Report of “ Recent results on the Kobayashi and Green-Griffiths-Lang Conjectures”

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The present article surveys the development of the titled conjectures and related topics, especially the Kobayashi conjecture which claims the Kobayashi hyperbolicity of generic hypersurfaces of high degree in complex projective space. The main theme is the notion of “directed manifolds” due to the present author and its applications to the hyperbolicity problem centered around the Kobayashi Conjecture. In the last year the Kobayashi Conjecture was finally solved affirmatively by D. Brotbek [Brot17], and then some improving works have been taken on since then. The present author also gave some simplification of the proof in §12.

As for §§1–11, the contents are mainly based on [Dem97], [Dem11], [Dem12] and [Dem14], which were already published: it is good to find a nice summarization of them in one place here.

The content of §12 is new and devoted to a simplification of Brotbek’s proof of the Kobayashi Conjecture [Brot17]. Here, unfortunately the presentation of this section is rather rough, and some more clarifications should be expected, as the referee thinks:

Comments to §12.A :

(1) The general impression is that readers could be lost. Some statements are too compact. In fact, everything in this §12 is too condensed. While the first 60 pages of the paper are well-presented in detail, these new 10 pages should be written in a more reader-friendly style to guarantee a readership (for what concerns the technical core). For instance, ”We can therefore summarize this *discussion* by the following statement” leaves the impression that a more rigorous presentation could have been done. Instead, it would be nice to anticipate why and how such a desingularization is used. Otherwise, except people who already read and understood Brotbek’s first complete solution of the Kobayashi conjecture, only very few people (who know well the trick of reducing to the nefness/ampleness of a certain auxiliary line bundle) will follow. Also, some words why tautological very ample line bundle $\mathcal{O}_{\mathrm{Gr}}(1)$ is anyway used/needed — although the rest diverges from Brotbek’s approach and does not use a large collection of Wronskians — would be welcome.

(2) Important: A rigorous text should be set up for pp. 63–64 and its Proposition 12.9 directly in the context of parameterized complex manifolds $\{X_a\}_a$, because this is what is really used, *e.g.*, on page 67.

Comments to §12.B:

(3) Even if the referee masters the way how special families of hypersurfaces can be used, he was a bit lost in the presentation of the special family introduced here. A lot of letters and quantities appear, without explanations about why and how. This could look technical, but with a reader-friendly guide for understanding.

(4) The philosophy behind is that the smaller the number of parameters is, easier it is to construct negatively twisted jet differentials, and to control their base locus (*e.g.* for plain Fermat equations). In some sense, Brotbek's proof of Kobayashi's conjecture is more advanced, because the family of equations inspired from Masuda-Noguchi he dealt with contains much more parameters, whence the construction of negatively twisted jet differentials is more difficult. In any case, Brotbek embraces more hypersurfaces.

(5) The idea of the present author is to decrease the number of parameters, and to set up a family of equations for which just standard Wronskians can be used, allowing the jet order to be larger than n, equal to $n^3 + n^2 + 1$. In some sense, his family of special equations (12.10) is set up in order to be able to use only plain Wronskians (instead of tackling the delicate problem of constructing jet differentials in the full Masuda-Noguchi family dealt with by Brotbek). Indeed, with (12.10), for the control of the base locus, the standard old easy Lemma 12.25 about the codimension of the zero-set of all matrix minors of a certain size can be applied. This is a new point of this paper.

(6) If some explicit examples of equations with coefficients in $\mathbf{Z}$ can be found from the present approach (but nobody is at present able to produce explicit examples like the ones of Masuda-Noguchi for some deep reason), it will be quite interesting from the view point of *Lang's Conjecture* mentioned at the beginning of Introduction. In fact, an example of a projective hyperbolic hypersurface defined over $\mathbb{Z}$ which has only finitely many rational points over any given algebraic number field was given by

Noguchi, J., An arithmetic property of Shirosaki's hyperbolic projective hypersurface, Forum Math. **15** (2003), 935–941.

