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Hermitian-Yang-Mills approach to the conjecture of Griffiths on the positivity of ample vector bundles

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Abstract. Given a vector bundle of arbitrary rank with ample determinant line bundle on a projective manifold, we propose a new elliptic system of differential equations of Hermitian-Yang-Mills type for the curvature tensor. The system is designed so that solutions provide Hermitian metrics with positive curvature in the sense of Griffiths—and even in the dual Nakano sense. As a consequence, if an existence result could be obtained for every ample vector bundle, the Griffiths conjecture on the equivalence between ampleness and positivity of vector bundles would be settled.

Bibliography: 15 titles.

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§ 1. Introduction

Let X be a projective n-dimensional manifold. A conjecture due to Griffiths [5] states that a holomorphic vector bundle $E\to X$ is ample in the sense of Hartshorne, meaning that the associated line bundle $\mathscr{O}_{\mathbb{P}(E)}(1)$ is ample, if and only if E possesses a Hermitian metric h such that the Chern curvature tensor $\Theta_{E,h}=i\nabla^2_{E,h}$ is Griffiths positive. In other words, if we let $\operatorname{rank}E=r$ and

$$\Theta_{E,h}=i\sum_{1\leqslant j,k\leqslant n,\,1\leqslant\lambda,\mu\leqslant r}c_{jk\lambda\mu}\,dz_j\wedge d\overline{z}_k\otimes e_\lambda^*\otimes e_\mu \tag{1.1}$$

in terms of holomorphic coordinates $(z_1,\dots,z_n)$ on X and an orthonormal frame $(e_\lambda)_{1\leqslant\lambda\leqslant r}$ of E, the associated quadratic form

$$\widetilde{\Theta}_{E,h}(\xi\otimes v):=\langle\Theta_{E,h}(\xi,\overline{\xi})\cdot v,v\rangle_h=\sum_{1\leqslant j,k\leqslant n,\,1\leqslant\lambda,\mu\leqslant r}c_{jk\lambda\mu}\xi_j\overline{\xi}_k v_\lambda\overline{v}_\mu \tag{1.2}$$

should take positive values at nonzero tensors $\xi\otimes v\in T_X\otimes E$. A stronger concept is Nakano positivity (see [8]), asserting that

$$\widetilde{\Theta}_{E,h}(\tau):=\sum_{1\leqslant j,k\leqslant n,\,1\leqslant\lambda,\mu\leqslant r}c_{jk\lambda\mu}\tau_{j\lambda}\overline{\tau}_{k\mu}>0 \tag{1.3}$$

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for all nonzero tensors $\tau = \sum_{j,\lambda} \tau_{j\lambda} \frac{\partial}{\partial z_j} \otimes e_\lambda \in T_X \otimes E$. It is in fact interesting to consider also the curvature tensor of the dual bundle E^*, which happens to be given by the opposite of the transpose of $\Theta_{E,h}$, that is,

$$\Theta_{E^*,h^*} = -{}^T\Theta_{E,h} = -\sum_{1\leqslant j,k\leqslant n,\, 1\leqslant \lambda,\mu\leqslant r} c_{jk\mu\lambda}\, dz_j \wedge d\overline{z}_k \otimes (e_\lambda^*)^* \otimes e_\mu^*. \tag{1.4}$$

This leads to the concept of dual Nakano positivity, stipulating that

$$-\widetilde{\Theta}_{E^*,h^*}(\tau) = \sum_{1\leqslant j,k\leqslant n,\, 1\leqslant \lambda,\mu\leqslant r} c_{jk\mu\lambda}\tau_{j\lambda}\overline{\tau}_{k\mu} > 0 \tag{1.5}$$

for all nonzero tensors $\tau = \sum_{j,\lambda} \tau_{j\lambda} \frac{\partial}{\partial z_j} \otimes e_\lambda^* \in T_X \otimes E^*$. On the other hand, Griffiths positivity of $\Theta_{E,h}$ is equivalent to Griffiths negativity of Θ_{E^*,h^*}, and implies the positivity of the induced metric on the tautological line bundle $\mathscr{O}_{\mathbb{P}(E)(1)}$. By the Kodaira embedding theorem [6], the positivity of $\mathscr{O}_{\mathbb{P}(E)(1)}$ is equivalent to its ampleness, hence we see immediately from the definitions that

$$\widetilde{\Theta}_{E,h} \text{ dual Nakano positive} \quad\Longrightarrow\quad \widetilde{\Theta}_{E,h} \text{ Griffiths positive} \quad\Longrightarrow\quad E \text{ ample.} \tag{1.6}$$

In this short note, we consider the following converse problem.

Basic question 1.1. Does it hold that

$$E \text{ ample} \quad\Longrightarrow\quad \widetilde{\Theta}_{E,h} \text{ dual Nakano positive?}$$

A positive answer would clearly settle the Griffiths conjecture, in an even stronger form. We observe that Nakano positivity implies Griffiths positivity, but in general is a more restrictive condition. As a consequence, we cannot expect ampleness to imply Nakano positivity. For instance, $T_{\mathbb{P}^n}$ is easily shown to be ample (and Nakano semi-positive for the Fubini-Study metric), but it is not Nakano positive, as the Nakano vanishing theorem [8] would then yield

$$H^{n-1,n-1}(\mathbb{P}^n, \mathbb{C}) = H^{n-1}(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}) = H^{n-1}(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes T_{\mathbb{P}^n}) = 0. \tag{1.7}$$

On the other hand, it does not seem that there are any examples of ample vector bundles that are not dual Nakano positive, thus the above basic question is still legitimate, even though it might look very optimistic. We should mention here that subtle relations between ampleness, and Griffiths and Nakano positivity are known to hold—for instance, Berndtsson [1] has proved that the ampleness of E implies the Nakano positivity of $S^m E \otimes \det E$ for every $m \in \mathbb{N}$. See also [3] for an earlier direct and elementary proof of the much weaker result that the Griffiths positivity of E implies the Nakano positivity of $E \otimes \det E$, and [7] for further results analogous to those of [1].

So far, Griffiths' conjecture is known to hold when $n = \dim X = 1$ or $r = \operatorname{rank} E = 1$ (in which cases, Nakano and dual Nakano positivity coincide with Griffiths positivity). Proofs can be found in [14], Theorem 2.6, and [2]. In both cases, the proof is based on the existence of Harder-Narasimhan filtrations and on the Narasimhan-Seshadri theorem [9] for stable vector bundles—the 1-dimensional case of the Donaldson-Uhlenbeck-Yau theorem [4], [13]. It is tempting to investigate whether gauge theory techniques could be used to approach the Griffiths conjecture. In this direction, Naumann [10] proposed a Kähler-Ricci flow method that starts with a given Finsler metric of positive curvature, and converges to a Hermitian metric. It is however unclear whether the flow introduced in [10] preserves positivity, so it might very well produce in the limit a Hermitian metric that does not have positive curvature. Another related suggestion is Pingali's proposal made in [11] to study the vector bundle Monge-Ampère equation $(\Theta_{E,h})^n = \eta\,\mathrm{Id}_E$, where η is a positive volume form on X. Solving such an equation requires polystability in dimension $n = 1$, and, in general, a positivity property of (E, h) that is even stronger than Nakano positivity (and thus much stronger than ampleness).

