quantifier and order of quantifiers.

unnecessarily weak assertion   Students are often uncomfortable when faced with an
assertion such as

" Either $x > 0$ or $x < 2$ "

because one could obviously make a stronger statement. The statement is nevertheless
true.

Example 1   Students have problems both with "$2 \leq 2$" and with "$2 \leq 3$". This may
be compounded by problems with inclusive and exclusive or.

Remark 1   It appears to me that unnecessarily weak statements occur primarily in
these contexts:

a) When the statement is what follows formally from the preceding argument.
b) When the statement is made in that form because it allows one to deduce a desired
   result.

I believe students are uncomfortable primarily in the case of (b), and that their
discomfort is an instance of walking blindfolded. ***Information needed.***

Acknowledgments   Michael Barr.

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unwind   A typical definition in mathematics may make use of a number of previously defined concepts. To **unwind** such a definition is to replace the defined terms with explicit, spelled-out requirements. This may change a conceptual definition into an elementary definition. An example is given under elementary. See rewrite using definitions.

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up to   Let E be an equivalence relation. To say that a definition or description of a type of mathematical object determines the object **up to** E means that any two objects satisfying the description are equivalent with respect to E.

Example 1   An indefinite integral $\int f(x)\,dx$ is determined up to a constant. In this case the equivalence relation is that of differing by a constant.

The objects are often described in terms of parameters, in which case any two objects satisfying the description are equivalent once the parameters are instantiated.

Example 2   The statement "G is a finite group of order n containing an element of order n" forces G to be the cyclic group of order n, so that the statement defines G up to isomorphism once n is instantiated.

See copy.

Citation (Fri95.29), (MacBir93.182).

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uppercase　See case.

vacuous implication   A conditional assertion "If A then B" is true if A happens to be false. This is not usually the interesting case and so this phenomenon is called **vacuous implication**.

Difficulties Students have a tendency to forget about it even if reminded of it. For example, if I note that the less-than relation on the set of all reals is antisymmetric, a student will often ask, "How can less-than be antisymmetric? It's impossible to have $r < s$ and $s < r$!"

vanish   A function f **vanishes** at a point a if $f(a) = 0$.

> *Example 1*   "Consider the collection of all continuous functions that vanish at 0."
>
> *Citations* (BouGriKac62.717), (New67.912).

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variable   A variate symbol.

Example 1   In the expression

" Let f be a function for which $f(x) > 0$ for $x > 2$. "

the x and the f are both variate symbols.

Remark 1   In common mathematical parlance only x in the preceding expression would be called a variable. In the terminology of mathematical logic, both x and f are variables.

See bound variable, free variable and the discussion after Example 2 under variable mathematical object.

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variable clash   A substitution of an expression containing a free variable into an expression that contains and binds the same literal variable.

Example 1   A student must solve an integral $\int_0^9 r^3 A\,dr$, where she knows that A is the area of a certain circle. She therefore rewrites it as $\int_0^9 r^3 \pi r^2\,dr$; this will give the wrong answer.

variate   A free identifier, either in the symbolic language or in English, is **variate** if it is intended to refer to a variable mathematical object. A variate identifier, at least in intent, has more than one interpretation in the universe of discourse. These two points of view — the identifier names a variable mathematical object and the identifier has more than one interpretation — are discussed at length in section (b) under mathematical object.

  Example 1   In the assertion, "If the quantity a is positive, then a^x is positive for all real x", x and a are both variate. In contrast, in the phrase "the exponential function a^x", a is variate but x is not an identifier, it is a dummy variable. In this case, in common usage, x is a variable and a is a parameter.

  Example 2   In the passage
      "Let G be a group with identity element e."
"G" and "e" are variate.

  Example 3   "Let G be a group and $g \in G$. Suppose the group G is commutative . . . ." This illustrates the fact that variable mathematical structures are commonly referred to using definite noun phrases.

  Remark 1   Being determinate or variate is a matter of the current interpretation; it is not an inherent property of the symbol, even though some symbols such as π are conventionally determinate and others such as x are conventionally variate. For example, π is sometimes used as the name of a projection function.

  Remark 2   The distinction between determinate and variate is not the same as the grammatical distinction between definite description and indefinite description. See Example 1 under definite description.

  Remark 3   The distinction between determinate and variate is not the same as the grammatical distinction between common and proper nouns. Indeed, all symbolic expressions seem to use syntax very similar to that of proper nouns. See Remark 2 under symbol.

  Remark 4   Note that variate and determinate identifiers are free by definition. Asking whether a bound variable is variate or determinate does not in any obvious way make sense. See Remark 2 under bound identifier.

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Remark 5 In the passage

"Suppose x is a real variable and $3x + 1 = 7$."

Then one deduces that $x = 2$. Its use in that sentence is nevertheless variate. The *intent* is that it be a variable. The conditions imposed force it to denote just one number. (It is easy to think of examples where, unlike this one, it is very difficult to determine whether the conditions force a unique value.) It is the intent that matters.

Remark 6 Apparently [ISO, 1982] , quoted in [Beccari, 1997] , recommends a practice which in my terminology would be: Use upright typographic characters for determinate symbols and slanted typographical characters for variate symbols. This recommendation was carried out in two research papers I participated in [Bagchi and Wells, 1997a] , [Bagchi and Wells, 1997b] . I have not seen [ISO, 1982] .

Terminology The names "determinate" and "variate" are my own coinages. I felt it important not to use the phrase "variable identifier" because it is ambiguous.

Acknowledgments Owen Thomas.

verify   To verify an assertion is to check that the statement holds for all possible instantiations of the variables in the assertion. The word "verify" is used particularly when this check is performed by considering the possibilities case by case.

Example 1   One might prove an assertion about all finite simple groups by checking each family of finite simple groups and each sporadic one separately.

Mathematicians tend to find such proofs unsatisfying.

References   I got this idea from [Rota, 1996] , page 136.

Remark 1   It is my impression that scientists sometimes say an equation is "verified" if it is true for some example instantiations; no claim is made that all cases have been verified. I do not have a citation for this. ***Citations needed.***

vinculum   See bar.

Vulcanism   This is the theory, espoused, usually subconsciously, by many mathematicians and logicians, that the English language should be forced to mirror the notation, syntax and rules of one or another of the common forms of mathematical logic. This is a special kind of prescriptivism.

Example 1   The statement "All that glitters is not gold", translated into logical notation the way the syntax indicates, gives

$$\forall x(\text{glitters}(x) \Rightarrow (\text{not gold}(x)))$$

However, its *meaning* is

$$\text{not } \forall x(\text{glitters}(x) \Rightarrow \text{gold}(x))$$

The "not" modifies the whole sentence, not the phrase "is gold". Many, including perhaps most mathematicians, would regard this sentence as "wrong" in spite of the fact that native English speakers use sentences like it all the time and rarely misunderstand them.

Another example is given under order of quantifiers.

Remark 1   Vulcanism has succeeded in ruling out the use of double negatives in educated discourse in English, but not in colloquial use in some dialects. See [Huddleston and Pullum, 2002] , Chapter 9. It has not succeeded in ruling out the phenomenon described in Example 1.

Remark 2   Natural language has been around for thousands of years and has evolved into a wonderfully subtle tool for communication. First order logic is about a century old (although it has older precursors dating back to Aristotle) and represents an artificial form of reasoning suited to mathematics, but to little else. There is more about the suitability of mathematical logic in Remark 2 under mathematical logic, in subsection (c) of mathematical object and under only if.

Remark 3   The name is my own.

Vulcanize   To **Vulcanize** an English sentence in the mathematical register is to restate it in a form that can be mindlessly translated into one of the usual forms of symbolic logic in a way that retains the intended meaning.

Example 1   "Every element has an inverse" could be Vulcanized into "For each element x there is an element y that is inverse to x", which translates more or less directly into $\forall x \exists y \, (\text{Inverse}(y, x))$.

Remark 1   A style manual for mathematical writing should address the issue of how much Vulcanizing is appropriate. Thus the Vulcanizing in Example 1 is surely unnecessary, but one should avoid saying "There is an inverse for every element", which reverses the quantifiers. (See Example 1 under order of quantifiers.)

See also fundamentalist.

walking blindfolded   Sometimes a lecturer lists steps in an argument that will indeed culminate in a valid proof, but the reason for the steps is not apparent to the student. The student may feel like someone who is walking straight ahead with a blindfold on: how do you know you won't bump into a wall or fall off a cliff? That is **walking blindfolded** (my name). This is closely related to the attitude described in section (a) under attitudes.

It is my observation that many students find it difficult or impossible to follow a proof when they cannot see where it is going.

See also look ahead.

well-defined   Suppose you try to define a function F on a partition Π of a set A by specifying its value on a class C of Π in terms of an element $x \in C$ (a case of synecdoche). For this to work, one must have $F(x) = F(x')$ whenever x is equivalent to x'. In that case the function F is said to be **well-defined**. (Of course, it is not defined at all if it is not well-defined!).

     Example 1   Let $\mathbb{Z}_2$ be the group of congruence class of integers mod 2, with the class of n denoted $[n]$. Define $F : \mathbb{Z}_2 \to \mathbb{Z}_2$ by $F[n] = [n^2]$. Then F is well-defined (in fact, it is the identity function), because an integer is even if and only if its square is even. If you say $G[n] = $ the number of primes dividing n, then G would not be well-defined, since $G[2] = [1]$, $G[6] = [2]$, and $[2] = [6]$ but $[1] \neq [2]$.

     Definition on equivalence classes is perhaps the most common use of "well-defined", but there are other situations in which it is used as well.

     Example 2   Let $\mathbb{P}$ denote the set of all nonempty subsets of the set of nonnegative integers. Define $F : \mathbb{P} \to \mathbb{Z}$ by: $F(A)$ is the smallest element of A. Since the nonnegative integers are well-ordered, F is well-defined. This is a case where there might have been doubt that the object exists, rather than worry about whether it is ambiguous.

     Example 3   Conway defined a sequence on the positive integers by $a(1) = a(2) = 1$ and

$$a(n) = a(a(n-1)) + a(n - a(n-1))$$

for $n \geq 3$. This is well-defined because one can show by induction that $a(n) < n$ for $n \geq 3$ (otherwise the term $a(n - a(n-1))$ could cause trouble). This example comes from [Mallows, 1991] .

     See radial concept and fundamentalist.

Variations   Many authors omit the hyphen in "well-defined".

     Remark 1   There is a subtlety in Example 3. The observation that $a(n) < n$ for $n \geq 3$ does indeed show that the sequence is well-defined, but a sequence can sometimes be well-defined even if the function calls in the definition of the value at n refer to larger

integers than n. An example is the function

$$F(n) = \begin{cases} F\Big(F(n+11)\Big) & (n \leq 100) \\ n - 10 & (n > 100) \end{cases}$$

Citations needed.

when   Often used to mean "if".

    Example 1

    "When a function has a derivative, it is necessarily continuous."

    Remark 1   Modern dictionaries [Neufeldt, 1988]   record this meaning of "when", but the original Oxford English Dictionary does not.

    One occasionally comes across elaborations of this usage, such as "when and only when", "exactly when", "precisely when" and so on, all apparently meaning "if and only if".

    The usage "if whenever" evidently is motivated by the desire to avoid two if's in a row, for example in the sentence, "A relation α is symmetric if whenever $x\alpha y$ then $y\alpha x$".

References   This discussion follows [Bagchi and Wells, 1998a] .

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where   **Where** is used in two special ways in the mathematical register.

(a) To state a postcondition

Example 1   " **Definition** An element a of a group is **involutive** if $a^2 = e$, where e is the identity element of the group." Here the statement "where e is the identity element of the group" is a postcondition.

Citations (Bar96.626), (Ros93.223).

Remark 1 [Krantz, 1997] , page 44 and [Steenrod *et al.*, 1975] , page 38, both deprecate this usage.

Acknowledgments Michael Barr for references.

(b) Used to introduce a constraint

Example 2   "A point x where $f'(x) = 0$ is a critical point." In contrast to the first usage, I have not found citations where this usage doesn't carry a connotation of location.

Citation (Law96.243).

without loss of generality   A proof of an assertion involving two elements x and y of some mathematical structure S might ostensibly require consideration of two cases in which x and y are related in different ways to each other; for example for some predicate P, $P(x, y)$ or $P(y, x)$ could hold. However, if there is a symmetry of S that interchanges x and y, one may need to consider only one case. In that case, the proof may begin with a remark such as,

"Without loss of generality, we may assume $P(x, y)$."

Citations needed.

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witness   If $P(x)$ is a predicate with just the one variable x, a particular object c for which $P(c)$ is true is a **witness** to the fact that $\exists x P(x)$ is true.

WLOG Without loss of generality.

yes it's weird   Students sometimes express discomfort at examples that seem arbitrary in some sense.

Example 1   Try using the set $\{1, 3, 5, 6, 7, 9, 11\}$ in an example; you may get some question such as "Why did you put a 6 in there?"

Example 2   A different sort of example is a heterogenous set such as the set $\{3, \{2, 3\}, \emptyset\}$, which has both numbers and sets as elements.

Example 3   Thom [1992] objects to the use of or between adjectives when the qualities are heterogenous. Thus for him "Find all the balls that are red or white" is acceptable, but not "Find all the balls that are red or large". He was discussing the use of such examples with children in school. I have not had a student express discomfort or confusion at such usage; this may be because they have been brainwashed/educated (take your pick) by the American school system.

Remark 1   In teaching abstract mathematics I have adopted the practice of explicitly recognizing the students' discomfort in situations such as in Examples 1 and 2 ("yes, it's weird"). I generally say something such as: allowing such constructions is necessary to do abstract mathematics. As far as I can tell this satisfies nearly everyone. I have no basis for doing this from the mathematical education literature, but it appears to me that the discomfort is real and may very well contribute to the common attitude expressed by the phrase "I don't have a mathematical mind".

When a teacher takes the point of view that the student should have known that such arbitrary constructions were legitimate, or otherwise engages in put-down behavior, it can only contribute to the student's feeling of not being cut out for mathematics.

References needed.

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you don't know shriek   This is the indignant shriek that begins, "You mean you don't know ... !?" (Or "never heard of ... ") This is often directed at young college students who may be very bright but who simply have not lived long enough to pick up all the information a middle aged college professor has. I remember emitting this shriek when I discovered as a young teacher that about half my freshman calculus students didn't know what a lathe is. In my fifties the shriek was emitted at me when two of my colleagues discovered that I had never heard of the prestigious private liberal arts college they sent their offspring to.

This phenomenon should be distinguished from the annoyance expressed at someone who isn't paying attention to what is happening or to what someone is saying.

Terminology   The name is mine. However, this phenomenon needs a more insulting name guaranteed to embarrass anyone who thinks of using it.

