Hermitian-Yang-Mills approach to the conjecture of Griffiths on the positivity of ample vector bundles

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Abstract. Given a vector bundle of arbitrary rank with ample determinant line bundle on a projective manifold, we propose a new elliptic system of differential equations of Hermitian-Yang-Mills type for the curvature tensor. The system is designed so that solutions provide Hermitian metrics with positive curvature in the sense of Griffiths – and even in the dual Nakano sense. As a consequence, if an existence result could be obtained for every ample vector bundle, the Griffiths conjecture on the equivalence between ampleness and positivity of vector bundles would be settled.

Keywords. Ample vector bundle, Hermitian metric, Griffths positivity, Nakano positivity, Hermitian-Yang-Mills equation, Monge-Ampère equation, elliptic operator.

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Dedicated to Professor Vyacheslav Shokurov on the occasion of his 70^{th} birthday

1. Introduction

Let X be a projective n-dimensional manifold. A conjecture due to Griffiths [Gri69] stipulates that a holomorphic vector bundle $E \to X$ is ample in the sense of Hartshorne, meaning that the associated line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample, if and only if E possesses a Hermitian metric h such that the Chern curvature tensor $\Theta_{E,h} = i\nabla^2_{E,h}$ is Griffiths positive. In other words, if we let $\operatorname{rank} E = r$ and

$$\Theta_{E,h} = i \sum_{1 \le j,k \le n,\, 1 \le \lambda,\mu \le r} c_{jk\lambda\mu} dz_j \wedge d\overline{z}_k \otimes e^*_\lambda \otimes e_\mu \tag{1.1}$$

in terms of holomorphic coordinates $(z_1, \ldots, z_n)$ on X and of an orthonormal frame $(e_\lambda)_{1 \le \lambda \le r}$ of E, the associated quadratic form

$$\widetilde{\Theta}_{E,h}(\xi \otimes v) := \langle \Theta_{E,h}(\xi, \overline{\xi}) \cdot v, v \rangle_h = \sum_{1 \le j,k \le n,\, 1 \le \lambda,\mu \le r} c_{jk\lambda\mu} \xi_j \overline{\xi}_k v_\lambda \overline{v}_\mu \tag{1.2}$$

should take positive values on non zero tensors $\xi \otimes v \in T_X \otimes E$. A stronger concept is Nakano positivity (cf. [Nak55]), asserting that

$$\widetilde{\Theta}_{E,h}(\tau) := \sum_{1 \le j,k \le n,\, 1 \le \lambda,\mu \le r} c_{jk\lambda\mu} \tau_{j\lambda} \overline{\tau}_{k\mu} > 0 \tag{1.3}$$

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for all non zero tensors $\tau = \sum_{j,\lambda} \tau_{j\lambda} \frac{\partial}{\partial z_j} \otimes e_\lambda \in T_X \otimes E$. It is in fact interesting to consider also the curvature tensor of the dual bundle E^*, which happens to be given by the opposite of the transpose of $\Theta_{E,h}$, i.e.

$$\Theta_{E^*,h} = -{}^T\Theta_{E,h} = -\sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\mu\lambda} dz_j \wedge d\overline{z}_k \otimes (e^*_\lambda)^* \otimes e^*_\mu. \tag{1.4}$$

This leads to the concept of dual Nakano positivity, stipulating that

$$-\widetilde{\Theta}_{E^*,h}(\tau) = \sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\mu\lambda}\tau_{j\lambda}\overline{\tau}_{k\mu} > 0 \tag{1.5}$$

for all non zero tensors $\tau = \sum_{j,\lambda} \tau_{j\lambda} \frac{\partial}{\partial z_j} \otimes e^*_\lambda \in T_X \otimes E^*$. On the other hand, Griffiths positivity of $\Theta_{E,h}$ is equivalent to Griffiths negativity of $\Theta_{E^*,h}$, and implies the positivity of the induced metric on the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)(1)}$. By the Kodaira embedding theorem [Kod54], the positivity of $\mathcal{O}_{\mathbb{P}(E)(1)}$ is equivalent to its ampleness, hence we see immediately from the definitions that

$$\widetilde{\Theta}_{E,h} \text{ dual Nakano positive} \;\Rightarrow\; \widetilde{\Theta}_{E,h} \text{ Griffiths positive} \;\Rightarrow\; E \text{ ample.} \tag{1.6}$$

In this short note, we consider the following converse problem:

1.7. Basic question. *Does it hold that*

$$E \text{ ample} \;\Rightarrow\; \widetilde{\Theta}_{E,h} \text{ dual Nakano positive ?}$$

A positive answer would clearly settle the Griffiths conjecture, in an even stronger form. One should observe that Nakano positivity implies Griffiths positivity, but is in general a more restrictive condition. As a consequence, one cannot expect ampleness to imply Nakano positivity. For instance, $T_{\mathbb{P}^n}$ is easy shown to be ample (and Nakano semi-positive for the Fubini-Study metric), but it is not Nakano positive, as the Nakano vanishing theorem [Nak55] would then yield

$$H^{n-1,n-1}(\mathbb{P}^n, \mathbb{C}) = H^{n-1}(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}) = H^{n-1}(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes T_{\mathbb{P}^n}) = 0. \tag{1.8}$$

On the other hand, it does not seem that there are any examples of ample vector bundles that are not dual Nakano positive, thus the above basic question is still legitimate, even though it might look very optimistic. We should mention here that subtle relations between ampleness, Griffiths and Nakano positivity are known to hold – for instance, B. Berndtsson[Ber09] has proved that the ampleness of E implies the Nakano positivity of $S^mE\otimes\det E$ for every $m \in \mathbb{N}$. See also [DeS79] for an earlier direct and elementary proof of the much weaker result that the Griffiths positivity of E implies the Nakano positivity of $E \otimes \det E$, and [MoT07] for further results analogue to those of [Ber09].

