MORSE COHOMOLOGY ESTIMATES FOR SEMPLE JET BUNDLES

JEAN-PIERRE DEMAILLY

Abstract. In 2016, the conjecture has been settled in a different way by Damian Brotbek, making a more direct use of Wronskian differential operators and associated multiplier ideals; shortly afterwards, Ya Deng showed how the proof could be modified to yield an explicit value of d_n. We give here a short proof based on a drastic simplification of their ideas, along with a further improvement of Deng's bound, namely $d_n = \lfloor \frac{1}{5}(en)^{2n+2} \rfloor$.

Key words: Kobayashi hyperbolic variety, directed manifold, genus of a curve, jet bundle, jet differential, jet metric, Chern connection and curvature, negativity of jet curvature, variety of general type, Kobayashi conjecture, Green-Griffiths conjecture, Lang conjecture.

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Contents

0. Introduction

The goal of these lectures is to study the conjecture of Kobayashi [Kob70, Kob78] on the hyperbolicity of generic hypersurfaces of high degree in projective space, and the related conjecture by Green-Griffiths [GG79] and Lang [Lan86] on the structure of entire curve loci.

1. Semple tower associated to a directed manifold

1.A. Category of directed manifolds

We start by recalling the main definitions concerning the category of directed varieties. For the sake of simplicity, we first assume that the objects under consideration are nonsingular.

1.1. Definition. *A (complex)* directed manifold *is a pair (X, V) consisting of a n-dimensional complex manifold X equipped with a A morphism $\Phi : (X, V) \to (Y, W)$ in the category of directed manifolds is a holomorphic map such that $\Phi_*(V) \subset W$.*

It is eventually interesting to allow singularities for V. We then assume that there exists a dense Zariski open set $X' = X \smallsetminus Y \subset X$ such that $V_{|X'}$ is a subbundle of $(T_X))|X'$ and the closure $\overline{V_{|X'}}$ in the total space of T_X is an anaytic subset. The rank $r \in \{0, 1, \ldots, n\}$ of V is by definition the dimension of V_x at points $x \in X'$; the dimension may be larger at points $x \in Y$. This happens e.g. on $X = \mathbb{C}^n$ for the rank 1 linear space V generated by the Euler vector field: $V_z = \mathbb{C} \sum_{1 \leqslant j \leqslant n} z_j \frac{\partial}{\partial z_j}$ for $z \neq 0$, and $V_0 = \mathbb{C}^n$. The absolute situation is the case $V = T_X$ and the relative situation is the case when $V = T_{X/S}$ is the relative tangent space to a smooth holomorphic map $X \to S$. In general, we can associate to V a sheaf $\mathcal{V} = \mathcal{O}(V) \subset \mathcal{O}(T_X)$ of holomorphic sections. No assumption need be made on the Lie bracket tensor $[\bullet,\bullet] : \mathcal{V}\times\mathcal{V}\to\mathcal{O}(T_X)/\mathcal{V}$, i.e. we do not assume any kind of integrability for $\mathcal{V}$. One of the most central conjectures in the theory is the

1.2. Generalized Green-Griffiths-Lang conjecture. *Let (X,V) be a projective directed manifold where $V\subset T_X$ is nonsingular (i.e. a subbundle of T_X). Assume that (X,V) is of "general type" in the sense that $K_V:=\det V^*$ is a big line bundle. Then there should exist a proper algebraic subvariety $Y\subsetneq X$ containing the images $f(\mathbb{C})$ of all entire curves $f:\mathbb{C}\to X$ tangent to V.*

A similar statement can be made when V is singular, but then K_V has to be replaced by a certain (nonnecessarily invertible) rank 1 sheaf of "locally bounded" forms of $\mathcal{O}(\det V^*)$, with respect to a smooth hermitian form ω on T_X. The reader will find a more precise definition in [Dem18].

1.B. The 1-jet fonctor

The basic idea is to introduce a fonctorial process which produces a new complex directed manifold $(\widetilde X,\widetilde V)$ from a given one (X,V). The new structure $(\widetilde X,\widetilde V)$ plays the role of a space of 1-jets over X. We let

$$\widetilde X = P(V),\qquad \widetilde V\subset T_{\widetilde X} \tag{1.3}$$

be the projectivized bundle of lines of V, together with a subbundle $\widetilde V$ of $T_{\widetilde X}$ defined as follows: for every point $(x,[v])\in\widetilde X$ associated with a vector $v\in V_x\smallsetminus\{0\}$,

$$\widetilde V_{(x,[v])} = \big\{\eta\in T_{\widetilde X,\,(x,[v])}\,;\,d\pi_x(\eta)\in\mathbb{C}v\big\},\qquad \mathbb{C}v\subset V_x\subset T_{X,x}, \tag{1.3$'$}$$

where $\pi:\widetilde X=P(V)\to X$ is the natural projection and $\pi_*:T_{\widetilde X}\to\pi^*T_X$ is its differential. On $\widetilde X=P(V)$ we have a tautological line bundle $\mathcal{O}_{\widetilde X}(-1)\subset\pi^*V\subset\pi^*T_X$ such that $\mathcal{O}_{\widetilde X}(-1)_{(x,[v])}=\mathbb{C}v$. The bundle $\widetilde V$ is characterized by the exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_{\widetilde X/X} & \longrightarrow & T_{\widetilde X} & \xrightarrow{d\pi} & \pi^*T_X & \longrightarrow & 0,\\
 & & \| & & \cup & & \cup & & \\
0 & \longrightarrow & T_{\widetilde X/X} & \longrightarrow & \widetilde V & \xrightarrow{d\pi} & \mathcal{O}_{\widetilde X}(-1) & \longrightarrow & 0,
\end{array} \tag{1.4}$$

$$0\longrightarrow\mathcal{O}_{\widetilde X}\longrightarrow\pi^*V\otimes\mathcal{O}_{\widetilde X}(1)\longrightarrow T_{\widetilde X/X}\longrightarrow 0, \tag{1.4$'$}$$

where $T_{\widetilde X/X}$ denotes the relative tangent bundle of the fibration $\pi:\widetilde X\to X$. The first sequence is a direct consequence of the definition of $\widetilde V$, whereas the second is a relative version of the Euler exact sequence describing the tangent bundle of the fibers $P(V_x)$. From these exact sequences we infer

$$\dim\widetilde X = n+r-1,\qquad \operatorname{rank}\widetilde V=\operatorname{rank}V=r, \tag{1.5}$$

and by taking determinants we find $\det(T_{\widetilde X/X})=\pi^*\det V\otimes\mathcal{O}_{\widetilde X}(r)$, hence

$$\det\widetilde V=\pi^*\det V\otimes\mathcal{O}_{\widetilde X}(r-1). \tag{1.6}$$

Clearly $\pi:(\widetilde X,\widetilde V)\to(X,V)$ is a morphism of complex directed manifolds and this construction is fonctorial with respect to morphisms $\Phi:(X,V)\to(Y,W)$ for which Φ_* is injective.

1.C. Lifting of curves to the 1-jet bundle

Suppose that we are given a holomorphic curve $f:D(0,R)\to X$ parametrized by the disk $D(0,R)$ of centre 0 and radius R in the complex plane, and that f is a tangent curve of the directed manifold, i.e., $f'(t)\in V_{f(t)}$ for every $t\in D(0,R)$. If f is nonconstant, there is a well defined and unique tangent line $[f'(t)]$ for every t, even at stationary points, and the map

$$\widetilde f:D(0,R)\to\widetilde X,\qquad t\mapsto\widetilde f(t):=(f(t),[f'(t)]) \tag{1.7}$$

is holomorphic (at a stationary point t_0, we just write $f'(t)=(t-t_0)^s u(t)$ with $s\in\mathbb{N}^*$ and $u(t_0)\neq 0$, and we define the tangent line at t_0 to be $[u(t_0)]$, hence $\widetilde f(t)=(f(t),[u(t)])$ near

t_0 ; even for $t = t_0$, we still denote $[f'(t_0)] = [u(t_0)]$ for simplicity of notation). By definition $f'(t) \in \mathcal{O}_{\widetilde{X}}(-1)_{\widetilde{f}(t)} = \mathbb{C}\,u(t)$, hence the derivative f' defines a section

$$f' : T_{D(0,R)} \to \widetilde{f}^*\mathcal{O}_{\widetilde{X}}(-1). \tag{1.8}$$

Moreover $\pi \circ \widetilde{f} = f$, therefore

$$\pi_*\widetilde{f}'(t) = f'(t) \in \mathbb{C}u(t) \Longrightarrow \widetilde{f}'(t) \in \widetilde{V}_{(f(t),u(t))} = \widetilde{V}_{\widetilde{f}(t)}$$

and we see that $\widetilde{f}$ is a tangent trajectory of $(\widetilde{X},\widetilde{V})$. We say that $\widetilde{f}$ is the *canonical lifting* of f to $\widetilde{X}$. Conversely, if $g : D(0,R) \to \widetilde{X}$ is a tangent trajectory of $(\widetilde{X},\widetilde{V})$, then by definition of $\widetilde{V}$ we see that $f = \pi \circ g$ is a tangent trajectory of (X,V) and that $g = \widetilde{f}$ (unless g is contained in a vertical fiber $P(V_x)$, in which case f is constant).

For any point $x_0 \in X$, there are local coordinates $(z_1,\ldots,z_n)$ on a neighborhood Ω of x_0 such that the fibers $(V_z)_{z\in\Omega}$ can be defined by linear equations

$$V_z = \Big\{v = \sum_{1\leqslant j\leqslant n} v_j\frac{\partial}{\partial z_j}\,;\, v_j = \sum_{1\leqslant k\leqslant r} a_{jk}(z)v_k \text{ for } j = r+1,\ldots,n\Big\}, \tag{1.9}$$

where (a_{jk}) is a holomorphic $(n-r)\times r$ matrix. It follows that a vector $v \in V_z$ is completely determined by its first r components $(v_1,\ldots,v_r)$, and the affine chart $v_j \neq 0$ of $P(V)_{\restriction\Omega}$ can be described by the coordinate system

$$\Big(z_1,\ldots,z_n;\frac{v_1}{v_j},\ldots,\frac{v_{j-1}}{v_j},\frac{v_{j+1}}{v_j},\ldots,\frac{v_r}{v_j}\Big). \tag{1.10}$$

Let $f \simeq (f_1,\ldots,f_n)$ be the components of f in the coordinates $(z_1,\ldots,z_n)$ (we suppose here R so small that $f(D(0,R)) \subset \Omega$). It should be observed that f is uniquely determined by its initial value x and by the first r components $(f_1,\ldots,f_r)$. Indeed, as $f'(t) \in V_{f(t)}$, we can recover the other components by integrating the system of ordinary differential equations

$$f_j'(t) = \sum_{1\leqslant k\leqslant r} a_{jk}(f(t))f_k'(t), \qquad j > r, \tag{1.11}$$

on a neighborhood of 0, with initial data $f(0) = x$. We denote by $m = m(f,t_0)$ the *multiplicity* of f at any point $t_0 \in D(0,R)$, that is, $m(f,t_0)$ is the smallest integer $m \in \mathbb{N}^*$ such that $f_j^{(m)}(t_0) \neq 0$ for some j. By (1.11), we can always suppose $j \in \{1,\ldots,r\}$, for example $f_r^{(m)}(t_0) \neq 0$. Then $f'(t) = (t-t_0)^{m-1}u(t)$ with $u_r(t_0) \neq 0$, and the lifting $\widetilde{f}$ is described in the coordinates of the affine chart $v_r \neq 0$ of $P(V)_{\restriction\Omega}$ by

$$\widetilde{f} \simeq \Big(f_1,\ldots,f_n;\frac{f_1'}{f_r'},\ldots,\frac{f_{r-1}'}{f_r'}\Big). \tag{1.12}$$

1.D. The Semple tower

Following [Dem95], we define inductively the *projectivized k-jet bundle* X_k (or *Semple k-jet bundle*) and the associated subbundle $V_k \subset T_{X_k}$ by

$$(X_0,V_0) = (X,V), \qquad (X_k,V_k) = (\widetilde{X}_{k-1},\widetilde{V}_{k-1}). \tag{1.13}$$

In other words, (X_k,V_k) is obtained from (X,V) by iterating k-times the lifting construction $(X,V) \mapsto (\widetilde{X},\widetilde{V})$ described in § 1.B. By (1.3–1.5), we find

$$\dim X_k = n + k(r-1), \qquad \operatorname{rank} V_k = r, \tag{1.14}$$

together with exact sequences

$$0 \longrightarrow T_{X_k/X_{k-1}} \longrightarrow V_k \xrightarrow{(\pi_k)_*} \mathcal{O}_{X_k}(-1) \longrightarrow 0, \tag{1.15}$$

$$0 \longrightarrow \mathcal{O}_{X_k} \longrightarrow \pi_k^*V_{k-1}\otimes\mathcal{O}_{X_k}(1) \longrightarrow T_{X_k/X_{k-1}} \longrightarrow 0. \tag{1.15$'$}$$

where π_k is the natural projection $\pi_k : X_k \to X_{k-1}$ and $(\pi_k)_*$ its differential. Formula (1.6) yields

$$\det V_k = \pi_k^* \det V_{k-1} \otimes \mathcal{O}_{X_k}(r-1). \tag{1.16}$$

Every nonconstant tangent trajectory $f : D(0,R) \to X$ of (X,V) lifts to a well defined and unique tangent trajectory $f_{[k]} : D(0,R) \to X_k$ of (X_k, V_k). Moreover, the derivative $f'_{[k-1]}$ gives rise to a section

$$f'_{[k-1]} : T_{D(0,R)} \to f^*_{[k]}\mathcal{O}_{X_k}(-1). \tag{1.17}$$

In coordinates, one can compute $f_{[k]}$ in terms of its components in the various affine charts (1.10) occurring at each step: we get inductively

$$f_{[k]} = (F_1, \ldots, F_N), \qquad f_{[k+1]} = \Big(F_1, \ldots, F_N, \frac{F'_{s_1}}{F'_{s_r}}, \ldots, \frac{F'_{s_{r-1}}}{F'_{s_r}}\Big) \tag{1.18}$$

where $N = n + k(r-1)$ and $\{s_1, \ldots, s_r\} \subset \{1, \ldots, N\}$. If $k \geqslant 1$, $\{s_1, \ldots, s_r\}$ contains the last $r-1$ indices of $\{1, \ldots, N\}$ corresponding to the "vertical" components of the projection $X_k \to X_{k-1}$, and in general, s_r is an index such that $m(F_{s_r}, 0) = m(f_{[k]}, 0)$, that is, F_{s_r} has the smallest vanishing order among all components F_s (s_r may be vertical or not, and the choice of $\{s_1, \ldots, s_r\}$ need not be unique).

By definition, there is a canonical injection $\mathcal{O}_{X_k}(-1) \hookrightarrow \pi_k^* V_{k-1}$, and a composition with the projection $(\pi_{k-1})_*$ (analogue for order $k-1$ of the arrow $(\pi_k)_*$ in the sequence (1.15)) yields for all $k \geqslant 2$ a canonical line bundle morphism

$$\mathcal{O}_{X_k}(-1) \hookrightarrow \pi_k^* V_{k-1} \xrightarrow{(\pi_k)^*(d\pi_{k-1})} \pi_k^* \mathcal{O}_{X_{k-1}}(-1), \tag{1.19}$$

which admits precisely $D_k = P(T_{X_{k-1}/X_{k-2}}) \subset P(V_{k-1}) = X_k$ as its zero divisor (clearly, D_k is a hyperplane subbundle of X_k). Hence we find

$$\mathcal{O}_{X_k}(1) = \pi_k^* \mathcal{O}_{X_{k-1}}(1) \otimes \mathcal{O}(D_k). \tag{1.20}$$

Now, we consider the composition of projections

$$\pi_{k,j} = \pi_{j+1} \circ \cdots \circ \pi_{k-1} \circ \pi_k : X_k \longrightarrow X_j. \tag{1.21}$$

Then $\pi_{k,0} : X_k \to X_0 = X$ is a locally trivial holomorphic fiber bundle over X, and the fibers $X_{k,x} = \pi_{k,0}^{-1}(x)$ are k-stage towers of $\mathbb{P}^{r-1}$-bundles. Since we have (in both directions) morphisms $(\mathbb{C}^r, T_{\mathbb{C}^r}) \leftrightarrow (X,V)$ of directed manifolds which are bijective on the level of bundle morphisms, the fibers are all isomorphic to a "universal" non singular projective algebraic variety of dimension $k(r-1)$ which we will denote by $\mathbb{R}_{r,k}$; it is not hard to see that $\mathbb{R}_{r,k}$ is rational, since (1.18) provides affine charts of $\mathbb{R}_{r,k}$ that are isomorphic to $\mathbb{C}^{k(r-1)}$.

1.22. Remark. When (X,V) is singular, one can easily extend the construction of the Semple tower by fonctoriality. In fact, assume that X is a closed analytic subset of some open set $Z \subset \mathbb{C}^N$, and that $X' \subset X$ is a Zariski open subset on which $V_{\restriction X'}$ is a subbundle of $T_{X'}$. Then we consider the injection of the nonsingular directed manifold (X', V') into the absolute structure (Z, W), $W = T_Z$. This yields an injection $(X'_k, V'_k) \hookrightarrow (Z_k, W_k)$, and we simply define (X_k, V_k) to be the closure of (X'_k, V'_k) into (Z_k, W_k). It is not hard to see that this is indeed a closed analytic subset of the same dimension $n + k(r-1)$, where $r = \operatorname{rank} V'$.