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Z

(a) The letter The letter Z is pronounced "zee" in the USA and "zed" in the United Kingdom and in much of the ex-British Empire.

Remark 1   The specification language Z was invented in Britain. Some American computer scientists call it "zed" as a result, although they say "zee" when referring to the letter.

(b) The integers The symbol $\mathbb{Z}$ usually denotes the set of all integers. Some authors use $\mathbb{I}$.
   Citations (DavPri90.3).

Remark 2   Some authors strongly object to the use of the blackboard bold type style exemplified by $\mathbb{R}$ and $\mathbb{Z}$.

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zero   See root.

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Citations

Links are given to every location where the text is cited.

1. (316)
Adem, A. (1997), 'Recent developments in the cohomology of finite groups'. *Notices of the American Mathematical Society*, volume , pages 806–812.

[p. 806. Lines 20–22, second column.]
Now taking generators for IG as a $\mathbb{Z}G$-module, we can map a free $\mathbb{Z}G$-module of finite rank onto IG.

2. (294)
Adler, A. and S.-Y. R. Li (1977), 'Magic cubes and Prouhet sequences'. *American Mathematical Monthly*, volume **84**, pages 618–627.

[p. 619. Lines 12–13.]
DEFINITIONS. By an N-*cube of order T*, we mean an N-dimensional array of T^N N-dimensional boxes.

3. (188)
Akin, E. and M. Davis (1985), 'Bulgarian solitaire'. *American Mathematical Monthly*, volume **92**, pages 237–250.

[p. 243. Line 3.]
We have already proved (a) and the formal proof of (b) is very similar.

4. (40, 462)
Aldous, D. and P. Diaconis (1986), 'Shuffling cards and stopping times'. *American Mathematical Monthly*, volume **93**, pages 333–348.

[p. 333. Lines 3–2 above Figure 1.]
By an inductive argument, all $(n-1)!$ arrangements of the lower cards are equally likely.

5. (405)
Antman, S. S. (1980), 'The equations for large vibrations of strings'. *American Mathematical Monthly*, volume **87**, pages 359–370.

[p. 364. Formula 3.10..]
Let us set
$$\eta(x,t) = \phi(x)\psi(t)\mathbf{e}$$

6. (230)
Anton, H. (1984), *Elementary Linear Algebra, Fourth Edition*. John Wiley and Sons.

[p. 90. Lines 4–7.]

7. (327)
Anton, H. (1984), *Elementary Linear Algebra, Fourth Edition*. John Wiley and Sons.

[p. 91. Lines 8–4 from bottom.]

8. (367)
Anton, H. (1984), *Elementary Linear Algebra, Fourth Edition*. John Wiley and Sons.
[p. 113. Lines 22–24.]

9. (64, 111, 121, 265, 334, 358)
Anton, H. (1984), *Elementary Linear Algebra, Fourth Edition*. John Wiley and Sons.
[p. 121. Lines 10–13.]

10.
Anton, H. (1984), *Elementary Linear Algebra, Fourth Edition*. John Wiley and Sons.
[p. 176. Lines 8–10 of Section 4.7.]

11. (123, 217)
Applebaum, C. H. (1971), 'ω-homomorphisms and ω-groups'. *Journal of Symbolic Logic*, volume **36**, pages 55–65.
[p. 56. Lines 9–8 from the bottom.]
DEFINITION. The ω-group G_1 is ω-homomorphic to the ω-group G_2 [written: $G - 1 \geq_\omega G_2$] if there is at least one ω-homomorphism from G_1 onto G_2.

12. (41)
Arbib, M. A. and E. G. Manes (1974), 'Machines in a category: An expository introduction'. *SIAM Review*, volume **16**, pages 163–192.
[p. 169. Lines 1–4.]
A group may be thought of as a set with 3

operators, a binary operation labeled · (we say the label · has *arity* 2 since · labels a 2-*ary* operator); a unary operation labeled [1] (which has arity 1), and a constant labeled e (we say e has arity 0, and refer to constants as nullary operators).

13.
Arratia, R., A. D. Barbour, and S. Tavaré (1997), 'Random combinatorial structures and prime factorizations'. *Notices of the American Mathematical Society*, volume , pages –.
[p. –. .]
We call combinatorial structures that have this property "logarithmic".

14. (183)
Axler, S. (1995), 'Down with determinants!'. *American Mathematical Monthly*, volume **102**, pages 139–154.
[p. 140. Lines 12–11 from bottom.]
To show that T (our linear operator on V) has an eigenvalue, fix any non-zero vector $v \in V$.

15.
Axler, S. (1995), 'Down with determinants!'. *American Mathematical Monthly*, volume **102**, pages 139–154.
[p. 142. Proposition 3.4.]
The generalized eigenvectors of T span V.

16. (123)

Baer, R. (1955), 'Supersoluble groups'.
*Proceedings of the American Mathematical
Society*, volume **6**, pages 16–32.
        [p. 16. Lines 18–19.]
DEFINITION. *The group G is supersoluble if
every homomorphic image $H \neq 1$ of G contains
a cyclic normal subgroup different from* 1.

17. (30, 462)

Balinski, M. L. and H. P. Young (1977),
'Apportionment schemes and the quota
method'. *American Mathematical Monthly*,
volume **84**, pages 450.
        [p. 451. Theorem 1.]
An apportionment method M is
house-monotone and consistent if and only if it
is a Huntington method.

18. (214)

**Baker, H. H., A. K. Dewdney, and A. L. Szilard
(1974)**, 'Generating the nine-point graphs'.
Mathematics of Computation, volume **28**, pages
833–838.
        [p. 835. Lines 4–3 above Figure 2.]
In Fig. 2(a) below, a nine-point graph is shown.

19. (273)

Barnes, C. W. (1984), 'Euler's constant and e'.
American Mathematical Monthly, volume **91**,
pages 428–430.
        [p. 429. Lilnes 10–11.]

We follow the customary approach in
elementary calculus courses by using the
definition $\ln(x) = \int_1^x t^{-1}\, dt$.

20. (144, 216, 483)

Bartle, R. C. (1996), 'Return to the Riemann
integral'. *American Mathematical Monthly*,
volume **103**, pages 625–632.
        [p. 626. Lines 17–19 and 36.]
Usually the partition is ordered and the
intervals are specified by their end points; thus
$I_i := [x_{i-1}, x_i]$, where

$$a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_n = b.$$

$\ldots$ A strictly positive function δ on I is called
a *gauge* on I.

[Shouldn't the $[0, 1]$ be a superscript below? (And similarly in the following citation).]

21. (53)

Bartle, R. C. (1996), 'Return to the Riemann
integral'. *American Mathematical Monthly*,
volume **103**, pages 625–632.
        [p. 627. Lines 23–24.]
(3.2) If $h : [0, 1] \to \mathbf{R}$ is Dirichlet's function (=
the characteristic function of the rational
numbers in $[0, 1]$), then $h \in \mathcal{R}^*([0, 1])$ and
$\int_0^1 h = 0$.

505

22. (80, 224, 339, 421, 425, 446, 462)
Bartle, R. C. (1996), 'Return to the Riemann integral'. *American Mathematical Monthly*, volume **103**, pages 625–632.
          [p. 631. Lines 3–6, 13–14, 19.]
(8.2) Dominated Convergence Theorem. *Let (f_n) be a sequence in $\mathcal{R}^*([a,b])$, let g, $h \in \mathcal{R}^*([a,b])$ be such that*

$$g(x) \leq f_n(x) \leq h(x) \quad \text{for all} \quad x \in [a,b],$$

and let $f(x) = \lim_n f_n(x) \in \mathbf{R}$ for all $x \in [a,b]$. Then $f \in \mathcal{R}^*([a,b])$ and (8a) holds.
 ... As usual, we define a *null set* in $I := [a,b]$ to be a set that can be covered by a countable union of intervals with arbitrarily small total length.
 ... *Every $f \in \mathcal{R}^*(I)$ is measurable on I.*

23. (147, 184)
Bell, H., J. R. Blum, J. V. Lewis, and J. Rosenblatt (1966), *Modern University Calculus with Coordinate Geometry*. Holden-Day, Inc.
          [p. 48. Proof.]
Proof. Let N be the set of those positive integers which satisfy the following conditions:
(a) 1 is a member of N,
(b) whenever x is a member of N, then $x \geq 1$. We need to show that N is precisely the set of all positive integers to prove our result.

24. (66, 373)
Bass, H. and R. Kulkarni (1990), 'Uniform tree lattices'. *Journal of the American Mathematical Society*, volume **3**, pages 843–902.
          [p. 845. Lines 22–23.]
We put $i(e) = [\mathcal{A}_{\partial_0 e} : \alpha_e \mathcal{A}_e]$ and call $(A, i) = I(\mathbf{A})$ the corresponding *edge-indexed graph.*

25. (349)
Bauer, F. L. (1977), 'Angstl's mechanism for checking wellformedness of parenthesis-free formulae'. *Mathematics of Computation*, volume **31**, pages 318–320.
          [p. 318. Just below the first figure.]
The formula in parenthesis-free (polish) form [2] is now written over the fixed bars ...

26. (214, 319)
Bauer, H. (1978), 'Approximation and abstract boundaries'. *American Mathematical Monthly*, volume **85**, pages 632–647.
          [p. 644. Lines 4–2 above the figure.]
We obtain

$$\Phi(X) = \left\{ \big(x, u(x)\big) \mid x \in [a,b] \right\}$$

which is the graph of the function u.

27. (122, 123)
Bellamy, D. P. (1975), 'Weak chainability of pseudocones'. *Proceedings of the American*

Mathematical Society, volume **48**, pages 476–478.

[p. 476. Lines 1–4.]

A continuum is a compact metric space. $I = [0, 1]$; $A = (0, 1]$; S is the unit circle in the complex numbers. If X is a continuum, a *pseudocone over X* is a compactification of A with remainder X.

28. (199)

Bell, H., J. R. Blum, J. V. Lewis, and J. Rosenblatt (1966), *Modern University Calculus with Coordinate Geometry.* Holden-Day, Inc.

[p. 186. Theorem 3.4.]

3.4. Theorem: Alternative definition of monotone. The function f is monotone increasing on A if and only if, whenever x and $x + h > x$ are in $A \subseteq \operatorname{dmn} f$, we have

$$f(x + h) - f(x) > 0.$$

29. (117, 199)

Belna, C. L., M. J. Evans, and P. D. Humke (1979), 'Symmetric and strong differentiation'. *American Mathematical Monthly*, volume **86**, pages 121–123.

[p. 121. Line 1.]

Throughout we let f denote a real valued function defined on the real line **R**.

30. (63)

Bendersky, M. and D. M. Davis (1994), '3-primary v_1-periodic homotopy groups of F_4 and E_6'. *Transactions of the American Mathematical Society*, volume **344**, pages 291–306.

[p. 295. Lines 5–4 from bottom.]

The Toda bracket $\langle -, 3, \alpha_1 \rangle$ is essentially multiplication by v_1, which acts nontrivially from $v_{4i+2} S^{23}$ to $v_{4i+6} S^{23}$.

31. (260)

Bezem, M. (1989), 'Compact and majorizable functionals of finite type'. *Journal of Symbolic Logic*, volume **54**, pages 271–280.

[p. 271. Lines 8–6 from bottom.]

We shall occasionally use lambda-notation to specify functionals, ie $\lambda X.$— specifies a functional F such that $FX =$ — for all X.

32. (265, 462)

Bhatia, R. and P. Šemrl (1997), 'Approximate isometries on Euclidean spaces'. *American Mathematical Monthly*, volume **104**, pages 497–504.

[p. 502. Lemma 3.]

Lemma 3. *Let f and g be as in Lemma 2, and let $u \in E_n$ be a unit vector. Then for every $x \in E_n$ orthogonal to u we have* $|\langle f(x), g(u) \rangle| \leq 3\varepsilon$.

33. (328)
Bhatia, R. and P. Šemrl (1997), 'Approximate isometries on Euclidean spaces'. *American Mathematical Monthly*, volume **104**, pages 497–504.

[p. 503. Line 9 from bottom.]
Since f is an ε-isometry, we have

$$m - \varepsilon < \|f(x + my) - f(x)\| < m + \varepsilon,$$

or equivalently,

$$m - \varepsilon < \|(m - a + b_m)y + u_m\| < m + \varepsilon.$$

34. (91, 216)
Bieri, R. and J. R. J. Groves (1986), 'A rigidity property for the set of all characters induced by valuations'. *Transactions of the American Mathematical Society*, volume **294**, pages 425–434.

[p. 425. Abstract.]
We prove that $\Delta(G)$ satisfies a certain rigidity property and apply this to give a new and conceptual proof of the Brewster-Roseblade result [**4**] on the group of automorphisms stabilizing G.

35. (216, 216, 233)
Billingsley, P. (1973), 'Prime numbers and Brownian motion'. *American Mathematical Monthly*, volume **80**, pages 1099–1115.

[p. 1107. Lines 3–2 from bottom.]

For an illustration of this theorem, suppose the A in (10) is the set $[x : \alpha \leq x(1) \leq \beta]$ of paths in $C_0[0, 1]$ that over the point $t = 1$ have a height between α and β.

36. (181)
Birman, J. S. (1993), 'New points of view in knot theory'. *Bulletin of the American Mathematical Society (N.S.)*, volume **28**, pages 253–287.

[p. 279. Lines 23–24.]
Using Lemma 2, we find a closed braid representative K_β of BK, $\beta \in B_n$.

37. (109, 462)
Blass, A. (1979), 'Natural endomorphisms of Burnside rings'. *Transactions of the American Mathematical Society*, volume **253**, pages 121–137.

[p. 122. Lines 19–20.]
For any two G-sets X and Y, there are obvious actions of G on the disjoint union $X + Y$ and (componentwise) on the product $X \times Y$.

38. (122, 124)
Blecksmith, R., M. McCallum, and J. L. Selfridge (1998), '3-smooth representations of integers'. *American Mathematical Monthly*, volume , pages 529–543.

[p. 529. Lines 6–7 of Introduction.]
A 3-*smooth number* is a positive integer whose prime divisors are only 2 or 3.

39. (113, 113, 339)

Blecksmith, R., M. McCallum, and J. L. Selfridge (1998), '3-smooth representations of integers'. *American Mathematical Monthly*, volume , pages 529–543.

[p. 535. Lines 10–9 from bottom.]

Corollary. Assume that $n > 1$ is prime to 6. Then n has a unique representation if and only if the 2-rep and the 3-rep of n agree.

40. (470)

Boudreau, P. E., J. J. S. Griffin, and M. Kac (1962), 'An elementary queueing problem'. *American Mathematical Monthly*, volume **69**, pages 713–724.

[p. 717. Lines 19–21.]

But the analyticity of $G(z, w)$ requires that the numerator of the fraction vanish whenever the denominator does.