So far, the Griffiths conjecture is known to hold when $n = \dim X = 1$ or $r = \operatorname{rank} E = 1$ (in which cases, Nakano and dual Nakano positivity coincide with Griffiths positivity). Proofs can be found in [Ume73, Theorem 2.6] and [CaF90]. In both cases, the proof is based on the existence of Harder-Narasimhan filtrations and on the Narasimhan-Seshadri theorem [NaS65] for stable vector bundles – the 1-dimensional case of the Donaldson-Uhlenbeck-Yau theorem [Don85], [UhY86]. It is tempting to investigate whether techniques of gauge theory could be used to approach the Griffiths conjecture. In this direction, P. Naumann [Nau17] proposed a Kähler-Ricci flow method that starts with a given Finsler metric of positive curvature, and converges to a Hermitian metric. It is however unclear whether the flow introduced in [Nau17] preserves positivity, so it might very well produce in the limit a Hermitian metric that does not have positive curvature.

Here, we describe a different continuity method based on a natural determinantal equation, that is designed to enforce positivity of the curvature, actually in the dual Nakano sense. It would however remain to check whether one can obtain long time existence of the flow for the said equation or one of its variants.

2. Approach via a combination of Monge-Ampère and Hermitian-Yang-Mills equations

Let $E \to X$ be a holomorphic vector bundle equipped with a smooth Hermitian metric h. If the Chern curvature tensor $\Theta_{E,h}$ is dual Nakano positive, then the $\frac{1}{r}$-power of the $(n \times r)$-dimensional determinant of the corresponding Hermitian quadratic form on $T_X \otimes E^*$ can be seen as a positive (n,n)-form

$$\det_{T_X \otimes E^*}({}^T\Theta_{E,h})^{1/r} := \det(c_{jk\mu\lambda})^{1/r}_{(j,\lambda),(k,\mu)}\, idz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge idz_n \wedge d\overline{z}_n. \tag{2.1}$$

Moreover, this (n,n)-form does not depend on the choice of coordinates (z_j) on X, nor on the choice of the orthonormal frame (e_λ) on E (but the orthonormality of (e_λ) is required). Conversely, given a Kähler metric ω_0 on X, the basic idea is that assigning a "matrix Monge-Ampère equation"

$$\det_{T_X \otimes E^*}({}^T\Theta_{E,h})^{1/r} = f\,\omega_0^n, \tag{2.2}$$

where f is a smooth positive function, may enforce the dual Nakano positivity of $\Theta_{E,h}$ if that assignment is combined with a continuity technique from an initial starting point where positivity is known. For $r = 1$, we have ${}^T\Theta_{E,h} = \Theta_{E,h} = -i\partial\overline{\partial}\log h$, and equation (2.2) is a standard Monge-Ampère equation. If f is given and independent of h, Yau's theorem [Yau78] guarantees the existence of a unique solution $\theta = \Theta_{E,h} > 0$, provided E is an ample line bundle and $\int_X f\,\omega_0^n = c_1(E)^n$. One then gets a smoothly varying solution $\theta_t = \Theta_{E,h_t} > 0$ when the right hand side f_t of (2.2) varies smoothly with respect to some parameter t.

Now, assuming E to be ample of rank $r > 1$, equation (2.2) becomes underdetermined, since the real rank of the space of hermitian matrices h on E is equal to r^2, while (2.2) provides only one scalar equation. If $E = \bigoplus_{1\le j\le r} E_j$ splits as a direct sum of ample line bundles and we take a diagonal Hermitian structure $h = \bigoplus h_j$ on E, the $nr \times nr$ determinant splits as a product of blocks, and equation (2.2) reduces to

$$\Bigg(\prod_{1\le j\le r} \Theta^n_{E_j,h_j}\Bigg)^{1/r} = f\,\omega_0^n. \tag{2.2_s}$$

This "split equation" can be solved for any $f = \prod f_j^{1/r}$ with $\int_X f_j\,\omega_0^n = c_1(E_j)^n$, just by solving the individual equations $\Theta^n_{E_j,h_j} = f_j\,\omega_0^n$, $f_j > 0$, but the decomposition need not be unique. In this case, the Hölder inequality requires $\int_X f\omega_0^n \le (\prod c_1(E_j)^n)^{1/r}$, and the equality can be reached by taking all f_j's to be proportional to f.

In general, solutions might still exist, but the lack of uniqueness prevents us from getting a priori bounds. In order to recover a well determined system of equations, one needs to introduce (r^2-1) additional scalar equations, or rather a matrix equation of real rank (r^2-1). If E is ample, the determinant line bundle $\det E$ is also ample. By the Kodaira embedding theorem, we can find a smooth Hermitian metric η_0 on $\det E$ so that $\omega_0 := \Theta_{\det E,\eta_0} > 0$ is a Kähler metric on X. In case E is ω_0-stable or ω_0-polystable, we know by the Donaldson-Uhlenbeck-Yau theorem that there exists a Hermitian metric h on E satisfying the Hermite-Einstein condition

$$\omega_0^{n-1} \wedge \Theta_{E,h} = \frac{1}{r}\,\omega_0^n \otimes \mathrm{Id}_E, \tag{2.3}$$

since the slope of E with respect to $\omega_0 \in c_1(E)$ is equal to $\frac{1}{r}$.