In § 2, we describe a more flexible differential system based on a combination of a huge determinantal equation and a trace free Hermite-Einstein condition. It relies on the well-known continuity method, and is designed to enforce positivity of the curvature, actually in the dual Nakano sense—a condition that could eventually still be equivalent to ampleness. We show that it is possible to design a nonlinear differential system that is elliptic and invertible, at least near the origin of time. It would, however, remain to check whether we can obtain long time existence of the solution for the said equation or one of its variants. Section 3 is devoted to a discussion of a related extremal problem, and a concept of volume for vector bundles.

§ 2. An approach via a combination of Monge-Ampère and Hermitian-Yang-Mills equations

Let $E \to X$ be a holomorphic vector bundle equipped with a smooth Hermitian metric h. If the Chern curvature tensor $\Theta_{E,h}$ is dual Nakano positive, then the $\frac{1}{r}$th power of the $(n \times r)$-dimensional determinant of the corresponding Hermitian quadratic form on $T_X \otimes E^*$ can be seen as a positive (n, n)-form

$$\det{}_{T_X\otimes E^*}({}^T\Theta_{E,h})^{1/r} := \det(c_{jk\mu\lambda})^{1/r}_{(j,\lambda),(k,\mu)} i\,dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge i\,dz_n \wedge d\overline{z}_n. \quad (2.1)$$

Moreover, this (n, n)-form does not depend on the choice of coordinates (z_j) on X, nor on the choice of the orthonormal frame (e_λ) on E (but (e_λ) must be orthonormal). Conversely, given a Kähler metric ω_0 on X, the basic idea is that assigning a 'matrix Monge-Ampère equation'

$$\det{}_{T_X\otimes E^*}({}^T\Theta_{E,h})^{1/r} = f\omega_0^n, \quad (2.2)$$

where f is a smooth positive function, may enforce the dual Nakano positivity of $\Theta_{E,h}$ if that assignment is combined with a continuity technique starting from an initial point where positivity is known. For $r = 1$, we have ${}^T\Theta_{E,h} = \Theta_{E,h} = -i\,\partial\overline{\partial}\log h$, and equation (2.2) is a standard Monge-Ampère equation. If f is given and independent of h, Yau's theorem [15] guarantees the existence of a unique solution $\theta = \Theta_{E,h} > 0$, provided E is an ample line bundle and $\int_X f\omega_0^n = c_1(E)^n$. We then get a smoothly varying solution $\theta_t = \Theta_{E,h_t} > 0$ when the right-hand side f_t of (2.2) varies smoothly with respect to some parameter t.

Now, assuming E to be ample of rank $r > 1$, equation (2.2) becomes underdetermined, since the real rank of the space of Hermitian matrices h on E is equal to r^2, while (2.2) provides only one scalar equation. If $E = \bigoplus_{1\leqslant j\leqslant r} E_j$ splits as a direct sum of ample line bundles and we take a diagonal Hermitian structure $h = \bigoplus h_j$ on E, the $(nr \times nr)$-determinant splits as a product of blocks, and equation (2.2) reduces to

$$\Big(\prod_{1\leqslant j\leqslant r} \Theta^n_{E_j,h_j}\Big)^{1/r} = f\omega_0^n. \tag{2.2$_s$}$$

This 'split equation' can be solved for any $f = \prod f_j^{1/r}$ with $\int_X f_j\omega_0^n = c_1(E_j)^n$, just by solving the individual equations $\Theta^n_{E_j,h_j} = f_j\omega_0^n$, $f_j > 0$, but the decomposition need not be unique. In this case, the Hölder inequality requires that $\int_X f\omega_0^n \leqslant \big(\prod c_1(E_j)^n\big)^{1/r}$, and equality can be achieved by taking all the f_j to be proportional to f.

In general, solutions might still exist, but the lack of uniqueness prevents us from getting a priori bounds. In order to recover a well-determined system of equations, we need to introduce r^2-1 additional scalar equations, or rather a matrix equation of real rank (r^2-1). If E is ample, the determinant line bundle $\det E$ is also ample. By the Kodaira embedding theorem, we can find a smooth Hermitian metric η_0 on $\det E$ so that $\omega_0 := \Theta_{\det E,\eta_0} > 0$ is a Kähler metric on X. In case E is ω_0-stable or ω_0-polystable, we know by the Donaldson-Uhlenbeck-Yau theorem that there exists a Hermitian metric h on E satisfying the Hermite-Einstein condition

$$\omega_0^{n-1} \wedge \Theta_{E,h} = \frac{1}{r}\omega_0^n \otimes \mathrm{Id}_E, \tag{2.3}$$

since the slope of E with respect to $\omega_0 \in c_1(E)$ is equal to $1/r$.

In general, we cannot expect E to be ω_0-polystable, but Uhlenbeck and Yau have shown that there always exist smooth solutions to a certain 'cushioned' Hermite-Einstein equation. To make things more precise, let $\mathrm{Herm}(E)$ be the space of Hermitian (not necessarily positive) forms on E, and given a Hermitian metric $h > 0$, let $\mathrm{Herm}_h(E,E)$ be the space of h-Hermitian endomorphisms $u \in \mathrm{Hom}(E,E)$; we let

$$\mathrm{Herm}(E) \to \mathrm{Herm}_h(E,E), \qquad q \mapsto \widetilde{q} \quad \text{such that} \quad q(v,w) = \langle v,w\rangle_q = \langle \widetilde{q}(v),w\rangle_h \tag{2.4}$$

denote the natural isomorphism between Hermitian quadratic forms and Hermitian endomorphisms, which of course depends on h. We also let

$$\mathrm{Herm}^\circ_h(E,E) = \big\{u \in \mathrm{Herm}_h(E,E);\ \mathrm{tr}\,u = 0\big\} \tag{2.5}$$

be the subspace of 'trace free' Hermitian endomorphisms. In the sequel, we fix a reference Hermitian metric H_0 on E such that $\det H_0 = \eta_0$, so that $\Theta_{\det E,\det H_0} = \omega_0 > 0$. By [13], Theorem 3.1, for every $\varepsilon > 0$, there exists a smooth Hermitian metric q_ε on E such that

$$\omega_0^{n-1} \wedge \Theta_{E,q_\varepsilon} = \omega_0^n \otimes \Big(\frac{1}{r}\,\mathrm{Id}_E - \varepsilon \log \widetilde{q}_\varepsilon\Big), \tag{2.6}$$