41. (393)

Bredon, G. E. (1971), 'Counterexamples on the rank of a manifold'. *Proceedings of the American Mathematical Society*, volume **27**, pages 592–594.

[p. 592. Lines 3–5.]

The Poincaré polynomial of M is $P_M(t) = \sum b_i t^i$ where b_i is the ith Betti number of M. Thus rank $M \geq 1$ iff -1 is a root of $P_M(t)$.

42. (453)

Bridges, D. S. (1977), 'A constructive look at orthonormal bases in Hilbert space'. *American Mathematical Monthly*, volume **84**, pages 189–191.

[p. 189. End of second paragraph.]

We also consider the trivial space $\{0\}$ to be of finite dimension 0.

43. (76)

Britt, J. (1985), 'The anatomy of low dimensional stable singularities'. *American Mathematical Monthly*, volume **92**, pages 183–201.

[p. 184. Lines 7–9.]

The mapping behaves very differently at singular points from the way it does at regular ones, where locally it behaves as if it were a mapping onto its codomain.

44. (53, 62, 146, 407, 459)

Brickman, L. (1993), 'The symmetry principle for Möbius transformations'. *American Mathematical Monthly*, volume **100**, pages 781–782.

[p. 782. Lemma 2.]

Lemma 2.*For each circle or extended line E, there is a unique $\bar{T} \in \bar{\mathcal{M}}$ such that*

$$E = \{z \in \hat{\mathbf{C}} : \bar{T}(z) = z\}.$$

(E is exactly the set of fixed points of $\bar{T}$.) This

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$\bar{T}$ is an involution of $\hat{\mathbf{C}}$; that is, $\bar{T} \circ \bar{T}$ is the identity.

45. (109, 156, 458)
Brink, C. and J. Pretorius (1992), 'Boolean circulants, groups, and relation algebras'. *American Mathematical Monthly*, volume **99**, pages 146–152.
        [p. 146. Lines 18–22.]
Let $\mathcal{B}_n$ be the algebra which has as the base set all n-square Boolean matrices and is endowed with the componentwise Boolean operations of complementation $'$, meet $\cdot$ and join $+$ (under which it forms a Boolean algebra) as well as the matrix operations of transposition and multiplication ; , and the identity matrix I.

46. (94, 265)
Bruce, J. W. (1993), 'A really trivial proof of the Lucas-Lehmer test'. *American Mathematical Monthly*, volume **100**, pages 370–371.
        [p. 370. Theorem 1.]
Theorem 1 (LUCAS-LEHMER). Let p be a prime number. Then $M_p = 2^p - 1$ is a prime if M_p divides S_{p-1}.

47. (184, 233)
Bruckner, A. M., J. Marik, and C. E. Weil (1002), 'Some aspects of products of derivatives'. *American Mathematical Monthly*, volume **99**, pages 134–145.

        [p. 140. Lines 2–3.]
As an illustration of this theorem let us consider a function u with the following properties . . .

48. (462)
Bryant, V. (1993), *Aspects of Combinatorics*. Cambridge University Press.
        [p. 30. Lines 14–15.]

49. (150, 365)
Bryant, V. (1993), *Aspects of Combinatorics*. Cambridge University Press.
        [p. 42. Line 6 from bottom.]

50. (46, 462)
Bryant, V. (1993), *Aspects of Combinatorics*. Cambridge University Press.
        [p. 62. Lines 10–18.]

51. (326)
Buckholtz, D. (1997), 'Inverting the difference of Hilbert space projections'. *American Mathematical Monthly*, volume **104**, pages 60–61.
        [p. 60. Lines 1–3.]
Let R and K be subspaces of a Hilbert space H, and let P_R and P_K denote the orthogonal projections of H onto these subspaces. When is the operator $P_R - P_K$ invertible? . . .

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52. (94, 94)

Buhler, J., D. Eisenbud, R. Graham, and C. Wright (1994), 'Juggling drops and descents'. *American Mathematical Monthly*, volume **101**, pages 507–519.

      [p. 513. Lines 8–9.]

Which finite sequences correspond to juggling patterns? Certainly a necessary condition is that the average must be an integer. However this isn't sufficient.

53. (181)

Bumby, R. T., F. Kochman, and D. B. West, editors (1993a), 'Problems and solutions'. *American Mathematical Monthly*, volume **100**, pages 796–809.

      [p. 796. Problem 10331.]

Find all positive integers n such that $n!$ is multiply perfect; i.e., a divisor of the sum of its positive divisors.

54. (58, 327, 425)

Bumby, R. T., F. Kochman, and D. B. West, editors (1993b), 'Problems and solutions'. *American Mathematical Monthly*, volume **100**, pages 498–505.

      [p. 499. Problem 10311.]

It is well-known that if g is a primitive root modulo p, where $p > 2$ is prime, either g or $g + p$ (or both) is a primitive root modulo p^2 (indeed modulo p^k for all $k \geq 1$.)

55. (21)

Burgstahler, S. (1986), 'An algorithm for solving polynomial equations'. *American Mathematical Monthly*, volume **93**, pages 421–430.

      [p. 423. Lines 7–11.]

The new algorithm can now be described. To approximate roots of $P(x) = 0$ (which, without loss of generality, is assumed not to have a root at $x = 0$):

Step 1: If the desired root is known to be near the origin, solve $P(1/z) = 0$ for $z = 1/x$.

Step 2: Determine a preliminary root estimate $R \neq 0$.

Step 3: Use R and formula (6) to find x_i and replace R by this number.

Step 4: If R is unsatisfactory as a root estimate, repeat step 3. (Or repeat step 1 if $|R| \ll 1$.)

56. (209, 388)

Burton, D. M. (1994), *Elementary Number Theory, Third Edition*. Wm C. Brown Publishers.

      [p. 17. Theorem 2–1.]

57. (94, 113, 446)

Burton, D. M. (1994), *Elementary Number Theory, Third Edition*. Wm C. Brown Publishers.

      [p. 24. Corollary 2.]

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58. (389)

Burckel, R. B. (1997), 'Three secrets about harmonic functions'. *American Mathematical Monthly*, volume **104**, pages 52–56.
[p. 55. Lines 4–6.]
In fact, this result is part of a large subject called quadrature problems that interested readers can find more about in . . .

59. (18)

Busenberg, S., D. C. Fisher, and M. Martelli (1989), 'Minimal periods of discrete and smooth orbits'. *American Mathematical Monthly*, volume **96**, pages 5–17.
[p. 8. Lines 2–4.]
Therefore, a normed linear space is really a pair $(\mathbf{E}, \|\cdot\|)$ where $\mathbf{E}$ is a linear vector space and $\|\cdot\| : \mathbf{E} \to (0, \infty)$ is a norm. In speaking of normed spaces, we will frequently abuse this notation and write $\mathbf{E}$ instead of the pair $(\mathbf{E}, \|\cdot\|)$.

60. (379)

Call, G. S. and D. J. Velleman (1993), 'Pascal's matrices'. *American Mathematical Monthly*, volume **100**, pages 372–376.
[p. 373. Theorem 2.]
Theorem 2. *For any real numbers x and y,* $P[x]P[y] = P[x + y].$

61.

Call, G. S. and D. J. Velleman (1993), 'Pascal's

matrices'. *American Mathematical Monthly*, volume **100**, pages 372–376.
[p. 375. Theorem 5.]
Theorem 5. *For every real number x,* $P[x] = e^{xL}.$

62. (216, 216)

Carr, D. M. (1982), 'The minimal normal filter on $p_\kappa \lambda$'. *Proceedings of the American Mathematical Society*, volume **86**, pages 316–320.
[p. 316. Lines 2–3.]
Unless specified otherwise, κ denotes an uncountable regular cardinal and λ is a cardinal $\geq \kappa$.

63. (53, 462)

Chow, T. Y. (1999), 'What is a closed-form number?'. *American Mathematical Monthly*, volume **106**, pages 440–448.
[p. 444. Lines 12–14.]
Definition. A *tower* is a finite sequence $A = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ of nonzero complex numbers such that for all $i \in \{1, 2, \ldots n\}$ there exists some integer $m_i > 0$ such that $\alpha_i^{m_i} \in A_{i-1}$ or $e^{\alpha_i m_i} \in A_{i-1}$ (or both).

64. (258)

Clifford, A. H. (1959), 'Connected ordered topological semigroups with idempotent endpoints II'. *Transactions of the American*

Mathematical Society, volume **91**, pages 193–208.

[p. 106. Lines 4–1 from bottom.]
The binary operation in S, which we now denote by $\circ$, is completely determined by that in T, which we denote by juxtaposition, and the mappings ψ and θ as follows (wherein $x, y \in T^0$; $t \in T$; $\kappa, \lambda \in K$):

65. (94)
Copper, M. (1993), 'Graph theory and the game of Sprouts'. *American Mathematical Monthly*, volume **100**, pages 478–482.

[p. 480. Lemma 1.]
Suppose that the cubic graph G arises as just described from a complete game of Sprouts played on m vertices in p plays. Then

$$f = 2 + p - m$$

66. (161)
Culler, M. and P. B. Shalen (1992), 'Paradoxical decompositions, 2-generator Kleinian groups, and volumes of hyperbolic 3-manifolds'. *Journal of the American Mathematical Society*, volume **5**, pages 231–288.

[p. 235. Lines 7–6 from bottom.]
We establish some notation and conventions that will be used throughout the paper.

67. (214)

Curjel, C. R. (1990), 'Understanding vector fields'. *American Mathematical Monthly*, volume **97**, pages 524–527.

[p. 524. Lines 17–16 from bottom.]
In the following exercises students have to use ruler and pencil to work on graphs and curves given by drawings.

68. (66, 120, 132, 217)
Currie, J. (1993), 'Open problems in pattern avoidance'. *American Mathematical Monthly*, volume **100**, pages 790–793.

[p. 790. Lines 8–3 from bottom.]
A **word** is a finite sequence of elements of some finite set Σ. We call the set Σ the **alphabet**, the elements of Σ **letters**. The set of all words over Σ is written Σ^*.
. . . The **empty word**, with no letters, is denoted by ε.

69. (214)
Dankner, A. (1978), 'On Smale's axiom A dynamical systems'. *Annals of Mathematics*, volume **107**, pages 517–553.

[p. 539. 5.2.7.]
On $\{r_4 \leq r \leq r_5$ and $z \leq z_2\}$ the action of f on z-coordinates is multiplication by $e(r)$, where $e(r)$ is a smooth bump function with graph shown in Figure 4.

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70. (130, 216, 414)

Darst, R. and C. Goffman (1970), 'A Borel set which contains no rectangles'. *American Mathematical Monthly*, volume **77**, pages 728–729.

[p. 729. Line 23.]

Then $\phi(0) = m_1 \left([U \cap F] \cap [U \cap G] \right) > .8\epsilon$.

71. (78, 306, 310, 329, 374, 489)

Davey, B. A. and H. A. Priestley (1990), *Introduction to Lattices and Order.* Cambridge University Press.

[p. 3. Beginning of 1.4.]

Each of $\mathbb{N}$ (the natural numbers $\{1, 2, 3, \ldots\}$), $\mathbb{Z}$ (the integers) and $\mathbb{Q}$ (the rational numbers) also has a natural order making it a chain.

72. (216)

de Boor, C. and K. Höllig (1991), 'Box-spline tilings'. *American Mathematical Monthly*, volume **98**, pages 793–802.

[p. 795. Line 6 from bottom.]

In this situation, it is convenient to introduce the new variables

$$(u, v) := \Xi^T x = (\xi^T x, \eta^T x).$$

73. (147)

Devaney, R. L. and M. B. Durkin (1991), 'The exploding exponential and other chaotic bursts in complex dynamics'. *American Mathematical Monthly*, volume **98**, pages 217–233.

[p. 222. Lines 10–9 from bottom.]

That is, if a point is contained in $J(F)$, then so are all of its images and all of its preimages.

74. (265)

Ž. Djoković, D. (1982), 'Closures of conjugacy classes in classical real linear Lie groups. II'. *Transactions of the American Mathematical Society*, volume **270**, pages 217–252.

[p. 233. Lines 1 and 8.]

Let first $k = 0$, i.e, $m = n$. ... Now let $k > 0$. ...

75. (146, 232)

Dornhoff, L. L. and F. E. Hohn (1978), *Applied Modern Algebra.* Macmillan.

[p. 166. Lines 1–3.]

A monoid $[M, \circ]$ with identity element 1 is a *group* iff for each $m \in M$ there is an inverse element $m^{-1} \in M$ such that

$$m^{-1} \circ m = m \circ m^{-1} = 1$$

76. (110)

Downey, R. and J. F. Knight (1992), 'Orderings with αth jump degree 0^α'. *Proceedings of the American Mathematical Society*, volume **114**, pages 545–552.

[p. 546. Lemma 1.1.]

If C is r.e. in X but not recursive in X, then there is an ordering **A** such that **A** is recursive in C and no copy of **A** is recursive in C.

77. (82, 339)

Drasin, D. (1995), 'Review of (a) *Normal families of meromorphic functions*, by Chi-tai Chung, and (b) *Normal families*, by Joel L. Schiff'. *Bulletin of the American Mathematical Society (N.S.)*, volume **32**, pages 257–260.

[p. 258. First line of Theorem.]
Let D be a domain, $a, b \neq 0$ two complex numbers and $k \geq 1$ an integer.

78.

Dubins, L. E. (1977), 'Group decision devices'. *American Mathematical Monthly*, volume **84**, pages 350–356.

[p. 353. Line 6.]
Proof. For each d, $u(d, \cdot)$ has an inverse function $u^{-1}(d, \cdot)$, where

$$u(d, c) = t \leftrightarrow u^{-1}(d, t) = c.$$

79. (53, 238, 313)

Duke, W. (1997), 'Some old problems and new results about quadratic forms'. *Notices of the American Mathematical Society*, volume **44**, pages 190–196.

[p. 193. Lines 9–13, second column.]
...it can be seen that the number of representations of n as a sum of four squares is eight times the sum of those divisors of n which are not multiples of four. In particular, it is never zero!

80. (414)

Dummigan, N. (1995), 'The determinants of certain Mordell-Weil lattices'. *American Journal of Mathematics*, volume **117**, pages 1409–1429.

[p. 1419. Last line of Definition 1..]
We denote the circle diagram by enclosing the string diagram in square brackets, e.g. $[XOXOX]$.

81. (414)

Feeman, T. G. and O. Marrero (2000), 'Sequences of chords and of parabolic segments enclosing proportional areas'. *The College Mathematics Journal*, volume **31**, pages 379–382.