In general, one cannot expect E to be ω_0-polystable, but Uhlenbeck-Yau have shown that there always exist smooth solutions to a certain "cushioned" Hermite-Einstein equation. To make things more precise, let $\mathrm{Herm}(E)$ be the space of Hermitian (non necessarily positive) forms on E, and given a Hermitian metric $h > 0$, let $\mathrm{Herm}_h(E, E)$ be the space of h-Hermitian endomorphisms $u \in \mathrm{Hom}(E, E)$; we denote by

$$\mathrm{Herm}(E) \to \mathrm{Herm}_h(E,E), \quad q \mapsto \widetilde{q} \;\text{ such that }\; q(v,w) = \langle v,w\rangle_q = \langle \widetilde{q}(v), w\rangle_h \tag{2.4}$$

the natural isomorphism between Hermitian quadratic forms and Hermitian endomorphisms, which depends of course on h. We also let

$$\mathrm{Herm}^\circ_h(E,E) = \big\{u \in \mathrm{Herm}_h(E,E)\,;\ \mathrm{tr}\,u = 0\big\} \tag{2.5}$$

be the subspace of "trace free" Hermitian endomorphisms. In the sequel, we fix a reference Hermitian metric H_0 on E such that $\det H_0 = \eta_0$, so that $\Theta_{\det E,\det H_0} = \omega_0 > 0$. By [UhY86, Theorem 3.1], for every $\varepsilon > 0$, there exists a smooth Hermitian metric q_ε on E such that

$$\omega_0^{n-1} \wedge \Theta_{E,q_\varepsilon} = \omega_0^n \otimes \Big(\frac{1}{r}\mathrm{Id}_E - \varepsilon\,\log \widetilde{q}_\varepsilon\Big), \tag{2.6}$$

where $\widetilde{q}_\varepsilon$ is computed with respect to H_0, and $\log u$ denotes the logarithm of a positive Hermitian endomorphism u. The intuitive reason is that the term $\log \widetilde{q}_\varepsilon$ introduces sufficient "friction" to avoid any explosion of approximating solutions when using a standard continuity method (see sections 2,3 in [UhY86]). On the other hand, when $\varepsilon \to 0$, the metrics q_ε become "more and more distorted" and yield asymptotically a splitting of E in weakly holomorphic subbundles corresponding to the Harder-Narasimhan filtration of E with respect to ω_0. If we write $\det q_\varepsilon = e^{-\varphi} \det H_0$ and take the trace in (2.6), we find $\omega_0^{n-1}\wedge(\omega_0 + i\partial\overline{\partial}\varphi) = \omega_0^n(1+\varepsilon\varphi)$, hence $\omega_0^{n-1} \wedge i\partial\overline{\partial}\varphi - \varepsilon\varphi\,\omega_0^n = 0$. A standard application of the maximum principle shows that $\varphi = 0$, thus (2.6) implies $\det q_\varepsilon = \det H_0$ and $\log \widetilde{q}_\varepsilon \in \mathrm{Herm}^\circ_{H_0}(E, E)$. In general, for an arbitrary Hermitian metric h, we let

$$\Theta^\circ_{E,h} = \Theta_{E,h} - \frac{1}{r}\Theta_{\det E,\det h} \otimes \mathrm{Id}_E \in C^\infty(X, \Lambda^{1,1}_{\mathbb{R}}T^*_X \otimes \mathrm{Herm}^\circ_h(E,E)) \tag{2.7}$$

be the curvature tensor of $E \otimes (\det E)^{-1/r}$ with respect to the trivial determinant metric $h^\circ := h \otimes (\det h)^{-1/r}$. Equation (2.6) is equivalent to prescribing $\det q_\varepsilon = \det H_0$ and

$$\omega_0^{n-1} \wedge \Theta^\circ_{E,q_\varepsilon} = -\varepsilon\,\omega_0^n \otimes \log \widetilde{q}_\varepsilon. \tag{2.8}$$

This is a matrix equation of rank (r^2-1) that involves only q_ε° and does not depend on $\det q_\varepsilon$. Notice that we have here $\log\widetilde{q}_\varepsilon \in \mathrm{Herm}^\circ_{H_0}(E,E)$, but also $\log\widetilde{q}_\varepsilon \in \mathrm{Herm}^\circ_{q_\varepsilon}(E,E)$.

In this context, given $\alpha>0$ large enough, it seems natural to search for a time dependent family of metrics $h_t(z)$ on the fibers E_z of E, $t\in[0,1]$, satisfying a generalized Monge-Ampère equation

$$\text{(2.9)}\qquad \det{}_{T_X\otimes E^*}\big({}^T\Theta_{E,h_t}+(1-t)\alpha\,\omega_0\otimes\mathrm{Id}_{E^*}\big)^{1/r}=f_t\,\omega_0^n,\qquad f_t>0,$$

and trace free Hermite-Einstein conditions

$$\text{(2.9°)}\qquad \omega_t^{n-1}\wedge\Theta^\circ_{E,h_t}=g_t,$$

with smoothly varying families of functions $f_t\in C^\infty(X,\mathbb{R})$, Hermitian metrics $\omega_t>0$ on X and sections $g_t\in C^\infty(X,\Lambda^{n,n}_{\mathbb{R}}T_X^*\otimes\mathrm{Herm}^\circ_{h_t}(E,E))$, $t\in[0,1]$. Here, we start e.g. with the Yau-Uhlenbeck solution $h_0=q_\varepsilon$ of (2.6) (so that $\det h_0=\det H_0$), and take $\alpha>0$ so large that ${}^T\Theta_{E,h_0}+\alpha\,\omega_0\otimes\mathrm{Id}_{E^*}>0$ in the sense of Nakano. If these conditions can be met for all $t\in[0,1]$ without any explosion of the solutions h_t, we infer from (2.9) that