1.E. Jet bundles and jet differentials

Following Green-Griffiths [GrGr79], we consider the bundle $J_k X \to X$ of k-jets of germs of parametrized curves in X, i.e., the set of equivalence classes of holomorphic maps $f : (\mathbb{C}, 0) \to (X, x)$, with the equivalence relation $f \sim g$ if and only if all derivatives $f^{(j)}(0) = g^{(j)}(0)$ coincide for $0 \leqslant j \leqslant k$, when computed in some local coordinate system of X near x. The projection map

$J_kX \to X$ is simply $f \mapsto f(0)$. If $(z_1, \ldots, z_n)$ are local holomorphic coordinates on an open set $\Omega \subset X$, the elements f of any fiber J_kX_x, $x \in \Omega$, can be seen as $\mathbb{C}^n$-valued maps

$$f = (f_1, \ldots, f_n) : (\mathbb{C}, 0) \to \Omega \subset \mathbb{C}^n,$$

and they are completetely determined by their Taylor expansion of order k at $t = 0$

$$f(t) = x + t\, f'(0) + \frac{t^2}{2!} f''(0) + \cdots + \frac{t^k}{k!} f^{(k)}(0) + O(t^{k+1}).$$

In these coordinates, the fiber J_kX_x can thus be identified with the set of k-tuples of vectors $(\xi_1, \ldots, \xi_k) = (f'(0), \ldots, f^{(k)}(0)) \in (\mathbb{C}^n)^k$. It follows that J_kX is a holomorphic fiber bundle with typical fiber $(\mathbb{C}^n)^k$ over X. However, J_kX is not a vector bundle for $k \geqslant 2$, because of the nonlinearity of coordinate changes: a coordinate change $z \mapsto w = \Psi(z)$ on X induces a polynomial transition automorphism on the fibers of J_kX, given by a formula

$$(\Psi \circ f)^{(j)} = \Psi'(f) \cdot f^{(j)} + \sum_{s=2}^{s=j} \sum_{j_1+j_2+\cdots+j_s=j} c_{j_1\ldots j_s} \Psi^{(s)}(f) \cdot (f^{(j_1)}, \ldots, f^{(j_s)}) \tag{1.23}$$

with suitable integer constants $c_{j_1\ldots j_s}$ (this is easily checked by induction on s). According to the above philosophy, we introduce the concept of jet bundle in the general situation of complex directed manifolds.

1.24. Definition. *Let (X, V) be a complex directed manifold. We define $J_kV \to X$ to be the bundle of k-jets of curves $f : (\mathbb{C}, 0) \to X$ which are tangent to V, i.e., such that $f'(t) \in V_{f(t)}$ for all t in a neighborhood of 0, together with the projection map $f \mapsto f(0)$ onto X.*

It is easy to check that J_kV is actually a subbundle of J_kX. In fact, by using (1.11), we see that the fibers J_kV_x are parametrized by

$$\big((f_1'(0), \ldots, f_r'(0)); (f_1''(0), \ldots, f_r''(0)); \ldots; (f_1^{(k)}(0), \ldots, f_r^{(k)}(0))\big) \in (\mathbb{C}^r)^k$$

for all $x \in \Omega$, hence J_kV is a locally trivial $(\mathbb{C}^r)^k$-subbundle of J_kX. Alternatively, we can pick a local holomorphic connection ∇ on V such that for any germs $w = \sum_{1\leqslant j\leqslant n} w_j \frac{\partial}{\partial z_j} \in \mathcal{O}(T_{X,x})$ and $v = \sum_{1\leqslant\lambda\leqslant r} v_\lambda e_\lambda \in \mathcal{O}(V)_x$ in a local trivializing frame $(e_1, \ldots, e_r)$ of $V_{\restriction\Omega}$ we have

$$\nabla_w v(x) = \sum_{1\leqslant j\leqslant n,\, 1\leqslant\lambda\leqslant r} w_j \frac{\partial v_\lambda}{\partial z_j} e_\lambda(x) + \sum_{1\leqslant j\leqslant n,\, 1\leqslant\lambda,\mu\leqslant r} \Gamma^\mu_{j\lambda}(x) w_j v_\lambda\, e_\mu(x). \tag{1.25}$$

We can of course take the frame obtained from (1.9) by lifting the vector fields $\partial/\partial z_1, \ldots, \partial/\partial z_r$, and the "trivial connection" given by the zero Christoffel symbolds $\Gamma = 0$. One then obtains a trivialization $J^kV_{\restriction\Omega} \simeq V_{\restriction\Omega}^{\oplus k}$ by considering

$$J_kV_x \ni f \mapsto (\xi_1, \xi_2, \ldots, \xi_k) = (\nabla f(0), \nabla^2 f(0), \ldots, \nabla^k f(0)) \in V_x^{\oplus k} \tag{1.26}$$

and computing inductively the successive derivatives $\nabla f(t) = f'(t)$ and $\nabla^s f(t)$ via

$$\nabla^s f = (f^*\nabla)_{d/dt}(\nabla^{s-1} f) = \sum_{1\leqslant\lambda\leqslant r} \frac{d}{dt}\Big(\nabla^{s-1} f\Big)_\lambda e_\lambda(f) + \sum_{1\leqslant j\leqslant n,\, 1\leqslant\lambda,\mu\leqslant r} \Gamma^\mu_{j\lambda}(f) f_j' \Big(\nabla^{s-1} f\Big)_\lambda e_\mu(f).$$

This identification depends of course on the choice of ∇ and cannot be defined globally in general (unless we are in the rare situation where V has a global holomorphic connection). □

Let $\mathbb{G}_k$ be the group of germs of k-jets of biholomorphisms of $(\mathbb{C}, 0)$, that is, the group of germs of biholomorphic maps

$$t \mapsto \varphi(t) = a_1 t + a_2 t^2 + \cdots + a_k t^k, \qquad a_1 \in \mathbb{C}^*,\ a_j \in \mathbb{C},\ j \geqslant 2,$$

in which the composition law is taken modulo terms t^j of degree $j > k$. Then $\mathbb{G}_k$ is a k-dimensional nilpotent complex Lie group, which admits a natural fiberwise right action on J_kV

$$J_kV \times \mathbb{G}_k \to J_kV, \quad (f, \varphi) \mapsto f \circ \varphi. \tag{1.27}$$

There is a semidirect decomposition $\mathbb{G}_k = \mathbb{G}'_k \ltimes \mathbb{C}^*$ given by a split exact sequence

$$1 \to \mathbb{G}'_k \to \mathbb{G}_k \to \mathbb{C}^* \to 1$$

where $\mathbb{G}_k \to \mathbb{C}^*$ is the obvious morphism $\varphi \mapsto \varphi'(0)$, the commutator group $\mathbb{G}'_k = [\mathbb{G}_k, \mathbb{G}_k]$ is the group of k-jets of biholomorphisms tangent to the identity, and $\mathbb{C}^* \subset \mathbb{G}_k$ is the (nonnormal) subgroup of homotheties $\varphi(t) = \lambda t$. The corresponding action of $\mathbb{C}^*$ on k-jets is described in coordinates by

$$\lambda \cdot (\xi_1, \xi_2, \ldots, \xi_k) = (\lambda\xi_1, \lambda^2\xi_2, \ldots, \lambda^k\xi_k), \quad \xi_s = \nabla^s f(0).$$

Following [GrGr79], we introduce the bundle $E^{\mathrm{GG}}_{k,m}V^* \to X$ of polynomials $P(x;\xi_1,\ldots,\xi_k)$ that are homogeneous on the fibers of J_kV of weighted degree m with respect to the $\mathbb{C}^*$ action, i.e.

$$P(x;\lambda\xi_1,\ldots,\lambda^k\xi_k) = \lambda^m P(x;\xi_1,\ldots\xi_k), \tag{1.28}$$

in other words they are polynomials of the form

$$P(x;\xi_1,\ldots\xi_k) = \sum_{|\alpha_1|+2|\alpha_2|+\cdots+k|\alpha_k|=m} a_{\alpha_1\ldots\alpha_k}(x)\,\xi_1^{\alpha_1}\xi_2^{\alpha_2}\cdots\xi_k^{\alpha_k} \tag{1.29}$$

where $\xi_s = (\xi_{s,1},\ldots,\xi_{s,r}) \in \mathbb{C}^r \simeq V_x$ and $\xi_s^{\alpha_s} = \xi_{s,1}^{\alpha_{s,1}}\ldots\xi_{s,r}^{\alpha_{s,r}}$, $|\alpha_s| = \sum_{1\leqslant j\leqslant r}\alpha_{s,j}$. Sections of the sheaf $\mathcal{O}(E^{\mathrm{GG}}_{k,m}V^*)$ can also be viewed as algebraic differential operators acting on germs of curves $f : (\mathbb{C},0) \to X$ tangent to V, by putting

$$P(f)(t) = \sum_{|\alpha_1|+2|\alpha_2|+\cdots+k|\alpha_k|=m} a_{\alpha_1\ldots\alpha_k}(f(t))\,(\nabla f(t))^{\alpha_1}(\nabla^2 f(t))^{\alpha_2}\cdots(\nabla^k f(t))^{\alpha_k} \tag{1.29$'$}$$

where the $a_{\alpha_1\ldots\alpha_k}(x)$ are holomorphic in x. With the graded algebra bundle $E^{\mathrm{GG}}_{k,\bullet}V^* = \bigoplus_m E^{\mathrm{GG}}_{k,m}V^*$ we associate an analytic fiber bundle

$$X^{\mathrm{GG}}_k := \mathrm{Proj}(E^{\mathrm{GG}}_{k,\bullet}V^*) = (J_kV \smallsetminus \{0\})/\mathbb{C}^* \tag{1.30}$$

over X, which has weighted projective spaces $\mathbb{P}(1^{[r]}, 2^{[r]}, \ldots, k^{[r]})$ as fibers; here $J_kV \smallsetminus \{0\}$ is the set of nonconstant jets of order k. As such, it possesses a tautological sheaf $\mathcal{O}_{X^{\mathrm{GG}}_k}(1)$ [the reader should observe however that $\mathcal{O}_{X^{\mathrm{GG}}_k}(m)$ is invertible only when m is a multiple of $\mathrm{lcm}(1,2,\ldots,k)$].

1.31. Proposition. *By construction, if $\pi_k : X^{\mathrm{GG}}_k \to X$ is the natural projection, we have the direct image formula*

$$(\pi_k)_*\mathcal{O}_{X^{\mathrm{GG}}_k}(m) = \mathcal{O}(E^{\mathrm{GG}}_{k,m}V^*)$$

for all k and m.

In the geometric context, we are not really interested in the bundles $(J_kV \smallsetminus \{0\})/\mathbb{C}^*$ themselves, but rather on their quotients $(J_kV \smallsetminus \{0\}) /\!/ \mathbb{G}_k$ (would such nice complex space quotients exist!). In fact the following fundamental result from [Dem95] shows that the Semple bundle X_k constructed above plays the role of such a quotient.

1.32. Theorem and Definition. *Let $E_{k,m}V^* \subset E^{\mathrm{GG}}_{k,m}V^*$ be the set of polynomial differential operators $f \mapsto P(f)$ that are invariant under arbitrary changes of parametrization, i.e., such that for every $\varphi \in \mathbb{G}_k$*

$$P(f\circ\varphi) = (\varphi')^m P(f)\circ\varphi \tag{$*$}$$

[the weighted degree condition (1.28) *being the special case when $\varphi(t) = \lambda t$, $\lambda \in \mathbb{C}^*$].*

Let $\pi_{k,0} : X_k \to X$ be the Semple jet bundles defined above and let J_kV^{reg} be the bundle of regular k-jets of maps $f : (\mathbb{C},0) \to (X,V)$, that is, jets f such that $f'(0) \neq 0$. Then

(i) *The quotient $J_kV^{\mathrm{reg}}/\mathbb{G}_k$ has the structure of a locally trivial bundle over X, and there is a holomorphic embedding $J_kV^{\mathrm{reg}}/\mathbb{G}_k \hookrightarrow X_k$ over X, which identifies $J_kV^{\mathrm{reg}}/\mathbb{G}_k$ with X^{reg}_k (thus X_k is a relative compactification of $J_kV^{\mathrm{reg}}/\mathbb{G}_k$ over X).*

(ii) *The direct image sheaf*

$$(\pi_{k,0})_*\mathcal{O}_{X_k}(m) \simeq \mathcal{O}(E_{k,m}V^*)$$

can be identified with the sheaf of holomorphic sections of $E_{k,m}V^*$.

(iii) *For every* $m \geqslant 1$, *the relative base locus of the linear system* $|\mathcal{O}_{X_k}(m)|$ *is equal to the set* $X_k^{\rm sing}$ *of singular* k*-jets [one has* $X_k^{\rm sing} = \emptyset$ *for* $k = 1$*]. Moreover,* $\mathcal{O}_{X_k}(1)$ *is relatively big over* X.

Sketch of proof. We refer to [Dem95] for details. In order to prove (i) and (ii), the main point is that the lifts $f_{[k]} : D(0,R) \to (X_k, V_k)$ of a curve $f : D(0,R) \to (X,V)$ are defined inductively by $f_{[k]} = (f_{[k-1]}, [f'_{[k-1]}])$, hence for any change of variable $\varphi : D(0,R') \to D(0,R)$, they satisfy the relations

$$(f\circ\varphi)_{[k]} = f_{[k]}\circ\varphi, \quad (f\circ\varphi)'_{[k-1]} = \varphi' f'_{[k-1]}\circ\varphi \in \mathcal{O}_{X_k}(-1) \subset \pi^*_{k,k-1}V_{k-1}.$$

We conclude that there is a well defined set-theoretic map

$$J_kV^{\rm reg}/\mathbb{G}_k \to X_k^{\rm reg}, \qquad f \mod \mathbb{G}_k \mapsto f_{[k]}(0). \tag{1.33}$$

Given a holomorphic section $\sigma \in H^0(\pi^{-1}_{k,0}(U), \mathcal{O}_{X_k}(m))$, we can then associate a differential operator

$$P(f) = \sigma(f_{[k]})\cdot(f'_{[k-1]})^m. \tag{1.34}$$

Clearly, condition $(*)$ is satisfied and in particular P is homogeneous of degree m on $J_kV^{\rm reg}$; such a holomorphic function must be a homogeneous polynomial on the fibers. □

2. Algebraic properties of the algebra of differential operators

2.A. Green-Griffiths and Semple algebras

By construction, the *Green-Griffiths graded algebra*

$$\mathcal{A}_k^{\rm GG}V^\star = \bigoplus_{m\in\mathbb{Z}} E^{\rm GG}_{k,m}V^\star \tag{2.1}$$

of differential operators is fiberwise isomorphic to the polynomial ring

$$\mathbb{C}[f'_1,\ldots,f'_r,f''_1,\ldots,f''_r,\ldots,f_1^{(k)},\ldots,f_r^{(k)}]$$

and in particular it is finitely generated. More geometrically, we get a holomorphic filtration of $E^{\rm GG}_{k,m}V^\star$ by considering the partial degree of $P(f)$ in terms of the last derivative $f^{(k)}$ and putting

$$F^a(E^{\rm GG}_{k,m}V^\star) = \big\{P \in E^{\rm GG}_{k,m}V^\star\,;\ \deg_{f^{(k)}} P(f) \leqslant a\big\}.$$

Then the graded pieces are polynomials of the form $Q(f',\ldots,f^{(k-1)})(f^{(k)})^{\alpha_k}$, $|\alpha_k| = a$, i.e.

$$G^a(E^{\rm GG}_{k,m}V^\star) \simeq E^{\rm GG}_{k-1,m-ka}V^\star \otimes S^aV^*.$$

We can then inductively combine the successive filtrations obtained via the partial degrees in $f^{(k)}$, $f^{(k-1)},\ldots,f^{(1)} = f'$ to get a full decomposition

$$G^\bullet(E^{\rm GG}_{k,m}V^\star) \simeq \bigoplus_{\substack{a=(a_1,\ldots,a_k)\in\mathbb{N}^k\\ a_1+2a_2+\cdots+ka_k=m}} S^{a_1}V^*\otimes\cdots\otimes S^{a_k}V^*. \tag{2.2}$$

Hence $\mathcal{A}_k^{\rm GG}V^\star$ is just locally isomorphic to a k-fold tensor product of symmetric algebras $S^\bullet V^*$. We define the *Semple algebra* to be the graded subalgebra of $\mathcal{A}_k^{\rm GG}$ such that

$$\mathcal{A}_kV^\star = (\mathcal{A}_k^{\rm GG})^{\mathbb{G}'_k} = \bigoplus_{m\in\mathbb{Z}} E_{k,m}V^\star, \tag{2.3}$$

in particular $\mathcal{A}_1V^\star = \mathcal{A}_1^{\rm GG}V^\star = S^\bullet V^*$. As $\mathbb{G}'_k$ is a non reductive group, it is a priori unclear whether $\mathcal{A}_kV^\star$ is finitely generated for $k \geqslant 2$.

the subalgebra of $\mathbb{G}'_k$-invariant differential operators is finitely generated. This can be checked by hand ([Dem07a], [Dem07b]) for $n = 2$ and $k \leqslant 4$. Rousseau [Rou06] also checked the case $n = 3$, $k = 3$, and then Merker [Mer08, Mer10] proved the finiteness for $n = 2, 3, 4$, $k \leqslant 4$ and $n = 2$, $k = 5$. Recently, Bérczi and Kirwan [BeKi12] made an attempt to prove the finiteness in full generality, but it appears that the general case is still unsettled.