[p. 381. Second to last displayed formula.]

$$1 \leq \frac{L_n^2}{d_n^2} \leq \frac{\left[1 + c\left(a_{n+1} + a_n\right)\right]^2}{1 + c^2\left(a_{n+1} + a_n\right)^2} \to \frac{c^2}{c^2} = 1$$

82. (65, 216)

Edelman, A. and E. Kostlan (1995), 'How many zeros of a random polynomial are real?'. *Bulletin of the American Mathematical Society (N.S.)*, volume **32**, pages 1–37.

[p. 7. Lines 6–5 from the bottom.]
If $v(t)$ is the moment curve, then we may calculate $\|\gamma'(t)\|$ with the help of the following observations and some messy algebra:

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83. ([408](#), [462](#))
Edgar, G. A., D. H. Ullman, and D. B. West, editors (1997), 'Problems and solutions'. *American Mathematical Monthly*, volume **104**, pages 566–576.

[p. 574. Problem 10426.]

Show that any integer can be expressed as a sum of two squares and a cube. Note that the integer being represented and the cube are both allowed to be negative.

84. ([64](#), [370](#))
Epp, S. S. (1995), *Discrete Mathematics with Applications, 2nd Ed.* Brooks/Cole.

[p. 2. Blue box.]

85.
Epp, S. S. (1995), *Discrete Mathematics with Applications, 2nd Ed.* Brooks/Cole.

[p. 30. Lines 11.]

86. ([19](#), [184](#))
Epp, S. S. (1995), *Discrete Mathematics with Applications, 2nd Ed.* Brooks/Cole.

[p. 36. Lines 21–24.]

87. ([132](#), [306](#), [310](#))
Epp, S. S. (1995), *Discrete Mathematics with Applications, 2nd Ed.* Brooks/Cole.

[p. 76. Lines 12–11 from bottom.]

88. ([124](#), [147](#), [209](#), [230](#), [385](#))

Epp, S. S. (1995), *Discrete Mathematics with Applications, 2nd Ed.* Brooks/Cole.

[p. 534. Blue box.]

89. ([160](#), [458](#))
Exner, G. R. (1996), *An Accompaniment to Higher Mathematics.* Springer-Verlag.

[p. 35. Theorem 1.115.]

Suppose E is an equivalence relation on a set S. For any x in S, denote by E_x the set of all y in S equivalent under E to x. Then the collection of all E_x is a partition of S.

90. ([184](#), [216](#), [224](#))
Farrell, F. T. and L. E. Jones (1989), 'A topological analogue of Mostow's rigidity theorem'. *Journal of the American Mathematical Society*, volume , pages 257–370.

[p. 272. Lemma 2.1.]

The following inequalities hold for any vector η tangent to a leaf of the foliation $\mathcal{F}$ of FM.

$$2h\left(dq(\eta), dq(\eta)\right) \geq \hat{h}(\eta, \eta) \geq h\left(dq(\eta), dq(\eta)\right)$$

where $q : \mathcal{D}M \to M$ denotes the bundle projections.

91. ([258](#))
Fearnley-Sander, D. (1982), 'Hermann Grassmann and the prehistory of universal algebra'. *American Mathematical Monthly*, volume **89**, pages 161–166.

[p. 161. Lines 16–12 from bottom.]

Given a set of symbols $S = \{x_1, x_2, \ldots, x_n\}$, the set

$$G = \{x_1, x_2, \ldots, x_n, (x_1 x_1), (x_1 x_2), \ldots ,$$
$$(x_n x_n), x(x_1(x_1 x_1)), (x_1(x_1 x_2)), \ldots ,$$
$$(x_1(x_n x_n)), (x_2(x_1 x_1)), \ldots ,$$
$$(x_n(x_n x_n)), ((x_1 x_1)x_1), ((x_1 x_1)x_2), \ldots \}$$

obtained by repeated juxtaposition of the symbols already written down, forms a groupoid in a natural way (the binary operation being juxtaposition).

92. (217, 367)
Finn, R. (1999), 'Capillary surface interfaces'. *Notices of the American Mathematical Society*, volume **46**, pages 770–781.
[p. 774. Formula (11) and the line above.]
The same procedure with $\Omega^* = \Omega$ yields

$$2H = \frac{|\Sigma| \cos \gamma}{|\Omega|}$$

93. (144, 144, 147, 198)
Fisher, D. (1982), 'Extending functions to infinitesimals of finite order'. *American Mathematical Monthly*, volume **89**, pages 443–449.
[p. 445. Lines 9–11.]
(i) if f is in the domain of T, and $\text{dom}(f)$ is the domain of f, then the function with domain

$\text{dom}(f)$ and constant value 1 is in the domain of T ...

94. (380)
Fournelle, T. A. (1993), 'Symmetries of the cube and outer automorphisms of S$_6$'. *American Mathematical Monthly*, volume **100**, pages 377–380.
[p. 377. Line 9 above picture.]
To begin, recall that an *isometry* of $\mathbb{R}^3$ is a bijection which preserves distance.

95. (325)
Fraleigh, J. B. (1982), *A First Course in Abstract Algebra*. Addison-Wesley.
[p. 11. Lines 4–5 of Section 1.2.]

96. (124, 354, 407, 458)
Fraleigh, J. B. (1982), *A First Course in Abstract Algebra*. Addison-Wesley.
[p. 41. Lines 6–7.]

97. (338, 338, 425)
Frantz, M. (1998), 'Two functions whose powers make fractals'. *American Mathematical Monthly*, volume **105**, pages 609–617.
[p. 609. Lines 4 and 5.]
...Richard Darst and Gerald Taylor investigated the differentiability of functions f^p (which for our purposes we will restrict to

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$(0,1))$ defined for each $p \geq 1$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1/n^p & \text{if } x = m/n \text{ with } (m,n) = 1. \end{cases}$$

98. (396)
Frantz, M. (1998), 'Two functions whose powers make fractals'. *American Mathematical Monthly*, volume **105**, pages 609–617.
        [p. 614. Formula (4).]
...if $p > 1$ and B is the set of numbers x that are not dyadic rationals and satisfy

$$|x - m/2^n| \leq (2^n)^{-p}$$

for infinitely many dyadic rationals $m/2^n$, then...

99. (66, 101, 123, 147)
Freiling, C. (1990), 'Symmetric derivates, scattered, and semi-scattered sets'. *Transactions of the American Mathematical Society*, volume **318**, pages 705–720.
        [p. 705. Abstract.]
We call a set right scattered (left scattered) if every nonempty subset contains a point isolated on the right (left).

100. (137)
Freiling, C. (1990), 'Symmetric derivates, scattered, and semi-scattered sets'. *Transactions of the American Mathematical*

Society, volume **318**, pages 705–720.
        [p. 715. Lines 7–6 from bottom.]
Let F be as stated and let A, B and C be disjoint sets where $A \cup B \cup C = I \ldots$

101. (380, 467)
Friedman, R. (1995), 'Vector bundles and $SO(3)$-invariants for elliptic surfaces'. *Journal of the American Mathematical Society*, volume **8**, pages 29–139.
        [p. 29. Lines 4–6.]
Recall that a relatively minimal simply connected elliptic surface S is specified up to deformation type by its geometric genus $p_g(S)$ and by two relatively prime integers m_1, m_2, the multiplicities of its multiple fibers.

102. (319)
Furi, M. and M. Martelli (1991), 'The teaching of mathematics'. *American Mathematical Monthly*, volume **98**, pages 835–846.
        [p. 842. Lines 17–20.]
By applying the Mean Value Theorem to f on $[a,d]$ and $[d,b]$ respectively, we obtain

$$f(d) - f(a) = f'(c_1)(d-a), \qquad c_1 \in (a,d),$$
$$f(b) - f(d) = f'(c_2)(b-d), \qquad c_1 \in (d,b).$$

103. (19)
Fulda, J. S. (1989), 'Material implication revisited'. *American Mathematical Monthly*, volume **96**, pages 247–250.
[p. 248. Third paragraph.]
It is the thesis of this paper that this uneasiness is none other than the familiar temptation to commit the fallacy of conversion $(p \Rightarrow q \vdash q \Rightarrow p)$, also known as the fallacy of affirming the consequence $(p \Rightarrow q, q \vdash p)$ ...

104. (53, 120)
Galvin, F. (1994), 'A proof of Dilworth's chain decomposition theorem'. *American Mathematical Monthly*, volume **101**, pages 352–353.
[p. 352. Lines 1–2.]

105. (209, 425)
Gelbaum, B. R. and J. M. H. Olmsted (1990), *Theorems and Counterexamples in Mathematics*. Springer-Verlag.
[p. 65. Exercise 2.1.2.14.]
Show that for f in **Exercise 2.1.1.13. 49**, if g is given by
$$g(x) \stackrel{\text{def}}{=} \int_0^x f(t)dt$$
then: ...

106. (319)
Gelbaum, B. R. and J. M. H. Olmsted (1990), *Theorems and Counterexamples in*

Mathematics. Springer-Verlag.
[p. 85. Theorem 2.1.3.5..]
Let (2.1.3.3) obtain everywhere on a measurable set E of positive measure. Then ...

107. (27, 265)
Giblin, P. J. and S. A. Brassett (1985), 'Local symmetry of plane curves'. *American Mathematical Monthly*, volume **92**, pages 689–707.
[p. 691. Line 6 under the figure.]
Suppose we are given a smooth simple closed plane curve and a point p on it. Is there always a circle or straight line tangent to the curve at p and at another point $p' \neq p$?

108. (123, 124, 241)
Giesy, D. P. (1971), 'The general solution of a differential-functional equation'. *American Mathematical Monthly*, volume **78**, pages 37–42.
[p. 37. Lines 10–1.]
2DEFINITION. A function g is *of type* (A) *on* I if g has an antiderivative v on I and
$g(a) = g(b)$ *implies* $v(a) = v(b)$ *for all a and b
in I*

109. (216)
Gilmore, P. C. (1960), 'An alternative to set theory'. *American Mathematical Monthly*, volume **67**, pages 621–632.
[p. 622. Lines 5–3 from bottom.]

The nu relation between symbols is such that the following sentences are true:

$2\,\nu$ odd, $\quad 4\,\nu$ odd, $\quad 1\,\nu$ even, $\quad 3\,\nu$ even.

110. (94)
Graham, N., R. C. Entringer, and L. A. Szekely (1994), 'New tricks for old trees: Maps and the pigeonhole principle'. *American Mathematical Monthly*, volume **101**, pages 664–667.
    [p. 665. Lines 4–3 from bottom.]
Assume ϕ has no fixed vertex. Then, for every vertex v, there is a unique non-trivial path in T from v to $\phi(v)$.

111. (198, 199)
Graham, R. L., D. E. Knuth, and O. Patashnik (1989), *Concrete Mathematics*. Addison-Wesley.
    [p. 71. Lines 15–13 from bottom.]
Let $f(x)$ be a continuous, monotonically increasing function with the property that
$$f(x) = \text{integer} \Rightarrow x = \text{integer}$$
(The symbol '$\Rightarrow$' means "implies.")

112. (63)
Grayson, M., C. Pugh, and M. Shub (1994), 'Stably ergodic diffeomorphisms'. *Annals of Mathematics, 2nd Ser.*, volume **140**, pages 295–329.
    [p. 304. Lines 7–6 from bottom.]
This is the usual picture of the Lie bracket

$[X, Y] = Z$, and becomes especially clear if drawn in a flowbox for X.

113. (230, 324, 358, 397, 463)
Grassman, W. K. and J.-P. Tremblay (1996), *Logic and Discrete Mathematics: A Computer Science Perspective*. Prentice-Hall.
    [p. 105. Definition 2.7.]

114. (34, 368)
Grassman, W. K. and J.-P. Tremblay (1996), *Logic and Discrete Mathematics: A Computer Science Perspective*. Prentice-Hall.
    [p. 234. Definition 5.3.]

115. (273)
Greenlaw, R. and H. J. Hoover (1998), *Fundamentals of the Theory of Computation*. Morgan Kaufmann Publishers, Inc.
    [p. 36. Footnote.]
All logarithms in this text are base two unless noted otherwise. We use the now quite common notation $\lg x$ instead of $\log_2 x$.

116. (82)
Grimaldi, R. P. (1999), *Discrete and Combinatorial Mathematics, An Applied Introduction, Fourth Edition*. Addison-Wesley.
    [p. 128. Example 3.2(a).]

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117. (369)
Gross, K. L. (1978), 'On the evolution of noncommutative harmonic analysis'. *American Mathematical Monthly*, volume **85**, pages 525–548.

[p. 537. Lines 16–17.]

1. A connected compact *abelian* Lie group is necessarily a *torus*, by which is meant a direct product of circles.

118. (101)
Guckenheimer, J. and S. Johnson (1990), 'Distortion of S-unimodal maps'. *Annals of Mathematics, 2nd Ser.*, volume **132**, pages 71–130.

[p. 72. Lines 16–18.]

Our third result examines maps that have "sensitive dependence to initial conditions". These are maps whose non-wandering set contains an interval.

119. (216, 216)
Harris, M. (1993), '*L*-functions of 2×2 unitary groups and factorization of periods of Hilbert modular forms'. *Journal of the American Mathematical Society*, volume **6**, pages 637–719.

[p. 639. Lines 9–8 from bottom.]

Following the pattern first observed by Waldspurger, the vanishing of $\Theta(\pi, \omega)$ to $GU_{\mathcal{H}}(D)$ either vanishes or equals $\left(\check{\pi}^D, \omega^{-1} \right)$, . . .

120. (227)
Hardman, N. R. and J. H. Jordan (1967), 'A minimum problem connected with complete residue systems in the Gaussian integers'. *American Mathematical Monthly*, volume **74**, pages 559–561.

[p. 559. Lines 1–2.]

A Gaussian integer, γ, is a complex number that can be expressed as $\gamma = a + bi$, where a and b are real integers and i is the so-called imaginary unit.

121. (117)
Hassell, C. and E. Rees (1993), 'The index of a constrained critical point'. *American Mathematical Monthly*, volume **100**, pages 772–778.

[p. 772. Lines 4–6.]

To find the critical points of a smooth function f defined on $M^n \subset \mathbb{R}^{n+m}$, a smooth submanifold given as the common zero-set of m smooth functions $g_i : \mathbb{R}^{n+m} \to \mathbb{R}$.

122. (310)
Hathaway, A. S. (1887), 'A memoir in the theory of numbers'. *American Journal of Mathematics*, volume **9**, pages 162–179.

[p. 162. Lines 4–6.]

The labors of Gauss, Kummer, Dirichlet, Kronecker, Dedekind, and others, have extended the scope of the theory of numbers far

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beyond its original limit of the science of the natural numbers $0, \pm 1, \pm 2, \pm 3, \ldots$.

123. (329)
Haws, L. and T. Kiser (1995), 'Exploring the brachistochrone problem'. *American Mathematical Monthly*, volume **102**, pages 328–336.