$$\text{(2.9}^+\text{)}\qquad {}^T\Theta_{E,h_t}+(1-t)\alpha\,\omega_0\otimes\mathrm{Id}_{E^*}>0\quad\text{in the sense of Nakano}$$

for all $t\in[0,1]$. At time $t=1$, we will then get a Hermitian metric h_1 on E such that Θ_{E,h_1} is dual Nakano positive. We still have the freedom of adjusting f_t, ω_t and g_t in equations (2.9) and (2.9°). We have a system of differential equations of order 2, and any choice of the right hand sides of the form

$$\text{(2.10)}\qquad f_t(z)=F(t,z,h_t(z),D_zh_t(z),D_z^2h_t(z))>0,$$

$$\text{(2.10°)}\qquad g_t(z)=G(t,z,h_t(z),D_zh_t(z),D_z^2h_t(z))\in C^\infty(X,\Lambda^{1,1}_{\mathbb{R}}T_X^*\otimes\mathrm{Herm}^\circ(E,E))$$

is a priori acceptable for the sake of enforcing the positivity condition (2.9$^+$), although the presence of second order terms $D_z^2h_t(z))$ might affect the principal symbol of the equations. In equation (2.9°), the metrics ω_t could possibly be taken to depend on t, but unless some commodity reason would appear in next stages of the analysis, it seems simpler to set $\omega_t=\omega_0$ independent of t. At this stage, we have the following

2.11. Theorem. *Let (E,H_0) be a smooth Hermitian holomorphic vector bundle such that E is ample and $\omega_t=\omega_0=\Theta_{\det E,\det H_0}>0$. Then the system of equations* (2.7, 2.7°) *is a well determined* (*essentially non linear*) *elliptic system of equations for all choices of smooth right hand sides*

$$f_t=F(t,z,h_t,D_zh_t)>0,\qquad g_t=G(t,z,h_t,D_zh_t,D_z^2h_t)\in\mathrm{Herm}^\circ(E,E),$$

provided that the symbol η_h of the linearized operator $u\mapsto DG_{D^2h}(t,z,h,Dh,D^2h)\cdot D^2u$ has an Hilbert-Schmidt norm $\sup_{\xi\in T_X^*,|\xi|_{\omega_0}=1}\|\eta_h(\xi)\|_h\le(r^2+1)^{-1/2}\,n^{-1}$ for any of the metrics $h=h_t$ involved. If a smooth solution h_t exists on the whole time interval $[0,1]$, *then E is dual Nakano positive.*

Proof. If we write a hermitian metric h on E under the form $h(v,w)=\langle\widetilde{h}(v),w\rangle_{H_0}$ with $\widetilde{h}\in\mathrm{Herm}_{h_0}(E,E)$, we have $h=H_0\widetilde{h}$ in terms of matrices. The curvature tensor is given by the usual formula $\Theta_{E,h} = i\overline\partial(h^{-1}\partial h) = i\overline\partial(\widetilde h^{-1}\partial_{H_0}\widetilde h)$, where $\partial_{H_0}s = H_0^{-1}\partial(H_0 s)$ is the $(1,0)$-component of the Chern connection associated with H_0 on E. For simplicity of notation, we put

$$M := \mathrm{Herm}(E), \qquad M_h = \mathrm{Herm}_h(E,E), \qquad M_h^\circ = \mathrm{Herm}_h^\circ(E,E).$$

The system of equations $(2.9, 2.9^\circ)$ is associated with the non linear differential operator

$$P : C^\infty(X,M) \to C^\infty(X, \mathbb{R}\oplus M_h^\circ), \qquad h \mapsto P(h)$$

defined by

$$P(h) = \omega_0^{-n}\Big(\det{}_{T_X\otimes E^*}\big({}^T\Theta_{E,h} + (1-t)\alpha\,\omega_0\otimes \mathrm{Id}_{E^*}\big)^{1/r},\ \omega_0^{n-1}\wedge\Theta_{E^\circ,h} - G(t,z,h,Dh,D^2h)\Big).$$

It is by definition elliptic at h if its linearization $u \mapsto (dP)_h(u)$ is an elliptic linear operator, a crucial fact being that M and $\mathbb{R}\oplus M_h^\circ$ have the same rank r^2 over the field $\mathbb{R}$. Our goal is to compute the symbol $\sigma_{dP} \in C^\infty(X, S^2T_X^{\mathbb{R}}\otimes \mathrm{Hom}(M, \mathbb{R}\oplus M_h^\circ))$ of dP, and to check that $u \mapsto \sigma_{dP}(\xi)\cdot u$ is invertible for every non zero vector $\xi \in T_X^*$. We pick an infinitesimal variation δh of h in $C^\infty(X,M)$, and represent it as $\delta h = \langle u\,\bullet, \bullet\rangle_h$ with $u \in M_h = \mathrm{Herm}_h(E,E)$. In terms of matrices, we have $\delta h = hu$, i.e. $u = (u_{\lambda\mu}) = h^{-1}\delta h$ is the "logarithmic variation of h". In this setting, we evaluate $(dP)_h(u)$ in orthonormal coordinates $(z_j)_{1\le j\le n}$ on X relatively to ω_0. We have $h + \delta h = h(\mathrm{Id}+u)$ and $(h+\delta h)^{-1} = (\mathrm{Id}-u)h^{-1}$ modulo $O(u^2)$, thus

