CURVATURE OF METRICS ON DEMAILLY-SEMPLE TOWER

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ABSTRACT. We compute the curvature of several metrics on Demailly-Semple projective tower on a projective variety. The holomorphic Morse inequalities will then provide estimates to prove the Green-Griffiths-Lang conjecture. Our work is along the generalization and completion of the work of Demailly on this conjecture.

1. INTRODUCTION

Demailly-Semple tower associated to a pair (X, V) where X is a complex projective manifold (may be singular) and V is a holomorphic subbundle of T_X the tangent bundle of X of rank r, is a prolongation sequence of projective bundles

$$(1) \qquad \mathbb{P}^{r-1} \to X_k = P(V_{k-1}) \to X_{k-1}, \qquad k \geq 1$$

obtained inductively making X_k a weighted projective bundle over X. The sequence provides a tool to study the locus of nonconstant holomorphic maps $f : \mathbb{C} \to X$ such that $f'(t) \in V$. It is a conjecture due to Green-Griffiths-Lang that the total image of all these curves is included in a proper subvariety of X; provided (X, V) is of general type. By general type we mean K_V the canonical bundle of V is big.

Conjecture: (Green-Griffiths-Lang) [GG]
Let (X, V) be a pair where X is a projective variety of general type, and V is a holomorphic subbundle of the tangent bundle T_X. Then there should exists an algebraic subvariety $Y \subsetneqq X$ such that every nonconstant entire curve $f : \mathbb{C} \to X$ tangent to V is contained in Y. It is a basic fact by J. Semple that the map f can be lifted to maps $f_k : \mathbb{C} \to X_k$.

An alternative definition is to consider the vector bundles $E_{k,m}^{GG} V^*$ of germs of weighted homogeneous polynomials along the fibers in $X_k \to X_{k-1} \to \dots \to X$. These polynomials can be definitely considered in the germs of indeterminates $z = f$, $\xi_1 = f'$, ..., $\xi_k = f^{(k)}$, where $f^{(k)}$ is the k-th derivative of f. Then a motivation to approach the above conjecture is the following.

Theorem 1.1. *(Demailly vanishing Theorem)* [D1] *Any section of $P \in H^0(X, E_{k,m}^{GG} V^* \otimes L^{-1}) = H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_{0k}^* L^{-1})$ with L being an ample line bundle on X, satisfies an algebraic differential equation $P(f, f', ..., f^{(k)}) = 0$.*

Definition 1.2. *A smooth k-jet metric h_k on (X, V) is a hermitian metric on the line bundle $\mathcal{O}_{X_k}(-1)$ over X_k in the Green-Griffiths tower. Say h_k has negative jet curvature along V_k, if*

$$(2) \qquad \Theta_{h_k^{-1}}(\xi) \geq \epsilon \mid \xi \mid_{w_k}^2, \qquad \exists \epsilon > 0, w_k \text{ on } T_{X_k}$$

The k-jet metrics can be constructed inductively on the Green-Griffiths bundles, starting from some metric h on $V \subset T_X$. A basic example is,

$$(3) \qquad \mid (z; \xi) \mid = (\sum_s \parallel \xi_s \parallel_h^{2p/s})^{1/p}$$

where p should be chosen as a multiple of $lcm(1, 2, ...k)$

$$(4) \qquad \parallel \xi_s \parallel^2 = \sum_\alpha \mid \xi_{s\alpha} \mid^2 + \sum_{ij\alpha\beta} c_{ij\alpha\beta} z_i \bar{z}_j \xi_{s\alpha} \bar{\xi}_{s\beta}$$

Here p should be chosen as a multiple of $lcm(1, 2, ...k)$. It gives the Fubini-Study metric with its curvature in level s of the jet bundle. Denote the second term in (5) by Θ_s^{FS}. Therefore

$$(5)$$
$$\begin{aligned}
\log \mid (z, \xi) \mid &= \frac{1}{p} \log \big(\sum_s (\parallel \xi_s \parallel^2 + \Theta_s^{FS})^{p/s} = \frac{1}{p} \log \big(\sum_s \parallel \xi_s \parallel^2 (1 + \frac{\Theta_s^{FS}}{\parallel \xi_s \parallel^2})^{p/s} \big) \\
&= \frac{1}{p} \log \big(\sum_s \parallel \xi_s \parallel^2 (1 + \frac{p}{s}.\frac{\Theta_s^{FS}}{\parallel \xi_s \parallel^2} + ...) \big) \\
&= \frac{1}{p} \log \big(\sum_s \parallel \xi_s \parallel^2 + (\sum_s \parallel \xi_s \parallel^2 \frac{p}{s}.\frac{\Theta_s^{FS}}{\parallel \xi_s \parallel^2}) \big)
\end{aligned}$$