Fix coordinates $(z_1, \ldots, z_n)$ near a point $x_0 \in X$, such that $V_{x_0} = \mathrm{Vect}(\partial/\partial z_1, \ldots, \partial/\partial z_r)$. Let $f = (f_1, \ldots, f_n)$ be a regular k-jet tangent to V. Then there exists $i \in \{1, 2, \ldots, r\}$ such that $f'_i(0) \neq 0$, and there is a unique reparametrization $t = \varphi(\tau)$ such that $f \circ \varphi = g = (g_1, g_2, \ldots, g_n)$ with $g_i(\tau) = \tau$ (we just express the curve as a graph over the z_i-axis, by means of a change of parameter $\tau = f_i(t)$, i.e. $t = \varphi(\tau) = f_i^{-1}(\tau)$). Suppose $i = r$ for the simplicity of notation. The space X_k is a k-stage tower of $\mathbb{P}^{r-1}$-bundles. In the corresponding inhomogeneous coordinates on these $\mathbb{P}^{r-1}$'s, the point $f_{[k]}(0)$ is given by the collection of derivatives

$$\big((g'_1(0), \ldots, g'_{r-1}(0)); (g''_1(0), \ldots, g''_{r-1}(0)); \ldots; (g_1^{(k)}(0), \ldots, g_{r-1}^{(k)}(0))\big).$$

[Recall that the other components $(g_{r+1}, \ldots, g_n)$ can be recovered from $(g_1, \ldots, g_r)$ by integrating the differential system (5.10)]. Thus the map $J_k V^{\mathrm{reg}}/\mathbb{G}_k \to X_k$ is a bijection onto X_k^{reg}, and the fibers of these isomorphic bundles can be seen as unions of r affine charts $\simeq (\mathbb{C}^{r-1})^k$, associated with each choice of the axis z_i used to describe the curve as a graph. The change of parameter formula $\frac{d}{d\tau} = \frac{1}{f'_r(t)} \frac{d}{dt}$ expresses all derivatives $g_i^{(j)}(\tau) = d^j g_i/d\tau^j$ in terms of the derivatives $f_i^{(j)}(t) = d^j f_i/dt^j$

$$\begin{aligned}
&(g'_1, \ldots, g'_{r-1}) = \Big(\frac{f'_1}{f'_r}, \ldots, \frac{f'_{r-1}}{f'_r}\Big);\\
&(g''_1, \ldots, g''_{r-1}) = \Big(\frac{f''_1 f'_r - f''_r f'_1}{f'^3_r}, \ldots, \frac{f''_{r-1} f'_r - f''_r f'_{r-1}}{f'^3_r}\Big); \ \ldots\ ;\\
&(g_1^{(k)}, \ldots, g_{r-1}^{(k)}) = \Big(\frac{f_1^{(k)} f'_r - f_r^{(k)} f'_1}{f'^{k+1}_r}, \ldots, \frac{f_{r-1}^{(k)} f'_r - f_r^{(k)} f'_{r-1}}{f'^{k+1}_r}\Big) + (\text{order} < k).
\end{aligned} \tag{3.12}$$

2.B. Wronskians

Let U be an open set of X, $\dim X = n$, and $s_0, \ldots, s_k \in \mathcal{O}_X(U)$ be holomorphic functions. To these functions, we can associate a Wronskian operator of order k defined by

$$W_k(s_0, \ldots, s_k)(f) = \begin{vmatrix} s_0(f) & s_1(f) & \ldots & s_k(f) \\ D(s_0(f)) & D(s_1(f)) & \ldots & D(s_k(f)) \\ \vdots & & & \vdots \\ D^k(s_0(f)) & D^k(s_1(f)) & \ldots & D^k(s_k(f)) \end{vmatrix} \tag{4.1}$$

where $f : t \mapsto f(t) \in U \subset X$ is a germ of holomorphic curve (or a k-jet of curve), and $D = \frac{d}{dt}$. For a biholomorphic change of variable φ of $(\mathbb{C}, 0)$, we find by induction on ℓ a polynomial differential operator $Q_{\ell,s}$ of order $\leqslant \ell$ acting on φ satisfying

$$D^\ell(s_j(f \circ \varphi)) = \varphi'^\ell D^\ell(s_j(f)) \circ \varphi + \sum_{s<\ell} p_{\ell,s}(\varphi) D^s(s_j(f)) \circ \varphi.$$

It follows easily from there that

$$W_k(s_0, \ldots, s_k)(f \circ \varphi) = (\varphi')^{1+2+\cdots+k} W_k(s_0, \ldots, s_k)(f) \circ \varphi,$$

hence $W_k(s_0, \ldots, s_k)(f)$ is an invariant differential operator of degree $k' = \frac{1}{2}k(k+1)$. Especially, we get in this way a section that we denote

$$W_k(s_0, \ldots, s_k) = \begin{vmatrix} s_0 & s_1 & \ldots & s_k \\ D(s_0) & D(s_1) & \ldots & D(s_k) \\ \vdots & & & \vdots \\ D^k(s_0) & D^k(s_1) & \ldots & D^k(s_k) \end{vmatrix} \in H^0(U, E_{k,k'} T^*_X). \tag{4.2}$$

2.2. Proposition. *These Wronskian operators satisfy the following properties.*

(a) $W_k(s_0, \ldots, s_k)$ *is* $\mathbb{C}$*-multilinear and alternate in* $(s_0, \ldots, s_k)$.

(b) *For any* $g \in \mathcal{O}_X(U)$*, we have*

$$W_k(gs_0, \ldots, gs_k) = g^{k+1} W_k(s_0, \ldots, s_k).$$

Property 2.2 (b) is an easy consequence of the Leibniz formula

$$D^\ell(g(f)s_j(f)) = \sum_{k=0}^{\ell} \binom{\ell}{k} D^k(g(f)) D^{\ell-k}(s_j(f)),$$

by performing linear combinations of rows in the determinants. This property implies in its turn that one can define more generally an operator

$$W_k(s_0, \ldots, s_k) \in H^0(U, E_{k,k'}T^*_X \otimes L^{k+1}) \tag{2.3}$$

for any $(k+1)$-tuple of sections $s_0, \ldots, s_k \in H^0(U, L)$ of a holomorphic line bundle $L \to X$. In fact, when we compute the Wronskian in a local trivialization of $L_{\restriction U}$, Property 4.3 (b) shows that the determinant is independent of the trivialization. Moreover, if $g \in H^0(U, G)$ for some line bundle $G \to X$, we have

$$W_k(gs_0, \ldots, gs_k) = g^{k+1} W_k(s_0, \ldots, s_k) \in H^0(U, E_{k,k'}T^*_X \otimes L^{k+1} \otimes G^{k+1}). \tag{2.4}$$

2.C. Brackets

If P is a differential operator given by a section of $\mathcal{O}(E^{\mathrm{GG}}_{k,m}V^\star)$, we define DP to be its "obvious" derivative

$$DP(f) = P(f)', \quad \text{i.e.} \quad DP(f)(t) = \frac{d}{dt} P(f)(t). \tag{2.5}$$

The operator DP is then a section of $\mathcal{O}(E^{\mathrm{GG}}_{k,m}V^\star)$. If P is a $\mathbb{G}'_k$-invariant operator in $\mathcal{O}(E_{k,m}V^\star)$, the relation $P(f \circ \varphi) = \varphi'^m P(f) \circ \varphi$ implies

$$DP(f \circ \varphi) = (\varphi'^m P(f) \circ \varphi)' = \varphi'^{m+1} DP(f) \circ \varphi + m\varphi'^{m-1}\varphi'' P(f) \circ \varphi,$$

therefore DP is no longer an invariant operator (unless $m = 0$, in which case DP is of degree 1). However, if P, Q are $\mathbb{G}'_k$-invariant operators of respective degrees δ_P, δ_Q; it is easy to check that their *bracket* defined as

$$[P, Q] = \delta_P P(DQ) - \delta_Q Q(DP) \tag{2.6}$$

is again $\mathbb{G}'_k$-invariant, of degree $\delta_P + \delta_Q + 1$. This can be seen by observing that the terms $\varphi''(\ldots)$ coming from DP and DQ cancel, or by noticing that

$$[P, Q] = P^{\delta_Q+1} Q^{-\delta_P+1} D\left(\frac{Q^{\delta_P}}{P^{\delta_Q}}\right) \tag{2.6$'$}$$

where $Q^{\delta_P}/P^{\delta_Q}$ is homogeneous of degree 0 (i.e. $\mathbb{G}_k$-invariant). A straightforward albeit tedious calculation shows that this bracket satisfies the usual Jacobi identity

$$[P, [Q, R]] + [Q, [R, P]] + [R, [P, Q]] = 0. \tag{2.7}$$

Formula (2.6$'$) has the advantage that for any line bundle L (or even any $\mathbb{Q}$-line bundle L) and

$$P \in H^0(X, E_{k,\delta_P}V^* \otimes L^{\delta_P}), \; Q \in H^0(X, E_{k,\delta_Q}V^* \otimes L^{\delta_Q})$$

we also get globally defined brackets

$$[P, Q] \in H^0(X, E_{k+1,\delta_P+\delta_Q+1}V^* \otimes L^{\delta_P+\delta_Q}).$$

Another special case is the "degree 0" bracket associated with two sections $\sigma_1, \sigma_2 \in H^0(X, L)$

$$\tau = [\sigma_1, \sigma_2] = \sigma_1 D\sigma_2 - \sigma_2 D\sigma_1 = \sigma_1^2 D\left(\frac{\sigma_2}{\sigma_1}\right) \in H^0(X, E_{1,1}T_X^* \otimes L^2) = H^0(X, T_X^* \otimes L^2). \tag{2.8}$$

In a similar way, given integers $a, b \geqslant 1$ and families of sections $\sigma_j \in H^0(X, L^a)$, $1 \leqslant j \leqslant k-2$, $\tau_j \in H^0(X, V^* \otimes L^b) = H^0(X, E_{1,1}V^* \otimes L^b)$, $1 \leqslant j \leqslant k$, we define inductively iterated brackets

$$B_k(\sigma_1, \ldots, \sigma_{k-2}\,; \tau_1, \ldots, \tau_k) \in H^0(X, E_{k,2k-1}V^* \otimes L^{(k-2)a+kb}), \quad k \geqslant 2, \tag{2.9}$$

by putting

$$B_2(\tau_1, \tau_2) = [\tau_1, \tau_2] = \tau_1^2 D\left(\frac{\tau_2}{\tau_1}\right), \tag{2.9'}$$

and, for $k \geqslant 3$ and $c_k = (k-3) - (k-2)b/a \in \mathbb{Q}$,

$$B_k(\sigma_1, \ldots, \sigma_{k-2}\,; \tau_1, \ldots, \tau_k) = \sigma_1^{c_k+1}\tau_1^{2k-2}\, D\left(\frac{B_{k-1}(\sigma_2, \ldots, \sigma_{k-2}\,; \tau_2, \ldots, \tau_k)}{\sigma_1^{c_k}\tau_1^{2k-3}}\right). \tag{2.9''}$$

If L is very ample, by (2.8), there are many such sections when we take $a = 1$, $b = 2$, $c_k = 1-k$, and we then get sections $B_k(\sigma_1, \ldots, \sigma_{k-2}\,; \tau_1, \ldots, \tau_k) \in H^0(X, E_{k,2k-1}V^* \otimes L^{3k-2})$ whose degrees $m = 2k-1$ grow linearly with k, as well as the correcting twist L^{3k-2}.

3. Morse inequalities and the Green-Griffiths-Lang conjecture

3.A. Statement of Morse inequalities

One of the main purpose of holomorphic Morse inequalities is to provide estimates of cohomology groups with values in high tensor powers of a given line bundle L, once a smooth hermitian metric h on L is given. We denote by $\Theta_{L,h} = -\frac{i}{2\pi}\partial\overline{\partial}\log h$ the $(1,1)$-curvature form of h.

3.1. Holomorphic Morse inequalities ([Dem85]). *Let X be a compact complex manifolds, $E \to X$ a holomorphic vector bundle of rank r, and (L, h) a hermitian line bundle. The dimensions $h^q(X, E \otimes L^m)$ of cohomology groups of the tensor powers $E \otimes L^m$ satisfy the following asymptotic estimates as $m \to +\infty$:*

(3.1 WM) *Weak Morse inequalities*:

$$h^q(X, E \otimes L^m) \leqslant r\frac{m^n}{n!}\int_{X(L,h,q)} (-1)^q \Theta_{L,h}^n + o(m^n)\ .$$

(3.1 SM) *Strong Morse inequalities*:

$$\sum_{0 \leqslant j \leqslant q} (-1)^{q-j} h^j(X, E \otimes L^m) \leqslant r\frac{m^n}{n!}\int_{X(L,h,\leqslant q)} (-1)^q \Theta_{L,h}^n + o(m^n)\ .$$

(3.1 RR) *Asymptotic Riemann-Roch formula*:

$$\chi(X, E \otimes L^m) := \sum_{0 \leqslant j \leqslant n} (-1)^j h^j(X, E \otimes L^m) = r\frac{m^n}{n!}\int_X \Theta_{L,h}^n + o(m^n)\ .$$

Moreover (cf. Bonavero's PhD thesis [Bon93]), if $h = e^{-\varphi}$ is a singular hermitian metric with analytic singularities of pole set $\Sigma = \varphi^{-1}(-\infty)$, the estimates still hold provided all cohomology groups are replaced by cohomology groups $H^q(X, E \otimes L^m \otimes \mathcal{I}(h^m))$ twisted with the corresponding L^2 multiplier ideal sheaves

$$\mathcal{I}(h^m) = \mathcal{I}(k\varphi) = \big\{f \in \mathcal{O}_{X,x},\ \ \exists V \ni x,\ \int_V |f(z)|^2 e^{-m\varphi(z)} d\lambda(z) < +\infty\big\},$$

and provided the Morse integrals are computed on the regular locus of h, namely restricted to $X(L,h,q) \smallsetminus \Sigma$:

$$\int_{X(L,h,q)\smallsetminus\Sigma} (-1)^q \Theta^n_{L,h}.$$

The special case of 3.1 (SM) when $q = 1$ yields a very useful criterion for the existence of sections of large multiples of L.

3.2. Corollary. *Let $L \to X$ be a holomorphic line bundle equipped with a singular hermitian metric $h = e^{-\varphi}$ with analytic singularities of pole set $\Sigma = \varphi^{-1}(-\infty)$. Then we have the following lower bounds*

(a) *at the h^0 level :*

$$\begin{aligned} h^0(X, E \otimes L^m) &\geqslant h^0(X, E \otimes L^m \otimes \mathcal{I}(h^m)) \\ &\geqslant h^0(X, E \otimes L^m \otimes \mathcal{I}(h^m)) - h^1(X, E \otimes L^m \otimes \mathcal{I}(h^m)) \\ &\geqslant r \frac{k^n}{n!} \int_{X(L,h,\leqslant 1)\smallsetminus\Sigma} \Theta^n_{L,h} - o(k^n) \ . \end{aligned}$$

Especially L is big as soon as $\int_{X(L,h,\leqslant 1)\smallsetminus\Sigma} \Theta^n_{L,h} > 0$ for some singular hermitian metric h on L.

(b) *at the h^q level :*

$$h^q(X, E \otimes L^m \otimes \mathcal{I}(h^m)) \geqslant r \frac{k^n}{n!} \sum_{j=q-1,q,q+1} (-1)^q \int_{X(L,h,j)\smallsetminus\Sigma} \Theta^n_{L,h} - o(k^n) \ .$$

The goal of this section is to study the existence and properties of entire curves $f : \mathbb{C} \to X$ drawn in a complex irreducible n-dimensional variety X, and more specifically to show that they must satisfy certain global algebraic or differential equations as soon as X is projective of general type. By means of holomorphic Morse inequalities and a probabilistic analysis of the cohomology of jet spaces, it is possible to prove a significant step of the generalized Green-Griffiths-Lang conjecture. The use of holomorphic Morse inequalities was first suggested in [Dem07a], and then carried out in an algebraic context by S. Diverio in his PhD work ([Div08, Div09]). The general more analytic and more powerful results presented here first appeared in [Dem11, Dem12].