$$\begin{aligned} d\Theta_{E,h}(u) &= i\overline\partial(h^{-1}\partial(hu)) - i\overline\partial(uh^{-1}\partial h) = i\overline\partial\partial u + i\overline\partial(h^{-1}\partial h\,u) - i\overline\partial(uh^{-1}\partial h)\\ &= i\overline\partial\partial_{h^*\otimes h}u = -i\partial_{h^*\otimes h}\overline\partial u, \end{aligned} \tag{2.12}$$

where $\partial_{h^*\otimes h}$ denotes here the $(1,0)$-component of the Chern connection on the holomorphic vector bundle $\mathrm{Hom}(E,E) = E^*\otimes E$ induced by the metric $h^*\otimes h$. As a consequence, the order 2 term of the linearized operator is just

$$d\Theta_{E,h}(u)^{[2]} = -i\partial\overline\partial u,$$

and the logarithmic differential of the first scalar component $P_{\mathbb{R}}(h)$ of $P(h)$ has order 2 terms given by

$$P_{\mathbb{R}}(h)^{-1}\,dP_{\mathbb{R},h}(u)^{[2]} = \frac{1}{r}\,\mathrm{tr}(-\theta^{-1}\cdot{}^T i\partial\overline\partial u) = -\frac{1}{r}\,(\det\theta)^{-1}\sum_{j,k,\lambda,\mu}\widetilde\theta_{jk\lambda\mu}\,\frac{\partial^2 u_{\lambda\mu}}{\partial z_j\partial\overline z_k}, \tag{2.13}$$

where θ is the $(n\times r)$-matrix of $\theta = \theta(t,h) = {}^T\Theta_{E,h} + (1-t)\alpha\omega_0\otimes\mathrm{Id}_{E^*} > 0$, $\widetilde\theta$ its co-adjoint and $\theta^{-1} = (\det\theta)^{-1}\,{}^T\widetilde\theta$, so that $P_{\mathbb{R}}(h) = \omega_0^{-n}\,(\det\theta)^{1/r}$. We also have to compute the order 2 terms in the differential of the second component

$$h \mapsto P^\circ(h) = \omega_0^{-n}\big(\omega_0^{n-1}\wedge\Theta^\circ_{E,h} - G(t,z,h,Dh,D^2h)\big).$$

Let us set $u = \frac{1}{r}\,\mathrm{tr}\,u\otimes\mathrm{Id}_E + u^\circ$, $u^\circ \in M^\circ$, and $\mathrm{tr}\,u = \sum_\lambda u_{\lambda\lambda} \in \mathbb{R}$. Putting $\tau = \frac{1}{r}\,\mathrm{tr}\,u$, this actually gives an isomorphism $M_h \to \mathbb{R}\oplus M_h^\circ$, $u \mapsto (\tau, u^\circ)$. Since u° is the logarithmic variation of $h^\circ = h(\det h)^{-1/r}$, we get

$$(dP^\circ)_h(u)^{[2]} = \omega_0^{-n}\big(-\omega_0^{n-1}\wedge i\partial\overline\partial u^\circ - DG_{D^2h}\cdot D^2u\big). \tag{2.14}$$

If we fix a Hermitian metric h and take a non zero cotangent vector $0 \neq \xi \in T^*_X$, the symbol σ_{dP} is given by an expression of the form

$$\text{(2.15)} \qquad \sigma_{(dP)_h}(\xi)\cdot u = -\left(\frac{(\det\theta)^{-1+1/r}}{r\,\omega_0^n}\sum_{j,k,\lambda,\mu}\tilde\theta_{jk\lambda\ mu}\,\xi_j\overline\xi_k\,u_{\lambda\mu}\,,\ \frac{1}{n}\,|\xi|^2u^\circ+\widetilde\sigma_G(\xi)\cdot u\right)$$

where $\widetilde\sigma_G$ is the principal symbol of the operator $DG_{D^2h}\cdot D^2$. If $g_t = G(t,z,h_t,Dh_t)$ is independent of D^2h_t, the latter symbol $\widetilde\sigma_G$ is equal to 0 and it is easy to see from (2.13) that $u\mapsto\sigma_{(dP)_h}(\xi)\cdot u$ is an isomorphism in $\mathrm{Hom}(M_h,\mathbb{R}\oplus M_h^\circ)$. In fact, the first summation yields

$$\sum_{j,k,\lambda,\mu}\tilde\theta_{jk\lambda\mu}\,\xi_j\overline\xi_k\,u_{\lambda\mu}=\sum_{j,k,\lambda,\mu}\tilde\theta_{jk\lambda\mu}\,\xi_j\overline\xi_k\,u^\circ_{\lambda\mu}+\frac{1}{r}\sum_{j,k,\lambda}\tilde\theta_{jk\lambda\lambda}\,\xi_j\overline\xi_k\ \mathrm{tr}\,u.$$