Now Apply the operator $\frac{i}{2\pi} \partial \bar{\partial}$ to obtain

$$(6) \qquad \Theta = w_{a,r,p}(\xi) + \frac{i}{2\pi} \sum_s \frac{1}{s} \frac{\mid \xi_s \mid^{2p/s}}{\sum_t \mid \xi_t \mid^{2p/t}} \sum c_{ij\alpha\beta} \frac{\xi_{s\alpha} \bar{\xi}_{s\beta}}{\mid \xi_s \mid^2} dz_i \wedge d\bar{z}_j + ...$$

(We have used the identity $\log(f + g) = \log(f) + (g/f) - ....$). The next motivation due to Demailly is the holomorphic Morse inequalities;

Theorem 1.3. *(Holomorphic Morse inequalities-J. Demaily) If $E \to X$ is a holomorphic vector bundle over a compact complex manifold X, and (L, h) a hermitian line bundle, then*

$$(7) \qquad \sum_{0 \le j \le q} (-1)^{q-j} h^j(X, E \otimes L^k) \le r \frac{k^n}{n!} \int_{X(L,h,\le q)} (-1)^q \Theta_{L,h}^n + o(k^n)$$

where Θ denotes the curvature of the k-jet metric associated to h, and $X(q,L) = \{x \in X \mid \frac{i}{2\pi} \Theta(L) \text{ has exactly } q \text{ negative eigenvalues }\}$.

For example, from the theorem one deduces that

$$(8) \qquad h^0(X, E \otimes A^p) \ge h^0(X, E \otimes A^p) - h^1(X, E \otimes A^p) \ge r \frac{k^n}{n!} \int_{X(A,h,\le 1)} \Theta_{A,h}^n$$

If we could show that

$$(9) \qquad \int_{X(\le 1, L)} (\frac{i}{2\pi} \Theta(L))^n > 0$$

then some high power of L twisted by E has many non-zero sections. An example of this application is the following.

Theorem 1.4. [D1] *Lets (X, V) be a directed projective variety such that K_V is big, and let A be an ample divisor. Then for $k >> 1$ and $\delta \in \mathbb{Q}_+$ small enough, and $\delta \le c(\log k)/k$, the number of sections $h^0(X, E_{k,m}^{GG} \otimes \mathcal{O}(-m\delta A))$ has maximal growth, i.e is larger than $c_k m^{n+kr-1}$ for some $m \ge m_k$, where $c, c_k > 0, n = \dim(X), r = \mathrm{rank}(V)$. In particular entire curves $f : \mathbb{C} \to X$ satisfy many algebraic differential equations.*

The next motivation toward the GGL-conjecture is the singularity locus of the k-jet metrics which we denote by Σ_{h_k}. A general fact about this locus proved in [D1] is,

$$(10) \qquad \Sigma_{h_k} \subset \pi_k^{-1}(\Sigma_{h_{k-1}}) \cup D_k$$

where $D_k = P(T_{X_{k-1}/X_{k-2}}) \subset X_k$. The divisors D_k are the singularity locus of the projective jet bundle X_k and their relation with the singularity of the the k-jet metric is

$$(11) \qquad \mathcal{O}_{X_k}(1) = \pi_k^* \mathcal{O}_{X_{k-1}}(1) \otimes \mathcal{O}(D_k)$$

There are canonical inclusions $\mathcal{O}_{X_k}(-1) \subset \pi_k^* V_{k-1} \subset \pi_k^* \mathcal{O}_{X_{k-1}}(-1)$ which admit D_k as a zero divisor.