3.B. Positively curved hermitian metric on the Semple tautological bundle

To start with, we consider the directed variety $(\mathbb{C}^r, T_{\mathbb{C}^r})$ when $\mathbb{C}^r$ is equipped with its standard hermitian metric. Pick a random k-jet

$$f(t) = x + t\xi_1 + t^2\xi_2 + \cdots + t^k\xi_k + O(t^{k+1}), \quad \xi_s \in \mathbb{C}^r,\ 1 \leqslant s \leqslant k. \tag{3.3}$$

Given $\varepsilon_1, \ldots, \varepsilon_k > 0$, we get a natural Finsler metric on the tautological Green-Griffiths bundle $\mathcal{O}_{\mathrm{GG},(\mathbb{C}^r)_k}(-1)$ of $(\mathbb{C}^r, T_{\mathbb{C}^r})$ by putting

$$\|f\|^{\mathrm{GG}}_{\varepsilon,p} = \bigg(\sum_{s=1}^k (\varepsilon_s|\xi_s|)^{2p/s}\bigg)^{1/2p}. \tag{3.4}$$

In fact if $\lambda \cdot f$ denotes the $t \mapsto f(\lambda t)$, we do have the required homogeneity property under the $\mathbb{C}^*$ action, namely $\|\lambda \cdot f\|^{\mathrm{GG}}_{\varepsilon,p} = |\lambda|\, \|f\|^{\mathrm{GG}}_{\varepsilon,p}$. The choice of a suitable integer p (e.g. $p = \mathrm{lcm}(1, 2, \ldots, k)$ or a multiple) yields a smooth metric on $\mathcal{O}_{\mathrm{GG},(\mathbb{C}^r)_k}(-1)$. However, formula (3.4) is not $\mathbb{G}'_k$ invariant and therefore cannot be used to construct a metric on the Semple tautological bundle $\mathcal{O}_{(\mathbb{C}^r)_k}(-1)$ of $(\mathbb{C}^r, T_{\mathbb{C}^r})$. Let us assume that we have a regular k-jet, i.e. that $\xi_1 \neq 0$. By composing f with a suitable element $\varphi(t) = t + a_2t^2 + \ldots + a_kt^k + O(t^{k+1})$ and applying a Gram-Schmidt orthogonalization argument, we can always obtain $\xi_s \in (\xi_1)^\perp$ for $s \geqslant 2$ (proceeding inductively with changes of variables $t \mapsto t + a_s t^s$). In fact, there is a unique $\varphi \in \mathbb{G}'_k$ achieving this condition. In view of (??), it is natural to define a Finsler metric on f (taken $\mathrm{mod}\,\mathbb{G}'_k$) by

$$\|f\|_{\varepsilon,\beta,p} = \left(\sum_{s=1}^{k} \big(\varepsilon_s|\xi_1|^{1-s\beta_s}|\xi_s|^{\beta_s}\big)^{2p}\right)^{1/2p}, \tag{3.5}$$

under the assumption that $\xi_s \in (\xi_1)^\perp$ for $s \geqslant 2$; here $\beta = (\beta_1, \ldots, \beta_s)$ is a positive weight with $\beta_1 = 1$ and $0 < \beta_s \leqslant 1/s$ for $s \geqslant 2$ (the first term in the sum is equal to $(\varepsilon_1|\xi_1|)^{2p}$ and we will in fact take $\varepsilon_1 = 1$). The curvature of $\mathcal{O}_{(\mathbb{C}^r)_k}(1)$ fiberwise (i.e. on the rational variety $\mathbb{R}_{r,k}$) is given by $\frac{i}{2\pi}\partial\overline\partial \log \|f\|^2_{\varepsilon,p}$, provided holomorphic coordinates are used. We compute the curvature form at a point $\xi^0 = (\xi^0_s)_{1\leqslant s\leqslant k}$ such that $\xi^0_1 \neq 0$. By applying a dilation $t \mapsto f(\lambda t)$, $\lambda \in \mathbb{C}^*$, we can assume that $|\xi^0_1| = 1$. A nearby point $\xi = (\xi_s)_{1\leqslant s\leqslant k}$ can be written as $\xi_1 = \xi^0_1 + \zeta$ and ξ_s, $s \geqslant 2$, with $\zeta, \xi_s \in (\xi^0_1)^\perp$. Notice that $((\xi^0_1)^\perp)^k$ is a $k(r-1)$-dimensional complex subspace that defines an affine chart of $\mathbb{R}_{r,k}$ containing ξ^0. The difficulty is that we do not necessarily have $\langle \xi_s, \xi_1\rangle = 0$ any more, however $\langle \xi_s, \xi_1\rangle = \langle \xi_s, \zeta\rangle = O(|\zeta|)$ for $s \geqslant 2$, and we have to correct this by applying an element $\varphi \in \mathbb{G}'_k$ close to identity. When computing $i\partial\overline\partial(...)$ at $\zeta = 0$, all terms $O(\zeta^2, \overline\zeta^2, |\zeta|^3)$ can be neglected, and the calculations performed below will be made modulo such terms. In particular, higher powers of $\langle \xi_s, \zeta\rangle$ can be be neglected as they are of the form $O(\overline\zeta^2)$. A suitable choice is

$$\varphi(t) = t - \sum_{s=2}^{k} \langle \xi_s, \zeta\rangle t^s.$$

Then

$$f \circ \varphi(t) = x + \sum_{s=1}^{k} t^s \tilde\xi_s + O(t^{k+1})$$

where $\tilde\xi_1 = \xi_1$ and

$$\tilde\xi_s = \xi_s - \langle \xi_s, \zeta\rangle \xi_1 - \sum_{j=2}^{s-1} j\langle \xi_{s-j+1}, \zeta\rangle \xi_j \mod O(\zeta^2, \overline\zeta^2, |\zeta|^3), \quad s \geqslant 2$$

(the final summation is obtained by expanding the terms $(t - \sum_{\ell\geqslant 2}\langle \xi_\ell, \zeta\rangle t^\ell)^j \xi_j$ for $2 \leqslant j \leqslant s-1$ and $\ell = s - j + 1$). Observe that $|\xi_1|^2 = 1 + |\zeta|^2$ and $\langle \xi_j, \xi_1\rangle = \langle \xi_j, \xi^0_1\rangle + \langle \xi_j, \zeta\rangle = \langle \xi_j, \zeta\rangle$ for $j \geqslant 2$, hence

$$\langle \tilde\xi_s, \xi_1\rangle = \langle \xi_s, \zeta\rangle - \langle \xi_s, \zeta\rangle(1 + |\zeta|^2) - \sum_{j=2}^{s-1} j\langle \xi_{s-j+1}, \zeta\rangle\langle \xi_j, \zeta\rangle = O(\zeta^2, \overline\zeta^2, |\zeta|^3),$$

so we do not need any more accurate correction. Moreover, by a straightforward calculation

$$|\tilde\xi_s|^2 = \left|\xi_s - \sum_{j=2}^{s-1} j\langle \xi_{s-j+1}, \zeta\rangle \xi_j\right|^2 - |\langle \xi_s, \zeta\rangle|^2 \mod O(\zeta^2, \overline\zeta^2, |\zeta|^3).$$

Therefore, for $\varepsilon_1 = 1$, we find

$$\begin{aligned}\|f\|_{\varepsilon,\beta,p} &= \left(|\xi_1|^{2p} + \sum_{s=2}^{k} \big(\varepsilon_s|\xi_1|^{1-s\beta_s}|\tilde\xi_s|^{\beta_s}\big)^{2p}\right)^{1/2p}\\ &= \left((1 + |\zeta|^2)^p + \sum_{s=2}^{k} \big(\varepsilon_s^2(1 + |\zeta|^2)^{1-s\beta_s}|\tilde\xi_s|^{2\beta_s}\big)^p\right)^{1/2p} \qquad (3.6)\end{aligned}$$

We take profit of the constants ε_s to perform a rescaling and set $u_s = \varepsilon_s \xi_s$. Then our formula becomes

$$\|f\|^2_{\varepsilon,\beta,p} = \Big((1+|\zeta|^2)^p + \sum_{s=2}^{k} \big((1+|\zeta|^2)^{1-s\beta_s} |\tilde u_s|^{2\beta_s} \big)^p \Big)^{1/2p} \tag{3.7}$$

where (modulo negligible terms) the u_s and $\tilde u_s$ are defined by $u_1 = u_1^0 + \zeta$, $|u_1^0|^2 = 1$,

$$\tilde u_s = u_s - \langle u_s, \zeta\rangle u_1 - \sum_{j=2}^{s-1} \frac{\varepsilon_s}{\varepsilon_j \varepsilon_{s-j+1}} j \langle u_{s-j+1}, \zeta\rangle u_j$$

$$|\tilde u_s|^2 = \Big| u_s - \sum_{j=2}^{s-1} \frac{\varepsilon_s}{\varepsilon_j \varepsilon_{s-j+1}} j \langle u_{s-j+1}, \zeta\rangle u_j \Big|^2 - |\langle u_s, \zeta\rangle|^2.$$

By taking $\varepsilon_k \ll \varepsilon_{k-1} \ll \cdots \ll \varepsilon_2 \ll 1$, all terms $\varepsilon_s/(\varepsilon_j \varepsilon_{s-j+1})$ will become negligible (e.g. equal to ε if we put $\varepsilon_s = \varepsilon^{2s-3}$, $s \geqslant 2$). However, we have to estimate the errors more precisely and for this, we observe that for all complex numbers x, y and all $\gamma > 0$, we have

$$|xy| \leqslant |x|^{1+\gamma} + |y|^{1+1/\gamma}$$

(consider the disjoint cases $|y| \leqslant |x|^\gamma$ and $|x| < |y|^{1/\gamma}$). In particular

$$(|u_{s_j+1}||u_j|)^{\beta_s} \leqslant |u_{s-j+1}|^{\beta_s(1+\gamma)} + |u_j|^{\beta_s(1+1/\gamma)}.$$

In this circumstance, we want that $\beta_s(1+\gamma) = \beta_{s-j+1}$ and $\beta_s(1+1/\gamma) = \beta_j$, whence $\gamma = \beta_{s-j+1}/\beta_j$, and so we get the condition

$$\beta_s(1 + \beta_{s-j+1}/\beta_j) = \beta_{s-j+1}, \qquad \text{i.e} \quad \frac{1}{\beta_j} + \frac{1}{\beta_{s-j+1}} = \frac{1}{\beta_s}$$

for all $j, s \geqslant 2$. This leads us to take $1/\beta_s = \mathrm{Const}\,(s-1)$, i.e. $\beta_s = \alpha/(s-1)$ for $s \geqslant 2$, and we want to take the largest possible value of $\alpha > 0$. In any case, with the above choise of the β_s and with $|\zeta| \leqslant 1$, we find

$$\|f\|^2_{\varepsilon,\beta,p} = \Big((1+|\zeta|^2)^p + \sum_{s=2}^{k} (1+|\zeta|^2)^{p(1-s\beta_s)} \big(|u_s|^2 - |\langle u_s, \zeta\rangle|^2 \big)^{p\beta_s} \Big)^{1/2p} (1 \pm O(\varepsilon)) \tag{3.8}$$

by expanding the ε terms in the squares $|\tilde u_s|^2$. Modulo negligible terms $O(\zeta^2, \overline{\zeta}^2, |\zeta|^3)$, we have

$$|u_s|^2 - |\langle u_s, \zeta\rangle|^2 = |u_s|^2 + |u_s \wedge \zeta|^2 - |u_s|^2|\zeta|^2 = \frac{|u_s|^2 + |u_s \wedge \zeta|^2}{1+|\zeta|^2}.$$

hence

$$\|f\|^2_{\varepsilon,\beta,p} = \Big((1+|\zeta|^2)^p + \sum_{s=2}^{k} (1+|\zeta|^2)^{p(1-(s+1)\beta_s)} \big(|u_s|^2 + |u_s \wedge \zeta|^2 \big)^{p\beta_s} \Big)^{1/2p} (1 \pm O(\varepsilon)). \tag{3.8$'$}$$

In order to ensure the plurisubharmonicity of the summation between the big parentheses (so as to get a singular hermitian metric of $\mathcal{O}_{X_k}(1)$ of nonnegative relative curvature form), we want $\beta_s = \alpha/(s-1) \leqslant 1/(s+1)$. For $s \geqslant 2$, this leads us to take $\alpha = 1/3$, hence $\beta_s = 1/3(s-1)$.

A suitable choice is

$$\varphi(t) = t - |\xi_1^0|^{-2} \sum_{s=2}^{k} \langle \xi_s, \zeta\rangle t^s.$$

Then

$$f \circ \varphi(t) = x + \sum_{s=1}^{k} t^s \tilde\xi_s + O(t^{k+1})$$

where $\tilde{\xi}_1 = \xi_1$ and

$$\tilde{\xi}_s = \xi_s - |\xi_1^0|^{-2}\left(\langle\xi_s,\zeta\rangle\xi_1 + \sum_{j=2}^{s-1} j\,\langle\xi_{s-j+1},\zeta\rangle\xi_j\right) \mod O(\zeta^2,\overline{\zeta}^2,|\zeta|^3), \quad s \geqslant 2$$

(the final summation is obtained by expanding the terms $(t-\langle\xi_\ell,\zeta\rangle t^\ell)^j\xi_j$ for $\ell = s-j+1$, $j \leqslant s-1$. As $\langle\tilde{\xi}_s,\xi_1\rangle = |\xi_1^0|^2 + |\zeta|^2$, we have

$$\langle\tilde{\xi}_s,\xi_1\rangle = \langle\xi_s,\zeta\rangle - \langle\xi_s,\zeta\rangle\frac{|\xi_1^0|^2+|\zeta|^2}{|\xi_1^0|^2} - |\xi_1^0|^{-2}\sum_{j=2}^{s-1} j\,\langle\xi_{s-j+1},\zeta\rangle\langle\xi_j,\zeta\rangle = O(\zeta^2,\overline{\zeta}^2,|\zeta|^3),$$

so we do not need any more accurate correction. Moreover, by a straightforward calculation

$$|\tilde{\xi}_s|^2 = \left|\xi_s - \sum_{j=2}^{s-1}\frac{j\langle\xi_{s-j+1},\zeta\rangle}{|\xi_1^0|^2}\xi_j\right|^2 - \frac{|\langle\xi_s,\zeta\rangle|^2}{|\xi_1^0|^2} \mod O(\zeta^2,\overline{\zeta}^2,|\zeta|^3).$$

Therefore, we find

$$\begin{aligned}\|f\|_{\varepsilon,p} &= \left(\sum_{s=1}^k \varepsilon_s\big(|\xi_1|^{s-1}|\tilde{\xi}_s|\big)^{2p/(2s-1)}\right)^{1/2p}\\ &= \left(\sum_{s=1}^k \varepsilon_s\big((1+|\zeta|^2)^{s-1}|\tilde{\xi}_s|\big)^{p/(2s-1)}\right)^{1/2p} \qquad (3.6)\end{aligned}$$

By rotating the coordinates and using a dilation, we can always reduce ourselves to computing the curvature at a point $\xi^0 = (\xi_s^0)_{1\leqslant s\leqslant k}$ such that $\xi_1^0 = (1,0,\ldots,0)$ and $\langle\xi_s^0,\xi_1^0\rangle = 0$, i.e. $\xi_{s,1}^0 = 0$ for $s \geqslant 2$. A nearby point $(\xi_s)_{1\leqslant s\leqslant k}$ is given by $\xi_1 = \xi_1^0 + \zeta = (1,\zeta_2,\ldots,\zeta_r)$ and $\xi_s = (0,\xi_{s,2},\ldots,\xi_{s,r})$, and the $k(r-1)$ coordinates ζ_j and $\xi_{s,j}$, $2 \leqslant s,j \leqslant r$, precisely define the holomorphic structure. The difficulty is that we do not necessarily have $\langle\xi_s,\xi_1\rangle = 0$ any more, however $\langle\xi_s,\xi_1\rangle = \langle\xi_s,\zeta\rangle = O(|\zeta|)$ for $s \geqslant 2$, and we have to correct this by applying an element $\varphi \in \mathbb{G}'_k$ close to identity. When computing $i\partial\overline{\partial}(...)$ at $\zeta = 0$, all terms $O(\zeta^2,\overline{\zeta}^2,|\zeta|^3)$ can be neglected, and the calculations performed below will be made modulo such terms. In particular, higher powers of $\langle\xi_2,\zeta\rangle$ can be be neglected (they are of the form $O(\overline{\zeta}^2)$).

9.A. Introduction

Let (X,V) be a directed variety. By definition, proving the algebraic degeneracy of an entire curve $f;(\mathbb{C},T_\mathbb{C}) \to (X,V)$ means finding a non zero polynomial P on X such that $P(f) = 0$. As already explained in § 8, all known methods of proof are based on establishing first the existence of certain algebraic differential equations $P(f\,;\,f',f'',\ldots,f^{(k)}) = 0$ of some order k, and then trying to find enough such equations so that they cut out a proper algebraic locus $Y \subsetneq X$. We use for this global sections of $H^0(X,E_{k,m}^{\rm GG}V^*\otimes\mathcal{O}(-A))$ where A is ample, and apply the fundamental vanishing theorem 8.15. It is expected that the global sections of $H^0(X,E_{k,m}^{\rm GG}V^*\otimes\mathcal{O}(-A))$ are precisely those which ultimately define the algebraic locus $Y \subsetneq X$ where the curve f should lie. The problem is then reduced to (i) showing that there are many non zero sections of $H^0(X,E_{k,m}^{\rm GG}V^*\otimes\mathcal{O}(-A))$ and (ii) understanding what is their joint base locus. The first part of this program is the main result of this section.

