Towards the Green-Griffiths-Lang conjecture

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Abstract. The Green-Griffiths-Lang conjecture stipulates that for every projective variety X of general type over $\mathbb{C}$, there exists a proper algebraic subvariety of X containing all non constant entire curves $f : \mathbb{C} \to X$. Using the formalism of directed varieties, we prove here that this assertion holds true in case X satisfies a strong general type condition that is related to a certain jet-semistability property of the tangent bundle T_X.

0. Introduction

The goal of this note is to study the Green-Griffiths-Lang conjecture, as stated in [GG79] and [Lan86]. It will be useful to work here in a more general context and consider the category of directed projective manifolds (or varieties). Since the problems we consider are birationally invariant, varieties can in fact always be replaced by nonsingular models whenever this is needed. A directed projective manifold is a pair (X, V) where X is a projective manifold equipped with an analytic linear subspace $V \subset T_X$, i.e. a closed irreducible complex analytic subset V of the total space of T_X, such that each fiber $V_x = V \cap T_{X,x}$ is a complex vector space. If X is not connected, V should rather be assumed to be irreducible merely over each connected component of X, but we will hereafter assume that our manifolds are connected. A morphism $\Phi : (X, V) \to (Y, W)$ in the category of directed manifolds is an analytic map $\Phi : X \to Y$ such that $\Phi_* V \subset W$. We refer to the case $V = T_X$ as being the *absolute case*, and to the case $V = T_{X/S} = \mathrm{Ker}\, d\pi$ for a fibration $\pi : X \to S$, as being the *relative case*; V may also be taken to be the tangent space to the leaves of a singular analytic foliation on X, or maybe even a non integrable linear subspace of T_X.

We are especially interested in *entire curves* that are tangent to V, namely non constant holomorphic morphisms $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ of directed manifolds. In the absolute case, these are just arbitrary entire curves $f : \mathbb{C} \to X$. The strong form of the Green-Griffiths-Lang conjecture stipulates

0.1. GGL conjecture. *Let X be a projective variety of general type. Then there exists a proper algebraic variety $Y \subsetneq X$ such that every entire curve $f : \mathbb{C} \to X$ satisfies $f(\mathbb{C}) \subset Y$.*

[The weaker form would state that entire curves are algebraically degenerate, so that $f(\mathbb{C}) \subset Y_f \subsetneq X$ where Y_f might depend on f]. The smallest admissible algebraic set $Y \subset X$

is by definition the *entire curve locus* of X, defined as the Zariski closure

$$\mathrm{ECL}(X)=\overline{\bigcup_f f(\mathbb{C})}^{\mathrm{Zar}}. \tag{0.2}$$

If $X\subset\mathbb{P}^N_{\mathbb{C}}$ is defined over a number field $\mathbb{K}_0$ (i.e. by polynomial equations with equations with coefficients in $\mathbb{K}_0$) and $Y=\mathrm{ECL}(X)$, it is expected that for every number field $\mathbb{K}\supset\mathbb{K}_0$ the set of $\mathbb{K}$-points in $X(\mathbb{K})\smallsetminus Y$ is finite, and that this property characterizes $\mathrm{ECL}(X)$ as the smallest algebraic subset Y of X that has the above property for all $\mathbb{K}$ ([Lan86]). This conjectural arithmetical statement would be a vast generalization of the Mordell-Faltings theorem, and is one of the strong motivations to study the purely geometric GGL conjecture as a first step.

0.3. Problem (generalized GGL conjecture). *Let (X,V) be a projective directed manifold. Find geometric conditions on V ensuring that all entire curves $f:(\mathbb{C},T_{\mathbb{C}})\to(X,V)$ are contained in a proper algebraic subvariety $Y\subsetneq X$. Does this hold when (X,V) is of general type, in the sense that the canonical sheaf K_V is big ?*

As above, we define the entire curve locus set of a pair (X,V) to be the smallest admissible algebraic set $Y\subset X$, i.e.

$$\mathrm{ECL}(X,V)=\overline{\bigcup_{f:(\mathbb{C},T_{\mathbb{C}})\to(X,V)} f(\mathbb{C})}^{\mathrm{Zar}}. \tag{0.4}$$

In case V has no singularities, the *canonical sheaf* K_V is defined to be $(\det\mathcal{O}(V))^*$ where $\mathcal{O}(V)$ is the sheaf of holomorphic sections of V, but in general this naive definition would not work. Take for instance a generic pencil of elliptic curves $\lambda P(z)+\mu Q(z)=0$ of degree 3 in $\mathbb{P}^2_{\mathbb{C}}$, and the linear space V consisting of the tangents to the fibers of the rational map $\mathbb{P}^2_{\mathbb{C}}\dashrightarrow\mathbb{P}^1_{\mathbb{C}}$ defined by $z\mapsto Q(z)/P(z)$. Then V is given by

$$0\longrightarrow\mathcal{O}(V)\longrightarrow\mathcal{O}(T_{\mathbb{P}^2_{\mathbb{C}}})\xrightarrow{PdQ-QdP}\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(6)\otimes\mathcal{J}_S\longrightarrow 0$$

where $S=\mathrm{Sing}(V)$ consists of the 9 points $\{P(z)=0\}\cap\{Q(z)=0\}$, and $\mathcal{J}_S$ is the corresponding ideal sheaf of S. Since $\det\mathcal{O}(T_{\mathbb{P}^2})=\mathcal{O}(3)$, we see that $(\det(\mathcal{O}(V))^*=\mathcal{O}(3)$ is ample, thus Problem 0.3 would not have a positive answer (all leaves are elliptic or singular rational curves and thus covered by entire curves). An even more “degenerate” example is obtained with a generic pencil of conics, in which case $(\det(\mathcal{O}(V))^*=\mathcal{O}(1)$ and $\#S=4$.