[p. 333. Lines 10–9 from bottom.]
... to obtain the 2nd order differential equation

$$\left(1 + \left(y'\right)^2\right)\left(1 + \mu y'\right) + 2\left(y - \mu x\right)y'' = 0.$$

124. (94, 94, 94, 150, 169)
Henriksen, M., S. Larson, J. Martinez, and R. G. Woods (1994), 'Lattice-ordered algebras that are subdirect products of valuation domains'. *Transactions of the American Mathematical Society*, volume **345**, pages 195–221.

[p. 213. Lines 6–10 under Proposition 5.1.]
... $QF(X)$ is an F-space if and only if:
(*) If C_1, C_2 are disjoint cozero sers, then there are zero sets Z_1, Z_2 such that $C_1 \subseteq Z_1$, $C_2 \subseteq Z_2$ and int $(Z_1 \cap Z_2) = \emptyset$.
Thus (*) is a sufficient but not necessary condition for $QF(X)$ to be an SV-space.

125. (141, 147)
Herstein, I. N. (1964), *Topics in Algebra*. Blaisdell.

[p. 2. Lines 11–14.]
The set A will be said to be a *subset* of the set S if every element in A is an element of S, that is, if $a \in A$ implies that $a \in S$. We shall write this as $A \subset S$ ... This notation is not meant to preclude the possibility that $A = S$.

126. (94)
Hofmann, K. H. and C. Terp (1994), 'Compact subgroups of Lie groups and locally compact groups'. *Proceedings of the American Mathematical Society*, volume **120**, pages 623–634.

[p. 630. Lines 19–18 from bottom.]
... and thus G_0/K_0 is homeomorphic to a euclidean space only if $C = K_0$.

127. (64)
Holland, Jr., S. S. (1995), 'Orthomodularity in infinite dimensions; a theorem of M. Solèr'. *Bulletin of the American Mathematical Society (N.S.)*, volume **32**, pages 205–234.

[p. 206. Lines 5–4 from the bottom.]
Thus $(M + M^\perp)^{\perp\perp} = 0^\perp = E$. But $M + M^\perp$ is closed, so $M + M^\perp = E$.

128. (74)
Holland, Jr., S. S. (1995), 'Orthomodularity in infinite dimensions; a theorem of M. Solèr'. *Bulletin of the American Mathematical Society (N.S.)*, volume **32**, pages 205–234.

[p. 222. line 10 from bottom.]

The set of those elements of L which are the join of finitely many atoms is closed under the operations $\vee$ and $\wedge$ ...

129. (86)
Howe, R. E. and E.-C. Tan (1993), 'Homogeneous functions on light cones: the infinitesimal structure of some degenerate principal series representations'. *Bulletin of the American Mathematical Society (N.S.)*, volume **28**, pages 1–74.

[p. 8. Lines 7–9.]
Thus our computation of the action of $\mathfrak{p}$ on individual K-types is in principle (and will turn out to be in practice) sufficient for understanding the submodule structure of $S^a(X^0)$.

130. (124, 216)
Ipsen, I. C. F. and C. D. Meyer (1995), 'The angle between complementary subspaces'. *American Mathematical Monthly*, volume **102**, pages 904–911.

[p. 905. First lines.]
Definition 2.1. For nonzero subspaces $\mathcal{R}, \mathcal{N} \subseteq \mathfrak{R}^n$, the minimal angle between $\mathcal{R}$ and $\mathcal{N}$ is defined to be the number $0 \leq \theta \leq \pi/2$ that satisfies

$$\cos\theta = \max_{\substack{\mathbf{u} \in \mathcal{R}, \mathbf{v} \in \mathcal{N} \\ \|\mathbf{u}\|_2 = \|\mathbf{v}\|_2 = 1}} \mathbf{v}^T \mathbf{u}$$

131. (198)
Jackson, D. (1934), 'The convergence of Fourier series'. *American Mathematical Monthly*, volume **41**, pages 67–84.

[p. 70. Lines 13–12 from bottom.]
It must be recognized however that the series has not yet been proved to converge *to the value $f(x)$.*

132. (342)
Jackson, B. W. and D. Thoro (1990), *Applied Combinatorics with Problem Solving.* Addison-Wesley.

[p. 55. Definition of permutation.]
Let X be a set with n different objects. An arrangement of all the elements of X in a sequence of length n is called a permutation.

133. (146)
Jackson, B. W. and D. Thoro (1990), *Applied Combinatorics with Problem Solving.* Addison-Wesley.

[p. 71. Problem 8.]
A small library contains 15 different books. If five different students simultaneously check out one book each, ...

134. (113, 113)
Jankovic, D. and T. R. Hamlet (1990), 'New topologies from old via ideals'. *American Mathematical Monthly*, volume **97**, pages 295–310.

[p. 300. lines 17–14 from bottom.]
By taking $\mathcal{I} = \mathcal{J}$ in the above theorem, the following corollary answers the question about the relationship between τ^* and τ^{**}.

135. (209)
Jenkyns, T. and E. Muller (2000), 'Triangular triples from ceilings to floors'. *American Mathematical Monthly*, volume **107**, pages 634–639.
[p. 634. Line 1.]
A *triangular triple* is a sequence of non-negative integers (i, j, k) that gives the lengths of the sides of a triangle.

136. (123)
Jones, R. and J. Pearce (2000), 'A postmodern view of fractions and the reciprocals of Fermat primes'. *Mathematics Magazine*, volume **73**, pages 83–97.
[p. 95. Lines 16–17.]
DEFINITION. *A positive integer $n > 1$ is perfectly symmetric if its reciprocal is symmetric in any base b provided $b \not\equiv 0$ (mod n) and $b \not\equiv 1$ (mod n).*

137. (184)
Karlin, S. (1972), 'Some mathematical models of population genetics'. *American Mathematical Monthly*, volume **79**, pages 699–739.
[p. 706. Lines 12–16.]
Under these conditions adding the relations in

(2.6) using obvious inequalities produces

$$(2.8) \qquad x' + y' < 2\frac{xy + \frac{1}{2}(x + y)}{1 + \frac{1}{2}(x + y)}$$

Since $4xy \leq (x + y)^2$ we see that $x' + y' < x + y$. It follows that $x^{(n)} + y^{(n)}$ decreases in n and its limit is necessarily zero indicating that 0 is globally stable.

138. (112)
Kang, M.-c. (1997), 'Minimal polynomials over cyclotomic fields'. *American Mathematical Monthly*, volume **104**, pages 260.
[p. 260. Lines 23–24.]
From the definition of T, we may interpret the elements in T as those invertible elements in $\mathbb{Z}/e'\mathbb{Z}$ that are of the form $1 + kd$ for some k because . . .

139. (462)
Kaufman, R. (1974), 'Sets of multiplicity and differentiable functions. II'. *Transactions of the American Mathematical Society*, volume **200**, pages 427–435.
[p. 429. Lines 8–10.]
Because sets of small Lebesgue measure have small μ_k-measure, a proper choice of $Y = \eta_{k+1}$ enables us to obtain the necessary estimates.

140. (344)

Kendig, K. M. (1983), 'Algebra, geometry, and algebraic geometry: Some interconnections'. *American Mathematical Monthly*, volume **90**, pages 161–174.

[p. 166. Fourth paragraph.]
Parametrize $y = x^2$ by $x = t$, $y = t^2$, and plug into $Ax + By$.

141.

Klambauer, G. (1978), 'Integration by parts and inverse functions'. *American Mathematical Monthly*, volume **85**, pages 668–669.

[p. 668. Line 1.]
Let f be a strictly increasing function with continuous derivative on a compact interval $[a, b]$.

142.

Klebanoff, A. and J. Rickert (1998), 'Studying the cantor dust at the edge of feigenbaum diagrams'. *The College Mathematics Journal*, volume **29**, pages 189–198.

[p. 196. Second line below figure.]
The largest region leaves in one iteration and is bounded by the curves

$$x_{1_i} = \frac{1}{2} + (-1)^i \sqrt{\frac{1}{4} - \frac{1}{a}}, \qquad i = 1, 2$$

which satisfy $f_a(x) = 1$.

143.

Kleiner, I. (1999), 'Field theory: From equations to axiomatization'. *American Mathematical Monthly*, volume **106**, pages 677–684.

[p. 678. Lines 18–20.]
This says (in our terminology) that if E is the splitting field of a polynomial $f(x)$ over a field F, then $E = F(V)$ for some rational function V of the roots of $f(x)$.

144. (124, 425)

Klosinski, L. F., G. L. Alexanderson, and L. C. Larson (1993), 'The fifty-third William Lowell Putnam Mathematical Competition'. *American Mathematical Monthly*, volume **100**, pages 755–767.

[p. 757. Problem A-2.]
Define $C(\alpha)$ to be the coefficient of x^{1992} in the power series expansion about $x = 0$ of $(1 + x)^\alpha$. Evaluate . . .

145. (141)

Klosinski, L. F., G. L. Alexanderson, and L. C. Larson (1993), 'The fifty-third William Lowell Putnam Mathematical Competition'. *American Mathematical Monthly*, volume **100**, pages 755–767.

[p. 758. Problem B-1.]
Let S be a set of n distinct real numbers.

525

146. (134)
Knoebel, R. A. (1981), 'Exponentials reiterated'. *American Mathematical Monthly*, volume **88**, pages 235–252.
  [p. 235. Lines 1–3.]
When is x^y less than y^x?For what kind of numbers does $x^y = y^x$? And is there a formula for y as a function of x?

147. (230, 407, 448)
Kolman, B., R. C. Busby, and S. Ross (1996), *Discrete Mathematical Structures, 3rd Edition.* Prentice-Hall.
  [p. 109. Lines 15–11 from the bottom.]

148. (68)
Kolman, B., R. C. Busby, and S. Ross (1996), *Discrete Mathematical Structures, 3rd Edition.* Prentice-Hall.
  [p. 111. Theorem 2.]

149. (124)
Konvalina, J. (2000), 'A unified interpretation of the binomial coefficients, the Stirling numbers, and the Gaussian coefficients'. *American Mathematical Monthly*, volume **107**, pages 901–910.
  [p. 902. Lines 2–4.]
Define *the binomial coefficient of the first kind* $\binom{n}{k}$ to be the number of k-element subsets of S; that is, the number of ways to choose k

distinct objects from S with the order of selection not important.

150. (109)
Kopperman, R. (1988), 'All topologies come from generalized metrics'. *American Mathematical Monthly*, volume **95**, pages 89–97.
  [p. 93. Lines 9–11.]
If for each $i \in I$, A_i is a value semigroup (together with $+_i$, 0_i, ∞_i), then so is their product (with $+$, 0 defined coordinatewise; 1/2 and inf are also taken coordinatewise.)

151. (58, 94, 147, 169, 238, 265, 294, 316, 365, 443)
Krantz, S. G. (1995), *The Elements of Advanced Mathematics.* CRC Press.
  [p. 40. Proposition 3.1.]

152. (156)
Krantz, S. G. (1995), *The Elements of Advanced Mathematics.* CRC Press.
  [p. 55. Line 18.]

153. (265, 388)
Krantz, S. G. (1995), *The Elements of Advanced Mathematics.* CRC Press.
  [p. 57. Definition 4.16.]

154. (38, 216, 369)
Kupka, J. and K. Prikry (1984), 'The measurability of uncountable unions'.

American Mathematical Monthly, volume **91**, pages 85–97.

[p. 86. Lines 5–6.]

Conversely, if an arbitrary function $f : X \to \mathbb{R}$ satisfies (1.6) and if the (finite Radon) measure μ is complete, then f is $\mathcal{A}$-measurable.

155. (329)

Lam, C. W. H. (1991), 'The search for a finite projective plane of order 10'. *American Mathematical Monthly*, volume **98**, pages 305–318.

[p. 305. Lines 11–12.]

A *finite projective plane of order n*, with $n > 0$, is a collection of $n^2 + n + 1$ lines and $n^2 + n + 1$ points such that . . .

156. (483)

Lawlor, G. (1996), 'A new minimization proof for the brachistochrone'. *American Mathematical Monthly*, volume **103**, pages 242–249.

[p. 243. Proposition 1.2.]

The velocity of a marble rolling without friction down a ramp is proportional to $\sqrt{|y|}$, if the marble starts at rest at a point where $y = 0$.

157. (368)

Leep, D. B. and G. Myerson (1999), 'Marriage, magic and solitaire'. *American Mathematical Monthly*, volume **106**, pages 419–429.

[p. 428. Corollary 13.]

Corollary 13. No vector space V over an infinite field F is a finite union of proper subspaces.

158. (216, 217)

Lefton, P. (1977), 'Galois resolvents of permutation groups'. *American Mathematical Monthly*, volume **84**, pages 642–644.

[p. 643. Lines 1–2.]

DEFINITION. Let $\Phi(z, y)$ be the minimnal polynomial for $F(x)$ over $Q(y)$. We call $\Phi(z, y)$ the Galois resolvent of Π corresponding to $F(x)$.

159. (209, 453)

Lenstra, Jr., H. W. (1992), 'Algorithms in algebraic number theory'. *Bulletin of the American Mathematical Society (N.S.)*, volume **26**, pages 211–244.

[p. 216. Lines 21–23.]

Other algorithms may even give a nontrivial factor of p, . . .

160. (53, 66, 124, 169, 230)

Lewis, H. R. and C. H. Papadimitriou (1998), *Elements of the Theory of Computation, 2nd Edition.* Prentice-Hall.

[p. 20. Lines 11–10 from the bottom.]

161. (273)

Llewellyn, D. C., C. Tovey, and M. Trick (1988), 'Finding saddlepoints of two-person,

zero sum games'. *American Mathematical Monthly*, volume **95**, pages 912–918.
[p. 913. Lines 19–18 from bottom.]
For ease of notation, we will denote $\log_2 x$ by $\lg x$.

162. (62)
Loeb, P. A. (1991), 'A note on Dixon's proof of Cauchy's Integral Theorem'. *American Mathematical Monthly*, volume **98**, pages 242–244.
[p. 243. Lines 4–5.]
The trace of the curve γ in the complex plane is denoted by $\{\gamma\}$.

163. (67)
Lorch, E. R. (1971), 'Continuity and Baire functions'. *American Mathematical Monthly*, volume **78**, pages 748–762.
[p. 753. Item (2) under **III**.]
There are three cases:
(1) *E* has finite cardinality.
(2) *E* has denumerable cardinality.
(3) *E* has the cardinality c of the continuum.

164. (66)
St. Luke (1949), 'The Gospel according to St. Luke'. In *The Holy Bible, Authorized King James Version*, pages 55–89. Collins Clear-Type Press.
[p. 56. Chapter 1, verse 48.]