9.1. Theorem. *Let (X,V) be a directed projective variety such that K_V is big and let A be an ample divisor. Then for $k \gg 1$ and $\delta \in \mathbb{Q}_+$ small enough, $\delta \leqslant c(\log k)/k$, the number of sections $h^0(X,E_{k,m}^{\rm GG}V^*\otimes\mathcal{O}(-m\delta A))$ has maximal growth, i.e. is larger that $c_k m^{n+kr-1}$ for some $m \geqslant m_k$, where c, $c_k > 0$, $n = \dim X$ and $r = \operatorname{rank} V$. In particular, entire curves $f : (\mathbb{C},T_\mathbb{C}) \to (X,V)$ satisfy (many) algebraic differential equations.*

The statement is very elementary to check when $r = \operatorname{rank} V = 1$, and therefore when $n = \dim X = 1$. In higher dimensions $n \geqslant 2$, only very partial results were known before Theorem 9.1 was obtained in [Dem11], [and they dealt merely with the absolute case $V = T_X$]. In dimension 2, Theorem 9.1 is a consequence of the Riemann-Roch calculation of Green-Griffiths [GrGr79], combined with a vanishing theorem due to Bogomolov [Bog79] – the latter actually only applies to the top cohomology group H^n, and things become much more delicate when extimates of intermediate cohomology groups are needed. In higher dimensions, Diverio [Div08, Div09] proved the existence of sections of $H^0(X, E^{\mathrm{GG}}_{k,m}V^* \otimes \mathcal{O}(-1))$ whenever X is a hypersurface of $\mathbb{P}^{n+1}_{\mathbb{C}}$ of high degree $d \geqslant d_n$, assuming $k \geqslant n$ and $m \geqslant m_n$. More recently, Merker [Mer15] was able to treat the case of arbitrary hypersurfaces of general type, i.e. $d \geqslant n + 3$, assuming this time k to be very large. The latter result is obtained through explicit algebraic calculations of the spaces of sections, and the proof is computationally very intensive. Bérczi [Ber15, Ber18] also obtained related results with a different approach based on residue formulas, assuming e.g. $d \geqslant n^{9n}$.

All these approaches are algebraic in nature. Here, however, our techniques are based on more elaborate curvature estimates in the spirit of Cowen-Griffiths [CoGr76]. They require holomorphic Morse inequalities (see 9.10 below) – and we do not know how to translate our method in an algebraic setting. Notice that holomorphic Morse inequalities are essentially insensitive to singularities, as we can pass to non singular models and blow-up X as much as we want: if $\mu : \widetilde{X} \to X$ is a modification then $\mu_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$ and $R^q\mu_*\mathcal{O}_{\widetilde{X}}$ is supported on a codimension 1 analytic subset (even codimension 2 if X is smooth). It follows from the Leray spectral sequence that the cohomology estimates for L on X or for $\widetilde{L} = \mu^*L$ on $\widetilde{X}$ differ by negligible terms, i.e.

$$h^q(\widetilde{X}, \widetilde{L}^{\otimes m}) - h^q(X, L^{\otimes m}) = O(m^{n-1}). \tag{9.2}$$

Finally, singular holomorphic Morse inequalities (in the form obtained by L. Bonavero [Bon93]) allow us to work with singular Hermitian metrics h; this is the reason why we will only require to have big line bundles rather than ample line bundles. In the case of linear subspaces $V \subset T_X$, we introduce singular Hermitian metrics as follows.

9.3. Definition. *A singular hermitian metric on a linear subspace $V \subset T_X$ is a metric h on the fibers of V such that the function $\log h : \xi \mapsto \log |\xi|^2_h$ is locally integrable on the total space of V.*

Such a metric can also be viewed as a singular Hermitian metric on the tautological line bundle $\mathcal{O}_{P(V)}(-1)$ on the projectivized bundle $P(V) = V \smallsetminus \{0\}/\mathbb{C}^*$, and therefore its dual metric h^* defines a curvature current $\Theta_{\mathcal{O}_{P(V)}(1),h^*}$ of type $(1,1)$ on $P(V) \subset P(T_X)$, such that

$$p^*\Theta_{\mathcal{O}_{P(V)}(1),h^*} = \frac{i}{2\pi}\partial\overline{\partial}\log h, \qquad \text{where } p : V \smallsetminus \{0\} \to P(V). \tag{9.4}$$

If $\log h$ is quasi-plurisubharmonic (or quasi-psh, which means psh modulo addition of a smooth function) on V, then $\log h$ is indeed locally integrable, and we have moreover

$$\Theta_{\mathcal{O}_{P(V)}(1),h^*} \geqslant -C\omega \tag{9.5}$$

for some smooth positive $(1,1)$-form on $P(V)$ and some constant $C > 0$; conversely, if (9.5) holds, then $\log h$ is quasi-psh.

9.6. Definition. *We will say that a singular Hermitian metric h on V is* admissible *if h can be written as $h = e^{\varphi}h_{0|V}$ where h_0 is a smooth positive definite Hermitian on T_X and φ is a quasi-psh weight with analytic singularities on X, as in Definition* 9.3*. Then h can be seen as a singular hermitian metric on $\mathcal{O}_{P(V)}(1)$, with the property that it induces a smooth positive definite metric on a Zariski open set $X' \subset X \smallsetminus \operatorname{Sing}(V)$; we will denote by $\operatorname{Sing}(h) \supset \operatorname{Sing}(V)$ the complement of the largest such Zariski open set X'.*

If h is an admissible metric, we define $\mathcal{O}_h(V^*)$ to be the sheaf of germs of holomorphic sections sections of $V^*_{|X\smallsetminus\operatorname{Sing}(h)}$ which are h^*-bounded near $\operatorname{Sing}(h)$; by the assumption on the analytic singularities, this is a coherent sheaf (as the direct image of some coherent sheaf on $P(V)$), and actually, since $h^* = e^{-\varphi} h_0^*$, it is a subsheaf of the sheaf $\mathcal{O}(V^*) := \mathcal{O}_{h_0}(V^*)$ associated with a smooth positive definite metric h_0 on T_X. If r is the generic rank of V and m a positive integer, we define similarly

$$
\begin{aligned}
(9.7)\qquad {}^bK_{V,h}^{[m]} &= \text{sheaf of germs of holomorphic sections of } (\det V^*_{|X'})^{\otimes m} = (\Lambda^r V^*_{|X'})^{\otimes m} \\
&\ \text{which are } \det h^*\text{-bounded},
\end{aligned}
$$

so that ${}^bK_V^{[m]} := {}^bK_{V,h_0}^{[m]}$ according to Def. 2.7. For a given admissible Hermitian structure (V,h), we define similarly the sheaf $E_{k,m}^{\mathrm{GG}} V_h^*$ to be the sheaf of polynomials defined over $X \smallsetminus \mathrm{Sing}(h)$ which are "h-bounded". This means that when they are viewed as polynomials $P(z\,;\,\xi_1,\ldots,\xi_k)$ in terms of $\xi_j = (\nabla_{h_0}^{1,0})^j f(0)$ where $\nabla_{h_0}^{1,0}$ is the $(1,0)$-component of the induced Chern connection on (V,h_0), there is a uniform bound

$$
(9.8)\qquad \big|P(z\,;\,\xi_1,\ldots,\xi_k)\big| \leqslant C\Big(\sum \|\xi_j\|_h^{1/j}\Big)^m
$$

near points of $X \smallsetminus X'$ (see section 2 for more details on this). Again, by a direct image argument, one sees that $E_{k,m}^{\mathrm{GG}} V_h^*$ is always a coherent sheaf. The sheaf $E_{k,m}^{\mathrm{GG}} V^*$ is defined to be $E_{k,m}^{\mathrm{GG}} V_h^*$ when $h = h_0$ (it is actually independent of the choice of h_0, as follows from arguments similar to those given in section 2). Notice that this is exactly what is needed to extend the proof of the vanishing theorem 8.15 to the case of a singular linear space V; the value distribution theory argument can only work when the functions $P(f\,;\ f',\ldots,f^{(k)})(t)$ do not exhibit poles, and this is guaranteed here by the boundedness assumption.

Our strategy can be described as follows. We consider the Green-Griffiths bundle of k-jets $X_k^{\mathrm{GG}} = J^k V \smallsetminus \{0\}/\mathbb{C}^*$, which by (9.3) consists of a fibration in *weighted projective spaces*, and its associated tautological sheaf

$$
L = \mathcal{O}_{X_k^{\mathrm{GG}}}(1),
$$

viewed rather as a virtual $\mathbb{Q}$-line bundle $\mathcal{O}_{X_k^{\mathrm{GG}}}(m_0)^{1/m_0}$ with $m_0 = \mathrm{lcm}(1,2,\ldots,k)$. Then, if $\pi_k : X_k^{\mathrm{GG}} \to X$ is the natural projection, we have

$$
E_{k,m}^{\mathrm{GG}} = (\pi_k)_* \mathcal{O}_{X_k^{\mathrm{GG}}}(m) \quad \text{and} \quad R^q(\pi_k)_* \mathcal{O}_{X_k^{\mathrm{GG}}}(m) = 0 \text{ for } q \geqslant 1.
$$

Hence, by the Leray spectral sequence we get for every invertible sheaf F on X the isomorphism

$$
(9.9)\qquad H^q(X, E_{k,m}^{\mathrm{GG}} V^* \otimes F) \simeq H^q(X_k^{\mathrm{GG}}, \mathcal{O}_{X_k^{\mathrm{GG}}}(m) \otimes \pi_k^* F).
$$

The latter group can be evaluated thanks to holomorphic Morse inequalities. Let us recall the main statement.

9.10. Holomorphic Morse inequalities ([Dem85])**.** *Let X be a compact complex manifolds, $E \to X$ a holomorphic vector bundle of rank r, and (L,h) a hermitian line bundle. The dimensions $h^q(X, E \otimes L^m)$ of cohomology groups of the tensor powers $E \otimes L^m$ satisfy the following asymptotic estimates as $m \to +\infty$:*

(WM) *Weak Morse inequalities*:

$$
h^q(X, E \otimes L^m) \leqslant r\frac{m^n}{n!}\int_{X(L,h,q)} (-1)^q \Theta_{L,h}^n + o(m^n)\ .
$$

(SM) *Strong Morse inequalities*:

$$
\sum_{0 \leqslant j \leqslant q} (-1)^{q-j} h^j(X, E \otimes L^m) \leqslant r\frac{m^n}{n!}\int_{X(L,h,\leqslant q)} (-1)^q \Theta_{L,h}^n + o(m^n)\ .
$$

(RR) *Asymptotic Riemann-Roch formula*:

$$
\chi(X, E \otimes L^m) := \sum_{0 \leqslant j \leqslant n} (-1)^j h^j(X, E \otimes L^m) = r\frac{m^n}{n!}\int_X \Theta_{L,h}^n + o(m^n)\ .
$$

Moreover (cf. Bonavero's PhD thesis [Bon93]), if $h = e^{-\varphi}$ is a singular hermitian metric with analytic singularities of pole set $P = \varphi^{-1}(-\infty)$, the estimates still hold provided all cohomology groups are replaced by cohomology groups $H^q(X, E \otimes L^m \otimes \mathcal{I}(h^m))$ twisted with the corresponding L^2 multiplier ideal sheaves

$$\mathcal{I}(h^m) = \mathcal{I}(k\varphi) = \big\{f \in \mathcal{O}_{X,x},\ \ \exists V \ni x,\ \int_V |f(z)|^2 e^{-m\varphi(z)} d\lambda(z) < +\infty\big\},$$

and provided the Morse integrals are computed on the regular locus of h, namely restricted to $X(L,h,q) \smallsetminus \Sigma$:

$$\int_{X(L,h,q)\smallsetminus\Sigma} (-1)^q \Theta^n_{L,h}.$$

The special case of 9.10 (SM) when $q = 1$ yields a very useful criterion for the existence of sections of large multiples of L.

9.11. Corollary. *Let $L \to X$ be a holomorphic line bundle equipped with a singular hermitian metric $h = e^{-\varphi}$ with analytic singularities of pole set $\Sigma = \varphi^{-1}(-\infty)$. Then we have the following lower bounds*

(a) *at the h^0 level* :

$$\begin{aligned} h^0(X, E \otimes L^m) &\geqslant h^0(X, E \otimes L^m \otimes \mathcal{I}(h^m)) \\ &\geqslant h^0(X, E \otimes L^m \otimes \mathcal{I}(h^m)) - h^1(X, E \otimes L^m \otimes \mathcal{I}(h^m)) \\ &\geqslant r\frac{k^n}{n!}\int_{X(L,h,\leqslant 1)\smallsetminus\Sigma} \Theta^n_{L,h} - o(k^n)\ . \end{aligned}$$

Especially L is big as soon as $\int_{X(L,h,\leqslant 1)\smallsetminus\Sigma} \Theta^n_{L,h} > 0$ for some singular hermitian metric h on L.

(b) *at the h^q level* :

$$h^q(X, E \otimes L^m \otimes \mathcal{I}(h^m)) \geqslant r\frac{k^n}{n!}\sum_{j=q-1,q,q+1} (-1)^q \int_{X(L,h,j)\smallsetminus\Sigma} \Theta^n_{L,h} - o(k^n)\ .$$

Now, given a directed manifold (X, V), we can associate with any admissible metric h on V a metric (or rather a natural family) of metrics on $L = \mathcal{O}_{X_k^{\mathrm{GG}}}(1)$. The space X_k^{GG} always possesses quotient singularities if $k \geqslant 2$ (and even some more if V is singular), but we do not really care since Morse inequalities still work in this setting thanks to Bonavero's generalization. As we will see, it is then possible to get nice asymptotic formulas as $m \to +\infty$. They appear to be of a *probabilistic nature* if we take the components of the k-jet (i.e. the successive derivatives $\xi_j = f^{(j)}(0)$, $1 \leqslant j \leqslant k$) as random variables. This probabilistic behaviour was somehow already visible in the Riemann-Roch calculation of [GrGr79]. In this way, assuming K_V big, we produce a lot of sections $\sigma_j = H^0(X_k^{\mathrm{GG}}, \mathcal{O}_{X_k^{\mathrm{GG}}}(m) \otimes \pi_k^* F)$, corresponding to certain divisors $Z_j \subset X_k^{\mathrm{GG}}$. The hard problem which is left in order to complete a proof of the generalized Green-Griffiths-Lang conjecture is to compute the base locus $Z = \bigcap Z_j$ and to show that $Y = \pi_k(Z) \subset X$ must be a proper algebraic variety.

9.B. Hermitian geometry of weighted projective spaces

The goal of this section is to introduce natural Kähler metrics on weighted projective spaces, and to evaluate the corresponding volume forms. Here we put $d^c = \frac{i}{4\pi}(\overline{\partial} - \partial)$ so that $dd^c = \frac{i}{2\pi}\partial\overline{\partial}$. The normalization of the d^c operator is chosen such that we have precisely $(dd^c \log |z|^2)^n = \delta_0$ for the Monge-Ampère operator in $\mathbb{C}^n$. Given a k-tuple of "weights" $a = (a_1, \ldots, a_k)$, i.e. of integers $a_s > 0$ with $\gcd(a_1, \ldots, a_k) = 1$, we introduce the weighted projective space $P(a_1, \ldots, a_k)$ to be the quotient of $\mathbb{C}^k \smallsetminus \{0\}$ by the corresponding weighted $\mathbb{C}^*$ action:

$$P(a_1, \ldots, a_k) = \mathbb{C}^k \smallsetminus \{0\}/\mathbb{C}^*, \qquad \lambda \cdot z = (\lambda^{a_1} z_1, \ldots, \lambda^{a_k} z_k). \tag{9.12}$$

As is well known, this defines a toric $(k-1)$-dimensional algebraic variety with quotient singularities. On this variety, we introduce the possibly singular (but almost everywhere smooth and non degenerate) Kähler form $\omega_{a,p}$ defined by

$$\pi_a^*\omega_{a,p} = dd^c\varphi_{a,p}, \qquad \varphi_{a,p}(z) = \frac{1}{p}\log\sum_{1\leqslant s\leqslant k}|z_s|^{2p/a_s}, \tag{9.13}$$

where $\pi_a : \mathbb{C}^k \smallsetminus \{0\} \to P(a_1,\ldots,a_k)$ is the canonical projection and $p > 0$ is a positive constant. It is clear that $\varphi_{p,a}$ is real analytic on $\mathbb{C}^k \smallsetminus \{0\}$ if p is an integer and a common multiple of all weights a_s, and we will implicitly pick such a p later on to avoid any difficulty. Elementary calculations give the following well-known formula for the volume

$$\int_{P(a_1,\ldots,a_k)} \omega_{a,p}^{k-1} = \frac{1}{a_1\ldots a_k} \tag{9.14}$$

(notice that this is independent of p, as it is obvious by Stokes theorem, since the cohomology class of $\omega_{a,p}$ does not depend on p).