If we want to get a positive answer to Problem 0.3, it is therefore indispensable to give a definition of K_V that incorporates in a suitable way the singularities of V; this will be done in Def. 1.2. The goal is then to give a positive answer to Problem 0.3 under some possibly more restrictive conditions for the pair (X,V). These conditions will be expressed in terms of the tower of Semple jet bundles

$$(X_k,V_k)\to(X_{k-1},V_{k-1})\to\ldots\to(X_1,V_1)\to(X_0,V_0):=(X,V) \tag{0.5}$$

which we define more precisely in Section 1, following [Dem95]. It is constructed inductively by setting $X_k=P(V_{k-1})$ (projective bundle of *lines* of V_{k-1}), and all V_k have the same rank $r=\mathrm{rank}\,V$, so that $\dim X_k=n+k(r-1)$ where $n=\dim X$. If $\mathcal{O}_{X_k}(1)$ is the tautological line bundle over X_k associated with the projective structure, we define the k-stage Green-Griffiths locus of (X,V) to be

$$\mathrm{GG}_k(X,V)=\bigcap_{m\in\mathbb{N}}\big(\text{base locus of } \mathcal{O}_{X_k}(m)\otimes\pi_{k,0}^*A^{-1}\big) \tag{0.6}$$

where A is any ample line bundle on X and $\pi_{k,\ell}:X_k\to X_\ell$ is the natural projection from X_k to X_ℓ, $0\leqslant\ell\leqslant k$. Clearly, $\mathrm{GG}_k(X,V)$ does not depend on the choice of A. The basic vanishing theorem for entire curves (cf. [GG79], [SY96] and [Dem95]) asserts that for every entire curve $f:(\mathbb{C},T_{\mathbb{C}})\to(X,V)$, then its k-jet $f_{[k]}:(\mathbb{C},T_{\mathbb{C}})\to(X_k,V_k)$ satisfies

$$f_{[k]}(\mathbb{C})\subset\mathrm{GG}_k(X,V),\quad\text{hence}\quad f(\mathbb{C})\subset\pi_{k,0}\left(\mathrm{GG}_k(X,V)\right). \tag{0.7}$$

It is therefore natural to define the global Green-Griffiths locus of (X,V) to be

$$\mathrm{GG}(X,V)=\bigcap_{k\in\mathbb{N}}\pi_{k,0}\left(\mathrm{GG}_k(X,V)\right). \tag{0.8}$$

By (0.7) we infer that

$$\mathrm{ECL}(X,V)\subset\mathrm{GG}(X,V). \tag{0.9}$$

The main result of [Dem11] (Theorem 2.37 and Cor. 2.38) implies the following very useful information:

0.10. Theorem. *Assume that (X,V) is of "general type", i.e. that the canonical sheaf K_V is big on X. Then there exists an integer k_0 such that $\mathrm{GG}_k(X,V)$ is a proper algebraic subset of X_k for $k\geqslant k_0$ [though $\pi_{k,0}(\mathrm{GG}_k(X,V))$ might still be equal to X for all k].*

In fact, if F is an invertible sheaf on X such that $K_V\otimes F$ is big, the probabilistic estimates of [Dem11, Cor. 2.38] produce sections of

$$\mathcal{O}_{X_k}(m)\otimes\pi_{k,0}^*\mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big) \tag{0.11}$$

for $m\gg k\gg 1$. The (long and involved) proof uses a curvature computation and singular holomorphic Morse inequalities to show that the line bundles involved in (0.11) are big on X_k for $k\gg 1$. One applies this to $F=A^{-1}$ with A ample on X to produce sections and conclude that $\mathrm{GG}_k(X,V)\subsetneq X_k$.

Thanks to (0.9), the GGL conjecture is satisfied whenever $\mathrm{GG}(X,V)\subsetneq X$. By [DMR10], this happens for instance in the absolute case when X is a generic hypersurface of degree $d\geqslant 2^{n^5}$ in $\mathbb{P}^{n+1}$. However, as [Lan86] already mentioned, very simple examples show that one can have $\mathrm{GG}(X,V)=X$ even when (X,V) is of general type, and this already occurs in the absolute case as soon as $\dim X\geqslant 2$. A typical example is a product of directed manifolds

$$(X,V)=(X',V')\times(X'',V''),\qquad V=\mathrm{pr}'^*V'\oplus\mathrm{pr}''^*V''. \tag{0.12}$$

The absolute case $V=T_X$, $V'=T_{X'}$, $V''=T_{X''}$ on a product of curves is the simplest instance. It is then easy to check that $\mathrm{GG}(X,V)=X$, cf. (3.2). Diverio and Rousseau [DR13] have given many more such examples, including the case of indecomposable varieties (X,T_X), e.g. Hilbert modular surfaces, or more generally compact quotients of bounded symmetric domains of rank $\geqslant 2$. The problem here is the failure of some sort of stability condition that is introduced in Section 3. This leads to a somewhat technical concept of more manageable directed pairs (X,V) that we call *strongly of general type*, see Def. 3.1. Our main result can be stated

0.13. Theorem (partial solution to the GGL conjecture). *Let (X,V) be a directed pair that is strongly of general type. Then the Green-Griffiths-Lang conjecture holds true for (X,V), namely $\mathrm{ECL}(X,V)$ is a proper algebraic subvariety of X.*

The proof proceeds through a complicated induction on $n=\dim X$ and $k=\operatorname{rank} V$, which is the main the main reason why we have to introduce directed varieties, even in the absolute case. An interesting feature of this result is that the conclusion on $\mathrm{ECL}(X,V)$ is reached without implying to know anything about the Green-Griffiths locus $\mathrm{GG}(X,V)$, even a posteriori. Nevetherless, this is not yet enough to confirm the GGL conjecture. Our hope is that pairs (X,V) that are of general type without being strongly of general type – and thus exhibit some sort of "jet-instability" – can be investigated by different methods, e.g. by the diophantine approximation techniques of McQuillan [McQ98].

1. Semple jet bundles and associated canonical sheaves

Let (X,V) be a directed projective manifold and $r=\operatorname{rank} V$, that is, the dimension of generic fibers. Then V is actually a holomorphic subbundle of T_X on the complement $X\smallsetminus V_{\rm sing}$ of a certain minimal analytic set $V_{\rm sing}\subsetneq X$, called hereafter the singular set of V. If $\mu:\widehat X\to X$ is a proper modification (a composition of blow-ups with smooth centers, say), we get a directed manifold $(\widehat X,\widehat V)$ by taking $\widehat V$ to be the closure of $\mu_*^{-1}(V')$, where $V'=V_{|X'}$ is the restriction of V over a Zariski open set $X'\subset X\smallsetminus V_{\rm sing}$ such that $\mu:\mu^{-1}(X')\to X'$ is a biholomorphism. We will be interested in taking modifications realized by iterated blow-ups of certain nonsingular subvarieties of the singular set $V_{\rm sing}$, so as to eventually "improve" the singularities of V; outside of $V_{\rm sing}$ the effect of blowing-up will be irrelevant, as one can see easily.