Theorem 1.5. *(J. P. Demaily [D1], [D2]) Let (X, V) be a compact directed manifold. If (X, V) has a k-jet metric h_k with negative jet curvature, then every entire curve $f : \mathbb{C} \to X$ tangent to V satisfies $f_k(\mathbb{C}) \subset \Sigma_{h_k}$, where Σ_{h_k} is the singularity locus of h_k.*

Assume k, m be large enough such that $H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_{0k}^* L^{-1})$ has non-zero sections $\sigma_1, ..., \sigma_N$ with base locus Z. Assume L has a smooth metric h_L with positive curvature. Then we can define a singular metric h_k' on $\mathcal{O}_{X_k}(-1)$ by

$$(12) \qquad h_k'(w, \xi) = (\sum_j |\sigma_j(w).\xi^m|^2)^{1/m}, \qquad w \in X_k, \ \xi \in \mathcal{O}_{X_k}(-1)_w$$

and $\Sigma_{h_k'} = Z$. The calculation shows

$$(13) \qquad \frac{i}{2\pi} \partial \bar{\partial} \log h_k' = \Theta_{h_k'^{-1}}(\mathcal{O}_{X_k}(-1)) \geq \frac{1}{m} \pi_{0k}^* \Theta_L$$

Summarizing all we obtain;

Corollary 1.6. *(J. P. Demaily [D1], [D2]) Assume that there exists $k, m > 0$ and an ample line bundle L on X such that $H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_{0k}^* L^{-1}) = H^0(X, E_{k,m}(V^*) \otimes L^{-1})$ has non-zero sections $\sigma_1, ..., \sigma_n$. Let $Z \subset X_k$ be the base locus of these sections. then every entire curve tangent to V is such that $f_k(\mathbb{C}) \subset \Sigma_{h_k}$. In other words, for every global $G_k$invariant polynomial differential operator P of weighted degree m and values in L^{-1}, every entire curve f must satisfy $P(f) = 0$.*

In case of use the line bundle L is defined via $\mathcal{O}_{X_k}(1)$ by a rational twist denoted by δ in Theorem 1.1. It can specifically be given by

$$(14) \qquad L_k = \mathcal{O}_{X_k}(1) \otimes \pi_k^* \mathcal{O}(\frac{1}{kr}(1 + \frac{1}{2} + ... + \frac{1}{k})L^{-1})$$

In [D1] Demailly uses the k-jet metric

$$(15) \ | \ (\xi) \ |= (\sum_s \epsilon_s \parallel \xi_s \parallel_h^{2p/s})^{1/p} = (\sum_s (\sum_\alpha | \ \xi_{s\alpha} \ |^2 + \sum_{ij\alpha\beta} c_{ij\alpha\beta} z_i \bar{z}_j \xi_{s\alpha} \bar{\xi}_{s\beta})^{p/s})^{1/p} + ...$$

on the Green-Griffiths bundle of jets (here the metric is written on a orthonormal frame of V; see the Remark below). The exponent p should be chosen as a multiple of $lcm(1, 2, ...k)$ to obtain integer exponents, and $1 >> \epsilon_1 >> \epsilon_2 >> ... >> \epsilon_k > 0$ are scaled according to a partition of unity on X such that as $\epsilon_s/\epsilon_{s-1} \to 0$ the metric converges to (3). The scaling with the epsilons also guarantees the orthogonality in the components $(\xi_1, ..., \xi_k)$ in the limit (see the reference). The curvature of this metric is calculated in [D1] as (5) and (6). Then for the volume estimation, we need to look at the integral

$$(16) \qquad \int_{X_k^{GG,q}} \Theta^{n+kr-1} = \frac{(n+kr-1)!}{n!(kr-1)!} \int_X \int_{P(1^r,...,k^r)} w_{a,r,p}^{kr-1}(\xi) 1_{\gamma_k,q}(z, \xi) \gamma_k(z, \xi)^n$$