Our later calculations will require a slightly more general setting. Instead of looking at $\mathbb{C}^k$, we consider the weighted $\mathbb{C}^*$ action defined by

$$\mathbb{C}^{|r|} = \mathbb{C}^{r_1}\times\ldots\times\mathbb{C}^{r_k}, \qquad \lambda\cdot z = (\lambda^{a_1}z_1,\ldots,\lambda^{a_k}z_k). \tag{9.15}$$

Here $z_s \in \mathbb{C}^{r_s}$ for some k-tuple $r = (r_1,\ldots,r_k)$ and $|r| = r_1+\ldots+r_k$. This gives rise to a weighted projective space

$$\begin{aligned} &P(a_1^{[r_1]},\ldots,a_k^{[r_k]}) = P(a_1,\ldots,a_1,\ldots,a_k,\ldots,a_k),\\ &\pi_{a,r} : \mathbb{C}^{r_1}\times\ldots\times\mathbb{C}^{r_k}\smallsetminus\{0\} \longrightarrow P(a_1^{[r_1]},\ldots,a_k^{[r_k]}) \end{aligned} \tag{9.16}$$

obtained by repeating r_s times each weight a_s. On this space, we introduce the degenerate Kähler metric $\omega_{a,r,p}$ such that

$$\pi_{a,r}^*\omega_{a,r,p} = dd^c\varphi_{a,r,p}, \qquad \varphi_{a,r,p}(z) = \frac{1}{p}\log\sum_{1\leqslant s\leqslant k}|z_s|^{2p/a_s} \tag{9.17}$$

where $|z_s|$ stands now for the standard Hermitian norm $(\sum_{1\leqslant j\leqslant r_s}|z_{s,j}|^2)^{1/2}$ on $\mathbb{C}^{r_s}$. This metric is cohomologous to the corresponding "polydisc-like" metric $\omega_{a,p}$ already defined, and therefore Stokes theorem implies

$$\int_{P(a_1^{[r_1]},\ldots,a_k^{[r_k]})} \omega_{a,r,p}^{|r|-1} = \frac{1}{a_1^{r_1}\ldots a_k^{r_k}}. \tag{9.18}$$

Using standard results of integration theory (Fubini, change of variable formula...), one obtains:

9.19. Proposition. *Let $f(z)$ be a bounded function on $P(a_1^{[r_1]},\ldots,a_k^{[r_k]})$ which is continuous outside of the hyperplane sections $z_s = 0$. We also view f as a $\mathbb{C}^*$-invariant continuous function on $\prod(\mathbb{C}^{r_s}\smallsetminus\{0\})$. Then*

$$\begin{aligned} &\int_{P(a_1^{[r_1]},\ldots,a_k^{[r_k]})} f(z)\,\omega_{a,r,p}^{|r|-1}\\ &= \frac{(|r|-1)!}{\prod_s a_s^{r_s}}\int_{(x,u)\in\Delta_{k-1}\times\prod S^{2r_s-1}} f(x_1^{a_1/2p}u_1,\ldots,x_k^{a_k/2p}u_k)\prod_{1\leqslant s\leqslant k}\frac{x_s^{r_s-1}}{(r_s-1)!}\,dx\,d\mu(u) \end{aligned}$$

where Δ_{k-1} is the $(k-1)$-simplex $\{x_s \geqslant 0, \sum x_s = 1\}$, $dx = dx_1\wedge\ldots\wedge dx_{k-1}$ its standard measure, and where $d\mu(u) = d\mu_1(u_1)\ldots d\mu_k(u_k)$ is the rotation invariant probability measure on the product $\prod_s S^{2r_s-1}$ of unit spheres in $\mathbb{C}^{r_1}\times\ldots\times\mathbb{C}^{r_k}$. As a consequence

$$\lim_{p\to+\infty}\int_{P(a_1^{[r_1]},\ldots,a_k^{[r_k]})} f(z)\,\omega_{a,r,p}^{|r|-1} = \frac{1}{\prod_s a_s^{r_s}}\int_{\prod S^{2r_s-1}} f(u)\,d\mu(u).$$

Also, by elementary integrations by parts and induction on k, $r_1, \ldots, r_k$, it can be checked that

$$\int_{x\in\Delta_{k-1}} \prod_{1\leqslant s\leqslant k} x_s^{r_s-1} dx_1 \ldots dx_{k-1} = \frac{1}{(|r|-1)!} \prod_{1\leqslant s\leqslant k} (r_s-1)!\ . \tag{9.20}$$

This implies that $(|r|-1)! \prod_{1\leqslant s\leqslant k} \frac{x_s^{r_s-1}}{(r_s-1)!}\, dx$ is a probability measure on Δ_{k-1}.

9.C. Probabilistic estimate of the curvature of k-jet bundles

Let (X, V) be a compact complex directed non singular variety. To avoid any technical difficulty at this point, we first assume that V is a holomorphic vector subbundle of T_X, equipped with a smooth Hermitian metric h.

According to the notation already specified in § 7, we denote by J^kV the bundle of k-jets of holomorphic curves $f : (\mathbb{C}, 0) \to X$ tangent to V at each point. Let us set $n = \dim_{\mathbb{C}} X$ and $r = \operatorname{rank}_{\mathbb{C}} V$. Then $J^kV \to X$ is an algebraic fiber bundle with typical fiber $\mathbb{C}^{rk}$, and we get a projectivized k-jet bundle

$$X_k^{\mathrm{GG}} := (J^kV \smallsetminus \{0\})/\mathbb{C}^*, \qquad \pi_k : X_k^{\mathrm{GG}} \to X \tag{9.21}$$

which is a $P(1^{[r]}, 2^{[r]}, \ldots, k^{[r]})$ weighted projective bundle over X, and we have the direct image formula $(\pi_k)_*\mathcal{O}_{X_k^{\mathrm{GG}}}(m) = \mathcal{O}(E_{k,m}^{\mathrm{GG}}V^*)$ (cf. Proposition 7.9). In the sequel, we do not make a direct use of coordinates, because they need not be related in any way to the Hermitian metric h of V. Instead, we choose a local holomorphic coordinate frame $(e_\alpha(z))_{1\leqslant\alpha\leqslant r}$ of V on a neighborhood U of x_0, such that

$$\langle e_\alpha(z), e_\beta(z)\rangle = \delta_{\alpha\beta} + \sum_{1\leqslant i,j\leqslant n,\, 1\leqslant\alpha,\beta\leqslant r} c_{ij\alpha\beta} z_i \overline{z}_j + O(|z|^3) \tag{9.22}$$

for suitable complex coefficients $(c_{ij\alpha\beta})$. It is a standard fact that such a normalized coordinate system always exists, and that the Chern curvature tensor $\frac{i}{2\pi} D_{V,h}^2$ of (V, h) at x_0 is then given by

$$\Theta_{V,h}(x_0) = -\frac{i}{2\pi} \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}\, dz_i \wedge d\overline{z}_j \otimes e_\alpha^* \otimes e_\beta. \tag{9.23}$$

Consider a local holomorphic connection ∇ on $V_{|U}$ (e.g. the one which turns (e_α) into a parallel frame), and take $\xi_k = \nabla^k f(0) \in V_x$ defined inductively by $\nabla^1 f = f'$ and $\nabla^s f = \nabla_{f'}(\nabla^{s-1} f)$. This gives a local identification

$$J_kV_{|U} \to V_{|U}^{\oplus k}, \qquad f \mapsto (\xi_1, \ldots, \xi_k) = (\nabla f(0), \ldots, \nabla f^k(0)),$$

and the weighted $\mathbb{C}^*$ action on J_kV is expressed in this setting by

$$\lambda \cdot (\xi_1, \xi_2, \ldots, \xi_k) = (\lambda\xi_1, \lambda^2\xi_2, \ldots, \lambda^k\xi_k).$$

Now, we fix a finite open covering $(U_\alpha)_{\alpha\in I}$ of X by open coordinate charts such that $V_{|U_\alpha}$ is trivial, along with holomorphic connections ∇_α on $V_{|U_\alpha}$. Let θ_α be a partition of unity of X subordinate to the covering (U_α). Let us fix $p > 0$ and small parameters $1 = \varepsilon_1 \gg \varepsilon_2 \gg \ldots \gg \varepsilon_k > 0$. Then we define a global weighted Finsler metric on J^kV by putting for any k-jet $f \in J_x^kV$

$$\Psi_{h,p,\varepsilon}(f) := \Big(\sum_{\alpha\in I} \theta_\alpha(x) \sum_{1\leqslant s\leqslant k} \varepsilon_s^{2p} \|\nabla_\alpha^s f(0)\|_{h(x)}^{2p/s}\Big)^{1/p} \tag{9.24}$$

where $\|\ \|_{h(x)}$ is the Hermitian metric h of V evaluated on the fiber V_x, $x = f(0)$. The function $\Psi_{h,p,\varepsilon}$ satisfies the fundamental homogeneity property

$$\Psi_{h,p,\varepsilon}(\lambda \cdot f) = \Psi_{h,p,\varepsilon}(f)\, |\lambda|^2 \tag{9.25}$$

with respect to the $\mathbb{C}^*$ action on J^kV, in other words, it induces a Hermitian metric on the dual L^* of the tautological $\mathbb{Q}$-line bundle $L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1)$ over X_k^{GG}. The curvature of L_k is given by

$$\pi_k^*\Theta_{L_k,\Psi_{h,p,\varepsilon}^*} = dd^c \log \Psi_{h,p,\varepsilon} \tag{9.26}$$

Our next goal is to compute precisely the curvature and to apply holomorphic Morse inequalities to $L \to X_k^{\rm GG}$ with the above metric. It might look a priori like an untractable problem, since the definition of $\Psi_{h,p,\varepsilon}$ is a rather unnatural one. However, the "miracle" is that the asymptotic behavior of $\Psi_{h,p,\varepsilon}$ as $\varepsilon_s/\varepsilon_{s-1} \to 0$ is in some sense uniquely defined and very natural. It will lead to a computable asymptotic formula, which is moreover simple enough to produce useful results.

9.27. Lemma. *On each coordinate chart U equipped with a holomorphic connection ∇ of $V_{|U}$, let us define the components of a k-jet $f \in J^kV$ by $\xi_s = \nabla^s f(0)$, and consider the rescaling transformation*

$$\rho_{\nabla,\varepsilon}(\xi_1,\xi_2,\ldots,\xi_k) = (\varepsilon_1^1\xi_1, \varepsilon_2^2\xi_2, \ldots, \varepsilon_k^k\xi_k) \quad \textit{on } J^k_xV,\ x \in U$$

(it commutes with the $\mathbb{C}^*$-action but is otherwise unrelated and not canonically defined over X as it depends on the choice of ∇). Then, if p is a multiple of $\mathrm{lcm}(1,2,\ldots,k)$ and $\varepsilon_s/\varepsilon_{s-1} \to 0$ for all $s = 2,\ldots,k$, the rescaled function $\Psi_{h,p,\varepsilon} \circ \rho_{\nabla,\varepsilon}^{-1}(\xi_1,\ldots,\xi_k)$ converges towards

$$\Big(\sum_{1\leqslant s\leqslant k} \|\xi_s\|_h^{2p/s}\Big)^{1/p}$$

on every compact subset of $J^kV_{|U} \smallsetminus \{0\}$, uniformly in C^∞ topology.

Proof. Let $U \subset X$ be an open set on which $V_{|U}$ is trivial and equipped with some holomorphic connection ∇. Let us pick another holomorphic connection $\widetilde\nabla = \nabla + \Gamma$ where $\Gamma \in H^0(U, \Omega^1_X \otimes \mathrm{Hom}(V,V)$. Then $\widetilde\nabla^2 f = \nabla^2 f + \Gamma(f)(f')\cdot f'$, and inductively we get

$$\widetilde\nabla^s f = \nabla^s f + P_s(f\,;\nabla^1 f,\ldots,\nabla^{s-1}f)$$

where $P(x\,;\xi_1,\ldots,\xi_{s-1})$ is a polynomial with holomorphic coefficients in $x \in U$ which is of weighted homogeneous degree s in $(\xi_1,\ldots,\xi_{s-1})$. In other words, the corresponding change in the parametrization of $J^kV_{|U}$ is given by a $\mathbb{C}^*$-homogeneous transformation

$$\widetilde\xi_s = \xi_s + P_s(x\,;\xi_1,\ldots,\xi_{s-1}).$$

Let us introduce the corresponding rescaled components

$$(\xi_{1,\varepsilon},\ldots,\xi_{k,\varepsilon}) = (\varepsilon_1^1\xi_1,\ldots,\varepsilon_k^k\xi_k), \qquad (\widetilde\xi_{1,\varepsilon},\ldots,\widetilde\xi_{k,\varepsilon}) = (\varepsilon_1^1\widetilde\xi_1,\ldots,\varepsilon_k^k\widetilde\xi_k).$$

Then

$$\begin{aligned}\widetilde\xi_{s,\varepsilon} &= \xi_{s,\varepsilon} + \varepsilon_s^s\,P_s(x\,;\varepsilon_1^{-1}\xi_{1,\varepsilon},\ldots,\varepsilon_{s-1}^{-(s-1)}\xi_{s-1,\varepsilon})\\ &= \xi_{s,\varepsilon} + O(\varepsilon_s/\varepsilon_{s-1})^s\,O(\|\xi_{1,\varepsilon}\| + \ldots + \|\xi_{s-1,\varepsilon}\|^{1/(s-1)})^s\end{aligned}$$

and the error terms are thus polynomials of fixed degree with arbitrarily small coefficients as $\varepsilon_s/\varepsilon_{s-1} \to 0$. Now, the definition of $\Psi_{h,p,\varepsilon}$ consists of glueing the sums

$$\sum_{1\leqslant s\leqslant k} \varepsilon_s^{2p}\|\xi_k\|_h^{2p/s} = \sum_{1\leqslant s\leqslant k} \|\xi_{k,\varepsilon}\|_h^{2p/s}$$

corresponding to $\xi_k = \nabla_\alpha^s f(0)$ by means of the partition of unity $\sum\theta_\alpha(x) = 1$. We see that by using the rescaled variables $\xi_{s,\varepsilon}$ the changes occurring when replacing a connection ∇_α by an alternative one ∇_β are arbitrary small in C^∞ topology, with error terms uniformly controlled in terms of the ratios $\varepsilon_s/\varepsilon_{s-1}$ on all compact subsets of $V^k \smallsetminus \{0\}$. This shows that in C^∞ topology, $\Psi_{h,p,\varepsilon}\circ\rho_{\nabla,\varepsilon}^{-1}(\xi_1,\ldots,\xi_k)$ converges uniformly towards $(\sum_{1\leqslant s\leqslant k}\|\xi_k\|_h^{2p/s})^{1/p}$, whatever the trivializing open set U and the holomorphic connection ∇ used to evaluate the components and perform the rescaling are. □

Now, we fix a point $x_0 \in X$ and a local holomorphic frame $(e_\alpha(z))_{1\leqslant\alpha\leqslant r}$ satisfying (9.22) on a neighborhood U of x_0. We introduce the rescaled components $\xi_s = \varepsilon_s^s\nabla^s f(0)$ on $J^kV_{|U}$ and compute the curvature of

$$\Psi_{h,p,\varepsilon} \circ \rho_{\nabla,\varepsilon}^{-1}(z\,;\,\xi_1,\ldots,\xi_k) \simeq \bigg(\sum_{1\leqslant s\leqslant k} \|\xi_s\|_h^{2p/s}\bigg)^{1/p}$$

(by Lemma 9.27, the errors can be taken arbitrary small in C^∞ topology). We write $\xi_s = \sum_{1\leqslant\alpha\leqslant r}\xi_{s\alpha}e_\alpha$. By (9.22) we have

$$\|\xi_s\|_h^2 = \sum_\alpha |\xi_{s\alpha}|^2 + \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}z_i\overline{z}_j\xi_{s\alpha}\overline{\xi}_{s\beta} + O(|z|^3|\xi|^2).$$

The question is to evaluate the curvature of the weighted metric defined by

$$\begin{aligned}\Psi(z\,;\,\xi_1,\ldots,\xi_k) &= \bigg(\sum_{1\leqslant s\leqslant k} \|\xi_s\|_h^{2p/s}\bigg)^{1/p}\\ &= \bigg(\sum_{1\leqslant s\leqslant k}\Big(\sum_\alpha |\xi_{s\alpha}|^2 + \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}z_i\overline{z}_j\xi_{s\alpha}\overline{\xi}_{s\beta}\Big)^{p/s}\bigg)^{1/p} + O(|z|^3).\end{aligned}$$

We set $|\xi_s|^2 = \sum_\alpha |\xi_{s\alpha}|^2$. A straightforward calculation yields

$$\begin{aligned}&\log\Psi(z\,;\,\xi_1,\ldots,\xi_k) =\\ &= \frac{1}{p}\log\sum_{1\leqslant s\leqslant k}|\xi_s|^{2p/s} + \sum_{1\leqslant s\leqslant k}\frac{1}{s}\frac{|\xi_s|^{2p/s}}{\sum_t|\xi_t|^{2p/t}}\sum_{i,j,\alpha,\beta}c_{ij\alpha\beta}z_i\overline{z}_j\frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2} + O(|z|^3).\end{aligned}$$

By (9.26), the curvature form of $L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1)$ is given at the central point x_0 by the following formula.