Following [Dem11], the canonical sheaf K_V is defined to be the rank 1 analytic sheaf such that

$$K_V(U)=\text{sheaf of locally bounded sections of }\ \mathcal O_X(\Lambda^r V'^*)(U\cap X') \tag{1.1}$$

where $r=\operatorname{rank}(V)$, $X'=X\smallsetminus V_{\rm sing}$, $V'=V_{|X'}$, and "bounded" means bounded with respect to a smooth hermitian metric h on T_X. This is easily seen to be the same as $\mathcal L_V\otimes\overline{\mathcal J}_V$, where $\mathcal L_V$ is the invertible sheaf $\mathcal O_X(\Lambda^rV^*)^{**}$, $\mathcal J_V\subset\mathcal O_X$ is the ideal sheaf such that $\mathcal L_V\otimes\mathcal J_V$ is the image of the natural morphism $\mathcal O(\Lambda^rT^*_X)\to\mathcal O(\Lambda^rV^*)\to\mathcal L_V$, and $\overline{\mathcal J}_V$ is the integral closure of $\mathcal J_V$ in $\mathcal O_X$. The reason is that the integral closure $\overline{\mathcal G}$ of a coherent ideal $\mathcal G=(g_k)\subset\mathcal O_X$ can be defined analytically as the set of germs f such that $|f|\leqslant C\sum|g_k|$ for some constant $C\geqslant 0$. It follows in particular that K_V is a coherent sheaf. By blowing-up $\mathcal J_V$ and taking a desingularization $\widehat X$, one can always find a *log-resolution* of $\mathcal J_V$ (or K_V), i.e. a modification $\mu:\widehat X\to X$ such that $\mu^*\mathcal J_V\subset\mathcal O_{\widehat X}$ is an invertible ideal sheaf (hence integrally closed); it follows that $\mu^*\overline{\mathcal J}_V=\mu^*\mathcal J_V$ and $\mu^*K_V=\mu^*\mathcal L_V\otimes\mu^*\mathcal J_V$ are invertible sheaves on $\widehat X$. Notice that for any modification $\mu':(X',V')\to(X,V)$, there is always a well defined natural morphism

$$\mu'^*K_V\to K_{V'} \tag{1.2}$$

(though it need not be an isomorphism, and $K_{V'}$ is possibly non invertible even when μ' is taken to be a log-resolution of K_V). Indeed $(\mu')_* = d\mu' : V' \to \mu^* V$ is continuous with respect to ambient hermitian metrics on X and X', and going to the dual reverses the arrows while preserving boundedness with respect to the metrics. If $\mu'' : X'' \to X'$ provides a simultaneous log-resolution of $K_{V'}$ and $\mu'^* K_V$, we get a non trivial morphism of invertible sheaves

$$(\mu' \circ \mu'')^* K_V = \mu''^* \mu'^* K_V \longrightarrow \mu''^* K_{V'}, \tag{1.3}$$

hence the bigness of $\mu'^* K_V$ with imply that of $\mu''^* K_{V'}$.

1.4. Definition. *We say that the rank 1 sheaf K_V is "big" if the invertible sheaf $\mu^* K_V$ is big in the usual sense for any log resolution $\mu : \widehat{X} \to X$ of K_V. Finally, we say that (X, V) is of general type if there exists a modification $\mu' : (X', V') \to (X, V)$ such that $K_{V'}$ is big ; any higher blow-up $\mu'' : (X'', V'') \to (X', V')$ then also yields a big canonical sheaf by* (1.2).

Clearly, "general type" is a birationally (or bimeromorphically) invariant concept, by the very definition. When $\dim X = n$ and $V \subset T_X$ is a subbundle of rank r, one constructs a tower of "Semple k-jet bundles" $\pi_{k,k-1} : (X_k, V_k) \to (X_{k-1}, V_{k-1})$ that are $\mathbb{P}^{r-1}$-bundles, with $\dim X_k = n + k(r-1)$ and $\operatorname{rank}(V_k) = r$. For this, we take $(X_0, V_0) = (X, V)$, and for every $k \geqslant 1$, we set inductively $X_k := P(V_{k-1})$ and

$$V_k := (\pi_{k,k-1})_*^{-1} \mathcal{O}_{X_k}(-1) \subset T_{X_k},$$

where $\mathcal{O}_{X_k}(1)$ is the tautological line bundle on X_k, $\pi_{k,k-1} : X_k = P(V_{k-1}) \to X_{k-1}$ the natural projection and $(\pi_{k,k-1})_* = d\pi_{k,k-1} : T_{X_k} \to \pi_{k,k-1}^* T_{X_{k-1}}$ its differential (cf. [Dem95]). In other terms, we have exact sequences

$$0 \longrightarrow T_{X_k/X_{k-1}} \longrightarrow V_k \stackrel{(\pi_{k,k-1})_*}{\longrightarrow} \mathcal{O}_{X_k}(-1) \longrightarrow 0, \tag{1.5}$$

$$0 \longrightarrow \mathcal{O}_{X_k} \longrightarrow \pi^* V_{k-1} \otimes \mathcal{O}_{X_k}(1) \longrightarrow T_{X_k/X_{k-1}} \longrightarrow 0, \tag{1.6}$$

where the last line is the Euler exact sequence associated with the relative tangent bundle of $P(V_{k-1}) \to X_{k-1}$. Notice that we by definition of the tautological line bundle we have

$$\mathcal{O}_{X_k}(-1) \subset \pi_{k,k-1}^* V_{k-1} \subset \pi_{k,k-1}^* T_{X_{k-1}},$$

and also $\operatorname{rank}(V_k) = r$. Let us recall also that for $k \geqslant 2$, there are "vertical divisors" $D_k = P(T_{X_{k-1}/X_{k-2}}) \subset P(V_{k-1}) = X_k$, and that D_k is the zero divisor of the section of $\mathcal{O}_{X_k}(1) \otimes \pi_{k,k-1}^* \mathcal{O}_{X_{k-1}}(-1)$ induced by the second arrow of the first exact sequence (1.5), when k is replaced by $k-1$. This yields in particular

$$\mathcal{O}_{X_k}(1) = \pi_{k,k-1}^* \mathcal{O}_{X_{k-1}}(1) \otimes \mathcal{O}(D_k). \tag{1.7}$$

By composing the projections we get for all pairs of indices $0 \leqslant j \leqslant k$ natural morphisms

$$\pi_{k,j} : X_k \to X_j, \quad (\pi_{k,j})_* = (d\pi_{k,j})_{|V_k} : V_k \to (\pi_{k,j})^* V_j,$$

and for every k-tuple $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k$ we define

$$\mathcal{O}_{X_k}(\mathbf{a}) = \bigotimes_{1 \leqslant j \leqslant k} \pi_{k,j}^* \mathcal{O}_{X_j}(a_j), \quad \pi_{k,j} : X_k \to X_j.$$

We extend this definition to all weights $\mathbf{a} \in \mathbb{Q}^k$ to get a $\mathbb{Q}$-line bundle in $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Now, Formula (1.7) yields

$$\mathcal{O}_{X_k}(\mathbf{a}) = \mathcal{O}_{X_k}(m) \otimes \mathcal{O}(-\mathbf{b} \cdot D) \quad \text{where} \quad m = |\mathbf{a}| = \sum a_j,\ \mathbf{b} = (0, b_2, \ldots, b_k) \tag{1.8}$$

and $b_j = a_1 + \ldots + a_{j-1}$, $2 \leqslant j \leqslant k$.