The factor $w_{a,r,p}^{kr-1}$ does not depend to $z \in X$ and only depend universally on fibers. The expression of the curvature can be made simpler by re-scaling $x_s = |\xi_s|_h^{2p/s}$, $u_s = \xi_s/|\xi_s|_h$. The curvature finds the following formula,

$$(17) \qquad \Theta_{L_k, h_k} = w_{FS,p,k}(\xi) + \frac{i}{2\pi} \sum_s \frac{1}{s} \sum c_{ij\alpha\beta} u_{s\alpha} \bar{u}_{s\beta} dz_i \wedge d\bar{z}_j$$

where $w_{FS,p,k}(\xi)$ is positive definite on ξ. The other terms are weighted average of the values of the curvature $\Theta_{V,h}$ on vectors u_s in the sphere bundle $S(V)$. The weighted projective space is a quotient of the pseudo-sphere $\sum |\xi_s|^{2p/s} = 1$. Thus the second sum in the curvature is of the form $\sum \frac{1}{s} \gamma(u_s)$ where u_s are essentially random variables on the sphere. Because γ is quadratic

$$(18) \qquad \int_{u \in S(V)} \gamma(u)^n = \left(\sum_s \frac{1}{s}\right)^n Tr(\gamma)$$

The integral becomes

$$(19) \qquad \frac{(n+kr-1)!}{n!(k!)^r(kr-1)!} \frac{1}{(kr)^n}\left(1 + \frac{1}{2} + ... + \frac{1}{k}\right)^n \int_X 1_{\eta,q} \eta^n + ...$$

where $\eta = \Theta_{det(V^*),h^*} + \Theta_{L,h_L}$.

Remark 1.7. *If X is a projective variety and E a hermitian vector bundle on X with hermitian metric h, such that in an orthonormal frame (e_λ) around $x \in X$ the metric can be written as*

$$(20) \qquad h(e_\lambda, e_\mu) = \delta_{\lambda,\mu} - \sum_{j,k} c_{jk\lambda\mu} z_j \bar{z}_k + O(|z^3|)$$

This fact has been used in (17). We will also employ this formula for the test metrics on invariant jets in the next section.

Remark 1.8. *Forgetting about the complications in the metrics in order to define positive k-jet metrics on Green-Griffiths bundle one can proceed as follows. It is possible to choose the metric h on X such that the curvature of the canonical bundle $K_V = \bigwedge^r V^*$ is given by a Kahler form, i.e.*

$$(21) \qquad \Theta_{K_V, det(h^*)} = w > 0$$

Then $\Theta_{det(V),det(h)} = -w$. In the fibration sequence

$$\mathcal{O}_{X_1}(-1) \;\hookrightarrow\; \pi_1^* V$$
$$\downarrow$$
(22) $\qquad\qquad\qquad\qquad\qquad X_1$
$$\downarrow$$
$$X$$

One can consider a metric with curvature $\epsilon_1 \Theta_{\mathcal{O}_X(-1)} + \pi_1^* w > 0$ for $\epsilon_1 << 1$ on $\mathcal{O}_{X_1(-1)}$. Repeating this argument inductively, one obtains a metric on $\mathcal{O}_{X_k}(-1)$ with curvature

$$(23) \quad \Theta_{\mathcal{O}_{X_k}(-1),\vartheta} = \sum_j \epsilon_1 ... \epsilon_j \pi_{jk}^* \Theta_{\mathcal{O}_{X_j}(-1)} + \pi_{0k}^* w > 0 \;, \qquad ... << \epsilon_2 << \epsilon_1 << 1$$

However the metric ϑ is hard to analyze, because of adding the curvature to the metric at each step.

2. INVARIANT JETS VS INVARIANT METRICS

J. P. Demaily [D2] develops the ideas in [GG] and considers the jets of differentials that are also invariant under change of coordinate on $\mathbb{C}$. In fact if we consider the bundle $J_k \to X$ of germs of parametrized curves in X. Its fiber at $x \in X$ is the set of equivalence classes of germs of holomorphic maps $f : (\mathbb{C}, 0) \to (X, x)$ with equivalence relation $f^{(j)}(0) = g^{(j)}(0)$, $0 \le j \le k$. By choosing local holomorphic coordinates around x, the elements of the fiber $J_{k,x}$ can be represented by the Taylor expansion

$$(24) \qquad\qquad f(t) = x + t f'(0) + \frac{t^2}{2!} f''(0) + ... + \frac{t^k}{k!} f^{(k)}(0) + O(t^{k+1})$$