But it is singular, and so it is a very interesting problem to find a non-singular such example.

(7) A recent (June 2018) paper of Riedl-Yang shows that in some sense, the study of special families (quite small within the full universal family) is useless now, since it suffices to improve results on algebraic degeneracy of entire curves in the various papers which provided solutions of Green-Griffiths conjecture for hypersurfaces.

In what follows, the comments are for stylistic matters, wordings, and an addition of references for improvements and/or fairness. In the list, "L.xxx" stands for the line number put at the left side of each page, and "L.xxx±y" means the line below or above "y'th" line.

(1) p.1, in the footnote: at the University of Tokyo

(2) L.22+3: any non-constant

(3) L.22+4: ... ("Brody hyperbolicity"). In this paper entire holomorphic curves are assumed to be non-constant and simply called entire curves. If ....
Cf. L.210. It is better to put this kind of convention at the beginning, somewhere, e.g., here.

(4) L.29: Delete "with equations".

(5) L.32: [Lan86]
This writing varies place to place, and must be uniformized all through the manuscript; the same for [Lan87].

(6) L.42: 0.3 (b)

(7) L.46: algebraic curves

(8) L.48—49: "These observations led Zaidenberg .... for $n \geqslant 1$" does not seem to represent the realm. In fact, Zaidenberg was based, I think, on a rather more classical result such that the complement of $2n + 1$ hyperplanes of $\mathbb{P}^n$ in general position is hyperbolic and hyperbolically embedded into $\mathbb{P}^n$ (first stated by Fujimoto, Nagoya Math. J., 1972).

(9) L.55: ($\mathbb{P}^n$,

(10) L.61: [McQ96], and Noguchi-Winkelmann :
Noguchi, J., Winkelmann, J.: Nevanlinna Theory in Several Complex Variables and Diophantine Approximation, GrundL. der Math. Wiss. VoL. 350, Springer, Tokyo-Heidelberg-New York-Dordrecht-London, 2014.
This should also be cited here (as another survey article).

(11) At the end of (0.6), put ",".

(12) L.125-1: Put "," at the end of the equation.

(13) L.128+6: [GG80] (also in the bibliography)
The number is used to represent the publication year, and this should be applied uniformly.

(14) L.129-3: "," at the end (not ".").

(15) L.154: Nevertheless

(16) At the end of (1.1'), put ",".

(17) At 2 lines below (1,1"), delete "positive".

(18) L.199-3: that $\mathbf{k}_X(\xi)$ is upper semi-continuous on T_X and d_X^K

(19) L.206+1: Kobayashi pseudometric

(20) L.209—211: Therefore, if there is an entire curve $\Phi : \mathbb{C} \to X$, then by monotonicity

(21) L.224: (a), (b), (c)

(22) L.232: by the choice of R. □
Using ....

(23) L.264: however, an entire curve

(24) L.268: affine linear

(25) L.293: an n-dimensional

(26) L.294: The subject of the sentence "assuming X ....." is "this", so it appears to be unnatural, since "this" has no function to "assume" something. It may be better to say, e.g., "with the assumption of X being connected, ...".

(27) L.306: Some comment on the coherence of $\mathcal{V}$ is needed with a suitable reference, or with giving some clarification.

(28) L.331-1: $\}$,

(29) L.335: a locally

(30) L.374-2, -1: Put "," at the ends of the equations.

(31) L.374: \mathrm{rank}

(32) L.388: functor
This should be applied all through the text; also for "fonctorial", etc.

(33) L.391: Hartogs extension
The analytic continuation over an analytic subsets of codimension $\geqslant 2$ is usually called "Hartogs' extension". The same appears below in several places.

(34) L.397: coherent ideals

(35) L.398-1: Delete "." at the end of the equation.

(36) L.400: a sort of

(37) L.451-1: Put "," at the end of the equation.