When $V_{\mathrm{sing}} \neq \emptyset$, one can always define X_k and V_k to be the respective closures of X'_k, V'_k associated with $X' = X \smallsetminus V_{\mathrm{sing}}$ and $V' = V_{|X'}$, where the closure is taken in the nonsingular "absolute" Semple tower $(\mathcal{X}_k, \mathcal{A}_k)$ obtained from $(\mathcal{X}_0, \mathcal{A}_0) = (X, T_X)$. We can then replace (X_k, V_k) by a modification $(\widehat{X}_k, \widehat{V}_k)$ if we want to work with a non singular model $\widehat{X}_k$ of X_k. The exceptional set of $\widehat{X}_k$ over X_k can be chosen to lie above $V_{\mathrm{sing}} \subset X$, and proceeding inductively with respect to k, we can also arrange the modifications in such a way that we get a tower structure $(\widehat{X}_{k+1}, \widehat{V}_{k+1}) \to (\widehat{X}_k, \widehat{V}_k)$; however, in general, it will not be possible to achieve that $\widehat{V}_k$ is a subbundle of T_{X_k}.

It is not true that $K_{\widehat{V}_k}$ is big in case (X, V) is of general type (especially since the fibers of $X_k \to X$ are towers of $\mathbb{P}^{r-1}$ bundles, and the canonical bundles of projective spaces are always negative !). However, a "twisted version" holds true.

1.9. Lemma. *If (X, V) is of general type, then all pairs (X_k, V_k) have a twisted canonical bundle $K_{V_k} \otimes \mathcal{O}_{X_k}(p)$ that is still big when one multiplies K_{V_k} by a suitable $\mathbb{Q}$-line bundle $\mathcal{O}_{X_k}(p)$, $p \in \mathbb{Q}_+$.*

Proof. First assume that V has no singularities. The exact sequences (1.5) and (1.6) provide

$$K_{V_k} := \det V_k^* = \det(T^*_{X_k/X_{k-1}}) \otimes \mathcal{O}_{X_k}(1) = \pi^*_{k,k-1} K_{V_{k-1}} \otimes \mathcal{O}_{X_k}(-(r-1))$$

where $r = \mathrm{rank}(V)$. Inductively we get

$$K_{V_k} = \pi^*_{k,0} K_V \otimes \mathcal{O}_{X_k}(-(r-1)\mathbf{1}), \qquad \mathbf{1} = (1, ..., 1) \in \mathbb{N}^k. \tag{1.10}$$

We know by [Dem95] that $\mathcal{O}_{X_k}(\mathbf{c})$ is relatively ample over X when we take the special weight $\mathbf{c} = (2\,3^{k-2}, ..., 2\,3^{k-j-1}, ..., 6, 2, 1)$, hence

$$K_{V_k} \otimes \mathcal{O}_{X_k}((r-1)\mathbf{1} + \varepsilon\mathbf{c}) = \pi^*_{k,0} K_V \otimes \mathcal{O}_{X_k}(\varepsilon\mathbf{c})$$

is big over X_k for any sufficiently small positive rational number $\varepsilon \in \mathbb{Q}^*_+$. Thanks to Formula (1.8), we can in fact replace the weight $(r-1)\mathbf{1}+\varepsilon\mathbf{c}$ by its total degree $p = (r-1)k+\varepsilon|\mathbf{c}| \in \mathbb{Q}_+$. The general case of a singular linear space follows by considering suitable modifications of X (notice that the condition for a section of $\widehat{\mathcal{V}}^*_k$ to be bounded with respect to the dual of a metric on $\mathcal{X}_k$ is less restrictive than saying that is is bounded with respect to the dual of the pull-back of a metric taken downstairs on a birational model $\mathcal{X}$ of X, so we eventually get even more sections than those coming from downstairs in the above formula). □

2. Induced directed structure on a subvariety of a jet space.

Let Z be an irreducible algebraic subset of some k-jet bundle X_k over X, such that Z projects onto X_{k-1}, i.e. $\pi_{k,k-1}(Z) = X_{k-1}$. We define the linear subspace $W \subset T_Z \subset T_{X_k|Z}$ to be the closure

$$W := \overline{T_{Z'} \cap V_k} \tag{2.1}$$

taken on a suitable Zariski open set $Z' \subset Z_{\rm reg}$ where the intersection $T_{Z'} \cap V_k$ has constant rank and is a subbundle of $T_{Z'}$. Alternatively, we could also take W to be the closure of $T_{Z'} \cap V_k$ in the k-th stage $(\mathcal{X}_k, \mathcal{A}_k)$ of the absolute Semple tower. We say that (Z, W) is the *induced* directed variety structure. In the sequel, we always consider such a subvariety Z of X_k as a directed pair (Z, W) by taking the induced structure described above. Let us first quote the following easy observation.

2.2. Observation. *For $k \geqslant 1$, let $Z \subsetneq X_k$ be an irreducible algebraic subset of X_k that projects onto X_{k-1}. Then the induced directed structure $(Z, W) \subset (X_k, V_k)$, satisfies*

$$1 \leqslant \operatorname{rank} W < r := \operatorname{rank}(V_k).$$

Proof. Take a Zariski open subset $Z' \subset Z_{\rm reg}$ such that $W' = T_{Z'} \cap V_k$ is a vector bundle over Z'. Since $X_k \to X_{k-1}$ is a $\mathbb{P}^{r-1}$-bundle, Z has codimension at most $r-1$ in X_k. Therefore $\operatorname{rank} W \geqslant \operatorname{rank} V_k - (r-1) \geqslant 1$. On the other hand, if we had $\operatorname{rank} W = \operatorname{rank} V_k$ generically, then $T_{Z'}$ would contain $V_{k|Z'}$, in particular it would contain all vertical directions $T_{X_k/X_{k-1}} \subset V_k$ that are tangent to the fibers of $X_k \to X_{k-1}$. By taking the flow along vertical vector fields, we would conclude that Z' is a union of fibers of $X_k \to X_{k-1}$ up to an algebraic set of smaller dimension, but this is excluded since Z projects onto X_{k-1} and $Z \subsetneq X_k$. □

2.3. Definition. *For $k \geqslant 1$, let $Z \subset X_k$ be an irreducible algebraic subset of X_k that projects onto X_{k-1}. We assume moreover that $Z \not\subset D_k = P(T_{X_{k-1}/X_{k-2}})$ (and put here $D_1 = \emptyset$ in what follows to avoid to have to single out the case $k = 1$). In this situation we say that (Z, W) is of general type modulo $X_k \to X$ if there exists $p \in \mathbb{Q}_+$ such that $K_W \otimes \mathcal{O}_{X_k}(p)_{|Z}$ is big over Z, possibly after replacing Z by a suitable nonsingular model $\widehat{Z}$ (and pulling-back W and $\mathcal{O}_{X_k}(p)_{|Z}$ to the non singular variety $\widehat{Z}$).*

The main result of [Dem11] mentioned in the introduction as Theorem 0.10 implies the following important “induction step”.