where $\widetilde{q}_\varepsilon$ is computed with respect to H_0, and $\log u$ denotes the logarithm of a positive Hermitian endomorphism u. The intuitive reason is that the term $\log\widetilde{q}_\varepsilon$ introduces sufficient 'friction' to avoid any explosion of approximating solutions when using a standard continuity method (see §§ 2 and 3 in [13]). On the other hand, when $\varepsilon\to 0$, the metrics q_ε become 'more and more distorted' and, asymptotically, yield a splitting of E into weakly holomorphic subbundles corresponding to the Harder-Narasimhan filtration of E with respect to ω_0. If we write $\det q_\varepsilon = e^{-\varphi}\det H_0$ and take the trace in (2.6), we find $\omega_0^{n-1}\wedge(\omega_0+i\,\partial\overline{\partial}\varphi)=\omega_0^n(1+\varepsilon\varphi)$, hence $\omega_0^{n-1}\wedge i\,\partial\overline{\partial}\varphi-\varepsilon\varphi\omega_0^n=0$. A standard application of the maximum principle shows that $\varphi=0$, thus (2.6) implies $\det q_\varepsilon=\det H_0$ and $\log\widetilde{q}_\varepsilon\in\mathrm{Herm}^\circ_{H_0}(E,E)$. In general, for an arbitrary Hermitian metric h, we let

$$\Theta^\circ_{E,h}=\Theta_{E,h}-\frac{1}{r}\Theta_{\det E,\det h}\otimes\mathrm{Id}_E\in C^\infty(X,\Lambda^{1,1}_{\mathbb{R}}T^*_X\otimes\mathrm{Herm}^\circ_h(E,E)) \tag{2.7}$$

be the curvature tensor of $E\otimes(\det E)^{-1/r}$ with respect to the trivial determinant metric $h^\circ:=h\otimes(\det h)^{-1/r}$. Equation (2.6) is equivalent to prescribing $\det q_\varepsilon=\det H_0$ and

$$\omega_0^{n-1}\wedge\Theta^\circ_{E,q_\varepsilon}=-\varepsilon\omega_0^n\otimes\log\widetilde{q}_\varepsilon. \tag{2.8}$$

This is a matrix equation of rank r^2-1 that involves only q°_ε and does not depend on $\det q_\varepsilon$. Notice that here we have $\log\widetilde{q}_\varepsilon\in\mathrm{Herm}^\circ_{H_0}(E,E)$, and also $\log\widetilde{q}_\varepsilon\in\mathrm{Herm}^\circ_{q_\varepsilon}(E,E)$.

In this context, given $\alpha>0$ large enough, it seems natural to search for a time dependent family of metrics $h_t(z)$ on the fibres E_z of E, $t\in[0,1]$, satisfying a generalized Monge-Ampère equation

$$\det{}_{T_X\otimes E^*}({}^T\Theta_{E,h_t}+(1-t)\alpha\omega_0\otimes\mathrm{Id}_{E^*})^{1/r}=f_t\omega_0^n,\qquad f_t>0, \tag{2.9}$$

and trace-free Hermite-Einstein conditions

$$\omega_t^{n-1}\wedge\Theta^\circ_{E,h_t}=g_t \tag{2.9°}$$

with smoothly varying families of functions $f_t\in C^\infty(X,\mathbb{R})$, Hermitian metrics $\omega_t>0$ on X and sections $g_t\in C^\infty(X,\Lambda^{n,n}_{\mathbb{R}}T^*_X\otimes\mathrm{Herm}^\circ_{h_t}(E,E))$, $t\in[0,1]$. Here, we start, for example, with the Yau-Uhlenbeck solution $h_0=q_\varepsilon$ of (2.6) (so that $\det h_0=\det H_0$), and take $\alpha>0$ so large that ${}^T\Theta_{E,h_0}+\alpha\omega_0\otimes\mathrm{Id}_{E^*}>0$ in the sense of Nakano. If these conditions can be met for all $t\in[0,1]$ without any 'explosion' of the solutions h_t, we infer from (2.9) that

$${}^T\Theta_{E,h_t}+(1-t)\alpha\omega_0\otimes\mathrm{Id}_{E^*}>0\quad\text{in the sense of Nakano} \tag{2.9$^+$}$$

for all $t\in[0,1]$. At time $t=1$ we then get a Hermitian metric h_1 on E such that Θ_{E,h_1} is dual Nakano positive. We still have the freedom of adjusting f_t, ω_t and g_t in equations (2.9) and (2.9°). We have a system of differential equations of order 2, and any choice of the right-hand sides of the form

$$f_t(z)=F\big(t,z,h_t(z),D_zh_t(z),D^2_zh_t(z)\big)>0 \tag{2.10}$$

and

$$g_t(z)=G\big(t,z,h_t(z),D_zh_t(z),D^2_zh_t(z)\big)\in C^\infty(X,\Lambda^{1,1}_{\mathbb{R}}T^*_X\otimes\mathrm{Herm}^\circ(E,E)) \tag{2.10°}$$

is a priori acceptable for the sake of enforcing the positivity condition (2.9^+), although the presence of second order terms $D_z^2 h_t(z)$) might affect the principal symbol of the equations. In equation (2.9°), the metrics ω_t could possibly be taken to depend on t, but unless some cogent reason appears in the next stages of the analysis, it seems simpler to set $\omega_t = \omega_0$ independent of t. At this stage, we have the following.

Theorem 2.1. *Let (E, H_0) be a smooth Hermitian holomorphic vector bundle such that E is ample and $\omega_t = \omega_0 = \Theta_{\det E, \det H_0} > 0$. Then the system of equations (2.9), (2.9°) is a well-defined (essentially nonlinear) elliptic system of equations for any choice of the smooth right-hand sides*

$$f_t = F(t, z, h_t, D_z h_t) > 0 \quad \textit{and} \quad g_t = G(t, z, h_t, D_z h_t, D_z^2 h_t) \in \mathrm{Herm}^\circ(E, E)$$

provided that, for any of the metrics $h = h_t$ involved, the Hilbert-Schmidt norm of the symbol η_h of the linearized operator $u \mapsto DG_{D^2 h}(t, z, h, Dh, D^2 h) \cdot D^2 u$ satisfies $\sup_{\xi \in T_X^*, |\xi|_{\omega_0} = 1} \|\eta_h(\xi)\|_h \leqslant (r^2 + 1)^{-1/2} n^{-1}$. If a smooth solution h_t exists on the whole time interval $[0, 1]$, then E is dual Nakano positive.