Setting $f = (f_1, ..., f_n)$ on open neighborhoods of $0 \in \mathbb{C}$, the fiber is

$$(25) \qquad\qquad\qquad J_{k,x} = \{(f'(0), ..., f^{(k)}(0))\} = \mathbb{C}^{nk}$$

Lets G_k be the group of local reparametrizations of $(\mathbb{C}, 0)$

$$(26) \qquad\qquad t \longmapsto \phi(t) = \alpha_1 t + \alpha_2 t^2 + ... + \alpha_k t^k, \qquad \alpha_1 \in \mathbb{C}^*$$

Its action on the k-jets is given by the following matrix multiplication

$$(27) \quad [f'(0), f''(0)/2!, ..., f^{(k)}(0)/k!]. \begin{bmatrix} a_1 & a_2 & a_3 & ... & a_k \\ 0 & a_1^2 & 2a_1a_2 & ... & a_1a_{k-1} + ...a_{k-1}a_1 \\ 0 & 0 & a_1^3 & ... & 3a_1^2 a_{k-2} + ... \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & a_1^k \end{bmatrix}$$

Let $E_{k,m}$ be the Demaily-Semple bundle whose fiber at x consists of U_k-invariant polynomials on the fiber coordinates of J_k at x of weighted degree m. Set $E_k = \bigoplus_m E_{k,m}$, the Demaily-Semple bundle of graded algebras of invariants. Then one needs to work out the aforementioned calculations with an invariant metric. Toward this we examine the following metrics

Test 1: We replace the metric in (17) or (18) by

(28)
$$| (z; \xi) | = (\sum_s \epsilon_s \| \mathbf{W}_s \|_h^{2p/s(s+1)})^{1/p} = (\sum_s (\sum_{l=1}^s | W_l |^2 + \sum_l \sum_{ij\alpha\beta} c_{ij\alpha\beta} z_i \bar{z}_j \xi_{l\alpha} \bar{\xi}_{l\beta})^{p/s(s+1)})^{1/p}$$

where $\| \mathbf{W}_s \| = \sum_{l=1}^s | W_l |^2$. We have used the Remark 1.4 for the replacement of the curvature at each stage and the formula $W_l = \xi_1 \wedge ... \wedge \xi_l$. Taking the logarithm and using the identity $\log(f - g) = \log f + f/g + ...$ and applying $\frac{i}{2\pi} \partial \bar{\partial}$ similar to the former case we obtain

(29)
$$\Theta = \sum_s | \mathbf{W}_s |^{p/s(s+1)} + \frac{i}{2\pi} \sum_s \frac{1}{s(s+1)} \frac{| \mathbf{W}_s |^{2p/s(s+1)}}{\sum_t | \mathbf{W}_t |^{2p/t(t+1)}} \sum_l \sum c_{ij\alpha\beta} \xi_{s\alpha} \bar{\xi}_{s\beta} dz_i \wedge d\bar{z}_j$$

In this case one estimates

$$(30) \quad \int_{X_k^{GG},q} \Theta^{n+k(r-1)} = \frac{(n+k(r-1))!}{n!(k(r-1))!} \int_X \int_{P(1^r,...,k^r)} Q(\xi) 1_{\gamma_k,q}(z,\xi) \gamma_k(z,\xi)^n$$

where γ is the second term in the sum above and Q is function depending only to ξ. Doing the same change of coordinates and re-scaling one receives to consider expected values for

$$(31) \quad \int \gamma_k^n = \frac{1}{kr} \sum_s \frac{1}{s(s+1)} \frac{i}{2\pi} \sum_{l=1}^s \sum_{ij\alpha} c_{ij\alpha\alpha}(z) dz_i d\bar{z}_j$$

which gives similar estimates as before. We formulate it below.