(38) L.462+7—+8: The part of "*Gamma* in a $1:1$ way .... $\dim Z = 2$). is non-trivial. Give a reference or more clarifications.

(39) L.465—466: ; contradiction.

(40) L.479: ; thus
The word "thus" is adv. and has no function of conj. This should be applied all through the text; the same for "hence" (cf., e.g., l:575-6, L.590+4) and "therefore" (cf., e.g., L.1113-5).

(41) L.532: e.g. (cf. Carlson [Ca72], Noguchi [Nog77b])
[Ca72] Carlson, J.A.: Some degeneracy theorems for entire functions with values in an algebraic variety, Trans. Amer. Math. Soc. **168** (1972), 273–301.
[Nog77b] Noguchi, J.: Meromorphic mappings into a compact complex space, Hiroshima Math. J. **7** (1977), 411–425.

(42) L.574+5: $[f'(t)] \in P(V_{f(t)})$

(43) p.22, +1 after the subsection title "5.C. ....": Is V non-singular?

(44) L.579-3: $|z|^3$

(45) p.24: Change "." to "," at the end of (6.4').

(46) L.604+3: the natural line bundle morphisms
The wording "canonical line bundle" may cause a misleading or a confusion with the canonical line bundle of a smooth variety.

(47) L.609: The symbol "$\mathbb{R}_{r,k}$" is confusing, as if it is an object of a real analytic one. For $\mathbb{C}$ is representing complex numbers and then $\mathbb{R}$ is naturally expected to represent real numbers. It is certainly better to use another notation: e.g., how about $\mathcal{R}_{r,k}$?

(48) L.613-1: Put "." at the end of (6.11).

(49) p.26, 5: Put "," at the end of the line.

(50) p.26, -1 from (7.1): defined by $\mathbb{H}$,

(51) L.634: and in particular

(52) p.26, -4 — -1: It is unclear what amounts to by "every monomial .... otherwise.". Is it sufficient to say "every monomial $(f^{(\bullet)})^\ell = (f')^{\ell_1}(f'')^{\ell_2}\cdots(f^{(s)})^{\ell_s}$ of partial order s $(1 \leqslant s \leqslant k)$ and weighted degree $|\ell|_s = \ell_1 + 2\ell_2 + \cdots + s\ell_s$ into a polynomial $((\Psi \circ f)^{(\bullet)})^\ell$ in $(f', f'', \ldots, f^{(s)})$ which has the same partial order s and weighted degree."?

(53) L.644: rather in

(54) L.646: a canonical subbundle algebra
To avoid a confusion with "canonical bundle".

(55) L.664-5: codimension r

(56) L.664-4: Hartogs' extension

(57) L.674-1: Put "," at the end of the equation.

(58) L.677-2: Is $\mathbb{N}$ the set of positive numbers?
I think that here is the first appearance of the symbol, so that it is better to give a comment to specify whether it represents positive integers or non-negative integers.

(59) L.679-2: Equivalently, for all $k \geqslant 1$,

(60) L.679: X_k *and generated by sections.*

(61) L.679+1: to be nef
It is suggested to move the part "Let us recall ..... details)." just below (7.18).

(62) L.680: Is the proof finished? The statement seems to remain yet under "if".
It is better to break the line at "All".

(63) L.681+1: $A^{\otimes p}$ with $a_\bullet$

(64) L.681+5: (*resp.*

(65) L.681+6: $\sum a_j$.)

(66) L.691: Even if (X, V)

(67) L.695: I think, it is better to delete "still": With this the meaning becomes unclear.

(68) L.696+2: Put "," at the end of the equation.

(69) L.697-1: Put "," at the end of the equation.

(70) L.701-6: has a constant rank

(71) L.701-2: (X, V) satisfies $f_{[k]}(\mathbb{C}) \subset Z$,
To avoid a double subordinate clause structure for a very short one.