2.4. Proposition. *Let (X, V) be a directed pair where X is projective algebraic. Take an irreducible algebraic subset $Z \not\subset D_k$ of the associated k-jet Semple bundle X_k that projects onto X_{k-1}, $k \geqslant 1$, and assume that the induced directed space $(Z, W) \subset (X_k, V_k)$ is of general type modulo $X_k \to X$. Then there exists a divisor $\Sigma \subset Z_\ell$ in a sufficiently high stage of the Semple tower (Z_ℓ, W_ℓ) associated with (Z, W), such that every non constant holomorphic map $f : \mathbb{C} \to X$ tangent to V that satisfies $f_{[k]}(\mathbb{C}) \subset Z$ also satisfies $f_{[k+\ell]}(\mathbb{C}) \subset \Sigma$.*

Proof. Let $E \subset Z$ be a divisor containing $Z_{\rm sing} \cup (Z \cap \pi_{k,0}^{-1}(V_{\rm sing}))$, chosen so that on the nonsingular Zariski open set $Z' = Z \smallsetminus E$ all linear spaces $T_{Z'}$, $V_{k|Z'}$ and $W' = T_{Z'} \cap V_k$ are subbundles of $T_{X_k|Z'}$, the first two having a transverse intersection on Z'. By taking closures over Z' in the absolute Semple tower of X, we get (singular) directed pairs $(Z_\ell, W_\ell) \subset (X_{k+\ell}, V_{k+\ell})$, which we eventually resolve into $(\widehat{Z}_\ell, \widehat{W}_\ell) \subset (\widehat{X}_{k+\ell}, \widehat{V}_{k+\ell})$ over nonsingular bases. By construction, locally bounded sections of $\mathcal{O}_{\widehat{X}_{k+\ell}}(m)$ restrict to locally bounded sections of $\mathcal{O}_{\widehat{Z}_\ell}(m)$ over $\widehat{Z}_\ell$.

Since Theorem 0.10 and the related estimate (0.11) are universal in the category of directed varieties, we can apply them by replacing X with $\widehat{Z} \subset \widehat{X}_k$, the order k by a new index ℓ, and F by

$$F_k = \mu^*\Big(\big(\mathcal{O}_{X_k}(p) \otimes \pi_{k,0}^* \mathcal{O}_X(-\varepsilon A)\big)_{|Z}\Big)$$

where $\mu : \widehat{Z} \to Z$ is the desingularization, $p \in \mathbb{Q}_+$ is chosen such that $K_W \otimes \mathcal{O}_{x_k}(p)_{|Z}$ is big, A is an ample bundle on X and $\varepsilon \in \mathbb{Q}_+^*$ is small enough. The assumptions show that $K_{\widehat{W}} \otimes F_k$ is big on $\widehat{Z}$, therefore, by applying our theorem and taking $m \gg \ell \gg 1$, we get in fine a large number of (metric bounded) sections of

$$\mathcal{O}_{\widehat{Z}_\ell}(m) \otimes \widehat{\pi}^*_{k+\ell,k}\mathcal{O}\Big(\frac{m}{\ell r'}\Big(1+\frac{1}{2}+\ldots+\frac{1}{\ell}\Big)F_k\Big)$$
$$= \mathcal{O}_{\widehat{X}_{k+\ell}}(m\mathbf{a}') \otimes \widehat{\pi}^*_{k+\ell,0}\mathcal{O}\Big(-\frac{m\varepsilon}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)A\Big)_{|\widehat{Z}_\ell}$$

where $\mathbf{a}' \in \mathbb{Q}_+^{k+\ell}$ is a positive weight (of the form $(0,\ldots,\lambda,\ldots,0,1)$ with some non zero component $\lambda \in \mathbb{Q}_+$ at index k). These sections descend to metric bounded sections of

$$\mathcal{O}_{X_{k+\ell}}((1+\lambda)m) \otimes \widehat{\pi}^*_{k+\ell,0}\mathcal{O}\Big(-\frac{m\varepsilon}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)A\Big)_{|Z_\ell}.$$

Since A is ample on X, we can apply the fundamental vanishing theorem (see e.g. [Dem97] or [Dem11], Statement 8.15), or rather an "embedded" version for curves satisfying $f_{[k]}(\mathbb{C}) \subset Z$, proved exactly by the same arguments. The vanishing theorem implies that the divisor Σ of any such section satisfies the conclusions of Proposition 2.4, possibly modulo exceptional divisors of $\widehat{Z} \to Z$; to take care of these, it is enough to add to Σ the inverse image of the divisor $E = Z \smallsetminus Z'$ initially selected. □

3. Strong general type condition for directed manifolds

Our main result is the following partial solution to the Green-Griffiths-Lang conjecture, providing a sufficient algebraic condition for the analytic conclusion to hold true. We first give an ad hoc definition.

3.1. Definition. *Let (X,V) be a directed pair where X is projective algebraic. We say that that (X,V) is "strongly of general type" if it is of general type and for every irreducible algebraic set $Z \subsetneq X_k$, $Z \not\subset D_k$, that projects onto X_{k-1}, $k \geqslant 1$, the induced directed structure $(Z,W) \subset (X_k,V_k)$ is of general type modulo $X_k \to X$.*

3.2. Example. The situation of a product $(X,V) = (X',V') \times (X'',V'')$ described in (0.12) shows that (X,V) can be of general type without being strongly of general type. In fact, if (X',V') and (X'',V'') are of general type, then $K_V = \mathrm{pr}'^*\,K_{V'} \otimes \mathrm{pr}''^*\,K_{V''}$ is big, so (X,V) is again of general type. However

$$Z = P(\mathrm{pr}'^*\,V') = X'_1 \times X'' \subset X_1$$

has a directed structure $W = \mathrm{pr}'^*\,V'_1$ which does not possess a big canonical bundle over Z, since the restriction of K_W to any fiber $\{x'\} \times X''$ is trivial. The higher stages (Z_k,W_k) of the Semple tower of (Z,W) are given by $Z_k = X'_{k+1} \times X''$ and $W_k = \mathrm{pr}'^*\,V'_{k+1}$, so it is easy to see that $\mathrm{GG}_k(X,V)$ contains Z_{k-1}. Since Z_k projects onto X, we have here $\mathrm{GG}(X,V) = X$ (see [DR13] for more sophisticated indecomposable examples).