165. (448)
Mac Lane, S. (1971), *Categories for the Working Mathematician*. Springer-Verlag.
[p. 58. Line 15.]
Thus a universal element $\langle r, e \rangle$ for H is exactly a universal arrow from $*$ to H.

166. (308)
MacEachern, S. N. and L. M. Berliner (1993), 'Aperiodic chaotic orbit'. *American Mathematical Monthly*, volume **100**, pages 237–241.
[p. 237. Lines 12–16.]
For ease of exposition, the presentation here is specialized to a familiar example for $J = [0, 1]$, namely the tent map defined by the function

$$f(x) = 2x \qquad 0 \le x < .5$$
$$2 - 2x \qquad .5 \le x \le 1$$

167. (308)
MacEachern, S. N. and L. M. Berliner (1993), 'Aperiodic chaotic orbit'. *American Mathematical Monthly*, volume **100**, pages 237–241.
[p. 241. Lines 13–11 from bottom.]
For any $y \in S_1$, witn $y \ne 1$, there are exactly two values of z (namely $z = y/2$ or $1 - y/2$) for which $\sigma(\mathbf{t}_z) = \mathbf{t}_y$.

168. (52, 237, 325, 462)
Mac Lane, S. and G. Birkhoff (1993), *Algebra, third edition.* Chelsea.

> [p. 43. Lines 5–12.]

A *group* is a set G together with a binary operation $G \times G \to G$, written $(a, b) \mapsto ab$, such that: ... In other words, a group is a monoid in which every element is invertible.

169. (467)
Mac Lane, S. and G. Birkhoff (1993), *Algebra, third edition.* Chelsea.

> [p. 182. Lines 4–3 from the bottom.]

Therefore these identities characterize the biproduct $A_1 \oplus A_2$ up to isomorphism ...

170. (66, 141)
Marchisotto, E. A. (1992), 'Lines without order'. *American Mathematical Monthly*, volume **99**, pages 738–745.

> [p. 739. Lines 23–25.]

Given two distinct points a and b, the class of all points p such that there exists a motion that leaves a fixed and transforms p into b is called a *sphere* of center a and passing through b.

171. (183)
Marchisotto, E. A. (1992), 'Lines without order'. *American Mathematical Monthly*, volume **99**, pages 738–745.

> [p. 741. Lines 2–4.]

Notice that the rotation in Euclidean space described above not only fixes a and b, but also fixes all points collinear with a and b.

172. (69)
Martelli, M., M. Deng, and T. Seph (1998), 'Defining chaos'. *Mathematics Magazine*, volume **72**, pages 112–122.

> [p. 116. Example 3.1.]

Let $f : [0, 1] \to [0, 1]$ be defined by

$$f(x) = \begin{cases} x + .5 & 0 \le x \le .5 \\ 0 & .5 \le x \le 1. \end{cases}$$

173. (46, 238)
Mauldon, J. G. (1978), 'Num, a variant of Nim with no first-player win'. *American Mathematical Monthly*, volume **85**, pages 575–578.

> [p. 575. Lines 3–5.]

These constraints are such that, if one player could in his turn leave (say) n matchsticks in a particular heap, then the other player could not. In particular, at most one of the players is entitled to clear any particular heap.

174. (458)
Maxson, C. J. (1003), 'Near-rings of invariants. II'. *Proceedings of the American Mathematical Society*, volume **117**, pages 27–35.

> [p. 27. Line 4.]

Under function addition, $(M, +)$ is a group.

175. (228)
Mazur, B. (1993), 'On the passage from local to global in number theory'. *Bulletin of the American Mathematical Society (N.S.)*, volume **29**, pages 14–50.

[p. 29. Lines 8–6 from the bottom.]
From now on we assume n to be good, and we will freely identify elements of $H^1(G_K, E[n])$ with the Δ_n-equivariant homomorphisms $G_M \to E[n]$ to which they give rise.

176. (217, 344, 389)
McColm, G. L. (1989), 'Some restrictions on simple fixed points of the integers'. *Journal of Symbolic Logic*, volume **54**, pages 1324–1345.

[p. 1328. Lines 10–13.]
Simultaneously, we define the *evaluation function*

$$\text{ev} : \text{terms} \times \text{assignments} \to \text{values}$$

so that if θ is a term and **x** is an assignment of elements from ω and perhaps partial functions on ω, then $\text{ev}(\theta, \mathbf{x})$ is the result of plugging **x** into θ.

177. (349)
Mckenzie, R. (1975), 'On spectra, and the negative solution of the decision problem for identities having a finite nontrivial model'. *Journal of Symbolic Logic*, volume **40**, pages 186–196.

[p. 187. First sentence of fifth paragraph.]
We use Polish notation (sometimes with added parentheses for clarity) to denote the *terms* build up from function symbols and variables of a first order language.

178. (32, 99, 169, 217, 425)
Mead, D. G. (1993), 'Generators for the algebra of symmetric polynomials'. *American Mathematical Monthly*, volume **100**, pages 386–388.

[p. 387. Lines 12–16.]
Consider the monomial symmetric function $\langle a_1, a_2, \ldots, a_k \rangle$ in $Q[x_1, x_2, \ldots, x_n]$ and let $t = \Sigma_{i=1}^k a_i$. Then there is a positive rational number c and an element B in $Q[p_1, p_2, \ldots, p_{i-1}]$ such that

$$\langle a_1, a_2, \ldots, a_k \rangle = (-1)^{k_1} c p_i + B.$$

179. (99, 199, 200, 327, 369, 397)
Mollin, R. A. (1997), 'Prime-producing quadratics'. *American Mathematical Monthly*, volume **104**, pages 529–544.

[p. 531. Definition 2.1.]
Consider $F(x) = ax^2 + bx + c$ $(a, b, c \in \mathbb{Z})$, $a \neq 0$, and suppose $|F(x)|$ is prime for all integers $= 0, 1, \ldots, l-1$. If $l \in \mathbb{N}$ is the smallest value such that $|F(l)|$ is composite, $|F(l)| = 1$, or $|F(l)| = |F(x)|$ for some

$x = 0, 1, \ldots, l - 1$, *then* $F(x)$ *is said to have prime-production length* l.

180. (216, 228)
Morgan, J. W. (1992), 'λ-trees and their applications'. *Bulletin of the American Mathematical Society (N.S.)*, volume , pages 87–112.

[p. 91. Lines 10–6 from the bottom.]
We form the topological space $X(\Gamma)$ by beginning with the disjoint union

[Check that the repeate coproduct symbol is correct]

$$\coprod_{v \in V} X_v \coprod \coprod_{e \in E} X_e \times I,$$

and (a) identifying $X_e \times I$ with $X_{\bar{e}} \times I$ via $(x, t) \Leftrightarrow (x, 1 - t)$ and (b) gluing $X_e \times \{0\}$ to $X_{i(e)}$ via the given inclusion.

181. (256)
Morgan, F. (1988), 'Area-minimizing surfaces, faces of Grassmannians, and calibrations'. *American Mathematical Monthly*, volume **95**, pages 813–822.

[p. 814. Lines 2–5 below first figure.]
The coordinates of the line at angle θ to the x-axis are the oriented projections of a unit length from that line onto the axes, namely $(\cos\theta, \sin\theta)$. Hence $G(1, \mathbb{R}^2)$ is just the unit circle in $\mathbb{R}^2$.

182. (198)
Morrison, K. E. (1995), 'Cosine products, Fourier transforms, and random sums'. *American Mathematical Monthly*, volume **102**, pages 716–724.

[p. 716. Line 1.]
The function $\sin x/x$ is endlessly fascinating.

183. (38)
Mornhinweg, D., D. B. Shapiro, and K. G. Valente (1993), 'The principal axis theorem over arbitrary fields'. *American Mathematical Monthly*, volume **100**, pages 749–754.

[p. 749. Title of Article.]
The Principal Axis Theorem Over Arbitrary Fields

184. (185, 462)
Mornhinweg, D., D. B. Shapiro, and K. G. Valente (1993), 'The principal axis theorem over arbitrary fields'. *American Mathematical Monthly*, volume **100**, pages 749–754.

[p. 751. Theorem 2.]
Let F be a formally real pythagorean field. The following are equivalent:
(i) F has the Principal Axis Property,
(ii) Every symmetric matrix over F is diagonalizable over F, and
(iii) Every symmetric matrix over F has an eigenvalue in F.

531

185. (158)
Neidinger, R. D. and R. J. Annen III (1996), 'The road to chaos is filled with polynomial curves'. *American Mathematical Monthly*, volume **103**, pages 640–653.

[p. 642. Lines 15–23.]

Superattracting Root Theorem. *Let* $n \in \mathbf{N}$. *The parameter* r *satisfies* $Q_n(r) = 0$ *and* $Q_j(r) \neq 0$ *for* $0 < j < n$ *if and only if iteration of* $f_r(x)$ *has a superattracting periodic point of period* n.

186. (308)
Nevo, A. (1994), 'Harmonic analysis and pointwise ergodic theorems for noncommuting transformations'. *Journal of the American Mathematical Society*, volume **7**, pages 875–902.

[p. 875. Lines 10–11.]

Let $(X, \mathcal{B}, m)$ be a standard Lebesgue measure space, namely a measure space whose σ-algebra is countably generated and countably separate.

187. (179, 216)
Newns, W. F. (1967), 'Functional dependence'. *American Mathematical Monthly*, volume **74**, pages 911–920.

[p. 911. Lines 13–12 from bottom.]

Let X, $\mathbf{I}$ be sets, $(Y_\iota)_{\iota \in \mathbf{I}}$ a family of sets, and for each $\iota \in \mathbf{I}$ let $f_\iota : X \to Y_\iota$.

188. (310)
Newcomb, S. (1881), 'Note on the frequency of use of the different digits in natural numbers'. *American Journal of Mathematics*, volume **4**, pages 39–40.

[p. 39. Lines 8–10, lines 2–1 from the bottom, and lines 1–2 on page 40.]

The question we have to consider is, what is the probability that if a natural number be taken at random its first significant digit will be n, its second n', etc. . . .

Our problem is thus reduced to the following: We have a series of numbers between 1 and i, . . .

189. (470)
Newns, W. F. (1967), 'Functional dependence'. *American Mathematical Monthly*, volume **74**, pages 911–920.

[p. 912. Lines 3–4.]

The support of F is the smallest closed set outside which F vanishes identically.

190. (82, 113, 113, 199, 388, 397)
Niven, I. (1956), *Irrational Numbers.* Mathematical Association of America.

[p. 41. Corollary 3.12.]

If θ *is rational in degrees, say* $\theta = 2\pi r$ *for some rational number* r, *then the only rational values of the trigonometric functions of* θ *are as follows:* $\sin \theta, \cos \theta = 0, \pm\frac{1}{2}, \pm 1$; $\sec \theta, \csc \theta = \pm 1, \pm 2$; $\tan \theta, \cot \theta = 0, \pm 1$.

532

191. (78, 462)
Niven, I. (1956), *Irrational Numbers.*
Mathematical Association of America.
　　　　[p. 83. Lemma 7.1.]
Any $r + 1$ linear forms in r indeterminates with
rational coefficients are linearly dependent over
the rationals.

192. (68, 82, 199, 217)
Osofsky, B. L. (1994), 'Noether Lasker primary
decomposition revisited'. *American
Mathematical Monthly*, volume **101**, pages
759–768.
　　　　[p. 760. Lines 14–16.]
With this convention on sides, the defining
property of a module homomorphism
$\phi : M \to N$ is that
$(r \cdot x + x \cdot y)\phi = r \cdot (x)\phi_s \cdot (y)\phi$ for all $x, y \in M$
and $r, s \in R$.

193. (40)
Osserman, R. (1979), 'Bonnesen-style
isoperimetric inequalities'. *American
Mathematical Monthly*, volume **86**, pages 1–29.
　　　　[p. 5. Line 10.]
Before proceeding further with the argument,
let us prove (27).

194. (216)
Osserman, R. (1979), 'Bonnesen-style
isoperimetric inequalities'. *American
Mathematical Monthly*, volume **86**, pages 1–29.

　　　　[p. 18. Lines 9–7 from bottom.]
The isoperimetric inequality (36) gives

$$L^2 \geq 4\pi A + \alpha^2 A^2 > \alpha^2 A^2$$

for simply connected domains with $K \leq -\alpha^2$,
so that (77) holds in that case.

195. (217, 217, 462)
Ostrowski, A. M. (1971), 'Some properties of
reduced polynomial equations'. *SIAM Journal
on Numerical Analysis*, volume **8**, pages
623–638.
　　　　[p. 624. Lines 14–16.]
Taking $x = 2$ in (6), we obtain
$\psi_n(2) = 1 + 2^{n-1}$, so that

$$(7) \qquad \rho_n < 2 \quad (n \geq 2)$$

and we see that all roots of a reduced equation
lie in the disk $|z| < 2$.

196. (137, 179, 199)
Oxtoby, J. C. (1977), 'Diameters of arcs and the
gerrymandering problem'. *American
Mathematical Monthly*, volume **84**, pages
155–162.
　　　　[p. 155. Lines 24–25.]
For what values of c (if any) is it true that for
every finite family of disjoint finite sets F_j with
diam $F_j > 0$ there exist disjoint polygonal arcs
A_i such that $F_j \subseteq A_i$ and diam $A_i \leq c$ diam F_j
for all j?

197. (232, 393)
Pincus, J. D. (1964), 'On the spectral theory of singular integral operators'. *Transactions of the American Mathematical Society*, volume **113**, pages 101–128.

[p. 108. Lines 1–4.]
Therefore, we may deduce that

$$F(\xi, z) = \frac{T(\xi, z) - S(\xi, z)}{\sqrt{(S(\xi, z)T(\xi, z))}} \; .$$
$$\exp\left\{ \frac{1}{2\pi i} \int_a^b \log \left| \frac{A(\mu) - \xi - \epsilon k(\mu)}{A(\mu) - \xi + \epsilon k(\mu)} \right| \frac{d\mu}{\mu - z} \right\}$$

and it is now clear that the roots of $F(\xi, z)$ are the roots of the function

$$W(\xi, z) = \frac{T(\xi, z) - S(\xi, z)}{\sqrt{(S(\xi, z)T(\xi, z))}}.$$

198. (367)
Pólya, G. (1965), *Mathematical Discovery, Volume II.* John Wiley and Sons, Inc.

[p. 7. Lines 19–23.]
. . . from the last equation of sect. 7.4 we obtain

$$x = \frac{ah}{b - a}$$

Then we substitute this value for x in the two foregoing equations of sect 7.4, obtaining. . .

199.
Pomerance, C. (1996), 'A tale of two sieves'. *Notices of the American Mathematical Society*, volume **43**, pages 1473–1485.

[p. 1482. Last sentence of second column.]
This discrepancy was due to fewer computers being used on the project and some "down time" while code for the final stages of the algorithm was being written.