By an easy calculation, we get an inverse operator $\mathbb{R}\oplus M_h^\circ\to M_h$, $(\tau,v)\mapsto u$ where

$$-r\,\omega_0^n\,(\det\theta)^{1-1/r}\,\tau=\sum_{j,k,\lambda,\mu}\tilde\theta_{jk\lambda\mu}\,\xi_j\overline\xi_k\,u^\circ_{\lambda\mu}+\frac{1}{r}\sum_{j,k,\lambda}\tilde\theta_{jk\lambda\lambda}\,\xi_j\overline\xi_k\ \mathrm{tr}\,u,\qquad -v=\frac{1}{n}\,|\xi|^2u^\circ,$$

hence $u^\circ=-\frac{n}{|\xi|^2}v$ and

$$\sigma_{(dP)_h}(\xi)^{-1}\cdot(\tau,v)=\frac{\frac{n}{|\xi|^2}\sum_{j,k,\lambda,\mu}\tilde\theta_{jk\lambda\mu}\,\xi_j\overline\xi_k\,v_{\lambda\mu}-r\,\omega_0^n\,(\det\theta)^{1-1/r}\,\tau}{\sum_{j,k,\lambda}\tilde\theta_{jk\lambda\lambda}\xi_j\overline\xi_k}\ \mathrm{Id}_E-\frac{n}{|\xi|^2}\,v.$$

Let us take the Hilbert-Schmidt norms $|u|^2=\sum_{\lambda,\mu}|u_{\lambda\mu}|^2$ on $M_h=\mathrm{Herm}_h(E,E)$, and $c|\tau|^2+|v|^2$ on $\mathbb{R}\oplus M_h^\circ$ (h being the reference metric, and $C>0$ a constant). By homogeneity, we can also assume $|\xi|=|\xi|_{\omega_0}=1$. Since $(\sum_{j,k}\tilde\theta_{jk\lambda\mu}\,\xi_j\overline\xi_k)_{1\le\lambda,\mu\le r}$ is a positive Hermitian matrix by the Nakano positivity property, its trace is a strict upper bound for the largest eigenvalue, and we get

$$\left|\sum_{j,k,\lambda}\tilde\theta_{jk\lambda\mu}\,\xi_j\overline\xi_k\,v_{\lambda\mu}\right|^2\le(1-\delta)\left(\sum_{j,k,\lambda}\tilde\theta_{jk\lambda\lambda}\xi_j\overline\xi_k\right)^2\sum_\lambda|v_{\lambda\mu}|^2.$$

The Cauchy-Schwarz inequality yields

$$\left|\sum_{j,k,\lambda,\mu}\tilde\theta_{jk\lambda\mu}\,\xi_j\overline\xi_k\,v_{\lambda\mu}\right|^2\le r(1-\delta)\left(\sum_{j,k,\lambda}\tilde\theta_{jk\lambda\lambda}\xi_j\overline\xi_k\right)^2\sum_{\lambda,\mu}|v_{\lambda\mu}|^2.$$

For $|\xi|=1$, as $\mathrm{Id}_E\perp M^\circ$ and $|\mathrm{Id}_E|^2=r$, this implies

$$\begin{aligned}\left|\sigma_{(dP)_h}(\xi)^{-1}\cdot(\tau,v)\right|^2&\le\left(nr^{1/2}(1-\delta)^{1/2}|v|+\frac{r\,\omega_0^n\,(\det\theta)^{1-1/r}}{\sum_{j,k,\lambda}\tilde\theta_{jk\lambda\lambda}\xi_j\overline\xi_k}\,|\tau|\right)^2r+n^2\,|v|^2\\&<(n^2r^2+n^2)(C|\tau|^2+|v|^2)\end{aligned}$$

for C large enough. By a standard pertubation argument, (2.13) remains bijective if $|\widetilde\sigma_G(\xi)|_h$ is less than the inverse of the norm of $\sigma_{(dP)_h}(\xi)^{-1}$, i.e. $(r^2+1)^{-1/2}\,n^{-1}$. Similarly, one could also allow the scalar right hand side F to have a "small dependence" on D^2h_t, but this seems less useful. □

Our next concern is to ensure that the existence of solutions holds on an open interval of time $[0,t_0[$ (and hopefully on the whole interval $[0,1]$). In the case of a rank one metric $h=e^{-\varphi}$, it is well-known that the Kähler-Einstein equation $(\omega_0+i\partial\overline{\partial}\varphi_t)^n=e^{tf+\lambda\varphi_t}\omega_0^n$ more easily results in getting openness and closedness of solutions when applying the continuity method for $\lambda>0$, as the linearized operator $\psi\mapsto\Delta_{\omega_{\varphi_t}}\psi-\lambda\psi$ is always invertible. One way to generalize the Kähler-Einstein condition to the case of higher ranks $r\geq 1$ is to take

$$f_t(z)=(\det H_0(z)/\det h_t(z))^\lambda\,a_0(z),\qquad \lambda\geq 0,\tag{2.16}$$

where $a_0(z)=\omega_0^{-n}\det({}^T\Theta_{E,h_0}+\alpha\omega_0\otimes \mathrm{Id}_{E^*})^{1/r}>0$ is chosen so that the equation is satisfied by h_0 at $t=0$ (the choice $\lambda>0$ has the interest that f_t gets automatically rescaled by multiplying h_t by a constant, thus ensuring strict invertibility). For the trace free part, what is a needed is to introduce a friction term g_t that helps again in getting invertibility of the linearized operator, and could possibly avoid the explosion of solutions when t grows to 1. A choice compatible with the Yau-Uhlenbeck solution (2.8) at $t=0$ is to take

$$g_t=-\varepsilon\,(\det H_0(z)/\det h_t(z))^\mu\,\omega_0^n\otimes\log\widetilde{h}_t^\circ,\qquad \varepsilon>0,\ \mu\in\mathbb{R},\tag{2.16°}$$

if one remembers that $\det h_0=\det H_0$. These right hand sides do not depend on higher derivatives of h_t, so Theorem 2.11 ensures the ellipticity of the differential system. Moreover:

2.17. Theorem. *For $\varepsilon\geq\varepsilon_0(h_t)$ and $\lambda\geq\lambda_0(h_t)(1+\mu^2)$ with $\varepsilon_0(h_t)$ and $\lambda_0(h_t)$ large enough, the elliptic differential system defined by* (2.9, 2.9°) *and* (2.16, 2.16°)*, namely*

$$\omega_0^{-n}\,\det{}_{T_X\otimes E^*}\big({}^T\Theta_{E,h_t}+(1-t)\alpha\,\omega_0\otimes\mathrm{Id}_{E^*}\big)^{1/r}=\left(\frac{\det H_0(z)}{\det h_t(z)}\right)^\lambda a_0(z)$$
$$\omega_0^{-n}\big(\omega_0^{n-1}\wedge\Theta^\circ_{E,h_t}\big)=-\varepsilon\left(\frac{\det H_0(z)}{\det h_t(z)}\right)^\mu\log\widetilde{h}_t^\circ,$$

possesses an invertible elliptic linearization. As a consequence, for such values of ε and λ, there exists an open interval $[0,t_0)\subset[0,1]$ on which the solution h_t exists.

Proof. We replace the operator $P:C^\infty(X,M)\to C^\infty(X,\mathbb{R}\oplus M_h^\circ)$ used in the proof of Theorem 2.9 by $\widetilde{P}=(\widetilde{P}_\mathbb{R},\widetilde{P}^\circ)$ defined by

$$\widetilde{P}_\mathbb{R}(h)=\omega_0^{-n}(\det h(z)/\det H_0(z))^\lambda\,\det{}_{T_X\otimes E^*}\big({}^T\Theta_{E,h}+(1-t)\alpha\,\omega_0\otimes\mathrm{Id}_{E^*}\big)^{1/r},$$
$$\widetilde{P}^\circ(h)=\omega_0^{-n}\big(\omega_0^{n-1}\wedge\Theta^\circ_{E,h}\big)+\varepsilon\,(\det h(z)/\det H_0(z))^{-\mu}\,\log\widetilde{h}^\circ.$$

Here, we have to care about the linearized operator dP itself, and not just with its principal symbol. We let again $u=h^{-1}\delta h\in\mathrm{Herm}_h(E,E)$ and use formula (2.12) for $d\Theta_{E,h}(u)$. This implies

$$\widetilde{P}_\mathbb{R}(h)^{-1}\,d\widetilde{P}_{\mathbb{R},h}(u)=\lambda\,\mathrm{tr}\,u-\frac{1}{r}\,\mathrm{tr}_{T_X\otimes E^*}\Big(\theta^{-1}\cdot{}^T\big(i\partial_{h^*\otimes h}\overline{\partial}u\big)\Big).$$

We need the fact that $h^\circ=h\cdot(\det h)^{-1/r}$ possesses, when viewed as a Hermitian endomorphism, a logarithmic variation

$$(\widetilde{h}^\circ)^{-1}\delta\widetilde{h}^\circ=u^\circ=u-\frac{1}{r}\,\mathrm{tr}\,u\cdot\mathrm{Id}_E.$$

By the classical formula expressing the differential of the logarithm of a matrix, we have

$$d\log g(\delta g) = \int_0^1 \big((1-t)\mathrm{Id} + tg\big)^{-1}\delta g\,\big((1-t)\mathrm{Id} + tg\big)^{-1}\,dt,$$

which implies

$$d\log \widetilde{h}^\circ(u) = \int_0^1 \big((1-t)\mathrm{Id} + t\,\widetilde{h}^\circ\big)^{-1}\,\widetilde{h}^\circ u^\circ\,\big((1-t)\mathrm{Id} + t\,\widetilde{h}^\circ\big)^{-1}\,dt.$$

In the end, we obtain

$$\begin{aligned}&(d\widetilde{P}^\circ)_h(u) = -\omega_0^{-n}\Big(\omega_0^{n-1}\wedge i\partial_{h^*\otimes h}\overline{\partial}u^\circ\Big) + \\ &\varepsilon\left(\frac{\det h(z)}{\det H_0(z)}\right)^{-\mu}\left(\int_0^1 \big((1-t)\mathrm{Id} + t\,\widetilde{h}^\circ\big)^{-1}\,\widetilde{h}^\circ u^\circ\,\big((1-t)\mathrm{Id} + t\,\widetilde{h}^\circ\big)^{-1}\,dt - \mu\,\mathrm{tr}\,u\,\log\widetilde{h}^\circ\right).\end{aligned}$$

In order to check the invertibility, we use the norm $|\tau|^2 + C|v|^2$ on $\mathbb{R}\oplus M_h^\circ$ and compute the L^2 inner product $\langle\!\langle (d\widetilde{P})_h(u), (\tau, u^\circ)\rangle\!\rangle$ over X, where $\tau = \frac{1}{r}\,\mathrm{tr}\,u$. The ellipticity of operators $-i\partial_H\overline{\partial}$ implies that it has a discrete sequence of eigenvalues converging to $+\infty$, and that we get Gårding type inequalities of the form $\langle\!\langle -i\partial_H\overline{\partial}v, v\rangle\!\rangle_H \geq c_1\|\nabla v\|_H^2 - c_2\|v\|_H^2$ where $c_1, c_2 > 0$ depend on H. We apply such inequalities to $v = \tau$, $H = 1$, and $v = u^\circ$, $H = h^*\otimes h$, replacing u with $u = \tau\,\mathrm{Id} + u^\circ$. From this, we infer