3.3. Remark. It follows from Definition 2.3 that $(Z,W) \subset (X_k,V_k)$ is automatically of general type modulo $X_k \to X$ if $\mathcal{O}_{X_k}(1)_{|Z}$ is big. Notice further that

$$\mathcal{O}_{X_k}(1+\varepsilon)_{|Z} = \big(\mathcal{O}_{X_k}(\varepsilon) \otimes \pi^*_{k,k-1}\mathcal{O}_{X_{k-1}}(1) \otimes \mathcal{O}(D_k)\big)_{|Z}$$

where $\mathcal{O}(D_k)_{|Z}$ is effective and $\mathcal{O}_{X_k}(1)$ is relatively ample with respect to the projection $X_k \to X_{k-1}$. Therefore the bigness of $\mathcal{O}_{X_{k-1}}(1)$ on X_{k-1} also implies that every directed subvariety $(Z, W) \subset (X_k, V_k)$ is of general type modulo $X_k \to X$. If (X, V) is of general type, we know by the main result of [Dem11] that $\mathcal{O}_{X_k}(1)$ is big for $k \geqslant k_0$ large enough, and actually the precise estimates obtained therein give explicit bounds for such a k_0. The above observations show that we need to check the condition of Definition 3.1 only for $Z \subset X_k$, $k \leqslant k_0$. Moreover, at least in the case where V, Z, and $W = T_Z \cap V_k$ are nonsingular, we have

$$K_W \simeq K_Z \otimes \det(T_Z/W) \simeq K_Z \otimes \det(T_{X_k}/V_k)_{|Z} \simeq K_{Z/X_{k-1}} \otimes \mathcal{O}_{X_k}(1)_{|Z}.$$

Thus we see that, in some sense, it is only needed to check the bigness of K_W modulo $X_k \to X$ for "rather special subvarieties" $Z \subset X_k$ over X_{k-1}, such that $K_{Z/X_{k-1}}$ is not relatively big over X_{k-1}. □

3.4. Hypersurface case. Assume that $Z \neq D_k$ is an irreducible hypersurface of X_k that projects onto X_{k-1}. To simplify things further, also assume that V is nonsingular. Since the Semple jet-bundles X_k form a tower of $\mathbb{P}^{r-1}$-bundles, their Picard groups satisfy $\mathrm{Pic}(X_k) \simeq \mathrm{Pic}(X) \oplus \mathbb{Z}^k$ and we have $\mathcal{O}_{X_k}(Z) \simeq \mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{k,0}^* B$ for some $\mathbf{a} \in \mathbb{Z}^k$ and $B \in \mathrm{Pic}(X)$, where $a_k = d > 0$ is the relative degree of the hypersurface over X_{k-1}. Let $\sigma \in H^0(X_k, \mathcal{O}_{X_k}(Z))$ be the section defining Z in X_k. The induced directed variety (Z, W) has $\operatorname{rank} W = r-1 = \operatorname{rank} V - 1$ and formula (1.10) yields $K_{V_k} = \mathcal{O}_{X_k}(-(r-1)\mathbf{1}) \otimes \pi_{k,0}^*(K_V)$. We claim that

$$K_W \supset \big(K_{V_k} \otimes \mathcal{O}_{X_k}(Z)\big)_{|Z} \otimes \mathcal{J}_S = \big(\mathcal{O}_{X_k}(\mathbf{a} - (r-1)\mathbf{1}) \otimes \pi_{k,0}^*(B \otimes K_V)\big)_{|Z} \otimes \mathcal{J}_S \tag{3.5}$$

where $S \subsetneq Z$ is the set (containing Z_{sing}) where σ and $d\sigma_{|V_k}$ both vanish, and $\mathcal{J}_S$ is the ideal locally generated by the coefficients of $d\sigma_{|V_k}$ along $Z = \sigma^{-1}(0)$. In fact, the intersection $W = T_Z \cap V_k$ is transverse on $Z \smallsetminus S$; then (3.5) can be seen by looking at the morphism

$$V_{k|Z} \xrightarrow{d\sigma_{|V_k}} \mathcal{O}_{X_k}(Z)_{|Z},$$

and observing that the contraction by $K_{V_k} = \Lambda^r V_k^*$ provides a metric bounded section of the canonical sheaf K_W. In order to investigate the positivity properties of K_W, one has to show that B cannot be too negative, and in addition to control the singularity set S. The second point is a priori very challenging, but we get useful information for the first point by observing that σ provides a morphism $\pi_{k,0}^*\mathcal{O}_X(-B) \to \mathcal{O}_{X_k}(\mathbf{a})$, hence a nontrivial morphism

$$\mathcal{O}_X(-B) \to E_{\mathbf{a}} := (\pi_{k,0})_*\mathcal{O}_{X_k}(\mathbf{a})$$

By [Dem95, Section 12], there exists a filtration on $E_{\mathbf{a}}$ such that the graded pieces are irreducible representations of $\mathrm{GL}(V)$ contained in $(V^*)^{\otimes \ell}$, $\ell \leqslant |\mathbf{a}|$. Therefore we get a nontrivial morphism

$$\mathcal{O}_X(-B) \to (V^*)^{\otimes \ell}, \qquad \ell \leqslant |\mathbf{a}|. \tag{3.6}$$

If we know about certain (semi-)stability properties of V, this can be used to control the negativity of B. □

We further need the following useful concept that generalizes entire curve loci.

3.7. Definition. *If Z is an algebraic set contained in some stage X_k of the Semple tower of (X,V), we define its "induced entire curve locus" $\mathrm{IEL}(Z) \subset Z$ to be the Zariski closure of the union $\bigcup f_{[k]}(\mathbb{C})$ of all jets of entire curves $f:(\mathbb{C},T_{\mathbb{C}}) \to (X,V)$ such that $f_{[k]}(\mathbb{C}) \subset Z$.*

We have of course $\mathrm{IEL}(\mathrm{IEL}(Z)) = \mathrm{IEL}(Z)$ by definition. It is not hard to check that modulo certain "vertical divisors" of X_k, the $\mathrm{IEL}(Z)$ locus is essentially the same as the entire curve locus $\mathrm{ECL}(Z,W)$ of the induced directed variety, but we will not use this fact here. Since $\mathrm{IEL}(X) = \mathrm{ECL}(X,V)$, Proving the Green-Griffiths-Lang property amounts to showing that $\mathrm{IEL}(X) \subsetneq X$ in the $k=0$ stage of the tower.