Proof. If we write a Hermitian metric h on E under the form $h(v, w) = \langle \widetilde{h}(v), w \rangle_{H_0}$ with $\widetilde{h} \in \mathrm{Herm}_{h_0}(E, E)$, we have $h = H_0 \widetilde{h}$ in terms of matrices. The curvature tensor is given by the usual formula $\Theta_{E,h} = i\,\overline{\partial}(h^{-1}\partial h) = i\,\overline{\partial}(\widetilde{h}^{-1}\,\partial_{H_0}\widetilde{h})$, where $\partial_{H_0} s = H_0^{-1}\,\partial(H_0 s)$ is the $(1,0)$-component of the Chern connection associated with H_0 on E. For simplicity of notation, we put

$$M := \mathrm{Herm}(E), \qquad M_h = \mathrm{Herm}_h(E, E) \quad \text{and} \quad M_h^\circ = \mathrm{Herm}_h^\circ(E, E).$$

The system of equations (2.9), (2.9°) is associated with the nonlinear differential operator

$$P\colon C^\infty(X, M) \to C^\infty(X, \mathbb{R} \oplus M_h^\circ), \qquad h \mapsto P(h),$$

defined by

$$P(h) = \omega_0^{-n}\big(\det_{T_X \otimes E^*}({}^T\Theta_{E,h} + (1-t)\alpha\omega_0 \otimes \mathrm{Id}_{E^*})^{1/r},\\ \omega_0^{n-1} \wedge \Theta_{E^\circ,h} - G(t, z, h, Dh, D^2 h)\big).$$

By definition, it is elliptic at h if its linearization $u \mapsto (dP)_h(u)$ is an elliptic linear operator, a crucial fact being that M and $\mathbb{R} \oplus M_h^\circ$ have the same rank r^2 over the field $\mathbb{R}$. Our goal is to compute the symbol $\sigma_{dP} \in C^\infty(X, S^2 T_X^{\mathbb{R}} \otimes \mathrm{Hom}(M, \mathbb{R} \oplus M_h^\circ))$ of dP, and to check that $u \mapsto \sigma_{dP}(\xi) \cdot u$ is invertible for every nonzero vector $\xi \in T_X^*$. We pick an infinitesimal variation δh of h in $C^\infty(X, M)$, and represent it as $\delta h = \langle u\,\bullet\,, \bullet\,\rangle_h$ with $u \in M_h = \mathrm{Herm}_h(E, E)$. In terms of matrices, we have $\delta h = hu$, that is, $u = (u_{\lambda\mu}) = h^{-1}\delta h$ is the 'logarithmic variation of h'. In this setting, we evaluate $(dP)_h(u)$ in orthonormal coordinates $(z_j)_{1 \leqslant j \leqslant n}$ on X relative to ω_0. We have $h + \delta h = h(\mathrm{Id} + u)$ and $(h + \delta h)^{-1} = (\mathrm{Id} - u)h^{-1}$ modulo $O(u^2)$, thus

$$d\Theta_{E,h}(u) = i\,\overline{\partial}(h^{-1}\,\partial(hu)) - i\,\overline{\partial}(uh^{-1}\,\partial h) = i\,\overline{\partial}\partial u + i\,\overline{\partial}(h^{-1}\,\partial h u) - i\,\overline{\partial}(uh^{-1}\,\partial h)\\ = i\,\overline{\partial}\partial_{h^*\otimes h} u = -i\,\partial_{h^*\otimes h}\overline{\partial} u, \tag{2.11}$$

where, here, $\partial_{h^*\otimes h}$ denotes the $(1,0)$-component of the Chern connection on the holomorphic vector bundle $\mathrm{Hom}(E,E)=E^*\otimes E$ induced by the metric $h^*\otimes h$. As a consequence, the order 2 term of the linearized operator is just

$$d\Theta_{E,h}(u)^{[2]}=-i\,\partial\overline{\partial}u,$$

and the logarithmic differential of the first scalar component $P_{\mathbb{R}}(h)$ of $P(h)$ has order 2 terms given by

$$P_{\mathbb{R}}(h)^{-1}\,dP_{\mathbb{R},h}(u)^{[2]}=\frac{1}{r}\,\mathrm{tr}(-\theta^{-1}\cdot{}^{T}i\,\partial\overline{\partial}u)=-\frac{1}{r}(\det\theta)^{-1}\sum_{j,k,\lambda,\mu}\widetilde{\theta}_{jk\lambda\mu}\frac{\partial^2u_{\lambda\mu}}{\partial z_j\,\partial\overline{z}_k},\tag{2.12}$$

where θ is the $(n\times r)$-matrix of $\theta=\theta(t,h)={}^{T}\Theta_{E,h}+(1-t)\alpha\omega_0\otimes\mathrm{Id}_{E^*}>0$, $\widetilde{\theta}$ is its co-adjoint and $\theta^{-1}=(\det\theta)^{-1}\,{}^{T}\widetilde{\theta}$, so that $P_{\mathbb{R}}(h)=\omega_0^{-n}(\det\theta)^{1/r}$. We also have to compute the order 2 terms in the differential of the second component

$$h\mapsto P^{\circ}(h)=\omega_0^{-n}\big(\omega_0^{n-1}\wedge\Theta^{\circ}_{E,h}-G(t,z,h,Dh,D^2h)\big).$$

We set $u=(1/r)\,\mathrm{tr}\,u\otimes\mathrm{Id}_E+u^{\circ}$, $u^{\circ}\in M^{\circ}$, and $\mathrm{tr}\,u=\sum_\lambda u_{\lambda\lambda}\in\mathbb{R}$. Putting $\tau=(1/r)\,\mathrm{tr}\,u$, this actually gives an isomorphism $M_h\to\mathbb{R}\oplus M_h^{\circ}$, $u\mapsto(\tau,u^{\circ})$. Since u° is the logarithmic variation of $h^{\circ}=h(\det h)^{-1/r}$, we get

$$(dP^{\circ})_h(u)^{[2]}=\omega_0^{-n}(-\omega_0^{n-1}\wedge i\,\partial\overline{\partial}u^{\circ}-DG_{D^2h}\cdot D^2u).\tag{2.13}$$

If we fix a Hermitian metric h and take a nonzero cotangent vector $0\neq\xi\in T^*_X$, then the symbol σ_{dP} is given by an expression of the form

$$\sigma_{(dP)_h}(\xi)\cdot u=-\bigg(\frac{(\det\theta)^{-1+1/r}}{r\omega_0^n}\sum_{j,k,\lambda,\mu}\widetilde{\theta}_{jk\lambda\mu}\xi_j\overline{\xi}_ku_{\lambda\mu},\ \frac{1}{n}|\xi|^2u^{\circ}+\widetilde{\sigma}_G(\xi)\cdot u\bigg),\tag{2.14}$$

where $\widetilde{\sigma}_G$ is the principal symbol of the operator $DG_{D^2h}\cdot D^2$. If $g_t=G(t,z,h_t,Dh_t)$ is independent of D^2h_t, then the latter symbol $\widetilde{\sigma}_G$ is equal to 0 and it is easy to see from (2.12) that $u\mapsto\sigma_{(dP)_h}(\xi)\cdot u$ is an isomorphism in $\mathrm{Hom}(M_h,\mathbb{R}\oplus M_h^{\circ})$. In fact, the first summation yields

$$\sum_{j,k,\lambda,\mu}\widetilde{\theta}_{jk\lambda\mu}\xi_j\overline{\xi}_ku_{\lambda\mu}=\sum_{j,k,\lambda,\mu}\widetilde{\theta}_{jk\lambda\mu}\xi_j\overline{\xi}_ku^{\circ}_{\lambda\mu}+\frac{1}{r}\sum_{j,k,\lambda}\widetilde{\theta}_{jk\lambda\lambda}\xi_j\overline{\xi}_k\,\mathrm{tr}\,u.$$