(72) L.710: $V_{k|Z'}$; in particular

(73) L.715+5: exist

(74) L.716-2: if there exists $p \geqslant 0$
"if ... assume that ... such that ...," sounds too much.

(75) L.718: weaker than what?

(76) L.720: *exist*

(77) L.722: *of general type modulo* $X_\bullet \to X$ *for every*

(78) L.753–754: "the case of general complex tori in §10" —— Is this really correct?

(79) L.773: neighborhood where the weight

(80) L.802+8: Delete "$\geqslant$" at the end of the line.

(81) L.809: (iii) What is the "assumptions"?

(82) L.812, L.828, L.830, L.846: Delete $\Box$ at the end of the line.

(83) L.871+7: Delete one "hypothesis".

(84) L.912-1: What is the range of q?

(85) L.924: ($\supset$ Sing(V))

(86) L.925+1: metric on V

(87) L.925+2: Delete "sections".

(88) L.938+2: What is the definition of $X(L.h, q)$?

(89) L.938+4: What is the definition of $X(L.h, \leqslant q)$?

(90) L.956+4: δ_0 (the Dirac measure at 0)

(91) L.958+4: Here $\lambda \in \mathbb{C}^*$ and

(92) p.44, (9.17): Put "," at the end of the equation.

(93) L.959+5, (9.21), (9.24): Put "," at the end of the equation.

(94) (9.26): Put "." at the end of the equation.

(95) L.969+4: Put "," at the end of the equation.

(96) L.969+5—+6: polynomial of weighted degree s in $(\xi_1, \ldots, \xi_{s-1})$ with holomorphic coefficients in $x \in U$.
To avoid an unnecessary double subordinate clause structure.

(97) L.975: to perform

(98) L.978+3: Put "," at the end of the equation.

(99) L.969+4: Put "," at the end of the equation.

(100) L.977-2: Delete =.

(101) L.981+8: Delete =.

(102) L.981+9: Put "," at the end of the equation.

(103) L.981+12: (9.20)

(104) (9.35) : Put "," at the end.

(105) p.49, -2: true exactly in the same form
"exactly" is an adv. for "is".

(106) L.988+2: structures

(107) L.989+4: have only

(108) L.990-5: Put "," at the end.

(109) L.990-3: What is $X(\eta, q)$?

(110) L.990-3: $\setminus S$ with S

(111) L.990-1: h^0 (in two places)

(112) L.994: ; in particular
It is unclear if the statement after "in particular" is inside the subordinate clause formed by "when", or if it is a particular consequence of the claimed statement of the Corollary.

(113) L.995+9: Put "," at the end.

(114) L.995+10: effective with

(115) L.997+2: $\frac{1}{k}$) and $r = \operatorname{rank} V$,

(116) L.1005: Delete $\square$ at the end.

(117) p.53, -6: Put "," at the end.

(118) p.53, -3: Put ";" in stead of "," at the end.
This is because the sentence continues with "hence", an adv. (not conj.).

(119) L.1025-1: Put "," at the end.

(120) L.1050-2: ; thus

(121) L.1079: a very generic

(122) L.1083: based on

(123) L.1108: D_k (cf. (6.9))

(124) L.1113-1: Put "," at the end.

(125) p.59, (11.5.2): Put "," at the end.

(126) +1 below (11.5.2): set $\{\sigma = d\sigma_{|V_k} = 0\}$ ($\supset Z_{\text{sing}}$) and
To avoid "where ..... where ....".

(127) L.1129: Delete □ at the end.

(128) L.1140+1—+4: The statement "If ...... $= p_0$" is logically unclear. What does amount to by "if we can prove the result for p_0'"? .....

(129) L.1146+8: Put "," at the end of the line.

(130) L.1147-4: ; otherwise

(131) L.1152: ; in other words,

(132) L.1164: Delete □ at the end.