9.28. Proposition. *With the above choice of coordinates and with respect to the rescaled components $\xi_s = \varepsilon_s^s\nabla^s f(0)$ at $x_0 \in X$, we have the approximate expression*

$$\Theta_{L_k,\Psi^*_{h,p,\varepsilon}}(x_0,[\xi]) \simeq \omega_{a,r,p}(\xi) + \frac{i}{2\pi}\sum_{1\leqslant s\leqslant k}\frac{1}{s}\frac{|\xi_s|^{2p/s}}{\sum_t|\xi_t|^{2p/t}}\sum_{i,j,\alpha,\beta}c_{ij\alpha\beta}\frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2}\,dz_i\wedge d\overline{z}_j$$

where the error terms are $O(\max_{2\leqslant s\leqslant k}(\varepsilon_s/\varepsilon_{s-1})^s)$ uniformly on the compact variety X_k^{GG}. Here $\omega_{a,r,p}$ is the (degenerate) Kähler metric associated with the weight $a = (1^{[r]}, 2^{[r]},\ldots,k^{[r]})$ of the canonical $\mathbb{C}^*$ action on J^kV.

Thanks to the uniform approximation, we can (and will) neglect the error terms in the calculations below. Since $\omega_{a,r,p}$ is positive definite on the fibers of $X_k^{\mathrm{GG}} \to X$ (at least outside of the axes $\xi_s = 0$), the index of the $(1,1)$ curvature form $\Theta_{L_k,\Psi^*_{h,p,\varepsilon}}(z,[\xi])$ is equal to the index of the $(1,1)$-form

$$\gamma_k(z,\xi) := \frac{i}{2\pi}\sum_{1\leqslant s\leqslant k}\frac{1}{s}\frac{|\xi_s|^{2p/s}}{\sum_t|\xi_t|^{2p/t}}\sum_{i,j,\alpha,\beta}c_{ij\alpha\beta}(z)\frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2}\,dz_i\wedge d\overline{z}_j \tag{9.29}$$

depending only on the differentials $(dz_j)_{1\leqslant j\leqslant n}$ on X. The q-index integral of $(L_k,\Psi^*_{h,p,\varepsilon})$ on X_k^{GG} is therefore equal to

$$\begin{aligned}&\int_{X_k^{\mathrm{GG}}(L_k,q)}\Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}} =\\ &= \frac{(n+kr-1)!}{n!(kr-1)!}\int_{z\in X}\int_{\xi\in P(1^{[r]},\ldots,k^{[r]})}\omega_{a,r,p}^{kr-1}(\xi)\mathbb{1}_{\gamma_k,q}(z,\xi)\gamma_k(z,\xi)^n\end{aligned}$$

where $\mathbb{1}_{\gamma_k,q}(z,\xi)$ is the characteristic function of the open set of points where $\gamma_k(z,\xi)$ has signature $(n-q,q)$ in terms of the dz_j's. Notice that since $\gamma_k(z,\xi)^n$ is a determinant, the product

$\mathbb{1}_{\gamma_k,q}(z,\xi)\gamma_k(z,\xi)^n$ gives rise to a continuous function on X_k^{GG}. Formula 9.20 with $r_1 = \ldots = r_k = r$ and $a_s = s$ yields the slightly more explicit integral

$$\int_{X_k^{\mathrm{GG}}(L_k,q)} \Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}} = \frac{(n+kr-1)!}{n!(k!)^r} \times$$
$$\int_{z\in X}\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k} \mathbb{1}_{g_k,q}(z,x,u) g_k(z,x,u)^n \,\frac{(x_1\ldots x_k)^{r-1}}{(r-1)!^k}\,dx\,d\mu(u),$$

where $g_k(z,x,u) = \gamma_k(z, x_1^{1/2p}u_1,\ldots,x_k^{k/2p}u_k)$ is given by

$$g_k(z,x,u) = \frac{i}{2\pi}\sum_{1\leqslant s\leqslant k}\frac{1}{s}x_s\sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}(z)\,u_{s\alpha}\overline{u}_{s\beta}\,dz_i\wedge d\overline{z}_j \tag{9.30}$$

and $\mathbb{1}_{g_k,q}(z,x,u)$ is the characteristic function of its q-index set. Here

$$d\nu_{k,r}(x) = (kr-1)!\,\frac{(x_1\ldots x_k)^{r-1}}{(r-1)!^k}\,dx \tag{9.31}$$

is a probability measure on Δ_{k-1}, and we can rewrite

$$\int_{X_k^{\mathrm{GG}}(L_k,q)} \Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}} = \frac{(n+kr-1)!}{n!(k!)^r(kr-1)!} \times$$
$$\int_{z\in X}\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k} \mathbb{1}_{g_k,q}(z,x,u) g_k(z,x,u)^n \,d\nu_{k,r}(x)\,d\mu(u). \tag{9.32}$$

Now, formula (9.30) shows that $g_k(z,x,u)$ is a "Monte Carlo" evaluation of the curvature tensor, obtained by averaging the curvature at random points $u_s \in S^{2r-1}$ with certain positive weights x_s/s ; we should then think of the k-jet f as some sort of random variable such that the derivatives $\nabla^k f(0)$ are uniformly distributed in all directions. Let us compute the expected value of $(x,u)\mapsto g_k(z,x,u)$ with respect to the probability measure $d\nu_{k,r}(x)\,d\mu(u)$. Since $\int_{S^{2r-1}} u_{s\alpha}\overline{u}_{s\beta}d\mu(u_s) = \frac{1}{r}\delta_{\alpha\beta}$ and $\int_{\Delta_{k-1}} x_s\,d\nu_{k,r}(x) = \frac{1}{k}$, we find

$$\mathbf{E}(g_k(z,\bullet,\bullet)) = \frac{1}{kr}\sum_{1\leqslant s\leqslant k}\frac{1}{s}\cdot\frac{i}{2\pi}\sum_{i,j,\alpha} c_{ij\alpha\alpha}(z)\,dz_i\wedge d\overline{z}_j.$$

In other words, we get the normalized trace of the curvature, i.e.

$$\mathbf{E}(g_k(z,\bullet,\bullet)) = \frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)\Theta_{\det(V^*),\det h^*}, \tag{9.33}$$

where $\Theta_{\det(V^*),\det h^*}$ is the $(1,1)$-curvature form of $\det(V^*)$ with the metric induced by h. It is natural to guess that $g_k(z,x,u)$ behaves asymptotically as its expected value $\mathbf{E}(g_k(z,\bullet,\bullet))$ when k tends to infinity. If we replace brutally g_k by its expected value in (9.32), we get the integral

$$\frac{(n+kr-1)!}{n!(k!)^r(kr-1)!}\,\frac{1}{(kr)^n}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^n\int_X \mathbb{1}_{\eta,q}\eta^n,$$

where $\eta := \Theta_{\det(V^*),\det h^*}$ and $\mathbb{1}_{\eta,q}$ is the characteristic function of its q-index set in X. The leading constant is equivalent to $(\log k)^n/n!(k!)^r$ modulo a multiplicative factor $1+O(1/\log k)$. By working out a more precise analysis of the deviation, the following result has been proved in [Dem11] and [Dem12].

9.34. Probabilistic estimate. *Fix smooth Hermitian metrics h on V and $\omega = \frac{i}{2\pi}\sum\omega_{ij}dz_i\wedge d\overline{z}_j$ on X. Denote by $\Theta_{V,h} = -\frac{i}{2\pi}\sum c_{ij\alpha\beta}dz_i\wedge d\overline{z}_j\otimes e^*_\alpha\otimes e_\beta$ the curvature tensor of V with respect to an h-orthonormal frame (e_α), and put*

$$\eta(z) = \Theta_{\det(V^*),\det h^*} = \frac{i}{2\pi}\sum_{1\leqslant i,j\leqslant n}\eta_{ij}dz_i\wedge d\overline{z}_j,\qquad \eta_{ij} = \sum_{1\leqslant\alpha\leqslant r} c_{ij\alpha\alpha}.$$

Finally consider the k-jet line bundle $L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1) \to X_k^{\mathrm{GG}}$ equipped with the induced metric $\Psi^*_{h,p,\varepsilon}$ (as defined above, with $1 = \varepsilon_1 \gg \varepsilon_2 \gg \ldots \gg \varepsilon_k > 0$). When k tends to infinity, the integral of the top power of the curvature of L_k on its q-index set $X_k^{\mathrm{GG}}(L_k, q)$ is given by

$$\int_{X_k^{\mathrm{GG}}(L_k,q)} \Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}} = \frac{(\log k)^n}{n!\,(k!)^r}\bigg(\int_X \mathbb{1}_{\eta,q}\eta^n + O((\log k)^{-1})\bigg)$$

for all $q = 0, 1, \ldots, n$, and the error term $O((\log k)^{-1})$ can be bounded explicitly in terms of Θ_V, η and ω. Moreover, the left hand side is identically zero for $q > n$.

The final statement follows from the observation that the curvature of L_k is positive along the fibers of $X_k^{\mathrm{GG}} \to X$, by the plurisubharmonicity of the weight (this is true even when the partition of unity terms are taken into account, since they depend only on the base); therefore the q-index sets are empty for $q > n$. It will be useful to extend the above estimates to the case of sections of

$$L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1) \otimes \pi_k^*\mathcal{O}\Big(\frac{1}{kr}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big)F\Big) \tag{9.35}$$

where $F \in \mathrm{Pic}_{\mathbb{Q}}(X)$ is an arbitrary $\mathbb{Q}$-line bundle on X and $\pi_k : X_k^{\mathrm{GG}} \to X$ is the natural projection. We assume here that F is also equipped with a smooth Hermitian metric h_F. In formulas (9.32–9.34), the renormalized curvature $\eta_k(z, x, u)$ of L_k takes the form

$$\eta_k(z, x, u) = \frac{1}{\frac{1}{kr}(1 + \frac{1}{2} + \ldots + \frac{1}{k})} g_k(z, x, u) + \Theta_{F,h_F}(z), \tag{9.36}$$

and by the same calculations its expected value is

$$\eta(z) := \mathbf{E}(\eta_k(z, \bullet, \bullet)) = \Theta_{\det V^*, \det h^*}(z) + \Theta_{F,h_F}(z). \tag{9.37}$$

Then the variance estimate for $\eta_k - \eta$ is unchanged, and the L^p bounds for η_k are still valid, since our forms are just shifted by adding the constant smooth term $\Theta_{F,h_F}(z)$. The probabilistic estimate 9.34 is therefore still true in exactly the same form, provided we use (9.35 – 9.37) instead of the previously defined L_k, η_k and η. An application of holomorphic Morse inequalities gives the desired cohomology estimates for

$$\begin{aligned} &h^q\Big(X, E^{\mathrm{GG}}_{k,m}V^* \otimes \mathcal{O}\Big(\frac{m}{kr}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big)F\Big)\Big) \\ &\qquad = h^q(X_k^{\mathrm{GG}}, \mathcal{O}_{X_k^{\mathrm{GG}}}(m) \otimes \pi_k^*\mathcal{O}\Big(\frac{m}{kr}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big)F\Big)\Big), \end{aligned}$$

provided m is sufficiently divisible to give a multiple of F which is a $\mathbb{Z}$-line bundle.

9.38. Theorem. *Let (X, V) be a directed manifold, $F \to X$ a $\mathbb{Q}$-line bundle, (V, h) and (F, h_F) smooth Hermitian structure on V and F respectively. We define*

$$\begin{aligned} L_k &= \mathcal{O}_{X_k^{\mathrm{GG}}}(1) \otimes \pi_k^*\mathcal{O}\Big(\frac{1}{kr}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big)F\Big), \\ \eta &= \Theta_{\det V^*, \det h^*} + \Theta_{F,h_F}. \end{aligned}$$

Then for all $q \geqslant 0$ and all $m \gg k \gg 1$ such that m is sufficiently divisible, we have

$$h^q(X_k^{\mathrm{GG}}, \mathcal{O}(L_k^{\otimes m})) \leqslant \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\bigg(\int_{X(\eta,q)} (-1)^q\eta^n + O((\log k)^{-1})\bigg), \tag{a}$$

$$h^0(X_k^{\mathrm{GG}}, \mathcal{O}(L_k^{\otimes m})) \geqslant \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\bigg(\int_{X(\eta,\leqslant 1)} \eta^n - O((\log k)^{-1})\bigg), \tag{b}$$

$$\chi(X_k^{\mathrm{GG}}, \mathcal{O}(L_k^{\otimes m})) = \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\big(c_1(V^* \otimes F)^n + O((\log k)^{-1})\big). \tag{c}$$

Green and Griffiths [GrGr79] already checked the Riemann-Roch calculation (9.38 c) in the special case $V = T_X^*$ and $F = \mathcal{O}_X$. Their proof is much simpler since it relies only on Chern class calculations, but it cannot provide any information on the individual cohomology groups, except in very special cases where vanishing theorems can be applied; in fact in dimension 2, the Euler characteristic satisfies $\chi = h^0 - h^1 + h^2 \leqslant h^0 + h^2$, hence it is enough to get the vanishing of the top cohomology group H^2 to infer $h^0 \geqslant \chi$; this works for surfaces by means of a well-known vanishing theorem of Bogomolov which implies in general

$$H^n\Big(X, E_{k,m}^{\mathrm{GG}} T_X^* \otimes \mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)\Big)\Big) = 0$$

as soon as $K_X \otimes F$ is big and $m \gg 1$.

In fact, thanks to Bonavero's singular holomorphic Morse inequalities [Bon93], everything works almost unchanged in the case where $V \subset T_X$ has singularities and h is an admissible metric on V (see Definition 9.6). We only have to find a blow-up $\mu : \widetilde{X}_k \to X_k$ so that the resulting pull-backs $\mu^* L_k$ and $\mu^* V$ are locally free, and $\mu^* \det h^*$, $\mu^* \Psi_{h,p,\varepsilon}$ only have divisorial singularities. Then η is a $(1,1)$-current with logarithmic poles, and we have to deal with smooth metrics on $\mu^* L_k^{\otimes m} \otimes \mathcal{O}(-mE_k)$ where E_k is a certain effective divisor on X_k (which, by our assumption in 9.6, does not project onto X). The cohomology groups involved are then the twisted cohomology groups

$$H^q(X_k^{\mathrm{GG}}, \mathcal{O}(L_k^{\otimes m}) \otimes \mathcal{J}_{k,m})$$

where $\mathcal{J}_{k,m} = \mu_*(\mathcal{O}(-mE_k))$ is the corresponding multiplier ideal sheaf, and the Morse integrals need only be evaluated in the complement of the poles, that is on $X(\eta, q) \smallsetminus S$ where $S = \mathrm{Sing}(V) \cup \mathrm{Sing}(h)$. Since

$$(\pi_k)_*\big(\mathcal{O}(L_k^{\otimes m}) \otimes \mathcal{J}_{k,m}\big) \subset E_{k,m}^{\mathrm{GG}} V^* \otimes \mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)\Big)$$

we still get a lower bound for the H^0 of the latter sheaf (or for the H^0 of the un-twisted line bundle $\mathcal{O}(L_k^{\otimes m})$ on X_k^{GG}). If we assume that $K_V \otimes F$ is big, these considerations also allow us to obtain a strong estimate in terms of the volume, by using an approximate Zariski decomposition on a suitable blow-up of (X, V). The following corollary implies in particular Theorem 9.1.

9.39. Corollary. *If F is an arbitrary $\mathbb{Q}$-line bundle over X, one has*

$$\begin{aligned} h^0\Big(X_k^{\mathrm{GG}}, \mathcal{O}_{X_k^{\mathrm{GG}}}(m) \otimes \pi_k^* \mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)\Big) \\ \geqslant \frac{m^{n+kr-1}}{(n+kr-1)!} \frac{(\log k)^n}{n!\,(k!)^r}\Big(\mathrm{Vol}(K_V \otimes F) - O((\log k)^{-1})\Big) - o(m^{n+kr-1}), \end{aligned}$$

when $m \gg k \gg 1$, in particular there are many sections of the k-jet differentials of degree m twisted by the appropriate power of F if $K_V \otimes F$ is big.