By an easy calculation, we get an inverse operator $\mathbb{R}\oplus M_h^{\circ}\to M_h$, $(\tau,v)\mapsto u$, where

$$-r\omega_0^n(\det\theta)^{1-1/r}\tau=\sum_{j,k,\lambda,\mu}\widetilde{\theta}_{jk\lambda\mu}\xi_j\overline{\xi}_ku^{\circ}_{\lambda\mu}+\frac{1}{r}\sum_{j,k,\lambda}\widetilde{\theta}_{jk\lambda\lambda}\xi_j\overline{\xi}_k\,\mathrm{tr}\,u\ \text{ and }\ -v=\frac{1}{n}|\xi|^2u^{\circ},$$

hence $u^{\circ}=-(n/|\xi|^2)v$ and

$$\sigma_{(dP)_h}(\xi)^{-1}\cdot(\tau,v)=\frac{(n/|\xi|^2)\sum_{j,k,\lambda,\mu}\widetilde{\theta}_{jk\lambda\mu}\xi_j\overline{\xi}_kv_{\lambda\mu}-r\omega_0^n(\det\theta)^{1-1/r}\tau}{\sum_{j,k,\lambda}\widetilde{\theta}_{jk\lambda\lambda}\xi_j\overline{\xi}_k}\,\mathrm{Id}_E-\frac{n}{|\xi|^2}v.$$

Now take the Hilbert-Schmidt norms $|u|^2=\sum_{\lambda,\mu}|u_{\lambda\mu}|^2$ on $M_h=\mathrm{Herm}_h(E,E)$, and $c|\tau|^2+|v|^2$ on $\mathbb{R}\oplus M_h^\circ$ (h being the reference metric, and $C>0$ a constant). By homogeneity, we can also assume that $|\xi|=|\xi|_{\omega_0}=1$. Since $\big(\sum_{j,k}\widetilde\theta_{jk\lambda\mu}\xi_j\overline\xi_k\big)_{1\leqslant\lambda,\mu\leqslant r}$ is a positive Hermitian matrix by the Nakano positivity property, its trace is a strict upper bound for the largest eigenvalue, and we get

$$\Big|\sum_{j,k,\lambda}\widetilde\theta_{jk\lambda\mu}\xi_j\overline\xi_k v_{\lambda\mu}\Big|^2\leqslant(1-\delta)\Big(\sum_{j,k,\lambda}\widetilde\theta_{jk\lambda\lambda}\xi_j\overline\xi_k\Big)^2\sum_{\lambda}|v_{\lambda\mu}|^2.$$

The Cauchy-Schwarz inequality yields

$$\Big|\sum_{j,k,\lambda,\mu}\widetilde\theta_{jk\lambda\mu}\xi_j\overline\xi_k v_{\lambda\mu}\Big|^2\leqslant r(1-\delta)\Big(\sum_{j,k,\lambda}\widetilde\theta_{jk\lambda\lambda}\xi_j\overline\xi_k\Big)^2\sum_{\lambda,\mu}|v_{\lambda\mu}|^2.$$

As $\mathrm{Id}_E\perp M^\circ$ and $|\,\mathrm{Id}_E|^2=r$, for $|\xi|=1$ this implies that

$$\begin{aligned}\big|\sigma_{(dP)_h}(\xi)^{-1}\cdot(\tau,v)\big|^2&\leqslant\bigg(nr^{1/2}(1-\delta)^{1/2}|v|+\frac{r\omega_0^n(\det\theta)^{1-1/r}}{\sum_{j,k,\lambda}\widetilde\theta_{jk\lambda\lambda}\xi_j\overline\xi_k}|\tau|\bigg)^2r+n^2|v|^2\\&<(n^2r^2+n^2)(C|\tau|^2+|v|^2)\end{aligned}$$

for C large enough. By a standard perturbation argument, (2.12) remains bijective if $|\widetilde\sigma_G(\xi)|_h$ is less than the inverse of the norm of $\sigma_{(dP)_h}(\xi)^{-1}$, that is, $(r^2+1)^{-1/2}n^{-1}$. Similarly, we could also allow the scalar right-hand side F to have a 'small dependence' on D^2h_t, but this seems less useful. The theorem is proved.

Our next concern is to ensure that the existence of solutions holds on an open interval of time $[0,t_0)$ (and hopefully on the whole interval $[0,1]$). In the case of a rank one metric $h=e^{-\varphi}$, it is well known that the Kähler-Einstein equation $(\omega_0+i\,\partial\overline\partial\varphi_t)^n=e^{tf+\lambda\varphi_t}\omega_0^n$ lets us obtain results on openness and closedness of solutions more easily when applying the continuity method for $\lambda>0$, as the linearized operator $\psi\mapsto\Delta_{\omega_{\varphi_t}}\psi-\lambda\psi$ is always invertible. One way to generalize the Kähler-Einstein condition to the case of higher ranks $r\geqslant1$ is to take

$$f_t(z)=\bigg(\frac{\det H_0(z)}{\det h_t(z)}\bigg)^\lambda a_0(z),\qquad\lambda\geqslant0,\tag{2.15}$$

where $a_0(z)=\omega_0^{-n}\det({}^T\Theta_{E,h_0}+\alpha\omega_0\otimes\mathrm{Id}_{E^*})^{1/r}>0$ is chosen so that the equation is satisfied by h_0 at $t=0$ (the choice of $\lambda>0$ means that f_t automatically gets rescaled by multiplying h_t by a constant, thus ensuring strict invertibility). For the trace-free part, what is needed is to introduce a friction term g_t that again helps in getting invertibility of the linearized operator, and could possibly avoid an explosion of solutions when t increases to 1. A choice compatible with the Yau-Uhlenbeck solution (2.8) at $t=0$ is to take

$$g_t=-\varepsilon\bigg(\frac{\det H_0(z)}{\det h_t(z)}\bigg)^\mu\omega_0^n\otimes\log\widetilde h_t^\circ,\qquad\varepsilon>0,\quad\mu\in\mathbb{R},\tag{2.15°}$$

if we recall that $\det h_0=\det H_0$. These right-hand sides do not depend on higher derivatives of h_t, so Theorem 2.1 ensures the ellipticity of the differential system. Moreover,

