REPORT JEAN-PIERRE DEMAILLY'S MANUSCRIPT "RECENT RESULTS ON KOBAYASHI AND GREEN-GRIFFITHS-LANG CONJECTURES"

This manuscript is a survey article presenting the author's recent progress on the Green-Griffiths-Lang conjecture and the Kobayashi conjecture. It gives a comprehensive and complete (almost self contained) account on some of the author's work concerned with hyperbolicity, jet differentials and the Green-Griffiths-Lang conjectures (such as [Dem97], [Dem11], [Dem15]) but it also contains an original and important proof of the Kobayashi conjecture which seems to have not been published elsewhere.

The content of the manuscript can be roughly split into three parts which we will detailed below: foundational material on hyperbolicity and jet differentials, progress on the Green-Griffiths-Lang conjecture and progress on the Kobayashi conjecture. Most of the first two parts already appeared in other articles of the author whereas the last part is completely original.

The first part of the manuscript (sections 1 to 8), contains the foundations needed in the rest of the paper. It provides a short introduction to hyperbolicity and the theory of jet differentials. Let us just recall that if X is compact complex manifold, then X is hyperbolic if there doesn't exists any non-constant holomorphic map $f : \mathbb{C} \to X$. The theory of jet differentials is an important tool for the study of hyperbolicity problems. The point is that the existence of jet differential equations vanishing along some ample divisor provide obstructions to the existence of entire curve. Therefore, if one can understand enough those type of jet differentials, then, in some favorable situations, one can deduce some important consequences on the hyperbolicity properties of X. To produce and control jet differentials on different type of varieties is the leitmotiv of the rest of this manuscript.

The author presents more precisely the theory of *invariant* jet differentials and directed pair which he developed 20 years ago and which have been of great importance since then. Much of the content of this part already appeared in the author's article [Dem97]. However, in the present text, a more general version of directed pairs is developed (following the author's article [Dem15]). This generalization is critical for the author's work concerning the Green-Griffiths-Lang conjecture. Moreover, this generalization also yield many interesting and subtle considerations that certainly deserve to be explored further.

Sections 9 and 11 are concerned with the Green-Griffiths-Lang conjecture, which is one of the deepest and most important open problem in the study of entire curves.

Conjecture 0.1 (Green-Griffiths-Lang)**.** *Let X be a projective variety of general type, then there exists a proper Zariski closed subset $Y \subsetneq X$ such that any non-constant morphism $f : \mathbb{C} \to X$ satisfies*

$$f(\mathbb{C}) \subset Y.$$

This conjecture is still widely open in general and the author presents here arguably the most important general result towards this conjecture, namely the existence of jet differential equations vanishing along some ample divisor on any projective variety of general type (which he proved in [Dem11]). The author uses this theorem (and the formalism of the previous part of the manuscript) to give a positive answer to this conjecture under the hypothesis is *strongly* of general type, an ad hoc notion introduced by the author.

Section 10 also provides the main ideas of the proof of the Green-Griffiths-Lang conjecture for generic hypersurfaces of large degrees in $\mathbb{P}^{n+1}$ following the work of Siu, Diverio-Merker-Rousseau, the author and others.

Section 12 is concerned with the Kobayashi conjecture.

Conjecture 0.2 (Kobayashi)**.** *A (very) generic hypersurface $H \subset \mathbb{P}^{n+1}$ of degree $d \geqslant d_n$ large enough is hyperbolic.*

The Kobayashi conjecture has been an open problem for over 40 years and was only proven a few years ago by Siu (with no effective bound on d_n). More recently, the work of Brotbek combined with the work of Deng yield the first proof of this conjecture with an effective bound on d_n. In Section 12 of the manuscript under review, the author provides a new proof of this conjecture.

The author's argument is much simpler and shorter than the proof of Siu and the proof of Brotbek and Deng. It shares with Brotbek's approach the use of Wronskian jet differentials and the reduction to the construction of certain explicit examples. However, the examples provided by the author are outstanding. They are very cleverly chosen so that the author is able to construct a large amount of very explicit Wronskian jet differential equations in a simple way. The proof of the Kobayashi conjecture is then reduced to the study of the base locus of those jet differential equations, but their simplicity (basically determinants) allows the author to reduce this problem to a rather elementary linear algebra problem. Moreover, this proof provides a bound $d_n \sim \lfloor \frac{1}{5}(en)^{2n+2} \rfloor$ slightly better bound than the previous bound $d_n \sim (n+2)^{3n+9}$ obtained by Ya Deng. Although those bounds are still very large and far from optimal (d_n is expected to be linear in n), it is consistent with the previous work on the hyperbolicity of generic hypersurface (Diverio-Merker-Rousseau / Demailly / Berczi / Darondeau), where all the results involve similar bounds.

This new proof is remarkable and of great importance since it shorter and easier to follow than the previous proofs. It will therefore most likely become the reference text concerning the Kobayashi conjecture and the starting point for further improvements on the bound d_n.

Altogether, this very well written article gives a state of art on the Green-Griffiths-Lang and the Kobayashi conjectures. In our opinion, this text will certainly be wildly used by any mathematician interested in the Green-Griffiths or the Kobayashi conjecture. Therefore this manuscript should without a doubt be published in the Japanese Journal of Mathematics as a part of the 16^{th} Takagi lectures in celebration of the 100^{th} anniversary of K. Kodaira's birth.

Let us just point out a few typos we've noticed.

(1) Last line of the abstract. Is ".em" a typo ?

(2) Page 33 line 6 from below. "$f \mapsto f \circ g$" should be replaced "$(f, g) \mapsto f \circ g$".
(3) Page 39 lines 20 to 22 it is not clear to which part of the manuscript the author is referring to.
(4) Page 56 line 17 from below, "generic" should probably be replaced by "very generic".
(5) Page 59 line 19 from below. "Irreducible divisor" should probably be replaced by "irreducible components".
(6) Page 62 line 14 and 15 from above, is $Q_{\ell,s} = p_{\ell,s}$?
(7) Page 63 line 17 from above, q_I seems to be undefined,
(8) Page 65 line 22 from above. Is $q = k$?
(9) Page 67, A minor suggestion in the presentation of Lemma 12.25. It might be easier for the reader if the notation of the Lemma would coincide with those used when the lemma is applied, by replacing k with $k+1$ and letting $N \in \{0, \ldots, k\}$ and so on.
(10) Page 68 line 16 from above. Shouldn't $e_A(x)^{|\alpha_J|+|\beta_J|}$ be replaced by $e_A(x)^{|\alpha_J|+|\beta_J|+|\gamma_J|}$?
(11) Page 69 in the proof of lemma 12.29, shouldn't r be replaced by r' ?

References

[Dem97] Jean-Pierre Demailly. Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 285–360. Amer. Math. Soc., Providence, RI, 1997.

[Dem11] Jean-Pierre Demailly. Holomorphic Morse inequalities and the Green-Griffiths-Lang conjecture. *Pure Appl. Math. Q.*, 7(4, Special Issue: In memory of Eckart Viehweg):1165–1207, 2011.

[Dem15] Jean-Pierre Demailly. Towards the Green-Griffiths-Lang conjecture. In *Analysis and geometry*, volume 127 of *Springer Proc. Math. Stat.*, pages 141–159. Springer, Cham, 2015.