Corollary 2.1. *The lifts of the entire curve locus for projective directed (X, V) along the Demaily-Semple bundles are contained in the zero set of all Wronskians*

$$(32) \qquad\qquad\qquad W_1 = W_2 = ... = W_k = ... = 0$$

Proof. The theorem follows from Theorem 1.2 and the convergence in (32), followed by the methodology in [D1] which is exactly the same as that of Demailly metric in Section 1. □

Test 2: We test the following metric

$$(33) \qquad\qquad\qquad \sum_{s=1}^{k} \epsilon_s (\sum_{\alpha} \mid P_\alpha(\xi) \mid^2)^{p/w(P_\alpha)})^{1/p}$$

where the P_α are a set of invariant polynomials in jet coordinates. The effect of this is then, the Demailly-Semple locus of the lifts of entire curves should be contained in

$$(34) \qquad\qquad\qquad P_\alpha = 0, \qquad \forall \alpha$$

For instance a choice of P_α's could be the Wronskians (Test 1). However we slightly try to do some better choice. First lets make some correspondence between invariant jets with non-invariant ones. Lets consider a change of coordinates on the $\mathbb{C}$ by

$$(35) \qquad \xi = (f_1, ..., f_r) \longmapsto (f_1 \circ f_1^{-1}, ..., f_r \circ f_1^{-1}) = (t, g_2, ..., g_r) = \eta$$

This makes the first coordinate to be the identity and the other components to be invariant by any change of coordinates on $\mathbb{C}$. If we differentiate in the new coordinates successively, then all the resulting fractions are invariant of degree 0

$$(36) \qquad g_2' = \frac{f_2'}{f_1'^2}, \qquad g_2'' = \frac{P_2 = f_1' f_2'' - f_2' f_1'' = W(f_1, f_2)}{f_1'^3}, ...$$

We take the P_α's to be all the polynomials that appear in the numerators of the components when we successively differentiate (36) with respect to t. An invariant metric in the first coordinates corresponds to a usual metric in the second one subject to the condition that we need to make the average under the unitary change of coordinates in V. This corresponds to change of coordinates on the manifold X as

$$(37) \qquad (\psi \circ f)^{(k)}(0) = \psi'(0).f^{(k)} + \text{higher order terms according to epsilons}$$

That is the effect of the change of variables in X has only effect as the first derivative by composition with a linear map, up to the epsilon factors. Therefore the above metric becomes similar to the metric used in [D1] in the new coordinates produced by g's,

$$(38) \quad |(z;\xi)| \sim \left(\sum_s \epsilon_s \parallel \eta_s.(\eta_{11})^{2s-1} \parallel_h^{p/(2s-1)}\right)^{1/p} = \left(\sum_s \epsilon_s \parallel \eta_s \parallel_h^{p/(2s-1)}\right)^{1/p} |\eta_{11}|$$

where the weight of η_s can be seen by differentiating (36) to be equal $(2s-1)$ inductively. We need to modify the metric in (39) slightly to be invariant under hermitian transformations of the vector bundle V. In fact the role of η_{11} can be done by any other η_{1i} or even any other non-sero vector. To fix this we consider

$$(39) \quad |(z;\xi)| = \int_{\parallel v \parallel_1 = 1} \sum_s \epsilon_s \parallel \eta_s \parallel_h^{p/(2s-1)})^{1/p} |< \eta_1.v >|^2$$

where the integration only affects the last factor making average over all vectors in $v \in V$. This will remove the the former difficulty. The curvature is the same as for the metric in (17) but only an extra contribution from the last factor,

$$(40)$$
$$\gamma_k(z,\eta) = \frac{i}{2\pi}\left(w_{r,p}(\eta) + \sum_{lm\alpha} b_{lm\alpha}\left(\int_{\parallel v \parallel_1 = 1} v_\alpha\right)dz_l \wedge d\bar{z}_m + \sum_s \frac{1}{s}\frac{|\eta_s|^{2p/s}}{\sum_t |\eta_t|^{2p/t}}\sum c_{ij\lambda\mu}\frac{\eta_{s\alpha}\bar{\eta}_{s\beta}}{|\eta_s|^2}dz_i \wedge d\bar{z}_j\right)$$