$$\begin{aligned}\langle\!\langle (d\widetilde{P})_h(u), (\tau, u^\circ)\rangle\!\rangle \geq c_1\|d\tau\|^2 - c_2\|\tau\|^2 + \lambda r\,\|\tau\|^2 - \frac{1}{r}\langle\!\langle \mathrm{tr}_{T_X\otimes E^*}\Big(\theta^{-1}\cdot{}^T\big(i\partial_{h^*\otimes h}\overline{\partial}u^\circ\big)\Big), \tau\rangle\!\rangle \\ + C\Big(c_1^\circ\,\|\nabla u^\circ\|^2 - c_2^\circ\|u^\circ\|^2 + c_3\varepsilon\,\|u^\circ\|^2 - c_4\varepsilon\,|\mu|\,\|\tau\|\,\|u^\circ\|\Big)\end{aligned}$$

where all constants c_j may possibly depend on h. An integration by parts yields

$$\begin{aligned}\left|\frac{1}{r}\langle\!\langle \mathrm{tr}_{T_X\otimes E^*}\Big(\theta^{-1}\cdot{}^T\big(i\partial_{h^*\otimes h}\overline{\partial}u^\circ\big)\Big), \tau\rangle\!\rangle\right| &\leq c_5\|\nabla u^\circ\|\,(\|d\tau\| + \|\tau\|) \\ &\leq \frac{1}{2}c_1\left(\|d\tau\|^2 + \|\tau\|^2\right) + c_6\|\nabla u^\circ\|^2\end{aligned}$$

and we have

$$c_4\varepsilon\,|\mu|\,\|\tau\|\,\|u^\circ\| \leq \frac{1}{2}c_3\varepsilon\,\|u^\circ\|^2 + c_7\varepsilon\mu^2\,\|\tau\|^2.$$

If we choose $\varepsilon \geq 2c_2^\circ/c_3 + 1$, $C \geq c_6/c_1^\circ + 1$ and $\lambda r \geq c_2 + \frac{1}{2}c_1 + Cc_7\varepsilon\mu^2 + 1$, we finally get

$$\langle\!\langle (d\widetilde{P})_h(u), (\tau, u^\circ)\rangle\!\rangle \geq \frac{1}{2}c_1\|d\tau\|^2 + \|\tau\|^2 + c_1^\circ\|\nabla u^\circ\|^2 + \frac{1}{2}Cc_3\varepsilon\|u^\circ\|^2$$

and conclude that $(d\widetilde{P})_h$ is an invertible elliptic operator. The openness property at $t = 0$ then follows from standard results on elliptic PDE's. □

2.18. Remarks. (a) Theorem 2.17 is not very satisfactory since the constants $\varepsilon_0(h_t)$ and $\lambda_0(h_t)$ depend on the solution h_t. It would be important to know if one can get sufficiently uniform estimates to make these constants independent of h_t, thereby guaranteeing the long time existence of solutions. This might require modifying somewhat the right hand side of our equations, especially the trace free part, while taking a similar determinantal Monge-Ampère equation that still enforces the dual Nakano positivity of the curvature tensor. The Yau iteration technique used in [Yau78] to get 0 order estimates for Monge-Ampère equations will probably have to be adapted to this situation.

(b) The non explosion of solutions when $t \to 1$ does not come for free, since this property cannot hold when $\det E$ is ample, but E is not. One possibility would be to show that an explosion at time $t_0 < 1$ produces a "destabilizing subsheaf" $\mathcal{S}$ contradicting the ampleness of $E/\mathcal{S}$, similarly to what was done in [UhY86] to contradict the stability hypothesis.

2.19. Variants. (a) The determinantal equation always yields a Kähler metric

$$\beta_t := \mathrm{tr}_E\big(\Theta_{E,h_t} + (1-t)\alpha\,\omega_0 \otimes \mathrm{Id}_E\big) = \Theta_{\det E, \det h_t} + r(1-t)\alpha\,\omega_0 > 0.$$

An interesting variant of the trace free equation is

$$\omega_t^{-n}\left(\omega_t^{n-1} \wedge \Theta^\circ_{E,h_t}\right) = -\varepsilon\left(\frac{\det H_0(z)}{\det h_t(z)}\right)^{\mu} \log \widetilde{h}^\circ_t$$

with $\omega_t = \beta_t$, although the ellipticity of the differential system is less obvious in that case. However, it can be shown that the ellipticity is preserved, at least near $t = 0$, for $\omega_t = \omega_0 + \delta\beta_t$ with $\delta > 0$ small.

(b) In a first step towards solving (2.6), [UhY86] consider equations that have even stronger friction terms, taking the right hand side to be of the form

$$\omega_0^{n-1} \wedge \Theta_{E,h} = \omega_0^n \otimes \big(-\varepsilon \log \widetilde{h} + \sigma\,\widetilde{h}^{-1/2}\,\Gamma_0\,\widetilde{h}^{1/2} - \Gamma_0\big), \quad \sigma > 0,$$

and letting $\sigma \to 0$ at the end of the analysis. Here we can do just the same, for instance by adding a term equal to a multiple of $(\widetilde{h}^\circ_t)^{-1/2}\,\Gamma_t\,(\widetilde{h}^\circ_t)^{1/2} - \Gamma_t$ in the trace free equation, as such terms are precisely trace free for any $\Gamma_t \in C^\infty(X, \mathrm{Hom}(E,E))$.

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