3.8. Theorem. *Let (X,V) be a directed pair of general type. Assume that there is an integer $k_0 \geqslant 0$ such that for every $k > k_0$ and every irreducible algebraic set $Z \subsetneq X_k$, $Z \not\subset D_k$, that projects onto X_{k-1}, the induced directed structure $(Z,W) \subset (X_k,V_k)$ is of general type modulo $X_k \to X$. Then $\mathrm{IEL}(X_{k_0}) \subsetneq X_{k_0}$.*

Proof. We argue here by contradiction, assuming that $\mathrm{IEL}(X_{k_0}) = X_{k_0}$. The main argument consists of producing inductively an increasing sequence of integers

$$k_0 < k_1 < \ldots < k_j < \ldots$$

and directed varieties $(Z^j, W^j) \subset (X_{k_j}, V_{k_j})$ satisfying the following properties :

(a) $(Z^0, W^0) = (X_{k_0}, V_{k_0})$;

(b) for all $j \geqslant 0$, $\mathrm{IEL}(Z^j) = Z^j$;

(c) Z^j is an irreducible algebraic variety such that $Z^j \subsetneq X_{k_j}$ for $j \geqslant 1$, Z^j is not contained in the vertical divisor $D_{k_j} = P(T_{X_{k_j-1}/X_{k_j-2}})$ of X_{k_j}, and (Z^j, W^j) is of general type modulo $X_{k_j} \to X$ (i.e. some nonsingular model is) ;

(d) for all $j \geqslant 0$, the directed variety (Z^{j+1}, W^{j+1}) is contained in some stage (of order $\ell_j = k_{j+1} - k_j$) of the Semple tower of (Z^j, W^j), namely

$$(Z^{j+1}, W^{j+1}) \subset (Z^j_{\ell_j}, W^j_{\ell_j}) \subset (X_{k_{j+1}}, V_{k_{j+1}})$$

and

$$W^{j+1} = \overline{T_{Z^{j+1\prime}} \cap W^j_{\ell_j}} = \overline{T_{Z^{j+1\prime}} \cap V_{k_j}}$$

is the induced directed structure.

(e) for all $j \geqslant 0$, we have $Z^{j+1} \subsetneq Z^j_{\ell_j}$ but $\pi_{k_{j+1},k_{j+1}-1}(Z^{j+1}) = Z^j_{\ell_j-1}$.

For $j=0$, we have nothing to do by our hypotheses. Assume that (Z^j, W^j) has been constructed. By Proposition 2.4, we get an algebraic subset $\Sigma \subsetneq Z^j_\ell$ in some stage of the semple tower (Z^j_ℓ) of Z^j such that every entire curve $f:(\mathbb{C},T_{\mathbb{C}}) \to (X,V)$ satisfying $f_{[k_j]}(\mathbb{C}) \subset Z^j$ also satisfies $f_{[k_j+\ell]}(\mathbb{C}) \subset \Sigma$. By definition, this implies the first inclusion in the sequence

$$Z^j = \mathrm{IEL}(Z^j) \subset \pi_{k_j+\ell,k_j}(\mathrm{IEL}(\Sigma)) \subset \pi_{k_j+\ell,k_j}(\Sigma) \subset Z^j$$

(the other ones being obvious), so we have in fact an equality throughout. Let (S_α) be the irreducible components of $\mathrm{IEL}(\Sigma)$. We have $\mathrm{IEL}(S_\alpha) = S_\alpha$ and one of the components S_α must already satisfy $\pi_{k_j+\ell,k_j}(S_\alpha) = Z^j = Z^j_0$. We take $\ell_j \in [1,\ell]$ to be the smallest order such that $Z^{j+1} := \pi_{k_j+\ell,k_j+\ell_j}(S_\alpha) \subsetneq Z^j_{\ell_j}$, and set $k_{j+1} = k_j + \ell_j > k_j$. By definition of ℓ_j, we have $\pi_{k_{j+1},k_{j+1}-1}(Z^{j+1}) = Z^j_{\ell_j-1}$, otherwise ℓ_j would not be minimal. The fact that

$\mathrm{IEL}(S_\alpha) = S_\alpha$ immediately implies $\mathrm{IEL}(Z^{j+1}) = Z^{j+1}$. Also Z^{j+1} cannot be contained in the vertical divisor $D_{k_{j+1}}$. In fact no irreducible algebraic set Z such that $\mathrm{IEL}(Z) = Z$ can be contained in a vertical divisor D_k, because $\pi_{k,k-2}(D_k)$ corresponds to stationary jets in X_{k-2} ; as every non constant curve f has non stationary points, its k-jet $f_{[k]}$ cannot be entirely contained in D_k. Finally, the induced directed structure (Z^{j+1}, W^{j+1}) must be of general type modulo $X_{k_{j+1}} \to X$, by the assumption that (X, V) is strongly of general type. The inductive procedure is therefore complete.

By Observation 2.2, we have

$$\mathrm{rank}\, W^j < \mathrm{rank}\, W^{j-1} < \ldots < \mathrm{rank}\, W^1 < \mathrm{rank}\, W^0 = \mathrm{rank}\, V.$$

After a sufficient number of iterations we reach $\mathrm{rank}\, W^j = 1$. In this situation the Semple tower of Z^j is trivial, $K_{W^j} = W^{j\,*} \otimes \overline{\mathcal{J}}_{W^j}$ is big, and Proposition 2.4 produces a divisor $\Sigma \subsetneq Z^j_\ell = Z^j$ containing all jets of entire curves with $f_{[k_j]}(\mathbb{C}) \subset Z^j$. This contradicts the fact that $\mathrm{IEL}(Z^j) = Z^j$. We have reached a contradiction, and the theorem is thus proved. □

3.9. Remark. As it proceeds by contradiction, the proof is unfortunately non constructive – especially it does not give any information on the degree of the locus $Y \subsetneq X_{k_0}$ whose existence is asserted. On the other hand, and this is a bit surprising, the conclusion is obtained even though the conditions to be checked do not involve cutting down the dimensions of the base loci of jet differentials; in fact, the contradiction is obtained even though the integers k_j may increase and $\dim Z^j$ may become very large.