The contribution of the factor $|\eta_{11}|$ can be understood as the curvature of the sub-bundle of V which is orthogonal complement to the remainder. Thus

$$(41) \quad\quad\quad\quad\quad\quad b_{lm\alpha} = c_{lm\alpha\alpha}$$

where c_{lm11} is read from the coefficients of the curvature tensor of (V, w^{FS}) the Fubini-Study metric on V (the second factor in (40)). Then we need to look at the integral

$$(42) \quad \int_{X_{k,q}} \Theta^{n+k(r-1)} = \frac{(n+k(r-1))!}{n!(k(r-1))!}\int_X \int_{P(1^r,...,k^r)} w_{a,r,p}^{k(r-1)}(\eta) 1_{\gamma_k,q}(z,\eta)\gamma_k(z,\eta)^n$$

In the course of evaluating with the Morse inequalities the curvature form is replaced by the trace of the above tensor in raising to the power $n = \dim X$, then if we use polar coordinates

$$(43) \quad\quad\quad\quad x_s = \parallel \eta_s \parallel^{2p/s}, \quad\quad u_s = \eta_s / \parallel \eta_s \parallel$$

Then the value of the curvature when integrating over the sphere yields the following

$$(44) \qquad \gamma_k = \frac{i}{2\pi}\Big(\sum_{lm} b_{lm\alpha} dz_l \wedge d\bar{z}_m + \sum_s \frac{1}{s}\sum c_{ij\lambda\lambda} u_{s\lambda}\bar{u}_{s\lambda} dz_i \wedge d\bar{z}_j\Big)$$

Because the first term is a finite sum with respect to s, the estimates for this new form would be essentially the same as those in [D1]. Therefore one expects

$$(45) \qquad \int_{X_{k,q}} \Theta^{n+k(r-1)} = \frac{(\log k)^n}{n!(k!)^r}\Big(\int_X 1_{\gamma,q}\gamma^n + O((\log k)^{-1})\Big)$$

similar to [D1], for non-invariant case. We have proved the following.

Corollary 2.2. *The analogue of Theorem 1.4 holds for the bundle $E_{k,m}^{GG}$ replaced by $E_{k,m}$.*

3. Existence of global dual differential operators

In [M] J. Merker proves the Green-Griffiths-Lang conjecture for a generic hypersurface in $\mathbb{P}^{n+1}$. He proves for $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ of degree d as a generic member in the universal family

$$(46) \qquad \mathfrak{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{\left(\frac{n+1+d)!}{(n+1)!d!}\right)-1}$$

parametrizing all such hypersurfaces, the GGL-conjecture holds. His method uses a theorem of Y. T. Siu, as the following.

Theorem 3.1. *(Y. T. Siu (2004) [S], [?]) Let X be a general hypersurface in $\mathbb{P}^{n+1}$ as explained above. Then, there are two constants $c_n \geq 1$ and $c'_n \geq 1$ such that the twisted tangent bundle*

$$(47) \qquad T_{J_{vert}^n(\mathfrak{X})} \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^{n+1}}(c_n) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^{\left(\frac{n+1+d)!}{(n+1)!d!}\right)-1}}(c'_n)$$

where π_1, π_2 are the projections, is generated at every point by its global sections.

The proof by Merker, the above Theorem is established outside a certain exceptional algebraic subset $\Sigma \subset J_{\text{vert}}^n(\mathfrak{X})$ defined by vanishing of certain Wronskians. In order to give a similar proof of GGL conjecture for general X one may use the following generalization.

Question: If $X \subset \mathbb{P}^{n+1}$ be a generic member of a family $\mathfrak{X}$ of projective varieties, then there are constants c_n and c'_n such that

$$(48) \qquad T_{J_{\text{vert}}^n(\mathfrak{X})} \otimes \mathcal{O}_{\mathfrak{X}_k}(c_n) \otimes \pi_{0k}^* L^{c'_n}$$

is generated at every point by its global sections, where L is an ample line bundle on $\mathfrak{X}$. By the analogy between microlocal differential operators and formal polynomials on the symmetric tensor algebra it suffices to show

$$(49) \qquad H^0(X_k, Sym^{\leq m'}\tilde{V}_k \otimes \mathcal{O}_{X_k}(m) \otimes \pi_{0k}^* B) \neq 0, \qquad m' >> m >> k$$

where $\tilde{V}_k$ is the in-homogenized V_k as acting as differential operators in first order. We also wish to work over the Demailly-Semple bundle of invariant jets. To this end by a similar procedure as the former case one may check the holomorphic Morse estimates applied to the following metric on the symmetric powers.