Proof. The volume is computed here as usual, i.e. after performing a suitable log-resolution $\mu : \widetilde{X} \to X$ which converts K_V into an invertible sheaf. There is of course nothing to prove if $K_V \otimes F$ is not big, so we can assume $\mathrm{Vol}(K_V \otimes F) > 0$. Let us fix smooth Hermitian metrics h_0 on T_X and h_F on F. They induce a metric $\mu^*(\det h_0^{-1} \otimes h_F)$ on $\mu^*(K_V \otimes F)$ which, by our definition of K_V, is a smooth metric (the divisor produced by the log-resolution gets simplified with the degeneration divisor of the pull-back of the quotient metric on $\det V^*$ induced by $\mathcal{O}(\Lambda^r T_X^*) \to \mathcal{O}(\Lambda^r V^*)$). By the result of Fujita [Fuj94] on approximate Zariski decomposition, for every $\delta > 0$, one can find a modification $\mu_\delta : \widetilde{X}_\delta \to X$ dominating μ such that

$$\mu_\delta^*(K_V \otimes F) = \mathcal{O}_{\widetilde{X}_\delta}(A + E)$$

where A and E are $\mathbb{Q}$-divisors, A ample and E effective, with

$$\mathrm{Vol}(A) = A^n \geqslant \mathrm{Vol}(K_V \otimes F) - \delta.$$

If we take a smooth metric h_A with positive definite curvature form Θ_{A,h_A}, then we get a singular Hermitian metric $h_A h_E$ on $\mu_\delta^*(K_V \otimes F)$ with poles along E, i.e. the quotient $h_A h_E/\mu_\delta^*(\det h_0^{-1} \otimes h_F)$ is of the form $e^{-\varphi}$ where φ is quasi-psh with log poles $\log|\sigma_E|^2 \pmod{C^\infty(\widetilde{X}_\delta)}$ precisely given by the divisor E. We then only need to take the singular metric h on T_X defined by

$$h = h_0 e^{\frac{1}{r}(\mu_\delta)_*\varphi}$$

(the choice of the factor $\frac{1}{r}$ is there to correct adequately the metric on $\det V$). By construction h induces an admissible metric on V and the resulting curvature current $\eta = \Theta_{K_V,\det h^*} + \Theta_{F,h_F}$ is such that

$$\mu_\delta^*\eta = \Theta_{A,h_A} + [E], \qquad [E] = \text{current of integration on } E.$$

Then the 0-index Morse integral in the complement of the poles is given by

$$\int_{X(\eta,0)\smallsetminus S} \eta^n = \int_{\widetilde{X}_\delta} \Theta_{A,h_A}^n = A^n \geqslant \mathrm{Vol}(K_V \otimes F) - \delta$$

and (9.39) follows from the fact that δ can be taken arbitrary small. □

The following corollary implies Theorem 0.12.

9.40. Corollary. *Let (X,V) be a projective directed manifold such that $K_V^\bullet$ is big, and A an ample $\mathbb{Q}$-divisor on X such that $K_V^\bullet \otimes \mathcal{O}(-A)$ is still big. Then, if we put $\delta_k = \frac{1}{kr}(1 + \frac{1}{2} + \ldots + \frac{1}{k})$, $r = \operatorname{rank} V$, the space of global invariant jet differentials*

$$H^0(X, E_{k,m}V^* \otimes \mathcal{O}(-m\delta_k A))$$

has (many) non zero sections for $m \gg k \gg 1$ and m sufficiently divisible.

Proof. Corollary 9.39 produces a non zero section $P \in H^0(E_{k,m}^{\mathrm{GG}}V^* \otimes \mathcal{O}_X(-m\delta_k A))$ for $m \gg k \gg 1$, and the arguments given in subsection 7.D (cf. (7.30)) yield a non zero section

$$Q \in H^0(E_{k,m'}V^* \otimes \mathcal{O}_X(-m\delta_k A)), \quad m' \leqslant m.$$

By raising Q to some power p and using a section $\sigma \in H^0(X, \mathcal{O}_X(dA))$, we obtain a section

$$Q^p \sigma^{mq} \in H^0(X, E_{k,pm'}V^* \otimes \mathcal{O}(-m(p\delta_k - qd)A)).$$

One can adjust p and q so that $m(p\delta_k - qd) = pm'\delta_k$ and $pm'\delta_k A$ is an integral divisor. □

9.41. Example. In some simple cases, the above estimates can lead to very explicit results. Take for instance X to be a smooth complete intersection of multidegree $(d_1, d_2, \ldots, d_s)$ in $\mathbb{P}_{\mathbb{C}}^{n+s}$ and consider the absolute case $V = T_X$. Then $K_X = \mathcal{O}_X(d_1 + \ldots + d_s - n - s - 1)$ and one can check via explicit bounds of the error terms (cf. [Dem11], [Dem12]) that a sufficient condition for the existence of sections is

$$k \geqslant \exp\Big(7.38\, n^{n+1/2}\Big(\frac{\sum d_j + 1}{\sum d_j - n - s - a - 1}\Big)^n\Big).$$

This is good in view of the fact that we can cover arbitrary smooth complete intersections of general type. On the other hand, even when the degrees d_j tend to $+\infty$, we still get a large lower bound $k \sim \exp(7.38\, n^{n+1/2})$ on the order of jets, and this is far from being optimal: Diverio [Div08, Div09] has shown e.g. that one can take $k = n$ for smooth hypersurfaces of high degree, using the algebraic Morse inequalities of Trapani [Tra95]. The next paragraph uses essentially the same idea, in our more analytic setting. □

9.D. Non probabilistic estimate of the Morse integrals

We assume here that the curvature tensor $(c_{ij\alpha\beta})$ satisfies a lower bound

$$\sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}\xi_i\overline{\xi}_j u_\alpha \overline{u}_\beta \geqslant -\sum \gamma_{ij}\xi_i\overline{\xi}_j\;|u|^2, \qquad \forall \xi \in T_X,\; u\in V \tag{9.42}$$

for some semipositive $(1,1)$-form $\gamma = \frac{i}{2\pi}\sum \gamma_{ij}(z)\,dz_i\wedge d\overline{z}_j$ on X. This is the same as assuming that the curvature tensor of (V^*,h^*) satisfies the semipositivity condition

$$\Theta_{V^*,h^*} + \gamma\otimes \mathrm{Id}_{V^*} \geqslant 0 \tag{9.42$'$}$$

in the sense of Griffiths, or equivalently $\Theta_{V,h} - \gamma\otimes\mathrm{Id}_V \leqslant 0$. Thanks to the compactness of X, such a form γ always exists if h is an admissible metric on V. Now, instead of replacing Θ_V with its trace free part $\widetilde{\Theta}_V$ and exploiting a Monte Carlo convergence process, we replace Θ_V with $\Theta_V^\gamma = \Theta_V - \gamma\otimes\mathrm{Id}_V \leqslant 0$, i.e. $c_{ij\alpha\beta}$ by $c^\gamma_{ij\alpha\beta} = c_{ij\alpha\beta} + \gamma_{ij}\delta_{\alpha\beta}$. Also, we take a line bundle $F = A^{-1}$ with $\Theta_{A,h_A}\geqslant 0$, i.e. F seminegative. Then our earlier formulas (9.28), (9.35), (9.36) become instead

$$g_k^\gamma(z,x,u) = \frac{i}{2\pi}\sum_{1\leqslant s\leqslant k}\frac{1}{s}x_s\sum_{i,j,\alpha,\beta} c^\gamma_{ij\alpha\beta}(z)\,u_{s\alpha}\overline{u}_{s\beta}\,dz_i\wedge d\overline{z}_j \geqslant 0, \tag{9.43}$$

$$L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1)\otimes \pi_k^*\mathcal{O}\Big(-\frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)A\Big), \tag{9.44}$$

$$\Theta_{L_k} = \eta_k(z,x,u) = \frac{1}{\frac{1}{kr}(1+\frac{1}{2}+\ldots+\frac{1}{k})} g_k^\gamma(z,x,u) - (\Theta_{A,h_A}(z) + r\gamma(z)). \tag{9.45}$$

In fact, replacing Θ_V by $\Theta_V - \gamma\otimes\mathrm{Id}_V$ has the effect of replacing $\Theta_{\det V^*} = \mathrm{Tr}\,\Theta_{V^*}$ by $\Theta_{\det V^*} + r\gamma$. The major gain that we have is that $\eta_k = \Theta_{L_k}$ is now expressed as a difference of semipositive $(1,1)$-forms, and we can exploit the following simple lemma, which is the key to derive algebraic Morse inequalities from their analytic form (cf. [Dem94], Theorem 12.3).

9.46. Lemma. *Let $\eta = \alpha - \beta$ be a difference of semipositive $(1,1)$-forms on an n-dimensional complex manifold X, and let $\mathbb{1}_{\eta,\leqslant q}$ be the characteristic function of the open set where η is non degenerate with a number of negative eigenvalues at most equal to q. Then*

$$(-1)^q\mathbb{1}_{\eta,\leqslant q}\;\eta^n \leqslant \sum_{0\leqslant j\leqslant q}(-1)^{q-j}\alpha^{n-j}\beta^j,$$

in particular

$$\mathbb{1}_{\eta,\leqslant 1}\;\eta^n \geqslant \alpha^n - n\alpha^{n-1}\wedge\beta \qquad \textit{for } q=1.$$

Proof. Without loss of generality, we can assume $\alpha > 0$ positive definite, so that α can be taken as the base hermitian metric on X. Let us denote by

$$\lambda_1\geqslant\lambda_2\geqslant\ldots\geqslant\lambda_n\geqslant 0$$

the eigenvalues of β with respect to α. The eigenvalues of $\eta = \alpha-\beta$ are then given by

$$1-\lambda_1\leqslant\ldots\leqslant 1-\lambda_q\leqslant 1-\lambda_{q+1}\leqslant\ldots\leqslant 1-\lambda_n,$$

hence the open set $\{\lambda_{q+1}<1\}$ coincides with the support of $\mathbb{1}_{\eta,\leqslant q}$, except that it may also contain a part of the degeneration set $\eta^n = 0$. On the other hand we have

$$\binom{n}{j}\alpha^{n-j}\wedge\beta^j = \sigma_n^j(\lambda)\,\alpha^n,$$

where $\sigma_n^j(\lambda)$ is the j-th elementary symmetric function in the λ_j's. Thus, to prove the lemma, we only have to check that

$$\sum_{0\leqslant j\leqslant q}(-1)^{q-j}\sigma_n^j(\lambda) - \mathbb{1}_{\{\lambda_{q+1}<1\}}(-1)^q\prod_{1\leqslant j\leqslant n}(1-\lambda_j)\geqslant 0.$$

This is easily done by induction on n (just split apart the parameter λ_n and write $\sigma_n^j(\lambda) = \sigma_{n-1}^j(\lambda) + \sigma_{n-1}^{j-1}(\lambda)\,\lambda_n$). □

We apply here Lemma 9.46 with

$$\alpha = g_k^\gamma(z,x,u), \qquad \beta = \beta_k = \frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)(\Theta_{A,h_A}+r\gamma),$$

which are both semipositive by our assumption. The analogue of (9.32) leads to

$$\begin{aligned}&\int_{X_k^{\mathrm{GG}}(L_k,\leqslant 1)} \Theta_{L_k,\Psi^*_{h,p,\varepsilon}}^{n+kr-1}\\ &\quad= \frac{(n+kr-1)!}{n!(k!)^r(kr-1)!}\int_{z\in X}\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k} \mathbb{1}_{g_k^\gamma-\beta_k,\leqslant 1}\,(g_k^\gamma-\beta_k)^n\,d\nu_{k,r}(x)\,d\mu(u)\\ &\quad\geqslant \frac{(n+kr-1)!}{n!(k!)^r(kr-1)!}\int_{z\in X}\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k} ((g_k^\gamma)^n - n(g_k^\gamma)^{n-1}\wedge\beta_k)\,d\nu_{k,r}(x)\,d\mu(u).\end{aligned}$$

The resulting integral now produces a "closed formula" which can be expressed solely in terms of Chern classes (at least if we assume that γ is the Chern form of some semipositive line bundle). It is just a matter of routine to find a sufficient condition for the positivity of the integral. One can first observe that g_k^γ is bounded from above by taking the trace of $(c_{ij\alpha\beta})$, in this way we get

$$0\leqslant g_k^\gamma \leqslant \Big(\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\Big)(\Theta_{\det V^*}+r\gamma)$$

where the right hand side no longer depends on $u\in(S^{2r-1})^k$. Also, g_k^γ can be written as a sum of semipositive $(1,1)$-forms

$$g_k^\gamma = \sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\theta^\gamma(u_s), \qquad \theta^\gamma(u) = \sum_{i,j,\alpha,\beta} c^\gamma_{ij\alpha\beta}u_\alpha\overline{u}_\beta\,dz_i\wedge d\overline{z}_j,$$

hence for $k\geqslant n$ we have

$$(g_k^\gamma)^n \geqslant n!\sum_{1\leqslant s_1<\ldots<s_n\leqslant k}\frac{x_{s_1}\ldots x_{s_n}}{s_1\ldots s_n}\,\theta^\gamma(u_{s_1})\wedge\theta^\gamma(u_{s_2})\wedge\ldots\wedge\theta^\gamma(u_{s_n}).$$

Since $\int_{S^{2r-1}}\theta^\gamma(u)\,d\mu(u) = \frac{1}{r}\,\mathrm{Tr}(\Theta_{V^*}+\gamma) = \frac{1}{r}\Theta_{\det V^*}+\gamma$, we infer from this

$$\begin{aligned}&\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k}(g_k^\gamma)^n\,d\nu_{k,r}(x)\,d\mu(u)\\ &\qquad\geqslant n!\sum_{1\leqslant s_1<\ldots<s_n\leqslant k}\frac{1}{s_1\ldots s_n}\Big(\int_{\Delta_{k-1}}x_1\ldots x_n\,d\nu_{k,r}(x)\Big)\Big(\frac{1}{r}\Theta_{\det V^*}+\gamma\Big)^n.\end{aligned}$$

By putting everything together, we conclude:

9.47. Theorem. *Assume that $\Theta_{V^*}\geqslant -\gamma\otimes\mathrm{Id}_{V^*}$ with a semipositive $(1,1)$-form γ on X. Then the Morse integral of the line bundle*

$$L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1)\otimes\pi_k^*\mathcal{O}\Big(-\frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)A\Big), \qquad A\geqslant 0$$

satisfies for $k\geqslant n$ the inequality

$$\begin{aligned}&\frac{1}{(n+kr-1)!}\int_{X_k^{\mathrm{GG}}(L_k,\leqslant 1)}\Theta_{L_k,\Psi^*_{h,p,\varepsilon}}^{n+kr-1}\\ (*)\qquad &\geqslant \frac{1}{n!(k!)^r(kr-1)!}\int_X c_{n,r,k}\big(\Theta_{\det V^*}+r\gamma\big)^n - c'_{n,r,k}\big(\Theta_{\det V^*}+r\gamma\big)^{n-1}\wedge\big(\Theta_{A,h_A}+r\gamma\big)\end{aligned}$$

where

$$c_{n,r,k} = \frac{n!}{r^n}\Big(\sum_{1\leqslant s_1<\ldots<s_n\leqslant k}\frac{1}{s_1\ldots s_n}\Big)\int_{\Delta_{k-1}} x_1\ldots x_n\,d\nu_{k,r}(x),$$

$$c'_{n,r,k} = \frac{n}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)\int_{\Delta_{k-1}}\Big(\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\Big)^{n-1}d\nu_{k,r}(x).$$

Especially we have a lot of sections in $H^0(X_k^{\rm GG}, mL_k)$, $m \gg 1$, as soon as the difference occurring in $(*)$ is positive.

The statement is also true for $k < n$, but then $c_{n,r,k} = 0$ and the lower bound $(*)$ cannot be positive. By Corollary 9.11, it still provides a non trivial lower bound for $h^0(X_k^{\rm GG}, mL_k) - h^1(X_k^{\rm GG}, mL_k)$, though. For $k \geqslant n$ we have $c_{n,r,k} > 0$ and $(*)$ will be positive if $\Theta_{\det V^*}$ is large enough. By Formula 9.20 we have

$$c_{n,r,k} = \frac{n!\,(kr-1)!}{(n+kr-1)!}\sum_{1\leqslant s_1<\ldots<s_n\leqslant k}\frac{1}{s_1\ldots s_n} \geqslant \frac{(kr-1)!}{(n+kr-1)!}, \tag{9.48}$$

(with equality for $k = n$), and by ([Dem11], Lemma 2.20 (b)) we get the upper bound

$$\frac{c'_{n,k,r}}{c_{n,k,r}} \leqslant \frac{(kr+n-1)r^{n-2}}{k/n}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^n\Bigg[1+\frac{1}{3}\sum_{m=2}^{n-1}\frac{2^m(n-1)!}{(n-1-m)!}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^{-m}\Bigg].$$

The case $k = n$ is especially interesting. For $k = n \geqslant 2$ one can show (with $r \leqslant n$ and H_n denoting the harmonic sequence) that

$$\frac{c'_{n,k,r}}{c_{n,k,r}} \leqslant \frac{n^2+n-1}{3}n^{n-2}\exp\Big(\frac{2(n-1)}{H_n}+n\log H_n\Big) \leqslant \frac{1}{3}\big(n\log(n\log 24n)\big)^n. \tag{9.49}$$

We will later need the particular values that can be obtained by direct calculations (cf. Formula (9.20 and [Dem11, Lemma 2.20]).

$$c_{2,2,2} = \frac{1}{20},\qquad c'_{2,2,2} = \frac{9}{16},\qquad \frac{c'_{2,2,2}}{c_{2,2,2}} = \frac{45}{4}, \tag{9.50$_2$}$$

$$c_{3,3,3} = \frac{1}{990},\qquad c'_{3,3,3} = \frac{451}{4860},\qquad \frac{c'_{3,3,3}}{c_{3,3,3}} = \frac{4961}{54}. \tag{9.50$_3$}$$

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Jean-Pierre Demailly
Université de Grenoble-Alpes, Institut Fourier (Mathématiques)
UMR 5582 du C.N.R.S., 100 rue des Maths, 38610 Gières, France
e-mail: jean-pierre.demailly@univ-grenoble-alpes.fr