Theorem 2.2. *For $\varepsilon \geqslant \varepsilon_0(h_t)$ and $\lambda \geqslant \lambda_0(h_t)(1+\mu^2)$ with $\varepsilon_0(h_t)$ and $\lambda_0(h_t)$ large enough, the elliptic differential system defined by* (2.9), (2.9°) *and* (2.15), (2.15°), *namely*

$$\omega_0^{-n}\det_{T_X\otimes E^*}({}^T\Theta_{E,h_t}+(1-t)\alpha\omega_0\otimes \mathrm{Id}_{E^*})^{1/r}=\left(\frac{\det H_0(z)}{\det h_t(z)}\right)^{\lambda}a_0(z),$$

$$\omega_0^{-n}(\omega_0^{n-1}\wedge\Theta^\circ_{E,h_t})=-\varepsilon\left(\frac{\det H_0(z)}{\det h_t(z)}\right)^{\mu}\log\widetilde{h}^\circ_t,$$

possesses an invertible elliptic linearization. As a consequence, for such values of ε and λ, there exists an open interval $[0,t_0)\subset[0,1]$ on which the solution h_t exists.

Proof. We replace the operator $P\colon C^\infty(X,M)\to C^\infty(X,\mathbb{R}\oplus M_h^\circ)$ used in the proof of Theorem 2.1 by $\widetilde{P}=(\widetilde{P}_{\mathbb{R}},\widetilde{P}^\circ)$ defined by

$$\widetilde{P}_{\mathbb{R}}(h)=\omega_0^{-n}\left(\frac{\det h(z)}{\det H_0(z)}\right)^{\lambda}\det_{T_X\otimes E^*}({}^T\Theta_{E,h}+(1-t)\alpha\omega_0\otimes\mathrm{Id}_{E^*})^{1/r},$$

$$\widetilde{P}^\circ(h)=\omega_0^{-n}(\omega_0^{n-1}\wedge\Theta^\circ_{E,h})+\varepsilon\left(\frac{\det h(z)}{\det H_0(z)}\right)^{-\mu}\log\widetilde{h}^\circ.$$

Here, we have to keep an eye on the linearized operator dP itself, and not just its principal symbol. We again let $u=h^{-1}\delta h\in\mathrm{Herm}_h(E,E)$ and use formula (2.11) for $d\Theta_{E,h}(u)$. This implies that

$$\widetilde{P}_{\mathbb{R}}(h)^{-1}\,d\widetilde{P}_{\mathbb{R},h}(u)=\lambda\,\mathrm{tr}\,u-\frac{1}{r}\,\mathrm{tr}_{T_X\otimes E^*}\big(\theta^{-1}\cdot{}^T(i\,\partial_{h^*\otimes h}\overline{\partial}u)\big).$$

We need the fact that, when viewed as a Hermitian endomorphism, $h^\circ=h\cdot(\det h)^{-1/r}$ possesses a logarithmic variation

$$(\widetilde{h}^\circ)^{-1}\delta\widetilde{h}^\circ=u^\circ=u-\frac{1}{r}\,\mathrm{tr}\,u\cdot\mathrm{Id}_E\,.$$

By the classical formula expressing the differential of the logarithm of a matrix, we have

$$d\log g(\delta g)=\int_0^1((1-t)\,\mathrm{Id}+tg)^{-1}\delta g((1-t)\,\mathrm{Id}+tg)^{-1}\,dt,$$

which implies that

$$d\log\widetilde{h}^\circ(u)=\int_0^1((1-t)\,\mathrm{Id}+t\widetilde{h}^\circ)^{-1}\widetilde{h}^\circ u^\circ((1-t)\,\mathrm{Id}+t\widetilde{h}^\circ)^{-1}\,dt.$$

In the end, we obtain

$$(d\widetilde{P}^\circ)_h(u)=-\omega_0^{-n}(\omega_0^{n-1}\wedge i\,\partial_{h^*\otimes h}\overline{\partial}u^\circ)+\varepsilon\left(\frac{\det h(z)}{\det H_0(z)}\right)^{-\mu}$$
$$\times\left(\int_0^1((1-t)\,\mathrm{Id}+t\widetilde{h}^\circ)^{-1}\widetilde{h}^\circ u^\circ((1-t)\,\mathrm{Id}+t\widetilde{h}^\circ)^{-1}\,dt-\mu\,\mathrm{tr}\,u\log\widetilde{h}^\circ\right).$$

In order to check the invertibility, we use the norm $|\tau|^2 + C|v|^2$ on $\mathbb{R} \oplus M_h^\circ$ and compute the L^2 inner product $\langle\!\langle (d\widetilde{P})_h(u), (\tau, u^\circ)\rangle\!\rangle$ over X, where $\tau = (1/r)\operatorname{tr} u$. The ellipticity of the operator $-i\,\partial_H\overline{\partial}$ implies that it has a discrete sequence of eigenvalues converging to $+\infty$, and that we get Gårding type inequalities of the form $\langle\!\langle -i\,\partial_H\overline{\partial}v, v\rangle\!\rangle_H \geqslant c_1\|\nabla v\|_H^2 - c_2\|v\|_H^2$ where $c_1, c_2 > 0$ depend on H. We apply such inequalities to $v = \tau$, $H = 1$, and $v = u^\circ$, $H = h^*\otimes h$, replacing u with $u = \tau\operatorname{Id} + u^\circ$. From this we infer that

$$\begin{aligned}\langle\!\langle (d\widetilde{P})_h(u), (\tau, u^\circ)\rangle\!\rangle &\geqslant c_1\|d\tau\|^2 - c_2\|\tau\|^2 + \lambda r\|\tau\|^2\\ &\quad - \frac{1}{r}\langle\!\langle \operatorname{tr}_{T_X\otimes E^*}(\theta^{-1}\cdot {}^T(i\,\partial_{h^*\otimes h}\overline{\partial}u^\circ)), \tau\rangle\!\rangle\\ &\quad + C\big(c_1^\circ\|\nabla u^\circ\|^2 - c_2^\circ\|u^\circ\|^2 + c_3\varepsilon\|u^\circ\|^2 - c_4\varepsilon|\mu|\,\|\tau\|\,\|u^\circ\|\big),\end{aligned}$$

where all the constants c_j may possibly depend on h. Integrating by parts yields

$$\begin{aligned}\left|\frac{1}{r}\langle\!\langle \operatorname{tr}_{T_X\otimes E^*}(\theta^{-1}\cdot {}^T(i\,\partial_{h^*\otimes h}\overline{\partial}u^\circ)), \tau\rangle\!\rangle\right| &\leqslant c_5\|\nabla u^\circ\|\big(\|d\tau\| + \|\tau\|\big)\\ &\leqslant \frac{1}{2}c_1\big(\|d\tau\|^2 + \|\tau\|^2\big) + c_6\|\nabla u^\circ\|^2,\end{aligned}$$

and we have

$$c_4\varepsilon|\mu|\,\|\tau\|\,\|u^\circ\| \leqslant \frac{1}{2}c_3\varepsilon\|u^\circ\|^2 + c_7\varepsilon\mu^2\|\tau\|^2.$$