$$(50) \qquad |(z,\xi)| = \Big(\sum_{s=1}^{k} \epsilon_s \Big(\sum_{u_i \in S^s V^*} |W^s_{u_1,...,u_s}|^2 + \sum_{iju_\alpha u_\beta} C_{iju_\alpha u_\beta} z_i \bar{z}_j u_\alpha \bar{u}_\beta \Big)^{p/s(s+1)} \Big)^{1/p}$$

where $W^s_{u_1,...,u_s}$ is the Wronskian

$$(51) \qquad W^s_{u_1,...,u_s} = W(u_1 \circ f, ..., u_s \circ f)$$

and we regard the summand front the ϵ_s as a metric on $S^s V^*$. The coefficient $C_{iju_\alpha u_\beta}$ we are going to compute. Moreover the frame $\langle u_i \rangle$ is chosen of monomials to be holomorphic and orthonormal at 0 dual to the frame $\langle e^\alpha = \sqrt{l!/\alpha!}e_1^{\alpha_1}...e_r^{\alpha_r}\rangle$.

The scaling of the basis in $S^l V^*$ is to make the fame to be orthonormal and are calculated as follows;

$$(52)$$

$$\langle e^\alpha, e^\beta \rangle = \langle \sqrt{l!/\alpha!}e_1^{\alpha_1}...e_r^{\alpha_r}, \sqrt{l!/\alpha!}e_1^{\beta_1}...e_r^{\beta_r} \rangle = \sqrt{1/\alpha!\beta!}\langle \prod_{i=1}^{l} e_{\eta(i)}, \sum_{\sigma \in S_l} \prod_{i=1}^{l} e_{\eta \circ \sigma(i)} \rangle$$

via the embedding $S^l V^* \hookrightarrow V^{\otimes l}$ and the map $\eta : \{1....l\} \to \{1....r\}$ taking the value i the α_i times.

To compute the coefficients of the curvature tensor we proceed as follows. Because the frame $\langle e_\lambda \rangle$ of V were chosen to be orthonormal at the given point $x \in X$, substituting

$$(53) \qquad \langle e_\lambda, e_\mu \rangle = \delta_{\lambda\mu} + \sum_{ij\lambda\mu} c_{ij\lambda\mu} z_i \bar{z}_j + ...$$

It follows that

$$(54) \qquad \langle e^\alpha, e^\beta \rangle = \sqrt{1/\alpha!\beta!}(\delta_{\alpha\beta} + \sum_{\eta \circ \sigma(i)=\eta(i)} c_{ij\alpha_{\eta(i)}\beta_{\eta(i)}} z_i \bar{z}_j + ...$$

which explains the scalars $C_{iju_\alpha u_\beta}$ in terms of curvature of the metric on V. In the estimation of the volume

(55)
$$\int_{X_k} \Theta^{n+k(r-1)} = \int_{X_k} (\Theta_{\text{vert}} + \Theta_{\text{hor}})^{n+k(r-1)} = \frac{(n+k(r-1))!}{n!(k(r-1))!} \int_X \int_{\mathbb{P}(1^{[r]},...,k^{[r]})} \Theta_{\text{vert}}^{k(r-1)} \Theta_{hor}^n$$

However the calculations with Θ involves more complicated estimates. The above metric reflects some ideas of D. Brotbek [B], on Wronskian ideal sheaves. It would be interesting to check out if the Monte Carlo process would converge with this metric.

Lets see what is the conclusion of our discussions and the aforementioned metrics. As was mentioned the Theorem 1.1 has been stated and proved in [D1] and [D2] as an application of holomorphic Morse inequalities using the metric (18). Our methodology of using invariant metrics proves the same result for the invariant metrics on Demailly-Semple bundles. The convergence of the Monte-Carlo integrals in (32) and (45) implies the possibility to apply holomorphic Morse inequalities in the new case. The conclusion is that, the Theorem 1.1 is also true for Demailly-Semple jet differentials, i.e with $E_{k,m}^{GG}$ replaced by $E_{k,m}$ (see [D1] for definitions).

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