The special case $k_0 = 0$ of Theorem 3.8 yields the following

3.10. Partial solution to the GGL conjecture. *Let (X, V) be a directed pair that is strongly of general type. Then the Green-Griffiths-Lang conjecture holds true for (X, V), namely $\mathrm{ECL}(X, V) \subsetneq X$, in other words there exists a proper algebraic variety $Y \subsetneq X$ such that every non constant holomorphic curve $f : \mathbb{C} \to X$ tangent to V satisfies $f(\mathbb{C}) \subset Y$.*

3.11. Remark. The condition that (X, V) is strongly of general type seems to be related to some sort of stability condition. We are unsure what is the most appropriate definition, but here is one that makes sense. Fix an ample divisor A on X. For every irreducible subvariety $Z \subset X_k$ that projects onto X_{k-1} for $k \geqslant 1$, and $Z = X = X_0$ for $k = 0$, we define the slope $\mu_A(Z, W)$ of the corresponding directed variety (Z, W) to be

$$\mu_A(Z, W) = \frac{\inf \lambda}{\mathrm{rank}\, W},$$

where λ runs over all rational numbers such that there exists $m \in \mathbb{Q}_+$ for which

$$K_W \otimes \big(\mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^* \mathcal{O}(\lambda A)\big)_{|Z} \quad \text{is big on } Z$$

(again, we assume here that $Z \not\subset D_k$ for $k \geqslant 2$). Notice that (X, V) is of general type if and only if $\mu_A(X, V) < 0$, and that $\mu_A(Z, W) = -\infty$ if $\mathcal{O}_{X_k}(1)_{|A}$ is big. Also, the proof of Lemma 1.9 shows that

$$\mu_A(X_k, V_k) \leqslant \mu_A(X_{k-1}, V_{k-1}) \leqslant \ldots \leqslant \mu_A(X, V) \quad \text{for all } k$$

(with $\mu_A(X_k, V_k) = -\infty$ for $k \geqslant k_0 \gg 1$ if (X, V) is of general type). We say that (X, V) is *A-jet-stable* (resp. *A-jet-semi-stable*) if $\mu_A(Z, W) < \mu_A(X, V)$ (resp. $\mu_A(Z, W) \leqslant \mu_A(X, V)$)

for all $Z \subsetneq X_k$ as above. It is then clear that if (X,V) is of general type and A-jet-semi-stable, then it is strongly of general type in the sense of Definition 3.1. It would be useful to have a better understanding of this condition of stability (or any other one that would have better properties). $\square$

3.12. Example: case of surfaces. Assume that X is a minimal complex surface of general type and $V = T_X$ (absolute case). Then K_X is nef and big and the Chern classes of X satisfy $c_1 \leqslant 0$ ($-c_1$ is big and nef) and $c_2 \geqslant 0$. The Semple jet-bundles X_k form here a tower of $\mathbb{P}^1$-bundles and $\dim X_k = k+2$. Since $\det V^* = K_X$ is big, the strong general type assumption of 3.8 and 3.10 need only be checked for irreducible hypersurfaces $Z \subset X_k$ distinct from D_k that project onto X_{k-1}, of relative degree m. The projection $\pi_{k,k-1} : Z \to X_{k-1}$ is a ramified $m:1$ cover. Putting $\mathcal{O}_{X_k}(Z) \simeq \mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{k,0}(B)$, $B \in \mathrm{Pic}(X)$, we can apply (3.5) to get an inclusion

$$K_W \supset \big(\mathcal{O}_{X_k}(\mathbf{a}-\mathbf{1}) \otimes \pi_{k,0}^*(B \otimes K_X)\big)_{|Z} \otimes \mathcal{J}_S, \qquad \mathbf{a} \in \mathbb{Z}^k,\ \ a_k = m.$$

Let us assume $k=1$ and $S=\emptyset$ to make things even simpler, and let us perform numerical calculations in the cohomology ring

$$H^\bullet(X_1,\mathbb{Z}) = H^\bullet(X)[u]/(u^2 + c_1 u + c_2), \qquad u = c_1(O_{X_1}(1))$$

(cf. [DEG00, Section 2] for similar calculations and more details). We have

$$Z \equiv mu + b \quad \text{where} \quad b = c_1(B) \quad \text{and} \quad K_W \equiv (m-1)u + b - c_1.$$

We are allowed here to add to K_W an arbitrary multiple $\mathcal{O}_{X_1}(p)$, $p \geqslant 0$, which we rather write $p = mt + 1 - m$, $t \geqslant 1 - 1/m$. An evaluation of the Euler-Poincaré characteristic of $K_W + \mathcal{O}_{X_1}(p)_{|Z}$ requires computing the intersection number

$$\begin{aligned}\big(K_W + \mathcal{O}_{X_1}(p)_{|Z}\big)^2 \cdot Z &= \big(mt\,u + b - c_1\big)^2(mu+b)\\ &= m^2t^2\big(m(c_1^2 - c_2) - bc_1\big) + 2mt(b - mc_1)(b-c_1) + m(b-c_1)^2,\end{aligned}$$

taking into account that $u^3 \cdot X_1 = c_1^2 - c_2$. In case $S \neq \emptyset$, there is an additional (negative) contribution from the ideal $\mathcal{J}_S$ which is $O(t)$ since S is at most a curve. In any case, for $t \gg 1$, the leading term in the expansion is $m^2t^2(m(c_1^2 - c_2) - bc_1)$ and the other terms are negligible with respect to t^2, including the one coming from S. We know that T_X is semistable with respect to $c_1(K_X) = -c_1 \geqslant 0$. Multiplication by the section σ yields a morphism $\pi_{1,0}^*\mathcal{O}_X(-B) \to \mathcal{O}_{X_1}(m)$, hence by direct image, a morphism $\mathcal{O}_X(-B) \to S^m T_X^*$. Evaluating slopes against K_X (a big nef class), the semistability condition implies $bc_1 \leqslant \frac{m}{2}c_1^2$, and our leading term is bigger that $m^3t^2(\frac{1}{2}c_1^2 - c_2)$. We get a positive anwer in the well-known case where $c_1^2 > 2c_2$, corresponding to T_X being almost ample. Analyzing positivity for the full range of values (k,m,t) and of singular sets S seems an unsurmountable task at this point; in general, calculations made in [DEG00] and [McQ99] indicate that the Chern class and semistability conditions become less demanding for higher order jets (e.g. $c_1^2 > c_2$ is enough for $Z \subset X_2$, and $c_1^2 > \frac{9}{13}c_2$ suffices for $Z \subset X_3$). However, when $\mathrm{rank}\, V = 1$, the major gains come from the use of Ahlfors currents in combination with McQuillan's tautological inequalities [McQ98]. We therefore hope for a substantial strengthening of the above sufficient conditions, and a better understanding of the stability issues, possibly in combination with a use of Ahlfors currents and tautological inequalities.

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