If we choose

$$\varepsilon \geqslant 2\frac{c_2^\circ}{c_3} + 1, \qquad C \geqslant \frac{c_6}{c_1^\circ} + 1 \quad\text{and}\quad \lambda r \geqslant c_2 + \frac{1}{2}c_1 + Cc_7\varepsilon\mu^2 + 1,$$

we finally get

$$\langle\!\langle (d\widetilde{P})_h(u), (\tau, u^\circ)\rangle\!\rangle \geqslant \frac{1}{2}c_1\|d\tau\|^2 + \|\tau\|^2 + c_1^\circ\|\nabla u^\circ\|^2 + \frac{1}{2}Cc_3\varepsilon\|u^\circ\|^2$$

and conclude that $(d\widetilde{P})_h$ is an invertible elliptic operator. The openness property at $t = 0$ then follows from standard results on elliptic PDEs. The theorem is proved.

Remarks 2.3. (a) Theorem 2.2 is not very satisfactory since the constants $\varepsilon_0(h_t)$ and $\lambda_0(h_t)$ depend on the solution h_t. The ability to obtain sufficiently uniform estimates, so as to make these constants independent of h_t, could prove important, as it would guarantee the long time existence of solutions. This might require us to modify the right-hand side of our equations somewhat, especially the trace-free part, while taking a similar determinantal Monge-Ampère equation that still enforces the dual Nakano positivity of the curvature tensor. The Yau iteration technique used in [15] to get 0 order estimates for Monge-Ampère equations will probably have to be adapted to this situation.

(b) The non-explosion of solutions when $t \to 1$ does not come for free, since this property cannot hold when $\det E$ is ample, but E is not. One possibility would be to show that an explosion at time $t_0 < 1$ produces a 'destabilizing subsheaf' $\mathscr{S}$ contradicting the ampleness of $E/\mathscr{S}$, similarly to what was done in [13] to contradict the stability hypothesis.

Variants 2.4. (a) The determinantal equation always yields a Kähler metric

$$\beta_t := \mathrm{tr}_E(\Theta_{E,h_t} + (1-t)\alpha\omega_0 \otimes \mathrm{Id}_E) = \Theta_{\det E, \det h_t} + r(1-t)\alpha\omega_0 > 0.$$

An interesting variant of the trace free equation is

$$\omega_t^{-n}(\omega_t^{n-1} \wedge \Theta^\circ_{E,h_t}) = -\varepsilon\left(\frac{\det H_0(z)}{\det h_t(z)}\right)^\mu \log \widetilde{h}^\circ_t, \tag{2.16}$$

with $\omega_t = \dfrac{1}{r\alpha+1}\beta_t$ (notice that $\beta_0 = (r\alpha+1)\omega_0$). It is then important to know whether the corresponding differential system is still elliptic with an invertible linearization. According to equation (2.16), the $\mathrm{Herm}(E,E)^\circ$ part of the differential system depends on the functional

$$\widetilde{P}^\circ(h) = \omega_0^{-n}\big(\omega_t^{n-1} \wedge \Theta^\circ_{E,h}\big) + \varepsilon\left(\frac{\det h(z)}{\det H_0(z)}\right)^{-\mu} \log \widetilde{h}^\circ,$$

and, relative to the functional used in Theorem 2.2, the differential $d\widetilde{P}^\circ_h(u)$ acquires one additional term coming from the variation of ω_t^{n-1}. With the same notation as in our previous calculations, we have $\Theta_{\det E,\det h_t} = -i\,\partial\overline{\partial}\log\det(h_t)$ and $\delta(\beta_t)_h(u) = -i\,\partial\overline{\partial}\,\mathrm{tr}\,u$, hence

$$\begin{aligned}(d\widetilde{P}^\circ)_h(u) = -\omega_0^{-n}\bigg(&\omega_t^{n-1} \wedge i\,\partial_{h^*\otimes h}\overline{\partial}u^\circ + \frac{n-1}{r\alpha+1}\omega_t^{n-2} \wedge i\,\partial\overline{\partial}\,\mathrm{tr}\,u \wedge \Theta^\circ_{E,h}\bigg)\\ &+\varepsilon\left(\frac{\det h(z)}{\det H_0(z)}\right)^{-\mu}\bigg(\int_0^1 \big((1-t)\,\mathrm{Id} + t\widetilde{h}^\circ\big)^{-1}\widetilde{h}^\circ u^\circ \big((1-t)\,\mathrm{Id} + t\widetilde{h}^\circ\big)^{-1} dt\\ &- \mu\,\mathrm{tr}\,u \log \widetilde{h}^\circ\bigg).\end{aligned}$$

Again putting $\tau = \mathrm{tr}\,u$, this requires us to estimate one extra term appearing in the L^2-inner product $\langle\!\langle (d\widetilde{P})_h(u), (\tau, u^\circ)\rangle\!\rangle$, namely

$$\langle\!\langle (\omega_0^n)^{-1}\omega_t^{n-2} \wedge i\,\partial\overline{\partial}\tau \wedge \Theta^\circ_{E,h}, u^\circ\rangle\!\rangle.$$

We can apply the same integration by parts argument as before to conclude that $(d\widetilde{P})_h$ is again invertible, under a similar hypothesis $\lambda \geqslant \lambda_0(h_t)(1+\mu^2)$, at least for t small. A very recent note posted by Pingali [12] shows that when E is ω_0-stable and h_0 is taken to be the Hermite-Einstein metric, the trace free part of the differential system used in Theorem 2.2 has a solution of the form $h_t = h_0 e^{-\psi_t}$, thus it is always ‘conformal’ to h_0. There are cases where the dual Nakano positivity of h_0 is doubtful. As a consequence, even in that favourable case, it is unclear whether a long time existence result can hold for the total system, unless stronger restrictions on the Chern classes are made. Equation (2.16) does not seem to entail such constraints, and may thus be better suited to the problem under investigation.

(b) In its first step towards solving (2.6), [13] considers equations that have even stronger friction terms, taking the right-hand side to be of the form

$$\omega_0^{n-1} \wedge \Theta_{E,h} = \omega_0^n \otimes (-\varepsilon \log \widetilde{h} + \sigma\widetilde{h}^{-1/2}\Gamma_0\widetilde{h}^{1/2} - \Gamma_0), \qquad \sigma > 0,$$

and letting $\sigma \to 0$ at the end of the analysis. Here we can do just the same, for instance by adding a term equal to a multiple of $(\widetilde{h}^\circ_t)^{-1/2}\Gamma_t(\widetilde{h}^\circ_t)^{1/2} - \Gamma_t$ in the trace free equation, as such terms are precisely trace free for any $\Gamma_t \in C^\infty(X, \mathrm{Hom}(E,E))$.