200. (313)
Pomerance, C. (1996), 'A tale of two sieves'. *Notices of the American Mathematical Society*, volume **43**, pages 1473–1485.

[p. 1478. Lines 17–15 from the bottom of first column.]
If n is not a square modulo p, then $Q(x)$ is never divisible by p and no further computations with p need be done.

201. (106, 227)
Poor, H. V. (2000), 'Modulation and detection'. In [Dorf, 2000] , page 4 of Chapter 126. PDF files available at `http://www.engnetbase.com`.

[p. 4. Line 5.]
. . . where j denotes the imaginary unit.

202. (26, 132)
Powers, R. T. (1974), 'Selfadjoint algebras of unbounded operator. II'. *Transactions of the American Mathematical Society*, volume **187**, pages 261–293.

[p. 264. Line 5.]
All algebras in this section will have a unit denoted by 1.

203. (80, 198, 199, 339, 355, 425, 462)
Powers, V. (1996), 'Hilbert's 17th problem and the champagne problem'. *American Mathematical Monthly*, volume **103**, pages 879–887.

[p. 879. Lines 1–4 and 20–21.]
About 15 years ago, E. Becker gave a talk in which he proved that

$$B(t) := \frac{1+t^2}{2+t^2} \in \mathbf{Q}(t))$$

is a sum of $2n$-th powers of elements in $\mathbf{Q}(t)$ for all n.
... A rational function
$f \in \mathbf{R}(X) := \mathbf{R}(x_1, \ldots, x_k)$ is *positive semi-definite* (psd) if $f \geq 0$ at every point in $\mathbf{R}^k$ for which it is defined.

204. (256, 257)
Putnam, H. (1973), 'Recursive functions and hierarchies'. *American Mathematical Monthly*, volume **80**, pages 68–86.

[p. 82. Lines 1–6.]
If we adjoin to the above condition the further clause:

$$(Ex)(y)(z)\,(J(y,z) \in W_x \equiv J(y,z) \notin W_i)$$

then the definition becomes a definition of the class of recursive well-orderings (or, rather, of the corresponding set of indices), for this clause just says that the predicate W_i^2 has an r.e.

complement $\overline{W}_x^2$, and a predicate is recursive just in case it and its complement are both r.e.

205. (62, 132, 379, 463)
Rabinowitz, S. and P. Gilbert (1993), 'A nonlinear recurrence yielding binary digits'. *Mathematics Magazine*, volume **64**, pages 168–171.

[p. 168. Lines 10 and 3–1 from bottom.]
Let $\{x\}$ denote the fractional part of x, that is, $\{x\} = x - \lfloor x \rfloor$.
... 2. If k is an integer, a is a real number in the range $1 < a < 2$, and $x = k/(a-1)$, then

$$\left\lfloor a\lfloor x\rfloor + \frac{a}{2} \right\rfloor = \lfloor ax \rfloor$$

206. (129)
Ranum, D. L. (1995), 'On some applications of Fibonacci numbers'. *American Mathematical Monthly*, volume **102**, pages 640–645.

[p. 641. Lines 6–7 under Figure 2.]
At the other extreme, Figure 3 shows a worst case **degenerate** tree where each node has only 1 child except for the single leaf. [The trees here are binary trees.]

207. (217)
Reed, G. M. (1986), 'The intersection topology w.r.t. the real line and the countable ordinals'. *ransactions of the American Mathematical Society*, volume **297**, pages 509–520.

[p. 509. Lines 1–2.]

If Υ_1 and Υ_2 are topologies defined on the set X, then Υ is the *intersection topology* w.r.t. Υ_1 and Υ_2 defined on X, where $\{U_1 \cap U_2 \mid U_1 \in \Upsilon_1 \text{ and } U_2 \in \Upsilon_2\}$ is a basis for Υ.

208. (53, 453)

Ribet, K. A. (1995), 'Galois representations and modular forms'. *Bulletin of the American Mathematical Society (N.S.)*, volume **32**, pages 375–402.

[p. 391. Lines 12–13.]

Suppose that there is a non-trivial solution to Fermat's equation $X^\ell + Y^\ell = Z^\ell$.

209. (117)

Richmond, B. and T. Richmond (1993), 'The equal area zones property'. *American Mathematical Monthly*, volume **100**, pages 475–477.

[p. 475. Lines 7–8 below the figure.]

To state the problem precisely, suppose that $y = g(x)$ is a piecewise smooth nonnegative curve defined over $[a, b]$, and is revolved around the x-axis.

210. (150)

Rosenthal, P. (1987), 'The remarkable theorem of Levy and Steinitz'. *American Mathematical Monthly*, volume **94**, pages 342–351.

[p. 342. Lines 9–11.]

The theorem is the following: *the set of all sums of rearrangements of a given series of complex numbers is the empty set, a single point, a line in the complex plane, or the whole complex plane.*

211. (94, 370)

Rosen, K. (1991), *Discrete Mathematics and its Applications, Second Edition.* McGraw-Hill.

[p. 6. Definition 5.]

212. (94, 158, 241)

Rosen, K. (1993), *Elementary Number Theory and its Applications.* Addison-Wesley.

[p. 208. Theorems 6.2 and 6.3.]

213. (158, 241, 483)

Rosen, K. (1993), *Elementary Number Theory and its Applications.* Addison-Wesley.

[p. 223. Theorem 6.10..]

214. (304)

Rosen, K. (1993), *Elementary Number Theory and its Applications.* Addison-Wesley.

[p. 224. Theorem 6.11.]

215. (58, 169, 388)

Rosen, K. (1993), *Elementary Number Theory and its Applications.* Addison-Wesley.

[p. 293. Lines 6–8.]

216. (78, 329)

Rosen, M. (1995), 'Niels Hendrik Abel and equations of the fifth degree'. *American Mathematical Monthly*, volume **102**, pages 495–505.

[p. 504. Proposition 3.]

217. (280)

Ross, K. A. and C. R. B. Wright (1992), *Discrete Mathematics, 3rd Edition*. Prentice-Hall.

[p. 19. Lines 3–2 from bottom.]

We sometimes refer to a function as a **map** or **mapping** and say that f **maps** S into T.

218. (179)

Rota, G.-C. (1997), 'The many lives of lattice theory'. *Notices of the American Mathematical Society*, volume **44**, pages 1440–1445.

[p. 1440. .]

The family of all partitions of a set (also called equivalence relations) is a lattice when partitions are ordered by refinement.

219. (129)

Roth, B. (1981), 'Rigid and flexible frameworks'. *American Mathematical Monthly*, volume **88**, pages 6–21.

[p. 12. First two lines of Example 4.2.]

Consider the degenerate triangle $G(p)$ in $\mathbb{R}^2$ shown in Fig. 4 with collinear vertices ...

220. (91)

Rubel, L. A. (1989), 'The Editor's corner: Summability theory: A neglected tool of analysis'. *American Mathematical Monthly*, volume **96**, pages 421–423.

[p. 421. Lines 9–8 from bottom.]

We are now in a position to give a conceptual proof of Pringsheim's theorem ...

221. (147, 329)

Senechal, M. (1990), 'Finding the finite groups of symmetries of the sphere'. *American Mathematical Monthly*, volume **97**, pages 329–335.

[p. 330. Lines 5–6 of Section 3.]

Let H be a finite subgroup of $S0(3)$ of order n. To each element of H (other than the identity) there corresponds an axis that intersects the sphere in two points.

222. (256)

Shpilrain, V. (1995), 'On the rank of an element of a free Lie algebra'. *Proceedings of the American Mathematical Society*, volume **123**, pages 1303–1307.

[p. 1303. Lines 6–5 from bottom.]

If we have an element u of the free Lie algebra L and write $u = u(x_1, \ldots, x_n)$, this just means that no generators x_i with $i > n$ are involved in u.

537

223. (349, 390)

Singmaster, D. (1978), 'An elementary evaluation of the Catalan numbers'. *American Mathematical Monthly*, volume **85**, pages 366–368.

    [p. 366. Line 15–14 from the bottom.]
For example, when $n = 2$, the products above have the Polish form $XXabc$ and $XaXbc$ and the reverse Polish forms $abXcX$ and $abcXX$.

224. (176, 216)

Snyder, W. M. (1982), 'Factoring repunits'. *American Mathematical Monthly*, volume **89**, pages 462–466.

    [p. 463. Lines 11–14.]
We factor $\Phi_n(b)$ in the ring of algebraic integers of $\mathbb{Q}_n = \mathbb{Q}(\zeta)$. Then

$$\Phi_n(b) = \prod_{\substack{a=1 \\ (a,n)=1}}^{n} (b - \zeta^a) \qquad (1)$$

We now claim that if A is the ideal in R generated by two distinct factors $\zeta - a^{a_1}$ and $\zeta - a^{a_2}$ given in (1), then …

225. (176, 217)

Sogge, C. D. (1989), 'Oscillatory integrals and unique continuation for second order elliptic differential equations'. *Journal of the American Mathematical Society*, volume **2**, pages 491–515.
    [p. 494. Lines 2–6.]

… we let $B(x, \xi) = \sqrt[m]{P_m(x, \xi)}$ and notice that we can factor the symbol of the operator in (1.5) as follows

$$P_m(x,\xi) - \tau^m =$$
$$(B(x,\xi) - \tau) \cdot \left(B^{m-1} + B^{m-2}\tau + \cdots + \tau^{m-1}\right)$$

The second factor is uniformly elliptic in the sense that it is bounded below by a multiple of $\left(|\xi|^{m-1} + \tau^{m-1}\right)$, while the first factor vanishes for certain $|\xi| \approx \tau$.

226. (34, 230)

Solow, D. (1995), *The Keys to Advanced Mathematics: Recurrent Themes in Abstract Reasoning*. BookMasters Distribution Center.
    [p. 144. Definition 3.3.]
A set is a **strict subset** of a set B, written $A \subset B$, if and only if $A \subseteq B$ and $A \neq B$.

227. (217, 217)

Srinivasan, B. (1981), 'Characters of finite groups: Some uses and mathematical applications'. *American Mathematical Monthly*, volume **88**, pages 639–646.
    [p. 640. Line 7.]
The function $\chi : g \to \text{Trace}(\rho(g))$ of G into $\mathbb{C}$ is called the *character* of ρ.

228. (62, 130)

Starke, E. P., editor (1970a), 'Problems and solutions'. *American Mathematical Monthly*,

volume **77**, pages 765–783.

    [p. 774. Problem 5746.]

$$S(a) = \Sigma_{x,y,z} e \left\{ x + y + z + \frac{a}{yz + zx + xy} \right\}$$

229. (130, 414)

Starke, E. P., editor (1970b), 'Problems and solutions'. *American Mathematical Monthly*, volume **77**, pages 882–897.

    [p. 884. Problem E 2198.]

If $r > 1$ is an integer and x is real, define

$$f(x) = \Sigma_{k=0}^{\infty} \Sigma_{j=1}^{r-1} \left[\frac{x + jr^k}{r^{k+1}} \right],$$

where the brackets denote the greatest integer function.

230. (134, 144)

Stolarsky, K. B. (1995), 'Searching for common generalizations: The case of hyperbolic functions'. *American Mathematical Monthly*, volume **102**, pages 609–619.

    [p. 614. Lines 17–11 from bottom.]

Theorem. *If $y_1 = y_1(x)$ and $y_2 = y_2(x)$ are functions satisfying $0 \leq y_1(0) < y_2(0)$ and the differential equations*

$$\frac{dy_1}{dx} = Cy_2^\alpha, \qquad \frac{dy_2}{dx} = Cy_1^\alpha \qquad (4.4)$$

where $C > 0$, then for some constant c_0

$$y_2(x) - y_1(x) \to \begin{cases} 0 & \alpha > 0 \\ c_0 & \alpha = 0 \\ \infty & \alpha < 0 \end{cases} \qquad (4.5)$$

as x increases without limit in the (possibly infinite) domain of definition of $y_1(x)$ and $y_2(x)$.

231. (325)

Stolarsky, K. B. (1995), 'Searching for common generalizations: The case of hyperbolic functions'. *American Mathematical Monthly*, volume **102**, pages 609–619.

    [p. 619. Lines 9–13.]

. . . then any inequality

$$f(x_1, \ldots, x_n) \geq 0,$$

where the function F is formed by any finite number of rational operations and real exponentiations, is decidably true or false!

232. (94)

Strang, G. (1989), 'Patterns in linear algebra'. *American Mathematical Monthly*, volume **96**, pages 105–117.

    [p. 107. Lines 15–16.]

Certainly the expression $e^T Ce = c_1 e_1^2 + \cdots + c_4 e_4^2$ is not negative. It is zero only if $e = Ax = 0$.

233. (34, 66, 132, 372)
Straight, H. J. (1993), *Combinatorics: An Invitation.* Brooks/Cole.
    [p. 3. Definition 0.1.1.]

234. (38)
Straight, H. J. (1993), *Combinatorics: An Invitation.* Brooks/Cole.
    [p. 7. Line 9 from the bottom.]

235. (209, 459)
Straight, H. J. (1993), *Combinatorics: An Invitation.* Brooks/Cole.
    [p. 17. Definition 0.2.1.]

236. (342)
Straight, H. J. (1993), *Combinatorics: An Invitation.* Brooks/Cole.
    [p. 27. Definition 0.2.4.]

237. (81)
Straight, H. J. (1993), *Combinatorics: An Invitation.* Brooks/Cole.
    [p. 119. Line 7 from bottom.]

238. (176)
Suryanarayana, D. (1977), 'On a class of sequences of integers'. *American Mathematical Monthly*, volume **84**, pages 728–730.
    [p. 728. Lines 14–15.]
Let $\{a_n\}$ be an increasing sequence of positive integers such that $\log a_n / q_n \log q_n \to 0$ as $n \to \infty$, where q_n is the least prime factor of n.

239. (216, 216, 217)
Steve Surace, J. (1990), 'The Schrödinger equation with a quasi-periodic potential'. *Transactions of the American Mathematical Society*, volume **320**, pages 321–370.
    [p. 321. Abstract.]
We consider the Schrödinger equation

$$-\frac{d^2}{dx^2}\psi + \epsilon\left(\cos x + \cos\left(\alpha x + \vartheta\right)\right)\psi = E\psi$$

where ...

240. (217, 217, 217)
Talagrand, M. (1986), 'Derivations, L^Ψ-bounded martingales and covering conditions'. *Transactions of the American Mathematical Society*, volume **293**, pages 257–291.
    [p. 257. Abstract.]
Let (Ω, Σ, P) be a complete probability space. Let $(\Sigma_j)_{j \in J}$ be a directed family of sub-σ-algebras of Σ. Let (Φ, Ψ) be a pair of conjugate Young functions.