(133) L.1192: $\geqslant 1$; otherwise $W = 0$ and we can

(134) L.1198: [ShZa02]

(135) L.1199: The paper of Brotbek-Darondeau [BrDa17], 10 pages, appeared on arxiv.org in November 15, 2017, one month after the paper of Xie [Xie15], October 2015, 96 pages. Xie solved the Debarre ampleness conjecture first, in full generality, with an effective multidegree bound. Brotbek-Darondeau felt obliged to show a result. Their result does not (at all) cover all degrees, and was made effective, concerning degrees, only one year later, by Deng, with the help of Demailly. So in talks and in papers, to put at the same rank both papers is not the truth.

(136) p.62, (12.1): Put "," at the end. Missing vdots in the second column.

(137) p.62, -4: $f : (\mathbb{C}, 0) \ni t$
In the same line, is it better to delete "(or a k-jet of curve)? It is unclear what is intended.

(138) p.62,-1: Change the summation index in $s < \ell$ into another letter (confusion with s_j).

(139) p.63, 1: from this

(140) p.63, (12.2) : Independent jet coordinates are not introduced. It is a bit abusive, then, to write "$D(s_j)$", with as above $D = \frac{d}{dt}$, since this is meaningful only when composing with a map $t \longmapsto f(t) \in U \subset \mathbf{C}^n$.

(141) p.63, 3 below (12.6): "independent of the trivialization" $\longmapsto$ "independent of the trivialization and of coordinates charts"
For, in subsequent applications, changes of trivialization also concern changes of charts $U, V \subset X$.

(142) p.63, 1 below (12.6): Add the condition that $\dim \Sigma \geqslant k + 1$.

(143) p.63, after (12.6): Some $\pi_{k,0}^*$ are appearing, some are erased.

(144) p.63, 4 below (12.6): What is the notation $[W_k(.....)]_{I \subset J}$?
One may be able to "guess" it, but it is better to give some clear explanation.

(145) p.63, (12.7): Put "," at the end.

(146) p.63 (middle) – p.64: A full diagram with all maps, manifolds, bundles dealt with on this page 63 would help readers' understanding, hence widen audience.

(147) p.64, 2 below (12.8): "graph of Φ" $\longmapsto$ "graph of $\Phi|_{X_k \setminus B_k}$".

(148) p.64, 3 below (12.8): Referee does not understand "this construction is completely universal": it depends on the choice of $\Sigma \subset H^0(X, L)$, at minimum.

In any case, more rigorous assertions (with formal proofs) are expectable at this crucial passage.

(149) L.1205: Referee does not understand the phrase "(even though we might ...)".

(150) L.1207-1: Put "," at the end.

(151) L.1208+7: Here, [Xie15] should also be mentioned.

(152) p.64, -1 above (12.10): σ to be

(153) L.1242: [DeEG97] in relation with

(154) L.1246+4: Put "," at the end.

(155) L.1248: *vanishing theorem 8.15*

(156) L.1256: It is unclear what is "this".

(157) L.1258+7–+8 : In the phrase "By the universality of this construction", one has the impression that more rigorous details could be written.

(158) L.1258+9: , where

(159) L.1259, L.1260 : $\widehat{X}_{a,k}$

The phrase "(Here the fact..." looks as a commentary, not a formal proof, although there must be a formal proof here.

(160) L.1260, *Proof.*: In the papers of Brotbek, Xie, Brotbek-Darondeau, some explanations were given why one has to really treat separately the restrictions to intersections with hyperplanes. Here, the non-expert readers can be lost.

(161) L.1270-6: $X_{\mathcal{G},a}$ for $i \in$

(162) p.70, -2: and hence,

(163) L.1286-4: and therefore

(164) L.1286-3: Put "," at the end.

(165) L.1550: [ShZa02]

(166) L.1551: (2002) 2031–2035.

(167) L.1582: [Xie15] has been published online by Invent. Math. at https://doi.org/10.1007/s00222-017-0783-8.