§ 3. A concept of Monge-Ampère volume for vector bundles

If $E \to X$ is an ample vector bundle of rank r, the associated line bundle $\mathcal{O}_{\mathbb{P}(E)}(1) \to Y = \mathbb{P}(E)$ is ample, and we can consider its volume $c_1(\mathcal{O}_{\mathbb{P}(E)}(1))^{n+r-1}$. It is well known that this number (which is an integer) coincides with the Segre number $\int_X (-1)^n s_n(E)$, where $(-1)^n s_n(E)$ is the nth Segre class of E. Let us assume further that E is dual Nakano positive (if the solution of the Hermitian-Yang-Mills differential system of § 2 is unobstructed, this would follow from the ampleness of E). We can then introduce the following more involved concept of volume, which we will call the *Monge-Ampère volume of* E:

$$\mathrm{MAVol}(E) = \sup_h \int_X \det{}_{T_X \otimes E^*}((2\pi)^{-1} \cdot {}^T\Theta_{E,h})^{1/r}, \tag{3.1}$$

where the supremum is taken over all smooth metrics h on E such that ${}^T\Theta_{E,h}$ is Nakano positive. This supremum is always finite, and in fact we have the following.

Proposition 3.1. *For any dual Nakano positive vector bundle E,*

$$\mathrm{MAVol}(E) \leqslant r^{-n} c_1(E)^n.$$

Proof. We take h to be a Hermitian metric on E such that ${}^T\Theta_{E,h}$ is Nakano positive, and consider the Kähler metric

$$\omega = (2\pi)^{-1}\Theta_{\det E, \det h} = (2\pi)^{-1} \operatorname{tr}_{E^*} \cdot {}^T\Theta_{E,h} \in c_1(E).$$

If $(\lambda_j)_{1\leqslant j\leqslant nr}$ are the eigenvalues of the associated Hermitian form $(2\pi)^{-1} \cdot {}^T\widetilde{\Theta}_{E,h}$ with respect to $\omega \otimes h$, we have

$$\det{}_{T_X \otimes E^*}((2\pi)^{-1} \cdot {}^T\Theta_{E,h})^{1/r} = \Big(\prod_j \lambda_j\Big)^{1/r} \omega^n,$$

and

$$\Big(\prod_j \lambda_j\Big)^{1/(nr)} \leqslant \Big(\frac{1}{nr}\Big) \sum_j \lambda_j$$

by the inequality between the geometric and arithmetic means. Since

$$\sum_j \lambda_j = \operatorname{tr}_\omega\big(\operatorname{tr}_{E^*}((2\pi)^{-1} \cdot {}^T\Theta_{E,h})\big) = \operatorname{tr}_\omega \omega = n,$$

we conclude that

$$\int_X \det{}_{T_X \otimes E^*}((2\pi)^{-1} \cdot {}^T\Theta_{E,h})^{1/r} \leqslant \int_X \Big(\frac{1}{nr}\sum_j \lambda_j\Big)^n \omega^n = r^{-n}\int_X \omega^n = r^{-n} c_1(E)^n.$$

The proposition follows.

Remarks 3.2. (a) In case $E = \bigoplus_{1\leqslant j\leqslant r} E_j$ and $h = \bigoplus_{1\leqslant j\leqslant r} h_j$ are split, with all metrics h_j normalized to have proportional volume forms $((2\pi)^{-1}\Theta_{E_j,h_j})^n = \beta_j\omega^n$ with suitable constants $\beta_j > 0$, we get $\beta_j = c_1(E_j)^n/c_1(E)^n$, and the inequality reads

$$\Big(\prod_{1\leqslant j\leqslant r} c_1(E_j)^n\Big)^{1/r} \leqslant r^{-n}c_1(E)^n.$$

It is an equality when $E_1 = \cdots = E_r$, thus Proposition 3.1 is optimal as far as the constant r^{-n} is concerned. For $E = \bigoplus_{1\leqslant j\leqslant r} E_j$ split with distinct ample factors, it seems natural to conjecture that

$$\mathrm{MAVol}(E) = \Big(\prod_{1\leqslant j\leqslant r} c_1(E_j)^n\Big)^{1/r},$$

that is, that the supremum is reached for split metrics $h = \bigoplus h_j$. In case E is a non-split extension $0 \to A \to E \to A \to 0$ with A an ample line bundle—this is possible if $H^1(X, \mathscr{O}_X) \neq 0$, for example, on an abelian variety—we strongly suspect that $\mathrm{MAVol}(E) = c_1(A)^n$ but that the supremum is not reached by any smooth metric, as E is semi-stable but not polystable.

(b) It would be interesting to characterize the 'extremal metrics' h achieving the supremum in (3.1) when they exist. The calculations made in § 2 show that they satisfy some Euler-Lagrange equation

$$\int_X (\det\theta)^{1/r}\cdot \mathrm{tr}_{T_X\otimes E^*}\big(\theta^{-1}\cdot {}^T(i\,\partial_{h^*\otimes h}\overline{\partial}u)\big) = 0 \quad \forall\, u\in C^\infty(X, \mathrm{Herm}(E)),$$

where θ is the $(n\times r)$-matrix representing ${}^T\Theta_{E,h}$. After integrating by parts twice, freeing u from any differentiation, we get a fourth order nonlinear differential system that h has to satisfy. Remark 3.2, (a) leads us to suspect that this system is not always solvable, but the addition of suitable lower order 'friction terms' might make it universally solvable. This could possibly yield a better alternative to the more naive differential system of order 2 we proposed in § 2 to study the Griffiths conjecture.

(c) When $r > 1$, you might ask what is the infimum

$$\inf_h \int_X \det{}_{T_X\otimes E^*}((2\pi)^{-1}\cdot {}^T\Theta_{E,h})^{1/r}.$$

In the split case $(E,h) = \bigoplus(E_j, h_j)$, we can normalize Θ_{E_j,h_j} to satisfy $\Theta^n_{E_j,h_j} = f_j\omega^n$ with $\int_X f_j\omega^n = c_1(E_j)^n$, $f_j > 0$. Then

$$\int_X \det{}_{T_X\otimes E^*}((2\pi)^{-1}\cdot {}^T\Theta_{E,h})^{1/r} = \int_X (f_1\cdots f_r)^{1/r}\omega^n,$$

and this integral becomes arbitrarily small if we take the f_j to be large on disjoint open sets, and very small elsewhere. This example leads us to suspect that it is always the case that

$$\inf_h \int_X \det{}_{T_X\otimes E^*}((2\pi)^{-1}\cdot {}^T\Theta_{E,h})^{1/r} = 0$$

for $r > 1$. The 'friction terms' used in our differential systems should be chosen so as to prevent any such shrinking of the volume.

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