241. (110)
Talagrand, M. (1990), 'The three-space problem for L^1'. *Journal of the American Mathematical Society*, volume **3**, pages 9–29.
    [p. 9. Lines 9–8 from bottom.]
For simplicity, let us say that a Banach space *contains a copy of* L^1 if it contains a subspace

isomorphic to L^1.

242. (18)
Teitelbaum, J. T. (1991), 'The Poisson kernel for Drinfeld modular curves'. *Journal of the American Mathematical Society*, volume **4**, pages 491–511.

[p. 494. Lines 1–4.]
. . . may find a homeomorphism $x : E \to \mathbb{P}^1_k$ such that
$$x(\gamma u) = \frac{ax(u) + b}{cx(u) + d}.$$

We will tend to abuse notation and identify E with $\mathbb{P}^1_k$ by means of the function x.

243. (344)
Temple, B. and C. A. Tracy (1992), 'From Newton to Einstein'. *American Mathematical Monthly*, volume **99**, pages 507–521.

[p. 518. Line before (4.11).]
Plugging in we obtain:

244. (130, 414)
Tews, M. C. (1970), 'A continuous almost periodic function has every chord'. *American Mathematical Monthly*, volume **77**, pages 729–731.

[p. 730. Third line from the bottom.]

$$\left| \sin\left[\frac{2\pi}{a} \left(\frac{a}{4} + t \right) \right] - \sin\left(\frac{2\pi}{a} \cdot \frac{a}{4} \right) \right| < \sin \frac{2\pi}{a} p$$

245. (144)
Thielman, H. P. (1953), 'On the definition of functions'. *American Mathematical Monthly*, volume **60**, pages 259–262.

[p. 260. Lines 16–14 from bottom.]
A function f whose domain of definition is X, and whose range is Y is frequently denoted by $f : X \to Y$, and is referred to as a function *on* X *onto* Y.

246. (425)
Tits, J. (1964), 'Algebraic and abstract simple groups'. *The Annals of Mathematics, 2nd Ser.*, volume **80**, pages 313–329.

[p. 321. Lines 25–29.]
. . . (the symbols $A_2, A_3, \cdots, G_2$ have their usual meaning, and the left superscript denotes the degree of $\tilde{k}/k$ when $\tilde{k} \neq k$, i.e, when G does not split over k)
if $k = F_2$, groups of type A_2, $^2A_3$, $^2A_4$, B_3 and $^3D_4$;
if $k = F_3$, groups of type A_2, $^2A_2$, $^2A_3$, B_2 and $^3D_4$ and G_2.

247. (261)
van Lint, J. H. and R. M. Wilson (1992), *A Course in Combinatorics*. Cambridge University Press.

[p. 35. Lines 8–4 from bottom.]
. . . We shall show that a larger matching exists. (We mean larger in *cardinality*; we may

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not be able to find a complete matching containing these particular m edges.)

248. (30, 94, 462)
Van Douwen, E. K., D. J. Lutzer, and T. C. Przymusiński (1977), 'Some extensions of the Tietze-Urysohn Theorem'. *American Mathematical Monthly*, volume **84**, pages 435.
　　　[p. 435. Theorem A.]
If A is a closed subspace of the normal space X then there is a function $\eta : C^*(A) \to C^*(A)$ such that for every $f \in C^*(A)$, $\eta(F)$ extends F and has the same bounds as F.

249.
Vaught, R. L. (1973), 'Some aspects of the theory of models'. *American Mathematical Monthly*, volume **80**, pages 3–37.
　　　[p. 3. Lines 6–10.]
For example, each of the properties of being a group, an Abelian group, or a torsion-free Abelian group os expressible in the so-called elementary language (or first-order predicate calculus). Thus, instead of saying that the group $mathcalG$ is Abelian, we can sat it is a model of the elementary sentence $\forall x \forall y (x \circ y = y \circ x)$. Such properties are also called elementary.

250. (63)
Verner, J. H. (1991), 'Some Runge-Kutta formula pairs'. *SIAM Journal on Numerical*

Analysis, volume **28**, pages 496–511.
　　　[p. 501. Lines 1–2 under formula $(21'')$.]
This may be written as

$$\sum_{j=4}^{7} \left(\sum_i b_i a_{ij} \right) \cdot \left(\sum_k a_{jk} c_k^q - \frac{c_j^{q+1}}{q+1} \right) = 0$$

by invoking (15) to imply that the first bracket is zero for $j = 2, 3$. Since the second bracket is zero for $4 \le j \le 6$ by $(17'')$, and . . .

251. (63)
Wallach, N. R. (1993), 'Invariant differential operators on a reductive Lie algebra and Weyl group representations'. *Journal of the American Mathematical Society*, volume **6**, pages 779–816.
　　　[p. 786. Lines 8–7 from bottom.]
If $f, g \in \mathcal{P}(V_0 \times V_0^*)$ then let $\{f, g\}$ (the Poisson bracket of f and g) be as in Appendix 1.

252. (214)
Wilf, H. S. (1989), 'The editor's corner: The white screen problem'. *American Mathematical Monthly*, volume **96**, pages 704–707.
　　　[p. 704. Lines 11–8 from bottom.]
To translate the question into more precise mathematical language, we consider a grid of MN lattice points

$$G = \{(i, j) \mid 0 \le i \le M - 1; 0 \le j \le N - 1\}$$

and we regard them as being the vertices of a graph.

253. (94, 217)
Witte, D. (1990), 'Topological equivalence of foliations of homogeneous spaces'. *Transactions of the American Mathematical Society*, volume **317**, pages 143–166.
　　　[p. 144. Lines 16–13 from bottom.]
By composing $\tilde{f}$ with the inverse of σ, we may assume the restriction of $\tilde{f}$ to $\mathbf{Z}^n$ is the identity. Because
(**)$\mathbf{R}/\mathbf{Z}$ is compact,
this implies that $\tilde{f}$ moves points by a bounded amount . . .

254. (354, 358)
Yetter, D. N. (1990), 'Quantales and (noncommutative) linear logic'. *Journal of Symbolic Logic*, volume **55**, pages 41–64.
　　　[p. 44. Lines 14–13 from bottom.]
In our formalism we adopt prefix notation in preference to the infix/postfix notation used by Girard . . .

255. (338)
Yu, H. B. (1998), 'On the Diophantine equation $(x+1)^y - x^z = 1$'. *American Mathematical Monthly*, volume **105**, pages 656–657.
　　　[p. 656. Last line.]

$$\big((x+1)^{y_1} - 1\big)\big((x+1)^{y_1}) + 1\big) = x^z$$

256. (319)
Zabell, S. L. (1995), 'Alan Turing and the Central Limit Theorem'. *American Mathematical Monthly*, volume **102**, pages 483–494.
　　　[p. 486. Lines 4–6.]
Feller (1937) showed that if normal convergence occurs (that is, condition (2.2) holds), but condition (2.4) also obtains, then

$$\frac{1}{\rho}\frac{X_{m_k}}{s_{n_k}} \Rightarrow (0,1)\,.$$

257. (39)
Zalcman, L. (1975), 'A heuristic principle in complex function theory'. *American Mathematical Monthly*, volume **82**, pages 813–818.
　　　[p. 813. Lines 4–3 from the bottom.]
Here $f^\sharp$ is the *spherical derivative* of the function; the present notation . . . is better adapted for displaying the argument of the function explicitly.

258. (238)
Zalcman, L. (1980), 'Offbeat integral geometry'. *American Mathematical Monthly*, volume **87**, pages 161–175.
　　　[p. 162. Lines 6–10.]

Accordingly, let

$$\hat{f}(\xi,\eta) = \int\int f(x,y)e^{i(\xi x + \eta y)}dx\,dy$$

be the Fourier transform of f. We shall need only two facts about $\hat{f}$: it is continuous, and the correspondence between f and $\hat{f}$ is one-to-one. In particular, if $\hat{f} = 0$ then $f = 0$ (almost everywhere).

259. (217)
Zander, V. (1972), 'Fubini theorems for Orlicz spaces of Lebesgue-Bochner measurable functions'. *Proceedings of the American Mathematical Society*, volume **32**, pages 102–110.

[p. 102. Lines 1–2 of Abstract.]
Let (X, Y, v) be the volume space formed as the product of the volume spaces (X_i, Y_i, v_i) $(i = 1, 2)$.

260. (339, 425)
Zulli, L. (1996), 'Charting the 3-sphere—an exposition for undergraduates'. *American Mathematical Monthly*, volume **103**, pages 221–229.

[p. 227. Beginning of Section 5.]
Let us return for a moment to the circle $S^1 \subseteq \mathbf{C} = \mathbf{R}^2$.

Index

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pt

Left To Do

Listed here are the entries in the Handbook where more information is needed, along with the page number where each one begins.

Citations needed

evaluate, p. 162: Use of "evaluate" meaning give the value of a function.

existential quantifier, p. 169: Use of "Some X are Y" when there is in fact only one X.

function, p. 199: Use of f_x for value of function.

functional, p. 203: Use of the word "functional".

in general, p. 236: Uses of "generally" and "in general" with unambiguous meaning.

let, p. 266: Text fitting the pattern "Let an integer be **even** if it is divisible by 2" used by a native English speaker.

local, p. 271: Informal and formal usage of "local" and "global".

local identifier, p. 272: Explicit description of scope, as in "Throughout this chapter f will be a continuous function."

mathematical logic, p. 283: Use of "mathematical logic", "formal logic", "symbolic logic".

mental representation, p. 296: Use of "intuitively" and "you can think of ... ".

negation, p. 312: Uses of "Every X is not Y."

never, p. 313: "Never" referring to behavior of a function.

now, p. 316: "Now" used to to point out a fact that is already known or easily deduced and that will be used in the next step of the proof.

orthogonal, p. 332: Use of "orthogonal" to refer to a language or system of notation in which constructions are freely substitutable.

positive, p. 352: "Positive" defined to mean greater than or equal to 0.

postcondition, p. 353: Postcondition not using "where".

proof by instruction, p. 367: Example of proof instructing you to do something to a geometric figure.

Information needed

This lists all occurrences of the phrases **information needed** and **references needed**.

phrases such as "including all those with synthesizers".

definite description, p. 121: Discussion at expository level of definite and indefinite in linguistics.

empty set, p. 150: Origin of symbol $\emptyset$ for empty set.

indefinite description, p. 243: Elementary exposition of indefinite descriptions in linguistics.

mathematical object, p. 288: Have logicians ever explicated the notion of "variable object" as the set of all possible interpretations of the axioms for that kind of object?

order of quantifiers, p. 330: Linguistics literature about phrases such as "There is a perfect gift for every child" and its relationship if any to distributive plurals.

osmosis theory, p. 333: Letter to editor in Notices saying we should not have to teach students to understand the way mathematics is written, or the finer points of logic (for example how quantifiers are negated). They should be able to figure these things on their own.

pattern recognition, p. 341: References in math ed literature to pattern recognition, e.g. in substituting and in rules such as the chain rule.

Platonism, p. 343: Studies by linguists of the way mathematicians write about mathematical objects, in particular their use with articles (in contrast to concepts such as gravity).

pronunciation, p. 363: Math ed literature on need some student have of being able to pronounce a mathematical expression if they are going to read it.

root, p. 393: Statement that "root of a function" is incorrect.

syntax, p. 439: Math ed literature on students' use of syntax to decode mathematical expressions.

unnecessarily weak assertion, p. 465: Math ed literature about walking blindfolded.

yes it's weird, p. 487: Studies in math ed literature about value of explicitly recognizing students' sources of discomfort.

TeX Problems

1. (Hypertext version) If a link is broken across a line the second part does not work as a link.

2. (Hypertext version) The List of Words and Phrases is alphabetized incorrectly. In particular, if w is the first word in x, then w comes after x. For example, phrases beginning with "formal" come before the word "formal". I can probably fix this on the final run by

using Mathematica.

3. (Both versions) The string "pt" appears at the end of the Index; I have not been able to find out why.

4. (Both versions) Last page of index has no header.

5. (Paper version) In the bibliography, backreferences are to the page number. I have experimented with having the backreference refer to the entry it appears in, but the superscript always appears as "???" and generates an error message. I have not been able to make this work correctly.

6. (Paper version) Backreferences break after the parenthesis sometimes.

7. (Both versions) The spacing is funny before and after bibliographic citations. Also, sometimes they cause a line break after a parenthesis.

Things To Do

1. List of commonly mispronounced names such as "Riemann"? Make a table of symbols and names, together with how they are pronounced?

2. Talk about tuples, lists, sequences, multisets.

3. Entry for generic. This is a word with mathematical definitions that is also used informally. Generic in its nontechnical meaning may be same as prototype.

4. Talk about variable as role, as in [Lakoff and Núñez, 1997] . (Fundamental metonymy of algebra). Maybe "variable" should be a separate article.

5. Change "semantic contamination" to "metaphorical contamination" and include (some or all of) the "Difficulties" discussion now under metaphor.

6. Talk about the number zero (zero meaning root is already included). Use Chapter 3 of [Lakoff and Núñez, 1997] .

7. "Understand". Reference Sfard.

8. Attitudes: On some subjects, everyone thinks they are experts: "anyone can write", "anyone can program", etc. On others, few do (mathematics).

9. Mental representation may be incoherent (Tall and Vinner). Example of radial category in the sense of [Lakoff, 1986] ?

10. Put a separate entry for every concept that is defined elsewhere than under its own name. Thus converse error is defined under affirming the consequent, so there should be an entry **converse error** that says "See affirming the consequent". Some of these have been done.

11. Find more citations from distinct sources so that no one publication has more than one quote from it. This should make it possible to publish the citations with the book (perhaps in the CD-Rom version but not the printed version) without having to get copyright permissions. Right now only a few sources, mostly textbooks, have more than two citations.

12. Entry for composition of functions, tied to entry for function. Talk about order of composition, connection with writing functions on right or left, use of semicolon in c.s.

13. Separate entry for value, which is now under "function", including some things said under function. Talk about x goes to y under f, x becomes y, you get y etc. Distinguish value of a function from value of a variable. Some functions have one name for the function, another for the value:

function	result, value
addition	sum
subtraction	difference
multiplication	product
squaring	square
composition	composite

Other functions such as sin, log, etc. don't make this distinction. Many writers do not make the distinction with "composition". ***Citations needed.***

14. Definitions of these words. But in many cases I may decide not to include them.

a) a little thought

b) characterization

c) diagonal method

d) hand-waving

e) hence

f) induction

g) inhabits

h) inherit

i) integration (antidifferentiation), maybe under pattern recognition

j) interpolation
k) intuitionism
l) intuitively, intuition
m) inverse
n) lives in,
o) nothing but
p) ordinal

q) pairwise
r) paradox
s) size,
t) space (various meanings, related by common metaphor)
u) telegraphic style as in [Bagchi and Wells, 1998b] .